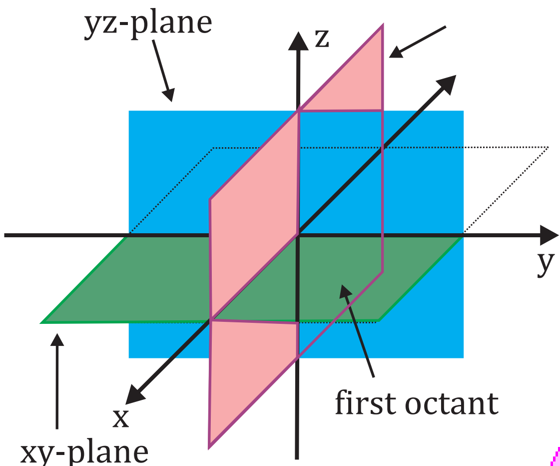

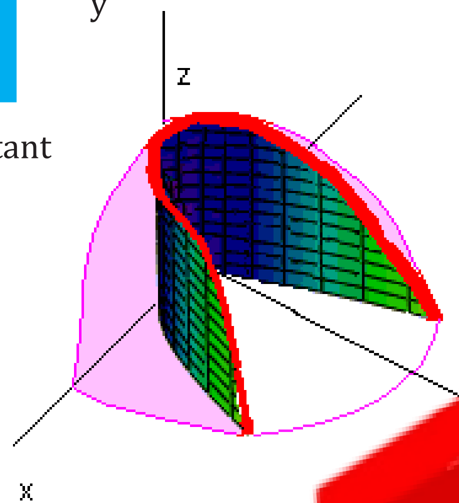

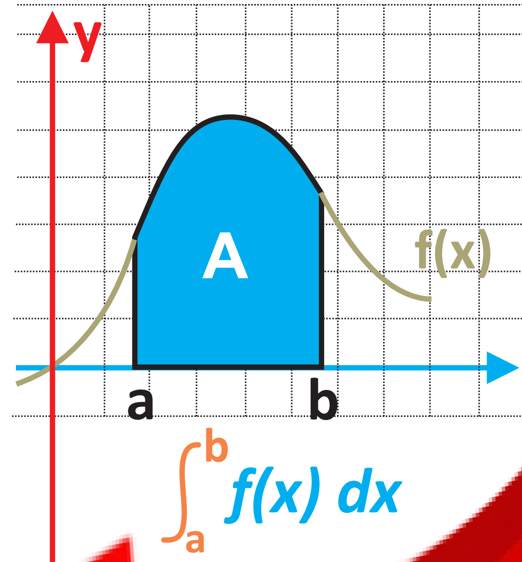

Integral Calculus, 3D Geometry and Vector Booster

with Problems & Solutions

for

JEE

Main and Advanced

McGraw Hill Education

Rejaul Makshud

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About the Author

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M. Sc. (Calcutta University, Kolkata)

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Dedicated to
My Beloved Mom and Dad

Preface

INTEGRAL CALCULUS, 3D GEOMETRY & VECTOR BOOSTER with Problems & Solutions for JEE Main and Advanced is meant for aspirants preparing for the entrance examinations of different technical institutions, especially NIT/IIT/BITSAT/IISc. In writing this book, I have drawn heavily from my long teaching experience at National Level Institutes. After many years of teaching I have realised the need of designing a book that will help the readers to build their base, improve their level of mathematical concepts and enjoy the subject.

This book is designed keeping in view the new pattern of questions asked in JEE Main and Advanced Exams. It has six chapters. Each chapter has the concept booster followed by a large number of exercises with the exact solutions to the problems as given below:

Level - I	:	Problems based on Fundamentals
Level - II	:	Mixed Problems (Objective Type Questions)
Level - III	:	Problems for JEE Advanced
Level - IV	:	Tougher problems for JEE Advanced
(0.......9)	:	Integer type Questions
Passages	:	Comprehensive Link Passages
Matching	:	Matrix Match
Reasoning	:	Assertion and Reason
Previous years' papers	:	Questions asked in Previous Years' IIT-JEE Exams

Remember friends, no problem in mathematics is difficult. Once you understand the concept, they will become easy. So please don't jump to exercise problems before you go through the Concept Booster and the objectives. Once you are confident in the theory part, attempt the exercises. The exercise problems are arranged in a manner that they gradually require advanced thinking.

I hope this book will help you to build your base, enjoy the subject and improve your confidence to tackle any type of problem easily and skillfully.

My special thanks goes to Mr. M.P. Singh (IISc. Bangalore), Mr. Yogesh Sindhwani (Head of School, Lancers International School, Gurugram), Mr. Manoj Kumar (IIT, Delhi), Mr. Nazre Hussain (B.Tech.), Dr. Syed Kashan Ali (MBBS) and Mr. Shahid Iyqbal, who have helped, inspired and motivated me to accomplish this task. As a matter of fact, teaching being the best learning process, I must thank all my students who inspired me most for writing this book.

I would like to convey my affectionate thanks to my wife, who helped me immensely and my children who bore with patience my neglect during the period I remained devoted to this book.

I also convey my sincere thanks to Mr Biswajit Das of McGraw Hill Education for publishing this book in such a beautiful format.

I owe a special debt of gratitude to my father and elder brother, who taught me the first lesson of Mathematics and to all my learned teachers—Mr. Swapan Halder, Mr. Jadunandan Mishra, Mr. Mahadev Roy and Mr. Dilip Bhattacharya, who instilled the value of quality teaching in me.

I have tried my best to keep this book error-free. I shall be grateful to the readers for their constructive suggestions toward the improvement of the book.

Rejaul Makshud
M.Sc. (Calcutta University, Kolkata)

Contents

CHAPTER

1 Indefinite Integrals

CONCEPT BOOSTER

1. Definition

The inverse process of differentiation is called integration.

Let $g(x)$ be a differentiable function of x such that $\frac{d}{dx}(g(x) + c) = f(x)$.

Then $\int f(x)\, dx = g(x) + c$.

Thus $g(x)$ is called a primitive or anti-derivative or an indefinite integral or simply integral of $f(x)$ with respect to x, where $f(x)$ is called the integrand, c is called the constant of integration.

2. Geometrical Interpretation of Integration

Let $f(x)$ be a given continuous function and $g(x)$ one of its anti-derivatives such that $y = \int f(x)\, dx = g(x) + c$.

If $y = \int f(x)\, dx = g(x) + c$, then $y = g(x) + c$ represents a family of parallel curves.

3. Basic Formulae on Integration

Formula 1

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2. $\int 0 . dx = c$
3. $\int k . dx = kx + c$
4. $\int \frac{1}{x} . dx = \log|x| + c$
5. $\int e^x dx = e^x + c$
6. $\int a^x dx = \frac{a^x}{\log a} + c$

Formula 2

1. $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$
2. $\int \sqrt{x}\, dx = \frac{2}{3} x^{3/2} + c$
3. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

Formula 3

1. $\int \sin x\, dx = -\cos x + c$
2. $\int \cos x\, dx = \sin x + c$
3. $\int \sec^2 x\, dx = \tan x + c$
4. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
5. $\int \sec x \cdot \tan x\, dx = \sec x + c$
6. $\int \operatorname{cosec} x \cdot \cot dx = -\operatorname{cosec} x + c$

Formula 4

1. $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + c$
2. $\int \frac{dx}{1 + x^2} = \tan^{-1} x + c$
3. $\int \frac{dx}{|x|\sqrt{x^2 - 1}} = \sec^{-1} x + c$

Type 1: Integrals of the form

$$\int \frac{dx}{1 \pm \sin x}$$

$$\int \left(\frac{\sin x}{1 \pm \sin x}\right) dx,$$

$$\int \frac{dx}{1 \pm \cos x},$$

$$\int \left(\frac{\cos x}{1 \pm \cos x}\right) dx,$$

$$\int \frac{\sec x\, dx}{\sec x \pm \tan x},$$

$$\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x \pm \cot x}\, dx$$

Rule: Simply rationalize the denominator.

Type 2: Integrals of the form $\int\left(\frac{f(x)}{g(x)}\right)dx$,

where $f(x)$ and $g(x)$ be two polynomials such that deg of $f(x) \geq$ deg of $g(x)$.

Rule: Simply divide the numerator by the denominator.

Type 3: An integral is related to inverse trigonometric functions.

Rule: Simply write the simplest form of the expression under the inverse term.

4. Standard Methods of Integration

There is no general method to find the integral of a function. If the integral is not a derivative of some known functions, the corresponding integrals cannot be determined. In general, we use the following three types of integration:

(i) Integration by substitution
(ii) Integration by parts
(iii) Integration by partial fractions.

Now we shall discuss about the integration by the substitution method.

Type 4: Integrals of the form $\int f(ax+b)\,dx$

Rule: Simply put $ax + b = t$.

Type 5: Integrals of the form

$$\int \frac{dx}{\sqrt{ax+b} \pm \sqrt{ax-d}}.$$

Rule: Simply rationalize the denominator.

Type 6: Integrals of the form

$$\int f(x)\sqrt{g(x)}\,dx,\ \int \frac{f(x)}{\sqrt{g(x)}}dx.$$

Rule: Put $g(x) = t^2$

Type 7: Integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$

Rule: Put $f(x) = t$

Formula 5

1. $\int \tan x\,dx = \log|\sec x| + c$
2. $\int \cot x\,dx = \log|\sec x| + c$
3. $\int \sec x\,dx = \log|\sec x + \tan x| + c$
$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$
4. $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c$
$= \log\left|\tan\left(\frac{x}{2}\right)\right| + c.$

Proof

1. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$
$= -\log|\cos x| + c$
$= \log|\sec x| + c$

2. $\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$
$= \log|\sin x| + c$

3. $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$
$= \log|\sec x + \tan x| + c$
$= \log\left|\frac{1+\sin x}{\cos x}\right| + c$
$= \log\left|\frac{1+\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}\right| + c$
$= \log\left|\frac{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right| + c$
$= \log\left|\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c$
$= \log\left|\tan\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{x}{2}\right)\right)\right| + c$
$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

4. $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\,dx$
$= \log|\operatorname{cosec} x - \cot x| + c$
$= \log\left|\frac{1-\cos x}{\sin x}\right| + c$
$= \log\left|\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right| + c$
$= \log\left|\tan\left(\frac{x}{2}\right)\right| + c$

Type 8: Integrals of the form

$$\int (f(x))^n \cdot f'(x)\,dx$$

Rule: Put $f(x) = t$

Type 9: Integrals of the form

$\int \sin^m x \cos^n x\,dx$, where m and n are real numbers.

Case I: When m is odd and n is even

Rule: Put $\cos x = t$

Case II: When m is even and n is odd

Rule: Put $\sin x = t$.

Case III: When m and n both are odd

Rule: Put either $\sin x = t$ or $\cos x = t$

Case IV: When m and n both are even

Rule: In this case, we shall use the following formulae:

$$1 + \cos 2x = 2\cos^2 x$$

or $$1 - \cos 2x = 2\sin^2 x.$$

Case V: When m is odd and n is zero

Rule: Put $\cos x = t$

Case VI: When m is zero and n is odd

Rule: Put $\sin x = t$

Case VII: When m is even and n is zero.

Rule: In this case we shall use the following formulae:

$$1 + \cos 2x = 2\cos^2 x$$

$$1 - \cos 2x = 2\sin^2 x.$$

Case VIII: When m is zero and n is even

Rule: In this case, we shall use the following formulae:

$$1 + \cos 2x = 2\cos^2 x$$

$$1 - \cos 2x = 2\sin^2 x.$$

Case IX: When $m + n = -$ve even integer

$$= -2k(\text{say}), \text{ where } k \in N$$

Rules

1. Divide the numerator and denominator by $\cos^{2k} x$
2. Put $\tan x = t$.

Formula 6

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c$
3. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$
4. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c.$
5. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$

Note: You should keep in mind that, the coefficient of x^2 must be unity every time. If not, first we make the co-efficient of x^2 be unity.

Type 10: Integrals of the form $\int \frac{dx}{ax^2 + bx + c}$

Rules

1. Make the co-efficient of x^2 unity
2. Express the denominator as a sum or difference of two perfect squares.

Type 11: Integrals is of the form

$$\int \frac{px + q}{ax^2 + bx + c} dx.$$

Rules

1. Reduce the denominator in such a way that

   d (Denominator) = Numerator + k
2. Then separate the Numerator into two terms and use

$$\int \frac{dx}{x^2 \pm a^2}.$$

Type 12: Integrals of the form

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

Rules

1. Make the coefficient of x^2 unity.
2. Express the term under the square root as a sum or difference of two perfect squares.
3. Use $\int \frac{dx}{x^2 \pm a^2}$ or $\int \frac{dx}{\sqrt{a^2 - x^2}}$

$$= \log\left|x + \sqrt{x^2 \pm a^2}\right| + c \text{ or } \sin^{-1}\left(\frac{x}{a}\right) + c$$

Type 13: Integrals of the form

$$\int \frac{f'(x)dx}{\sqrt{a\{f(x)\}^2 \pm b\{f(x)\} \pm c}},$$

i.e. reducible to $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Rules

1. Put $f(x) = t$
2. Apply (Type 12)

Type 14: Integrals of the form

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$$

Rules

1. Reduce the term under the square root in such a way that

$$d(ax^2 + bx + c) = \text{numerator} + k$$

2. Separate the numerator into two terms and use

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left|x + \sqrt{x^2 \pm a^2}\right| + c$$

Type 15: Integrals of the form

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x},$$

$$\int \frac{dx}{a \pm b\sin^2 x}, \int \frac{dx}{a \pm b\cos^2 x},$$

$$\int \frac{dx}{a\sin^2 x \pm b\cos^2 x \pm c},$$

$$\int \frac{dx}{(a\sin x \pm b\cos x)^2},$$

$$\int \frac{dx}{(a\sin x \pm b\cos x)(c\sin x \pm d\cos x)},$$

$$\int \frac{dx}{a + b\sin^2 x}.$$

Rules

1. Divide the numerator and denominator by the highest power of $\cos x$.
2. Put $\tan x = t$.

Type 16: Integrals of the form

$$\int \frac{\sin x}{\sin 3x\,dx}, \int \frac{\cos x}{\cos 3x\,dx}$$

Rules

1. First we cancel the common factors from the numerator and the denominator.
2. Divide the numerator and the denominator by $\cos^2 x$.
3. Put $\tan x = t$.

Type 17: Integrals of the form

$$\int \frac{dx}{a\cos x + b\sin x}, \int \frac{dx}{a + b\sin x},$$

$$\int \frac{dx}{a + b\cos x}, \int \frac{dx}{a\cos x + b\sin x + c}.$$

Rules

1. Replace $\sin x$ by $\dfrac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$,

   $\cos x$ by $\dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$.

2. Replace $1 + \tan^2\left(\frac{x}{2}\right)$ in the numerator by $\sec^2\left(\frac{x}{2}\right)$
3. Put $\tan\left(\frac{x}{2}\right) = t$
4. Use $\int \frac{dx}{x^2 + a^2}$ or $\int \frac{dx}{x^2 - a^2}$.

Type 18: Integrals of the form

$$\int \frac{dx}{a\cos x + b\sin x},$$

where $0 < \sqrt{a^2 + b^2} \le 2$

Rules

1. Divide the numerator and the denominator by $\sqrt{a^2 + b^2}$
2. Reduce the denominator as $\sin(A \pm B)$ or $\cos(A \pm B)$
3. Use

$$\int \sec x\,dx = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$$

or

$$\int \operatorname{cosec} x\,dx = \log\left|\tan\left(\frac{x}{2}\right)\right| + c$$

Type 19: Integrals of the form

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx.$$

Rules

1. Write the numerator = 1 (Denominator) + m(Denominator).
2. Compare the coefficients of $\sin x$ and $\cos x$
3. Solve for l and m.

Type 20: Integrals of the form

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx.$$

Rules

1. Write numerator = l (Denominator) + m(Denominator) + n
2. Compare the coefficient of $\sin x$, $\cos x$ and the contant term.
3. Find the values of l, m and n.

Type 21: Integrals of the form

$$\int \frac{(\cos x \pm \sin x)}{f(\sin 2x)}\,dx$$

Rules

1. Put $\sin x \pm \cos x = t$
2. Use $(\sin x \pm \cos x)^2 = 1 \pm \sin 2x$

Type 22: Integrals of the form

$$\int \frac{x^2 + 1}{x^4 \pm \lambda x^2 + 1}\,dx,$$

$$\int \frac{x^2 - 1}{x^4 \pm \lambda x^2 + 1}\,dx, \int \frac{dx}{x^4 \pm \lambda x^2 + 1},$$

where $\lambda \in R$

Rules

1. Divide the numerator and the denominator by x^2.
2. Reduce the denominator in such way that

$$d\left(x \pm \frac{1}{x}\right) = \text{Numerator} \pm k$$

3. Use $\int \frac{dx}{x^2 \pm a^2}$.

Type 23: Integrals of the form

$$\int \tan x \, dx, \int \cot x \, dx$$

$$\int (\tan x + \cot x) \, dx$$

$$\int (\tan x - \cot x) \, dx.$$

Rules

1. Put $\tan x$ or $\cot x = t^2$
2. Apply the form 24.

5. Integration by Parts

If u and v be two functions of x, then

$$\int (uv)\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v.dx\right\}dx$$

Proof: We know that,

$$\frac{d}{dx}(f(x).g(x)) = f(x).g'(x) + g(x).f'(x)$$

$$\Rightarrow \quad \int [f(x).g'(x) + g(x).f'(x)]\,dx = f(x).g(x)$$

$$\Rightarrow \quad \int (f(x).g'(x)dx$$

$$= f(x)g(x) - \int g(x).f'(x)dx$$

Put $\quad f(x) = u$, and $\frac{d}{dx}[g(x)] = v$

$$\Rightarrow \quad \int v\,dx = g(x)$$

Thus, $\quad \int (uv)\,dx = u\int v\,dx - \int\left\{\frac{du}{dv}\int v.dv\right\}dx.$

i.e. the integral of the product of two function

= (First function) × (Integral of the second function) – Integral of [(derivative of the first function) × (Integral of the second function)]

6. Choice of the First Function and the Second Function

1. We can choose the first fnction as the function which comes first in the word ILATE, where

   I stands for inverse trigonometric functions

   L stands for logarithmic functions

   A stands for algebraic functions

   T stands for trigonometric functions

   E stands for exponential functions.
2. If the integrand be logarithmic functions or inverse trigonometric functions alone, take the second function as unity.
3. If both the functions are trigonometric, consider the second function, whose integral being simpler and the other as the first function.

Type 24: Integrals of the form

$$\int e^x \{f(x) + f'(x)\}\,dx = e^x f(x) + c$$

Type 25: Integrals of the form

$$\int e^x \{(f + g) + (f' + g')\}\,dx = e^x (f + g) + c$$

Type 26: Integrals of the form

$$\int e^{kx} \{kf(x) + f'(x)\}\,dx = e^{kx} f(x) + c$$

Type 27: Integrals of the form

$$\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx.$$

Rule 1 $\quad \int e^{ax}\sin bx\,dx$

$$= \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$$

Rule 2 $\quad \int e^{ax}\cos bx\,dx.$

$$= \frac{e^{ax}}{a^2 + b^2}(a\cos bx - b\sin bx) + c$$

Proof: Let $\quad u = \int e^{ax}\cos bx\,dx$

and $\quad v = \int e^{ax}\sin bx\,dx$

Then $\quad u + iv$

$$= \int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx$$

$$= \int e^{ax}(\cos bx + i\sin bx)\,dx$$

$$= \int e^{ax}\cdot e^{ibx}dx$$

$$= \int e^{ax+ibx}dx$$

$$= \int e^{(a+ib)x}dx$$

$$= \frac{e^{(a+ib)x}}{a+ib} + c$$

$$= \frac{(a-ib)e^{(a+ib)x}}{(a^2+b^2)} + c$$

Comparing the real and imaginary part, we get the required result.

Formula 7

1. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$
2. $\int \sqrt{x^2 + a^2}\,dx$

   $= \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$
3. $\int \sqrt{x^2 - a^2}\,dx$

   $= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

Proof:

1. Let $I = \int \sqrt{a^2 - x^2}\, dx$

$\Rightarrow \quad I = \sqrt{a^2 - x^2} \int dx - \int \left(\frac{1 \times -2x}{2\sqrt{a^2 - x^2}} \cdot x \right) dx$

$\Rightarrow \quad I = x\sqrt{a^2 - x^2} + \int \left(x \frac{2}{\sqrt{a^2 - x^2}} \right) dx$

$\Rightarrow \quad I = x\sqrt{a^2 - x^2} - \int \left(\frac{(a^2 - x^2) - a^2}{\sqrt{a^2 - x^2}} \right) dx$

$\Rightarrow \quad I = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$

$\Rightarrow \quad I = x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$

$\Rightarrow \quad 2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$

$\Rightarrow \quad I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

2. Let $I = \int \sqrt{x^2 + a^2}\, dx$

$= \sqrt{x^2 + a^2} \int dx - \int \left(\frac{1 \times 2x}{2\sqrt{x^2 + a^2}} \cdot x \right) dx$

$= x\sqrt{x^2 + a^2} - \int \left(\frac{x^2}{\sqrt{x^2 - a^2}} \right) dx$

$= x\sqrt{x^2 + a^2} - \int \left(\frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} \right) dx$

$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + \int \frac{a^2\, dx}{\sqrt{x^2 + a^2}}$

$= x\sqrt{x^2 + a^2} - I + a^2 \log\left|x + \sqrt{x^2 + a^2}\right| + c$

$\Rightarrow \quad 2I = x\sqrt{x^2 + a^2} + a^2 \log\left|x + \sqrt{x^2 + a^2}\right| + c$

$\Rightarrow \quad I = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + c$

3. Let $I = \int \sqrt{x^2 - a^2}\, dx$

$= \sqrt{x^2 - a^2} \int dx - \int \left(\frac{1 \times 2x}{2\sqrt{x^2 - a^2}} \cdot x \right) dx$

$= x\sqrt{x^2 - a^2} - \int \left(\frac{x^2}{\sqrt{x^2 - a^2}} \right) dx$

$= x\sqrt{x^2 - a^2} - \int \left(\frac{(x^2 - a^2) + a^2}{\sqrt{x^2 - a^2}} \right) dx$

$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}\, dx$

$= x\sqrt{x^2 - a^2} - I - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c$

$\Rightarrow \quad 2I = x\sqrt{x^2 - a^2} - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c$

$\Rightarrow \quad I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c$

Type 28: Integrals of the form

$$\int \sqrt{ax^2 + bx + c}\, dx$$

Rules

1. First we make the coefficient of x^2 unity
2. Express the term under the square root as a sum or difference of two perfect squares.
3. Use $\int \sqrt{x^2 \pm a^2}\, dx$ or $\int \sqrt{a^2 - x^2}\, dx$

Type 29: Integrals of the form

$$\int (px + q)\sqrt{ax^2 + bx + c}\, dx$$

Rules

1. Reduce $(px + q)$ as a derivative of $(ax^2 + bx + c)$
2. Use

$$\int \sqrt{a^2 - x^2}\, dx \Big/ \int \sqrt{x^2 \pm a^2}\, dx.$$

7. Partial Fractions

A special type of Aational (proper) function is known as the partial fraction, where the degree of the numerator < degree of denominator.

Let $h(x) = \frac{f(x)}{g(x)}.$

Type I: When the denominator is expressible as the product of non-repeating linear factors.

Let $g(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$

Then $\frac{f(x)}{g(x)}$

$$= \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \frac{A_3}{(x - a_3)} + \ldots + \frac{A_n}{(x - a_n)}$$

where $A_1, A_2, \ldots, A_n$ are constant and can be determined by equating the numerator on RHS to the numerator on LHS and then substituting $x = a_1, a_2, a_3, \ldots, a_n$

Type II: When the denominator is expressible as the product of linear factors such that some of them are repeating.

Let $g(x) = (x - a_1)^k (x - a_2)(x - a_3) \ldots (x - a_n)$

Then $\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)} + \frac{A_1}{(x-a_1)^2} + \ldots + \frac{A_1}{(x-a_1)^k}$

$+ \frac{B_1}{(x-a_2)} + \frac{B_2}{(x-a_3)} + \ldots + \frac{B_n}{(x-a_n)}$

Type III: When the denominator is expressible as the product of linear and quadratic factors but non-repeating.

Let $g(x) = (x-a_1)(x-a_2)(x-a_3)\ \ldots\ (ax^2 - bx + c)$

Then $\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)}$

$+ \ldots + \frac{Bx + C}{ax^2 + bx + c}$

Type IV: Integrals of the form

$$\int \frac{x^2}{(x^2+a)(x^2+b)}\,dx$$

Rules

1. Put $x^2 = t$
2. Donot find its derivative
3. Use the concept of partial fractions.

Type V: Integrals of the form

$$\int \frac{(x^2+a)(x^2+b)}{(x^2+c)(x^2+d)}\,dx$$

Rules

1. Put $x^2 = t$
2. Do not find its derivative
3. Reduce it into a partial fraction (degree of the numerator < Degree of denominator)
4. Use the concept of partial fractions.

Advanced Level

Type 1: Integrals of the form

$$\int \tan^m x . \sec^n x\, dx$$

Rules

1. If m is even or odd integer and n is even positive integer, put $\tan x = t$
2. If m is odd positive integer and $n \in$ even positive integer, put $\sec x = t$.
3. If $m = 0$ and $n = 2r + 1$, $\forall\ r \in N$, write

$$\int \sec^{2r+1} x\ dx = \int \sec^{2r-1} x . \sec^2 x\ dx$$

and then integrate it by parts, where

Type 2: Integrals of the form

$$\int \cot^m x . \text{cosec}^n x\, dx$$

Rules

1. If m is even or odd integer and n is even positive integer, put $\cot x = t$.
2. If m is odd positive integer and $n \in$ even positive integer, put $\text{cosec}\, x = t$.
3. If $m = 0$ and $n = 2r + 1$, $\forall\ r \in N$, write

$$\int \text{cosec}^{2r+1} x\ dx = \int \text{cosec}^{2r-1} x . \text{cosec}^2 x\ dx$$

and then integrate it by parts, where consider $\text{cosec}^2 x$ as the first function. consider $\sec^2 x$ as the second function.

Type 3: Integrals of the form

$$\int \frac{dx}{x(x^n+1)},\ \int \frac{dx}{x^2(x^n+1)^{\frac{(n-1)}{n}}},$$

$$\int \frac{dx}{x^n\ (1+x^n)^{1/n}},\ \text{where } n \in N$$

Rules

1. Take common x^n from the denominator.
2. Put $1 + x^{-n} = t$.

Type 4: Integrals of the form

$$\int \frac{x^m}{(ax+b)^n}\,dx,\ \text{where } m, n \in N$$

Rule Put $ax + b = t$.

Type 5: Integrals of the form

$$\int \frac{dx}{x^m(ax+b)^n},\ \text{where } m, n \in N$$

Rule Put $\left(\frac{ax+b}{x}\right) = t$

Type 6: Integrals of the form

$$\int \frac{dx}{(x-a)^n(x-b)^n},\ \text{where } m, n \in N$$

Rules

1. $\left(\frac{x-a}{x-b}\right) = t$, when $m < n$
2. $\left(\frac{x-b}{x-a}\right) = t$, when $m > n$

Type 7: Integrals of the form

$$\int \frac{dx}{x(a+bx^n)},\ \text{where } n \in N$$

Rule: Put $x^n = \frac{1}{t}$

Type 8: Integrals of the form

$$\int \frac{x^{2m+1}}{(ax^2+b)^n}\,dx$$

Rule Put $(ax^2 + b) = t$

Type 9: Integrals of the form

$$\int \frac{(a\sin x + b)}{(a + b\sin x)^2} dx$$

Rules

1. Divide the numerator and the denominator by $\cos^2 x$
2. Put $a\sec x + b\tan x = t$.

Type 10: Integrals of the form

$$\int \frac{(a\cos x + b)}{(a + b\cos x)^2} dx$$

Rules

1. Divide the numerator and the denominator by $\sin^2 x$
2. Put $a\operatorname{cosec} x + b\cot x = t$.

Type 11: Integrals of the form

$$\int \frac{dx}{(a + b\sin x)^2}$$

Rule Put $\dfrac{a\sin x + b}{a + b\sin x} = t$

Type 12: Integrals of the form

$$\int \frac{dx}{(a + b\cos x)^2}$$

Rule Put $t = \left(\dfrac{a\cos x + b}{a + b\cos x}\right)$

Type 13: Integrals of the form

$$\int \left(\frac{ae^x + be^{-x}}{pe^x + qe^{-x}}\right) dx$$

Rules

1. Express the Numerator = l (Denominator) + m × derivative of (Denominator)
2. Compare the coefficients of e^x and e^{-x}
3. Find l and m.

8. Integration of Irrational Functions

Type 1: Integrals of the form

$$\int f\left(x, \left(\frac{ax + b}{cx + d}\right)^{\frac{\alpha}{n}}\right) dx,$$

where $a, b, c, d, \alpha, n \in R$

Rule Put $\left(\dfrac{ax + b}{cx + d}\right) = t^n$

Type 2: Integrals of the form

$$\int f\left(x, (ax + b)^{\alpha/n}, (ax + b)^{\beta/m}\right) dx,$$

where m, n are positive integers.

Rule Put $(ax + b) = t^p$, where p is the LCM of m and n.

Type 3: Integrals of the form

$$\int \frac{dx}{L_1(x)\sqrt{L_2(x)}}$$

Rule Put $L_2(x) = t^2$

Type 4: Integrals of the form

$$\int \frac{dx}{Q(x)\sqrt{L(x)}}$$

Rule Put $L(x) = t^2$

Type 5: Integrals of the form

$$\int \frac{dx}{L(x)\sqrt{Q(x)}}$$

Rule Put $L(x) = \dfrac{1}{t}$

Type 6: Integrals of the form

$$\int \frac{dx}{Q_1(x)\sqrt{Q_2(x)}}$$

Rule Put $x = \dfrac{1}{t}$ or $t^2 = \dfrac{Q_2(x)}{Q_1(x)}$

Type 7: Integrals of the form

$$\int \left\{f\left(x \pm \sqrt{x^2 + a^2}\right)^n\right\} dx$$

Rule Put $\left(x \pm \sqrt{x^2 + a^2}\right) = t$

Type 8: Integrals of the form

$$\int \frac{dx}{x^m (a + bx)^p}, \text{ where } m + p = N$$

and $m + p > 1$

Rule Put $a + bx = t\,x$.

Type 9: Integrals of the form

$$\int \frac{dx}{L_1(x))^m (L_2(x))^n}, \text{ where } m, n \in R$$

Rules

1. If $n > m$, Put $\dfrac{L_1(x)}{L_2(x)} = t$
2. If $m > n$, Put $\dfrac{L_2(x)}{L_1(x)} = t$

Type 10: Integrals of the form $\displaystyle\int \frac{dx}{x\sqrt{ax^n + b}}$

Rule Put $ax^n + b = t^2$

Type 11: Integrals of the form

$$\int \frac{dx}{(a + bx^2)^{3/2}} \text{ where } a, b \in R - \{0\}$$

Rule Put $x = \frac{1}{t}$

Type 12: Integrals of the form

$$\int \frac{dx}{(x - k)^r \sqrt{ax^2 + bx + c}},$$

where $r \in N$ and $k \in R - \{0\}$

Rule Put $x - k = \frac{1}{t}$

Type 13: Integrals of the form

$$\int \frac{(ax + b)}{(cx + d)\sqrt{px^2 + qx + r}}\, dx$$

Rules

1. Put $(ax + b) = A(cx + d) + B$
2. Find the values of A and B
3. Reduce the given integral into two separate integrals.

Type 14: Integrals of the form

$$\int \frac{(ax^2 + bx + c)}{(dx + e)\sqrt{px^2 + qx + r}}\, dx$$

Rules

1. Put $(ax^2 + bx + c)$
$$= L(dx + e)(2px + q) + M(dx + e) + N$$
2. Compare the co-efficents of the like terms of both the sides and find L, M and N.
3. Integrate the given integral.

Type 15: Integrals of the form

$$\int x^{\beta} (a + bx^{\gamma})^{\alpha}\, dx$$

Rules

1. If $\alpha \in I^+$, expand the integral by the concept of binomial expansion.
2. If $\alpha \in I^-$, we put $x = t^p$, where p is the LCM of the denominator of β and γ.
3. If $\frac{\beta + 1}{\gamma} \in I$ and α is a fraction, put $(a + bx^{\gamma}) = t^p$, where p is the denominator of α.
4. If $\frac{\beta + 1}{\gamma} + \alpha \in I$, put $(a + bx^{\gamma}) = t^p x^{\gamma}$, where p is the denominator of α.

9. Euler's Substitution

Type 16: Integrals of the form

$$\int R\left(x, \sqrt{ax^2 + bx + c}\right) dx$$

Rules

1. Put $\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}$, if $a > 0$
2. Put $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$, if $c > 0$,
3. Put $\sqrt{ax^2 + bx + c} = (x - \alpha)t$, or $(x - \beta)t$, where α and β are the real roots of $ax^2 + bx + c$.

10. Integration by Reduction Formula

A reduction formula is defined as a formula or a connection by means of which the power of the integrand is reduced, therefore, making the integration easier. The basic technique of obtaining a reduction formula is the integration by parts. In some cases the method of differentiation or other special devices is adopted.

Type I: Reduction formula for $\int \sin^n x\, dx$

Soln. Let $I_n = \int \sin^n x\, dx$

$$= \int \sin^{n-1} x \cdot \sin x\, dx$$

$$= \sin^{n-1} x \int \sin x\, dx - \int (n - 1)\sin^{n-2} x(-\cos^2 x)\, dx$$

$$= \sin^{n-1} x(-\cos x) + (n - 1) \int \sin^{n-2} x(1 - \sin^2 x)\, dx$$

$$= \sin^{n-1} x(-\cos x) + (n - 1)\, I_{n-2} - I_n$$

Thus, $(1 + n - 1)I_n = -\cos x \cdot \sin^{n-1} x + (n - 1)I_{n-2}$

$$\Rightarrow \quad nI_n = -\cos x \cdot \sin^{n-1} x + (n - 1)I_{n-2}$$

$$I_n = \frac{\cos x \cdot \sin^{n-1} x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}$$

which is the required reduction formula.

Type 2: Reduction formula for $\int \cos^n x\, dx$

Soln. Let $I_n = \int \cos^n x\, dx$

$$= \int \cos^{n-1} x \cdot \cos x\, dx$$

$$= \cos^{n-1} x \int \cos x\, dx + (n + 1)\int \cos^{n-2} x \cdot \sin^n x\, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n + 1)\int \cos^{n-2} x \cdot (1 - \cos^2 x)\, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n + 1)I_{n-2} - I_n$$

Thus, $(1 + n + 1)I_n = \cos^{n-1} x \cdot \sin x + (n + 1)I_{n-2} + C$

$$\Rightarrow I_n = \frac{\cos^{n-1} x \cdot \sin x}{(n + 2)} + \left(\frac{n + 1}{n + 2}\right) I_{n-2} + C$$

which is the required reduction formula.

Type 3: Reduction formula for $\int \tan^n x\, dx$

Soln. Let $I_n = \int \tan^n x\, dx$

$$= \int \tan^{n-2}x \cdot \tan^2 x\, dx$$

$$= \int \tan^{n-2}x \cdot (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2}x \cdot \sec^2 x\, dx - \int \tan^{n-2}x\, dx$$

$$= \left(\frac{\tan^{n-1}x}{n-1}\right) - I_{n-2} + C$$

which is the required reduction formula.

Type 4: Reduction formula for $\int \cot^n x\, dx$

Soln. Let $I_n = \int \cot^n x\, dx$

$$= \int \cot^{n-2}x \cdot \cot^2 x\, dx$$

$$= \int \cot^{n-2}x \cdot (\operatorname{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2}x \cdot \operatorname{cosec}^2 x\, dx - \int \cot^{n-2}x\, dx$$

$$= -\frac{\cot^{n-1}x}{n-1} - I_{n-2} + C$$

which is the required reduction formula.

Type 5: Reduction formula for $\int \sec^n x\, dx$

Soln. Let $I_n = \int \sec^n x\, dx$

$$= \int \sec^{n-2}x \cdot \sec^2 x\, dx$$

$$= \sec^{n-2}x \int \sec^2 x\, dx - \int (n-2)\sec^{n-2}x \cdot \tan^2 x\, dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \cdot (\sec^2 x - 1) dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int (\sec^n x - \sec^{n-2}x) dx$$

$$= \sec^{n-2}x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow (n-2) I_n = \sec^{n-2}x \tan x - (n-2) I_{n-2}$$

$$\Rightarrow I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \left(\frac{n-2}{n-1}\right) I_{n-2} + C$$

which is the required reduction formula.

Type 6: Reduction formula for $\int \frac{\sin nx}{\sin x}\, dx$

Soln. Let $I_n = \int \frac{\sin nx}{\sin x}\, dx$

$$= \int \left(2\cos(n-1)x + \frac{\sin(n-2)x}{\sin x}\right) dx$$

$$[\because \sin nx - \sin(n-2)x = 2\cos(n-1)x \sin x$$

$$\Rightarrow \frac{\sin nx}{\sin x} = 2\cos(n-1)x + \frac{\sin(n-2)x}{\sin x}]$$

$$= \frac{2\sin(n-1)x}{(n-1)} + I_{n-2}$$

which is the required reduction formula.

Type 7: Reduction formula for $\int \frac{dx}{(x^2 + k)^n}$

Soln. Let $I_n = \int \frac{dx}{(x^2 + k)^n}$

Then $I_{n-2} = \int \frac{dx}{(x^2 + k)^{n-1}}$

$$= \int \left(\frac{1}{(x^2 + k)^{n-1} \cdot 1}\right) dx$$

$$= \frac{1}{(x^2 + k)^{n-1}} \int 1 \cdot dx - \int \left(\frac{-(n-1)}{(x^2 + k)^n} \cdot 2x \cdot x\right) dx$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1) \int \left(\frac{x^2}{(x^2 + k)^n}\right) dx$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1) \int \left(\frac{(x^2 + k) - k}{(x^2 + k)^n}\right) dx$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1) \int \left(\frac{1}{(x^2 + k)^{n-1}} - \frac{k}{(x^2 + k)^n}\right) dx$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1)\, I_{n-1} - I_{n-1} - k \int \frac{dx}{(x^2 + k)^n}]$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1)(I_{n-1} - k I_n)$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1) I_{n-1} - 2(n-1) k I_n$$

$$\Rightarrow 2(n-1) k I_n$$

$$= \frac{x}{(x^2 + k)^{n-1}} + 2(n-1) I_{n-1} - I_{n-1}$$

$$= \frac{x}{(x^2 + k)^{n-1}} + (2n-3) I_{n-1}$$

$$\Rightarrow I_n = \frac{x}{2(n-1) k (x^2 + k)^{n-1}} + \frac{1}{k}\left(\frac{2n-3}{2n-2}\right) I_{n-1}$$

which is the required reduction formula.

Type 8: Reduction formula for $\int x^m (\log x)^n\, dx$

Soln. Let $I_{m,n} = \int x^m (\log x)^n\, dx$

$$= (\log x)^n \int x^m\, dx - \int \left(n(\log x)^{n-1}\left(\frac{1}{x}\right)\frac{x^{m+1}}{m+1}\right) dx$$

$$= (\log x)^n, \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int \left(x^m (\log x)^{n-1}\right) dx$$

$$= (\log x)^n, \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

$$\Rightarrow \quad I_{m,n-1} = (\log x)^n, \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

which is the required reduction formula.

Type 9: Reduction formula for

$$\int x^m (1-x)^n$$

Soln. Let $\quad I_{m,n} = \int x^m (1-x)^n dx$

$$= (1-x)^n \int x^m dx + \int \left(n(1-x)^{n-1} \cdot \frac{x^{m+1}}{m+1} \right) dx$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1} \int [x^{m+1} \cdot (1-x)^{n-1}] dx$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1} \int x^{m+1} . (1-x)^{n-1} dx$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1} \int [x^m . (1-x)^{n-1} \cdot x] dx$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1}$$

$$\int [x^m \cdot (1-x)^{n-1} \cdot \{-1(1-x)\}] dx$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1} [I_{m,\, n-1} - I_{m,\, n}]$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1} I_{m,n-1} - \frac{n}{m+1} - I_{m,n}$$

$$\Rightarrow \quad \left(1 + \frac{n}{m+1}\right) I_{m,n}$$

$$= (1-x)^n \frac{x^{m+1}}{m+1} + \frac{n}{m+1} I_{m,n-1}$$

$$\Rightarrow \quad I_{m,n} = \frac{x^{m+1}(1-x)^n}{(m+n+1)} + \frac{n}{m+n+1} I_{m,n-1}$$

which is the required reduction formula.

Type 10: Reduction formula for

$$\int \cos^m x \sin nx \, dx$$

Soln. Let $I_{m,n} = \int \cos^m x \sin nx \, dx$

$$= \cos^m x \int \sin nx \, dx$$

$$- \int \left(-\frac{\cos nx}{n} \cdot m \cdot \cos^{m-1} x \cdot - \sin x \right) dx$$

$$= \cos^m x \left(-\frac{\cos nx}{n} \right)$$

$$- \frac{m}{n} \int -\cos^{m-1} x (\cos nx \sin x) \, dx$$

$$= -\frac{\cos^m x \cdot \cos nx}{n}$$

$$-\frac{m}{n} \int -\cos^{m-1} x (\sin nx \cos x - \sin(n-1)x) \, dx$$

$$= -\frac{\cos^m x \cdot \cos nx}{n} - \frac{m}{n} \int \cos^m x \sin nx \, dx$$

$$+ \frac{m}{n} \int \cos^{m-1} x \sin(n-1) x \, dx$$

$$= -\frac{\cos^m x \cdot \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad \left(1 + \frac{m}{n}\right) I_{m,n}$$

$$= -\frac{\cos^m x \cdot \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x \cos nx}{(m+n)} + \frac{m}{(\mathrm{m}+n)} I_{m-1,n-1}$$

which is the required reduction formula.

> ***Note:*** Similarly, we can easily formulate the reduction formula for
>
> $$\int \cos^m x \cos nx \, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{(m+n)} I_{m-1,n-1}$$

Type 11: Reduction formula for

$$\int \sin^m x \sin nx \, dx$$

Soln. Let $I_{m,n} = \int \sin^m x \sin nx \, dx$

$$= \sin^m x \int \sin nx \, dx$$

$$- \int \left(m \sin^{m-1} x \cdot \cos x \cdot - \frac{\cos nx}{n} \right) dx$$

$$= -\frac{\sin^m x \cdot \cos nx}{n}$$

$$+ \frac{m}{n} \int \sin^{m-1} x \cos x \cos nx \, dx$$

$$= -\frac{\sin^m x \cdot \cos nx}{n}$$

$$+ \frac{m}{n} \int \sin^{m-1} x (\cos(n-1)x - \sin nx \sin x) dx$$

$$= -\frac{\sin^m x \cdot \cos nx}{n}$$

$$+ \frac{m}{n} \int \sin^{m-1} x (\cos(n-1) x \, dx$$

$$- \frac{m}{n} \int \sin^m x \sin nx \, dx$$

$$\Rightarrow \quad \left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{\sin^m x \cdot \cos nx}{n}$$

$$+ \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad I_{m,n} = -\frac{\sin^m x \cdot \cos nx}{(m+n)} + \frac{m}{(m+n)} I_{m-1,\,n-1}$$

> ***Note:*** Similarly, we can easily formulate the reduction formula for
> $$\int \sin^m x \cos nx\,dx = \sin^m x \frac{\sin nx}{(m+n)} + \frac{m}{(m+n)} I_{m-1,n-1}$$

Type 12: Reduction formula for

$$\int \sin^m x \cos^n x\,dx$$

Soln. Let $P = \sin^{m-1} x \cos^{n+1} x$

$$\Rightarrow \frac{dp}{dx} = (m-1)\sin^{m-2} x \cos^{n+2} x - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x \cdot \cos^2 x - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x \cdot (1-\sin^2 x) - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x \cdot (1-\sin^2 x) - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m-1)\sin^m x \cos^n x - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m-1+n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m+n)\sin^m x \cos^n x$$

On integration, we get

$$P = (m-1)\int \sin^{m-2} x \cos^n x\,dx - (m-n)\int \sin^m x \cos^n x\,dx$$

$$\Rightarrow \quad (m-n)\int \sin^m x \cos^n x\,dx = -P + (m-1)\int \sin^{m-2} x \cos^n x\,dx$$

$$\Rightarrow \quad \int \sin^m x \cos^n x\,dx = -\int \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

which is the required reduction formula.

11. Inexpressible Integrals

If an integral $\int f(x)\,dx$ as expressible in terms of elementary functions, the integral is known as computable. But if an integral $\int f(x)\,dx$ is not expressible in terms of elementary functions, the integral is known as inexpressible or 'cannot be found'.

Some inexpressible integrals are

(i) $\int e^{x^2}\,dx$

(ii) $\int e^{-x^2}\,dx$

(iii) $\int \sqrt{\sin x}\,dx$

(iv) $\int \sqrt{\cos x}\,dx$

(v) $\int x \tan x\,dx$

(vi) $\int \frac{\sin x}{x}\,dx$

(vii) $\int \frac{\cos x}{x}\,dx$

(viii) $\int \frac{dx}{\log x}$

(ix) $\int \sqrt{1+x^3}$

(x) $\int \sqrt[3]{1+x^3}\,dx$

(xi) $\int \sin(x^2)\,dx$

(xii) $\int \cos(x^2)\,dx$

(xiii) $\int \frac{x^2}{1+x^5}\,dx$

(xiv) $\int \sqrt{1-k^2 \sin^2 x}\,dx$

(xv) $\int \sqrt{x} \cos\sqrt{x}\,dx$.

Exercises

Level I

(Problems Based on Fundamentals)

ABC of Integration

1. Evaluate: $\int \log_x x\,dx$
2. Evaluate: $\int (3^{\log_x 2} - 2^{\log_x 3})$
3. Evaluate: $\int \left(x^m + m^x + m^m + \frac{m}{x}\right)dx$
4. Evaluate: $\int 2^x \cdot 3^x \cdot dx$
5. Evaluate: $\int \tan^2 x\,dx$
6. Evaluate: $\int \cot^2 x\,dx$
7. Evaluate: $\int \frac{dx}{\sin^2 x \cos^2 x}$

8. Evaluate: $\int \left(1+\tan\left(x+\frac{3\pi}{8}\right)\right)\left(1+\tan\left(\frac{\pi}{8}-x\right)\right)dx$

9. Evaluate: $\int (\tan x+\cot x)^2 dx$

10. Evaluate: $\int \frac{dx}{1+\cos^2 x}$

11. Evaluate: $\int (3^{\log_5 x}-2^{\log_5 x})\,dx$

12. Evaluate: $\int \left(1+\tan\left(\frac{\pi}{8}-x\right)\right)\left(1+\tan\left(\frac{\pi}{8}+x\right)\right)dx$

13. Evaluate: $\int \left(\frac{8^{1+x}+4^{1+x}}{2^{2x}}\right)dx$

14. Evaluate: $\int \left(\frac{x}{m}+\frac{m}{x}+x^m+m^x\right)dx$

15. Evaluate: $\int \frac{(a^x+b^x)^2}{a^x b^x}\,dx$

16. Evaluate: $\int \frac{(2^x+3^x)^2}{2^x\cdot 3^x dx}$

17. If $f'(x)=\frac{1}{x}+\frac{1}{\sqrt{1-x^2}}$ and $f(1)=\frac{\pi}{2}$, find $f(x)$.

18. If $f'(x)=a\cos x+b\sin x$ and $f'(0)=4, f'(0)=3, f\left(\frac{\pi}{2}\right)=5$, find $f(x)$.

Type 1

19. Evaluate: $\int \frac{dx}{1-\sin x}$

20. Evaluate: $\int \left(\frac{\sin^4 x+\cos^4 x}{\sin^2 x\cos^2 x}\right)dx$

21. Evaluate: $\int \left(\frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}\right)dx$

22. Evaluate: $\int \left(\frac{\cos 2x-\cos\alpha}{\cos x-\cos\alpha}\right)dx$

23. Evaluate: $\int \left(\frac{\cos^4 x-\sin^4 x}{\sqrt{1+\cos 4x}}\right)dx$

24. Evaluate: $\int \left(\frac{1+\tan^2 x}{1+\cot^2 x}\right)dx$

25. Evaluate: $\int \left(\frac{\cos x-\cos 2x}{1-\cos x}\right)dx$

26. Evaluate: $\int \left(\frac{\sqrt{x^4+x^{-4}+2}}{x^3}\right)dx$

27. Evaluate: $\int \left(\frac{5\cos^3 x+3\sin^3 x}{\cos^2 x\sin^2 x}\right)dx$

28. Evaluate: $\int \left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)(1+\sin 2x)\,dx$

29. Evaluate: $\int \left(\frac{\cos 5x+\cos 4x}{1+2\cos 3x}\right)dx$

30. Evaluate: $\int \left(\frac{\cos x-\cos 2x}{1-\cos x}\right)dx$

31. Evaluate: $\int \frac{dx}{(\tan x+\cot x+\sec x+\cos x)}$

Type 2

32. Evaluate: $\int \frac{x}{x+1}\,dx$

33. Evaluate: $\int \frac{(1+x)^2}{x(1+x^2)}\,dx$

34. Evaluate: $\int \frac{x^2-2}{x^2+1}\,dx$

35. Evaluate: $\int \frac{x-1}{(x^{2/3}+x^{1/3}+1)}\,dx$

36. Evaluate: $\int \left(\frac{x^4+2}{x^2+2}\right)dx$

37. Evaluate: $\int \left(\frac{x^4-3}{x^2+1}\right)dx$

38. Evaluate: $\int \left(\frac{x^6-1}{x^2+1}\right)dx$

39. Evaluate: $\int \left(\frac{x^8+x^4+1}{x^4+x^2+1}\right)dx$

40. Evaluate: $\int \left(\frac{x^4}{x^2+1}\right)dx$

41. Evaluate: $\int \left(\frac{x^4+x^2+1}{x^2+x+1}\right)dx$

42. Evaluate: $\int \left(\frac{x^6+1}{x^2+1}\right)dx$

Type 3

43. Evaluate: $\int \sin^{-1}(\sin x)\,dx$

44. Evaluate: $\int \sin^{-1}(\cos x)\,dx$

45. Evaluate: $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx$

46. Evaluate: $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$

47. Evaluate: $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx$

48. Evaluate: $\int \tan^{-1}\left(\frac{\sin x}{1-\cos x}\right)dx$

49. Evaluate: $\int \tan^{-1}\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)dx$

50. Evaluate: $\int \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)dx$

51. Evaluate: $\int \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)dx$

52. Evaluate: $\int \tan^{-1}\left(\frac{1-\sin x}{\cos x}\right)dx$

53. Evaluate: $\int \tan^{-1}\left(\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1+\sin x)}}\right)dx$

54. Evaluate: $\int \tan^{-1}(\sec x+\tan x)\,dx$

55. Evaluate: $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$

Type 4

56. Evaluate: $\int (3x+2)\,dx$

57. Evaluate: $\int \frac{dx}{2x-3}$

58. Evaluate: $\int \frac{dx}{5-2x}$

59. Evaluate: $\int e^{ax+b}\,dx$

60. Evaluate: $\int 3^{4x+5}\,dx$

61. Evaluate: $\int \cos(5x+3)\,dx$

62. Evaluate: $\int \sin 2x\,dx$

63. Evaluate: $\int \sqrt{3x+2}\,dx$

64. Evaluate: $\int \frac{dx}{\sqrt{3x+4}}$

Type 5

65. Evaluate: $\int \frac{dx}{\sqrt{x+2}-\sqrt{x+1}}$

66. Evaluate: $\int \frac{dx}{(\sqrt{2x+5}-\sqrt{2x+3})}$

67. Evaluate: $\int \frac{dx}{\sqrt{3x+4}-\sqrt{3x+1}}$

68. Evaluate: $\int \frac{dx}{\sqrt{2x+3}+\sqrt{2x-3}}$

69. Evaluate: $\int \frac{dx}{\sqrt{x+1}+\sqrt{x}}$

70. Evaluate: $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

71. Evaluate: $\int \frac{dx}{\sqrt{2x+2014}+\sqrt{2x+3013}}$

Type 6

72. Evaluate: $\int \frac{x}{\sqrt{x-1}}\,dx$

73. Evaluate: $\int \frac{\sqrt{x}}{x+1}\,dx$

74. Evaluate: $\int \frac{x}{\sqrt{3x+1}}\,dx$

75. Evaluate: $\int \frac{x+1}{\sqrt{2x-1}}\,dx$

76. Evaluate: $\int \frac{x-1}{\sqrt{x+4}}\,dx$

77. Evaluate: $\int \frac{x}{x^2+1}\,dx$

Type 7

78. Evaluate: $\int \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$

79. Evaluate: $\int \frac{3\cos x}{2\sin x+5}\,dx$

80. Evaluate: $\int \frac{\cos x-\sin x}{2+\sin 2x}\,dx$

81. Evaluate: $\int \frac{xe^x+e^x}{\cos^2(xe^x)}\,dx$

82. Evaluate: $\int \frac{dx}{x(1+\ln x)^2}$

83. Evaluate: $\int \frac{\cos x-\sin x+1-x}{e^x+\sin x+x}\,dx$

84. Evaluate: $\int \frac{dx}{1+e^x}$

85. Evaluate: $\int \frac{dx}{x(x^3+1)}$

86. Evaluate: $\int \frac{dx}{x(x^4+1)}$

87. Evaluate: $\int \frac{dx}{x(x^5-1)}$

88. Evaluate: $\int \frac{\sin 2x}{\sin 5x \sin 3x}\,dx$

89. Evaluate: $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

90. Evaluate: $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

91. Evaluate: $\int \frac{dx}{1+e^{-x}}$

92. Evaluate: $\int \frac{dx}{1+e^{x}}$

93. Evaluate: $\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x}\,dx$

94. Evaluate: $\int \frac{\sin(x-a)}{\sin x}\,dx$

95. Evaluate: $\int \frac{\sin x}{\sin(x-a)}\,dx$

96. Evaluate: $\int \frac{\sin(x+a)}{\sin(x+b)}\,dx$

97. Evaluate: $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

98. Evaluate: $\int \frac{1+\tan x}{x+\log\sec x}\,dx$

99. Evaluate: $\int \frac{\sin 2x}{\sin 5x.\sin 3x}\,dx$

100. Evaluate: $\int \frac{\cos x - \sin x}{1+\sin 2x}\,dx$

101. Evaluate: $\int \frac{dx}{\sin x.\cos^2 x}$

102. Evaluate: $\int \frac{dx}{\sin 2x.\cos^2 x}$.

103. Evaluate: $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

104. Evaluate: $\int \frac{dx}{\cos(x-a)\cos(x-b)}$

105. Evaluate: $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

106. Evaluate: $\int \frac{x^x(1+\ln x)}{x^x+1}\,dx$

107. Evaluate: $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx$

108. Evaluate:

$$\int \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}\,dx$$

Type 8

109. Evaluate: $\int 3x^2 \sin(x^3)\,dx$

110. Evaluate: $\int \frac{(1+\ln x)^3}{x}\,dx$

111. Evaluate: $\int \frac{dx}{x^2(1+x^4)^{3/4}}$

112. Evaluate: $\int \frac{dx}{\sqrt{x}(4+3\sqrt{x})^2}$

113. Evaluate: $\int 3^{3^{3^x}} \cdot 3^{3^x}\,dx$

114. Evaluate: $\int \tan^3 x \cdot \sec^2 x dx$

115. Evaluate: $\int \sin^3 x \cdot \cos x dx$

116. Evaluate: $\int \frac{(\log x)^3}{x}\,dx$

117. Evaluate: $\int \frac{\sin x}{\sqrt{3+2\cos x}}\,dx$

118. Evaluate: $\int \frac{\sqrt{2+\log x}}{x}\,dx$

119. Evaluate: $\int \frac{dx}{1+\sqrt{x}}$

120. Evaluate: $\int x^3 \sin x^4\,dx$

121. Evaluate: $\int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x\,dx$

122. Evaluate: $\int \frac{\sin x - \cos x}{e^x + \sin x}\,dx$

123. Evaluate: $\int \frac{dx}{x(1+x^3)}$

Type 9

Case I:

124. Evaluate: $\int \sin^3 x \cdot \cos^4 x\,dx$

125. Evaluate: $\int \sin x \cdot \cos^6 x\,dx$

126. Evaluate: $\int \sin^5 x \cdot \cos^9 x\,dx$

Case II:

127. Evaluate: $\int \sin^2 x \cdot \cos^3 x\,dx$

128. Evaluate: $\int \sin^4 x \cdot \cos^3 x\,dx$

129. Evaluate: $\int \sin^6 x \cdot \cos^5 x\,dx$

Case III:

130. Evaluate: $\int \sin^3 x \cdot \cos^3 x\,dx$

131. Evaluate: $\int \sin^5 x \cdot \cos^5 x\,dx$

132. Evaluate: $\int \sin^5 x \cdot \cos^7 x\,dx$

Case IV:

133. Evaluate: $\int \sin^2 x \cdot \cos^2 x\,dx$

134. Evaluate: $\int \sin^2 x \cdot \cos^4 x\,dx$

135. Evaluate: $\int \sin^4 x \cdot \cos^2 x \, dx$

Case V:

136. Evaluate: $\int \sin^3 x \, dx$

137. Evaluate: $\int \sin^5 x \, dx$

138. Evaluate: $\int \sin^7 x \, dx$

Case VI:

139. Evaluate: $\int \cos^5 x \, dx$

140. Evaluate: $\int \cos^3 x \, dx$

141. Evaluate: $\int \cos^7 x \, dx$

Case VII:

142. Evaluate: $\int \sin^4 x \, dx$

143. Evaluate: $\int \sin^2 x \, dx$

144. Evaluate: $\int \sin^6 x \, dx$

Case VIII:

145. Evaluate: $\int \cos^6 x \, dx$

146. Evaluate: $\int \cos^2 x \, dx$

147. Evaluate: $\int \cos^4 x \, dx$

Case IX:

148. Evaluate: $\int \frac{dx}{\sin^{1/2} x \cos^{3/2} x}$

149. Evaluate: $\int \frac{dx}{\sin^{3/2} x \cos^{5/2} x}$

150. Evaluate: $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

151. Evaluate: $\int \frac{\sin x}{\cos^5 x} dx.$

152. Evaluate: $\int \frac{dx}{\sin^3 x \cos^5 x}$

153. Evaluate: $\int \frac{\sin^2 x \, dx}{\cos^6 x}$

154. Evaluate: $\int \frac{dx}{\sin x \cos^3 x}$

155. Evaluate: $\int \frac{dx}{\sin^2 x \cos^4 x}$

156. Evaluate: $\int \frac{dx}{\sin^{\frac{1}{2}} x \cdot \cos^{\frac{7}{2}} x}$

157. Evaluate: $\int \frac{dx}{\sqrt{\sin^3 x \cdot \cos^5 x}}$

ABC of Formula 6

158. Evaluate: $\int \frac{dx}{x^2 + 4}$

159. Evaluate: $\int \frac{dx}{9x^2 + 1}$

160. Evaluate: $\int \frac{dx}{x^2 - 4}$

161. Evaluate: $\int \frac{x^4 - 1}{x^2 + 5dx}$

162. Evaluate: $\int \frac{dx}{\sqrt{x^4 + 4}}$

163. Evaluate: $\int \frac{dx}{\sqrt{4x^2 + 1}}$

164. Evaluate: $\int \left(\frac{x + 4}{x^3 + 4x} \right) dx$

165. Evaluate: $\int \left(\frac{x^4 + 1}{x^2 + 1} \right) dx$

166. Evaluate: $\int \frac{dx}{\sqrt{(2 - x)^2 + 1}}$

167. Evaluate: $\int \frac{x + 9}{x^3 + 9x} dx$

168. Evaluate: $\int \frac{1 + x}{1 + x^2} dx$

169. Evaluate: $\int \frac{1 + x}{x^3 + x} dx$

170. Evaluate: $\int \frac{dx}{x^4 + 1}$

171. Evaluate: $\int \frac{dx}{x^3 + x}$

Type 10

172. Evaluate: $\int \frac{dx}{x^3 + 4x + 4}$

173. Evaluate: $\int \frac{dx}{x^2 + 6x + 10}$

174. Evaluate: $\int \frac{dx}{2x^2 + 5x + 6}$

175. Evaluate: $\int \frac{dx}{x^2 + x + 1}$

176. Evaluate: $\int \frac{dx}{1 + x + x^2}$

177. Evaluate: $\int \frac{dx}{x^2 + 4x + 3}$

178. Evaluate: $\int \frac{dx}{4x^2 + 7x + 10}$

179. Evaluate: $\int \frac{dx}{x^2 - 2ax}$

180. Evaluate: $\int \frac{dx}{x^2 + 2ax}$

181. Evaluate: $\int \frac{dx}{a^2 + 2ax}$

182. Evaluate: $\int \frac{dx}{2ax - x^2}$

183. Evaluate: $\int \frac{dx}{(x^2 + 1)(x^2 + 4)}$

184. Evaluate: $\int \frac{x^2}{x^6 + 1dx}$

185. Evaluate: $\int \frac{\cos x\, dx}{\sin^2 x + 3\sin x + 2}$

186. Evaluate: $\int \frac{x^x(1 + \log x)}{x^{2x} + x^x + 1}\, dx$

187. Evaluate: $\int \frac{dx}{x(x^4 + 1)}$

188. Evaluate: $\int \frac{x\, dx}{x^4 + x^2 + 1}$

189. Evaluate: $\int \frac{e^x\, dx}{e^{2x} + 6e^x + 5}$

190. Evaluate: $\int \frac{3x^5}{1 + x^{12}}\, dx$

191. Evaluate: $\int \frac{dx}{x(x^4 + 1)}$

192. Evaluate: $\int \frac{dx}{x(x^3 + 1)}$

193. Evaluate: $\int \frac{dx}{x(x^n + 1)}$

Type 11

194. Evaluate: $\int \frac{2x + 3}{x^2 + 4x + 5}\, dx$

195. Evaluate: $\int \frac{3x + 2}{x^2 - 3x + 4}\, dx$

195. Evaluate: $\int \frac{x}{x^2 + x + 1}\, dx$

196. Evaluate: $\int \frac{4x + 1}{x^2 + 3x + 2}\, dx$

197. Evaluate: $\int \frac{dx}{2e^{2x} + 3e^x + 1}$

198. Evaluate: $\int \frac{(3\sin x - 2)\cos x\, dx}{(5 - \cos^2 x - 4\sin x)}$

199. Evaluate: $\int \frac{ax^3 + bx}{x^4 + c^2}\, dx.$

200. Evaluate: $\int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) \times (2 + 2\sin 2x)\, dx$

201. Evaluate: $\int \frac{\sin x + \cos x}{5 + 3\sin 2x}\, dx$

202. Evaluate: $\int \frac{\sin x - \cos x}{3 + 5\sin 2x}\, dx$

Type 12

203. Evaluate: $\int \frac{dx}{\sqrt{x^2 + x + 1}}$

204. Evaluate: $\int \frac{dx}{\sqrt{x^2 - 2ax}}$

205. Evaluate: $\int \frac{dx}{\sqrt{4x - x^2}}$

206. Evaluate: $\int \frac{dx}{\sqrt{6 - x - x^2}}$

207. Evaluate: $\int \frac{dx}{\sqrt{1 + x + x^2}}$

208. Evaluate: $\int \frac{dx}{\sqrt{1 + x - x^2}}$

209. Evaluate: $\int \frac{dx}{\sqrt{x^2 + 2ax}}$

210. Evaluate: $\int \frac{dx}{\sqrt{2ax - x^2}}$

Type 13

211. Evaluate: $\int \frac{x\, dx}{\sqrt{x^4 - x^2 + 1}}$

212. Evaluate: $\int \sqrt{\sec x - 1}\, dx$

213. Evaluate: $\int \frac{dx}{x^{3/4}\sqrt{x^{1/2} - 1}}$

214. Evaluate: $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\, dx$

215. Evaluate: $\int \frac{e^x}{\sqrt{4-e^{2x}}}\, dx$

216. Evaluate: $\int \frac{\sec^2 x}{\sqrt{16+\tan x}}\, dx$

217. Evaluate: $\int \sqrt{\sec x + 1}\, dx$

218. Evaluate: $\int \sqrt{\operatorname{cosec} x - 1}\, dx$

219. Evaluate: $\int \frac{dx}{\sqrt{1-e^{2x}}}$

220. Evaluate: $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\, dx$

221. Evaluate: $\int \frac{dx}{x^{2/3}\sqrt{x^{2/3}-4}}$

222. Evaluate: $\int \sqrt{\frac{x}{x^3-x^3}}\, dx$

223. Evaluate: $\int \left(\frac{\cos\theta+\sin\theta}{\sqrt{5+\sin 2\theta}}\right) d\theta$

224. Evaluate: $\int \frac{\sin\theta-\cos\theta}{\sqrt{2-\sin 2\theta}}\, d\theta$

Type 14

225. Evaluate: $\int \frac{x-1}{\sqrt{x^2-3x+2}}\, dx$

226. Evaluate: $\int \frac{3x+4}{\sqrt{x^2+5x+2}}\, dx$

227. Evaluate: $\int \frac{x+2}{\sqrt{x^2+5x+6}}\, dx$

228. Evaluate: $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}\, dx$

229. Evaluate: $\int \sqrt{\frac{a-x}{a+x}}\, dx$

230. Evaluate: $\int x^2\sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx$

231. Evaluate: $\int x^2\sqrt{\frac{4-x^3}{4+x^3}}\, dx$

Type 15

232. Evaluate: $\int \frac{dx}{3+4\sin^2 x}$

233. Evaluate: $\int \frac{dx}{3\sin^2 x+4\cos^2 x}$.

234. Evaluate: $\int \frac{dx}{(2\sin x+3\cos x)^2}$

235. Evaluate: $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\, dx$

236. Evaluate: $\int \frac{dx}{(\sin x+2\cos x)^2}$

237. Evaluate: $\int \frac{dx}{(\sin x+2\sec x)^2}$

238. Evaluate: $\int \frac{\sin 2x\, dx}{\sin^4 x+\cos^4 x}$

239. Evaluate: $\int \frac{dx}{(2\sin x+3\cos x)^2}$

240. Evaluate: $\int \frac{dx}{2+\cos^2 x}$

Type 16

241. Evaluate: $\int \frac{\sin x}{\sin 3x}\, dx$

242. Evaluate: $\int \frac{\operatorname{cosec} 3x}{\operatorname{cosec} x}\, dx$

243. Evaluate: $\int \frac{\sec 3x}{\sec x}\, dx$

Type 17

244. Evaluate: $\int \frac{dx}{1+2\sin x}$

245. Evaluate: $\int \frac{dx}{3\cos x+4}$

246. Evaluate: $\int \frac{dx}{1+\sin x+\cos x}$

247. Evaluate: $\int \frac{1+\sin x}{\sin x(1+\cos x)}\, dx$

248. Evaluate: $\int \frac{dx}{1+2\sin x}$

249. Evaluate: $\int \frac{dx}{3\sin x+4\cos x+5}$

250. Evaluate: $\int \frac{dx}{\cos x+\cos\alpha}$

251. Evaluate: $\int \frac{dx}{\tan x+4\cot x+4}$

Type 18

252. Evaluate: $\int \frac{dx}{\sin x+\cos x}$

253. Evaluate: $\int \frac{dx}{\sqrt{3}\sin x + \cos x}$

254. Evaluate: $\int \frac{dx}{\sin x + \sqrt{3}\cos x}$

255. Evaluate: $\int \frac{dx}{\sqrt{3}\sin x + \cos x}$

256. Evaluate: $\int \frac{dx}{\sqrt{3}\sin x - \cos x}$.

Type 19

257. Evaluate: $\int \frac{2\sin x + \cos x}{3\sin x + 2\cos x}\,dx$

258. Evaluate: $\int \frac{\sin x}{\sin x + \cos x}\,dx$

259. Evaluate: $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$

260. Evaluate: $\int \left(\frac{\sin x}{\sin x + \cos x}\right)dx$

261. Evaluate: $\int \frac{\cos x}{\sin x + \cos x}\,dx$

262. Evaluate: $\int \frac{1}{1 + \tan x}\,dx$

263. Evaluate: $\int \frac{1}{1 - \tan x}\,dx$

264. Evaluate: $\int \frac{1}{1 + \tan x}\,dx$

Type 20

265. Evaluate: $\int \frac{3\sin x + 2\cos x + 4}{3\cos x + 4\sin x + 5}\,dx$

266. Evaluate: $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

267. Evaluate: $\int \frac{2\cos x + 3\sin x}{2\sin x + 3\cos x + 5}\,dx$

Type 21

268. Evaluate: $\int \left(\frac{\sin x + 2\cos x}{9 + 16\sin 2x}\right)dx$

269. Evaluate: $\int (\cos x - \sin x)(2 + 3\sin 2x)\,dx$

270. Evaluate: $\int \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

271. Evaluate: $\int \frac{dx}{\cos x + \operatorname{cosec} x}$

272. Evaluate: $\int (\sin x + \cos x)(2 + 3\sin 2x)\,dx$

273. Evaluate: $\int (\sin x - \cos x)(3 - 4\sin 2x)\,dx$

274. Evaluate: $\int \left(\frac{\cos x - \sin x}{3 + 2\sin 2x}\right)dx$

275. Evaluate: $\int \left(\frac{\cos x - \sin x}{5 - 7\sin 2x}\right)dx$

276. Evaluate: $\int \left(\frac{2\cos x - \sin x}{9 + 16\sin 2x}\right)dx$

Type 22

277. Evaluate: $\int \frac{x^2 + 1}{x^4 + 1}\,dx$

278. Evaluate: $\int \frac{x^2 - 1}{x^4 + 1}\,dx$

279. Evaluate: $\int \frac{x^4 + 1}{x^6 + 1}\,dx$

280. Evaluate: $\int \frac{x^4 + 3x + 1}{x^4 + x^2 + 1}\,dx$

281. Evaluate: $\int \left(\frac{1 - x^2}{1 + x^2}\right)\frac{dx}{\sqrt{1 + x^2 + x^4}}$

282. Evaluate: $\int \left(\frac{x - 1}{x + 1}\right) \times \frac{dx}{\sqrt{x^3 + x^2 + x}}$

283. Evaluate: $\int \frac{x^2 - 1}{x^2 + 1} \times \frac{dx}{\sqrt{x^4 + 1}}$

284. Evaluate: $\int \frac{x^2 - 1}{x^3\sqrt{x^4 - 2x^2 + 1}}\,dx$

285. Evaluate: $\int \left(\frac{x^4 - 1}{x^2\sqrt{x^4 + x^2 + 1}}\right)dx$

286. Evaluate: $\int \frac{dx}{x^2(x + \sqrt{1 + x^2})}$

287. Evaluate: $\int \frac{x^2 - 3x - 1}{x^4 + x^2 + 1}\,dx$

288. Evaluate: $\int \frac{dx}{\sin^4 x + \cos^4 x}$

289. Evaluate: $\int \frac{dx}{x^4 + 1}$

290. Evaluate: $\int \frac{x^2 dx}{x^4 + 1}$

291. Evaluate: $\int \frac{2dx}{x^4 + 1}$.

292. Evaluate: $\int \left(\frac{x^4 + 1}{x^6 + 1}\right)dx$

293. Evaluate: $\int \frac{dx}{x^2(1 + x^4)^{3/4}}$

294. Evaluate: $\int \left(\frac{x^4 - 1}{x^2\sqrt{x^4 + x^2 + 1}}\right)dx$

295. Evaluate: $\int \frac{dx}{x^2(1+x^5)^{4/5}}\,dx$

296. Evaluate: $\int \left(\frac{x^2-1}{x\sqrt{1+x^4}}\right) dx$

297. Evaluate: $\int \frac{dx}{x\sqrt{x^4+3x^2+1}}$

298. Evaluate: $\int \left(\frac{x^2+1}{1-x^2}\right) \times \frac{dx}{\sqrt{x^4+x^2+1}}$

299. Evaluate: $\int \left(\frac{x^2+1}{x}\right) \times \frac{dx}{\sqrt{x^4+3x^2+1}}$

300. Evaluate: $\int \frac{x^x(x^{2x}+1)(\ln x+1)}{(x^{4x}+1)}\,dx$

301. Evaluate: $\int \frac{x^2-1}{x^3\sqrt{x^4-2x^2+1}}\,dx$

302. Evaluate: $\int \left(\frac{x^{2009}}{(1+x^2)^{1006}}\right) dx$

303. Evaluate: $\int \frac{(2x+1)\,dx}{(x^2+4x+1)^{3/2}}$

Type 23

304. Evaluate: $\int \left(\frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}\right) dx$

305. Evaluate: $\int \sqrt{\tan x}\,dx$

306. Evaluate: $\int (\sqrt{\tan x}-\sqrt{\cot x})\,dx$

307. Evaluate: $\int \sqrt{\cot x\,dx}$

308. Evaluate: $\int \left(\sqrt{\cot x}-\frac{1}{\sqrt{\cot x}}\right) dx$

309. Evaluate: $\int \left(\sqrt{\cot x}+\frac{1}{\sqrt{\cot x}}\right) dx$

310. Evaluate: $\int \left(\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}\right) dx$

311. Evaluate: $\int (\sqrt[4]{\tan x}+\sqrt[4]{\cot x})^2\,dx$

312. Evaluate: $\int \frac{dx}{(\sqrt{\sin x}+\sqrt{\cos x})^4}\,dx$

Integration by Parts

ABC of Integration by Parts

313. Evaluate: $\int x e^x\,dx$

314. Evaluate: $\int x\sin x\,dx$

315. Evaluate: $\int x^2\sin x\,dx$

316. Evaluate: $\int \log x\,dx$

317. Evaluate: $\int (\log x)^2 dx$

318. Evaluate: $\int \log(x^2+1)\,dx$

319. Evaluate: $\int \tan^{-1}x\,dx$

320. Evaluate: $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

321. Evaluate: $\int e^{\sqrt{x}}\,dx$

322. Evaluate: $\int \left(\frac{x+\sin x}{1+\cos x}\right) dx$

323. Evaluate: $\int \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) dx$

324. Evaluate: $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$

325. Evaluate: $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$

326. Evaluate: $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$

327. Evaluate: $\int \sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right) dx$

328. Evaluate: $\int \log(1+x)\,dx$

329. Evaluate: $\int \frac{x-\sin x}{1-\cos x}\,dx$

330. Evaluate: $\int \sin\sqrt{x\,dx}$

331. Evaluate: $\int x\log(1+x)\,dx$

332. Evaluate: $\int (\sin^{-1}x)^2\,dx$

333. Evaluate: $\int \frac{x-\sin x}{1-\cos x}\,dx$

334. Evaluate: $\int x\sin^3 x\,dx$

335. Evaluate: $\int x\left(\frac{\sec 2x-1}{\sec 2x+1}\right) dx$

336. Evaluate: $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

337. Evaluate: $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

338. Evaluate: $\int \frac{x^3\sin^{-1}(x^2)}{\sqrt{1-x^4}}\,dx$

339. Evaluate: $\int \frac{\sec x(2+\sec x)}{(1+2\sec x)^2}\,dx$

340. Evaluate: $\int \frac{1-x}{e^x+x}\,dx.$

Type 24

341. Evaluate: $\int e^x(\sin x+\cos x)\,dx$

342. Evaluate: $\int \frac{x e^x}{(x+1)^2}\,dx$

343. Evaluate: $\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$

344. Evaluate: $\int e^x\left(\frac{1+x+x^3}{(1+x^2)^{3/2}}\right)dx$

345. Evaluate: $\int e^x\left(\frac{x^2+1}{(x+1)^2}\right)dx$

346. Evaluate: $\int e^x\left\{\frac{1}{x}-\frac{1}{x^2}\right\}dx$

347. Evaluate: $\int e^x\left(\frac{x}{(x+1)^2}\right)dx$

348. Evaluate: $\int e^x\left\{\frac{1-\sin x}{1-\cos x}\right\}dx$

349. Evaluate: $\int e^x\left\{\frac{2+\sin 2x}{1+\cos 2x}\right\}dx$

350. Evaluate: $\int e^x\left(\frac{x^2+1}{(x+1)^2}\right)dx$

351. Evaluate: $\int \frac{\log x}{(1+\log x)^2}\,dx$

352. Evaluate: $\int e^x\left(\log x+\frac{1}{x}\right)dx$

353. Evaluate: $\int e^x\left(\frac{x-1}{(x+1)^3}\right)dx$

354. Evaluate: $\int\left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx$

355. Evaluate: $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$

356. Evaluate: $\int e^x\left(\frac{1-x}{1+x}\right)^2dx$

357. Evaluate: $\int e^x\left(\frac{x^2+1}{(x+1)^2}\right)dx$

358. Evaluate: $\int e^x\left(\frac{(x+1)+\sqrt{1-x^2}}{(x+1)^2\sqrt{1-x^2}}\right)dx$

359. Evaluate: $\int e^x(2\sec^2 x-1)\tan x\,dx$

360. Evaluate: $\int e^x(\log(\sec x+\tan x)+\sec x)\,dx$

361. Evaluate: $\int e^{\sin x}\left(\frac{x\cos^3 x-\sin x}{\cos^2 x}\right)dx$

Type 25

362. Evaluate: $\int e^x\left(\log x+\frac{1}{x^2}\right)dx$

363. Evaluate: $\int\left\{\log(\log x)+\frac{1}{(\log x)^2}\right\}dx$

364. Evaluate: $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$

365. Evaluate: $\int \frac{e^{\sin x}(x\cos^3 x-\sin x)}{\cos^2 x}\,dx$

366. Evaluate: $\int e^x\left(\frac{x^3-x+2}{(x^2-1)^2}\right)dx$

Type 26

367. Evaluate: $\int e^{3x}(3\sin x+\cos x)\,dx$

368. Evaluate: $\int e^{2x}(\sec^2 x+2\tan x)\,dx$

369. Evaluate: $\int e^{2x}\left(\frac{2\sin 4x-4}{1-\cos 4x}\right)dx$

370. Evaluate: $\int e^{2x}(-\sin x+2\cos x)\,dx$

371. Evaluate: $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$

372. Evaluate: $\int e^{-x/2}\left(\sqrt{\frac{1-\sin x}{1+\cos x}}\right)dx$

373. Evaluate: $\int e^{2x}(2\times\log(\sec x+\tan x)+\sec x)\,dx$

Type 27

374. Evaluate: $\int e^x\sin 3x\,dx$

375. Evaluate: $\int e^{4x}\cos 3x\,dx$

376. Evaluate: $\int e^{2x}\sin 3x\,dx$

377. Evaluate: $\int e^{-x}\cos x\,dx$

378. Evaluate: $\int e^{2x}\cos(3x+4)\,dx$

379. Evaluate: $\int e^x\cos 2x\,dx$

380. Evaluate: $\int \frac{1}{x^3}\sin(\log x)\,dx.$

Type 28

381. Evaluate: $\int \sqrt{4 - x^2}\, dx$

382. Evaluate: $\int \sqrt{1 - 9x^2}\, dx$

383. Evaluate: $\int \sqrt{x^2 + 1}\, dx$

384. Evaluate: $\int \sqrt{3x^2 + 1}\, dx$

385. Evaluate: $\int \sqrt{x^2 - 9}\, dx$

386. Evaluate: $\int \sqrt{4x^2 - 1}\, dx$

387. Evaluate: $\int \sqrt{x^2 + 2x + 3}\, dx$

388. Evaluate: $\int \sqrt{3 - 4x - x^2}\, dx$

389. Evaluate: $\int \sqrt{2ax - x^2}\, dx$

390. Evaluate: $\int \sqrt{2ax + x^2}\, dx$

391. Evaluate: $\int \sqrt{x - 4x^2}\, dx$

Type 29

392. Evaluate: $\int (2x + 1)\sqrt{x^2 + 3x + 4}\, dx$

393. Evaluate: $\int (x - 5)\sqrt{x^2 + x}\, dx$

394. Evaluate: $\int (3x - 2)\sqrt{x^2 + x + 1}\, dx$

395. Evaluate: $\int (4x + 1)\sqrt{x^2 - x - 2}\, dx$

396. Evaluate: $\int x\sqrt{1 + x - x^2}\, dx.$

Partial Fractions

Type 1

397. Evaluate: $\int \frac{(2x + 1)\,dx}{(x + 2)(x + 3)}$

398. Evaluate: $\int \frac{dx}{(x - 1)(x - 2)}$

399. Evaluate: $\int \frac{dx}{(x + 1)(x + 2)(x + 3)}$

400. Evaluate: $\int \frac{dx}{(x + 1)(x + 2)(x + 3)(x + 4)}$

401. Evaluate: $\int \frac{x - 1}{(x + 1)(x - 2)}\, dx$

402. Evaluate: $\int \frac{2x - 1}{(x + 1)(x + 2)(x + 3)}\, dx$

403. Evaluate: $\int \frac{x^3}{(x - 1)(x - 2)}\, dx$

404. Evaluate: $\int \frac{2x}{(x^2 + 1)(x^2 + 2)}\, dx$

405. Evaluate: $\int \frac{\cos\theta}{(2 + \cos\theta)(3 + \cos\theta)}\, d\theta$

406. Evaluate: $\int \frac{(1 - \cos x)}{\cos x(1 + \cos x)}\, dx$

407. Evaluate: $\int \frac{dx}{\sin x - \sin 2x}$

Type 2

408. Evaluate: $\int \frac{dx}{(x + 1)(x + 1)^2}$

409. Evaluate: $\int \frac{2x + 1}{(x + 2)(x - 3)^2}\, dx$

410. Evaluate: $\int \frac{3x + 1}{(x - 2)^2(x + 2)}\, dx$

411. Evaluate: $\int \frac{x^2 + 1}{(x - 1)^2(x + 3)}\, dx$

412. Evaluate: $\int \frac{x^2}{(x - 1)^3(x + 1)}\, dx$

413. Evaluate: $\int \frac{(x - 1)}{x^2(x + 4)}\, dx$

414. Evaluate: $\int \frac{(2x - 1)}{x^3(x - 2)}\, dx$

Type 3

415. Evaluate: $\int \frac{(2x + 3)\,dx}{(x + 1)(x^2 + 4)}$

416. Evaluate: $\int \frac{(3x - 2)\,dx}{(x - 1)(x^2 + 9)}$

417. Evaluate: $\int \frac{2x - 1}{(x + 1)(x^2 + 2)}\, dx$

418. Evaluate: $\int \frac{x}{(x + 1)(x^2 + 4)}\, dx$

419. Evaluate: $\int \frac{8}{(x + 2)(x^2 + 9)}\, dx$

420. Evaluate: $\int \frac{x}{(x + 1)(x^2 + 1)}\, dx$

Type 4

421. Evaluate: $\int \frac{x^2}{(x^2 - 1)(x^2 + 1)}\, dx$

422. Evaluate: $\int \frac{x^2}{(x^2 - 3)(x^2 + 4)}\, dx$

423. Evaluate: $\int \frac{(x^2+4)}{(x^2+5)(x^2+7)}\,dx$

424. Evaluate: $\int \frac{x^2}{(x^2-1)(x^2-2)}\,dx$

425. Evaluate: $\int \frac{x^2}{(x^2-1)(x^2-2)(x^2-3)}\,dx$

Type 5

426. Evaluate: $\int \left(\frac{(x^2+3)(x^2+1)}{(x^2-1)(x^2+2)}\right)dx$

427. Evaluate: $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$

428. Evaluate: $\int \frac{(x^2-1)(x^2+3)}{(x^2+2)(x^2+1)}\,dx.$

Level IIA

(For JEE-Advanced Examination)

Type 1

1. Evaluate: $\int \tan^4 x \cdot \sec^2 x\,dx$
2. Evaluate: $\int \sec^3 x\,dx$
3. Evaluate: $\int \sec^5 x\,dx$
4. Evaluate: $\int \tan^2 x \cdot \sec^4 x\,dx$
5. Evaluate: $\int \tan^3 x \cdot \sec^6 x\,dx$
6. Evaluate: $\int \tan^3 x \cdot \sec^5 x\,dx$
7. Evaluate: $\int \sec^7 x\,dx$
8. Evaluate: $\int \sec^9 x\,dx$

Type 2

9. Evaluate: $\int \operatorname{cosec}^2 x \cdot \cot^2 x\,dx$
10. Evaluate: $\int \cot^3 x \cdot \operatorname{cosec}^3 x\,dx$
11. Evaluate: $\int \operatorname{cosec}^3 x\,dx$
12. Evaluate: $\int \cot^2 x \cdot \operatorname{cosec}^4 x\,dx$
13. Evaluate: $\int \tan^{-5} x \cdot \sec^6 x\,dx$
14. Evaluate: $\int \cot^3 x \cdot \operatorname{cosec}^{-8} x\,dx$
15. Evaluate: $\int \operatorname{cosec}^5 x\,dx$
16. Evaluate: $\int \operatorname{cosec}^7 x\,dx$

Type 3

17. Evaluate: $\int \frac{dx}{x(x^7+1)}$
18. Evaluate: $\int \frac{dx}{x^2(x^4+1)^{3/4}}$
19. Evaluate: $\int \frac{dx}{x(x^5+1)}$
20. Evaluate: $\int \frac{dx}{x(x^4+1)}$
21. Evaluate: $\int \frac{dx}{x^2(x^7+1)^{6/7}}$
22. Evaluate: $\int \frac{dx}{x^3(1+x^3)^{1/3}}$

Type 4

23. Evaluate: $\int \frac{x^2}{(x+3)^2}\,dx$
24. Evaluate: $\int \frac{x^3\,dx}{(2x+3)^2}$
25. Evaluate: $\int \frac{x^2}{(x+2)^3}\,dx$
26. Evaluate: $\int \frac{x^2}{(ax+b)^2}\,dx$
27. Evaluate: $\int \frac{dx}{x(1+x^3)^2}$
28. Evaluate: $\int \frac{x^4}{(3x-2)^3}\,dx$

Type 5

29. Evaluate: $\int \frac{dx}{x^2(3x+2)^3}$
30. Evaluate: $\int \frac{dx}{x^3(b+ax)^2}$
31. Evaluate: $\int \frac{dx}{x^2(x+2)^3}$
32. Evaluate: $\int \frac{dx}{x^3(a+bx)^2}$
33. Evaluate: $\int \frac{dx}{x^2(1+x^2)^3}$
34. Evaluate: $\int \frac{dx}{x^2(a-bx)^2}$
35. Evaluate: $\int \frac{dx}{x^4(2x+1)^3}$

Type 6

36. Evaluate: $\int \frac{dx}{(x-1)^3(x-2)^4}$

37. Evaluate: $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

38. Evaluate: $\int \frac{dx}{(x-1)^3(x-2)^2}$

39. Evaluate: $\int \frac{dx}{(x-3)^4(x-2)^5}$

40. Evaluate: $\int \frac{dx}{(x-3)^{3/2}(x-2)^{7/2}}$

41. Evaluate: $\int \frac{dx}{\sqrt[5]{(x+1)^4(x+3)^6}}$

42. Evaluate: $\int \frac{dx}{\sqrt[5]{(x+1)^4(x+3)^7}}$

Type 7

43. Evaluate: $\int \frac{dx}{x(2+3x^3)}$

44. Evaluate: $\int \frac{dx}{x(3+5x^5)}$

45. Evaluate: $\int \frac{dx}{x(2+3x^2)}$

46. Evaluate: $\int \frac{dx}{x(3+4x^3)}$

47. Evaluate: $\int \frac{dx}{x(2-5x^3)}$

48. Evaluate: $\int \frac{dx}{x(1-4x^4)}$

49. Evaluate: $\int \frac{dx}{x(3x^4+1)}$

Type 8

50. Evaluate: $\int \frac{x^5}{(x^2-1)^4}\,dx$

51. Evaluate: $\int \frac{x^9}{(2x^2+3)^5}\,dx$

52. Evaluate: $\int \frac{x^3}{(x^2+1)^4}\,dx$

53. Evaluate: $\int \frac{x^5}{(x^2-3)^4}\,dx$

54. Evaluate: $\int \frac{x^7}{(3x^2-2)^4}\,dx$

55. Evaluate: $\int \frac{8x^9}{(3x^2-2)^5}\,dx$

56. Evaluate: $\int \frac{10x^{11}}{(3x^2+5)^4}\,dx$

Type 9

57. Evaluate: $\int \frac{2\sin x+3}{(3\sin x+2)^2}\,dx$

58. Evaluate: $\int \frac{(2\sin x+5)}{(2+5\sin x)^2}\,dx$

59. Evaluate: $\int \frac{(3\sin x-2)}{(2-3\sin x)^2}\,dx$

Type 10

60. Evaluate: $\int \left(\frac{4\cos x+3}{(3\cos x+4)^2}\right)dx$

61. Evaluate: $\int \frac{(\cos x+2)}{(1+2\cos x)^2}\,dx$

62. Evaluate: $\int \frac{3\cos x+4}{(3+4\cos x)^2}\,dx$

Type 11

63. Evaluate: $\int \frac{dx}{(3+4\sin x)^2}$

64. Evaluate: $\int \frac{dx}{(5+4\sin x)^2}$

65. Evaluate: $\int \frac{dx}{(1-2\sin x)^2}$

Type 12

66. Evaluate: $\int \frac{dx}{(2+3\cos x)^2}$

67. Evaluate: $\int \frac{dx}{(12+13\cos x)^2}$

68. Evaluate: $\int \frac{dx}{(3-4\cos x)^2}$

Type 13

69. Evaluate: $\int \left(\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\right)dx.$

70. Evaluate: $\int \frac{3e^x-2e^{-x}}{2e^x+5e^{-x}}\,dx$

71. Evaluate: $\int \frac{4e^x+3e^{-x}}{3e^x+7e^{-x}}\,dx$

Integration of Irrational Functions

Type 1

72. Evaluate: $\int \frac{dx}{\sqrt{(x+2)^5(x+1)^3}}$

73. Evaluate: $\int \left(\frac{x+2}{2x+3}\right)^{1/2} \cdot \frac{dx}{x}$

74. Evaluate: $\int \frac{(x+1)}{(x+2)(x+3)^{3/2}}\, dx$

75. Evaluate: $\int \frac{2}{(2-x)^2} \sqrt[3]{\frac{2-x}{2+x}}\, dx$

Type 2

76. Evaluate: $\int \frac{dx}{\sqrt{x+1} + \sqrt[3]{x+1}}$

77. Evaluate: $\int \frac{dx}{\sqrt{x+1} - (x+1)^{1/4}}$

78. Evaluate: $\int \frac{1 + x^{1/2} - x^{1/3}}{1 + x^{1/3}}\, dx$

79. Evaluate: $\int \frac{dx}{x^{1/2} + x^{1/3}}$

80. Evaluate: $\int \frac{\sqrt{x}}{\sqrt[4]{x^3} + 1}\, dx$

81. Evaluate: $\int \frac{\sqrt{x}}{\sqrt{x} + \sqrt[3]{x}}\, dx$

Type 3

82. Evaluate: $\int \frac{dx}{(x-1)\sqrt{x+3}}$

83. Evaluate: $\int \frac{dx}{(x+3)\sqrt{x+2}}$

84. Evaluate: $\int \frac{\sqrt{x}}{x+1}\, dx$

85 .Evaluate: $\int \frac{dx}{x\sqrt{x-2}}$

86. Evaluate: $\int \frac{dx}{(x+3)\sqrt{x}}$

87. Evaluate: $\int \frac{dx}{(x+3)\sqrt{2x+1}}$

Type 4

88. Evaluate: $\int \frac{dx}{x^2\sqrt{x-1}}$

89. Evaluate: $\int \frac{dx}{(x^2-4)\sqrt{x+1}}$

90. Evaluate: $\int \frac{dx}{(x^2+1)\sqrt{x}}$

91. Evaluate: $\int \frac{x\,dx}{(x^2+2x+2)\sqrt{x+1}}$

92. Evaluate: $\int \frac{dx}{(x^2-1)\sqrt{x}}$

93. Evaluate: $\int \frac{x\,dx}{(x^2-2x+2)\sqrt{x-1}}$

Type 5

94. Evaluate: $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$

95. Evaluate: $\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$

96. Evaluate: $\int \frac{dx}{x\sqrt{x^2+4}}$

97. Evaluate: $\int \frac{dx}{(x-1)\sqrt{x^2+4}}$

98. Evaluate: $\int \frac{dx}{(2x-1)\sqrt{x^2+1}}$

99. Evaluate: $\int \frac{dx}{(3x+2)\sqrt{x^2-4}}$

Type 6

100. Evaluate: $\int \frac{dx}{x^2\sqrt{x^2-1}}$

101. Evaluate: $\int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$

102. Evaluate: $\int \frac{dx}{(x^2-1)\sqrt{x^2+2}}$

103. Evaluate: $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

104. Evaluate: $\int \frac{x\,dx}{(x^4-1)\sqrt{x^4+3}}$

105. Evaluate: $\int \frac{dx}{(x^2-1)\sqrt{x^2+4x+5}}$

Type 7

106. Evaluate: $\int (x + \sqrt{x^2+1})^{10}\, dx$

107. Evaluate: $\int (x - \sqrt{x^2+4})^5\, dx$

108. Evaluate: $\int (x + \sqrt{1+x^2})^n\, dx$

109. Evaluate: $\int \frac{dx}{(x+\sqrt{x^2-4})^{5/3}}$

110. Evaluate: $\int \frac{dx}{(x-\sqrt{x^2+9})^{10}}$

111. Evaluate: $\int \frac{dx}{x^2(x-\sqrt{x^2+9})}$

Type 8

112. Evaluate: $\int \frac{dx}{x^{1/2}(2+3x)^{3/2}}$

113. Evaluate: $\int \frac{dx}{x^{2/3}(2+3x)^{4/3}}$

114. Evaluate: $\int \frac{dx}{x^{3/4}(3x-1)^{5/4}}$

115. Evaluate: $\int \frac{dx}{x^{1/3}(2x+1)^{5/3}}$

116. Evaluate: $\int \frac{dx}{x^{1/2}(2+3x)^{5/2}}$

117. Evaluate: $\int \frac{dx}{x^2(2+3x^2)^{5/2}}$

Type 9

118. Evaluate: $\int \frac{dx}{(x-1)^3(x+2)^4}$

119. Evaluate: $\int \frac{dx}{(x-1)^3(x-2)^2}$

120. Evaluate: $\int \frac{dx}{(x-1)^2(x-2)^3}$

121. Evaluate: $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

122. Evaluate: $\int \frac{dx}{x^2(x+5)^4}$

123. Evaluate: $\int \frac{dx}{(x-1)^{3/2}(x+1)^{5/2}}$

Type 10

124. Evaluate: $\int \frac{dx}{x\sqrt{3x^3+4}}$

125. Evaluate: $\int \frac{dx}{x\sqrt{5x^4+3}}$

126. Evaluate: $\int \frac{dx}{x\sqrt{2-5x^6}}$

127. Evaluate: $\int \frac{dx}{x\sqrt{3x-2}}$

128. Evaluate: $\int \frac{dx}{x\sqrt{3x^9-2}}$

129. Evaluate: $\int \frac{dx}{x\sqrt{2x^{10}-3}}$

Type 11

130. Evaluate: $\int \frac{dx}{(2+3x^2)^{3/2}}$

131. Evaluate: $\int \frac{dx}{(c+dx^2)^{3/2}}$

132. Evaluate: $\int \frac{dx}{(3+5x^2)^{3/2}}$

133. Evaluate: $\int \frac{dx}{(3-4x^2)^{3/2}}$

134. Evaluate: $\int \frac{x\,dx}{(2-5x^4)^{3/2}}$

135. Evaluate: $\int \frac{x^2\,dx}{(1-4x^6)^{3/2}}$

Type 12

136. Evaluate: $\int \frac{dx}{(x-2)^2\sqrt{x^2-4x+7}}$

137. Evaluate: $\int \frac{dx}{(x+1)^3\sqrt{x^2+2x+4}}$

138. Evaluate: $\int \frac{dx}{(x-2)^3\sqrt{4x^2-16x+20}}$

139. Evaluate: $\int \frac{dx}{(x^2-6x+9)\sqrt{4x^2-24x+20}}$

140. Evaluate: $\int \frac{dx}{(4x^2+4x+1)\sqrt{4x^2+4x+7}}$

141. Evaluate: $\int \frac{dx}{(x+1)^3\sqrt{x^2+2x-4}}$

Type 13

142. Evaluate: $\int \frac{(2x+3)}{(3x+4)\sqrt{x^2+2x+4}}\,dx$

143. Evaluate: $\int \frac{(2x+3)}{(x+1)\sqrt{x^2+2x+9}}\,dx$

144. Evaluate: $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+7}}\,dx$

Type 14

145. Evaluate: $\int \frac{x^2 + 4x + 2}{(x+1)\sqrt{x^2 + 2x + 3}}\, dx$

146. Evaluate: $\int \frac{x^2 + 5x + 6}{(x+2)\sqrt{x^2 + 5x + 4}}\, dx$

147. Evaluate: $\int \frac{x^2 + 10x + 6}{(x+2)\sqrt{x^2 + 4x + 9}}\, dx$

Type 15

148. Evaluate: $\int \sqrt[3]{x}\,(1 + \sqrt{x})^3\, dx$

149. Evaluate: $\int \sqrt[3]{x^2}\,(3 + x^{-2/3})^{-2}\, dx$

150. Evaluate: $\int \left(\frac{\sqrt{1 + \sqrt[4]{x}}}{\sqrt[3]{x^4}}\right) dx$

151. Evaluate: $\int \frac{dx}{x^7\sqrt[2]{1 + x^4}}$

152. Evaluate: $\int \frac{dx}{\sqrt{x}\,(\sqrt[4]{x} + 1)^{10}}$

153. Evaluate: $\int x^{-1/2}(2 + 3x^{1/3})^{-2}\, dx$

154. Evaluate: $\int \sqrt[3]{x} \times \sqrt[7]{\left(1 + \sqrt[3]{x^4}\right)}\, dx$

155. Evaluate: $\int x^{-6}(1 + 2x^3)^{2/3}\, dx$

156. Evaluate: $\int \frac{dx}{x\sqrt[3]{1 + x^5}}$

157. Evaluate: $\int \frac{dx}{x^{11}\sqrt[2]{1 + x^4}}$

158. Evaluate: $\int \left(\frac{\sqrt[3]{(1 + \sqrt[4]{x})}}{\sqrt[4]{x^3}}\right) dx$

Type 16

159. Evaluate: $\int \frac{dx}{(1 + \sqrt{x^2 + x + 1})}$

160. Evaluate: $\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$

161. Evaluate: $\int \frac{x\,dx}{\sqrt{7x - 10 - x^2}}$

162. Evaluate: $\int \frac{dx}{x - \sqrt{x^2 - x + 2}}$

163. Evaluate: $\int \frac{dx}{x - \sqrt{x^2 - 2x + 4}}$

164. Evaluate: $\int \frac{dx}{x\sqrt{x^2 - 3x + 2}}$

165. Evaluate: $\int \frac{dx}{x + \sqrt{x^2 - 1}}$

166. Evaluate: $\int \frac{x\,dx}{x + \sqrt{x^2 - 1}}$

Reduction Formulae

Type 1

167. Evaluate: $\int \sin^5 x\, dx$

168. Evaluate: $\int \sin^6 x\, dx$

Type 2

169. Evaluate: $\int \cos^7 x\, dx$

170. Evaluate: $\int \cos^8 x\, dx$

Type 3

171. Evaluate: $\int \tan^5 x\, dx$

172. Evaluate: $\int \tan^6 x\, dx$

Type 4

173. Evaluate: $\int \cot^7 x\, dx$

174. Evaluate: $\int \cot^6 x\, dx$

Type 5

175. Evaluate: $\int \sec^3 x\, dx$

176. Evaluate: $\int \sec^4 x\, dx$

177. Evaluate: $\int \sec^5 x\, dx$

178. Evaluate: $\int \sec^7 x\, dx$

Type 6

179. Evaluate: $\int \frac{\sin 3x}{\sin x}\, dx$

180. Evaluate: $\int \frac{\sin 5x}{\sin x}\, dx$

181. Evaluate: $\int \frac{\sin 6x}{\sin x}\, dx$

182. Evaluate: $\int \frac{\sin 8x}{\sin x}\, dx$

Type 7

183. Evaluate: $\int \frac{dx}{(x^2 + 2)^2}$

184. Evaluate: $\int \frac{dx}{(x^2+3)^3}$

185. Evaluate: $\int \frac{x+1}{(x^2+3x+2)^2}\,dx$

Type 8

186. Evaluate: $\int x^2 \log x\,dx$

187. Evaluate: $\int x^2 (\log x)^2\,dx$

188. Evaluate: $\int x^3 (\log x)^2\,dx$

189. Evaluate: $\int x^2 (1-x)^3\,dx$

Mixed Problems

190. Evaluate: $\int \sin^8 x\,dx$

191. Evaluate: $\int \cos^{10} x\,dx$

192. Evaluate: $\int \tan^8 x\,dx$

193. Evaluate: $\int \cot^8 x\,dx$

194. Evaluate: $\int \cot^9 x\,dx$

195. Evaluate: $\int \sec^9 x\,dx$

196. Evaluate: $\int x^n e^x\,dx$

197. Evaluate: $\int \frac{dx}{(x^2+2)^2}$

Level IIB

(Mixed Problems)

1. $\int x^2 \cos x\,dx =$

(a) $x^2 \sin x + 2x\cos x - 2\sin x + c$
(b) $x^2 \sin x + 2x\cos x + 2\sin x + c$
(c) $x^2 \sin x + 2x\cos x + \sin x + c$
(d) $x^2 \sin x + 2\cos x + \sin x + c$.

2. $\int e^x \cos^2 x\,dx =$

(a) $e^x + \frac{1}{10}e^x(\cos 2x + 2\sin 2x) + c$
(b) $\frac{1}{2}e^x + e^x(\cos 2x + 2\sin 2x) + c$
(c) $\frac{1}{2}e^x + (\cos 2x + 2\sin 2x) + c$
(d) $\frac{1}{2}e^x + \frac{1}{10}e^x(\cos 2x + 2\sin 2x) + c$

3. $\int \frac{\sin x}{\sin 3x}\,dx =$

(a) $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + c$
(b) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + c$
(c) $\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + c$
(d) $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + c$

4. $\int \frac{dx}{\sin(x-\alpha)\cos(x-\beta)} =$

(a) $\frac{1}{\cos(\alpha-\beta)}(\log|\sin(x-\alpha)| + \log|\sec(x-\beta)|) + c$
(b) $(\log|\sin(x-\alpha)| + \log|\sec(x-\beta)|) + c$
(c) $(\log|\sin(x-\alpha)| + c)$
(d) $(\log|\sec(x-\beta)| + c)$

5. $\int \tan^3 2x \cdot \sec 2x\,dx =$

(a) $\frac{1}{3}\sec^3 2x - \frac{1}{2}\sec 2x + c$
(b) $-\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + c$
(c) $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + c$
(d) $\frac{1}{3}\sec^3 2x + \frac{1}{2}\sec 2x + c$

6. If $\int f(x)\sin x \cdot \cos x\,dx = \frac{1}{(b^2-a^2)}\log(f(x)) + k$, then $f(x)$ is

(a) $\frac{1}{a^2\sin^2 x + b^2\cos^2 x} + c$
(b) $\frac{1}{a^2\sin^2 x - b^2\cos^2 x} + c$
(c) $\frac{1}{a^2\cos^2 x + b^2\sin^2 x} + c$
(d) $\frac{1}{a^2\cos^2 x - b^2\sin^2 x} + c$

7. If $\int \left(\frac{\cos 4x + 1}{\cot x - \tan x}\right)dx = k\cos 4x + c$, then

(a) $k = -\frac{1}{2}$ (b) $k = -\frac{1}{8}$
(c) $k = -\frac{1}{4}$ (d) $k = \frac{1}{6}$

8. $\int \frac{dx}{x^6 + x^4} =$

(a) $-\frac{1}{3x^2} + \frac{1}{x} + \text{cosec}^{-1}x + c$

(b) $-\frac{1}{3x^2} + \frac{1}{x} + \cot^{-1}x + c$

(c) $-\frac{1}{3x^3} + \frac{1}{x} + \tan^{-1}x + c$

(d) $\frac{1}{3x^2} - \frac{1}{x} + \sin^{-1}x + c$

9. $\int \frac{2\sin x + 5}{(2 + 5\sin x)^2} dx =$

(a) $\frac{\cos x}{2 + 5\sin x} + c$ (b) $\frac{-\cos x}{2 + 5\sin x} + c$

(c) $\frac{1}{2 + 5\sin x} + c$ (d) $\frac{\sin x}{2 + 5\sin x} + c$

10. $\int \left\{ \frac{(\log x - 1)}{1 + (\log x)^2} \right\} dx =$

(a) $\frac{xe^x}{1 + x^2} + c$ (b) $\frac{xe^x}{1 + (\log x)^2} + c$

(c) $\frac{\log x}{1 + (\log x)^2} + c$ (d) $\frac{x}{1 + x^2} + c$

11. $\int \frac{\text{cosec}^2 x - 2005}{\cos^{2005} x} dx =$

(a) $\frac{\cot x}{\cos^{2005} x} + c$ (b) $\frac{\tan x}{\cos^{2005} x} + c$

(c) $\frac{-\tan x}{\cos^{2005} x} + c$ (d) $\frac{-\cot x}{\cos^{2005} x} + c$

12. $\int \frac{\sin 2x + 2\tan x}{(\cos^6 x + 6\cos^2 x + 4)} dx =$

(a) $2\sqrt{\frac{1 + \cos^2 x}{\cos^7 x}} + c$

(b) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \times \sqrt{\frac{1 + \cos^2 x}{\cos^7 x}} + c$

(c) $\frac{1}{12}\log\left(\frac{1 + \cos^2 x}{\cos^7 x}\right) + c$

(d) $\frac{1}{12}\log\left(1 + \frac{6}{\cos^4 x} + \frac{4}{\cos^6 x}\right) + c$

13. $\int \frac{(1 + x)\sin x}{(x^2 + 2x)\cos^2 x - (1 + x)\sin 2x} dx =$

(a) $\frac{1}{2}\log\left|\frac{\sin x - (x + 1)\cos x - 1}{\sin x - (x + 1)\cos x + 1}\right| + c$

(b) $\frac{1}{2}\tan^{-1}\{\sin x - (x + 1)\cos x\} + c$

(c) $\frac{1}{2}\sin^{-1}\{\sin x - (x + 1)\cos x\} + c$

(d) $\frac{1}{2}\sin^{-1}(\sin x + \cos x) + c$

14. $\int e^x \left\{ \frac{1}{\sqrt{1 + x^2}} + 1 - \frac{2x^2}{\sqrt{(1 + x^2)^5}} \right\} dx =$

(a) $e^x \left(\frac{1}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{(1 + x^2)^3}} \right) + c$

(b) $e^x \left(\frac{1}{\sqrt{1 + x^2}} - \frac{x}{\sqrt{(1 + x^2)^3}} \right) + c$

(c) $e^x \left(\frac{1}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{(1 + x^2)^5}} \right) + \text{c}$

(d) $e^x \left(\frac{1}{\sqrt{1 - x^2}} - \frac{x}{\sqrt{(1 + x^2)}} \right) + c$

15. $\int \frac{x\cos x + 1}{\sqrt{2x^3 e^{\sin x} + x^2}} dx =$

(a) $\log\left(\frac{\sqrt{2xe^{\sin x} + 1} - 1}{\sqrt{2xe^{\sin x} + 1} + 1}\right) + c$

(b) $\log\left(\frac{\sqrt{2xe^{\sin x} - 1} + 1}{\sqrt{2xe^{\sin x} + 1} + 1}\right) + c$

(c) $\log\left(\frac{\sqrt{2xe^{\sin x} + 1} + 1}{\sqrt{2xe^{\sin x} - 1} + 1}\right) + c$

(d) $\log\left(\frac{\sqrt{2xe^{\sin x} + 1} + 1}{\sqrt{2xe^{\sin x} - 1} - 1}\right) + c$

16. Let $f(x)$ be a function such that $f(0) = f'(0) = 0, f''(x) = \sec^4 x + 4$, the function is

(a) $\log(\sin x) + \frac{1}{3}\tan^3 x + cx$

(b) $\frac{2}{3}\log(\sec) + \frac{1}{6}\tan^2 x + 2x^2$

(c) $\log(\cos x) + \frac{1}{6}\cos^2 x + \frac{x^2}{5}$

(d) none.

17. The value of the integral $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$, where

$x \in \left(0, \frac{\pi}{2}\right)$

(a) $\sqrt{2}\sin^{-1}(\cos x - \sin x) + c$

(b) $\sqrt{2}\sin^{-1}(\sin x - \cos x) + c$

(c) $\sqrt{2}\sin^{-1}(\sin x + \cos x) + c$

(d) $-\sqrt{2}\sin^{-1}(\sin x + \cos x) + c$

18. The value of the integral $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$, where $x \in \left(\pi, \frac{3\pi}{2}\right)$

(a) $\sqrt{2}\sin^{-1}(\cos x - \sin x) + c$

(b) $\sqrt{2}\sin^{-1}(\sin x - \cos x) + c$

(c) $\sqrt{2}\sin^{-1}(\sin x + \cos x) + c$

(d) $-\sqrt{2}\sin^{-1}(\sin x + \cos x) + c$

19. The value of the integral $I = \int (\sqrt{\tan x} + \sqrt{\cot x})dx$, where $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

(a) $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + c$

(b) $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + c$

(c) $-\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + c$

(d) $-\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + c$

20. $\int e^{(x\sin x + \cos x)}\left(\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right)dx$

(a) $e^{(x\sin x + \cos x)}\left(x - \frac{1}{x\cos x}\right) + c$

(b) $e^{(x\sin x + \cos x)}\left(\frac{1}{x\cos x}\right) + c$

(c) $e^{(x\sin x + \cos x)}\left(\frac{1}{x\cos x} - x\right) + c$

(d) $e^{(x\sin x + \cos x)}\left(\frac{1}{x\cos x} + x\right) + c$

21. Let $f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}$ and g be the inverse of f. Then the value of $g'(0)$

(a) 1 (b) 17

(c) $\sqrt{17}$ (d) None.

22. If $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t\,dt}{1 + t^4}$, then $f'(2)$ is

(a) 2/17 (b) 0

(c) 1 (d) cannot be determined.

23. $\int \frac{\ln|x|}{x \times \sqrt{1 + \ln|x|}}\,dx =$

(a) $\frac{2}{3}\sqrt{1 + \ln|x|} \times (\ln|x| - 2) + c$

(b) $\frac{2}{3}\sqrt{1 + \ln|x|} \times (\ln|x| + 2) + c$

(c) $\frac{1}{3}\sqrt{1 + \ln|x|} \times (\ln|x| - 2) + c$

(d) $\frac{1}{3}\sqrt{1 + \ln|x|} \times (\ln|x| + 2) + c$

24. If $\int_0^{f(x)} t^2\,dt = x\cos(\pi x)$, the value of $f'(9)$ is

(a) $-1/9$ (b) $-1/3$

(c) 1/3 (d) Non existent.

25. The number of values of x satisfying the equation $\int_{-1}^{x}\left(8t^2 + \frac{28}{3}t + 4\right)dt = \frac{\frac{3}{2}x + 1}{\log_{(x+1)}\left(\sqrt{(x + 1)}\right)}$, is

(a) 0 (b) 1

(c) 2 (d) 3.

26. $\int \sqrt{(1 + 2\cot x(\cot x + \operatorname{cosec} x))}\,dx =$

(a) $2\ln\left(\cos\left(\frac{x}{2}\right)\right) + c$

(b) $2\ln\left(\sin\left(\frac{x}{2}\right)\right) + c$

(c) $\frac{1}{2}\ln\left(\sin\left(\frac{x}{2}\right)\right) + c$

(d) None.

27. A differentiable function satisfies $3f^2(x)f'(x) = 2x$. Given $f(2) = 1$, the value of $f(3)$ is

(a) $\sqrt[3]{24}$ (b) $\sqrt[3]{6}$

(c) 6 (d) 2.

28. $\int \left(\frac{1}{x}\log\left(\frac{x}{e^x}\right)\right)dx =$

(a) $\frac{1}{2}e^x - \ln x + c$ (b) $\frac{1}{2}\ln x - e^x + c$

(c) $\frac{1}{2}\ln 2x - e^x + c$ (d) None.

29. $\int x \times 2\ln^{(x^2+1)}\,dx =$

(a) $\frac{2^{\ln(x^2+!)}}{2(x^2 + 1)} + c$

(b) $\frac{(x^2 + 1)2^{\ln(x^2+!)}}{2(x^2 + 1)} + c$

(c) $\left(\dfrac{(x^2+1)2^{\ln 2+1}}{2(\ln 2+1)}\right)+c$

(d) None.

30. $\int\left(\dfrac{2x+1}{(x^2+4x+1)^{3/2}}\right)dx =$

(a) $\left(\dfrac{x^3}{\sqrt{x^2+rx+1}}\right)+c$

(b) $\left(\dfrac{x}{\sqrt{x^2+rx+1}}\right)+c$

(c) $\left(\dfrac{x^2}{\sqrt{x^2+rx+1}}\right)+c$

(d) $\left(\dfrac{1}{\sqrt{x^2+rx+1}}\right)+c$

31. $\int e^x\left\{\dfrac{2\sin^2 x-1}{\cos x}+\dfrac{\cos x(2\sin x+1)}{1+\sin x}\right\}dx =$

(a) $e^x\tan x+c$ (b) $e^x\cot x+c$

(c) $e^x\text{cosec}^2 x+c$ (d) $e^x\sec^2 x+c$.

32. $\int\dfrac{e^{\tan^{-1}x}}{(1+x^2)}\left\{\left(\sec^{-1}\sqrt{1+x^2}\right)^2+\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\right\}dx$

(a) $e^{\tan^{-1}}x\cdot\tan^{-1}x+c$

(b) $\dfrac{e^{\tan^{-1}}x\cdot(\tan^{-1}x)^2}{2}+c$

(c) $e^{\tan^{-1}}x\cdot\left(\sec^{-1}\left(\sqrt{1+x^2}\right)\right)^2+c$

(d) $e^{\tan^{-1}}x\cdot\left(\text{cosec}^{-1}\left(\sqrt{1+x^2}\right)\right)^2+c$

33. $\int\dfrac{dx}{(1+\sqrt{x})\sqrt{x-x^2}} =$

(a) $\left(\dfrac{2(1+\sqrt{x})}{(1-x)^2}\right)+c$

(b) $\left(\dfrac{(1+\sqrt{x})}{(1-x)^2}\right)+c$

(c) $\left(\dfrac{(1-\sqrt{x})}{(1-x)^2}\right)+c$

(d) $\left(\dfrac{2(1-\sqrt{x})}{-\sqrt{1-x}}\right)+c$

34. $\int\left(\dfrac{\cos x-\sin x}{\sqrt{8-\sin 2x}}\right)dx =$

(a) $\sin^{-1}(\sin x+\cos x)+c$

(b) $\sin^{-1}\left(\dfrac{1}{3}(\sin x+\cos x)\right)+c$

(c) $\cos^{-1}((\sin x+\cos x))+c$

(d) $\cos^{-1}\left(\dfrac{1}{3}(\sin x+\cos x)\right)+c$.

35. If $\int\dfrac{\sin x}{\sin(x-\alpha)}dx = Ax+B\log|\sin(x-\alpha)|+c$, then

(a) $A=\sin\alpha$ (b) $B=\cos\alpha$

(c) $A=\cos\alpha$ (d) $B=\sin\alpha$

36. $\int x\log(x^2+1)\,dx = f(x)\log(x^2+1)+g(x)+c$, then

(a) $f(x)=\dfrac{1+x^2}{2}$

(b) $g(x)=\dfrac{1+x^2}{2}$

(c) $g(x)=-\dfrac{1+x^2}{2}$

(d) $f(x)=\dfrac{x^2-1}{2}$.

37. If $\int e^2x\left(\dfrac{1+\sin 2x}{1-\sin 2x}\right)dx = Ae^{2x}\cdot f(x)+c$, then

(a) $A=1/2$ (b) $A=1/3$

(c) $f(x)=\tan x$ (d) $f(x)=\tan 2x$.

38. If $\int\dfrac{dx}{\cos(x-a)\cos(x-b)}$

$=\dfrac{1}{A}(\log|f(x)|+\log|g(x)|+c$, then

(a) $A=\sin(a-b)$

(b) $f(x)=\cos(x-a)$

(c) $g(x)=\cos(x-b)$

(d) $A=\sin(b-a)$.

39. If $\int\tan^4 x\,dx = k\tan^3 x+L\tan x+f(x)$, then

(a) $k=1/3$ (b) $L=-1$

(c) $f(x)=x+c$ (d) $k=2/3$.

(More than one options are correct)

40. If $\int\left(\dfrac{x+\sin x}{1+\cos x}\right)dx = f(x)\tan(g(x))+c$, then

(a) $f(x)=x^2$ (b) $f(x)=x$

(c) $g(x)=\dfrac{x^2}{2}$ (d) $g(x)=\dfrac{x}{2}$

41. If $\int e^x\left(\dfrac{x-1}{(x+1)^3}\right)dx = \dfrac{e^x}{(g(x))^m}+c$, then

(a) $g(x)=x$ (b) $g(x)=x+1$

(c) $m=1$ (d) $m=2$.

42. If $\int \frac{dx}{x^2(x^4+1)^{3/4}} = A\left(1+\frac{1}{x^4}\right)^B + c$, then
(a) $A = -1$ (b) $B = 1/4$
(c) $A = 1/2$ (d) $B = 1/2$.

43. If $\int\left(\frac{\log_x e + \log_{ex} e + \log_{e^2x} e}{x}\right)dx$
$= A(\log(\log_e x)) + B\log_e(1+\log_e x)$
$+ C\log(2+\log_e x) + K$, then
(a) $A + B = 2$ (b) $A - C = 0$
(c) $A - B = 0$ (d) $A + B + C = 3$

44. If $\int\left(\frac{\sqrt{\tan x}}{\sin x\cos x}\right)dx = A(f(x))^{1/m} + c$, then
(a) $A = 2$ (b) $m = 2$
(c) $f(x) = \tan x$ (d) $A + m = 5$

45. If $\int\left(\frac{\sqrt{\cot x}}{\sin x\cos x}\right)dx = A\left(-\sqrt{f(x)}\right) + c$, then
(a) $A = 2$ (b) $f(x) = \cot x$
(c) $f(x) = \tan x$ (d) $A = 4$.

46. If $\int\left(\frac{x^{9/2}}{\sqrt{1+x^{11}}}\right)dx = \left(\frac{L}{M}\right)\log\left|x^{N/P} + \sqrt{1+x^{11}}\right| + c$, then
(a) $L + M = 13$
(b) $L - M = 9$
(c) $L + M + N + P = 26$
(d) $L + M + N = 15$.

47. If $\int\left(\frac{dx}{x^3\sqrt{1+x^4}}\right) = \frac{L}{M}\sqrt{N+\frac{P}{x^4}} + c$, then
(a) $L = 1$
(b) $M = 2$
(c) $L + M + N + P = 5$
(d) $L + M + N - P = 3$.

48. If $\int\left(\frac{\tan^{-1}x}{x^4}\right)dx =$
$-\frac{\tan^{-1}x}{Ax^3} + \frac{1}{B}\log\left|\frac{x^2+1}{x^2}\right| + \frac{1}{Cx^2} + k$,
then
(a) $A = 3$ (b) $B = 6$
(c) $C = 6$ (d) $A + B + C = 15$.

49. If $\int\left(\frac{2(\cos x + \sec x)\sin x}{\cos^6 x + 6\cos^2 x + 4}\right)dx$
$= \frac{1}{L}\left(\log\left(1+\frac{M}{\cos^4 x}+\frac{N}{\cos^6 x}\right)\right) + c$,
then
(a) $L + M = 18$ (b) $L - M = 6$
(c) $L + M + N = 22$ (d) $L + M - N = 14$.

50. If $\int\left(\frac{\cos^4 x}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{3/5}}\right)dx$
$= -\frac{1}{A}(B + \cot^5 x)^{\frac{m}{n}} + c$, then
(a) $A + B = 3$ (b) $m + n = 7$
(c) $m + n = 8$ (d) $A + B = 4$.

Level III

(Problems for JEE Advanced)

1. Evaluate: $\int \frac{\cos 2x - \cos x}{1 - \cos x}\,dx$
2. Evaluate: $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx$
3. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$
4. Evaluate: $\int \frac{\cos 2x}{\cos^2 x\cdot\sin^2 x}dx$
5. Evaluate: $\int \frac{\sin 2x}{\sin 5x\cdot\sin 3x}dx$
6. Evaluate: $\int \frac{\sin x}{\sin x + \cos x}\,dx$
7. Evaluate: $\int \frac{dx}{\sec x + \operatorname{cosec} x}$
8. Evaluate: $\int \tan 3x\cdot\tan 2x\cdot\tan x\,dx$
9. Evaluate: $\int \frac{b}{a + ce^x}dx$
10. Evaluate: $\int \frac{dx}{1 + e^x}$
11. Evaluate: $\int \frac{x + 9}{x^3 + 9x}\,dx$
12. Evaluate: $\int \frac{\sin x - \cos x}{e^x + \sin x}\,dx$
13. Evaluate: $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx$
14. Evaluate: $\int \frac{\sin(x+a)}{\sin(x+b)}\,dx$
15. Evaluate: $\int \frac{dx}{\sin(x-a)\sin(x-b)}$
16. Evaluate: $\int \frac{dx}{\sin(x-a)\cos(x-b)}$
17. Evaluate: $\int \frac{\sqrt{\tan x}}{\sin x\cdot\cos x\,dx}$

18. Evaluate: $\int \frac{dx}{x\sqrt{x^4 - 1}}$

19. Evaluate: $\int \frac{dx}{(x^2 + 1)^2}$

20. Evaluate: $\int \sqrt{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}}\, dx$

21. Evaluate: $\int \frac{x}{x^4 + x^2 + 1}\, dx$

22. Evaluate: $\int \sec^3 x\, dx$

23. Evaluate: $\int \operatorname{cosec}^3 x\, dx$

24. Evaluate: $\int \frac{x^2 - 1}{x\sqrt{x^4 + 3x^2 + 1}}\, dx$

25. Evaluate: $\int \frac{dx}{x^4 + 18x^2 + 81}$

26. Evaluate: $\int \left(\frac{x + \sin x}{1 + \cos x}\right) dx$

27. Evaluate: $\int \sin 4x\, e^{\tan 2} x\, dx$

28. Evaluate: $\int \sin^{-1}\left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}}\right) dx$

29. Evaluate: $\int \frac{x^2}{(x\sin x + \cos x)^2}\, dx$

30. Evaluate: $\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}\, dx$

31. Evaluate: $\int \frac{\sec x(2 + \sec x)}{(1 + 2\sec x)^2}\, dx$

32. Evaluate: $\int \cos(2\theta) \times \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$

33. Evaluate: $\int e^x\left(\frac{x^4 + 2}{(1 + x^2)^{5/2}}\right) dx$

34. Evaluate: $\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right) dx$

35. Evaluate: $\int \frac{dx}{\cos x + \operatorname{cosec} x}$

36. Evaluate: $\int \cot^{-1}(1 + x + x^2)\, dx$

37. Evaluate: $\int \tan^{-1}(1 + x + x^2)\, dx$

38. Evaluate: $\int \frac{\sin x}{\sin 4x\, dx}$

39. For any natural number m, evaluate

$\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx.$

40. Evaluate: $\int \sqrt[3]{\tan x}\, dx$

41. Evaluate: $\int \frac{dx}{(e^x - 1)^2}$

42. Evaluate: $\int \frac{\tan^{-1} x}{x^4} dx$

43. Evaluate: $\int \frac{dx}{(\sqrt{\cos x} + \sqrt{\sin x})^4}$

44. Evaluate: $\int \frac{1 + x^{-2/3}}{1 + x}\, dx$

45. Evaluate: $\int \frac{dx}{2\sin x + \sec x}$

46. Evaluate: $\int \frac{x^4 + 1}{x^6 + 1}\, dx$

47. Evaluate: $\int \frac{dx}{(e^x - 1)^2}$

48. Evaluate: $\int \frac{dx}{(x - 1)\sqrt{x + 2}}$

49. Evaluate: $\int \frac{dx}{x^2\sqrt{x + 1}}$

50. Evaluate: $\int \frac{x^4 - 1}{x^2\sqrt{x^4 + x^2 + 1}}\, dx$

51. Evaluate: $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}\, dx$

52. Evaluate: $\int \frac{\sqrt{\cos 2x}}{\sin x} dx$

53. Evaluate: $\int \frac{\cos\theta + \sin\theta}{\sqrt{5 + \sin\theta(2\theta)}}\, d\theta$

54. Evaluate: $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}\, dx$

55. Evaluate: $\int \frac{\tan 2\theta}{\sqrt{\sin^6\theta + \cos^6\theta}}\, d\theta$

Level III (Tougher Problems for JEE Advanced)

Q. Evaluate the following integrals.

1. $\int \frac{x^2 + 6}{(x\sin x + 3\cos x)^6} dx$

2. $\int \frac{\log(1 + \sin^2 x)}{\cos^2 x} dx$

3. $\int x^{-\frac{2}{3}}\left(1 + x^{\frac{2}{3}}\right)^{-1} dx$

4. $\int \left(1 + \tan\left(\frac{5\pi}{16} - x\right)\right)\left(\left(1 + \tan\left(-\frac{\pi}{16} + x\right)\right)\right) dx$

5. $\int \tan(x - \alpha) \cdot \tan(x + \alpha) \cdot \tan 2x \, dx$

6. $\int \cos\left(2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right) dx$

7. $\int \frac{(x-1)\sqrt{x^4 + 2x^3 - x^2 + 2x + 1}}{x^2(x+1)} dx$

8. $\int \frac{\tan\left(\frac{\pi}{4} - 4\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}} dx$

9. $\int \frac{\tan^{-1} x}{x^4} dx$

10. $\int \frac{(1 - x\sin x)}{x(1 - x^3 e^{3\cos x})} dx$

11. $\int \frac{(1 + x\cos x)}{x(1 - x^2 e^{2\sin x})} dx$

12. $\int \frac{dx}{2\sin x + \sec x}$

13. $\int \frac{dx}{\cos x\sqrt{\sin(2x + \alpha) + \sin\alpha}}$

14. $\int \left(3x^2 \tan\left(\frac{1}{x}\right) - x\sec^2\left(\frac{1}{x}\right)\right) dx$

15. $\int \frac{\sqrt{3\cos 2x - 1}}{\cos x} dx$

16. $\int \left(\frac{1}{1 - x^8}\right)\left(\cos^{-1}\left(\frac{2x}{1 + x^2}\right) + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)\right) dx$

17. $\int \frac{x^2 - x^3}{(x+1)(x^3 + x^2 + x)^{3/2}} dx$

18. $\int \frac{\cos^4 x}{\sin^3 x(\sin^5 x + \cos^5 x)^{3/5}}$

19. $\int \frac{\sin^3 x \, dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$

20. $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{4 + 3\sin^2 x} dx$

21. $\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2} dx$

22. $\int \frac{\sqrt{\sin x - \sin^3 x}}{1 - \sin^3 x} dx$

23. $\int \frac{e^x(2 - x^2)}{(1 - x)\sqrt{1 - x^2}} dx$

24. $\int \frac{dx}{\sqrt{\sin(x + \alpha)\cos^3(x - \beta)}}$

25. $\int e^x \left(\frac{x+2}{x+4}\right)^2 dx$

26. $\int \frac{\sqrt{1 + x^8}}{x^{13}} dx$

27. $\int \frac{dx}{\sin^3 x + \cos^3 x}$

28. $\int \frac{(x + \sqrt{1 + x^2})^3}{\sqrt{1 + x^2}} dx$

29. $\int \frac{(x+1)}{x(1 + xe^x)^2} dx$

30. $\int \sqrt{\frac{\sin x}{2\sin x + 3\cos x}} dx$

31. $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx$

32. $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$

33. $\int \frac{dx}{\sin^6 x + \cos^6 x}$

34. Evaluate $\int \frac{f(x)}{x^3 - 1} dx$, where $f(x)$ is a polynomial of degree 2 in x such that $f(0) = f(1) = -3 = 3f(2)$.

Integer Type Questions

1. If of $\int \frac{\cos x + \cos 2x}{2\cos x - 1} dx = A\sin x + Bx + c$, find the value of $A + B + 3$.

2. If $\int \frac{dx}{\sin^3 x + \cos^3 x}$

$$= \frac{\sqrt{2}}{L}\log\left|\tan\left(\frac{\pi}{M} + \frac{x}{N}\right)\right| + \frac{P}{3}\tan^{-1}(\sin x - \cos x) + c,$$

then find the value of $L + M - N - P$.

3. If $\int \frac{dx}{\sin^6 x + \cos^6 x}$

$= \tan^{-1}(L\tan x + M\cot x) + c,$

find the value of $L + M + N + 4$.

4. Let $\int \frac{x+1}{x^3 + x} dx$

$$= \tan^{-1}x + \frac{1}{L}\log\left|\frac{x^M}{x^N+1}\right| + c.$$

Find the value of $\left(\frac{L+M+N}{3}\right)$

5. Let $\int\left(\frac{\sin x}{\sin x + \cos x}\right)dx$

$$= Ax + B\log|\sin x + \cos x| + c$$

Find the value of $A + B + 1$

6. Let $\int\left(\frac{x^2}{(2x+3)^2}\right)dx$

$$= \frac{(3+2x)}{L} + \frac{1}{M}\log|3+2x| - \frac{9}{8(2x+3)} + C.$$

Find the value of $L + M + 4$

7. If $\int\left(\frac{3x^2+2x}{x^6+2x^5+x^4+2x^3+2x^2+5}\right)dx$

$$= \frac{1}{L}\tan^{-1}\left(\frac{x^M+x^N+P}{Q}\right) + k,$$

find the value of $L + M + N + P + Q$.

8. If $\int\left(\frac{\cos^4 x}{\sin^3 x(\sin^5 x + \cos^5 x)^{3/5}}\right)dx$

$$= -\frac{1}{L}\left(\frac{1+\tan^5 x}{\tan^5 x}\right)^{\frac{M}{N}} + c,$$

find the value of $L + M + N$.

9. If $\int x^{13/2}(1+x^{5/2})^{1/2}dx$

$$= \frac{L}{35}(1+x^{5/2})^{7/2} - \frac{8M}{25}(1+x^{5/2})^{5/2}$$

$$+ \frac{N}{15}(1+x^{5/2})^{3/2} + c,$$

find the value of $L + M + N$.

10. If $\int \cos^4 x\,dx = Lx + M\sin 2x + N\sin 4x + c$,

find the value of $8L + 4M + 32N$.

Comprehensive Link Passages

Passage I

In some of the cases we can split the integrand into the sum of the two functions such that the integration for which one of them integrate by parts and other one be fixed.

Consider $\int e^x\{f(x) + f'(x)\}\,dx$

$$= \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$

$$= e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)\,dx + c$$

$$= e^x f(x) + c$$

1. $\int\left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right)dx =$

(a) $\ln(\ln x) + c$ (b) $x(\ln x) + c$

(c) $\frac{x}{\ln x} + c$ (d) $x + \ln x + c$

2. $\int \frac{xe^x\,dx}{(1+x)^2} =$

(a) $xe^x + c$ (b) $\frac{e^x}{(1+x)^2 + c}$

(c) $e^x - \frac{1}{1+x} + c$ (d) $\frac{e^x}{1+x} + c$

3. $\int e^x\left(\log x - \frac{1}{x^2}\right)dx$

(a) $e^x\left(\log x - \frac{1}{x}\right) + c$

(b) $e^x\left(\log x + \frac{1}{x}\right) + c$

(c) $e^x\left(\log x + \frac{1}{x^2}\right) + c$

(d) $e^x\left(\log x + \frac{1}{\log x}\right) + c$

Passage II

For integrals

$\int f\left(x - \frac{a}{x}\right)\left(1 + \frac{a}{x^2}\right)dx$, put $x - \frac{a}{x} = t$,

$\int f\left(x + \frac{a}{x}\right)\left(1 - \frac{a}{x^2}\right)dx$, put $x + \frac{a}{x} = t$,

$\int f\left(x^2 - \frac{a}{x^2}\right)\left(x + \frac{a}{x^3}\right)dx$, put $x^2 - \frac{a}{x^2} = t$

$\int f\left(x^2 + \frac{a}{x^2}\right)\left(x - \frac{a}{x^3}\right)dx$, put $x^2 + \frac{a}{x^2} = t$

The following integrands can be brought into above forms by suitable transformations.

1. $\int \frac{x^4 - 2}{x^2\sqrt{x^4+x^2+2}}\,dx =$

(a) $\sqrt{x^2 + \frac{1}{x^2} + 1} + c$

(b) $\sqrt{x^2 + \frac{2}{x^2} + 1} + c$

(c) $\sqrt{x^2 + \frac{1}{x^2}} + c$

(d) $\sqrt{x^2 + \frac{2}{x^2}} + c.$

2. $\int \frac{(x-1)\,dx}{(x+1)\sqrt{x^3+x^2+x}} =$

(a) $\tan^{-1}\left(x + \frac{1}{x} + 1\right) + c$

(b) $\tan^{-1}\sqrt{\left(x + \frac{1}{x} + 1\right)} + c$

(c) $2\tan^{-1}\sqrt{\left(x + \frac{1}{x} + 1\right)} + c$

(d) $\tan^{-1}\sqrt{\left(x - \frac{1}{x} + 1\right)} + c$

3. $\int\left(\frac{5x^4 + 4x^5}{(x^5 + x + 1)^2}\right)dx =$

(a) $x^5 + x + 1 + c$

(b) $\left(\frac{x^5}{x^5 + x + 1}\right) + c$

(c) $\left(\frac{x^4}{x^5 + x + 1}\right) + c$

(d) $\left(\frac{x^4}{x^5 + x - 1}\right) + c$

Passage III

Let us consider the integrals of the form

$$\int R\left(x, x^{\frac{\alpha}{p}}, x^{\frac{\beta}{q}}, x^{\frac{\gamma}{r}}\right)dx$$

In this case, we shall put $x = t^m$, where m is the LCM of (p, q, r).

1. $\int\frac{\sqrt{x}}{\sqrt[4]{x^3} + 1}\,dx =$

(a) $\frac{4}{3}\left(\sqrt[4]{x^3} - \ln\left(\sqrt[4]{x^3} + 1\right)\right) + C$

(b) $\frac{4}{3}\left(\sqrt[2]{x^3} - \ln\left(\sqrt[2]{x^3} + 1\right)\right) + C$

(c) $\frac{5}{3}\left(\sqrt[2]{x^3} - \ln\left(\sqrt[2]{x^3} + 1\right)\right) + C$

(d) $\frac{8}{3}\left(\sqrt[4]{x^3} - \ln\left(\sqrt[4]{x^3} + 1\right)\right) + C$

2. $\int\left(\frac{\sqrt{x^3} - \sqrt[3]{x}}{6 - \sqrt[4]{x}}\right)dx =$

(a) $\frac{2}{27}\sqrt[4]{x^9} + \frac{2}{13}\sqrt[12]{x^{13}} + C$

(b) $\frac{2}{27}4Rx^9 - \frac{2}{13}\sqrt[12]{x^{13}} + C$

(c) $\frac{13}{27}4Rx^9 - \frac{2}{13}\sqrt[12]{x^{13}} + C$

(d) $-\frac{2}{27}4Rx^9 - \frac{2}{13}\sqrt[12]{x^{13}} + C$

3. $\int\left(\frac{x + \sqrt[3]{x^2} + \sqrt[4]{x}}{x + \sqrt[3]{x^4}}\right)dx$

(a) $\frac{3}{2}x^{2/3} + 6\tan^{-1}(\sqrt[6]{x}) + C$

(b) $\frac{3}{2}x^{2/3} - 6\tan^{-1}(\sqrt[6]{x}) + C$

(c) $-\frac{3}{2}x^{2/3} - 6\tan^{-1}(\sqrt[6]{x}) + C$

(d) $-\frac{3}{2}x^{2/3} + 6\tan^{-1}(\sqrt[6]{6}) + C$

Matrix Match
(For JEE-Advanced Examination only)

Q1. **Match the following columns:**

Column I		Column II	
(A)	$\int\frac{dx}{(x + 2)\sqrt{x}}$	(P)	$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$
(B)	$\int\frac{dx}{(\sqrt{x} + 2)\sqrt{x}}$	(Q)	$\sqrt{2}\tan^{-1}\left(\sqrt{\frac{x}{2}}\right) + c$
(C)	$\int\frac{dx}{x(x + 2)}$	(R)	$2\log(2 + \sqrt{x}) + c$
(D)	$\int\frac{dx}{(x^2 + 2)}$	(S)	$\frac{1}{2}\log\left(\frac{x}{x + 2}\right) + c$

Q.2 **Match the following columns:**

Column I		Column II	
(A)	$\int\sin^4 x\cdot\cos^3 x\,dx$	(P)	$\frac{1}{8}\left(3x + \frac{\sin 4x}{4} - 2\sin 2x\right) + c,$
(B)	$\int\sin^4 x\,dx$	(Q)	$-\frac{3}{8}(\tan x)^{-8/3} + \frac{3}{4}(\tan x)^{4/3} - \frac{3}{2}(\tan x)^{-2/3} + c$
(C)	$\int\sin^7 x\cos^3 x\,dx$	(R)	$\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$
(D)	$\int\operatorname{cosec}^{11/3}x\cdot\sec^{7/3}x\,dx$	(S)	$\frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + c$

Q.3 **Match the following columns:** (Write the suitable substitutions)

Column I		Column II	
(A)	$\int\frac{\sin x\cos x}{\sin^2 x + \cos^2 x}dx$	(P)	Put $\sin x = t$

(B)	$\int \frac{\sin x \cdot \cos x}{\sin^2 x + 2\cos^2 x + \cos x} dx$	(Q)	Put $\sin x = t$
(C)	$\int \frac{dx}{3 + \sin x \cos x}$	(R)	Put $\cos x = t$
(D)	$\int \frac{\cos^2 x - \sin^2 x}{1 - \sin^2 x \cos^2 x} dx$	(S)	Put $\sin x \cos x = t$

Q.4 **Match the following columns:**

Column I		Column II	
(A)	If $f(x) = \int \left(\frac{x + \sin x}{1 + \cos x}\right) dx$ and $f(0) = 0$, then $f\left(\frac{\pi}{2}\right)$	(P)	$\frac{\pi}{2}$
(B)	Let $f(x) = \int e^{\sin^{-1} x} \left(1 - \frac{x}{\sqrt{1 - x^2}}\right) dx$ and $f(0) = 1$, if $f\left(\frac{1}{2}\right) = \frac{k\sqrt{3}e^{\pi/6}}{\pi}$, then k is	(Q)	$\frac{\pi}{3}$
(C)	Let $f(x) = \int \frac{dx}{(x^2 + 1)(x^2 + 9)}$ and $f(0) = 0$, if $f(\sqrt{3}) = \frac{5}{56}k$, then k is	(R)	$\frac{\pi}{4}$
(D)	Let $f(x) = \int \left(\frac{\sqrt{\tan x}}{\sin x \cos x}\right) dx$ and $f(0) = 0$, if $f\left(\frac{\pi}{4}\right) = \frac{2k}{\pi}$, then k is	(S)	π.

Q.5 **Match the following columns:**

Column I		Column II	
(A)	$\int \frac{dx}{(x - 1)\sqrt{x + 4}}$	(P)	$-\frac{1}{\sqrt{x - 1}} - \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{\frac{x - 1}{2}}\right) + c$
(B)	$\int \frac{dx}{(x^2 - 1)\sqrt{x - 1}}$	(Q)	$-\frac{1}{\sqrt{3}} \log\left|\left(t - \frac{1}{3}\right) + \sqrt{t^2 - \frac{2}{3}t + \frac{1}{3}}\right| + c$, $t = \frac{1}{x + 2}$
(C)	$\int \frac{dx}{(x + 2)\sqrt{x^2 - 1}}$	(R)	$\frac{1}{3} \log\left|\frac{\sqrt{x + 4} - 3}{\sqrt{x + 4} + 3}\right| + c$
(D)	$\int \frac{dx}{(x^2 + 1)\sqrt{x^2 + 4}}$	(S)	$-\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{x^2 + 4}}{x\sqrt{3}}\right) + c$

Q.6 **Match the following columns:**

Column I		Column II	
(A)	$\int \frac{dx}{x^2 + x}$	(P)	$-\frac{1}{x} - \log\left|\frac{x}{x + 1}\right| + c$
(B)	$\int \frac{dx}{x^3 + x^2}$	(Q)	$\log\left|\frac{x}{x + 1}\right| + c$
(C)	$\int \frac{dx}{x^4 + x^3}$	(R)	$-\frac{1}{3x^3} + \frac{1}{2x^2} - \frac{1}{x} - \log\left|\frac{x}{x + 1}\right| + c$
(D)	$\int \frac{dx}{x^5 + x^4}$	(S)	$-\frac{1}{2x^2} + \frac{1}{x} + \log\left|\frac{x}{x + 1}\right| + c$

Q.7 **Match the following columns:**

Column I		Column II	
(A)	$\int \frac{dx}{x(x^2 + 1)}$	(P)	$\frac{1}{5} \log\left|\frac{x^5}{x^5 + 1}\right| + c$
(B)	$\int \frac{dx}{x(x^3 + 1)}$	(Q)	$\frac{1}{3} \log\left|\frac{x^3}{x^3 + 1}\right| + c$
(C)	$\int \frac{dx}{x(x^5 + 1)}$	(R)	$\frac{1}{2013} \log\left|\frac{x^{2013}}{x^{2013} + 1}\right| + c$
(D)	$\int dx/x(x^{2013} + 1)$	(S)	$\frac{1}{2} \log\left|\frac{x^2}{x^2 + 1}\right| + c$

Q.8 **Match the following columns:**

Column I		Column II	
(A)	$\int \frac{x^3 - 3x^2}{e^x + x^3} dx$	(P)	$x - \log\|x + e^x\| + c$
(B)	$\int \frac{x - 1}{e^x + x} dx$	(Q)	$\log\|\sin x + e^x\| - x + c$
(C)	$\int \frac{\cos x - \sin x}{e^x + \sin x} dx$	(R)	$x - \log\|\tan x + e^x\| + c$
(D)	$\int \frac{\tan x - \sec^2 x}{e^x + \tan x} dx$	(S)	$x - \log\|x^3 + e^x\| + c$

Q.9 **Match the following columns:**

Column I		Column II	
(A)	$\int\left(1 - \tan\left(\frac{5\pi}{4} - x\right)\right)\left(1 + \tan\left(\frac{3\pi}{2} - x\right)\right)dx$	(P)	$2x + c$
(B)	$\int\left(1 + \tan\left(\frac{\pi}{8} - x\right)\right)\left(1 + \tan\left(\frac{\pi}{8} + x\right)\right)dx$	(Q)	$3x + c$

(C)	$\int\left(1+\tan\left(\frac{5\pi}{4}-x\right)\right)$ $(1+\tan x)\,dx$	(R)	$4x + c$
(D)	$\int\left(1+\tan\left(\frac{7\pi}{8}-x\right)\right)$ $\left(1+\tan\left(\frac{3\pi}{8}+x\right)\right)dx$	(S)	$5x + c$

Questions asked in Previous Years' JEE-Advanced Examinations

1. $\int \frac{dx}{1-\cot x}$ **[IIT-JEE, 1978]**

2. $\int \frac{x\,dx}{1+x^4}$ **[IIT-JEE, 1979]**

3. $\int \frac{x^2\,dx}{(a+bx)^2}$ **[IIT-JEE, 1979]**

4. $\int \sqrt{1+\sin\left(\frac{x}{2}\right)}\,dx$ **[IIT-JEE, 1980]**

5. $\int \frac{x^2}{\sqrt{1-x}}\,dx$ **[IIT-JEE, 1980]**

6. $\int (e^{\log x}+\sin x)\cos x\,dx$ **[IIT-JEE, 1981]**

7. $\int e^x\left(\frac{x-1}{(x+1)^3}\right)dx$ **[IIT-JEE, 1983]**

8. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ **[IIT-JEE, 1984]**

9. $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ **[IIT-JEE, 1985]**

10. $\int \left(\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\right)dx$ **[IIT-JEE, 1986]**

11. $\int \frac{\sqrt{\cos 2x}}{\sin x\,dx}$ **[IIT-JEE, 1987]**

12. $\int (\sqrt{\tan x}+\sqrt{\cot x})\,dx$ **[IIT-JEE, 1988]**

13. If $I = \left(\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\right)dx = Ax + B\log|(9e^{2x}-4)| + C$ then $A = \ldots$, $B = \ldots$ and $C = \ldots$
[IIT-JEE, 1990]

14. $\int \left(\frac{1}{x^{1/3}+x^{1/4}} + \frac{\log(1+x^{1/6})}{\sqrt{x}+x^{1/3}}\right)dx$
[IIT-JEE, 1992]

15. $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$
[IIT-JEE, 1994]

16. The value of the integral $\int\left(\frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}\right)dx$ is
(a) $\sin x - 6\tan^{-1}(\sin x) + c$
(b) $\sin x - 2(\sin x)^{-1} + c$
(c) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$
(d) $\sin x - 2(\sin\ x)^{-1} - 5\tan^{-1}(\sin x) + c$
[IIT-JEE, 1995]

17. $\int \frac{x+1}{x(1+xe^x)^2}\,dx$ **[IIT-JEE, 1996]**

18. $\int \frac{dx}{\sqrt{(x-p)^3(x-q)}}$ is equal to
(a) $\frac{2}{(p-q)}\sqrt{\left(\frac{x-p}{x-q}\right)} + c$
(b) $-\frac{2}{(p-q)}\sqrt{\left(\frac{x-p}{x-q}\right)} + c$
(c) $\frac{1}{\sqrt{(x-p)(x-q)}} + c$
(d) none.
[IIT-JEE, 1996]

19. If $\int \frac{dx}{(\sin x+4)(\sin x-1)}$
$= \frac{A}{\left(\tan\left(\frac{x}{2}\right)-1\right)} + B\tan^{-1}(f(x)) + C,$
then
(a) $A = \frac{1}{5}$, $B = -\frac{2}{5\sqrt{15}}$, $f(x) = \frac{4\tan x+1}{5}$
(b) $A = -\frac{1}{5}$, $B = -\frac{2}{5\sqrt{15}}$, $f(x) = \frac{4\tan(x/2)+1}{\sqrt{15}}$
(c) $A = \frac{2}{5}$, $B = -\frac{2}{5\sqrt{15}}$, $f(x) = \frac{4\tan x+1}{5}$
(d) $A = \frac{2}{5}$, $B = -\frac{2}{5\sqrt{15}}$, $f(x) = \frac{4\tan(x/2)+1}{\sqrt{15}}$
[IIT-JEE, 1997]

20. $\int \left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)(2+2\sin 2x)$ is equal to
(a) $\sin 2x + c$ (b) $\cos 2x + c$
(c) $\tan 2x + c$ (d) None
[IIT-JEE, 1997]

21. $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \times \frac{dx}{x}$ **[IIT-JEE, 1997]**

22. $\int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}}$ is equal to

(a) $2\sec^{-1}(2x - 7) + c$
(b) $\sec^{-1}(2x - 7) + c$
(c) $1/2\sec^{-1}(2x - 7) + c$
(d) None of these.

[IIT-JEE, 1997]

23. $\int \frac{x^2 + 3x + 2}{(x^2 + 1)^2(x + 1)} dx$ **[IIT-JEE, 1999]**

24. $\int \text{in}^{-1}\left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 12}}\right)dx$ **[IIT-JEE, 2001]**

25. $(x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx$ where $m > 0$

[IIT-JEE, 2002]

26. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$

(a) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + c$

(b) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + c$

(c) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + c$

(d) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + c$ **[IIT, 2006]**

27. Let $f(x) = \frac{x}{(1 + x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = (f_O f_O \ldots {}_O f)(x)$, then $x^{n-2}g(x)dx$ is

(a) $\frac{1}{n(n - 1)}(1 + nx^n)^{1-\frac{1}{x}} + c$

(b) $\frac{1}{(n - 1)}(1 + nx^n)^{1-\frac{1}{x}} + c$

(c) $\frac{1}{n(n - 1)}(1 + nx^n)^{1+\frac{1}{n}} + c$

(d) $\frac{1}{(n + 1)}(1 + nx^n)^{1+\frac{1}{n}} + c$

[IIT, 2007]

28. Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx,$

$J = \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx.$

For an arbitrary constant C, the value of $J - I$ is

(a) $\frac{1}{2}\log\left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^x + 1}\right) + c$

(b) $\frac{1}{2}\log\left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right) + c$

(c) $\frac{1}{2}\log\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + c$

(d) $\frac{1}{2}\log\left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right) + c$

[IIT, 2008]

29. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals

(a) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(b) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(c) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(d) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

[IIT, 2012]

Answers

LEVEL IIB

1. (a)	2. (b)	3. (c)	4. (d)	5. (a)
6. (b)	7. (a)	8. (b)	9. (c)	10. (d)
11. (a)	12. (b)	13. (a)	14. (b)	15. (c)
16. (d)	17 (a)	18. (b)	19. (a)	20. (b)
21. (c)	22. (d)	23. (a)	24. (b)	25. (a)
26. (b)	27. (c)	28. (d)	29. (a)	30. (b)
31. (a)	32. (b)	33. (c)	34. (d)	35. (a)
36. (b)	37. (a)	38. (b)	39. (c)	40. (b, d)
41. (b, d)	42. (a, b)	43. (a, b, c, d)		44. (a, b, c)
45. (a, b)	46. (a, b, c)	47. (a, b, c, d)		48. (a, b)
49. (a, b, c, d)		50. (a, c)		

LEVEL IV

1. $\dfrac{-x\sec x}{x\sin x + 3\cos x} + \tan x + c$
2. $\tan x \log(1+\sin^2 x) - 2\sqrt{2}\tan^{-1}(t\sqrt{2}) + c$, where $t = \tan x$
3. $3\tan^{-1}(x^{1/3}) + c$
4. $-\dfrac{1}{2}\log|\cos(2x)| + \log|\cos(x-\alpha)| + \log|\cos(x+\alpha)| + c$
5. $-\dfrac{1}{2}\log|\cos(2x)| + \log|\cos(x-\alpha)| + \log|\cos(x+\alpha) + c$
6. $-\sqrt{1-x^2} + c$
7. $\dfrac{1}{2}\log|t^2+2t-3| - \dfrac{1}{2\sqrt{2}}\log\left|\dfrac{(t+1)-\sqrt{2}}{(t+1)+\sqrt{2}}\right| + -\dfrac{1}{\sqrt{3}}\sin^{-1}\left(\dfrac{3t+1}{2}\right) + c$

   where $t = \left(x + \dfrac{1}{x}\right)$
8. $-2\tan^{-1}\left(\sqrt{t + \dfrac{1}{t} + 1}\right) + c$

   where $t = x + \dfrac{1}{x}$
9. $-\dfrac{\theta}{3\tan^3\theta} - \dfrac{1}{3}\left(\dfrac{1}{2\sin^2\theta} + \log|\sin\theta|\right) + c$

   where $\tan\theta = x$
10. $\dfrac{1}{3}\log\left|\dfrac{(xe^x)^3 - 1}{(xe^x)^3}\right| + c$
11. $\dfrac{1}{2}\log\left|\dfrac{(xe^{\sin x})^2}{(xe^{\sin x})^2 - 1}\right| + c$
12. $\dfrac{1}{2\sqrt{2}}\log\left|\tan\left(\dfrac{x}{2} + \dfrac{\pi}{8}\right)\right| - \dfrac{1}{(\sin x + \cos x)} + c$
13. $\dfrac{\tan x\cos\alpha + \sin\alpha}{\sqrt{2}\cos\alpha} + c$
14. $x^3\tan\left(\dfrac{1}{x}\right) + c$
15. $\sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 2\sqrt{2}\tan^{-1}\left(\sqrt{\dfrac{\operatorname{cosec}^2 x - 3}{2}}\right) + c$
16. $\dfrac{\pi}{8}\left[\int\left(\dfrac{1}{1-x^2} + \dfrac{1}{1+x^2}\right) + \int\dfrac{(x^2+1)}{x^4+1} - \int\dfrac{(x^2+1)}{x^4+1}\right]dx$

    and then you do it.
17. $2\left(\dfrac{1}{\sqrt{x + \dfrac{1}{x} + 1}} + \tan^{-1}\left(\sqrt{x + \dfrac{1}{x} + 1}\right)\right) + c$
18. $-\dfrac{1}{2}(1 + \cot^5 x)^{2/5} + c$
19. $\log|\tan^{-1}(\cos x + \sec x)| + c$
20. $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{(\tan x + \cot x)\sqrt{2} - 1}{(\tan x + \cot x)\sqrt{2} + 1}\right| + c$
21. $-\dfrac{x^2}{(x\tan x + 1)} + 2\log|x\sin x + \cos x| + c$
22. $\dfrac{2}{3}\sin^{-1}(\sin^{3/2} x) + c$
23. $\left(e^x\sqrt{\dfrac{1+x}{1-x}}\right) + c$
24. $2\sec(\alpha+\beta)\sqrt{\cos(\alpha+\beta)\tan(x-\beta) + \sin(\alpha+\beta)} + c$
25. $e^x\left(\dfrac{x}{(x+4)}\right) + c$
26. $-\dfrac{1}{12}\left(\dfrac{1+x^8}{x^8}\right)^{3/2} + c$
27. $\dfrac{2}{3\sqrt{2}}\log\left|\tan\left(\dfrac{x}{2} + \dfrac{\pi}{8}\right)\right| - \dfrac{2}{3}\tan^{-1}(\sin x - \cos x) + c$
28. $\dfrac{(x + \sqrt{1+x^2})^3}{3} + c$
29. $\log\left|\dfrac{t}{t+1}\right| + \dfrac{1}{(t+1)} + c$, where $t = xe^x$
30. $-\dfrac{1}{3\sqrt{2}}\tan^{-1}\left(t - \dfrac{1}{t}\right) + \dfrac{1}{6\sqrt{2}}\log\left|\dfrac{\left(t + \dfrac{1}{t}\right) - \sqrt{2}}{\left(t + \dfrac{1}{t}\right) + \sqrt{2}}\right| + c$

    where $t = \sqrt{2 + 3\cot x}$
31. $-\dfrac{1}{8}\cos(4x) + c$
32. $\sin x - 2\operatorname{cosec} x - 6\tan^{-1}(\sin x) + c$
33. $\tan^{-1}(\tan x - \cot x) + c$
34. $\log\left|\dfrac{(x^2+x+1)}{(x-1)}\right| + \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + c$

INTEGER TYPE QUESTIONS

1. 5	2. 7	3. 5	4. 2	5. 1
6. 4	7. 9	8. 9	9. 7	10. 5

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (b)	2. (d)	3. (a)
Passage II:	1. (b)	2. (c)	3. (b)
Passage III:	1. (b)	2. (b)	3. (a)

MATRIX MATCH

1. (A)→(Q), (B)→(R), (C)→(S), (D)→(P)
2. (A)→(R), (B)→(P), (C)→(S), (D)→(Q)
3. (A)→(P, Q, R), (B)→(Q), (C)→(R), (D)→(R, S)
4. (A)→(P), (B)→(Q), (C)→(R), (D)→(S)
5. (A)→(R), (B)→(P), (C)→(S), (D)→(Q)
6. (A)→(Q), (B)→(P), (C)→(S), (D)→(R)
7. (A)→(S), (B)→(Q), (C)→(P), (D)→(R)
8. (A)→(S), (B)→(P), (C)→(Q), (D)→(R)
9. (A)→(P), (B)→(P), (C)→(P), (D)→(P)

HINTS AND SOLUTIONS

Level 1

1. We have,

$$\int \log_x x\,dx = \int 1\cdot dx = x + c$$

2. We have,

$$\int (3^{\log_x 2} - 2^{\log_x 3})\,dx = \int (2^{\log_x 3} - 2^{\log_x 3})dx$$
$$= \int 0\cdot dx = c$$

3. We have,

$$\int \left(x^m + m^x + m^m + \frac{m}{x}\right)dx$$
$$= \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + m^m x + m\log|x| + c$$

4. We have,

$$\int 2^x\cdot 3^x dx = \int (2\cdot 3)^x dx = \int 6^x dx$$
$$= \frac{6^x}{\log 6} + c$$

5. We have,

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$
$$= \tan x - x + c$$

6. We have,

$$\int \cot^2 x\,dx = \int (\text{cosec}^2 x - 1)\,dx$$
$$= -\cot x - x + c$$

7. We have,

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx$$
$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx$$
$$= \int (\sec^2 x + \text{cosec}^2 x)\,dx$$
$$= \tan x - \cot x + c$$

8. $\int \left(1 + \tan\left(x + \frac{3\pi}{8}\right)\right)\left(1 + \tan\left(\frac{\pi}{8} - x\right)\right)dx$

$$= \int 2\,dx \quad [\because \text{ if } A + B = \frac{\pi}{4},$$
then $(1 + \tan A)(1 + \tan B) = 2] = 2x + c$.

9. We have,

$$\int (\tan x + \cot x)^2 dx = \int (\tan^2 x + \cot^2 x + 2)\,dx$$
$$= \int (\sec^2 x + \text{cosec}^2 x)\,dx$$
$$= \tan x - \cot x + c$$

10. We have,

$$\int \frac{dx}{1 + \cos 2x} = \int \frac{dx}{2\cos^2 x} = \int \frac{1}{2}\sec^2 x\,dx$$
$$= \frac{1}{2}\tan x + c$$

11. The given integral is

$$\int (3^{\log_5 x} - x^{\log_5 3})\,dx = \int (3^{\log_6 x} - 3^{\log_6 x})\,dx$$
$$= \int (0)\cdot dx = c$$

12. The given integral is

$$\int \left(1 + \tan\left(\frac{\pi}{8} - x\right)\left(1 + \tan\left(\frac{\pi}{8} + x\right)\right)\right)dx$$
$$= \int (2)\cdot dx$$
$$= 2x + C$$

13. The given integral is

$$\int \left(\frac{8^{1+x} + 4^{1+x}}{2^{2x}}\right)dx = \int \left(\frac{2^{3+3x} + 2^{2+2x}}{2^{2x}}\right)dx$$
$$= \int (2^{3+x} + 2^2)\,dx$$
$$= 8\int 2^x dx + \int 4\,dx$$
$$= \frac{8\cdot 2^x}{\log 2} + 4x + c$$

14. The given integral is

$$\int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right)dx$$
$$= \frac{x^2}{2m} + m\log x + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + c$$

15. The given integral is

$$\int \frac{(a^x + b^x)^2}{a^x b^x} dx = \int \left(\frac{a^{2x} + b^{2x} + 2a^x b^x}{a^x b^x}\right) dx$$

$$= \int\left(\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right) dx$$

$$= \frac{(a/b)^x}{\log(a/b)} + \frac{(b/a)^x}{\log(b/a)} + 2x + c$$

16. The Given integral is

$$\int \frac{(2x + 3x)^2}{2^x \cdot 3^x} dx = \int \left(\left(\frac{2}{3}\right)^x + \left(\frac{3}{2}\right)^x + 2\right) dx$$

$$= \frac{(2/3)^x}{\log(2/3)} + \frac{(3/2)^x}{\log(3/2)} + 2x + c$$

17. It is given that

$$f'(x) = \frac{1}{x} + \frac{1}{\sqrt{1 - x^2}}$$

On integration, we get

$$f(x) = \log|x| + \sin^{-1} x + c$$

When $x = 1$ and $f(1) = \frac{\pi}{2}$, then $c = 0$

Hence, the function is

$$f(x) = \log|x| + \sin^{-1} x$$

18. It is given that

$$f'(x) = a\cos x + b\sin x$$

On integration, we get

$$f(x) = a\sin x - b\cos x + c$$

When $x = 0$ and $f'(0) = 4$, then $a = 4$

When $x = 0$, $f(0) = 3$ and $f\left(\frac{\pi}{2}\right) = 5$

then, we get

$$b = -2 \text{ and } c = 1$$

Hence, the function is

$$f(x)\ 4\cos x + 2\sin x + 1$$

19. We have,

$$\int \frac{dx}{1 - \sin x} = \int \frac{(1 + \sin x)}{(1 - \sin x)(1 + \sin x)} dx$$

$$= \int \frac{(1 + \sin x)}{(1 - \sin^2 x)} dx = \int \frac{1 + \sin x}{\cos^2 x} dx$$

$$= \int (\sec^2 x + \sec x \cdot \tan x)\, dx$$

$$= \tan x + \sec x + c$$

20. We have,

$$\int \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{((\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x)}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(1 - 2\sin^2 x \cos^2 x)}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x - 2\sin^2 x \cos^2 x)}{\sin^2 x \cos^2 x} dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 2)\, dx$$

$$= \tan x - \cot x - 2x + c$$

21. We have,

$$\int \left(\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x}\right) dx$$

$$= \int \left(\frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x}\right) dx$$

$$= \int \left(\frac{(\sin^4 x + \cos^4 x - \sin^2 x \cdot \cos^2 x)}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int \left(\frac{(1 - 3\sin^2 x \cdot \cos^2 x)}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int \left(\frac{(\sin^2 x + \cos^2 x - 3\sin^2 x \cos^2 x)}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 3) dx$$

$$= \tan x - \cot x - 3x + c$$

22. We have,

$$\int \left(\frac{\cos 2x - \cos\alpha}{\cos x - \cos\alpha}\right) dx$$

$$= \int \left(\frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{\cos x - \cos\alpha}\right) dx$$

$$= \int 2\left(\frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}\right) dx$$

$$= \int 2(\cos x + \cos\alpha)\, dx$$

$$= 2(\sin x + x\cos\alpha) + c$$

23. We have,

$$\int \left(\frac{\cos^4 x - \sin^4 x}{\sqrt{1 + \cos 4x}}\right) dx$$

$$= \int \left(\frac{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}{\sqrt{1 + \cos 4x}}\right) dx$$

$$= \int \left(\frac{(\cos^2 x - \sin^2 x)}{\sqrt{2}\cos 2x} \right) dx$$

$$= \int \left(\frac{\cos 2x}{\sqrt{2}\cos 2x} \right) dx$$

$$= \frac{x}{\sqrt{2}} + c$$

24. We have,

$$\int \left(\frac{1 + \tan^2 x}{1 + \cot^2 x} \right) dx = \int \left(\frac{1 + \tan^2 x}{1 + \frac{1}{\tan^2 x}} \right) dx$$

$$= \int \left(\frac{\tan^2 x (1 + \tan^2 x)}{(1 + \tan^2 x)} \right) dx$$

$$= \int (\tan^2 x)\, dx$$

$$= \int (\sec^2 x - 1)\, dx$$

$$= \tan x - x + c$$

25. We have,

$$\int \left(\frac{\cos x - \cos 2x}{1 - \cos x} \right) dx$$

$$= \int \left(\frac{\cos x - 2\cos^2 x + 1}{1 - \cos x} \right) dx$$

$$= -\int \left(\frac{2\cos^2 x - \cos x - 1}{1 - \cos x} \right) dx$$

$$= -\int \left(\frac{(\cos x - 1)(2\cos x + 1)}{1 - \cos x} \right) dx$$

$$= \int \left(\frac{(1 - \cos x)(2\cos x + 1)}{1 - \cos x} \right) dx$$

$$= \int ((2\cos x + 1))dx$$

$$= 2\sin x + x + c.$$

26. We have,

$$\int \left(\frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} \right) dx = \int \left(\sqrt{\frac{x^4 + x^{-4} + 2}{x^6}} \right) dx$$

$$= \int \left(\sqrt{\frac{x^8 + 2x^4 + 1}{x^{10}}} \right) dx$$

$$= \int \left(\frac{x^4 + 1}{x^5} \right) dx$$

$$= \int \left(x^{-5} + \frac{1}{x} \right) dx$$

$$= \log|x| - \frac{1}{4x^4} + c$$

27. We have,

$$\int \left(\frac{5\cos^3 x + 3\sin^3 x}{\cos^2 x \sin^2 x} \right) dx$$

$$= \int \left(\frac{5\cos^3 x}{\cos^2 x \sin^2 x} + \frac{3\sin^3 x}{\cos^2 x \sin^2 x} \right) dx$$

$$= 5\int \operatorname{cosec} x \cot x\, dx + 3\int \sec x \tan x\, dx$$

$$= -5\operatorname{cosec} x + 3\sec x + c$$

28. We have,

$$\int \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)(1 + \sin 2x)\, dx$$

$$= \int \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)(\cos x + \sin x)^2 dx$$

$$= \int (\cos x - \sin x)(\cos x + \sin x)\, dx$$

$$= \int (\cos^2 x - \sin^2 x)\, dx$$

$$= \int \cos 2x\, dx$$

$$= \frac{\sin 2x}{2} + c$$

29. The given integral is

$$\int \left(\frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \right) dx$$

$$= \int \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x - \sin 6x}\, dx$$

$$= \int \frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)}{-2\cos\left(\frac{9x}{2}\right)\sin\left(\frac{3x}{2}\right)}\, dx$$

$$= -\int 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) dx$$

$$= -\int (\cos 2x + \cos x)\, dx$$

$$= -\left(\frac{\sin 2x}{2} + \sin x \right) + c$$

30. The given integral is

$$\int \left(\frac{\cos x - \cos 2x}{1 - \cos x} \right) dx$$

$$= -\int \frac{2\cos^2 x - \cos x - 1}{1 - \cos x}\, dx$$

$$= -\int \frac{(\cos x - 1)(2\cos x + 1)}{1 - \cos x}\, dx$$

$$= \int (2\cos x + 1)\, dx$$

$$= (2\sin x + x) + C$$

31. The given integral is

$$\int \frac{dx}{(\tan x + \cot x + \sec x + \operatorname{cosec} x)}$$

$$= \int \frac{\sin x \cos x\, dx}{1 + \sin x + \cos x}$$

$$= \int \frac{\sin x\, dx}{\sec x + \tan x + 1}$$

$$= \int \frac{\sin x(1 + \tan x - \sec x)dx}{(1 + \tan x)^2 - \sec^2 x}$$

$$= \int \frac{\sin x(1 + \tan x - \sec x)dx}{2\tan x}$$

$$= \frac{1}{2}\int \cos x(1 + \tan x - \sec x)\, dx$$

$$= \frac{1}{2}\int (\cos x + \sin x - 1)\, dx$$

$$= \frac{1}{2}(\sin x + \cos x - x) + C$$

32. We have,

$$\int \frac{x}{x+1}\, dx = \int \frac{(x + 1 - 1)}{(x+1)}\, dx$$

$$= \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x - \log|x + 1| + C$$

33. We have,

$$\int \frac{(1+x)^2}{x(1+x^2)}\, dx = \int \frac{(1 + x^2 + 2x)}{x(1 + x^2)}\, dx$$

$$= \frac{(1 + x^2) + 2x}{x(1 + x^2)}\, dx$$

$$= \int \frac{dx}{x} + 2\int \frac{dx}{1 + x^2}$$

$$= \log|x| + 2\tan^{-1}x + c$$

34. We have,

$$\int \frac{x^2 - 2}{x^2 + 1}\, dx = \int \left(\frac{x^2 + 1 - 3}{x^2 + 1}\right) dx$$

$$= \int \left(1 - \frac{3}{x^2 + 1}\right) dx$$

$$= x - 3\tan^{-1}x + c$$

35. We have,

$$\int \frac{x - 1}{(x^{2/3} + x^{1/3} + 1)}\, dx = \int \frac{(x^{1/3})^3 - 1}{(x^{2/3} + x^{1/3} + 1)}\, dx$$

$$= \int \frac{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)}{(x^{2/3} + x^{1/3} + 1)}\, dx$$

$$= \int (x^{1/3} - 1)\, dx$$

$$= -\frac{3}{2} x^{-2/3} - x + c$$

36. We have,

$$\int \left(\frac{x^4 - 1 + 3}{x^2 + 1}\right) dx = \int \left(\frac{x^4 - 1}{x^2 + 1} + \frac{3}{x^2 + 1}\right) dx$$

$$= \int \left(\frac{(x^2 - 1)(x^2 + 1)}{x^2 + 1} + \frac{3}{x^2 + 1}\right) dx$$

$$= \int \left((x^2 - 1) + \frac{3}{x^2 + 1}\right) dx$$

$$= \frac{x^3}{3} - x + 3\tan^{-1}x + c$$

37. We have,

$$\int \left(\frac{x^4 - 1 - 2}{x^2 + 1}\right) dx = \int \left[\left(\frac{x^4 - 1}{x^2 + 1}\right) - \frac{2}{x^2 + 1}\right] dx$$

$$= \int \left[\left(\frac{(x^2 + 1)(x^2 - 1)}{x^2 + 1}\right) - \frac{2}{x^2 + 1}\right] dx$$

$$= \int \left((x^2 - x) - \frac{2}{x^2 + 1}\right) dx$$

$$= \frac{x^3}{3} - x - 2\tan^{-1}x + c$$

38. We have,

$$\int \left(\frac{x^6 + 1 - 2}{x^2 + 1}\right) dx = \int \left[\left(\frac{x^6 + 1}{x^2 + 1}\right) - \frac{2}{x^2 + 1}\right] dx$$

$$= \int \left[\left(\frac{(x^2 + 1)(x^4 - x^2 + 1)}{x^2 + 1}\right) - \frac{2}{x^2 + 1}\right] dx$$

$$= \int \left((x^4 - x^2 + 1) - \frac{2}{x^2 + 1}\right) dx$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1}x + c$$

39. We have,

$$\int \left(\frac{x^8 + x^4 + 1}{x^4 + x^2 + 1}\right) dx$$

$$= \int \left(\frac{(x^4 - x^2 + 1)(x^4 + x^2 + 1)}{(x^4 + x^2 + 1)}\right) dx$$

$$= \int ((x^4 + x^2 + 1))\, dx$$

$$= \frac{x^5}{5} + \frac{x^3}{3} + x + c$$

40. The given integral is

$$\int\left(\frac{x^4}{x^2+1}\right)dx = \int\frac{(x^4-1)+1}{x^2+1}\,dx$$

$$= \int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$$

$$= \left(\frac{x^3}{3} - x + \tan^{-1}x + c\right)$$

41. The given integral is

$$\int\left(\frac{x^4+x^2+1}{x^2+x+1}\right)dx$$

$$= \int\frac{(x^2-x+1)(x^2+x+1)}{(x^2+x+1)}\,dx$$

$$= \int(x^2-x+1)dx$$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2} + x + C\right)$$

42. The given integral is

$$\int\left(\frac{x^6+1}{x^2+1}\right)dx = \int\frac{(x^2+1)(x^4-x^2+1)}{(x^2+1)}\,dx$$

$$= \int(x^4-x^2+1)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{x^3}{3} + x + c\right)$$

43. We have,

$$\int\sin^{-1}(\sin x)\,dx = \int x\,dx = \frac{x^2}{2} + c$$

44. We have,

$$\int\sin^{-1}(\cos x)\,dx = \int\sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)dx\right]$$

$$= \int\left(\frac{\pi}{2} - x\right)dx$$

$$= \frac{\pi}{2}x - \frac{x^2}{2} + c$$

45. We have,

$$\int\tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx$$

$$= \int\tan^{-1}\left(\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right)dx$$

$$= \int\tan^{-1}(\tan x)\,dx = \int x\,dx$$

$$= \frac{x^2}{2} + c$$

46 We have,

$$\int\tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$$

$$= \int\tan^{-1}\left(\frac{2\sin x\cos x}{2\cos^2 x}\right)dx$$

$$= \int\tan^{-1}(\tan x)\,dx$$

$$= \int x\,dx$$

$$= \frac{x^2}{2} + c$$

47. The given integral is

$$\int\tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx = \int\tan^{-1}\left(\frac{\sin x}{\cos x}\right)dx$$

$$= \int\tan^{-1}(\tan x)\,dx$$

$$= \int x\,dx$$

$$= \frac{x^2}{2} + c$$

48. The given integral is

$$\int\tan^{-1}\left(\frac{\sin x}{1-\cos x}\right)dx$$

$$= \int\tan^{-1}\left(\frac{2\sin(x/2)\,\cos(x/2)}{2\sin^2(x/2)}\right)dx$$

$$= \int\tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right]dx$$

$$= \int\left(\frac{x}{2}\right)dx$$

$$= \frac{x^2}{4} + c$$

49. The given integral is

$$\int\tan^{-1}\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)dx$$

$$= \int\tan^{-1}\left(\frac{\cos(x/2) - \sin(x/2)}{\cos(x/2) + \sin(x/2)}\right)dx$$

$$= \int\tan^{-1}\left(\frac{1-\tan(x/2)}{1+\tan(x/2)}\right)dx$$

$$= \int\tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]dx$$

$$= \int\left(\frac{\pi}{4} - \frac{x}{2}\right)dx$$

$$= \left(\frac{\pi x}{4} - \frac{x^2}{4}\right) + c$$

50. The given integral is

$$\int \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right)dx$$

$$= \int \tan^{-1}[\tan(x/2)]\,dx$$

$$= \int \left(\frac{x}{2}\right)dx$$

$$= \frac{x^2}{4} + c$$

51. The given integral is

$$\int \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)dx$$

$$= \int \tan^{-1}\left(\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)dx$$

$$= \int \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)\right)dx$$

$$= \int \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)dx$$

$$= \int \left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

$$= \left(\frac{\pi x}{4}+\frac{x^2}{4}+c\right)$$

52. The given integral is

$$\int \tan^{-1}\left(\frac{1-\sin x}{\cos x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int \tan^{-1}\left(\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)dx$$

$$= \int \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)dx$$

$$= \int \left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= \frac{\pi x}{4}-\frac{x^2}{4}+c$$

53. The given integral is

$$\int \tan^{-1}\left(\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1+\sin x)}}\right)dx$$

$$= \int \tan^{-1}\left(\frac{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)+\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)-\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}\right)dx$$

$$= \int \tan^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= \int \tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]dx$$

$$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]dx$$

$$= \int \left(\frac{\pi}{2}-\frac{x}{2}\right)dx$$

$$= \frac{\pi x}{2}-\frac{x^2}{4}+c$$

54. The given integral is

$$\int \tan^{-1}(\sec x+\tan x)dx$$

$$= \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{1+\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int \tan^{-1}\left(\frac{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)dx$$

$$= \int \tan^{-1}\left[\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]dx$$

$$= \int \tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]dx$$

$$= \int \left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

$$= \frac{\pi x}{4}+\frac{x^2}{4}+c$$

55. The given integral is

$$\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{2\sin x\cos x}{2\cos^2 x}\right)dx$$

$$= \int \tan^{-1}(\tan x)\, dx$$

$$= \int x\, dx$$

$$= \frac{x^2}{2} + c$$

56. We have,

$$\int (3x + 2)\, dx = \int \frac{(3x + 2)^2}{2.3} + c$$

$$= \int \frac{(3x + 2)^2}{6} + c$$

57. We have,

$$\int \frac{dx}{2x - 3} = \frac{1}{2} \log |2x - 3| + c$$

58. We have,

$$\int \frac{dx}{5 - 2x} = -\frac{1}{2} \log |5 - 2x| + c$$

59. We have,

$$\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$$

60. We have,

$$\int 3^{4x+5} dx = \frac{3^{4x+5}}{4 \log 3} + \text{c}$$

61. We have,

$$\int \cos(5x + 3)\, dx = \frac{\sin(5x + 3)}{5} + c$$

62. We have,

$$\int \sin 2x\, dx = -\frac{\cos 2x}{2} + c$$

63. We have,

$$\int \sqrt{3x + 2}\, dx = \frac{(3x + 2)^{3/2}}{\frac{3}{2} \cdot 3} + c$$

$$= \frac{2}{9} (3x + 2)^{3/2} + c$$

64. We have,

$$\int (3x + 4)^{-\frac{1}{2}} dx = \frac{(3x + 4)^{\frac{1}{2}}}{\frac{1}{2} \cdot 3} + c$$

$$= \frac{2}{3} (3x + 4)^{\frac{1}{2}} + c$$

65. We have,

$$\int \frac{dx}{\sqrt{x + 2} - \sqrt{x + 1}}$$

$$= \int \frac{(\sqrt{x + 2} + \sqrt{x + 1})\, dx}{(\sqrt{x + 2} - \sqrt{x + 1})(\sqrt{x + 2} + \sqrt{x + 1})}$$

$$= \int \frac{(\sqrt{x + 2} + \sqrt{x + 1})\, dx}{(x + 2) - (x + 1)}$$

$$= \int (\sqrt{x + 2} + \sqrt{x + 1})\, dx$$

$$= \frac{2}{3} (x + 2)^{3/2} + \frac{2}{3} (x + 1)^{3/2} + c$$

66. We have,

$$\int \frac{dx}{(\sqrt{2x + 5} - \sqrt{2x + 3})}$$

$$= \int \frac{(\sqrt{2x + 5} + \sqrt{2x + 3}) dx}{(\sqrt{x + 5} - \sqrt{2x + 3})(\sqrt{2x + 5} + \sqrt{2x + 3})}$$

$$= \int \frac{(\sqrt{2x + 5} + \sqrt{2x + 3}) dx}{(2x + 5) - (2x + 3)}$$

$$= \frac{1}{2} \int (\sqrt{2x + 5} + \sqrt{2x + 3})\, dx$$

$$= \frac{1}{2} \left(\frac{2}{3} \times \frac{1}{2} (2x + 5)^{3/2} + \frac{2}{3} \times \frac{1}{2} (2x + 3)^{3/2} \right) + c$$

$$= \frac{1}{6} ((2x + 5)^{3/2} + (2x + 3)^{3/2}) + c$$

67. The given integral is

$$\int \frac{dx}{\sqrt{3x + 4} - \sqrt{3x + 1}}$$

$$= \int \frac{(\sqrt{3x + 2} + \sqrt{3x + 1})}{(3x + 4 - 3x - 1)} dx$$

$$= \frac{1}{3} \int (\sqrt{3x + 4} + \sqrt{3x + 1})\, dx$$

$$= \frac{1}{3} \left(\frac{(3x + 4)^{3/2}}{9/2} + \frac{(3x + 1)^{3/2}}{9/2} \right) + c$$

68. The given integral is

$$\int \frac{dx}{\sqrt{2x + 3} + \sqrt{2x - 3}}$$

$$= \int \frac{(\sqrt{2x + 3} - \sqrt{2x - 3})}{(2x + 3 - 2x + 3)} dx$$

$$= \frac{1}{6} \int (\sqrt{2x + 3} - \sqrt{2x - 3})\, dx$$

$$= \frac{1}{6} \left(\frac{(2x + 3)^{3/2}}{3} - \frac{(2x - 3)^{3/2}}{3} \right) + c$$

69. The given integral is

$$\int \frac{dx}{\sqrt{x + 1} + \sqrt{x}} = \int (\sqrt{x + 1} - \sqrt{x}) dx$$

$$= \frac{2}{3} ((x + 1)^{3/2} - (x)^{3/2}) + c$$

70. The given integral is

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$

$$= \int \left(\frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b} \right) dx$$

$$= \frac{1}{(a-b)} \int (\sqrt{x+a}-\sqrt{x+b})\, dx$$

$$= \frac{2}{3(a-b)} ((x+a)^{3/2}-(x+b)^{3/2}) + c$$

71. The given integral is

$$\int \frac{dx}{\sqrt{2x+2014}+\sqrt{2x+3013}}$$

$$= \int \frac{\sqrt{2x+2014}-\sqrt{2x+2013}}{(2x+2014-2x-2013)}\, dx$$

$$= \int (\sqrt{2x+2014}-\sqrt{2x+2013})\, dx$$

$$= \frac{1}{3}[(2x+2014)^{3/2}-(2x+2013)^{3/2}] + c$$

72. We have,

$$\int \frac{x}{\sqrt{x-1}}\, dx = \int \frac{(t^2+1)}{t} \times 2t\,dt.$$

Let $x-1=t^2$

$\Rightarrow\ dx = 2t\,dt$

$$= 2\int (t^2+1)\, dt$$

$$= 2\left(\frac{t^3}{3}+t\right) + c$$

$$= 2\left(\frac{(x-1)^{2/3}}{3}+\sqrt{x-1}\right) + c$$

73. We have,

$$\int \frac{\sqrt{x}}{x+1}\, dx = \int \frac{t}{t^2+1} \cdot 2t\,dt,$$ Let $x=t^2$

$\Rightarrow\ dx = 2t\,dt$

$$= 2\int \frac{t^2}{t^2+1}\, dt$$

$$= 2\int \frac{(t^2+1-1)}{t^2+1}\, dt$$

$$= 2\int \left(1-\frac{1}{t^2+1}\right) dt$$

$$= 2(t-\tan^{-1}t) + c$$

$$= 2(\sqrt{x}-\tan^{-1}(\sqrt{x})) + c$$

74. We have,

$$\int \frac{x}{\sqrt{3x+1}}\, dx$$

$$= \int \frac{\left(\frac{t^2-1}{3}\right)}{t} \times 2t\,dt,$$ Let $3x+1=t^2$

$\Rightarrow\ dx = \frac{2t\,dt}{3}$

$$= \frac{2}{3}\int (t^2-1)\, dt$$

$$= \frac{2}{3}\left(\frac{t^3}{3}-t\right) + c$$

$$= \frac{2}{3}\left(\frac{(3x+1)^{3/2}}{3}-\sqrt{3x+1}\right) + c$$

75. We have,

$$\int \frac{x+1}{\sqrt{2x+1}}\, dx$$

Let $2x-1=t^2$

$\Rightarrow\ 2\,dx = 2t\,dt$

$\Rightarrow\ dx = t\,dt$

$$\therefore \quad = \int \frac{(t^2+1)}{2t} t\,dt$$

$$= \frac{1}{2}\int (t^2+1)\, dt$$

$$= \frac{1}{2}\left(\frac{t^3}{3}+t\right) + c$$

$$= \frac{1}{2}\left(\frac{(2x-1)^{3/2}}{3}+(2x-1)^{1/2}\right) + c$$

76. We have,

$$\int \frac{x-1}{\sqrt{x+4}}\, dx$$

Let $x+4=t^2$

$\Rightarrow\ dx = 2t\,dt$

$$= \int \left(\frac{t^2-5}{t}\right) \times t\,dt$$

$$= \int (t^2-5)\, dt$$

$$= \left(\frac{t^3}{3}-5t\right) + c$$

$$= \frac{(x-1)^{3/2}}{3} - 5(x-1)^{1/2} + c$$

77. We have,

$$\int \frac{x}{x^2+1}\, dx$$

Let $x^2+1=t$

$$\Rightarrow \quad 2x\,dx = dt$$

$$\Rightarrow \quad x\,dx = \frac{1}{2}\,dt$$

$$= \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}\log|t| + c$$

$$= \frac{1}{2}\log|x^2 + 1| + c$$

78. We have,

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

Let $\sin x + \cos x = t$

$\Rightarrow \quad (\cos x - \sin x)\,dx = dt$

$$= \int \frac{dt}{t}$$

$$= \log|t| + c$$

$$= \log|\sin x + \cos x| + c$$

79. We have,

$$\int \frac{3\cos x}{2\sin x + 5}\,dx$$

Let $2\sin x + 5 = t$

$\Rightarrow \quad 2\cos x\,dx = dt$

$\Rightarrow \quad \cos x\,dx = \frac{1}{2}\,dt$

$$= \frac{3}{2}\int \frac{dt}{t}$$

$$= \frac{3}{2}\log|t| + c$$

$$= \frac{3}{2}\log|2\sin x + 5| + c$$

80. We have,

$$\int \frac{\cos x - \sin x}{2 + \sin 2x}\,dx = \int \frac{\cos x - \sin x}{1 + (1 + \sin 2x)}\,dx$$

$$= \int \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}\,dx$$

Let $\sin x + \cos x = t$

$\Rightarrow \quad (\cos x - \sin x)\,dx = dt$

$$= \int \frac{dt}{1 + t^2}$$

$$= \tan^{-1}(t) + c$$

$$= \tan^{-1}(t) + c$$

81. We have,

$$\int \frac{xe^x + e^x}{\cos^2(xe^x)}\,dx = \int \frac{(x + 1)e^x}{\cos^2(x\,e^x)}\,dx$$

Let $xe^x = t$

$\Rightarrow \quad (xe^x + e^x)\,dx = dt$

$\Rightarrow \quad (x + 1)\,e^x dx = dt$

$$= \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t\,dt$$

$$= \tan t + c$$

$$= \tan(xe^x) + c$$

82. We have,

$$\int \frac{dx}{x(1 + \ln x)^2}$$

Let $1 + \ln x = t$

$\Rightarrow \quad \frac{1}{x}dx = dt$

$$= \frac{dt}{t^2}$$

$$= -\frac{1}{t} + c$$

$$= -\frac{1}{t} + c$$

$$= -\frac{1}{1 + \ln x} + c$$

83. We have,

$$\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx$$

$$= \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x}\,dx$$

$$= \int \left(\frac{(e^x + \cos x + 1)}{e^x + \sin x + x} - \frac{(e^x + \sin x + x)}{(e^x + \cos x + 1)}\right)dx$$

$$= \int \left(\frac{(e^x + \cos x + 1)}{(e^x + \sin x + x)} - \frac{(e^x + \sin x + x)}{(e^x + \sin x + x)}\right)dx$$

$$= \int \left(\frac{(e^x + \cos x + 1)}{(e^x + \sin x + x)} - 1\right)dx$$

$$= \log|e^x + \sin x + x| - x + c$$

84. We have,

$$\int \frac{dx}{1 + e^x} = \int \frac{e^{-x}}{e^{-x} + 1}\,dx$$

Let $e^{-x} + 1 = t$

$\Rightarrow \quad -e^{-x}dx = dt$

$$= -\int \frac{dt}{t + 1}$$

$$= -\log|t + 1| + c$$

$$= -\log|e^{-x} + 1| + c$$

85. We have,

$$\int \frac{dx}{x(x^3+1)} = \int \frac{x^2 dx}{x^3(x^3+1)}$$

$$= \frac{1}{3}\int \frac{dt}{t(t+1)} \qquad \text{Put } x^3 = t$$

$$= \frac{1}{3}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{3}\log\left|\frac{t}{t+1}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + c$$

86. We have,

$$\int \frac{dx}{x(x^4+1)} = \int \frac{x^3 dx}{x^4(x^4+1)}$$

$$= \frac{1}{4}\int \frac{dt}{t(t+1)} \qquad \text{Let } x^4 = t \Rightarrow x^3 dx = \frac{dt}{4}$$

$$= \frac{1}{4}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{4}\log\left|\frac{t}{t+1}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{x^4}{x^4+1}\right| + c$$

87. We have,

$$\int \frac{dx}{x(x^5-1)} = \int \frac{x^4 dx}{x^5(x^5-1)}$$

$$= \frac{1}{5}\int \frac{dt}{t(t-1)} \qquad \text{Let } x^5 = t, \quad \frac{dx}{x} = \frac{dt}{5t}$$

$$= \frac{1}{5}\int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$$

$$= \frac{1}{5}\log\left|\frac{t-1}{t}\right| + c$$

$$= \frac{1}{5}\log\left|\frac{x^5-1}{x^5}\right| + c$$

88. We have,

$$\int \frac{\sin 2x}{\sin 5x \sin 3x}\,dx = \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x}\,dx$$

$$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x}\,dx$$

$$= \int \left(\frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} - \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x}\right) dx$$

$$= \int (\cot 3x - \cot 5x)\,dx$$

$$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$$

89. We have,

$$\int \frac{dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin((x-a)-(x-b))\,dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(b-a)}$$

$$\int \left(\frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}\right) dx$$

$$= \frac{1}{\sin(b-a)}\int [\cot(x-b) - \cot(x-a)]\,dx$$

$$= \frac{1}{\sin(b-a)}[\log|\sin(x-b)| - \log|\sin(x-a)|] + c$$

$$= \frac{1}{\sin(b-a)}\left(\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|\right) + c$$

90. The given integral is

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

Let $e^x + e^{-x} = t$

$\Rightarrow (e^x - e^{-x})\,dx = dt$

$$= \int \frac{dt}{t}$$

$$= \log|t| + c$$

$$= \log|e^x + e^{-x}| + c$$

91. The given integral is

$$\int \frac{dx}{1+e^{-x}} = \int \frac{e^x}{e^x+1}\,dx = \log|e^x + 1| + c$$

92. The given integral is

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}dx}{e^{-x}+1}$$

$$= -\log|e^{-x}+1| + c$$

$$= \log\left|\frac{e^x}{e^x+1}\right| + c$$

93. The given integral is

$$\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x}\,dx$$

Let $a\sin^2 x + b\cos^2 x = t$

$$(a-b)\sin 2x\,dx = dt$$

$$\sin 2x\,dx = \frac{dt}{(a-b)}$$

$$= \frac{1}{(a-b)}\int \frac{dt}{t}$$

$$= \frac{1}{(a-b)}\log|t| + c$$

$$= \frac{1}{(a-b)}\log|a\cos^2 x + b\sin^2 x| + c$$

94. The given integral is

$$\int \frac{\sin(x-a)}{\sin x}dx = \int \frac{\sin x\cos a - \cos x\sin a}{\sin x}dx$$

$$= \int (\cos a - \sin a\cot x)\,dx$$

$$= x\cos a = \sin a\log|\sin x| + c$$

95. The given integral is

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt,$$

Let $t = x - a$

$\Rightarrow\ dt = dx$

$$= t\cos a + \sin a\log|\sin t| + c$$

$$= (x-a)\cos a + \sin a\log|\sin(x-a)| + c$$

96. The given integral is

$$\int \frac{\sin(x+a)}{\sin(x+b)}dx$$

$$= \int \frac{\sin[t + (a-b)]}{\sin t}dt,$$

Let $t = (x+b)$

$\Rightarrow\ dt = dx$

$$= \int \frac{\sin(t)\cos(a-b) + \cos t\sin(a-b)}{\sin t}dt$$

$$= \int (\cos(a-b) + \sin(a-b)\cot t)\,dt$$

$$= t\cos(a-b) + \sin(a-b)\log|\sin t| + c$$

$$= (x+b)\cos(a-b) + \sin(a-b)\log|\sin(x+b)| + c$$

97. The given integral is

$$\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

Let $(\sqrt{x}+1) = t$

$\Rightarrow\ \dfrac{dx}{2\sqrt{x}} = dt$

$\Rightarrow\ \dfrac{dx}{\sqrt{x}} = 2dt$

$$= 2\int \frac{dt}{t}$$

$$= 2\log|t| + c$$

$$= 2\log|\sqrt{x}+1| + c$$

98. The given integral is

$$\int \frac{1+\tan x}{x + \log\sec x}dx$$

Let $x + \log\sec x = t$

$\Rightarrow\ (1+\tan x)\,dx = dt$

$$= \int \frac{dt}{t}$$

$$= \log|t| + c$$

$$= \log|x + \log(\sec x)| + c$$

99. The given integral is

$$\int \frac{\sin 2x}{\sin 3x\sin 5x}dx = \int \frac{\sin(5x-3x)}{\sin 3x\sin 5x}dx$$

$$= \int \frac{\sin 5x\cos 3x - \cos 5x\sin 3x}{\sin 3x\sin 5x}dx$$

$$= \int (\cot 3x - \cot 5x)\,dx$$

$$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$$

100. The given integral is

$$\int \frac{\cos x - \sin x}{1 + \sin 2x}dx = \int \frac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2}dx$$

Put $(\sin x + \cos x) = t$

$\Rightarrow\ (\cos x - \sin x)\,dx = dt$

$$= \int \frac{dt}{1+t^2}$$

$$= \tan^{-1}(t) + c$$

$$= \tan^{-1}(\sin x + \cos x) + c$$

101. The given integral is

$$\int \frac{dx}{\sin x\cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin x\cos^2 x}$$

$$= \int (\sec x\tan x + \operatorname{cosec} x)dx$$

$$= \sec x + \log\left|\tan\left(\frac{x}{2}\right)\right| + c$$

102. The given integral is

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x\cdot\cos^2 x}dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\,dx$$

$$= \tan x - \cot x + c.$$

103. The given integral is

$$\int \frac{dx}{\sin x(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)}$$

$$\int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)dx}$$

$$= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)]\,dx$$

$$= \frac{1}{\sin(a-\text{b})} [\log|\sin(x-a)| - \log|\sin(x-b)|] + c$$

$$= \frac{1}{\sin(a-b)} \left(\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|\right) + c$$

104. The given integral is

$$\int \frac{dx}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)]\,dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|] + c$$

$$= \frac{1}{\sin(a-b)} \left(\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right) + c$$

105. The given integral is

$$\int \frac{dx}{\sin(x-a)\cos(x-b)}$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos[(x-b)-(x-a)]}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int [\tan(x-b)\,\tan(x-b)]dx$$

$$= \frac{1}{\cos(a-b)} \left(\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right) + c$$

106. The given integral is

$$\int \frac{x^x(1+\ln x)}{x^x+1} dx = \int \frac{dt}{t}, \quad \text{Let } t = x^x$$

$$= \log|t| + c$$

$$= \log|x^x + 1| + c$$

107. The given integral is

$$\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x} dx$$

$$= \int \frac{(\cos x + 1) - (\sin x + x)}{(e^x + \sin x + x)} dx$$

$$= \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{(e^x + \sin x + x)} dx$$

$$= \int \left(\frac{(e^x + \cos x + 1)}{(e^x + \sin x + x) - 1}\right) dx$$

$$= \log|e^x + \sin x + x| - x + c$$

108. The given integral is

$$\int \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)} dx$$

Let $\tan^{-1}(\sec x + \cos x) = t$

$$\Rightarrow \frac{\sec x \tan x - \sin x}{1 + (\sec x + \cos x)^2} dx = dt$$

$$\Rightarrow \frac{\sin^3 x}{\cos^2 x(1 + (\sec x + \cos x)^2)} dx = dt$$

$$\Rightarrow \frac{\sin^3 x}{\cos^2 x(\sec^2 x + \cos^2 x + 3)} dx = dt$$

$$\Rightarrow \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)} dx = dt$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + c$$

$$= \log|\tan^{-1}(\sec x + \cos x)| + c$$

109. We have,

$$\int 3x^3 \sin(x^3)\,dx = \int \sin(t)\,dt \quad \text{where } x^3 = t$$

$$= -\cos(t) + c$$

$$= -\cos(x^3) + c$$

110. We have,

$$\int \frac{(1+\ln x)^3}{x} dx = \int t^3 dt \quad \text{Let } (1+\ln x) = t$$

$$\Rightarrow \frac{dx}{x} = dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{(1+\ln x)^4}{4} + c$$

111. We have,

$$\int \frac{dx}{x^2(1+x^4)^{3/4}} = \int \frac{dx}{x^2\left(x^4\left(\frac{1}{x^4}+1\right)\right)^{3/4}}$$

$$= \int \frac{dx}{x^5\left(\frac{1}{x^4}+1\right)^{3/4}}$$

$$= -\int \frac{4t^3}{t^3}\,dt, \text{ Let } \left(\frac{1}{x^4}+1\right) = t^4$$

$$\Rightarrow \frac{-4\,dx}{x^5} = t^3 dt$$

$$= -4\int dt$$

$$= -4t + c$$

$$= -4\left(\frac{1}{x^4}+1\right)^{1/4} + c$$

112. We have,

$$\int \frac{dx}{\sqrt{x}\,(4+3\sqrt{x})^2}$$

Let $4 + 3\sqrt{x} = t$

$$\Rightarrow \frac{3}{2\sqrt{x}}\,dx = dt$$

$$\Rightarrow \frac{dx}{\sqrt{x}} = \frac{2}{3}dt$$

$$= \frac{2}{3}\int \frac{dt}{t^2}$$

$$= -\frac{2}{3} \times \frac{1}{t} + c$$

$$= \frac{2}{3} \times \frac{1}{(4+3\sqrt{x})} + c$$

113. We have,

$$\int 3^{3^{3^x}} \cdot 3^{3^x} \cdot 3^x dx$$

Let $3^x = t$

$$\Rightarrow 3^x dx = \frac{1}{\log_e 3}dt$$

$$= \frac{1}{\log 3}\int 3^{3^t} \cdot 3^t dt$$

$$= \frac{1}{(\log 3)^2}\int 3^{3^t} \cdot 3^t dt, \quad \text{Let } 3^t = v$$

$$\Rightarrow \text{logs } 3dt = dv$$

$$= \frac{3^v}{(\log 3)^3} + c$$

$$= \frac{3^{3^{3^x}}}{(\log 3)^3} + c$$

114. The given integral is

$$\int \tan^3 x \cdot \sec^2 x\,dx$$

Let $\tan x = t$

$$\Rightarrow \sec^2 x\,dx = dt$$

$$= \int t^3 dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\tan^4 x}{4} + c$$

115. The given integral is

$$\int \sin^3 x \cdot \cos x\,dx$$

$$= \int t^3 dt, \quad \text{Let } t = \sin x$$

$$\Rightarrow dt = \cos x\,dx$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\sin^4 x}{4} + c$$

116. The given integral is

$$\int \frac{(\log x)^3}{x}\,dx$$

$$= \int t^3 dt, \quad \text{Let } t = \log x$$

$$\Rightarrow dt = \frac{dx}{x}$$

$$= \frac{t^4}{4} + c$$

$$= \frac{(\log x)^4}{4} + c$$

117. The given integral is

$$\int \frac{\sin x}{\sqrt{3+2\cos x}}\,dx$$

Let $3 + 2\cos x = t^2$

$$\Rightarrow -2\sin x\,dx = 2t\,dt$$

$$\Rightarrow \sin x\,dx = -t\,dt$$

$$= -\int \frac{t\,dt}{t}$$

$$= -\int dt$$

$$= -t + c$$

$$= -\sqrt{3+2\cos x} + c$$

118. The given integral is

$$\int \frac{\sqrt{2+\log x}}{x}\,dx$$

Let $2 + \log x = t^2$

$$\Rightarrow \frac{dx}{x} = 2t\,dt$$

$= \int (t \cdot 2t)\, dt$

$= 2\int t^2\, dt$

$= 2\left(\frac{t^3}{3}\right) + c$

$= \frac{2}{3}\,[(2 + \log x)^{3/2}] + c$

119. The given integral is

$\int \frac{dx}{1 + \sqrt{x}}$

Let $x = t^2$

$\Rightarrow\; dx = 2t\,dt$

$= \int \frac{2t}{1 + t}\, dt$

$= 2\int \left(1 - \frac{1}{1 + t}\right) dt$

$= 2(t - \log|1 + t|) + c$

$= 2(\sqrt{x} - \log|1 + \sqrt{x}|) + c$

120. The given integral is

$\int x^3 \sin x^4\, dx$

Let $x^4 = t$

$\Rightarrow\; 4x^3 dx = dt$

$\Rightarrow\; x^3 dx = \frac{dt}{4}$

$= \frac{1}{4}\int \sin(t) dt$

$= -\frac{1}{4}\cos(t) + c$

$= -\frac{1}{4}\cos(x^4) + c$

121. The given integral is

$\int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x dx$

Put $5^x = t$

$\Rightarrow\; 5^x dx = \frac{dt}{\log 5}$

$= \frac{1}{\log 5}\int 5^{5^t} \cdot 5^t dt$

$= \frac{1}{(\log 5)^2}\int 5^z \cdot dz,\; z = 5^t$

$= \frac{1}{(\log 5)^2}\int 5^z \cdot dz$

$= \frac{5^z}{(\log 5)^3} + c$

$= \frac{5^{5^{5^x}}}{(\log 5)^3} + c$

122. The given integral is

$\int \frac{\sin x - \cos x}{e^x + \sin x} dx$

$= \int \frac{(e^x + \sin x) - (e^x + \cos x)}{(e^x + \sin x)}\, dx$

$= \int \left(1 - \frac{(e^x + \cos x)}{(e^x + \sin x)}\right) dx$

$= (x - \log|e^x + \sin x| + c)$

123. The given integral is

$\int \frac{d}{x(1 + x^3)} = \int \frac{x^2 dx}{x^3(x^3 + 1)}$

$= \frac{1}{3}\int \frac{dt}{t(t + 1)},$ Let $t = x^3$

$\Rightarrow\; dt = 3x^2$

$= \frac{1}{3}\int \left(\frac{1}{t} - \frac{1}{t + 1}\right) dt$

$= \frac{1}{3}\log\left|\frac{t}{t + 1}\right| + \mathrm{C}$

$= \frac{1}{3}\log\left|\frac{x^3}{x^3 + 1}\right| + C$

124. We have,

$\int \sin^3 x \cos^4 x\, dx$

$= \int (1 - \cos^2 x)\cos^4 x \sin x\, dx$

$= -\int (1 - t^2)\, t^4 dt$ Let $\cos x = t$

$\Rightarrow\; \sin x dx = dt$

$= \int (t^6 - t^4)\, dt$

$= \left(\frac{t^7}{7} - \frac{t^5}{5}\right) dt$

$= \left(\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5}\right) + c$

127. We have,

$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x(1 - \sin^2 x)\cos x\, dx$

$= \int t^2(1 - t^2)\, dt,$ Let $\sin x = t$

$\Rightarrow\; \cos x dx = dt$

$= \int (t^2 - t^4)\, dt$

$= \left(\frac{t^3}{3} - \frac{t^5}{5}\right) + c$

$= \left(\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right) + c$

130. We have,

$$\int \sin^3 x \cos^3 x \, dx$$

$$= \int \sin^3 x (1 - \sin^2) \cos x \, dx$$

$$= \int t^3 (1 - t^2) \, dt \qquad \text{Let } \sin x = t$$

$$\Rightarrow \quad \cos x dx = dt$$

$$= \int (t^3 - t^5) \, dt$$

$$= \left(\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} \right) + c$$

134. We have,

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int (2\sin^2 x)(2\cos^2 x) dx$$

$$= \frac{1}{4} \int (1 - \cos 2x)(1 - \cos 2x) dx$$

$$= \frac{1}{4} \int (1 - \cos^2 2x) dx$$

$$= \frac{1}{4} \int \sin^2 2x \, dx$$

$$= \frac{1}{8} \int 2\sin^2 2x \, dx$$

$$= \frac{1}{8} \int (1 - \cos 4x) \, dx$$

$$= \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + c$$

135. We have,

$$\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cdot \sin x \, dx$$

$$= -\int (1 - t^2) dt, \qquad \text{Let } \cos x = t$$

$$\Rightarrow \quad -\sin x dx = dt$$

$$= \int (t^2 - 1) dt$$

$$= \left(\frac{t^3}{3} - t \right) + c$$

$$= \left(\frac{\cos^3 x}{3} - \cos x \right) + c$$

139. We have,

$$\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos dx$$

$$= \int (1 - \sin^2 x)^2 \cdot \cos dx$$

$$= \int (1 - t^2)^2 dt, \qquad \text{Let } \sin x = t$$

$$\Rightarrow \quad \cos x dx = dt$$

$$= \int (t^4 - 2t^2 + 1) dt$$

$$= \left(\frac{t^5}{5} - \frac{2}{3} t^3 + t \right) + c$$

$$= \left(\frac{\sin^5 x}{5} - \frac{2}{3} \sin^3 x + \sin x \right) + c$$

142. We have,

$$\int \sin^4 x \, dx = \frac{1}{4} \int (2\sin^2 x)^2 dx$$

$$= \frac{1}{4} \int (1 - \cos 2x)^2 dx$$

$$= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int (1 - 2\cos 2x) \, dx + \frac{1}{8} \int (2\cos^2 2x) \, dx$$

$$= \frac{1}{4} \int (1 - 2\cos 2x) \, dx + \frac{1}{8} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{4} \int \left(x - \frac{2\sin 2x}{2} \right) + \frac{1}{8} \left(x + \frac{\sin 4x}{4} \right) + c$$

$$= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$$

145. We have,

$$\int \cos^6 x \, dx = \frac{1}{8} \int (2\cos^2 x)^3 dx$$

$$= \frac{1}{8} \int (1 + \cos 2x)^3 dx$$

$$= \frac{1}{8} \int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x) \, dx$$

$$= \frac{1}{8} \int (1 + 3\cos 2x) \, dx + \frac{3}{16} \int (2\cos^2 2x) \, dx + \frac{1}{32} \int (4\cos^3 2x) \, dx$$

$$= \frac{1}{8} \int (1 + 3\cos 2x) dx + \frac{3}{16} \int (1 + \cos 4x) \, dx + \frac{1}{32} \int (\cos 6x + 3\cos 2x) \, dx$$

$$= \frac{1}{8} \left(x + \frac{3}{2} \sin 2x \right) + \frac{3}{16} \left(x + \frac{\sin 4x}{4} \right) + \frac{1}{32} \left(\frac{\sin 6x}{6} + \frac{3}{2} \sin 2x \right) + c$$

$$= \frac{5}{16} x + \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{192} \sin 6x + c$$

148. We have,

$$\int \frac{dx}{\sin^{1/2} x \cos^{3/2} x}$$

Divide the numerator and the denominator by $\cos^2 x$, we get

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx = \int \frac{2t\,dt}{t}, \quad \text{Let } \tan x = t^2$$
$$\Rightarrow \quad \sec^2 x dx = dt$$
$$= 2\int dt$$
$$= 2t + c$$
$$= 2\sqrt{\tan x} + c$$

149. We have,

$$\int \frac{dx}{\sin^{3/2} x \cos^{5/2} x}$$

Divide the numerator and the denominator by $\cos^4 x$, we get

$$\int \frac{\sec^4 x}{\tan^{3/2} x}\,dx = \int \frac{(1 + \tan^2 x)\sec^2 x}{\tan^{3/2} x}\,dx$$
$$= \int \frac{(1 + t^2)dt}{t^{3/2}} \quad \text{Let } \tan x = t^2$$
$$\Rightarrow \sec^2 xdx = dt$$
$$= \int (t^{-3/2} + t^{1/2})dt$$
$$= 2\sqrt{t} + \frac{2}{3}t^{3/2} + c$$
$$= 2\sqrt{\tan x} + \frac{2}{3}(\tan x)^{3/2} + c$$

150. The given integral is

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx = \int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx$$
$$= \int \frac{\sec^2 x\sqrt{\tan x}}{\tan x}\,dx$$
$$= \int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$$
$$= \int \frac{2t\,dt}{t}, \quad \text{Let } \tan x = t^2$$
$$\Rightarrow \sec^2 xdx = 2t\,dt$$
$$= \int 2\,dt$$
$$= 2t + c$$
$$= 2\sqrt{\tan x} + c$$

151. The given integral is

$$\int \frac{\sin x}{\cos^5 x}\,dx = \int \sec^4 x \tan x\,dx$$
$$= \int (1 + \tan^2 x)\tan x \sec^2 x\,dx$$
$$= \int (1 + t^2)t\,dt \quad \text{Let } \tan x = t$$
$$\Rightarrow \sec^2 xdx = dt$$
$$= \int (t + t^3)\,dt$$
$$= \frac{t^2}{2} + \frac{t^4}{4} + C$$
$$= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

152. The given integral is

$$\int \frac{dx}{\sin^3 x \cos^5 x} = \int \frac{\sec^8 x}{\tan^3 x}\,dx$$
$$= \int \frac{(1 + \tan^2 x)^3 \sec^2}{\tan^3 x}\,dx$$
$$= \int \frac{(1 + t^2)^3}{t^3}\,dt, \quad \text{Let } t = \tan x$$
$$\Rightarrow dt = \sec^2 xdx$$
$$= \int \frac{(1 + 3t^2 + 3t^4 + t^6)}{t^3}\,dt$$
$$= \int \left(\frac{1}{t^3} + \frac{3}{t} + 3t + t^3\right)dt$$
$$= -\frac{1}{2t^2} + 3\log|t| + \frac{3t^2}{2} + \frac{t^4}{4} + c$$

where $t = \tan x$

153. The given integral is

$$\int \frac{\sin^2 x\,dx}{\cos^6 x} = \int \tan^2 x \sec^4 x\,dx$$
$$= \int \tan^2 x(1 + \tan^2 x)\sec^2 x\,dx$$
$$= \int t^2(1 + t^2)\,dt \quad \text{Let } \tan x = t$$
$$\Rightarrow \sec^2 x\,dx = dt$$
$$= \int (t^4 + t^2)\,dt$$
$$= \frac{t^5}{5} + \frac{t^3}{3} + c$$
$$= \frac{\tan^5 x}{5} + \frac{\tan^3}{3} + c$$

154. The given integral is

$$\int \frac{dx}{\sin x \cos x^3 x} = \int \frac{\sec^4 x}{\tan x}\,dx$$
$$= \int \frac{(1 + \tan^2 x)\sec^2 x}{\tan x}\,dx$$
$$= \int \frac{(1 + t^2)}{t}\,dt \quad \text{Let } \tan x = t$$
$$= \int \left(1 + \frac{1}{t}\right)dt$$
$$= \frac{t^2}{2} + \log|t| + C$$
$$= \frac{\tan^2 x}{2} + \log|\tan x| + c$$

155. The given integral is

$$\int \frac{dx}{\sin^2 x \cos^4 x} = \int \frac{\sec^6 x}{\tan^2 x} dx$$

$$= \int \frac{(1 + \tan^2 x)^2 \sec^2 x}{\tan^2 x} dx$$

$$= \int \frac{(1 + t^2)^2}{t^2} dt \quad \text{Let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$= \int \frac{(t^4 + 2t^2 + 1)}{t^2} dt$$

$$= \int \left(t^2 + \frac{1}{t^2} + 2 \right) dt$$

$$= \frac{t^3}{3} - \frac{1}{t} + 2t + c$$

$$= \frac{\tan^3 x}{3} + 2\tan x - \cot x + c$$

156. The given integral is

$$\int \frac{dx}{\sin^{\frac{1}{2}} x \cos^{\frac{7}{2}} x} = \int \frac{\sec^4 x}{\sqrt{\tan x}} dx$$

$$= \int \frac{(1 + \tan^2 x)\sec^2 x}{\sqrt{\tan x}} dx$$

$$= \int \frac{(1 + t^2)2t}{t} dt \quad \text{Let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$= 2\int (1 + t^2) dt$$

$$= 2\left(t + \frac{t^3}{3} \right) + c$$

$$= 2\left(\tan x + \frac{\tan^3 x}{3} \right) + c$$

157. The given integral is

$$\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = \int \frac{\sec^4 x}{\tan^{3/2} x} dx$$

$$= \int \frac{(1 + \tan^2 x)\sec^2 x}{\tan^{3/2} x} dx$$

$$= \int \frac{(1 + t^2)2t}{t^3} dt, \quad \text{Let } \tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$$

$$= 2\int \frac{(1 + t^2)}{t} dt$$

$$= 2\int \left(t + \frac{1}{t} \right) dt$$

$$= 2\left(\frac{t^2}{2} + \log|t| \right) + C$$

$$= 2\left(\frac{\tan^2 x}{2} + \log|\tan x| \right) + C$$

158. We have,

$$\int \frac{dx}{x^2 + 4} = \int \frac{dx}{x^2 + 2^2} = \frac{1}{2} \tan^{-1}\left(\frac{x}{2} \right) + c$$

159. We have,

$$\int \frac{dx}{9x^2 + 1} = \frac{1}{9} \int \frac{dx}{x^2 + \left(\frac{1}{3} \right)^2} = \frac{1}{9} \times 3\tan^{-1}(3x) + c$$

160. We have,

$$\int \frac{dx}{x^2 - 4} = \int \frac{dx}{x^2 - (2)^2}$$

$$= \frac{1}{2.2} \log\left| \frac{x - 2}{x + 2} \right| + c$$

$$= \frac{1}{4} \log\left| \frac{x - 2}{x + 2} \right| + c$$

161. We have $\int \frac{x^4 - 1}{x^2 + 4} dx$

$$= \int \frac{(x^4 - 16) + 15}{x^2 + 4} dx$$

$$= \int \frac{((x^2 + 4)(x^2 - 4)) + 15}{x^2 + 4} dx$$

$$= \int \left(x^2 - 4 + \frac{5}{x^2 + 2^2} \right) dx$$

$$= \frac{x^3}{3} - 4x + \frac{5}{2} \tan^{-1}\left(\frac{x}{2} \right) + c$$

162. We have,

$$\int \frac{dx}{\sqrt{x^2 + 4}} = \int \frac{dx}{\sqrt{x^2 + 2^2}}$$

$$= \log(x + \sqrt{x^2 + 4}) + c$$

163. We have,

$$\int \frac{dx}{\sqrt{4x^2 + 1}} = \frac{1}{2} \int \frac{dx}{\sqrt{x^2 + \left(\frac{1}{2} \right)^2}}$$

$$= \frac{1}{2} \log\left| x + \sqrt{x^2 + \frac{1}{4}} \right| + c$$

164. We have,

$$\int \left(\frac{x + 4}{x^3 + 4x} \right) dx = \int \left(\frac{x}{x^3 + 4x} + \frac{4}{x^3 + 4x} \right) dx$$

$$= \int \left(\frac{1}{x^2 + 4} + \frac{4}{x(x^2 + 4)} \right) dx$$

$$= \int \frac{dx}{x^2 + 4} + \int \frac{4x}{x^2(x^2 + 4)} dx$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + 2 \int \frac{dt}{t(t + 4)}, \text{ Let } t = x^2$$

$$\Rightarrow dt = 2x^2 dx$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{2}{4} \int \left(\frac{1}{t} - \frac{1}{t + 4}\right) dt$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} \log\left|\frac{t}{t + 4}\right| + c$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} \log\left|\frac{x^2}{x^2 + 4}\right| + c$$

165. The given integral is

$$\int \frac{x^4 + 1}{x^2 + 1} dx = \int \frac{(x^4 - 1) + 2}{x^2 + 1} dx$$

$$= \int \left(x^2 - 1 + \frac{2}{x^2 + 1}\right) dx$$

$$= \left(\frac{x^3}{3} - x + 2\tan^{-1} x\right) + c$$

166. The given integral is

$$\int \frac{dx}{\sqrt{(2 - x)^2 + 1}} = \int \frac{dx}{\sqrt{(x - 2)^2 + 1}}$$

$$= \int \frac{dt}{\sqrt{t^2 + 1}}, \quad \text{Let } t = (x - 2) \Rightarrow dt = dx$$

$$= \log|t + \sqrt{t^2 + 1}| + c$$

$$= \log|(x - 2) + \sqrt{(x - 2)^2 + 1}| + c$$

167. The given integral is

$$\int \frac{x + 9}{x^3 + 9x} dx = \int \frac{x + 9}{x(x^2 + 9)} dx$$

$$= \int \frac{x + (9 + x^2) - x^2}{x(x^2 + 9)} dx$$

$$= \int \frac{dx}{(x^2 + 9)} + \int \frac{dx}{x} - \int \frac{x\,dx}{(x^2 + 9)}$$

$$= \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + \log|x| - \frac{1}{2} \log|x^2 + 9| + c$$

168. The given integral is

$$\int \frac{1 + x}{1 + x^2} dx = \int \frac{dx}{1 + x^2} + \int \frac{x}{1 + x^2} dx$$

$$= \tan^{-1}(x) + \frac{1}{2} \log|1 + x^2| + c$$

169. The given integral is

$$\int \frac{1 + x}{x^3 + x} dx = \int \frac{(1 + x)\,dx}{x(x^2 + 1)}$$

$$= \int \frac{(1 + x^2) + x - x^2}{x(x^2 + 1)} dx$$

$$= \int \frac{dx}{x} + \int \frac{dx}{x^2 + 1} - \int \frac{x}{x^2 + 1} dx$$

$$= \log|x| + \tan^{-1} x - \frac{1}{2} \log|x^2 + 1| + c$$

170. The given integral is

$$\int \frac{dx}{x^4 - 1} = \int \frac{dx}{(x^2 - 1)(x^2 + 1)}$$

$$= \frac{1}{2} \int \left(\frac{1}{x^2 - 1} - \frac{1}{x^2 + 1}\right) dx$$

$$= \frac{1}{2} \left(\frac{1}{2} \log\left|\frac{x - 1}{x + 1}\right| - \tan^{-1} x\right) + c$$

171. The given integral is

$$\int \frac{dx}{x^3 + x} = \int \frac{dx}{x(x^2 + 1)}$$

$$= \int \frac{x\,dx}{x^2(x^2 + 1)}$$

$$= \frac{1}{2} \int \frac{dt}{t(t + 1)}, \quad \text{Let } x^2 = t^2$$

$$2x\,dx = 2t\,dt$$

$$= \frac{1}{2} \int \left(\frac{1}{t} - \frac{1}{t + 1}\right) dt$$

$$= \frac{1}{2} \log\left|\frac{t}{t + 1}\right| + c$$

$$= \frac{1}{2} \log\left|\frac{x^2}{x^2 + 1}\right| + c$$

172. We have,

$$\int \frac{dx}{x^2 + 4x + 4} = \int \frac{dx}{(x^2 + 2)^2}$$

$$= -\frac{1}{(x + 2)} + c$$

173. We have,

$$\int \frac{dx}{x^2 + 6x + 10} = \int \frac{dx}{(x + 3)^2 + (10 - 9)}$$

$$= \int \frac{dx}{(x + 3)^2 + 1}$$

$$= \tan^{-1}(x + 3) + c$$

174. We have,

$$\int \frac{dx}{2x^2+5x+6} = \frac{1}{2}\int \frac{dx}{x^2+\frac{5}{2}x+3}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{5}{4}\right)^2+\left(3-\frac{25}{16}\right)}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{5}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}$$

$$= \frac{1}{2}\times\frac{4}{\sqrt{32}}\tan^{-1}\left(\frac{\left(x+\frac{5}{4}\right)}{\frac{\sqrt{23}}{4}}\right)+c$$

$$= \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{(4x+5)}{\sqrt{23}}\right)+c$$

175. The given integral is

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

176. The given integral is

$$\int \frac{dx}{1+x-x^2} = -\int \frac{dx}{x^2-x-1}$$

$$= -\int \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= -\frac{1}{\sqrt{5}}\log\left|\frac{\left(x-\frac{1}{2}\right)-\frac{\sqrt{5}}{2}}{\left(x-\frac{1}{2}\right)+\frac{\sqrt{5}}{2}}\right|+c$$

177. The given integral is

$$\int \frac{dx}{x^2+4x+3} = \int \frac{dx}{(x+2)^2-1}$$

$$= \frac{1}{2}\log\left|\frac{(x+2)-1}{(x+2)+1}\right|+c$$

$$= \frac{1}{2}\log\left|\frac{(x+1)}{(x+3)}\right|+c$$

178. The given integral is

$$\int \frac{dx}{4x^2+7x+10} = \frac{1}{4}\int \frac{dx}{\left(x^2+\frac{7}{4}x+\frac{5}{2}\right)}$$

$$= \frac{1}{4}\int \frac{dx}{\left(x+\frac{7}{8}\right)^2+\left(\frac{\sqrt{111}}{8}\right)^2}$$

$$= \frac{1}{4}\times\frac{8}{\sqrt{111}}\tan^{-1}\left(8x+\frac{7}{\sqrt{111}}\right)+c$$

$$= \frac{2}{\sqrt{111}}\tan^{-1}\left(8x+\frac{7}{\sqrt{111}}\right)+c$$

179. The given integral is

$$\int \frac{dx}{x^2-2ax} = \int \frac{dx}{(x^2-a)^2-a^2}$$

$$= \frac{1}{2a}\log\left|\frac{(x-a)-a}{(x-a)+a}\right|+c$$

$$= \frac{1}{2a}\log\left|\frac{(x-2a)}{x}\right|+c$$

180. The given integral is

$$\int \frac{dx}{x^2+2ax} = \int \frac{dx}{(x+a)^2-a^2}$$

$$= \frac{1}{2a}\log\left|\frac{x}{x+2a}\right|+c$$

181. The given integral is

$$\int \frac{dx}{a^2+2ax} = \frac{1}{2a}\log|a^2+2ax|+c$$

182. The given integral is

$$\int \frac{dx}{2ax-x^2} = -\int \frac{dx}{x^2-2ax}$$

$$= -\int \frac{dx}{(x-a)^2-a^2}$$

$$= -\frac{1}{2a}\log\left|\frac{x-a-a}{x-a+a}\right|+c$$

$$= \frac{1}{2a}\log\left|\frac{x}{x-2a}\right|+c$$

183. The given integral is

$$\int \frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{3a}\int\left(\frac{1}{x^2+1}-\frac{1}{x^2+4}\right)dx$$

$$= \frac{1}{3}\left(\tan^{-1}x-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right)+c$$

184. We have,

$$\int \frac{x^2}{x^6 + x}\,dx = \frac{1}{3}\,\frac{3x^2}{x^6 + 1}\,dx$$

$$= \frac{1}{3}\,\frac{dt}{t^2 + 1}, \quad \text{Let } t = x^3$$

$$\Rightarrow \quad dt = 3x^3\,dx$$

$$= \frac{1}{3}\tan^{-1}(t) + c$$

$$= \frac{1}{3}\tan^{-1}(x^3) + c$$

185. We have,

$$\int \frac{\cos x\,dx}{\sin^2 x + 3\sin x + 2}$$

$$= \int \frac{dt}{t^2 + 3t + 2} \qquad \text{Let } t = \sin x$$

$$\Rightarrow \quad dt = \cos x\,dx$$

$$= \int \frac{dt}{(t + 1)(t + 2)}$$

$$= \int \left(\frac{1}{t + 1} - \frac{1}{t + 2}\right)dt$$

$$= \log\left|\frac{t + 1}{t + 2}\right| + c$$

$$= \log\left|\frac{\sin x + 1}{\sin x + 2}\right| + c$$

186. We have,

$$\int \frac{x^x(1 + \log x)}{x^{2x} + x^x + 1}\,dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\left(t + \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^x + 1}{\sqrt{3}}\right) + c$$

187. We have,

$$\int \frac{dx}{x(x^4 + 1)} = \int \frac{x^3 dx}{x^4(x^4 + 1)}$$

$$= \frac{1}{4}\int \frac{4x^3 dx}{x^4(x^4 + 1)}$$

$$= \frac{1}{4}\int \frac{dt}{t(t + 1)}, \qquad \text{Let } t = x^4$$

$$\Rightarrow dt = 4x^3\,dx$$

$$= \frac{1}{4}\int\left(\frac{1}{t} - \frac{1}{t + 1}\right)$$

$$= \frac{1}{4}\log\left|\frac{t}{t + 1}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{x^4}{x^4 + 1}\right| + c$$

194. We have,

$$\int \frac{2x + 3}{x^2 + 4x + 5}\,dx = \int \frac{(2x + 4) - 1}{x^2 + 4x + 5}\,dx$$

$$= \int \frac{(2x + 4)}{x^2 + 4x + 5}\,dx - \int \frac{dx}{x^2 + 4x + 5}$$

$$= \int \frac{(2x + 4)}{x^2 + 4x + 5}\,dx - \int \frac{dx}{(x + 2)^2 + 1}$$

$$= \log|x^2 + 4x + 5| - \tan^{-1}(x + 2) + c$$

195. We have,

$$\int \frac{3x + 2}{x^2 - 3x + 4}\,dx$$

$$= \int \frac{\frac{3}{2}(2x - 3) + \left(2 + \frac{9}{2}\right)}{x^2 - 3x + 4}\,dx$$

$$= \frac{3}{2}\int \frac{(2x - 3)}{x^2 - 3x + 4}\,dx + \frac{13}{2}\int \frac{dx}{x^2 - 3x + 4}$$

$$= \frac{3}{2}\int \frac{(2x - 3)}{x^2 - 3x + 4}\,dx + \frac{13}{2}\int \frac{dx}{\left(x - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

$$= \frac{3}{2}\log|x^2 - 3x + 4| + \frac{13}{\sqrt{7}}\tan^{-1}\left(\frac{\left(x - \frac{3}{2}\right)}{\frac{\sqrt{7}}{2}}\right) + c$$

$$= \frac{3}{2}\log|x^2 - 3x + 4| + \frac{13}{\sqrt{7}}\tan^{-1}\left(\frac{2x - 3}{\sqrt{7}}\right) + c$$

195. The given integral is

$$\int \frac{x}{x^2 + x + 1}\,dx = \frac{1}{2}\int \frac{(2x + 1) - 1}{(x^2 + x + 1)}\,dx$$

$$= \frac{1}{2}\int \frac{(2x + 1)}{(x^2 + x + 1)}\,dx - \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\log|x^2 + x + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C$$

196. The given integral is

$$\int \frac{4x+1}{x^2+3x+2}\,dx = 2\int \frac{2x+\frac{1}{2}}{(x^2+3x+2)}\,dx$$

$$= 2\int \frac{(2x+3)-(5/2)}{(x^2+3x+2)}\,dx$$

$$= 2\int \frac{(2x+3)}{(x^2+3x+2)}\,dx - 5\int \frac{dx}{(x+1)(x+2)}$$

$$= 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right| + c$$

197. The given integral is

$$\int \frac{dx}{2e^{2x}+3e^x+1} = \int \frac{dx}{(2e^x+1)(e^x+1)}$$

$$= \int\left(\frac{2}{(2e^x+1)} - \frac{1}{(e^x+1)}\right)dx$$

$$= \int\left(\frac{2e^{-x}}{(2+e^{-x})} - \frac{e^{-x}}{(e^{-x}+1)}\right)dx$$

$$= \log|e^{-x}+1| - 2\log|e^{-x}+2| + c$$

198. The given integral is

$$\int \frac{(3\sin x-2)\cos x\,dx}{(5-\cos^2 x-4\sin x)}$$

$$= \int \frac{(3\sin x-2)\cos x}{\sin^2 x-4\sin x+4}\,dx$$

$$= \int \frac{(3t-2)}{t^2-4t+4}\,dt, \quad \text{Let } t = \sin x \Rightarrow dt = \cos x\,dx$$

$$= \int \frac{(3t-2)}{(t-2)^2}\,dt$$

$$= \int \frac{(3t-6+4)}{(t-2)^2}\,dt$$

$$= 3\int \frac{dt}{(t-2)} + 4\int \frac{dt}{(t-2)^2}$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + c$$

$$= 3\log|\sin x-2| - \frac{4}{(\sin x-2)} + c$$

199. The given integral is

$$\int \frac{ax^3+bx}{x^4+c^2}\,dx = a\int \frac{x^3}{x^4+c^2}\,dx + b\int \frac{x\,dx}{x^4+c^2}$$

$$= \frac{a}{4}\log|x^4+c^2| + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + c$$

200. The given integral is.

$$\int\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\times(2+2\sin 2x)\,dx$$

$$= \int\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)(1+(\sin x+\cos x)^2)\,dx$$

$$= \int\left(\frac{1+t^2}{t}\right)dt, \quad \text{Let } \cos x+\sin x = t \Rightarrow (-\sin x+\cos x)\,dx = dt$$

$$= \int\left(t+\frac{1}{t}\right)dt$$

$$= \frac{t^2}{2} + \log|t| + c$$

$$= \frac{(\sin x+\cos x)^2}{2} + \log|\sin x+\cos x| + c$$

201. The given integral is

$$\int \frac{\sin x+\cos x}{5+3\sin 2x}\,dx = \int \frac{(\sin x+\cos x)\,dx}{5+3(1-1+\sin 2x)}$$

$$= \int \frac{(\sin x+\cos x)\,dx}{8-3(1-\sin 2x)}$$

$$= \int \frac{(\sin x+\cos x)\,dx}{8-3(\sin x-\cos x)^2}$$

$$= \int \frac{dt}{8-3t^2}, \quad \text{Let } t = (\sin x-\cos x),\ dt = (\cos x+\sin x)\,dx$$

$$= \int \frac{dt}{8-3t^2}$$

$$= -\frac{1}{3}\int \frac{dt}{t^2-\left(\sqrt{\frac{8}{3}}\right)^2}$$

$$= -\frac{1}{2\sqrt{24}}\log\left|\frac{t-(\sqrt{8/3})}{t+(\sqrt{8/3})}\right| + c$$

where $t = \sin x + \cos x$

202. The given integral is

$$\int \frac{\sin x-\cos x}{3+5\sin 2x}\,dx = \int \frac{\sin x-\cos x}{5(1+\sin 2x)-2}\,dx$$

$$= \int \frac{\sin x-\cos x}{5(\sin x+\cos x)^2-2}\,dx$$

$$= \int \frac{\cos x-\sin x}{2-5(\sin x+\cos x)^2}\,dx$$

$$= \int \frac{dt}{2-5t^2}, \quad \text{Let } t = (\sin x+\cos x) \Rightarrow dt = (\cos x-\sin x)\,dx$$

$$= \int \frac{dt}{2 - 5t^2}$$

$$= -\frac{1}{5}\int \frac{dt}{t^2 - (\sqrt{2/5})^2}$$

$$= -\frac{1}{\sqrt{10}}\log\left|\frac{t - (\sqrt{2/5})}{t + (\sqrt{2/5})}\right| + c$$

$$= -\frac{1}{\sqrt{10}}\log\left|\frac{(\sin x + \cos x) - (\sqrt{2/5})}{(\sin x + \cos x) + (\sqrt{2/5})}\right| + c$$

203. We have,

$$\int \frac{dx}{\sqrt{x^2 + x + 1}} = \int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + c$$

204. We have,

$$\int \frac{dx}{\sqrt{x^2 - 2ax}} = \int \frac{dx}{\sqrt{x^2 - 2ax + a^2 - a^2}}$$

$$= \int \frac{dx}{\sqrt{(x - a)^2 - a^2}}$$

$$= \log\left|(x - a) + \sqrt{x^2 - 2ax}\right| + c$$

205. We have,

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - 4 + 4x - x^2}}$$

$$= \int \frac{dx}{\sqrt{2^2 - (x^2 - 4x + 4)}}$$

$$= \int \frac{dx}{\sqrt{2^2 - (x - 2)^2}}$$

$$= \sin^{-1}\left(\frac{x - 2}{2}\right) + c$$

205. The given integral is

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{-(x^2 - 4x)}}$$

$$= \int \frac{dx}{\sqrt{-\{(x - 2)^2 - 4\}}}$$

$$= \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$

$$= \sin^{-1}\left(\frac{x - 2}{2}\right) + c$$

206. The given integral is

$$\int \frac{dx}{\sqrt{6 - x - x^2}} = \int \frac{dx}{\sqrt{-(x^2 + x - 6)}}$$

$$= \int \frac{dx}{\sqrt{-\left\{\left(x + \frac{1}{2}\right)^2 - \frac{25}{4}\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{\left(x + \frac{1}{2}\right)}{\frac{5}{2}}\right) + c$$

$$= \sin^{-1}\left(\frac{2x + 1}{5}\right) + c$$

207. The given integral is

$$\int \frac{dx}{\sqrt{1 + x + x^2}} = \int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + c$$

208. The given integral is

$$\int \frac{dx}{\sqrt{1 + x - x^2}} = \int \frac{dx}{\sqrt{-\{x^2 - x - 1\}}}$$

$$= \int \frac{dx}{\sqrt{-\left\{\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{\left(x - \frac{1}{2}\right)}{\frac{\sqrt{5}}{2}}\right) + c$$

$$= \sin^{-1}\left(\frac{2x - 1}{\sqrt{5}}\right) + c$$

209. The given integral is

$$\int \frac{dx}{x^2 + 2ax} = \int \frac{dx}{\sqrt{(x + a)^2 - a^2}}$$

$$= \log\left|(x + a) + \sqrt{x^2 + 2ax}\right| + c$$

210. The given integral is

$$\int \frac{dx}{\sqrt{2ax - x^2}} = \int \frac{dx}{\sqrt{-\{x^2 - 2ax\}}}$$

$$= \int \frac{dx}{\sqrt{-\{(x-a)^2 - a^2\}}}$$

$$= \int \frac{dx}{\sqrt{a^2 - (x-a)^2}}$$

$$= \sin^{-1}\left(\frac{x-a}{a}\right) + c$$

211. We have,

$$\int \frac{x\,dx}{\sqrt{x^4 + x^2 + 1}}$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t^2 + t + 1}}, \quad \text{Let } x^2 = t \Rightarrow x\,dx = \frac{dt}{2}$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \frac{1}{2}\log\left|\left(t+\frac{1}{2}\right) + \sqrt{t^2+t+1}\right| + c$$

$$= \frac{1}{2}\log\left|\left(x^2+\frac{1}{2}\right) + \sqrt{x^4+x^2+1}\right| + c$$

212. We have,

$$\int \sqrt{\sec x - 1}\,dx$$

$$= \int \sqrt{\frac{1-\cos x}{\cos x} \times \frac{(1+\cos x)}{(1+\cos x)}}\,dx$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x(1+\cos x)}}\,dx$$

$$= \int \frac{\sin x}{\sqrt{\cos x(1+\cos x)}}\,dx$$

$$= -\int \frac{dt}{\sqrt{t(t+1)}} \quad \text{Let } \cos x = t \Rightarrow -\sin x\,dx = dt$$

$$= -\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{t^2+t}\right| + c$$

$$= -\log\left|\left(\cos x+\frac{1}{2}\right) + \sqrt{\cos^2 x+\cos x}\right| + c$$

213. We have,

$$\int \frac{dx}{x^{3/4}\sqrt{x^{1/2}-1}} = 4\int \frac{dt}{\sqrt{t^2-1}}, \quad \text{Let } x^{1/4} = t \Rightarrow \frac{1}{4}x^{-3/4} = dt$$

$$= 4\log\left|t + \sqrt{t^2-1}\right| + c$$

$$= 4 \times \log\left|x^{1/4} + \sqrt{x^{1/2}-1}\right| + c$$

214. We have,

$$\int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx = \int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx$$

$$= \int \sqrt{\frac{\sin(x-a)}{\sin(x+a)} \times \frac{\sin(x-a)}{\sin(x-a)}}\,dx$$

$$= \int \frac{\sin(x-a)}{\sqrt{\sin^2 x - \sin^2 a}}\,dx$$

$$= \int \frac{(\sin x\cos a - \cos x\sin a)}{\sqrt{\sin^2 x - \sin^2 a}}\,dx$$

$$= \cos a\int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2 a}}\,dx - \sin a\int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2 a}}\,dx$$

$$= \cos a\int \frac{\sin x}{\sqrt{\cos^2 a - \cos^2 x}}\,dx - \sin a\int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2 a}}\,dx$$

$$= -\cos a\sin^{-1}\left(\frac{\cos x}{\cos a}\right) - \sin a\log\left|\sin x + \sqrt{\sin^2 x - \sin 2a}\right| + c$$

215. The given integral is

$$\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx = \int \frac{e^x dx}{\sqrt{4-(e^x)^2}}$$

$$= \int \frac{dz}{\sqrt{4-z^2}}, \quad \text{Let } z = e^x \Rightarrow dx = e^x dt$$

$$= \sin^{-1}\left(\frac{z}{2}\right) + c$$

$$= \sin^{-1}\left(\frac{e^x}{2}\right) + c$$

216. The given integral is

$$\int \frac{\sec^2 x}{\sqrt{16+\tan x}}\,dx$$

$$= \int \frac{dt}{\sqrt{t^2+4}}, \quad \text{Let } t = \tan x \Rightarrow dt = \sec^2 x\,dt$$

$$= \log\left|t + \sqrt{t^2+4}\right| + c$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + c$$

217. The given integral is

$$\int \sqrt{\sec x - 1}\, dx = \int \sqrt{\frac{1 - \cos x}{\cos x}}\, dx$$

$$= \int \sqrt{\frac{(1 - \cos x)(1 + \cos x)}{\cos x (1 + \cos x)}}\, dx$$

$$= \int \frac{\sin x}{\sqrt{\cos x (1 + \cos x)}}\, dx$$

$$= -\int \frac{dt}{\sqrt{t^2 + t}}, \quad \text{Let } t = \cos x \Rightarrow dx = -\sin x\, dx$$

$$= -\int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= -\log\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right| + c$$

$$= -\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + c$$

218. The given integral is

$$\int \sqrt{\text{cosec} x - 1}\, dx = \int \sqrt{\frac{1 - \sin x}{\sin x}}\, dx$$

$$= \sqrt{\frac{(1 - \sin x)(1 + \sin x)}{\sin x (1 + \sin x)}}\, dx$$

$$= \int \frac{\cos x}{\sqrt{\sin x (1 + \sin x)}}\, dx$$

$$= \int \frac{dt}{\sqrt{t^2 + t}}, \quad \text{Let } t = \sin x \Rightarrow dt = \cos x\, dt$$

$$= \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right) - \left(\frac{1}{2}\right)^2}}$$

$$= \log\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right| + c$$

$$= \log\left|\left(\sin x + \frac{1}{2}\right) + \sqrt{\sin^2 x + \sin x}\right| + c$$

219. The given integral is

$$\int \frac{dx}{\sqrt{1 - e^{2x}}} = \int \frac{e^{-x}}{\sqrt{(e - x)^2 - 1}}\, dx$$

$$= -\int \frac{dt}{\sqrt{t^2 - 1}}, \quad \text{Let } t = e^{-x} \Rightarrow dt = -e^{-x} dt$$

$$= -\log\left|t + \sqrt{t^2 - 1}\right| + c$$

$$= -\log\left|e^{-x} + \sqrt{e^{-2x} - 1}\right| + c$$

220. The given integral is

$$\int \sqrt{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}}\, dx$$

$$= \int \sqrt{\frac{\sin(x - \alpha)\, \sin(x - \alpha)}{\sin(x - \alpha)\, \sin(x + \alpha)}}\, dx$$

$$= \int \frac{\sin(x - \alpha)}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx$$

$$= \int \frac{\sin x \cos\alpha - \cos x \sin\alpha}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx$$

$$= \cos\alpha \int \frac{\sin x\, dx}{\sqrt{\sin^2\alpha - \sin^2 x} - \sin\alpha} \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

$$= \cos\alpha \int \frac{\sin x\, dx}{\sqrt{\cos^2\alpha - \cos^2 x}} - \sin\alpha \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

$$= -\cos\alpha \sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right)$$

$$-\sin\alpha \log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right| + c$$

221. The given integral is

$$\int \frac{dx}{x^{2/3}\sqrt{x^{2/3} - 4}} = \int \frac{dx}{x^{2/3}\sqrt{(x^{1/3})^2 - 4}}$$

$$= \frac{1}{3}\int \frac{dt}{\sqrt{t^2 - 4}}, \quad \text{Let } t = x^{1/3} \Rightarrow dt = \frac{1}{3}x^{-2/3}\, dx$$

$$= \frac{1}{3}\int \frac{dt}{\sqrt{t^2 - 4}}$$

$$= \frac{1}{3}\log\left|t + \sqrt{t^2 - 4}\right| + c$$

$$= \frac{1}{3}\log\left|x^{1/3} + \sqrt{x^{2/3} - 4}\right| + c$$

222. The given integral is

$$\int \sqrt{\frac{x}{a^3 - x^3}}\, dx$$

$$\text{Let } x = a\sin^{2/3}\theta \Rightarrow dx = a \cdot \sin^{-1/3}\theta\, d\theta$$

$$= \int \frac{\sqrt{a}\sin^{1/3}\theta \sin^{-1/3}\theta \cos\theta}{a^{3/2}\cos\theta}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{\theta}{a} + c$$

$$= \frac{\sin^{-1}\left\{\left(\frac{x}{a}\right)^{3/2}\right\}}{a} + c$$

223. The given integral is

$$\int\left(\frac{\cos\theta + \sin\theta}{\sqrt{5 + \sin 2\theta}}\right)d\theta$$

$$= \int \frac{\cos\theta + \sin\theta}{\sqrt{6 - (1 - \sin 2\theta)}}\,d\theta$$

$$= \int \frac{\cos\theta + \sin\theta}{\sqrt{6 - (\sin\theta - \cos\theta)^2}}\,d\theta$$

Put $(\sin\theta - \cos\theta) = t$

$\Rightarrow \quad (\cos\theta + \sin\theta)d\theta = dt$

$$= \int \frac{dt}{\sqrt{6 - t^2}}$$

$$= \sin^{-1}\left(\frac{t}{\sqrt{6}}\right) + c$$

$$= \sin^{-1}\left(\frac{\sin\theta - \cos\theta}{\sqrt{6}}\right) + c$$

224. The given integral is

$$\int \frac{\sin\theta - \cos\theta}{\sqrt{2 - \sin 2\theta}}\,d\theta$$

$$= \int \frac{\sin\theta - \cos\theta}{\sqrt{3 - (1 + \sin 2\theta)}}\,d\theta$$

$$= \int \frac{\sin\theta - \cos\theta}{\sqrt{3 - (\sin\theta + \cos\theta)^2}}\,d\theta$$

$$= -\int \frac{dt}{\sqrt{3 - t^2}}, \quad \text{Let } t = (\sin\theta + \cos\theta)$$

$$dt = (\cos\theta - \sin\theta)\,d\theta$$

$$= \cos^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \cos^{-1}\left(\frac{\sin\theta + \cos\theta}{\sqrt{3}}\right) + c$$

225. We have,

$$\int \frac{x - 1}{\sqrt{x^2 - 3x + 2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x - 2}{\sqrt{x^2 - 3x + 2}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x - 3) + 1}{\sqrt{x^2 - 3x + 2}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x - 3)}{\sqrt{x^2 - 3x + 2}}\,dx + \frac{1}{2}\int \frac{dx}{\sqrt{x^2 - 3x + 2}}$$

$$= \frac{1}{2}\int \frac{2t\,dt}{t}\,dx + \frac{1}{2}\int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 + \left(2 - \frac{9}{2}\right)}}$$

Let $(x^2 - 3x + 2) = t^2$

$\Rightarrow \quad (2x + 3)\,dx = 2t\,dt$

$$= t + \frac{1}{2}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + c$$

$$= \sqrt{x^2 - 3x + 2} + \frac{1}{2}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + c$$

226. We have,

$$\int \frac{3x + 4}{\sqrt{x^2 + 5x + 2}}\,dx$$

$$= \int \frac{\frac{3}{2}(2x + 5) + \left(4 - \frac{15}{2}\right)}{\sqrt{x^2 + 5x + 2}}\,dx$$

$$= \frac{3}{2}\int \frac{(2x + 5)dx}{\sqrt{x^2 + 5x + 2}} - \frac{7}{2}\int \frac{dx}{\sqrt{x^2 + 5x + 2}}$$

$$= \frac{3}{2}\int \frac{2t\,dt}{t} - \frac{7}{2}\int \frac{dx}{\sqrt{\left(x + \frac{5}{2}\right)^2 + \left(2 - \frac{25}{4}\right)}}$$

Let $(x^2 + 5x + 2) = t^2$

$\Rightarrow \quad (2x + 5)\,dx = 2t\,dt$

$$= 3\int dt + \frac{7}{2}\int \frac{dx}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{17}}{2}\right)^2}}$$

$$= 3t + \frac{7}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 2}\right| + c$$

$$= 3\sqrt{x^2 + 5x + 2} + \frac{7}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 2}\right| + c$$

227. We have,

$$\int \frac{x + 2}{\sqrt{x^2 + 5x + 6}}\,dx = \frac{1}{2}\int \frac{(2x + 5) - 1}{\sqrt{x^2 + 5x + 6}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x + 5)}{\sqrt{x^2 + 5x + 6}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2 + 5x + 6}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x + 5)}{\sqrt{x^2 + 5x + 6}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$

$$= \sqrt{x^2 + 5x + 6}$$

$$- \frac{1}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 6x + 6}\right| + c$$

228. $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}\,dx$

Let $3x^2-5x+1$

$\Rightarrow (6x-5)\,dx = 2t\,dt$

$= \int \frac{2t\,dt}{t}$

$= 2\int dt$

$= 2t + c$

$= 2\sqrt{3x^2-5x+1} + c$

229. We have,

$$\int \sqrt{\frac{a-x}{a+x}}\,dx = \int \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\,dx$$

$$= \int \frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\int \frac{dx}{\sqrt{a^2-x^2}} - \int \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2} + c$$

230. $\int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

Let $x^2 = a^2 t$

$\Rightarrow 2x\,dx = a^2\,dt$

$\Rightarrow \frac{a^2}{2}\int \sqrt{\frac{a^2(1-t)}{a^2(1+t)}}\,dt$

$$= \frac{a^2}{2}\int \sqrt{\frac{(1-t)}{(1+t)}}\,dt$$

$$= \frac{a^2}{2}\int \sqrt{\frac{(1-t)}{(1+t)} \times \frac{1-t}{1-t}}\,dt$$

$$= \frac{a^2}{2}\int \sqrt{\frac{(1-t)^2}{(1-t^2)}}\,dt$$

$$= \frac{a^2}{2}\int \frac{(1-t)}{\sqrt{1-t^2}}\,dt$$

$$= \frac{a^2}{2}\left(\int \frac{dt}{\sqrt{1-t^2}} - \int \frac{t}{\sqrt{1-t^2}}\,dt\right)$$

$$= \frac{a^2}{2}\left(\sin^{-1}(t) + \sqrt{1-t^2}\right) + c$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x^2}{a^2}\right) + \sqrt{1-\left(\frac{x}{a}\right)^2}\right) + c$$

231. We have,

$$\int x^2\sqrt{\frac{4-x^3}{4+x^3}}\,dx$$

Let $x^3 = 4t$

$\Rightarrow x^2 dx = \frac{4}{3}dt$

$$= \frac{4}{3}\int \sqrt{\frac{4-4t}{4+4t}}\,dt$$

$$= \frac{4}{3}\int \sqrt{\frac{1-t}{1+t}}\,dt$$

$$= \frac{4}{3}\int \frac{(1-t)}{\sqrt{1-t^2}}\,dt$$

$$= \frac{4}{3}\left(\int \frac{dt}{\sqrt{1-t^2}} - \int \frac{t}{\sqrt{1-t^2}}\,dt\right)$$

$$= \frac{4}{3}\left(\sin^{-1}(t) + \sqrt{1-t^2}\right) + c$$

$$= \frac{4}{3}\left(\sin^{-1}\left(\frac{x^3}{4}\right) + \sqrt{1-\left(\frac{x^3}{4}\right)^2}\right) + c$$

232. We have,

$$\int \frac{dx}{3+4\sin^2 x} = \int \frac{\sec^2 x\,dx}{3\sec^2 x + 4\tan^2 x}$$

$$= \int \frac{\sec^2 x\,dx}{3+7\tan^2 x}$$

$$= \frac{1}{7}\int \frac{\sec^2 x\,dx}{\tan^2 x + \frac{3}{7}}$$

$$= \frac{1}{7}\int \frac{dt}{t^2 + \left(\sqrt{\frac{3}{2}}\right)^2}$$

$$= \frac{1}{7} \times \frac{\sqrt{7}}{3}\tan^{-1}\left(t\sqrt{\frac{7}{3}}\right) + c$$

$$= \frac{1}{\sqrt{21}} \times \tan^{-1}\left(\tan x\sqrt{\frac{7}{3}}\right) + c$$

233. We have,

$$\int \frac{dx}{3\sin^2 x + 4\cos^2 x} = \int \frac{\sec^2 x\,dx}{3\tan^2 x + 4}$$

$$= \int \frac{dt}{3t^2+4}$$

Let $\tan x = t$

$\Rightarrow \sec^2 x\,dx = dt$

$$= \frac{1}{3}\int \frac{dt}{t^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{3} \times \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{\sqrt{3}}{2} t\right) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{2} \tan x\right) + c$$

234. We have,

$$\int \frac{dx}{(2\sin x + \cos x)^2} = \int \frac{\sec^2 x\, dx}{(2\tan x + 3)^2}$$

$$= \int \frac{dt}{(2t + 3)^2}$$

$$= -\frac{1}{2(2t + 3)} + c$$

$$= -\frac{1}{2(2\tan x + 3)} + c$$

235. We have,

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$$= \int \frac{\tan x \sec^2 x}{\tan^4 x + 1}\, dx$$

$$= \frac{1}{2} \int \frac{2\tan x \sec^2 x}{\tan^4 x + 1}\, dx$$

$$= \frac{1}{2} \int \frac{dt}{t^2 + 1} \qquad \text{Let } \tan^2 x = t$$

$$\Rightarrow \quad 2\tan x \sec^2 x\, dx = dt$$

$$= \frac{1}{2} \tan^{-1}(t) + c$$

$$= \frac{1}{2} \tan^{-1}(\tan^2 x) + c$$

236. We have,

$$\int \frac{dx}{(\sin x + 2\cos x)^2} = \int \frac{\sec^2 x}{(\tan x + 2)^2}\, dx$$

$$= \int \frac{dt}{(t + 2)^2} \qquad \text{Let } \tan x = t$$

$$\Rightarrow \quad \sec^2 x\, dx = dt$$

$$= -\frac{1}{(t + 2)} + c$$

$$= -\frac{1}{(\tan x + 2)} + c$$

237. We have,

$$\int \frac{dx}{(\sin x + 2\sec x)^2} = \frac{1}{4} \int \frac{dx}{\left(\frac{\sin x}{2} + \sec x\right)^2}$$

$$= \frac{1}{4} \int \frac{\sec^2 x\, dx}{\left(\frac{\tan x}{2 + \sec^2 x}\right)^2}$$

$$= \frac{1}{4} \int \frac{\sec^2 x\, dx}{\left(\tan^2 x + \frac{\tan x}{2} + 1\right)^2}$$

$$= \frac{1}{4} \int \frac{\sec^2 x\, dx}{\left[\left(\tan x + \frac{1}{4}\right)^2 + \left(1 - \frac{1}{16}\right)^2\right]}$$

$$= \frac{1}{4} \int \frac{\sec^2 x\, dx}{\left[\left(\frac{\tan x + 1}{4}\right)^2 + \frac{15^2}{16}\right]}$$

$$\text{Let } \left(\tan x + \frac{1}{4}\right) = \frac{\sqrt{15}}{4} \tan\theta$$

$$\Rightarrow \quad \sec^2 x\, dx = \frac{\sqrt{15}}{4} \sec^2\theta\, d\theta$$

$$= \frac{1}{4} \int \frac{\frac{\sqrt{15}}{4} \sec^2\theta}{\left(\frac{15}{16}\right)^2 \sec^4\theta}$$

$$= \frac{1}{4} \times \frac{\sqrt{15}}{4} \times \left(\frac{16}{15}\right)^2 \int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{8}{(15)^{3/2}} \int (2\cos^2\theta) d\theta$$

$$= \frac{8}{(15)^{3/2}} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{8}{(15)^{3/2}} \left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{8}{(15)^{3/2}} (\theta + \sin\theta + \cos\theta) + c,$$

where $\theta = \tan^{-1}\left[\frac{4}{\sqrt{15}}\left(\tan x + \frac{1}{4}\right)\right]$

238. The given integral is

$$\int \frac{\sin^2 x\, dx}{\sin^4 x + \cos^4 x} = \int \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

Divide the numerator and the denominator by $\cos^4 x$ we get

$$\int \frac{2\tan x \sec^2 x}{\tan^4 x + 1}\, dx$$

$$\text{Let } \tan^2 x = t$$

$$\Rightarrow \quad 2\tan x \sec^2 x\, dx = dt$$

$$= \int \frac{dt}{t^2} + 1$$

$$= \tan^{-1}(t) + c$$

$$= \tan^{-1}(\tan^2 x) + c$$

239. The given integral is

$$\int \frac{dx}{(2\sin x + 3\cos x)^2}$$

Divide the numerator and the denominator by $\cos^2 x$, we get

$$\int \frac{\sec^2 x\, dx}{(2\tan x + 3)^2}$$

Let $2\tan x + 3 = t$

$\Rightarrow 2\sec^2 x dx = dt$

$$= \frac{1}{2}\int \frac{dt}{t^2}$$

$$= -\frac{1}{2t} + c$$

$$= -\frac{1}{2}(2\tan x + 3) + c$$

240. The given integral is

$$\int \frac{dx}{2+\cos^2 x} = \int \frac{\sec^2 x\, dx}{1 + 2\sec^2 x} = \int \frac{\sec^2 x\, dx}{2\tan^2 x + 3}$$

Let $\tan x = t$

$\Rightarrow \sec^2 x dx = dt$

$$= \int \frac{dt}{2t^2 + 3}$$

$$= \int \frac{dt}{t^2 + \left(\sqrt{\frac{3}{2}}\right)^2}$$

$$= \sqrt{\frac{2}{3}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\,t\right) + c$$

$$= \sqrt{\frac{2}{3}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right) + c$$

241. We have,

$$\int \frac{\sin x}{\sin 3x}dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x}dx$$

$$= \int \frac{dx}{3 - 4\sin^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{3\sec^2 x - 4\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{3 - \tan^2 x}$$

$$= -\int \frac{dt}{t^2 - 3}$$ Let $\tan x = t$

$\sec^2 x dx = dt$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\tan x-\sqrt{3}}{\tan x+\sqrt{3}}\right| + c$$

242. The given integral is

$$\int \frac{\operatorname{cosec}^3 x}{\operatorname{cosec} x}dx = \int \frac{\sin x}{\sin^3 x}dx$$

$$= \int \frac{\sin x}{3\sin x - 4\sin^3 x}dx$$

$$= \int \frac{dx}{3 - 4\sin^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{3\sec^2 x - 4\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{3 + 3\tan^2 x - 4\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{3 - \tan^2 x}$$

Let $\tan x = t$

$\Rightarrow \sec^2 x dx = dt$

$$= -\int \frac{dt}{t^2 - 3}$$

$$= \frac{-1}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{t+\sqrt{3}}{t-\sqrt{3}}\right|$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{\tan x+\sqrt{3}}{\tan x-\sqrt{3}}\right| + c$$

243. The given integral is

$$\int \frac{\sec 3x}{\sec x}dx = \int \frac{\cos x}{\cos^3 x}dx$$

$$= \int \frac{\cos x}{4\cos^3 x - 3\cos x}dx$$

$$= \int \frac{dx}{4\cos^2 x - 3}$$

Divide the numerator and the denominator by $\cos^2 x$ we get,

$$= \int \frac{\sec^2 x\, dx}{4 - 3\sec^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{4 - 3 - 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{1 - 3\tan^2 x}$$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$= \int \frac{dt}{1 - 3t^2}$$

$$= \frac{1}{3}\int \frac{dt}{t^2 - (1/\sqrt{3})^2}$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{t - \frac{1}{\sqrt{3}}}{t + \frac{1}{\sqrt{3}}}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}\tan x - 1}{\sqrt{3}\tan x + 1}\right| + c$$

244. We have,

$$\int \frac{dx}{1 + 2\sin x} = \int \frac{dx}{1 + 2\left(\frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right)}$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right) + 4\tan\left(\frac{x}{2}\right)}\,dx$$

Let $\tan\left(\frac{x}{2}\right) = t$

$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2dt$

$$= \int \frac{2dt}{t^2 + 4t + 1}$$

$$= 2\int \frac{dt}{(t + 2)^2 - (\sqrt{3})^2}$$

$$= \frac{2}{2\sqrt{3}}\log\left|\frac{(t + 2) - \sqrt{3}}{(t + 2) + \sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\left(\tan\left(\frac{x}{2}\right) + 2\right) - \sqrt{3}}{\left(\tan\left(\frac{x}{2}\right) + 2\right) + \sqrt{3}}\right| + c$$

245. We have,

$$\int \frac{dx}{\sin x + \cos x + 1}$$

$$= \int \frac{dx}{\left(\frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right) + \left(\frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right) + 1}$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{\left(2\tan\left(\frac{x}{2}\right) - \tan^2\left(\frac{x}{2}\right) + 2\right)}$$

$$= \frac{1}{2}\int \frac{dt}{2t - t^2 + 2} \quad \text{Let } \left(\frac{x}{2}\right) = t$$

$$= \frac{1}{2}\int \frac{-dt}{t^2 - 2t - 2} \quad \Rightarrow \frac{1}{2}\sec^2\left(\frac{3}{2}\right)dx = dt$$

$$= -\frac{1}{2}\int \frac{dt}{(t - 1)^2 - (\sqrt{3})^2}$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{(t - 1) - \sqrt{3}}{(t - 1) + \sqrt{3}}\right| + c$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\left(\tan\left(\frac{x}{2}\right) - 1\right) - \sqrt{3}}{\left(\tan\left(\frac{x}{2}\right) - 1\right) + \sqrt{3}}\right| + c$$

244. The given integral is

$$\int \frac{dx}{1 + 2\sin x}$$

$$= \int \frac{dx}{1 + 2\left(\frac{2\tan(x/2)}{1 + \tan^2(x/2)}\right)}$$

$$= \int \frac{(1 + \tan^2(x/2))dx}{1 + \tan^2(x/2) + 4\tan(x/2)}$$

$$= \int \frac{\sec^2(x/2)dx}{\tan^2(x/2) + 4\tan(x/2) + 1}$$

Let $\tan(x/2) = t$

$\Rightarrow \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$

$$= \int \frac{2dt}{t^2 + 4t + 1}$$

$$= 2\int \frac{dt}{(t + 2)^2 - (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{(t + 2) - \sqrt{3}}{(t + 2) + \sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{(\tan(x/2) + 2) - \sqrt{3}}{(\tan(x/2) + 2) + \sqrt{3}}\right| + c$$

245. The given integral is

$$\int \frac{dx}{3\cos x + 4} = \int \frac{dx}{3\left(\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} + 4\right)}$$

$$= \int \frac{(1 + \tan^2(x/2))}{4 + 4\tan^2(x/2) + 3 - 3\tan^2(x/2)}\,dx$$

$$= \int \frac{\sec^2(x/2)}{\tan^2(x/2) + 7}\,dx$$

Let $t = \tan(x/2)$

$\Rightarrow 2dt = \sec^2\left(\frac{x}{2}\right)dx$

$$= \int \frac{2dt}{t^2 + 7}$$

$$= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{t}{\sqrt{7}}\right) + c$$

$$= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{7}}\right) + c$$

246. The given integral is

$$\int \frac{dx}{\sin x + \cos x + 1}$$

$$= \int \frac{dx}{\frac{2\tan(x/2)}{1 + \tan^2(x/2)} + \frac{1 - \tan^2(x2)}{1 + \tan^2(x/2)} + 1}$$

$$= \int \frac{(1 + \tan^2(x/2))dx}{1 + \tan^2(x/2) + 2\tan(x/2) + 1 - \tan^2(x/2)}$$

$$= \int \frac{\sec^2(x/2)dx}{2\tan(x/2) + 2}$$

Let $\tan\left(\frac{x}{2}\right) = t$

$\Rightarrow \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$

$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2dt$

$$= \int \frac{2dt}{2t + 2}$$

$$= \int dt/t + 1$$

$$= \log|t + 1| + c$$

$$= \log\left|\tan\left(\frac{x}{2}\right) + 1\right| + c$$

247. The given integral is

$$\int \frac{1 + \sin x}{\sin x(1 + \cos x)} dx$$

$$= \int \frac{dx}{\sin x(1 + \cos x)} + \int \frac{dx}{(1 + \cos x)}$$

$$= \int \frac{\sin x\, dx}{(1 - \cos^2 x)(1 + \cos x)} + \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right)dx$$

$$= \int \frac{dt}{(t^2 - 1)(1 + t)} + \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right)dx$$

Let $t = \cos x$

$\Rightarrow dt = -\sin x\, dx$

$$= \int \frac{dt}{(t - 1)(t + 1)^2} + \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{1}{t^2 - 1} - \frac{1}{(t + 1)^2}\right)dt + \frac{1}{2}\int \sec^2(x/2)dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{t - 1}{t + 1}\right| + \frac{1}{(t + 1)}\right) + \tan(x/2) + c$$

$$= \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + \frac{1}{(\cos x + 1)}\right) + \tan(x/2) + c$$

248. The given integral is

$$\int \frac{dx}{1 - 2\sin x}$$

$$= \int \frac{dx}{1 - 2\left(\frac{2\tan(x/2)}{1 + \tan^2(x/2)}\right)}$$

$$= \int \frac{(1 + \tan^2(x/2))dx}{\tan^2(x/2) - 4\tan(x/2) + 1}$$

$$= \int \frac{\sec^2(x/2)dx}{\tan^2(x/2) - 4\tan(x/2) + 1}$$

Let $\tan(x/2) = t$

$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2dt$

$$= 2\int \frac{dt}{t^2 - 4t + 1}$$

$$= 2\int \frac{dt}{(t - 2)^2 - (\sqrt{3})^2}$$

$$= 2 \times \frac{1}{2}\log\left(\frac{(t - 2) - \sqrt{3}}{(t - 2) + \sqrt{3}}\right) + c$$

$$= \log\left|\frac{(\tan(x/2) - 2) - \sqrt{3}}{(\tan(x/2) - 2) + \sqrt{3}}\right| + c$$

249. The given integral is

$$\int \frac{dx}{3\sin x + 4\cos x + 5}$$

$$= \int \frac{dx}{3\left(\frac{2\tan(x/2)}{1 + \tan^2(x/2)}\right) + 4\left(\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}\right) + 5}$$

$$= \int \frac{(1 + \tan^2(x/2))dx}{\tan^2(x/2) + 6\tan(x/2) + 9}$$

$$= \int \frac{2dt}{t^2 + 6t + 9}$$

Let $\tan(x/2) = t$

$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2dt$

$$= \int \frac{2dt}{(t + 3)^2}$$

$$= -\frac{2}{(t+3)} + c$$

$$= -\frac{2}{(\tan(x/2)+3)} + c$$

250. The given integral is

$$\int \frac{dx}{\cos x + \cos\alpha}$$

$$= \int \frac{dx}{\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right) + k}, \text{ where } k = \cos\alpha$$

$$= \int \frac{[1+\tan^2(x/2)]dx}{1-\tan^2(x/2) + k[1+\tan^2(x/2)]}$$

$$= \int \frac{\sec^2(x/2)dx}{1-\tan^2(x/2) + k[1+\tan^2(x/2)]}$$

$$= \int \frac{2dt}{(1-t^2) + k(1+t^2)} \qquad \text{Let } \tan\left(\frac{x}{2}\right) = t$$

$$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2\,dt$$

$$= \int \frac{2dt}{(1+k) + (k-1)t^2}$$

$$= \frac{1}{(1-k)} \int \frac{2dt}{\frac{(1+k)}{(1-k)} - t^2}$$

$$= \frac{1}{(1-k)} \int \frac{2dt}{\left(\sqrt{\frac{(1+k)}{(1-k)}}\right)^2 - t^2}$$

$$= \frac{1}{(1-k)} \times \frac{1}{2\sqrt{\frac{(1+k)}{(1-k)}}} \log\left|\frac{\sqrt{\frac{(1+k)}{(1-k)}} + t}{\sqrt{\frac{(1+k)}{(1-k)}} - t}\right| + c$$

$$= \frac{1}{\sin\alpha}\log\left|\frac{\cot(\alpha/2) + \tan(x/2)}{\cot(\alpha/2) - \tan(x/2)}\right| + c$$

251. The given integral is

$$\int \frac{dx}{\tan x + 4\cot x + 4}$$

$$= \int \frac{4\sin x\cos x}{\sin^2 x + 4\cos^2 x + 4\sin x\cos x}\, dx$$

252. We have,

$$\int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + c$$

253. We have,

$$\int \frac{dx}{\sqrt{3}\sin x + \cos x}$$

$$= \frac{1}{2} \int \frac{dx}{\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x}$$

$$= \frac{1}{2} \int \frac{dx}{\sin\left(x + \frac{\pi}{6}\right)}$$

$$= \frac{1}{2} \int \operatorname{cosec}\left(x + \frac{\pi}{6}\right)dx$$

$$= \frac{1}{2} \log\left|\left|\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)\right|\right| + c$$

257. We have,

$$2\sin x + \cos x$$

$$= l(3\sin x + 2\cos x) + m(3\cos x - 2\sin x)$$

$$= (3l - 2m)\sin x + (2l + 3m)\cos x$$

Comparing the co-efficients of $\sin x$ and $\cos x$, we get

$$3l - 2m = 2 \text{ and } 2l + 3m = 1$$

On solving, we get,

$$l = 4/5 \text{ and } m = -3/5$$

$$\therefore \quad \frac{4}{5}\int \frac{3\sin x + 2\cos x}{3\sin x + 2\cos x}\, dx + \frac{3}{5}\int \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x}\, dx$$

$$= \frac{4}{5}\int dx - \frac{3}{5}\int \left(\frac{3\cos x - 2\sin x}{3\sin x + 2\cos x}\right)dx$$

$$= \frac{4}{5}x - \frac{3}{5}\log|3\sin x + 2\cos x| + c$$

258. We have,

$$\int \frac{\sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{(1 + \cos x - \sin x}{\sin x + \cos x)}\, dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + c$$

259. The given integral is

$$\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\, dx$$

we have $2\sin x + 3\cos x$

$$= l(3\sin x + 4\cos x) + m(3\cos x - 4\sin x)$$

$$= (3l - 4m)\sin x + (4l + 3m)\cos x$$

Comparing the co-efficients of $\sin x$ and $\cos x$, we get

$$(3l - 4m) = 2,\ (4l + 3m) = 3$$

On solving, we get

$$l = \frac{18}{25},\ m = \frac{1}{25}$$

The given integral reduces to

$$\int \frac{\frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)}{(3\sin x + 4\cos x)}\, dx$$

$$= \int \left(\frac{18}{25} + \frac{1}{25}\frac{(3\cos x - 4\sin x)}{(3\sin x + 4\cos x)}\right) dx$$

$$= \frac{18}{25}x + \frac{1}{25}\log|3\sin x + 4\cos x| + C$$

260. The given integral is

$$\int \frac{\sin x}{\sin x - \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x - \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{(\sin x - \cos x) + (\sin x + \cos x)}{\sin x - \cos x}\, dx$$

$$= \frac{1}{2}\int \left(1 + \frac{\cos x + \sin x}{\sin x - \cos x}\right) dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + c$$

261. The given integral is

$$\int \frac{\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \left(1 + \frac{(\cos x - \sin x)}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}(x + \log|\sin x + \cos x|) + c$$

262. The given integral is

$$\int \frac{1}{1 + \tan x}\, dx = \int \frac{\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}(x + \log|\sin x + \cos x|) + c$$

263. The given integral is

$$\int \frac{1}{1 - \tan x}\, dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2}\int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2}\int \left(1 + \frac{(\cos x + \sin x)}{\cos x - \sin x}\right) dx$$

$$= \frac{1}{2}(x - \log|\cos x - \sin x|) + c$$

264. The given integral is

$$\int \frac{1}{1 + \cot x}\, dx$$

$$= \int \frac{\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \left(1 + \frac{(\cos x - \sin x)}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}(x + \log|\sin x + \cos x|) + c$$

265. We have,

$$3\sin x + 2\cos x + 4$$

$$= l(3\cos x + 4\sin x + 5)$$

$$+ m(-3\sin x + 4\cos x) + n$$

$$= (4l - 3m)\sin x + (3l + 4m)\cos x$$

$$+ (5l + n)$$

Comparing the co-efficients of $\sin x$ and $\cos x$ we get

$$4l - 3m = 3,\ 3l + 4m = 2$$

and $5l + n = 4$

Solving, we get

$$l = \frac{18}{25},\ m = -\frac{1}{25},\ n = \frac{1}{5}$$

Thus, $\int \frac{3\sin x + 2\cos x + 4}{3\cos x + 4\sin x + 5}\,dx$

$$= \int\left(\frac{18}{25} + \frac{1}{25}\left(\frac{4\cos x - 3\sin x}{4\sin x + 3\cos x + 5}\right)\right)dx$$

$$+ \frac{2}{5}\int\frac{dx}{4\sin x + 3\cos x + 5}$$

$$= \frac{18}{25}x - \frac{1}{25}\log|4\sin x + 3\cos x + 5|$$

$$+ \frac{2}{25}\tan^{-1}\left(\frac{x - \tan^{-1}(4/3)}{2}\right) + c.$$

269. We have,

$$\int(\cos x - \sin x)(2 + 3\sin x\ 2x)dx$$

$$= \int(\cos x - \sin x)(2 + 3(-1 + 1 + \sin 2x))dx$$

$$= \int(\cos x - \sin x)(-1 + 3(\sin x + \cos x)^2)dx$$

$$= \int(3t^2 - 1)\,dt \quad \text{Let } t = \sin x + \cos x$$

$$\Rightarrow\ dt = (\cos x - \sin x)dx$$

$$= (t^3 - t) + c$$

$$= (\sin x + \cos x)^3 - (\sin x + \cos x) + c$$

270. We have,

$$\int\frac{\sin x + \cos x}{9 + 16\sin 2x}dx$$

$$= \int\frac{\sin x + \cos x}{9 + 16(1 - 1 + \sin 2x)}dx$$

$$= \int\frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2}dx$$

Let $\sin x - \cos x = t$

$\Rightarrow\ (\cos x + \sin x)dx = dt$

$$= \int\frac{dt}{25 - 16t^2}$$

$$= -\frac{1}{16}\int\frac{dt}{t^2 - \left(\frac{5}{4}\right)^2}$$

$$= -\frac{1}{16}\times\frac{1}{2\times\frac{5}{4}}\log\left|\frac{t - \frac{5}{4}}{t + \frac{5}{4}}\right| + c$$

$$= -\frac{1}{40}\log\left|\frac{4t - 5}{4t + 5}\right| + c$$

$$= -\frac{1}{40}\log\left|\frac{49(\sin x - \cos x) - 5}{4(\sin x - \cos x) + 5}\right| + c$$

271. We have,

$$\int\frac{dx}{\cos x + \operatorname{cosec} x}$$

$$= \int\frac{dx}{\cos x + \frac{1}{\sin x}}$$

$$= \int\frac{\sin x\,dx}{\cos x\sin x + 1}$$

$$= \int\frac{2\sin x\,dx}{2\cos x\sin x + 2}$$

$$= \int\frac{(\sin x + \cos x) + (\sin x - \cos x)}{2\cos x\sin x + 2}$$

$$= \int\frac{(\sin x + \cos x)}{2\cos x\sin x + 2}dx$$

$$+ \int\frac{(\sin x - \cos x)}{2\cos x\sin x + 2}dx$$

$$= \int\frac{(\sin x + \cos x)}{2 + \sin 2x}dx + \int\frac{(\sin x - \cos x)}{2 + \sin 2x}dx$$

$$= \int\frac{(\sin x + \cos x)}{3 - (\sin x - \cos x)^2}dx$$

$$+ \int\frac{(\sin x - \cos x)}{1 + (\sin x + \cos x)^2}dx$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{(\sin x - \cos x) - \sqrt{3}}{(\sin x - \cos x) + \sqrt{3}}\right|$$

$$- \tan^{-1}(\sin x + \cos x) + c$$

271. The given integral is

$$\int\frac{dx}{\cos x + \operatorname{cosec} x}$$

$$= \int\frac{\sin x\cos x}{\sin x + \cos x}dx$$

$$= \frac{1}{4}\int\frac{2\sin 2x}{\sin x + \cos x}dx$$

$$= \frac{1}{4}\int\frac{(\sin x + \cos x)^2 - (\sin x - \cos x)^2}{\sin x + \cos x}dx$$

$$= \frac{1}{4}\int\left[(\sin x + \cos x) - \frac{(\sin x - \cos x)}{\sin x + \cos x}\right]dx$$

$$= \frac{1}{4}\left[(\cos x - \sin x) - \frac{1}{3}(\sin x - \cos x)^3\right] + c$$

272. The given integral is

$$\int(\sin x + \cos x)(2 + 3\sin 2x)dx$$

$$= \int(\sin x + \cos x)[2 + 3\{1 - (1 - \sin 2x)\}]dx$$

$$= \int (\sin x + \cos x)(5 - 3(\sin x - \cos x)^2)dx$$

Let $\sin x - \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$

$$= \int (5 - 3t^2)dt$$

$$= (5t - t^3) + c$$

$$= (5(\sin x - \cos x) - (\sin x - \cos x)^3) + c$$

273. The given integral is

$$\int (\sin x - \cos x)(3 - 4\sin 2x)dx$$

$$= \int (\sin x - \cos x)(7 - 4(1 + \sin 2x))$$

$$= \int (\sin x - \cos x)(7 - 4(\sin x + \cos x)^2)dx$$

$$= \int (\cos x - \sin x)(4(\sin x + \cos x)^2 - 7)dx$$

Let $\sin x + \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$

$$= \int (4t^2 - 7)dt$$

$$= \left(\frac{4t^3}{3} - 7t\right) + c$$

$$= \left(\frac{4(\sin x + \cos x)^3}{3} - 7(\sin x + \cos x)\right) + c$$

274. The given integral is

$$\int \left(\frac{\cos x - \sin x}{3 + 2\sin 2x}\right)dx$$

$$= \int \frac{(\cos x - \sin x)}{1 + 2(1 + \sin 2x)}dx$$

$$= \int \frac{(\cos x - \sin x)}{1 + 2(\sin x + \cos x)^2}dx$$

Let $\sin x + \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$

$$= \int \frac{dt}{1 + 2t^2}$$

$$= \frac{1}{2}\int \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(t\sqrt{2}) + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}(\sin x + \cos x)) + c$$

275. The given integral is

$$\int \left(\frac{\cos x + \sin x}{5 - 4\sin 2x}\right)dx$$

$$= \int \left(\frac{\cos x + \sin x}{5 - 4(1 - (1 - \sin 2x))}\right)dx$$

$$= \int \frac{\cos x + \sin x}{1 - 4(\sin x - \cos x)^2}dx$$

Let $\sin x - \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$

$$= \int \frac{dt}{1 - 4t^2}$$

$$= -\frac{1}{4}\int \frac{dt}{t^2 - (1/2)^2}$$

$$= -\frac{1}{2}\log\left|\frac{2t - 1}{2t + 1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{2(\sin x - \cos x) - 1}{2(\sin x - \cos x) + 1}\right| + c$$

276. The given integral is

$$\int \left(\frac{2\cos x + \sin x}{9 + 16\sin 2x}\right)dx$$

$$= \int \frac{\frac{3}{2}(\cos x + \sin x) + \frac{1}{2}(\cos x - \sin x)}{9 + 6\sin 2x}dx$$

$$= \frac{3}{2}\int \frac{(\cos x + \sin x)}{9 + 6\sin 2x}dx + \frac{1}{2}\int \frac{(\cos x - \sin x)}{(9 + 6\sin 2x)}dx$$

$$= \frac{3}{2}\int \frac{(\cos x + \sin x)}{15 - 6(\sin x - \cos x)^2}dx$$

$$+ \frac{1}{2}\int \frac{(\cos x - \sin x)}{(3 + 6(\sin x + \cos x)^2)}dx$$

Let $\sin x - \cos x = t$ and $\sin x + \cos x = z$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$ and $(\cos x - \sin x)dx = dz$

$$= \frac{3}{2}\int \frac{dt}{15 - 6t^2} + \frac{1}{2}\int \frac{dz}{6z^2 + 3}$$

$$= \frac{1}{2}\int \frac{dt}{5 - 2t^2} + \frac{1}{6}\int \frac{dz}{2z^2 + 1}$$

$$= -\frac{1}{4}\int \frac{dt}{t^2 - (\sqrt{5}/2)^2} + \frac{1}{6}\int \frac{dz}{2z^2 + 1}$$

$$= -\frac{1}{4}\log\left|\frac{t - (\sqrt{5}/2)}{t + (\sqrt{5}/2)}\right| + \frac{1}{6\sqrt{2}}\tan^{-1}(z\sqrt{2}) + C$$

where $\quad \sin x - \cos x = t$ and $\sin x + \cos x = z$

277. We have,

$$\int \frac{x^2 + 1}{x^4 + 1}dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}}dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2 \cdot x \cdot \frac{1}{x}} dx$$

Let $\left(x - \frac{1}{x}\right) = t$

$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{dt}{t^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\left(x - \frac{1}{x}\right)\right) + c$$

278. We have,

$$\int \frac{x^2 - 1}{x^4 + 1} dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2 \cdot x \cdot \frac{1}{x}} dx$$

Let $\left(x + \frac{1}{x}\right) = t$

$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{dt}{t^2 - (\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{t - \sqrt{2}}{t + \sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{\left(x + \frac{1}{x}\right) - \sqrt{2}}{\left(x + \frac{1}{x}\right) + \sqrt{2}}\right| + c$$

279. We have,

$$\int \frac{x^4 + 1}{x^6 + 1} dx$$

$$= \int \frac{(x^2 + 1)^2 - 2x^2}{x^6 + 1} dx$$

$$= \int \frac{(x^2 + 1)^2 - 2x^2}{(x^2 + 1)(x^4 - x^2 + 1)} dx$$

$$= \int \frac{(x^2 + 1)}{(x^4 - x^2 + 1)} dx - 2\int \frac{x^2}{x^6 + 1} dx$$

$$= \int \frac{(x^2 + 1)}{(x^4 - x^2 + 1)} dx - \frac{2}{3} \int \frac{3x^2}{x^6 + 1} dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2} - 1\right)} dx - \frac{2}{3} \int \frac{3x^2}{x^6 + 1} dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 1} dx - \frac{2}{3} \int \frac{3x^2}{x^6 + 1} dx$$

$$= \int \tan^{-1}\left(x - \frac{1}{x}\right) - \frac{2}{3} \tan^{-1}(x^3) + c$$

280. We have,

$$\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$$

$$= \int \frac{(x^2 - x + 1) - 2x}{x^4 + x^2 + 1} dx$$

$$= \int \frac{(x^2 - x + 1)}{x^4 + x^2 + 1} dx \int \frac{2x}{x^4 + x^2 + 1} dx$$

$$= \int \frac{dx}{x^2 + x + 1} - \int \frac{2x}{x^4 + x^2 + 1} dx$$

$$= \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}, \int \frac{dt}{t^2 + t + 1}$$

Let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

281. We have,

$$\int \left(\frac{1 - x^2}{1 + x^2}\right) \frac{dx}{\sqrt{1 + x^2 + x^4}}$$

$$= \int \frac{-x^2\left(1 - \frac{1}{x^2}\right) dx}{x\left(x + \frac{1}{x}\right)\sqrt{x^2\left(x^2 + \frac{1}{x^2} + 1\right)}}$$

$$= -\int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)\sqrt{\left(x^2 + \frac{1}{x^2} + 1\right)}}$$

$$= -\int \frac{dt}{t\sqrt{t^2-1}} \qquad \text{Let } \left(x+\frac{1}{x}\right) = t$$

$$\Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$$

$$= \operatorname{cosec}^{-1}(t) + c$$

$$= \operatorname{cosec}^{-1}\left(x+\frac{1}{x}\right) + c$$

$$= \sin^{-1}\left(\frac{x}{x^2+1}\right) + c$$

282. We have,

$$\int \left(\frac{x-1}{x+1}\right) \times \frac{dx}{\sqrt{x^3+x^2+x}}$$

$$= \int \left(\frac{x^2-1}{(x+1)^2}\right) \times \frac{dx}{\sqrt{x^2\left(x+\frac{1}{x}+1\right)}}$$

$$= \int \left(\frac{x^2-1}{x(x+1)^2}\right) \times \frac{dx}{\sqrt{\left(x+\frac{1}{x}+1\right)}}$$

$$= \int \left(\frac{x^2-1}{x(x^2+2x+1)}\right) \times \frac{dx}{\sqrt{\left(x+\frac{1}{x}+1\right)}}$$

$$= \int \left(\frac{x^2-1}{x^2\left(x+\frac{1}{x}+2\right)}\right) \times \frac{dx}{\sqrt{\left(x+\frac{1}{x}+1\right)}}$$

$$= \int \left(\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}+2\right)}\right) \times \frac{dx}{\sqrt{\left(x+\frac{1}{x}+1\right)}}$$

$$\text{Let } \left(x+\frac{1}{x}+1\right) = t^2$$

$$\Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$$

$$= \int \frac{2t\,dt}{(t^2+1)t}$$

$$= \int \frac{2\,dt}{(t^2+1)}$$

$$= 2\tan^{-1} t + c$$

$$= 2\tan^{-1}\left(x+\frac{1}{x}+1\right)^{1/2} + c$$

283. We have,

$$\int \frac{x^2-1}{x^2+1} \times \frac{dx}{\sqrt{x^4+1}}$$

$$= \int \frac{x^2-1}{x\left(x+\frac{1}{x}\right)} \times \frac{dx}{\sqrt{x^2\left(x^2+\frac{1}{x^2}\right)}}$$

$$= \int \frac{x^2-1}{x^2\left(x+\frac{1}{x}\right)} \times \frac{dx}{\sqrt{x^2\left(x^2+\frac{1}{x^2}\right)}}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)} \times \frac{dx}{\sqrt{\left(x+\frac{1}{x}\right)^2-2}}$$

$$= \int \frac{dt}{t\sqrt{t^2-2}} \qquad \text{Let } \left(x+\frac{1}{x}\right) = t$$

$$\Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$$

$$= \int \frac{t\,dt}{t^2\sqrt{t^2-2}}$$

$$= \int \frac{v\,dv}{(v^2+2)v} \qquad \text{Let } t^2-2 = v^2$$

$$\Rightarrow 2t\,dt = 2v\,dv$$

$$= \int \frac{dv}{(v^2+2)}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-2}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{1}{\sqrt{2}}\left(x^2+\frac{1}{x^2}\right)}\right) + c$$

284. We have,

$$\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$$

$$= \int \frac{x^2-1}{\sqrt{x^3\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}\,dx$$

$$= \int \frac{x^2-1}{x^5\sqrt{\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}\,dx$$

$$= \int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)}{\sqrt{\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}\,dx$$

$$\text{Let } \left(2-\frac{2}{x^2}+\frac{1}{x^4}\right) = t^2$$

$$\Rightarrow \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx = 2t\,dt$$

$$= \frac{1}{4}\int \frac{2t\,dt}{t}$$

$$= \frac{1}{2} \times t + c$$

$$= \frac{1}{2} \times \left(2-\frac{2}{x^2}-\frac{1}{x^4}\right)^{1/2} + c$$

285. We have,

$$\int \left(\frac{x^4 - 1}{x^2\sqrt{x^4 + x^2 + 1}}\right)dx$$

$$= \int \left(\frac{x^4 - 1}{x^2\sqrt{x^2\left(x^2 + \frac{1}{x^2} + 1\right)}}\right)dx$$

$$= \int \left(\frac{\left(\frac{x^4 - 1}{x^3}\right)}{\sqrt{\left(x^2 + \frac{1}{x^2} + 1\right)}}\right)dx$$

$$= \int \left(\frac{\left(x - \frac{1}{x^3}\right)}{\sqrt{\left(x^2 + \frac{1}{x^2} + 1\right)}}\right)dx$$

Let $\left(x^2 + \frac{1}{x^2} + 1\right) = t^2$

$\Rightarrow \left(2x - \frac{2}{x^3}\right)dx = 2t\,dt$

$\Rightarrow \left(x - \frac{1}{x^3}\right)dx = t\,dt$

$$= \int \frac{t\,dt}{t}$$

$$= \int dt$$

$$= t + c$$

$$= \left(x^2 + \frac{1}{x^2} + 1\right)^{1/2} + c$$

286. We have,

$$\int \frac{dx}{x^2(x + \sqrt{1 + x^2})}$$

$$= \int \frac{dx}{x^2\left(x + \sqrt{x^2\left(1 + \frac{1}{x^2}\right)}\right)}$$

$$= \int \frac{dx}{x^2\left(x + x\sqrt{\left(1 + \frac{1}{x^2}\right)}\right)}$$

$$= \left(1 + \frac{1}{x^2}\right) = t^2$$

Put $\left(1 + \frac{1}{x^2}\right) = t^2$

$\Rightarrow -\frac{2}{x^3}dx = 2t\,dt$

$\Rightarrow \frac{dx}{x^3} = -t\,dt$

$$= \int \frac{-t\,dt}{1 + t}$$

$$= \int \left(\frac{1}{t + 1} - 1\right)dt$$

$$= \log|t + 1| - t + c$$

$$= \log\left|1 + \left(1 + \frac{1}{x^2}\right)^{1/2}\right| - \left(1 + \frac{1}{x^2}\right)^{1/2} + c$$

287. We have,

$$\int \frac{x^2 - 3x - 1}{x^4 + x^2 + 1}\,dx$$

$$= \int \frac{x^2 - 1}{x^4 + x^2 + 1}\,dx - \frac{3}{2}\int \frac{2x}{x^4 + x^2 + 1}\,dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2} + 1}\,dx - \frac{3}{2}\int \frac{2x}{x^4 + x^2 + 1}\,dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(1 - \frac{1}{x}\right)^2 - 1}\,dx - \frac{3}{2}\int \frac{2x}{x^4 + x^2 + 1}\,dx$$

Put $\left(x + \frac{1}{x}\right) = t$ and $x^2 = z$

$\Rightarrow \left(1 + \frac{1}{x_2}\right)dx = dt$ and $2x\,dx = dz$

$$= \int \frac{dt}{t^2 - 1} - \frac{3}{2}\int \frac{dz}{z^2 + z + 1}$$

$$= \frac{1}{2}\log\left|\frac{t - 1}{t + 1}\right| - \frac{3}{2} \times \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z + 1}{\sqrt{3}}\right) + c$$

$$= \frac{1}{2}\log\left|\frac{\left(x + \frac{1}{x}\right) - 1}{\left(x + \frac{1}{x}\right) + 1}\right| - \sqrt{3}\tan^{-1}\left(\frac{2(x^2) + 1}{\sqrt{3}}\right) + c$$

288. The given integral is

$$\int \frac{dx}{\sin^4 x + \cos^4 x} = \int \frac{\sec^4 x}{\tan^4 x + 1}\,dx$$

$$= \int \frac{(1 + \tan^2 x)\sec^2 x}{\tan^4 x + 1}\,dx$$

Put $\tan x = t$

$\Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{t^2 + 1}{t^4 + 1}\,dt$$

and then solve it.

289. The given integral is

$$\int \frac{dx}{x^4 + 1} = \frac{1}{2}\int \frac{2}{(x^4 + 1)}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{(x^4+1)}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)}{(x^4+1)}\,dx - \frac{1}{2}\int \frac{x^2-1}{(x^4+1)}\,dx$$

and then you do it.

290. The given integral is

$$\int \frac{x^2 dx}{x^4+1} = \frac{1}{2}\int \frac{2x^2}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)}{x^4+1}\,dx + \frac{1}{2}\int \frac{(x^2-1)}{x^4+1}\,dx$$

and then you do it.

291. The given integral is

$$\int \frac{2dx}{x^4+1} = \int \frac{(x^2+1)-(x^2-1)}{x^4+1}\,dx$$

$$= \int \frac{(x^2+1)}{x^4+1}\,dx - \int \frac{(x^2-1)}{x^4+1}\,dx$$

and then you do it.

292. The given integral is

$$\int \left(\frac{x^4+1}{x^6+1}\right) dx$$

$$= \int \frac{(x^2+1)^2-2x^2}{(x^6+1)}\,dx$$

$$= \int \frac{(x^2+1)^2-2x^2}{(x^2+1)\,(x^4-x^2+1)}\,dx$$

$$= \int \frac{(x^2+1)}{(x^4-x^2+1)}\,dx - \frac{2}{3}\int \frac{3x^2}{(x^3)^2+1}\,dx$$

$$= \int \frac{(x^2+1)}{(x^4-x^2+1)}\,dx - \frac{2}{3}\tan^{-1}(x^3) + c$$

and then you do it.

293. The given integral is

$$\int \frac{dx}{x^2(1+x^4)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Let $\left(1+\frac{1}{x^4}\right) = t^4$

$$\frac{-1}{x^5}\,dx = t^3\,dt$$

$$= -\int dt$$

$$= -t + c$$

$$= -\left(1+\frac{1}{x^4}\right)^{1/4} + c$$

294. The given integral is

$$\int \left(\frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\right) dx$$

$$= \int \frac{x^4-1}{x^2\sqrt{x^2\left(x^2+\frac{1}{x^2}+1\right)}}\,dx$$

$$= \int \frac{x^4-1}{x^3\sqrt{\left(x^2+\frac{1}{x^2}+1\right)}}\,dx$$

$$= \int \frac{x-\frac{1}{x^3}}{\sqrt{\left(x^2+\frac{1}{x^2}+1\right)}}\,dx$$

Put $\left(x^2+\frac{1}{x^2}+1\right) = t^2$

$$\left(x-\frac{1}{x^3}\right) dx = t\,dt$$

$$= \int dt$$

$$= t + c$$

$$= \sqrt{x^2+\frac{1}{x^2}+1} + c$$

295. The given integral is

$$\int \frac{dx}{x^2(1+x^5)^{4/5}} = \int \frac{dx}{x^6\left(1+\frac{1}{x^5}\right)^{4/5}}$$

Put $\left(1+\frac{1}{x^5}\right) = t^5$

$$\Rightarrow\ -\frac{5}{x^6}\,dx = 5t^4\,dt$$

$$\Rightarrow\ -\frac{1}{x^6}\,dx = t^4\,dt$$

$$= -\int dt$$

$$= -t + c$$

$$= -\left(1+\frac{1}{x^5}\right)^{1/5} + c$$

296. The given integral is

$$\int \left(\frac{x^2-1}{x\sqrt{1+x^4}}\right) dx$$

$$= \int \frac{(x^2-1)}{x^2\sqrt{x^2+\frac{1}{x^2}}}\,dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}\, dx$$

Let $\left(x + \frac{1}{x}\right) = t$

$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = t^4 dt$

$$= \int \frac{dt}{\sqrt{t^2 - 2}}$$

$$= \log\left|t + \sqrt{t^2 - 2}\right| + c$$

$$= \log\left|\left(x + \frac{1}{x}\right) + \sqrt{x^2 + \frac{1}{x^2}}\right| + c$$

297. The given integral is

$$\int \frac{dx}{x\sqrt{x^4 + 3x^2 + 1}}$$

$$= \int \frac{dx}{x^2\sqrt{x^2 + \frac{1}{x^2} + 3}}$$

$$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{\sqrt{x^2 + \frac{1}{x^2} + 3}}\, dx$$

$$= \frac{1}{2}\int \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{\sqrt{x^2 + \frac{1}{x^2} + 3}}\, dx$$

$$= \frac{1}{2}\int \frac{\left(1 + \frac{1}{x^2}\right)}{\sqrt{x^2 + \frac{1}{x^2} + 3}}\, dx - \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2}\right)}{\sqrt{x^2 + \frac{1}{x^2} + 3}}\, dx$$

$$= \frac{1}{2}\int \frac{\left(1 + \frac{1}{x^2}\right)}{\sqrt{\left(x - \frac{1}{x}\right)^2 + 5}}\, dx - \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2}\right)}{\sqrt{\left(x + \frac{1}{x}\right)^2 + 1}}\, dx$$

$$= \frac{1}{2}\log\left|\left(x - \frac{1}{x}\right) + \sqrt{x^2 + \frac{1}{x^2} + 3}\right|$$

$$-\frac{1}{2}\tan^{-1}\left(x + \frac{1}{x}\right) + c$$

298. The given integral is

$$\int \left(\frac{x^2 + 1}{1 - x^2}\right) \times \frac{dx}{\sqrt{x^4 + x^2 + 1}}$$

$$= -\int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)} \times \frac{dx}{\sqrt{\left(x - \frac{1}{x}\right)^2 + 3}}$$

Let $\left(x - \frac{1}{x}\right) = t$

$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

$$= -\int \frac{dt}{t\sqrt{t^2 + 3}}$$

$$= -\int \frac{t\,dt}{t^2\sqrt{t^2 + 3}}$$

Let $t^2 + 3 = v^2$

$\Rightarrow 2t\,dt = 2v\,dv$

$$= -\int \frac{v\,dv}{(v^2 - 3)v}$$

$$= -\int \frac{dv}{(v^2 - 3)}$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{v - \sqrt{3}}{v + \sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{t^2 + 3} - \sqrt{3}}{\sqrt{t^2 + 3} + \sqrt{3}}\right| + c$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^2 + \frac{1}{x^2} + 1} - \sqrt{3}}{\sqrt{x^2 + \frac{1}{x^2} + 1} + \sqrt{3}}\right| + c$$

299. The given integral is

$$\int \left(\frac{x^2 - 1}{x}\right) \times \frac{dx}{\sqrt{x^4 + 3x^2 + 1}}$$

$$= \int \left(1 - \frac{1}{x^2}\right) \times \frac{dx}{\sqrt{x^2 + \frac{1}{x^2} + 3}}$$

$$= \int \left(1 - \frac{1}{x^2}\right) \times \frac{dx}{\sqrt{\left(x + \frac{1}{x}\right)^2 + 1}}$$

$$= \log\left|\left(x + \frac{1}{x}\right) + \sqrt{x^2 + \frac{1}{x^2} + 3}\right| + c$$

300. The given integral is

$$\int \frac{x^x(x^{2x} + 1)(\ln x + 1)}{(x^{4x} + 1)}\, dx$$

Let $x^x = t$

$\Rightarrow x^x(1 + \ln x)\, dx = dt$

$$= \int \frac{t^2+1}{t^4+1}\, dt$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}\, dt$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\, dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}(x^x - x^{-x})\right) + c$$

301. The given integral is

$$\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\, dx$$

$$= \int \frac{(x^2-1)}{x^5\sqrt{\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}\, dx$$

$$= \int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)}{\sqrt{\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}\, dx$$

Let $\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right) = t^2$

$\Rightarrow \left(\frac{4}{x^2}-\frac{4}{x^4}\right) dx = 2t\,dt$

$\Rightarrow \left(\frac{1}{x^3}-\frac{1}{x^5}\right) dx = \frac{1}{2} t dt$

$$= \frac{1}{2}\int \frac{tdt}{t}$$

$$= \frac{1}{2}\int dt$$

$$= \frac{1}{2} t + c$$

$$= \frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}} + c$$

$$= \sqrt{\frac{2x^4-2x^2+1}{2x^2}} + c$$

302. The given integral is

$$\int \left(\frac{x^{2009}}{(1+x^2)^{1006}}\right) dx$$

$$= \int \frac{x^{2009}}{(1+x^2)^{1006}}\, dx$$

$$= \int \frac{x^{2009}}{x^{2012}\left(1+\frac{1}{x^2}\right)^{1006}}\, dx$$

$$= \int \frac{dx}{x^3\left(1+\frac{1}{x^2}\right)^{1006}}$$

Let $\left(1+\frac{1}{x^2}\right) = t$

$\Rightarrow -\frac{2}{x^3}\, dx = dt$

$\Rightarrow \frac{dx}{x^3} = \frac{dt}{-2}$

$$= -\frac{1}{2}\int \frac{dt}{t^{1006}}$$

$$= -\frac{1}{2} \times \frac{t^{-1005}}{-1005} + C$$

$$= \frac{1}{2010 \times t^{1005}} + C$$

$$= \frac{1}{2010 \times \left(1+\frac{1}{x^2}\right)^{1005}} + C$$

303. We have,

$$\int \frac{(2x+1)\,dx}{(x^2+4x+1)^{3/2}}$$

$$= \int \frac{(2x+1)\,dx}{x^3\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}$$

$$= \int \frac{\frac{2}{x^2}+\frac{1}{x^3}}{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}\, dx$$

Let $\left(1+\frac{4}{x}+\frac{1}{x^2}\right) = t^2$

$\Rightarrow -\frac{4}{x^2}-\frac{2}{x^3}\, dx = 2t\,dt$

$\Rightarrow \left(\frac{2}{x^2}+\frac{1}{x^3}\right) dx = -t\,dt$

$$= -\int \frac{t\,dt}{t^3}$$

$$= -\int \frac{dt}{t^2}$$

$$= \frac{1}{t} + c$$

$$= \frac{1}{\sqrt{1 + \frac{4}{x} + \frac{2}{x^2}}} + c$$

304. We have,

$$\int \left(\frac{\sqrt{\cot x} - \sqrt{\tan x}}{1 + 3\sin 2x}\right) dx$$

$$= \int \frac{1}{\sqrt{\tan x}}\left(\frac{1 - \tan x}{1 + 6\sin x \cos x}\right) dx$$

$$= \int \frac{1}{\sqrt{\tan x}}\left(\frac{1 - \tan x}{\sec^2 x + 6\tan x}\right) \sec^2 x\, dx$$

$$= \int \frac{1}{\sqrt{\tan x}}\left(\frac{1 - \tan x}{\tan^2 x + 6\tan x + 1}\right) \sec^2 x\, dx$$

$$= \int \frac{1}{t}\left(\frac{1 - t^2}{t^4\ 6t^2 + 1}\right) 2t\, dt$$

$$= 2\int \left(\frac{1 - t^2}{t^4 + 6t^2 + 1}\right) dt$$

$$= 2\int \left(\frac{\frac{1}{t^2} - 1}{t^2 + \frac{1}{t^2} + 6}\right) dt$$

$$= 2\int \left(\frac{\frac{1}{t^2} - 1}{\left(t + \frac{1}{t}\right)^2} + 4\right) dt$$

$$= 2\int \left(\frac{dz}{z^2 + 4}\right) \quad \text{Let } z = \left(t + \frac{1}{t}\right)$$

$$\Rightarrow \quad dx^2 = \left(1 - \frac{1}{t^2}\right) dx$$

$$= 2 \times \frac{1}{2} \tan^{-1}\left(\frac{z}{2}\right) + c$$

$$= \tan^{-1}\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right) + c$$

$$= \tan^{-1}\left(\frac{1}{2}\sqrt{\tan x} + \sqrt{\cot x}\right) + c$$

305. We have,

$$\int \sqrt{\tan x}\, dx$$

Put $\tan x = t^2$

$$\Rightarrow \quad \sec^2 x\, dx = 2t\, dt$$

$$\Rightarrow \quad dx = \frac{2t\, dt}{\sec^2 x}$$

$$= \frac{2t\, dt}{1 + \tan^2 x}$$

$$= \frac{2t\, dt}{1 + t^4}$$

$$= \int \frac{t \cdot 2t\, dt}{t^4 + 1}$$

$$= \int \frac{2t^2\, dt}{t^4 + 1}$$

$$= \int \frac{(t^2 - 1) + (t^2 + 1)\, dt}{t^4 + 1}$$

$$= \int \frac{(t^2 - 1)\, dt}{t^4 + 1} + \int \frac{(t^2 + 1)\, dt}{t^4 + 1}$$

$$= \int \frac{\left(1 - \frac{1}{t^2}\right) dt}{t^2 + \frac{1}{t^2}} + \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{t^2 + \frac{1}{t^2}}$$

$$= \int \frac{\left(1 - \frac{1}{t^2}\right) dt}{\left(t + \frac{1}{t}\right)^2 - 2} + \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2}$$

$$= \frac{1}{2\sqrt{2}} \log \left|\frac{\left(t + \frac{1}{t}\right) - \sqrt{2}}{\left(t + \frac{1}{t}\right) + \sqrt{2}}\right| + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}}\right) + c$$

$$= \frac{1}{2\sqrt{2}} \log \left|\frac{\left(x^2 + \frac{1}{x^2}\right) - \sqrt{2}}{\left(x^2 + \frac{1}{x^2}\right) + \sqrt{2}}\right|$$

$$+ \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\left(x^2 - \frac{1}{x^2}\right)}{\sqrt{2}}\right) + c$$

306. We have,

$$\int (\sqrt{\tan x} - \sqrt{\cot x})\, dx$$

$$= \int \left(\sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}\right) dx$$

$$= \int \left(\frac{\tan x - 1}{\sqrt{\tan x}}\right) dx$$

Put $\tan x = t^2$

$$\Rightarrow \quad \sec^2 x\, dx = 2t\, dt$$

$$\Rightarrow \quad dx = \frac{2t\, dt}{\sec^2 x}$$

$$= \frac{2t\, dt}{1 + \tan^2 x}$$

$$= \frac{2t\, dt}{1 + t^4}$$

$$= \int \left(\frac{t^2 - 1}{t^4 + 1}\right) 2t\,dt$$

$$= \int \left(\frac{t^2 - 1}{t^4 + 1}\right) \frac{2t\,dt}{t}$$

$$= \int \left(\frac{t^2 - 1}{t^4 + 1}\right) dt$$

$$= \int \left(\frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\right) dt$$

$$= \int \left(\frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 2}\right) dt$$

$$= \frac{1}{2\sqrt{2}} \log \left|\left(\frac{\left(t + \frac{1}{t}\right) - \sqrt{2}}{\left(t + \frac{1}{t}\right) + \sqrt{2}}\right)\right| + c$$

$$= \frac{1}{2\sqrt{2}} \log \left|\frac{(\sqrt{\tan x} + \sqrt{\cot x}) - \sqrt{2}}{(\sqrt{\tan x} + \sqrt{\cot x}) + \sqrt{2}}\right| + c$$

307. Do yourself

308. $\int \left(\sqrt{\cot x} - \frac{1}{\sqrt{\cot x}}\right) dx = \int \left(\frac{\cot x - 1}{\sqrt{\cot x}}\right) dx$

Let $\cot x = t^2$

$-\operatorname{cosec}^2 x\,dx = 2t\,dt$

$dx = -\frac{2t}{1 + t^4}\,dt$

$$= \int \left(t^2 - \frac{1}{t}\right) \times \frac{2t}{1 + t^4}\,dt$$

$$= 2\int \left(\frac{t^2 - 1}{t^4 + 1}\right) dt$$

$$= 2\int \left(\frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\right) dt$$

$$= 2\int \left(\frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 2}\right) dt$$

$$= 2 \times \frac{1}{\sqrt{2}} \log \left|\frac{\left(t + \frac{1}{t}\right) - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}}\right| + c$$

$$= \sqrt{2} \log \left|\frac{\left(t + \frac{1}{t}\right) - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}}\right| + c$$

$$= \sqrt{2} \log \left|\frac{(\sqrt{\cot x} + \sqrt{\tan x}) - \sqrt{2}}{(\sqrt{\cot x} + \sqrt{\tan x}) + \sqrt{2}}\right| + c$$

309. Do yourself

310. Do yourself

311. $\int (\sqrt[4]{\tan x} + \sqrt[4]{\cot x})^2\,dx$

$$= \int (\sqrt{\tan x} + \sqrt{\cot x} + 2)\,dx$$

$$= \int \left(\frac{\tan x + 1}{\sqrt{\tan x}} + 2\right) dx$$

$$= 2x + \int \left(\frac{\tan x + 1}{\sqrt{\tan x}}\right) dx$$

$$= 2x + 2\int \left(\frac{t^2 + 1}{t^4 + 1}\right) dt$$

Let $\tan x = t^2$

$\Rightarrow \sec^2 dx = 2t\,dt$

$\Rightarrow dx = \frac{2t}{1 + t^4} dt$

$$= 2x + 2\int \left(\frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\right) dt$$

$$= 2x + 2\int \left(\frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2}\right) dt$$

$$= 2x + 2 \times \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right)\right) + c$$

$$= 2x + \sqrt{2} \tan^{-1}\left(\frac{1}{\sqrt{2}}(\tan x - \cot x)\right) + c$$

312. We have,

$$\int \frac{dx}{(\sqrt{\sin x} + \sqrt{\cot x})^4} = \int \frac{\sec^2 x}{(\sqrt{\tan x} + 1)^4}\,dx$$

Put $\tan x = t^2$

$\Rightarrow \sec^2 x\,dx = 2t\,dt$

$$= \int \frac{dt}{(t + 1)^4}$$

$$= -\frac{4}{(t + 1)^3} + c$$

$$= -\frac{4}{(\sqrt{\tan x} + 1)^3} + c$$

Integration by Parts

313. We have,

$$\int x e^x dx = x \int e^x dx - \int (1 \cdot e^x)\ dx$$

$$= xe^x - e^x + c$$

314. We have,

$$\int x \sin x\, dx = x \int \sin x\, dx - \int (1 - \cos x) dx$$

$$= x \int \sin x\, dx + \int \cos x\, dx$$

$$= -x \cos x + \sin x + c$$

315. We have,

$$\int x^2 \sin x\, dx$$

$$= x^2 \int \sin x\, dx - \int (2x - \cos x)\ dx$$

$$= -x^2 \cos x + 2 \int \cos x\, dx$$

$$= -x^2 \cos x + 2x \int \cos x\, dx - 2 \int (1 \sin x)\ dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

316. We have,

$$\int \log x\, dx = \int (\log x \cdot 1)\ dx$$

$$= \log x \int dx - \int \left(\frac{1}{x} \cdot x\right) dx$$

$$= \log x \int dx - \int 1 \cdot dx$$

$$= x \log x - x + c$$

317. We have,

$$\int (\log x)^2\ dx = \int \{(\log x)^2 \cdot 1\}\ dx$$

$$= (\log x)^2 \int dx - \int \left(2 \log x \cdot \frac{1}{x} \cdot x\right) dx$$

$$= x (\log x)^2 - 2 \int \log x\, dx + c$$

$$= x (\log x)^2 - 2\ (x \log x - x) + c$$

318. We have,

$$\int \log (x^2 + 1)\ dx$$

$$= \int (\log (x^2 + 1) \cdot 1)\ dx$$

$$= \log (x^2 + 1) \int dx - \int \left(\frac{1}{x^2 + 1} \cdot 2x \cdot x\right) dx$$

$$= x \log (x^2 + 1) - 2 \int \left(\frac{x^2}{x^2 + 1}\right) dx$$

$$= x \log (x^2 + 1) - 2 \int \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= x \log (x^2 + 1) - 2(x - \tan^{-1} x) + c$$

319. We have,

$$\int \tan^{-1} x\, dx = \int \{\tan^{-1} x \cdot 1\} dx$$

$$= \tan^{-1}(x) \int dx - \int \frac{x}{1 + x^2}\ dx$$

$$= x \tan^{-1}(x) - \frac{1}{2} \log |1 + x^2| + c$$

320. We have,

$$\int \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) dx$$

$$= \int (2 \tan^{-1} x)\ dx$$

$$= 2 \int \tan^{-1} x\, dx$$

$$= 2\left(\tan^{-1} x \int dx \int \left(\frac{1}{1 + x^2} \cdot x\right) dx\right)$$

$$= 2\left(x \tan^{-1} x - \frac{1}{2} \int \left(\frac{2x}{1 + x^2}\right) dx\right) + c$$

$$= 2\left(x \tan^{-1} x - \frac{1}{2} \log |x^2 + 1|\right) + c$$

321. We have,

$$\int e^{\sqrt{x}}\ dx$$

Let $x = t^2$

$\Rightarrow$ $dx = 2t\, dt$

$$= 2 \int t e^t dt$$

$$= 2\left\lfloor t \int e^t dt - \int (1 \cdot e^t)\ dt \right\rfloor$$

$$= 2 (t e^t - e^t) + c$$

$$= 2 (\sqrt{x} e^{\sqrt{x}} - e^{\sqrt{x}}) + c$$

322. We have,

$$\int \left(\frac{x + \sin x}{1 + \cos x}\right) dx$$

$$= \int \frac{x}{1 + \cos x} dx + \int \frac{\sin x}{1 + \cos x} dx$$

$$= \int \frac{x}{2 \cos^2\left(\frac{x}{2}\right)} dx + \int \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{2 \cos^2\left(\frac{x}{2}\right)} dx$$

$$= \frac{1}{2} \int x \sec^2\left(\frac{x}{2}\right) dx + \int \tan\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left[x\int \sec^2\left(\frac{x}{2}\right) dx - \int\left(2\tan\left(\frac{x}{2}\right)\right) dx\right] + \int \tan\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left[2x\tan\left(\frac{x}{2}\right)\right] - \int \tan\left(\frac{x}{2}\right) dx + \int \tan\left(\frac{x}{2}\right) dx + c$$

$$= x\tan\left(\frac{x}{2}\right) + c$$

323. We have,

$$\int \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) dx$$

Let $x = \cos\theta$

$\Rightarrow \quad dx = -\sin\theta d\theta$

$$= -\int \tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right) \sin\theta d\theta$$

$$= -\int \tan^{-1}\left(\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\right) \sin\theta d\theta$$

$$= -\int \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right) \sin\theta d\theta$$

$$= -\frac{1}{2}\int \theta\sin\theta d\theta$$

$$= -\frac{1}{2}\left[\theta\int \sin\theta d\theta - \int(1-\cos\theta)\, d\theta\right]$$

$$= -\frac{1}{2}[-\theta\cos\theta + \sin\theta] + c$$

$$= \frac{1}{2}[\theta\cos\theta - \sin\theta] + c$$

$$= \frac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + c$$

324. We have,

$$\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}} dx$$

$$= \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int dx$$

Let $x = \sin^2\theta$

$\Rightarrow \quad dx = \sin 2\theta\, d\theta$

$$= \frac{4}{\pi}\int \sin^{-1}(\sin\theta)\sin 2\theta\, d\theta - x + c$$

$$= \frac{4}{\pi}\int \theta\sin 2\theta\, d\theta - x + c$$

$$= \frac{4}{\pi}\left[\theta\int \sin 2\theta d\theta - \int\left(1\cdot -\frac{\cos 2\theta}{2}\right) d\theta\right] - x + c$$

$$= \frac{4}{\pi}\left[\theta\int \sin 2\theta\, d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta\right] - x + c$$

$$= \frac{4}{\pi}\left[\left(\frac{-\theta\cos 2\theta}{2}\right) + \frac{1}{4}\sin 2\theta\right] - x + c$$

$$= \frac{4}{\pi}\left[-\frac{1}{2}(\theta(1-2\sin^2\theta)) + \frac{1}{2}\sin\theta\cos\theta\right] - x + c$$

$$= \frac{4}{\pi}\left[-\frac{1}{2}(\sin^{-1}x(1-2x^2)) + \frac{1}{2}x\sqrt{1-x^2}\right] - x + c$$

325. We have,

$$\int \frac{x^2}{(x\sin x + \cos x)^2} dx$$

$$= \int \frac{x\cos x}{x\sin x + \cos x}\cdot x\sec x\, dx$$

$$= x\sec x\int \frac{x\cos x}{x\sin x + \cos x} dx - \int\left(-\frac{(\sec x + x\sec x\tan x)}{x\sin x + \cos x}\right) dx$$

$$= \frac{x\sec x}{x\sin x + \cos x} + \int \sec x\, dx + c$$

$$= \frac{x\sec x}{x\sin x + \cos x} + \log|\sec x + \tan x| + c$$

326. We have,

$$\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}} dx$$

Let $x = \tan\theta$

$\Rightarrow \quad dx = 2\sec^2\theta d\theta$

$$= \int \frac{\tan\theta a\ \tan^{-1}(\tan\theta)}{\sec^3\theta}\cdot \sec^2\theta d\theta$$

$$= \int \theta\cos\theta d\theta$$

$$= \theta\int \cos\theta d\theta - \int(1.\sin\theta)d\theta$$

$$= \theta\sin\theta + \cos\theta + c$$

$$= \frac{x}{\sqrt{1+x^2}}\cdot\tan^{-1}x + \frac{1}{\sqrt{1-x^2}} + c.$$

327. We have,

$$\int \sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right)dx$$

Put $x = a\tan^2\theta$

$\Rightarrow \quad dx = 2a\tan\theta\sec^2\theta d\theta$

$$= \int \sin^{-1}\left(\sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}}\right)2a\tan\theta\sec^2\theta\, d\theta$$

$= 2a\int \sin^{-1}(\sin\theta)\tan\theta\sec^2\theta d\theta$

$= 2a\int \theta\tan\theta\sec^2\theta\, d\theta$

Let $\tan\theta = t$

$\Rightarrow \sec^2\theta d\theta = dt$

$= 2a\int t\tan^{-1}t\,dt$

$= 2a\left[\tan^{-1}t\int t\,dt - \int\left(\frac{1}{1+t^2}\right)\frac{t^2}{2}dt\right]$

$= 2a\left[\frac{t^2}{2}\tan^{-1}t - \frac{1}{2}\int\left(1 - \frac{1}{1+t^2}\right)dt\right]$

$= a[t^2\tan^{-1}t - t + \tan^{-1}t] + c$

$= a[(t^2+1)\tan^{-1}t - t] + c$

$= a[(\tan^2\theta + 1)\theta - \tan\theta] + c$

$= a\left[\left(\frac{x}{a}+1\right)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}}\right] + c$

328. The given integral is

$\int \log(1+x)dx$

$= \log(1+x)\int dx - \int\frac{x}{x+1}dx$

$= x\log(1+x) - \int\left(1 - \frac{1}{x+1}\right)$

$= x\log(1+x) - (x - \log|x+1|) + c$

329. The given integral is

$\int\frac{x - \sin x}{1-\cos x}dx$

$= \int\frac{x\,dx}{1-\cos x} - \int\frac{\sin x}{1-\cos x}dx$

$= \int\frac{x\,dx}{2\sin^2(x/2)} - \int\frac{2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)dx}$

$= \frac{1}{2}\int x\,\text{cosec}^2(x/2)dx - \int\cot\left(\frac{x}{2}\right)dx$

$= \frac{1}{2}\left[x\int\text{cosec}^2(x/2)dx + 2\int\cot(x/2)dx\right]$

$-\int\cot(x/2)dx + c$

$= \frac{1}{2}(-2x\cot(x/2)) + \int\cot(x/2)dx$

$-\int\cot(x/2)dx + c$

$= -\cot(x/2) + c$

330. The given integral is

$\int\sin\sqrt{x}\,dx$

Let $x = t^2$

$dx = 2t\,dt$

$= 2\int t\sin t\,dt$

$= 2\left\lfloor t\int\sin t\,dt + \int\cos t\,dt + c\right\rfloor$

$= 2[-t\cos t + \sin t] + c$

$= 2[-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}] + c$

331. The given integral is

$\int x\log(1+x)dx$

$= \log(1+x)\int x\,dx - \frac{1}{2}\int\frac{x^2}{(x+1)}dx$

$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\frac{(x^2-1)+1}{(x+1)}dx$

$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\left(x - 1 + \frac{1}{x+1}\right)dx$

$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\left(\frac{x^2}{2} - x + \log|x+1|\right) + c$

332. The given integral is

$\int(\sin^{-1}x)^2\,dx$

$= (\sin^{-1}x)^2\int dx - \int\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$= x(\sin^{-1}x)^2 - 2\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$= x(\sin^{-1}x)^2 - 2\int\theta\sin\theta d\theta$, Let $\sin^{-1}x = \theta$

$\frac{x}{\sqrt{1-x^2}}dx = d\theta$

$= x(\sin^{-1}x)^2 - 2\int\theta\sin\theta d\theta$

$= x(\sin^{-1}x)^2 - 2[\theta\sin\theta d\theta + \theta\cos\theta d\theta] + c$

$= x(\sin^{-1}x)^2 - 2[-\theta\cos\theta d\theta + \cos\theta]$

$= x(\sin^{-1}x)^2 - 2[-\sqrt{1-x^2}\sin^{-1}x + x] + c$

$= x(\sin^{-1}x)^2 + 2\lfloor\sqrt{1-x^2}\sin^{-1}x - x\rfloor + c$

333. The given integral is

$\int\frac{x - \sin x}{1-\cos x}dx$

$= \int\frac{x}{1-\cos x}dx - \int\frac{\sin x}{1-\cos x}dx$

$= \int\frac{x}{2\sin^2\left(\frac{x}{2}\right)}dx - \int\frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}dx$

$$= \frac{1}{2}\int x\,\mathrm{cosec}^2\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left[x\int \mathrm{cosec}^2\left(\frac{x}{2}\right)dx - 1\cdot\left(-2\cot\left(\frac{x}{2}\right)\right)dx\right] - \int \cot\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left[x\left(-2\cot\left(\frac{x}{2}\right)\right) + \int\left(2\cot\left(\frac{x}{2}\right)\right)dx\right] - \int \cot\left(\frac{x}{2}\right)dx + c$$

$$= -x\cot\left(\frac{x}{2}\right) + c$$

334. The given integral is

$$\int x\sin^3 x\,dx$$

$$= x\int \sin^3 x\,dx - \int\left(\frac{\cos^3 x}{3} - \cos x\right)dx$$

$$= x\left(\frac{\cos^3 x}{3} - \cos x\right) - \int\left(\frac{\cos^3 x}{3} - \cos x\right)dx$$

$$= x\left(\frac{\cos^3 x}{3} - \cos x\right) - \frac{1}{3}\left(\sin x - \frac{\sin^3 x}{3}\right) + \sin x + c$$

335. The given integral is

$$\int x\left(\frac{\sec 2x - 1}{\sec 2x + 1}\right)dx$$

$$= \int x\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)dx$$

$$= \int x\tan^2 x\,dx$$

$$= \int x(\sec^2 x - 1)\,dx$$

$$= \int x\sec^2 x\,dx - \int dx$$

$$= x\int \sec^2 x\,dx - \int \tan x\,dx - x + c$$

$$= x\tan x - \log|\sec x| - x + c$$

336. The given integral is

$$\int \sin^{-1}\left(\frac{2x}{1 + x^2}\right)dx$$

$$= \int (2\tan^{-1} x)\,dx$$

$$= 2\int \tan^{-1} x\,dx$$

$$= 2\left[\tan^{-1} x\int dx - \int \frac{x}{1 + x^2}dx\right]$$

$$= 2\left[x\tan^{-1} x - \frac{1}{2}\log|1 + x^2|\right] + c$$

337. The given integral is

$$\int \tan^{-1}\left(\frac{2x}{1 - x^2}\right)dx$$

$$= \int (2\tan^{-1} x)\,dx$$

$$= 2\int \tan^{-1} x\,dx$$

$$= 2\left[\tan^{-1} x\int dx - \int \frac{x}{1 + x^2}dx\right]$$

$$= 2\left[x\tan^{-1} x - \frac{1}{2}\log|1 + x^2|\right] + c$$

338. The given integral is

$$\int \frac{x^3\sin^{-1}(x^2)}{\sqrt{1 - x^4}}dx$$

$$= \frac{1}{2}\int \frac{2x\sin^{-1}(x^2) \times x^2}{\sqrt{1 - x^4}}dx$$

Let $\sin^{-1}(x^2) = t$

$$\Rightarrow \frac{2x}{\sqrt{1 - x^4}}dx = dt$$

$$= \frac{1}{2}\int t\sin t\,dt$$

$$= \int [t\int \sin t\,dt + \int \cos t\,dt] + c$$

$$= \frac{1}{2}[-t\cos t + \sin t] + c$$

$$= \frac{1}{2}\left[-\sin^{-1}(-x^2)\sqrt{1 - x^4} + x^2\right] + c$$

339. The given integral is

$$\int \frac{\sec x(2 + \sec x)}{(1 + 2\sec x)^2}dx$$

$$= \int \frac{2\cos x + 1}{(\cos x + 2)^2}dx$$

$$= \int \frac{\cos x(\cos x + 2) + \sin x}{(\cos x + 2)^2}dx$$

$$= \int \frac{\cos x}{(\cos x + 2)}dx + \int \frac{\sin x}{(\cos x + 2)^2}dx$$

$$= \frac{1}{(\cos x + 2)}\int \cos dx - \int \frac{\sin x}{(\cos x + 2)^2}dx + \int \frac{\sin x}{(\cos x + 2)^2}dx + c$$

$$= \frac{\sin x}{\cos x + 2} + c$$

340. The given integral is

$$\int \frac{1 - x}{e^x + x}dx.$$

$$= \int \frac{(1 + e^x) - (x + e^x)}{(e^x + x)}dx$$

$$= \int \left(\frac{(1 + e^x)}{(e^x + x)} - 1 \right) dx$$

$$= \log|(e^x + x)| - x + c$$

341. We have,

$$\int e^x (\sin x + \cos x)\, dx$$

We know that,

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$$

Here, $f(x) = \sin x \Rightarrow f'(x) = \cos x$

Thus,

$$\int e^x (\sin x + \cos x)\, dx = e^x \sin x + c$$

342. We have,

$$\int \frac{x e^x}{(x + 1)^2}\, dx$$

$$= \int \frac{e^x (x + 1 - 1)}{(x + 1)^2}\, dx$$

$$= \int e^x \left(\frac{x + 1}{(x + 1)^2} - \frac{1}{x + 1} \right) dx$$

$$= -\int e^x \left(\frac{1}{x + 1} - \frac{1}{(x + 1)^2} \right) dx$$

$$= -\frac{e^x}{x + 1} + c$$

343. We have,

$$\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$$

$$= \int e^x \left(\frac{2}{1 + \cos 2x} + \frac{\sin 2x}{1 + \cos 2x} \right) dx$$

$$= \int e^x \left(\frac{2}{2\cos^2 x} + \frac{2 \sin x \cos x}{2 \cos^2 x} \right) dx$$

$$= \int e^x (\sec^2 x + \tan x)\, dx$$

$$= \int e^x (\tan x + \sec^2 x)\, dx$$

$$= e^x \tan x + c$$

344. We have,

$$\int e^x \left(\frac{1 + x + x^3}{(1 + x^2)^{3/2}} \right) dx$$

$$= \int e^x \left(\frac{1 + x(1 + x^2)}{(1 + x^2)^{3/2}} \right) dx$$

$$= \int e^x \left(\frac{1}{(1 + x^2)^{3/2}} + \frac{x}{(1 + x^2)^{1/2}} \right) dx$$

$$= \int e^x \left(\frac{x}{(1 + x^2)^{1/2}} + \frac{1}{(1 + x^2)^{3/2}} \right) dx$$

$$= \frac{e^x x}{(1 + x^2)^{1/2}} + c$$

345. We have,

$$\int e^x \left(\frac{x^2 + 1}{(x + 1)^2} \right) dx$$

$$= \int e^x \left(\frac{x^2 - 1 + 2}{(x + 1)^2} \right) dx$$

$$= \int e^x \left(\frac{(x + 1)(x - 1) + 2}{(x + 1)^2} \right) dx$$

$$= \int e^x \left(\frac{(x - 1)}{(x + 1)} + \frac{2}{(x + 1)^2} \right) dx$$

$$= e^x \left(\frac{(x - 1)}{(x + 1)} \right) + c$$

346. The given integral is

$$\int e^x \left\{ \frac{1}{x} - \frac{1}{x^2} \right\} dx = \int e^x \left(\frac{1}{x} + \left(-\frac{1}{x^2} \right) \right) dx$$

$$= \frac{e^x}{x} + c$$

347. The given integral is

$$\int e^x \left(\frac{x}{(x + 1)^2} \right) dx$$

$$= \int e^x \left(\frac{(x + 1) - 1}{(x + 1)^2} \right) dx$$

$$= \int e^x \left(\frac{1}{(x + 1)} - \frac{1}{(x + 1)^2} \right) dx$$

$$= \int \frac{e^x}{(x + 1)} + c$$

348. The given integral is

$$\int e^x \left\{ \frac{1 - \sin x}{1 - \cos x} \right\} dx$$

$$= \int e^x \left(\frac{1}{1 - \cos x} - \frac{\sin x}{1 - \cos x} \right) dx$$

$$= \int e^x \left(\frac{1}{2 \sin^2 (x/2)} - \frac{2 \sin (x/2) \cos (x/2)}{2 \sin^2 (x/2)} \right) dx$$

$$= \int e^x \left(\frac{1}{2} \operatorname{cosec}^2 (x/2) - \cot (x/2) \right) dx$$

$$= \int e^x \left(\cot (x/2) + \left(-\frac{1}{2} \operatorname{cosec}^2 (x/2) \right) \right) dx$$

$$= -e^x \cot \left(\frac{x}{2} \right) + c$$

349. The given integral is

$$\int e^x \left\{\frac{2 + \sin 2x}{1 + \cos 2x}\right\} dx$$

$$= \int e^x \left(\frac{2}{1 + \cos 2x} + \frac{\sin 2x}{1 + \cos 2x}\right) dx$$

$$= \int e^x \left(\frac{2}{2\cos^2 x} + \frac{2 \sin x \cos x}{2\cos^2 x}\right) dx$$

$$= \int e^x (\sec^2 x + \tan x)\, dx$$

$$= \int e^x (\tan x + \sec^2 x)\, dx$$

$$= e^x \tan x + c$$

350. The given integral is

$$\int e^x \left(\frac{x^2 + 1}{(x+1)^2}\right) dx = \int e^x \left(\frac{(x^2 - 1) + 2}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{(x-1)}{(x+1)} + \frac{2}{(x+1)^2}\right) dx$$

$$= e^x \left(\frac{(x-1)}{(x+1)}\right) + c$$

351. The given integral is

$$\int \frac{\log x}{(1 + \log x)^2} dx$$

$$= \int \frac{t e^t}{(t+1)^2}\, dt$$

Let $\log x = t$

$\Rightarrow \quad dt = \dfrac{dx}{e^{\log x}}$

$$= \int \frac{[(t+1) - 1]e^t}{(t+1)^2}\, dt$$

$$= \int e^t \left[\frac{1}{t+1} + \left(-\frac{1}{(t+1)^2}\right)\right] dt$$

$$= \frac{e^t}{t+1} + c$$

$$= \frac{x}{\log x + 1} + c$$

352. The given integral is

$$\int e^x \left(\log x + \frac{1}{x}\right) dx = e^x \log x + c$$

353. The given integral is

$$\int e^x \left(\frac{x-1}{(x+1)^3}\right) dx$$

$$= \int e^x \left(\frac{(x+1) - 2}{(x+1)^3}\right) dx$$

$$= \int \left[e^x \frac{1}{(x+1)^2} + \left(-\frac{2}{(x+1)^3}\right)\right] dx$$

$$= \frac{e^x}{(x+1)^2} + c$$

354. The given integral is

$$\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx$$

$$= \int e^t \left(\frac{1}{t} - \frac{1}{t^2}\right) dt,$$

Let $t = \log x$

$\Rightarrow \quad dt = \dfrac{dx}{e^{\log x}}$

$$= \frac{e^t}{t} + c$$

$$= \frac{x}{\log x} + c$$

355. The given integral is

$$\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$$

$$= \int e^x \left(\frac{(1-x)^2}{(1+x^2)^2}\right) dx$$

$$= \int e^x \left(\frac{(1 + x^2 - 2x)}{(1+x^2)^2}\right) dx$$

$$= \int e^x \left[\frac{1}{(1+x^2)} + \left(\frac{-2x}{(1+x^2)^2}\right)\right] dx$$

$$= \frac{e^x}{(1+x^2)} + c$$

356. The given integral is

$$\int e^x \left(\frac{1-x}{1+x}\right)^2 dx$$

$$= \int e^x \left(\frac{(1-x)^2}{(1+x)^2}\right) dx$$

$$= \int e^x \left(\frac{(1+x^2) - 2x}{(1+x)^2}\right) dx$$

$$= \int e^x \left(\frac{x^2 - 1 - 2(x-1)}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{x^2 - 1}{(x+1)^2} - \frac{2(x-1)}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{x^2 - 1}{(x+1)^2} - \frac{2(x-1)}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\left(\frac{x-1}{x+1}\right) - \frac{2(x-1)}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\left(\frac{x-1}{x+1}\right) - \frac{2(x+1-2)}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\left(\frac{x-3}{x+1}\right) - \frac{4}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{x-3}{x+1}\right) + c$$

357. The given integral is

$$\int e^x \left(\frac{x^2+1}{(1+x)^2}\right) dx$$

$$= \int e^x \left(\frac{x^2-1+2}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\left(\frac{x-1}{x+1}\right) + \frac{2}{(1+x)^2}\right) dx$$

$$= e^x \left(\frac{x-1}{x+1}\right) + c$$

358. The given integral is

$$\int e^{\cos^{-1} x} \left(\frac{(x+1)+\sqrt{1-x^2})}{(x+1)^2 \sqrt{1-x^2}}\right) dx$$

Let $\cos^{-1} x = t$

$$\Rightarrow \quad \frac{dx}{\sqrt{1-x^2}} = -dt$$

$$= -\int e^t \left(\frac{1+\cos t+\sin t}{(1+\cos t)^2}\right) dt$$

$$= -\int e^t \left(\frac{1}{(1+\cos t)} + \frac{\sin t}{(1+\cos t)^2}\right) dt$$

$$= -\frac{e^t}{1+\cos t} + c$$

$$= -\frac{e^{\cos^{-1} x}}{1+x} + c$$

359. We have,

$$\int e^x (2\sec^2 x - 1)\tan x\, dx$$

$$= \int e^x (2\sec^2 x \tan x - \tan x)\, dx$$

$$= \int e^x [(\sec^2 x - \tan x) + (2\sec^2 x \tan x - \sec^2 x)]\, dx$$

$$= \int e^x (\sec^2 x - \tan x) + c$$

360. We have,

$$\int e^x (\log(\sec x + \tan x) + \sec x)\, dx$$

$$= e^x \log(\sec x + \tan x) + c$$

361. We have,

$$\int e^{\sin x} \left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right) dx$$

$$= \int e^{\sin x} (x\cos x - \sec x \tan x)\, dx$$

$$= \int e^{\sin x} \left(\frac{x - \tan x}{\cos^2 x}\right) \cos x\, dx$$

Let $\sin x = t$

$\Rightarrow \quad \cos x\, dx = dt$

$$= \int e^t \left(\sin^{-1} t - \frac{t}{(1-t^2)^{3/2}}\right) dt$$

$$= \int e^t \left(\left(\sin^{-1} t + \frac{1}{\sqrt{1-t^2}}\right) - \left(\frac{1}{\sqrt{1-t^2}} + \frac{t}{(1-t^2)^{3/2}}\right)\right) dt$$

$$= e^t \left(\sin^{-1} t + \frac{1}{\sqrt{1-t^2}}\right) + c$$

$$= e^{\sin x} (x + \sec x) + c$$

362. $\int e^x \left(\log x + \frac{1}{x^2}\right) dx$

$$= \int e^x \left\{\left(\log x - \frac{1}{x}\right) + \left(\frac{1}{x} + \frac{1}{x^2}\right)\right\} dx$$

$$= e^x \left(\log x - \frac{1}{x}\right) + c$$

363. $\int \left\{\log(\log x) + \frac{1}{(\log x)^2}\right\} dx$

$$= \int e^t \left(\log t + \frac{1}{t^2}\right) dt, \qquad \text{Let } t = \log x$$

$$\Rightarrow \quad dt = \frac{dx}{e^{\log x}}.$$

$$= \int e^t \left[\left(\log t - \frac{1}{t}\right) + \left(\frac{1}{t} + \frac{1}{t^2}\right)\right] dt$$

$$= e^t \left(\log t - \frac{1}{t}\right) + c$$

$$= x \left(\log(\log x) - \frac{1}{\log x}\right) + c$$

364. $\int e^x \left(\frac{x^4+2}{(1+x^2)^{5/2}}\right) dx$

$$= \int e^x \left(\frac{(1+x^2)^2 + 1 - 2x^2}{(1+x^2)^{5/2}}\right) dx$$

$$= \int e^x \left(\left\{\frac{1}{\sqrt{1+x^2}} - \frac{x}{(1-x^2)^{3/2}}\right\} + \left\{\frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right\}\right) dx$$

$$= e^x \left(\frac{1}{\sqrt{1+x^2}} + \frac{x}{(1-x^2)^{3/2}}\right) + c$$

365. $\int \frac{e^{\sin x}(x\cos^3 x - \sin x)}{\cos^2 x}\, dx$

$$= \int e^{\sin x} (x\cos x - \sec x \tan x)\, dx$$

Let $\sin x = t$

$\Rightarrow \cos x\, dx = dt$

$$\Rightarrow \quad dx = \frac{dt}{\cos x} = \frac{dt}{\sqrt{1-t^2}}$$

$$= \int e^t \left(\sin^{-1} t - \frac{t}{(1-t^2)^{3/2}}\right) dt$$

$$= \int e^t \left(\sin^{-1} t + \frac{1}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-t^2}} - \frac{t}{(1-t^2)^{3/2}}\right) dt$$

$$= e^t \left(\sin^{-1} t - \frac{1}{\sqrt{1-t^2}}\right) + c$$

$$= e^{\sin x}(x - \sec x) + c$$

366. We have,

$$\int e^x \left(\frac{x^3 - x - 2}{(x^2+1)^2}\right) dx$$

$$= \int e^x \left(\frac{(x^3 + x) + (x^2 + 1) + (1 - 2x^2) - 2x}{(x^2+1)^2}\right) dx$$

$$= \int e^x \left(\frac{x(x^2 + 1) + (x^2 + 1) + (1 - 2x^2) - 2x}{(x^2+1)^2}\right) dx$$

$$= \int e^x \left(\left\{\frac{x}{(x^2+1)} + \frac{1}{(x^2+1)}\right\} + \left\{\frac{1-x^2}{(x^2+1)^2} + \frac{-2x}{(x^2+1)^2}\right\}\right) dx$$

$$= \int e^x \left(\frac{x}{(x^2+1)} + \frac{1}{(x^2+1)}\right) + c$$

$$= e^x \left(\frac{x+1}{(x^2+1)}\right) + c$$

367. We have,

$$\int e^{3x}(3\sin x + \cos x)\, dx = e^{3x} \sin x + c$$

368. We have,

$$e^{2x}(\sec^2 x + 2\tan x)\, dx$$

$$= \int e^{2x}(2\tan x + \sec^2 x)\, dx$$

$$= e^{2x} 2\tan x + c$$

369. We have,

$$\int e^{2x} \left(\frac{2\sin 4x - 4}{1 - \cos 4x}\right) dx$$

$$= \int e^x \left(\frac{2\sin 4x}{1 - \cos 4x} - \frac{2}{1 - \cos 4x}\right) dx$$

$$= \int e^{2x} \left(\frac{4\sin 2x \cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right) dx$$

$$= \int e^{2x}(2\cot 2x + (-2\operatorname{cosec}^2 2x))\, dx$$

$$= e^{2x} \cot 2x + c$$

370. We have,

$$\int e^{2x}(-\sin x + 2\cos x)\, dx$$

$$= \int e^{2x}[2\cos x + (-\sin x)]\, dx$$

$$= e^{2x} 2\cos x + c$$

371. We have,

$$\int e^{2x} \left(\frac{1 + \sin 2x}{1 + \cos 2x}\right) dx$$

$$= \int e^{2x} \left(\frac{1 + \sin 2x}{1 + \cos 2x}\right) dx$$

$$= \int e^{2x} \left(\frac{1}{1 + \cos 2x} + \frac{\sin 2x}{1 + \cos 2x}\right) dx$$

$$= \int e^{2x} \left(\frac{1}{1 + 2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right) dx$$

$$= \int e^{2x} \left(\frac{1}{2}\sec^2 x + \tan x\right) dx$$

$$= \frac{1}{2}\int e^{2x}(\sec^2 x + 2\tan x)\, dx$$

$$= \frac{1}{2}\int e^{2x}(2\tan x + \sec^2 x)\, dx$$

$$= \frac{e^{2x} 2\tan x}{2} + c$$

372. We have,

$$\int e^{-\frac{x}{2}} \left(\frac{\sqrt{1 - \sin x}}{1 + \cos x}\right) dx$$

$$= \int e^{-\frac{x}{2}} \left(\frac{\cos(x/2) - \sin(x/2)}{2\cos^2(x/2)}\right) dx$$

$$= \int e^{-\frac{x}{2}} \left(\frac{1}{2}\sec\left(\frac{x}{2}\right) - \frac{1}{2}\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)\right) dx$$

$$= \int e^{-\frac{x}{2}} \left(-\frac{1}{2}\sec\left(\frac{x}{2}\right) + \frac{1}{2}\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)\right) dx$$

$$= \int e^{-\frac{x}{2}} \sec\left(\frac{x}{2}\right) + c$$

373. We have,

$$\int e^{2x}[2 \times \log(\sec x + \tan x) + \sec x]\, dx$$

$$= e^{2x}(\log(\sec x + \tan x)) + c$$

374. We have,

$$\int e^x \sin 3x\, dx$$
$$= \frac{e^x}{1^2 + 3^2}(1 \cdot \sin 3x - 3\cos 3x) + c$$
$$= \frac{e^x}{10}(\sin 3x - 3\cos 3x) + c$$

375. We have,

$$\int e^{4x} \cos 3x\, dx$$
$$= \frac{e^{4x}}{4^2 + 3^2}(4\cos 3x + 3\sin 3x) + c$$
$$= \frac{e^{4x}}{25}(4\cos 3x + 3\sin 3x) + c$$

376. We have,

$$\int e^{2x} \sin 3x\, dx$$
$$= \frac{e^{2x}}{(4 + 9)}(2\sin 3x - 3\cos 3x) + c$$
$$= \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + c$$

377. We have,

$$\int e^{-x} \cos x\, dx$$
$$= \frac{e^{-x}}{(1 + 1)}(-\cos x + \sin x) + c$$
$$= \frac{e^{-x}}{2}(-\cos x + \sin x) + c$$

378. We have,

$$\int e^{-x} \cos(3x + 4)\, dx$$
$$= \frac{e^{2x}}{3(4 + 1)}[2\cos(3x + 4) + \sin(3x + 4)] + c$$

379. We have,

$$\int e^x \cos^2 x\, dx$$
$$= \frac{1}{2}\int e^x (2\cos^2 x)\, dx$$
$$= \frac{1}{2}\int e^x (1 + \cos 2x)\, dx$$
$$= \frac{1}{2}\int (e^x + e^x \cos 2x)\, dx$$
$$= \frac{1}{2}\left[e^x + \frac{e^x}{5}(\cos 2x + 2\sin 2x)\right] + c$$

380. We have,

$$\int \frac{1}{x^3}\sin(\log x)\, dx$$
$$= \int e^{-2t} \sin t\, dt, \qquad \text{Let } t = \log x \Rightarrow dt = \frac{dx}{e^{\log x}}$$
$$= \int e^{-2t} \sin t\, dt$$
$$= \frac{e^{-2t}}{5}(-2\sin t - \cos t) + c$$
$$= -\frac{e^{-2t}}{5}(-2\sin t + \cos t) + c$$
$$= \frac{e^{-2\log x}}{5}[2\sin(\log x) + \cos(\log x)] + c$$
$$= -\frac{1}{5x^2}[2\sin(\log x) + \cos(\log x)] + c$$

381. We have,

$$\int \sqrt{4 - x^2}\, dx = \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) + c$$
$$= \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + c$$

382. We have,

$$\int \sqrt{1 - 9x^2}\, dx = 3\int \sqrt{\left(\frac{1}{3}\right)^2 - x^2}\, dx$$
$$= 3\left(\frac{x}{2}\sqrt{\frac{1}{9} - x^2} - \frac{\frac{1}{9}}{2}\sin^{-1}\left(\frac{x}{1/3}\right)\right) + c$$
$$= 3\left(\frac{x}{2}\sqrt{\frac{1}{9} - x^2} - \frac{1}{18}\sin^{-1}\left(\frac{x}{1/3}\right)\right) + c$$

383. We have,

$$\int \sqrt{x^2 + 1}\, dx = \frac{x}{2}\sqrt{x^2 + 1} + \frac{1}{2}\log\left|x + \sqrt{x^2 + 1}\right| + c$$

384. We have,

$$\int \sqrt{3x^2 + 1}\, dx$$
$$= \sqrt{3}\int \sqrt{x^2 + \frac{1}{3}}\, dx$$
$$= \sqrt{3}\left(\frac{x}{2}\sqrt{x^2 + \frac{1}{3}} + \frac{1/3}{2}\log\left|x + \sqrt{x^2 + \frac{1}{3}}\right|\right) + c$$

385. We have,

$$\int \sqrt{x^2 - 9}\, dx$$
$$= \frac{x}{2}\sqrt{x^2 - 9} - \frac{9}{2}\log\left|x + \sqrt{x^2 - 9}\right| + c$$

386. We have,

$$\int \sqrt{4x^2 - 1}\, dx$$
$$= 2\int \sqrt{x^2 - \left(\frac{1}{x}\right)^2}\, dx$$
$$= 2\left(\frac{x}{2}\sqrt{x^2 - \frac{1}{4}} - \frac{1/4}{2}\log\left|x + \sqrt{x^2 - \frac{1}{4}}\right|\right) + c$$

$$= \left(x\sqrt{x^2 - \frac{1}{4}} - \frac{1}{4}\log\left|x + \sqrt{x^2 - \frac{1}{4}}\right|\right) + c$$

387. We have,

$$\int \sqrt{x^2 + 2x + 3}\, dx$$

$$= \int \sqrt{(x^2 + 2x + 1) + 2}\, dx$$

$$= \int \sqrt{(x + 1)^2 + (\sqrt{2})^2}\, dx$$

$$= \log\left|(x + 1) + \sqrt{x^2 + 2x + 3}\right| + c$$

388. We have,

$$\int \sqrt{3 - 4x - x^2}\, dx$$

$$= \int \sqrt{-(x^2 + 4x - 3)\, dx}$$

$$= \int \sqrt{-\{(x^2 + 2)^2 - 7\}}\, dx$$

$$= \int \sqrt{7 - (x + 2)^2}\, dx$$

$$= \frac{(x + 2)}{2}\sqrt{3 - 4x - x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x + 2}{\sqrt{7}}\right) + c$$

387. We have,

$$\int \sqrt{x^2 + 2x + 3}\, dx$$

$$= \int \sqrt{(x + 1)^2 + (\sqrt{2})^2}\, dx$$

$$= \left(\frac{x + 1}{2}\right)\sqrt{x^2 + 2x + 3}$$

$$+ \frac{1}{2}\log\left|(x + 1) + \sqrt{x^2 + 2x + 3}\right| + c$$

388. We have,

$$\int \sqrt{3 - 4x - x^2}\, dx$$

$$= \int \sqrt{-(x^2 + 4x - 3)}\, dx$$

$$= \int \sqrt{-\{(x + 2)^2 - 7\}}\, dx$$

$$= \int \sqrt{7 - (x + 2)^2}\, dx$$

$$= \frac{(x + 2)}{2}\sqrt{3 - 4x - x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x + 2}{\sqrt{7}}\right) + c$$

389. We have,

$$\int \sqrt{2ax - x^2}\, dx$$

$$= \int \sqrt{-(x^2 - 2ax)}\, dx$$

$$= \int \sqrt{-\{(x - a)^2 - a^2\}}\, dx$$

$$= \int \sqrt{a^2 - (x - a)^2}\, dx$$

$$= \frac{(x - a)}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x - a}{a}\right) + c$$

390. We have,

$$\int \sqrt{2ax + x^2}\, dx$$

$$= \int \sqrt{(x + a)^2 - a^2}\, dx$$

$$= \frac{(x + a)}{2}\sqrt{x^2 + 2ax}$$

$$+ \frac{a^2}{2}\log\left|(x + a) + \sqrt{x^2 + 2ax}\right| + c$$

391. We have,

$$\int \sqrt{x - 4x^2}\, dx$$

$$= \int \sqrt{\left(\frac{1}{4}\right)^2 - 4\left(x - \frac{1}{8}\right)^2}\, dx$$

$$= \int \sqrt{\left(\frac{1}{4}\right)^2 - 4\left(2x - \frac{1}{4}\right)^2}\, dx$$

$$= \frac{1}{2}\int \sqrt{\left(\frac{1}{4}\right)^2 - t^2}\, dt, \quad \text{Let } \left(2x - \frac{1}{4}\right) = t$$

$$\Rightarrow \quad 2dx = dt$$

$$= \frac{1}{2}\int \sqrt{\left(\frac{1}{4}\right)^2 - t^2}\, dt$$

$$= \frac{t}{4}\sqrt{\frac{1}{16} - t^2} + \frac{1}{32}\sin^{-1}(4t) + c$$

$$= \frac{\left(2x - \frac{1}{4}\right)}{4}\sqrt{\frac{1}{16} - \left(2x - \frac{1}{4}\right)^2}$$

$$+ \frac{1}{32}\sin^{-1}\left[4\left(2x - \frac{1}{4}\right)\right] + c$$

392. We have,

$$\int (2x + 1)\sqrt{x^2 + 3x + 4}\, dx$$

$$= \int ((2x + 3) - 2)\sqrt{x^2 + 3x + 4}\, dx$$

$$= \int ((2x + 3))\sqrt{x^2 + 3x + 4}\, dx$$

$$- 2\int \sqrt{x^2 + 3x + 4}\, dx$$

Let $x^2 + 3x + 4 = t^2$

$\Rightarrow \quad (2x + 3)\, dx = 2t\, dt$

$$= \int (2t)\, t \cdot dt - 2\int \sqrt{\left(x + \frac{3}{2}\right)^2 + \left(4 - \frac{9}{4}\right)}dx$$

$$= 2\int t^2 dt - 2\int\sqrt{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\,dx$$

$$= 2\left(\frac{t^3}{3}\right) - 2\left[\frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x+4}\right.$$

$$\left. + \frac{7}{4}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x+4}\right|\right] + c$$

$$= \frac{2}{3}(x^2+3x+4)^{3/2}$$

$$-2\left[\frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x+4}\right.$$

$$\left. + \frac{7}{4}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x+4}\right|\right] + c$$

392. We have,

$$\int (2x+1)\sqrt{x^2+3x+4}\,dx$$

$$= \int (2x+3-2)\sqrt{x^2+3x+4}\,dx$$

$$= \int (2x+3)\sqrt{x^2+3x+4}\,dx - 2\int x^2+3x+4\,dx$$

Let $x^2 + 3x + 4 = t^2$

$\Rightarrow \quad (2x+3)\,dx = 2t\,dt$

$$= 2\int t^2\,dt - 2\int\sqrt{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\,dx$$

$$= \frac{2(x^2+3x+4)^{3/2}}{3} - \left(x+\frac{3}{2}\right)\sqrt{x^2+3x+4}$$

$$-\left[\frac{7}{4}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x+4}\right|\right] + c$$

393. We have,

$$\int (x-5)\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int (2x-10)\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int ((2x+1)-11)\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int (2x+1)\sqrt{x^2+x}\,dx$$

$$-\frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\,dx$$

and then you do.

394. We have,

$$\int (3x-2)\sqrt{x^2+x+1}\,dx$$

$$= 3\int\left(x-\frac{2}{3}\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int\left(2x-\frac{4}{3}\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int\left(\left(2x+\frac{4}{3}\right)-\frac{7}{3}\right)\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int\left(\left(2x+\frac{4}{3}\right)-\frac{7}{3}\right)\sqrt{x^2+x+1}\,dx$$

$$+\frac{7}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

and then you do.

395. We have,

$$\int (4x+1)\sqrt{x^2-x-2}\,dx$$

$$= 2\int\left(2x+\frac{1}{2}\right)\sqrt{x^2-x-2}\,dx$$

$$= 2\int\left((2x-1)+\frac{3}{2}\right)\sqrt{x^2-x-2}\,dx$$

$$= 2\int\left((2x-1)+\frac{3}{2}\right)\sqrt{x^2-x-2}\,dx$$

$$+ 3\int\sqrt{(x-1)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

and then you do it.

396. We have,

$$\int x\sqrt{1+x-x^2}\,dx$$

$$= \frac{1}{2}\int (2x)\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int (-2x)\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int ((-2x)-1)\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx$$

$$+\frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}\,dx$$

and then you do it.

Partial Fractions

397. We have,

$$\int\frac{(2x+1)dx}{(x+2)(x+3)}$$

$$\Rightarrow \quad \frac{(2x+1)}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$\Rightarrow \quad \frac{(2x+1)}{(x+2)(x+3)} = \frac{A(x+2)+B(x+3)}{(x+2)(x+3)}$$

$\Rightarrow \quad 2x+1 = A(x+3) + B(x+2)$

When $x = -2,\ A = -4 + 1 = -3$

and $x = -3,\ B = -6 + 1 = -5$

Thus, $\displaystyle\int \frac{(2x+1)dx}{(x+2)(x+3)}$

$$= -3\int\frac{dx}{x+2} - 5\int\frac{dx}{x+3}$$

$$= -3\log|x+2| - 5\log|x+3| + c$$

398. We have,

$$\int\frac{dx}{(x-1)(x-2)}$$

$$= \int\left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx$$

$$= \log|x-2| - \log|x-1| + c$$

$$= \log\left|\frac{x-2}{x-1}\right| + c$$

399. We have,

$$\int\frac{dx}{(x+1)(x+2)(x+3)}$$

$$= \frac{1}{2}\int\frac{((x+3)-(x+1))}{(x+1)(x+2)(x+3)}$$

$$= \frac{1}{2}\int\frac{dx}{(x+1)(x+2)} - \frac{1}{2}\int\frac{dx}{(x+2)(x+3)}$$

$$= \frac{1}{2}\int\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx - \frac{1}{2}\int\left(\frac{1}{x+2} - \frac{1}{x+3}\right)dx$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x+2}\right| - \frac{1}{2}\log\left|\frac{x+2}{x+3}\right| + c$$

$$= \frac{1}{2}\log\left|\left(\frac{x+1}{x+2}\right)\left(\frac{x+3}{x+2}\right)\right|$$

400. We have,

$$\int\frac{dx}{(x+1)(x+2)(x+3)(x+4)}$$

$$= \frac{1}{3}\int\frac{((x+4)-(x+1))\,dx}{(x+1)(x+2)(x+3)(x+4)}$$

$$= \frac{1}{3}\int\frac{dx}{(x+1)(x+2)(x+3)}$$

$$-\frac{1}{3}\int\frac{dx}{(x+1)(x+2)(x+3)}$$

$$= -\frac{1}{6}\int\frac{((x+4)-(x+2))\,dx}{(x+2)(x+3)(x+4)}$$

$$\frac{x}{(x^2+k)^{n-1}} + 2(n-1)\int\left(\frac{1}{(x^2+k)^{n-1}} - \frac{k}{(x^2+k)^n}\right)dx$$

$$= \frac{1}{6}\int\frac{dx}{(x+2)(x+3)} - \frac{1}{6}\int\frac{dx}{(x+3)(x+4)}$$

$$+ \frac{1}{6}\int\frac{dx}{(x+2)(x+3)} - \frac{1}{6}\int\frac{dx}{(x+1)(x+2)}$$

$$= \frac{1}{6}\log\left|\left(\frac{x+2}{x+3}\right)\left(\frac{x+4}{x+3}\right)\right|$$

$$+ \frac{1}{6}\log\left|\left(\frac{x+2}{x+3}\right)\left(\frac{x+2}{x+1}\right)\right| + c$$

401. We have,

$$\int\frac{x-1}{(x+1)(x-2)}dx$$

$$= \int\frac{(x+1)-2}{(x+1)(x-2)}dx$$

$$= \int\frac{dx}{(x-2)} - \int\frac{2}{(x+1)(x-2)}dx$$

$$= \int\frac{dx}{(x-2)} - 6\int\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx$$

$$= \log|x-2| - 6\log|x-2| + 6\log|x+1| + c$$

$$= 6\log|x+1| - 5\log|x-2| + c$$

402. We have,

$$\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$$

Now $\displaystyle\frac{(2x-1)}{(x-1)(x+2)(x-3)}$

$$\frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$$

$$(2x-1) = A(x+2)(x-3)$$

$$+B(x-1)(x-3) + C(x-1)(x+2)$$

When $x = 1$, then $A = -1/6$

$x = -2$, then $B = -1/3$

and $x = 3$, then $C = 1/2$

Thus, the given integral reduces to

$$-\frac{1}{6}\int\frac{dx}{(x-1)} + \frac{1}{3}\int\frac{dx}{(x+2)} + \frac{1}{2}\int\frac{dx}{(x+3)}$$

$$= -\frac{1}{6}\log|x-1| + \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x+3| + c$$

403. We have,

$$\int\frac{x^3}{(x-1)(x-2)}dx$$

$$= \int\frac{x^3}{x^2-3x+2}dx$$

$$= \int \left(x - 3 + \frac{7x - 6}{x^2 - 3x + 2} \right) dx$$

$$= \int \left(x - 3 + \frac{7x - 6}{(x-1)(x-2)} \right) dx$$

$$= \frac{x^2}{2} - 3x \int \left(\frac{7x - 6}{(x-1)(x-2)} \right) dx$$

Now, $\frac{7x - 6}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$(7x - 6) = A(x - 2) + B(x - 1)$

When $x = 1$, then $A = -1$

and $x = 2$, then $B = 8$

Thus, the given integral reduces to

$$\left(\frac{x^2}{2} - 3x \right) + \int \left(\frac{-1}{(x-1)} + \frac{8}{(x-2)} \right) dx$$

$$= \left(\frac{x^2}{2} - 3x \right) - \log|(x-1)| + 8\log|x - 2| + c$$

404. We have,

$$\int \frac{2x}{(x^2+1)(x^2+2)} dx$$

Let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

Thus, the given integral reduces to

$$2\int \frac{dt}{(t+1)(t+2)}$$

$$= 2\int \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dt$$

$$= 2\log\left| \frac{(t+1)}{(t+2)} \right| + c$$

$$= 2\log\left| \frac{(x^2+1)}{(x^2+2)} \right| + c$$

405. We have,

$$\int \frac{\cos\theta}{(2+\sin\theta)(3+\sin\theta)} d\theta$$

Let $\sin\theta = t$

$\Rightarrow \cos\theta d\theta = dt$

Thus, the given integral reduces to

$$\int \frac{dt}{(t+2)(t+3)}$$

$$= \int \left(\frac{1}{t+2} - \frac{1}{t+3} \right) dt$$

$$= \log\left| \frac{t+2}{t+3} \right| + c$$

$$= \log\left| \frac{\sin\theta + 2}{\sin\theta + 3} \right| + c$$

406. We have,

$$\int \frac{(1 - \cos x)}{\cos x(1 + \cos x)} dx$$

$$= \int \frac{(1 + \cos x - 2\cos x)}{\cos x(1 + \cos x)} dx$$

$$= \int \frac{dx}{\cos x} - 2\int \frac{dx}{(1 + \cos x)}$$

$$= \int \sec x\,dx - 2\int \left(\frac{1 - \cos x}{\sin^2 x} \right) dx$$

$$= \int \sec x\,dx - 2\int (1 - \text{cosec}^2 x - \text{cosec}\,x \cot x) dx$$

$$= \log|\sec x + \tan x| - 2(\text{cosec}\,x - \cot x) + c$$

407. We have,

$$\int \frac{dx}{\sin x - \sin 2x}$$

$$= \int \frac{dx}{\sin x - 2\sin x \cos x}$$

$$= \int \frac{dx}{\sin x(1 - 2\cos x)}$$

$$= \int \frac{\sin x\,dx}{\sin^2 x(1 - 2\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1 - \cos^2 x)(1 - 2\cos x)}$$

$$= \int \frac{dt}{(1 - t^2)(1 - 2t)}, \text{ where } \cos x = t$$

$$= -\int \frac{dt}{(t-1)(t+1)(2t-1)}$$

Now, $\frac{1}{(t-1)(t+1)(2t-1)}$

$$\frac{A}{(t-1)} + \frac{B}{(t+1)} + \frac{c}{(2t-1)}$$

$-1 = A(t+1)(2t-1) + B(t-1)(2t-1) + c(t^2 - 1)$

Put $t = 1$, then $A = -1/2$

$t = -1$, then $B = 1/10$

$t = 1/2$, then $C = 4/3$

The given integral reduces to

$$-\frac{1}{2}\int \frac{dt}{(t-1)} + \frac{1}{10}\int \frac{dt}{t+1} + \frac{4}{3}\int \frac{dt}{(2t-1)}$$

$$= -\frac{1}{2}\log|t-1| + \frac{1}{10}\log|t+1| + \frac{4}{3}\log|2t-1| + c$$

$$= -\frac{1}{2}\log|\cos x - 1| + \frac{1}{10}\log|\cos x + 1| + \frac{4}{3}\log|2\cos x - 1| + c$$

408. We have,

$$\int \frac{dx}{(x+1)(x+1)^2}$$

Now, $\dfrac{1}{(x+1)(x-1)^2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$

$\Rightarrow$ $\dfrac{1}{(x+1)(x-1)^2}$

$$= \frac{A(x-1)^2 + B(x^2-1) + C(x-1)}{(x+1)(x-1)^2}$$

$\Rightarrow$ $1 = A(x-1)^2 + B(x^2-1) + C(x+1)$

When $x = -1$, $A = 1/4$

$x = 1$, $C = 1/2$

and $x = 0$, $A - B + C = 1$

$\Rightarrow$ $B = 3/4 - 1 = -1/4$

Thus, $\displaystyle\int \frac{dx}{(x+1)(x+1)^2}$

$$= \frac{1}{4}\int \frac{dx}{(x+1)} - \frac{1}{4}\int \frac{dx}{(x-1)} + \frac{1}{2}\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{4}\log|x+1| - \frac{1}{4}\log|x-1| - \frac{1}{2(x-1)} + c$$

409. We have,

$$\int \frac{2x+1}{(x+2)(x-3)^2}\,dx$$

$$\frac{2x+1}{(x+2)(x-3)^2}$$

$$= \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$$

$$\frac{2x+1}{(x+2)(x-3)^2}$$

$$= \frac{A(x-3)^2 + B(x^2-9) + C(x+2)}{(x+2)(x-3)^2}$$

$\Rightarrow$ $2x + 1 = A(x-3)^2 + B(x^2-9) + C(x+2)$

When $x = -2$, $A = -3$

Put $x = 3$, $C = 7/5$

and $x = 0$, $9A - 6B + 2C = 1$

$\Rightarrow$ $6B = 14/5 - 27 - 1$

$\Rightarrow$ $B = -21/5$

Thus, $\dfrac{2x+1}{(x+2)(x-3)^2}\,dx$

$$= -3\int \frac{dx}{x+2} - \frac{21}{5}\int \frac{dx}{x-3} + \frac{7}{5}\int \frac{dx}{(x-3)^2}$$

$$= -3\log|x+2| - \frac{21}{5}\log|x-3| - \frac{7}{5(x-3)} + c$$

415. We have,

$$\int \frac{(2x+3)\,dx}{(x+1)(x^2+4)}$$

Now, $\dfrac{(2x+3)}{(x+1)(x^2+4)} = \dfrac{A}{(x+1)} + \dfrac{Bx+C}{(x^2+4)}$

$\Rightarrow$ $2x + 3 = A(x^2+4) + Bx(x+1) + C(x+1)$

When $x = -1$, $A = 1/5$

and $x = 0$, $4A + C = 3$

$\Rightarrow$ $C = 3 - 4A = 3 - 4/5 = 11/5$

Also $x = 1$, $5A + 2B + 2C = 5$

$\Rightarrow$ $2B = 5 - 5A - 2C = 5 - 1 - 22/5 = -2/5$

$\Rightarrow$ $B = -1/5$

Thus, $\displaystyle\int \frac{(2x+3)\,dx}{(x+1)(x^2+4)}$

$$= \frac{1}{5}\int \frac{dx}{x+1} - \frac{1}{5}\int \frac{x\,dx}{x^2+4} + \frac{11}{5}\int \frac{dx}{x^2+4}$$

$$= \frac{1}{5}\log|x+1| - \frac{1}{10}|x^2+1| + \frac{1}{5}\tan^{-1}(x) + c$$

416. We have,

$$\int \frac{(3x-2)\,dx}{(x-1)(x^2+9)}$$

Now, $\dfrac{(3x-2)}{(x-1)(x^2+9)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+9)}$

$\Rightarrow$ $(3x-2) = A(x^2+9) + Bx(x-1) + C(x-1)$

When $x = 1$, $A = 1/10$

and $x = 0$, $9A - C = -2$

$\Rightarrow$ $C = 2 + 9A = 2 + 9/10 = 29/10$

Also $x = -1,\ 10A + 2B - 2\ C = -5$

$\Rightarrow \quad 2B = 2C - 10A - 5 = 29/5 - 1 - 5$

$\Rightarrow \quad 2B = -1/5$

$\Rightarrow \quad B = -1/10$

Thus, $\int \frac{(3x-2)\,dx}{(x-1)(x^2+9)}$

$$= \frac{1}{10}\int \frac{dx}{x-1} - \frac{1}{10}\int \frac{xdx}{x^2+9} + \frac{29}{10}\int \frac{dx}{x^2+9}$$

$$= \frac{1}{10}\log|x-1| - \frac{1}{20}\log|x^2+9| + \frac{29}{30}\tan^{-1}\left(\frac{x}{3}\right) + c$$

417. We have,

$$\int \frac{2x-1}{(x+1)(x^2+2)}\,dx$$

Now, $\frac{2x-1}{(x+1)(x^2+2)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)}$

$$(2x-1) = A(x^2+2) + Bx(x+1) + C(x+1)$$

When $x = -1$, then $A = -1$

$x = 0$, then $C = 1$

and $x = 1$, then $B = 1$

Thus, the given integral reduces to

$$-\int \frac{dx}{(x+1)} + \int \frac{x}{(x^2+2)}\,dx + \int \frac{dx}{(x^2+2)}$$

$$= -\log|x+1| + \frac{1}{2}\log|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

418. $\int \frac{x}{(x+1)(x^2+4)}\,dx$

Now, $\int \frac{x}{(x+1)(x^2+4)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+4)}$

$$x = A(x^2+4) + Bx(x+1) + C(x+1)$$

Let $x = -1$, then $A = -1/5$

$x = 0$, then $C = 4/5$

and $x = 1$, then $B = 1/5$

Thus, the given integral reduces to

$$-\frac{1}{5}\int \frac{dx}{(x+1)} + \frac{1}{5}\int \frac{x}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{5}\log|(x+1)| + \frac{1}{5}\log|x^2+4| + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + c$$

419. Do yourself

420. Do yourself

421. We have,

$$\int \frac{x^2}{(x^2-1)(x^2+1)}\,dx$$

Put $x^2 = t$

Now, $\frac{t}{(t-1)(t+1)} = \frac{A}{(t-1) + B(t+1)}$

$\Rightarrow \quad \frac{t}{(t-1)(t+1)} = \frac{A(t+1) + B(t-1)}{(t-1)(t+1)}$

$\Rightarrow \quad t = A(t+1) + B(t-1)$

Put $t = 1,\ A = 1/2$

Put $t = -1,\ A = -1/2$

Thus, $\int \frac{x^2}{(x^2-1)(x^2+1)}\,dx$

$$= \frac{1}{2}\int \frac{dx}{x^2-1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}(x) + c$$

426. We have,

$$\int \frac{(x^2+3)(x^2+1)}{(x^2-1)(x^2+2)}dx$$

Put $x^2 = t$

Thus, $\frac{(t+3)(t+1)}{(t-1)(t+2)} = \frac{t^2+4t+c}{t^2+t-2}$

$$= \left(1 + \frac{-3t+5}{t^2+t-2}\right) = \left(1 + \frac{-3t+5}{(t+1)(t-2)}\right)$$

Now, $\frac{-3t+5}{(t+1)(t-2)} = \frac{A}{t-1} + \frac{B}{t-2}$

$\Rightarrow \quad (-3t+5) = A(t-2) + B(t-1)$

When $t = 1$, then $A = -2$

and $t = 2$, then $B = -1$

Therefore, the given integral reduces to

$$\int dx - 2\int \frac{dx}{x^2-1} - \int \frac{dx}{x^2-2}$$

$$= x - \log\left|\frac{x-1}{x+1}\right| - \frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + c$$

427. We have,

$$\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$$

Put $x^2 = t$

Now, $\frac{(t+1)(t+2)}{(t+3)(t+4)} = \frac{t^2+3t+2}{t^2+7t+12}$

$$= \left(1 - \frac{4t + 10}{t^2 + 7t + 12}\right) = \left(1 - \frac{4t + 10}{(t + 3)(t + 4)}\right)$$

Again, $\frac{4t + 10}{(t + 3)(t + 4)} = \frac{A}{(t + 3)} + \frac{B}{(t + 4)}$

$$(4t + 10) = A(t + 4) + B(t + 3)$$

When $t = -4$, then $B = 6$

and $t = -3$, then $A = -2$

Thus, $\int\left(1 + \frac{2}{(t^2 + 2)} - \frac{6}{(t^2 + 4)}\right)dt$

$$= \left(t + \sqrt{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - 3\tan^{-1}\left(\frac{t}{2}\right)\right) + c$$

$$= (x^2 + \sqrt{2}\tan^{-1}\left(\frac{x^2}{\sqrt{2}}\right) - 3\tan^{-1}\left(\frac{x^2}{2}\right) + c$$

428. Do yourself.

Hints and Solutions

Level IIA

1. We have,

$$\int \tan^4 x \cdot \sec^2 x\, dx$$

Let $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$= \int t^4 dt$$

$$= \frac{t^5}{5} + c$$

$$= \frac{(\tan x)^5}{5} + c$$

2. Let $I = \int \sec^3 x\, dx$

$$= \int \sec^2 x \cdot \sec x\, dx$$

$$= \sec x \int \sec^2 x\, dx - \int (\sec x \cdot \tan x)\tan x\, dx$$

$$= \sec x \cdot \tan x - \int (\sec x \cdot \tan^2 x)\, dx$$

$$= \sec x \cdot \tan x - \int (\sec x(\sec^2 x - 1))\, dx$$

$$= \sec x \cdot \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$= \sec x \cdot \tan x - I + \log|\sec x + \tan x| + c$$

$\Rightarrow 2I = \sec x \cdot \tan x + \log|\sec x + \tan x| + c$

$\Rightarrow \frac{1}{2}[\sec x \cdot \tan x + \log|\sec x + \tan x|] + c$

3. Let $I = \int \sec^5 x\, dx$

$$= \int \sec^3 x \cdot \sec^2 x\, dx$$

$$= \sec^3 x \int \sec^2 x\, dx - \int (3\sec^3 x \cdot \tan x) \cdot \tan x\, dx$$

$$= \sec^3 x \cdot \tan x - 3\int \sec^3 x(\sec^2 x - 1)\, dx$$

$$= \sec^3 x \cdot \tan x - 3\int \sec^5 x\, dx + 3\int \sec^3 x\, dx$$

$$= \sec^3 x \cdot \tan x - I$$

$$-\frac{1}{2}(\sec x \cdot \tan x + \log|\sec x + \tan x|) + c$$

$\Rightarrow 2I = \sec^3 x \tan x$

$$-\frac{1}{2}(\sec x \tan x + \log|\sec x + \tan x|) + c$$

$\Rightarrow I = \frac{1}{2}(\sec^3 x \tan x)$

$$-\frac{1}{4}(\sec x \tan x + \log|\sec x + \tan x|) + c$$

4. $\int \tan^2 x \sec^4 x\, dx = \int \tan^4 x(1 + \tan^2 x)\sec^2 x\, dx$

Let $\tan x = t$

$\Rightarrow \sec^2 x dx = dt$

$$= \int t^2(1 + t^2)\, dt$$

$$= \int (t^4 + t^2)\, dt$$

$$= \left(\frac{t^5}{5} + \frac{t^3}{3}\right) + c$$

$$= \left(\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3}\right) + c$$

5. $\int \tan^3 x \sec^6 x\, dx$

$$= \int \tan^3 x(1 + \tan^2 x)^2 \sec^2 x\, dx$$

$= \int t^3(1 + t^2)^2\, dt,$ Let $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$= \int t^3(t^4 + 2t^2 + 1)\, dt$$

$$= \int (t^7 + 2t^5 + t^3)\, dt$$

$$= \left(\frac{t^8}{8} + \frac{t^3}{3} + \frac{t^4}{4}\right) + c$$

$$= \left(\frac{\tan^8 x}{8} + \frac{\tan^3 x}{3} + \frac{\tan^4 x}{4}\right) + c$$

6. $\int \tan^3 x \sec^5 x\, dx$

$= \int \sec^4 x \cdot \tan^2 x \sec x \cdot \tan x\, dx$

$= \int \sec^4 x (\sec^2 x - 1) \sec x \cdot \tan x\, dx$

Let $\sec x = t$

$\Rightarrow \sec x \cdot \tan x dx = dt$

$= \int t^4 (t^2 - 1)\, dt,$

$= \int (t^6 - t^4)\, dt$

$= \left(\frac{t^7}{7} - \frac{t^5}{5}\right) + c$

$= \left(\frac{\sec^7 x}{7} - \frac{\sec^5 x}{5}\right) + c$

7. $\int \sec^4 x\, dx$

8. $\int \sec^9 x\, dx$

9. We have,

$\int \operatorname{cosec}^2 x \cdot \cot^2 x\, dx$

Let $\cot x = t$

$\Rightarrow \operatorname{cosec}^2 x\, dx = -dt$

$= -\int t^2 dx$

$= -\frac{t^3}{3} + c$

$= -\frac{(\cot x)^3}{3} + c$

10. Let $I = \int \cot^3 x \operatorname{cosec}^3 x\, dx$

Let $\operatorname{cosec} x = t$

$\Rightarrow -\operatorname{cosec} x \cdot \cot x\, dx = dt$

$\Rightarrow \operatorname{cosec} x \cdot \cot x\, dx = -dt$

$= \int \cot^3 x \cdot \operatorname{cosec}^3 x\, dx$

$= \int \cot^2 x \cdot \cot x \cdot \operatorname{cosec}^3 x\, dx$

$= \int (\operatorname{cosec}^2 x - 1) \cdot \operatorname{cosec}^2 x (\operatorname{cosec} x \cdot \cot x)\, dx$

$= -\int (t^2 - 1) t^2\, dt$

$= \int (t^4 - t^2)\, dt$

$= \left(\frac{t^5}{5} - \frac{t^3}{3}\right) + c$

$= \left(\frac{\operatorname{cosec}^5 x}{5} - \frac{\operatorname{cosec}^3 x}{3}\right) + c$

11. Let $I = \int \operatorname{cosec}^3 x\, dx$

$= \int \operatorname{cosec}^2 x \cdot \operatorname{cosec} x\, dx$

$= \operatorname{cosec} x \int \operatorname{cosec}^2 x\, dx$

$-\int (-\cot x)(-\operatorname{cosec} x \cdot \cot x)\, dx$

$= -\operatorname{cosec} x \cdot \cot x - \int (\operatorname{cosec}^2 x - 1) \operatorname{cosec} x\, dx$

$= -\operatorname{cosec} x \cdot \cot x - I - \int \operatorname{cosec}^3 x\, dx + \int \operatorname{cose} x\, dx$

$= -\operatorname{cosec} x \cdot \cot x - I + \log|\operatorname{cosec} x - \cot x| + c$

$\Rightarrow 2I = -\operatorname{cosec} x \cot x + \log|\operatorname{cosec} x - \cot x| + c$

$\Rightarrow I = -\frac{1}{2}\operatorname{cosec} x \cot x + \frac{1}{2}\log|\operatorname{cosec} x - \cot x| + c$

12. $\int \cot^2 x \operatorname{cosec}^4 x\, dx$

$= \int \cot^2 x (1 + \cot^2 x) \operatorname{cosec}^2 x\, dx$

Let $\cot x = t$

$\Rightarrow \operatorname{cosec}^2 x\, dx = dt$

$= -\int t^2 (1 + t^2)\, dt,$

$= -\int (t^4 + t^2)\, dt$

$= -\left(\frac{t^5}{5} + \frac{t^3}{3}\right) + c$

$= -\left(\cot^5 \frac{x}{5} + \cot^3 \frac{x}{3}\right) + c$

13. $\int \tan^{-5} x \cdot \sec^6 x\, dx$

$= \int \frac{\sec^6 x}{\tan^5 x} dx$

$= \int \frac{(1 + \tan^2 x)^2 \sec^2 x}{\tan^5 x} dx$

$= \int \left(\frac{(1 + t^2)^2}{t^5}\right) dt$

$= \int \left(\frac{t^4 + 2t^2 + 1}{t^5}\right) dt$

$= \int \left(\frac{1}{t} + \frac{2}{t^3} + \frac{1}{t^5}\right) dt$

$= \left(\log|t| - \frac{1}{t^2} - \frac{1}{4t^4}\right) + c$

$= \left(\log|\tan x| - \frac{1}{\tan^2 x} - \frac{1}{4\tan^4 x}\right) + c$

14. $\int \cot^3 x \cdot \operatorname{cosec}^{-8} x\, dx$

$= \int \frac{\cot^3 x}{\operatorname{cosec}^8 x} dx$

$= \int \frac{\cos^3 x}{\sin^5 x} dx$

$= \int \frac{(1 - \sin^2 x)\cos x}{\sin^5 x}\, dx$

$= \int \left(\frac{1-t^2}{t^5}\right) dt$, Let $t = \cos x$

$dt = -\sin x\, dx$

$= \int \left(\frac{1}{t^5} - \frac{1}{t^3}\right) dt$

$= \left(\frac{1}{2t^2} - \frac{1}{4t^4}\right) + c$

$= \left(\frac{1}{2\sin^2 x} - \frac{1}{4\sin^4 x}\right) + c$

15. $\int \operatorname{cosec}^5 x\, dx$

16. $\int \operatorname{cosec}^7 x\, dx$

17. We have,

$$\int \frac{dx}{x(x^7+1)} = \int \frac{dx}{x^8(1+x^{-7})}$$

Put $1 + x^{-7} = t$

$\Rightarrow -7^{x^{-8}} dx = dt$

$\Rightarrow \frac{dx}{x^8} = -\frac{1}{7} dt$

$= -\frac{1}{7}\int \frac{dt}{t}$

$= -\frac{1}{7}\log|t| + c$

$= -\frac{1}{7}\log|1 + x^{-7}| + c$

18. We have,

$$\int \frac{dx}{x^2(1+x)^{3/4}} = \int \frac{dx}{x^5(1+x^{-4})^{3/4}}$$

Let $1 + x^{-4} = t^4$

$\Rightarrow -4x^{-5} dx = 4t^3 dt$

$\Rightarrow \frac{dx}{x^5} = -t^3 dt$

$= -\int t^3 \frac{dt}{t^3}$

$= -\int dt$

$= -t + c$

$= -(1 + x^{-4})^{1/4} + c$

19. $\int \frac{dx}{x(x^5+1)} = \int \frac{x^4}{x^5(x^5+1)} dx$

$= \frac{1}{4}\int \frac{dt}{t(t-1)}$, Let $x^4 = t$

$4x^3 dx = dt$

$= \frac{1}{4}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$

$= \frac{1}{4}\log\left|\frac{t}{t+1}\right| + c$

$= \frac{1}{4}\log\left|\frac{x^4}{x^4+1}\right| + c$

20. $\int \frac{dx}{x(x^4+1)} = \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + c$

21. $\int \frac{dx}{x^2(x^7+1)^{6/7}} = \int \frac{dx}{x^8\left(1+\frac{1}{x^7}\right)^{6/7}}$

Let $\left(1 + \frac{1}{x^7}\right) = t^7$

$\Rightarrow -\frac{7}{x^6} dx = 7t^6 dt$

$\Rightarrow \frac{dx}{x^6} = -t^6 dt$

$= -\int \frac{t^6\, dt}{t^6}$

$= -t + c$

$= -\left(1 + \frac{1}{x^7}\right)^{1/7} + c$

22. $\int \frac{dx}{x^3(1+x^3)^{1/3}} = \int \frac{dx}{x^4}\left(1 + \frac{1}{x^3}\right)^{1/3}$

Put $\left(1 + \frac{1}{x^3}\right) = t^3$

$\Rightarrow -\frac{3}{x^4} dx = 3t^2 dt$

$\Rightarrow \frac{dx}{x^4} = -t^2 dt$

$= -\int \frac{t^2\, dt}{t}$

$= -\int t\, dt$

$= -\frac{t^2}{2} + c$

$= -\frac{1}{2}\left(1 + \frac{1}{x^3}\right)^{2/3} + c$

23. We have,

$$\int \frac{x^2}{(x+3)^2} dx$$

Let $x + 3 = t$

$\Rightarrow dx = dt$

$= \int \frac{(t-3)^2}{t^2} dt$

$= \int \frac{t^2 - 6t + 9}{t^2} dt$

$= \int \left(1 - \frac{6}{t} + \frac{9}{t^2}\right) dt$

$$= \left(t - 6\log|t| - \frac{9}{t}\right) + c$$

$$= \left((x+3) - 6\log|x+3| - \frac{9}{(x+3)}\right) + c$$

24. We have,

$$\int \frac{x^3\,dx}{(2x+3)^2}$$

Let $2x + 3 = t$

$\Rightarrow$ $2\,dx = dt$

$\Rightarrow$ $dx = 1/2\,dt$

$2x = 3 - t$

$\Rightarrow$ $x = \dfrac{3-t}{2}$

$$= \frac{1}{2}\int \frac{\left(\frac{3-t}{2}\right)^3}{t^2}dt$$

$$= \frac{1}{16}\int \frac{(3-t)^3}{t^2\,dt}$$

$$= \frac{1}{16}\int\left(\frac{(27 - 27t + 9t^2 - t^3)}{t^2}\right)dt$$

$$= \frac{1}{16}\int\left(\frac{27}{t^2} - \frac{27}{t} + 9 - t\right)dt$$

$$= \frac{1}{16}\int\left(-\frac{27}{t} - 27\log|t| + 9t - \frac{t^2}{2}\right) + c$$

where $t = 2x + 3$

25. $\displaystyle\int \frac{x^2}{(x+2)^3}dx$

Let $(x + 2) = t$

$\Rightarrow$ $dx = dt$

$$= \int \frac{(t-2)^2}{t^3}dt$$

$$= \int\left(\frac{t^2 - 4t + 4}{t^3}\right)dt$$

$$= \int\left(\frac{1}{t} - \frac{4}{t^2} + \frac{4}{t^3}\right)dt$$

$$= \log|t| + \frac{4}{t} - \frac{2}{t^2} + c$$

$$= \log|(x-2)| + \frac{4}{(x+2)} - \frac{2}{(x+2)^2} + c$$

26. $\displaystyle\int x^2(ax+b)^2\,dx$

Let $ax + b = t$

$\Rightarrow$ $dx = \dfrac{1}{a}dt$

$\Rightarrow$ $x = t - \dfrac{b}{a}$

$$= \frac{1}{a^2}\int \frac{(t-b)^2}{t^2}dt$$

$$= \frac{1}{a^2}\int \frac{(t^2 - 2bt + b^2}{t^2}dt$$

$$= \frac{1}{a^2}\int (1 - 2b/t + b^2/t^2)\,dt$$

$$= \frac{1}{a^2}\left(t - 2b\log|t| - \frac{b^2}{t}\right) + c$$

$$= \frac{1}{a^2}\left((ax+b) - 2\log|(ax+b)| - \frac{b^2}{(ax+b)}\right) + c$$

27. $\displaystyle\int \frac{dx}{x(1+x^3)^2} = \int \frac{x^2}{x^3(1+x^3)^2}dx$

$$= \frac{1}{3}\int \frac{dt}{t^2(t^3-1)} \text{ where } (x^3+1) = t$$

and then use partial fractions.

28. $\displaystyle\int \frac{x^4}{(3x-2)^3}dx$

Let $3x - 2 = t$

$\Rightarrow$ $dx = \dfrac{dt}{3}$

and $x = \dfrac{t+2}{3}$

$$= \frac{1}{243}\int \frac{(t+2)^4}{t^3}dt$$

$$= \frac{1}{243}\int\left(\frac{t^4 + 8t^2 + 24t^2 + 32t + 15}{t^3}\right)dt$$

$$= \frac{1}{243}\int\left(t + 8 + \frac{24}{t} + \frac{32}{t^2} + \frac{15}{t^3}\right)dt$$

$$= \frac{1}{243}\left(\frac{t^2}{2} + 8t + 24\log|t| - \frac{32}{t} - \frac{15}{2t^2}\right) + c$$

where $t = (3x - 2)$

29. We have,

$$\int \frac{dx}{x^3(3x+2)^3} = \int \frac{dx}{x^5\left(\frac{3x+2}{x}\right)^3}$$

Let $\left(3x + \dfrac{2}{x}\right) = t$

$\Rightarrow$ $\left(3 + \dfrac{2}{x}\right) = t$

$\Rightarrow$ $-\dfrac{2}{x^2}dx = dt$

$\Rightarrow$ $\dfrac{dx}{x^2} = -\dfrac{dt}{2}$

Also, $\dfrac{2}{x} = t - 3$

$\Rightarrow$ $\dfrac{1}{x} = \dfrac{t-3}{2}$

$$= \frac{1}{16}\int \frac{(t-3)^3}{t^3}dt$$

$$= -\frac{1}{16}\int \frac{(t^3 - 3t^2 + 3t - 27)}{t^3}dt$$

$$= -\frac{1}{16}\int \left(1 - \frac{3}{t} + \frac{3}{t^2} - \frac{27}{t^3}\right)dt$$

$$= -\frac{1}{16}\left(t - 3\log|t| - \frac{3}{t} + \frac{27}{2t^2}\right) + c$$

where, $\left(\frac{3x+2}{x}\right) = t$

30. We have,

$$\int \frac{dx}{x^3(b+ax)^2} = \int \frac{dx}{x^5\left(\frac{b+ax}{x}\right)^2}$$

Let $\left(\frac{b+ax}{x}\right) = t$

$\Rightarrow \left(a + \frac{b}{x}\right) = t$

$\Rightarrow -\frac{b}{x^2}dx = dt$

$\Rightarrow -\frac{dx}{x^2} = \frac{dt}{b}$

Also, $\frac{1}{x} = \frac{t-a}{b}$

$$= -\frac{1}{b^4}\int \frac{(t-a)^3}{t^3}dt$$

$$= -\frac{1}{b^4}\int \frac{(t^3 - 3t^2a + 3ta^2 - a^3)}{t^3}dt$$

$$= -\frac{1}{b^4}\int \left(1 - \frac{3a}{t} + \frac{3a^2}{t^2} - \frac{a^3}{t^3}\right)dt$$

$$= -\frac{1}{b^4}\left(t - 3a\log|t| - \frac{3a^2}{t} + \frac{a^3}{2t^2}\right) + c$$

where $\left(a + \frac{b}{x}\right) = t$

31. $\int \frac{dx}{x^2(x+2)^3} = \int \frac{dx}{x^5\left(\frac{x+2}{x}\right)^3}$

Put $\left(\frac{x+2}{x}\right) = t$

$\Rightarrow -\frac{2}{x^2}dx = dt$

$\Rightarrow \frac{dx}{x^2} = -\frac{dt}{2}$

Also, $1 + \frac{2}{x} = t$

$\Rightarrow x = \frac{2}{t-1}$

$$= -\frac{1}{2}\int \frac{dt}{\left(\frac{2}{t-1}\right)^3 t^3}$$

$$= -\frac{1}{16}\int \frac{(t-1)^3}{t^3}dt$$

$$= -\frac{1}{16}\int \left(\frac{t^3 - 3t^2 + 3t - 1}{t^3}\right)dt$$

$$= -\frac{1}{16}\int \left(1 - \frac{3}{t} + \frac{3}{t^2} - \frac{1}{t^3}\right)dt$$

$$= -\frac{1}{16}\left(t - 3\log|t| - \frac{3}{t} + \frac{1}{2t^2}\right) + c$$

where $t = \left(x + \frac{2}{x}\right)$

32. $\int \frac{dx}{x^3(a+bx)^2} = \int \frac{dx}{x^5\left(\frac{a+bx}{x}\right)^2}$

Let $\left(\frac{a+bx}{x}\right) = t$

$\Rightarrow -\frac{a}{x^2}dx = dt$

$\Rightarrow \frac{dx}{x^2} = -\frac{dt}{a}$

Also, $x = \frac{a}{t-b}$

$$= -\frac{1}{a}\int \frac{dt}{\left(\frac{a}{t-b}\right)^3 t^2}$$

$$= -\frac{1}{a^4}\int \frac{(t-b)^3}{t^2}dt$$

$$= -\frac{1}{a^4}\int \left(\frac{t^3 - 3t^2b + 3tb^2 - b^2}{t^2}\right)dt$$

$$= -\frac{1}{a^4}\int \left(t - 3b + \frac{3b^2}{t} - \frac{b^3}{t^2}\right)dt$$

$$= -\frac{1}{a^4}\left(\frac{t^2}{2} - 3bt + 3b^2\log|t| + \frac{b^2}{t}\right) + c$$

where $\left(\frac{a+ax}{x}\right) = t$

33. $\int \frac{dx}{x^2(1+x^2)^3}$

34. $\int \frac{dx}{x^2(a-bx)^2} = \int \frac{dx}{x^2}\left(\frac{a-bx}{x}\right)^2$

Let $\left(\frac{a-bx}{x}\right) = t$

$\Rightarrow \frac{a}{x^2}dx = -dt$

$$\Rightarrow \quad \frac{dx}{x^2} = -\frac{dt}{a}$$

Also, $\frac{a}{x} = t + b$

$$\Rightarrow \quad x = \frac{a}{t+b}$$

$$= -\frac{1}{a}\int \frac{dt}{\left(\frac{a}{t+b}\right)^2 t^2}$$

$$= -\frac{1}{a^3}\int \frac{(t+b)^2\,dt}{t^2}$$

$$= -\frac{1}{a^3}\int \frac{(t^2 + 2bt + b^2)}{t^2}dt$$

$$= -\frac{1}{a^3}\int\left(1 + \frac{2b}{t} + \frac{b^2}{t^2}\right)dt$$

$$= -\frac{1}{a^3}\left(t + 2b\log|t| - \frac{b^2}{t}\right) + c$$

where $t = \left(\frac{a - bx}{x}\right)$

35. $\int \frac{dx}{x^4(2x+1)^2} = \int \frac{dx}{x^7\left(\frac{2x+1}{x}\right)^3}$

Let $\left(\frac{2x+1}{x}\right) = t$

$$\Rightarrow \quad \frac{dx}{x^2} = -dt$$

Also, $2 + \frac{1}{x} = t$

$$\Rightarrow \quad x = \frac{1}{t-2}$$

$$= -\int \frac{dt}{\left(\frac{1}{t-2}\right)^5 t^3}$$

$$= -\int \frac{(t-2)^5\,dt}{t^3}$$

$$= \int \frac{(2-t)^5\,dt}{t^3}$$

$$= \int\left(\frac{32 - 80t + 80t^2 - 40t^3 + 10t^4 - t^5}{t^3}\right)dt$$

$$= \int\left(\frac{32}{t^3} - \frac{80}{t^2} + \frac{80}{t} - 40 + 10t - t^2\right)dt$$

$$= -\frac{16}{t^2} + \frac{80}{t} + 80\log|t| - 40t + 5t^2 - \frac{t^3}{3} + c$$

where $t = \frac{2x+1}{x}$

36. We have,

$$\int \frac{dx}{(x-1)^3(x-2)^4}$$

Let $\left(\frac{x-1}{x-2}\right) = t$

$$\Rightarrow \quad \frac{(x-2)\cdot 1\,(x-1)\cdot 1}{(x-2)^2}\,dx = dt$$

$$\Rightarrow \quad \frac{-dx}{(x-2)^2} = dt$$

Also, $x = \frac{2t-1}{t-1}$

$$\Rightarrow \quad (x-2) = \frac{1}{t-1}$$

Thus, $\int \frac{dx}{(x-1)^3(x-2)^4}$

$$= \int \frac{dx}{\left(\frac{x-1}{x-2}\right)^2 (x-2)^7}$$

$$= -\int \frac{dt}{t^3} \times (t-1)^5$$

$$= -\int \frac{(t-1)^5}{t^3}dt$$

$$= -\int \frac{(t^5 - 5t^4 + 10t^3 - 10t^2 + 5t - t^5)}{t^3}dt$$

$$= -\int\left(t^2 - 5t + 10 - \frac{10}{t} + \frac{5}{t^2} - \frac{1}{t^3}\right)dt$$

$$= -\int\left(\frac{t^3}{3} - \frac{5t^2}{2} + 10t - 10\log|t| - \frac{5}{t} + \frac{1}{2t^2}\right) + c$$

where, $t = \left(\frac{x-1}{x-2}\right)$

37. We have,

$$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$$

$$= \int \frac{dx}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}$$

Let $\frac{x-1}{x+2} = t$

$$\Rightarrow \quad \frac{dx}{(x+2)^2} = dt$$

Thus, $\frac{1}{3}\int \frac{dt}{t^{3/4}}$

$$= \frac{1}{3}\int t^{-3/4}\,dt$$

$$= \frac{1}{3}(4t^{1/4}) + c$$

$$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$$

38. $\int \frac{dx}{(x-1)^3(x-2)^2}$

$= \int \frac{dx}{(x-1)^5\left(\frac{x+2}{x-1}\right)^2}$

Put $\left(\frac{x+2}{x-1}\right) = t$

$\Rightarrow \frac{(x-1)\cdot 1 - (x+2)\cdot 1}{(x-1)^2} dx = dt$

$\Rightarrow -\frac{3}{(x-1)^2} dx = dt$

$\Rightarrow \frac{dx}{(x-1)^2} = \frac{dt}{-3}$

Also, $\left(\frac{x+2}{x-1}\right) = t$

$\Rightarrow x + 2 = tx - t$

$\Rightarrow x = \frac{t+2}{t-1}$

$\Rightarrow x - 1 = \frac{t+2}{t-1} - 1 = \frac{3}{(t-1)}$

$= -\frac{1}{3}\int \frac{dt}{\left(\frac{3}{t-1}\right)^3 t^2}$

$= -\frac{1}{3^4}\int \frac{(t-1)^3\,dt}{t^2}$

$= -\frac{1}{3^4}\int \frac{(t^3 - 3t^2 + 3t - 1)\,dt}{t^2}$

$= -\frac{1}{3^4}\int \left(t - 3 + \frac{3}{t} - \frac{1}{t^2}\right)dt$

$= -\frac{1}{3^4}\left(\frac{t^2}{2} - 3t + 3\log|t| + \frac{1}{t}\right) + c$

where $t = \frac{x+2}{x-1}$

39. $\int \frac{dx}{(x-3)^4(x-2)^5} = \int \frac{dx}{(x-2)^9\left(\frac{x-3}{x-2}\right)^4}$

Put $\left(\frac{x-3}{x-2}\right) = t$ and then you do it

40. $\int \frac{dx}{(x-3)^{3/2}(x-2)^{7/2}} = \int dx\left(\frac{x-3}{x-2}\right)^{3/2}(x-2)^5$

Put $(x - 3/x - 2) = t^2$ and then you do it.

41. $\int \frac{dx}{\sqrt{(x-2)^5(x-5)^7}} = \int \frac{dx}{(x-5)^6\left(\frac{x-2}{x-6}\right)^{5/2}}$

Put $\left(\frac{x-2}{x-6}\right) = t^2$ and then you do it.

42. $\int \frac{dx}{\sqrt[5]{(x+1)^4(x+3)^6}} = \int \frac{dx}{(x+1)^{4/5}(x+3)^{6/5}}$

$= \int \frac{dx}{(x+3)^2\left(\frac{x+1}{x+3}\right)^{4/5}}$

Put $\left(\frac{x+1}{x+3}\right) = t^5$ and then you do it.

43. We have,

$\int \frac{dx}{x(2+3x^3)} = \int \frac{x^2\,dx}{x^3(2+3x^3)}$

$= \frac{1}{3}\int \frac{3x^2\,dx}{x^3(2+3x^3)}$

$= \frac{1}{3}\int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\left(2+\frac{3}{t}\right)}$

$= -\frac{1}{3}\int \frac{dt}{2t+3}$

$= -\frac{1}{6}\int \log|2t+3| + c$

$= -\frac{1}{6}\log\left|\frac{2}{x^3} + 3\right| + c$

44. We have,

$\int \frac{dx}{x(3+5x^5)} = \frac{1}{5}\int \frac{5x^4\,dx}{x^5(3+5x^5)}$

Let $x^5 = \frac{1}{t}$

$\Rightarrow 5x^4\,dx = -\frac{dt}{t^2}$

$= -\frac{1}{5}\int \frac{dt}{(3t+5)}$

$= -\frac{1}{15}\log|3t+5| + c$

$= -\frac{1}{15}\log\left|\frac{3}{x^5} + 5\right| + c$

45. $\int \frac{dx}{x(2+3x^2)} = \int \frac{x\,dx}{x^2(2+3x^2)}$

Let $x^2 = t$

$\Rightarrow x\,dx = \frac{1}{2}dt$

$= \frac{1}{2}\int \frac{dt}{t(2+3t)}$

$= \frac{1}{4}\int \left(\frac{1}{t} - \frac{3}{2+3t}\right)dt$

$$= \frac{1}{4}(\log|t| - \log|2 + 3t|) + c$$

$$= \frac{1}{4}\left(\log\left|\frac{t}{2 + 3t}\right|\right) + c$$

$$= \frac{1}{4}\left(\log\left|\frac{x^2}{2 + 3x^2}\right|\right) + c$$

46. $$\int \frac{dx}{x(3 + 4x^3)} = \frac{1}{3}\int \frac{3x^2 dx}{x^3(3 + 4x^3)}$$

Let $x^3 = t$

$\Rightarrow \quad 3x^2\, dx = dt$

$$= \frac{1}{3}\int \frac{dt}{t(3 + 4t)}$$

$$= \frac{1}{9}\int\left(\frac{1}{t} - \frac{4}{3} + 4t\right)dt$$

$$= \frac{1}{9}(\log|t| - \log|3 + 4t|) + c$$

$$= \frac{1}{9}\left(\log\left|\frac{t}{3 + 4t}\right|\right) + c$$

$$= \frac{1}{9}\left(\log\left|\frac{x^3}{3 + 4x^3}\right|\right) + c$$

47. $$\int \frac{dx}{x(2 - 5x^3)} = \int \frac{3x^2 dx}{x^3(2 - 5x^3)}$$

Let $x^3 = t$

$\Rightarrow \quad 3x^2\, dx = dt$

$$= \frac{1}{3}\int \frac{dt}{t(2 - 5t)}$$

$$= -\frac{1}{3}\int \frac{dt}{t(5t - 2)}$$

$$= \frac{1}{6}\int\left(\frac{1}{t} - \frac{5}{5t} - 2\right)dt$$

$$= \frac{1}{6}\left(\log\left|\frac{t}{5t - 2}\right|\right) + c$$

$$= \frac{1}{6}\left(\log\left|\frac{x^3}{5x^3 - 2}\right|\right) + c$$

48. $$\int \frac{dx}{x(1 - 4x^4)} = \frac{1}{4}\int \frac{4x^3 dx}{x^4(1 - 4x^4)}$$

$$= \frac{1}{4}\int \frac{dt}{t(1 - 4t)}$$

$$= -\frac{1}{4}\int \frac{dt}{t(4t - 1)}$$

$$= \frac{1}{4}\int\left(\frac{1}{t} - \frac{4}{4t - 1}\right)dt$$

$$= \frac{1}{4}\log\left|\frac{t}{4t - 1}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{x^3}{4x^3 - 1}\right| + c$$

49. $$\int \frac{dx}{x(3x^4 + 1)} = \frac{1}{4}\int \frac{4x^3 dx}{x^4(3x^4 + 1)}$$

$$= \frac{1}{4}\int \frac{dt}{t(3t + 1)}$$

$$= \frac{1}{4}\int\left(\frac{1}{t} - \frac{3}{3t + 1}\right)dt$$

$$= \frac{1}{4}\log\left|\frac{t}{3t + 1}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{x^4}{3x^4 + 1}\right| + c$$

50. We have,

$$\int x^5 (x^2 - 1)^4 dx$$

Let $(x^2 - 1) = t$

$\Rightarrow \quad 2x\,dx = dt$

Also, $x^2 = t + 1$

Thus, $$\int \frac{x^5}{(x^2 - 1)^4} dx = \frac{1}{2}\int \frac{(x^2)^2 \cdot 2x}{(x^2 - 1)^4} dx$$

$$= \frac{1}{2}\int \frac{(t - 1)^2}{t^3} dt$$

$$= \frac{1}{2}\int\left(\frac{t^2 - 2t + 1}{t^3}\right)dt$$

$$= \frac{1}{2}\int\left(\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3}\right)dt$$

$$= \frac{1}{2}\left(\log|t| + \frac{2}{t} - \frac{1}{2t^2}\right) + c$$

$$= \frac{1}{2}\left(\log|x^2 - 1| + \frac{2}{(x^2 - 1)} - \frac{1}{2(x^2 - 1)^2}\right) + c$$

51. We have,

$$\int \frac{x^9}{(2x^2 + 3)^5 dx} = \frac{1}{4}\int \frac{(x^2)^4 4x\,dx}{(2x^2 + 3)^5}$$

Let $2x^2 + 3 = t$

$\Rightarrow \quad 4x\,dx = dt$

Thus,

$$\frac{1}{4}\int \frac{\left(\frac{t - 3}{2}\right)^4 dt}{t^5} = \frac{1}{64}\int \frac{(t - 3)^4 dt}{t^5}$$

$$= \frac{1}{64}\int \frac{(t^4 - 12t^3 + 54t^2 - 108t + 81}{t^5} dt$$

$$= \frac{1}{64}\int\left(\frac{1}{t} - \frac{12}{t^2} + \frac{54}{t^3} - \frac{108}{t^4} + \frac{81}{t^5}\right)dt$$

$$= \frac{1}{64}\left(\log|t| + \frac{12}{t} - \frac{27}{t^2} + \frac{36}{t^3} - \frac{81}{4t^4}\right) + c$$

where $2x^2 + 3 = t$

52. $\int \frac{x^3}{(x^2+1)^4}\,dx$

Let $(x^2 + 1) = t$

$\Rightarrow \quad 2x\,dx = dt$

$$= \frac{1}{2}\int \frac{x^2(2x)\,dx}{(x^2+1)^4}$$

$$= \frac{1}{2}\int \frac{(t-1)}{t^4\,dt}$$

$$= \frac{1}{2}\int \left(\frac{1}{t^3} - \frac{1}{t^4}\right)dt$$

$$= \frac{1}{2}\left(-\frac{1}{2t^2} + \frac{1}{3t^3}\right) + c$$

$$= \frac{1}{2}\left(-\frac{1}{2(x^2+1)^2} + \frac{1}{3(x^2+1)^3}\right) + c$$

53. $\int \frac{x^5}{(x^2-3)^4}\,dx$

Let $(x^2 - 3) = t$

$\Rightarrow \quad 2xdx = dt$

$$= \frac{1}{2}\int \frac{(x^2)^2(2x)\,dx}{(x^2-3)^4}$$

$$= \frac{1}{2}\int \frac{(t+3)^2\,dt}{t^4}$$

$$= \frac{1}{2}\int \frac{(t^2+6t+9)\,dt}{t^4}$$

$$= \frac{1}{2}\int \left(t^2 + \frac{6}{t^3} + \frac{9}{t^3}\right)dt$$

$$= \frac{1}{2}\left(\frac{t^3}{3} - \frac{3}{t^2} - \frac{3}{t^3}\right) + c$$

$$= \frac{1}{2}\left(\frac{(x^2-3)^3}{3} - \frac{3}{(x^2-3)^2} - \frac{3}{(x^2-3)^3)}\right) + c$$

54. $\int \frac{x^7}{(3x^2-2)^4}\,dx$

$$= \frac{1}{6}\int \frac{(x^2)^3(6x)\,dx}{(3x^2-2)^4}$$

$$= \frac{1}{6}\int \frac{\left(\frac{t+2}{3}\right)^3 dt}{t^4}$$

Let $3x^2 - 2 = \mathrm{t}$

$\Rightarrow \quad 6xdx = dt$

$$= \frac{1}{162}\int \frac{(t+2)^3\,dt}{t^4}$$

$$= \frac{1}{162}\int \frac{(t^3+6t^2+12t+8)\,dt}{t^4}$$

$$= \frac{1}{162}\int \left(\frac{1}{t} + \frac{6}{t^2} + \frac{12}{t^3} + \frac{8}{t^4}\right)dt$$

$$= \frac{1}{162}\left(\log|t| - \frac{6}{t} - \frac{6}{t^2} - \frac{2}{t^3}\right) + c$$

where $t = 3x^2 - 2$

55. $\int \frac{8x^6}{(3x^2-2)^5}\,dx$

$$= \frac{4}{3}\int \frac{(x^2)^4(6x)\,dx}{(3x^2-2)^5}$$

$$= \frac{4}{3}\int \frac{\left(\frac{t+2}{3}\right)^4 dt}{t^5}$$

Let $3x^2 - 2 = t$

$\Rightarrow \quad 6xdx = dt$

$$= \frac{4}{243}\int \frac{(t+2)^4\,dt}{t^5}$$

$$= \frac{4}{243}\int \left(\frac{t^4+8t^3+24t^2+32t+16}{t^5}\right)dt$$

$$= \frac{4}{243}\int \left(t + \frac{8}{t^2} + \frac{24}{t^3} + \frac{32}{t^4} + \frac{16}{t^5}\right)dt$$

$$= \frac{4}{243}\left(\frac{t^2}{2} - \frac{8}{t} - \frac{12}{t^2} - \frac{32}{2t^2} - \frac{4}{t^4}\right) + c$$

where $t = 3x^2 - 2$

56. Do yorself.

57. We have,

$$\int \frac{2\sin x + 3}{(3\sin x + 2)^2}\,dx$$

$$= \int \frac{2\sec x\tan x + 3\sec^2 x}{(3\tan x + 2\sec x)^2}\,dx$$

Let $2\sec x + 3\tan x = t$

$\Rightarrow \quad (2\sec x\tan x + 3\sec^2 x)\,dx = dt$

$$= \int \frac{dt}{t^2}$$

$$= -\frac{1}{t} + c$$

$$= -\frac{1}{(2\sec x + 3\tan x)} + c$$

58. Do yourself

59. Do yourself

60. We have,

$$\int\left(\frac{4\cos x+3}{(3\cos x+4)^2}\right)dx$$

$$=\int\left(\frac{(4\operatorname{cosec}x\cot x+3\operatorname{cosec}^2x}{(3\cot x+4\operatorname{cosec}x)^2}\right)dx$$

Let $3\cot x+4\operatorname{cosec}x=t$

$\Rightarrow -(3\operatorname{cosec}^2x+4\operatorname{cosec}x\cot x)\,dx=dt$

$$=-\int\frac{dt}{t^2}$$

$$=\frac{1}{t}+c$$

$$=\frac{1}{(3\cot x+4\operatorname{cosec}x)}+c$$

61. Do yorself.

62. Do yorself.

63. We have,

$$\int\frac{dx}{(3+4\sin x)^2}$$

Let $\dfrac{4+3\sin x}{3+4\sin x}=t$

$$\Rightarrow \frac{dx}{(3+4\sin x)^2}=-\frac{dt}{\cos x}$$

Also, $\sin x=\dfrac{3t-4}{3-4t}$

$$\Rightarrow \cos x=\frac{\sqrt{7}\sqrt{t^2-1}}{(3-4t)}$$

Thus,

$$\int\frac{dx}{(3+4\sin x)^2}=-\frac{1}{7}\int\frac{dt}{\cos x}$$

$$=-\frac{1}{7\sqrt{7}}\int\frac{3-4t}{\sqrt{t^2-1}}dt$$

$$=-\frac{1}{7\sqrt{7}}(3\log(t+\sqrt{t^2-1})-4\sqrt{t^2-1})+c$$

where, $\dfrac{4+3\sin x}{3+4\sin}x=t$

64. Do yorself.

65. Do yorself.

66. $\displaystyle\int\frac{dx}{(2+3\cos x)^2}=t$

Let $\left(\dfrac{dx}{(2+3\cos x)^2}\right)=t$

$$\Rightarrow \frac{5\sin x}{(2+3\cos x)^2}dx=dt$$

$$\Rightarrow \frac{dx}{(2+3\cos x)^2}=\frac{dt}{5\sin x}$$

Also, $\cos x=\dfrac{3-2t}{3t-2}$

$$\Rightarrow \sin x=\frac{\sqrt{5}\sqrt{t^2-1}}{(3t-2)}$$

$$=\frac{1}{5}\int\frac{dt}{\sin x}$$

$$=\frac{1}{5\sqrt{5}}\int\frac{(3t-2)}{\sqrt{t^2-1}}dt$$

$$=\frac{1}{5}\sqrt{5}\left(3\sqrt{t^2-1}-2\log\left|t+\sqrt{t^2-1}\right|\right)+c$$

where, $t=\left(\dfrac{2\cos x+3}{2+3\cos x}\right)$

67. Do yorself.

68. Do yorself.

69. Let $4e^x+6e^{-x}=l(9e^x-6^{-x})+m(9e^x+6e^{-x})$

$\Rightarrow 4e^x+6e^{-x}=e^x(9l+9m)+e^{-x}(-6l+6m)$

$\Rightarrow 9(l+m)=4$ and $-6(l-m)=6$

$\Rightarrow l=-\dfrac{5}{18},\ m=\dfrac{13}{18}$

Thus, $\displaystyle\int\left(\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\right)dx$

$$=-\frac{5}{18}\int dx+\frac{13}{18}\int\frac{9e^x+6e^{-x}}{(9e^x-6e^{-x})}dx$$

$$=-\frac{5}{18}x+\frac{13}{18}\log|(9e^x-6e^{-x})|+c$$

70. Do yorself.

71. Do yorself.

72. We have,

$$\int\frac{dx}{\sqrt{(x+2)^5(x+1)^3}}$$

$$=\int\frac{dx}{\sqrt{(x+2)^8\left(\frac{x+1}{x+2}\right)^3}}$$

$$=\int\frac{dx}{(x+2)^4\left(\frac{x+1}{x+2}\right)^{3/2}}$$

Put $\left(\dfrac{x+1}{x+2}\right)=t^2$

$$\Rightarrow \frac{dx}{(x+2)^2}=2t\,dt$$

Also, $x = \dfrac{2t^2 - 1}{1 - t^2}$

$\Rightarrow \quad x + 2 = \dfrac{2t^2 - 1}{1 - t^2} + 2 = \dfrac{1}{1 - t^2}$

Thus, $\displaystyle\int \frac{dx}{(x+2)^4\left(\frac{x+1}{x+2}\right)^{3/2}}$

$= \displaystyle\int \frac{(1-t^2)^4}{t^3} \times 2t\,dt$

$= 2\displaystyle\int \left(\frac{1 - 4t^2 + 6t^4 - 4t^6 + t^8)}{t^2}\right) dt$

$= 2\displaystyle\int \left(\frac{1}{t^2} - 4 + 6t^2 - 4t^4 + t^6\right) dt$

$= 2\left(-\dfrac{1}{t} - 4t + 2t^3 - \dfrac{4}{5t^5} + \dfrac{t^7}{7}\right) + c$

where $t = \sqrt{\dfrac{x+1}{x+2}}$

73. $\displaystyle\int \left(\frac{x+2}{2x+3}\right)^{1/2} \cdot \frac{dx}{x}$

Put $\dfrac{x+2}{2x+3} = t^2$

$\Rightarrow \quad \dfrac{(2x + 3 - 2x - 4)}{(2x+3)^2}\, dx = 2t\,dt$

$\Rightarrow \quad \dfrac{dx}{(2x+3)^2} = -2t\,dt$

$\Rightarrow \quad \dfrac{-dx}{(1-2t^2)^2} = -2t\,dt$

$\Rightarrow \quad \dfrac{dx}{(2x^2-1)^2} = 2t\,dt$

$\Rightarrow \quad dx = 2t(2t^2-1)^2\,dt$

Also, $x = \dfrac{3t^2 - 2}{1 - 2t^2}$

$= \displaystyle\int t \cdot \left(\frac{1-2t^2}{3t^2-2}\right) 2t(2t^2-1)^2\,dt$

$= 2\displaystyle\int \frac{t^2(1-2t^2)^3}{3t^2 - 2)}\, dt$

$= 2\displaystyle\int \frac{t^2(1 - 6t^2 + 12t^4 - 8t^6)}{(3t^2-2)}\, dt$

$= 2\displaystyle\int \frac{(t^2 - 6t^4 + 12t^6 - 8t^4)}{(3t^2-2)}\, dt$

$= -2\displaystyle\int \left(\frac{8t^8 - 12t^6 + 6t^4 - t^2}{(3t^2-2)}\right) dt$

and then you do it

74. $\displaystyle\int \frac{(x+1)}{(x+2)(x+3)^{3/2}}\, dx$

$= \displaystyle\int \frac{(x+2) - 1}{(x+2)(x+3)^{3/2}}\, dx$

$= \displaystyle\int \frac{dx}{(x+3)^{3/2}} - \int \frac{dx}{(x+2)(x+3)^{3/2}}$

$= \displaystyle\int \frac{dx}{(x+3)^{3/2}} - \int \frac{dx}{\left(\frac{x+2}{x+3}\right)(x+3)^2}$

Put $\left(\dfrac{x+2}{x+3}\right) = t$

$\Rightarrow \quad \left(\dfrac{(x+3)\cdot 1 - (x+2)\cdot 1}{(x+3)^2}\right) \cdot dx = dt$

$\Rightarrow \quad \dfrac{dx}{(x+3)^2} = dt$

$= -\dfrac{2}{\sqrt{x+3}} - \displaystyle\int \frac{dt}{t}$

$= -\dfrac{2}{\sqrt{x+3}} - \log\left|\dfrac{x+2}{x+3}\right| + c$

75. $\displaystyle\int \frac{2}{(2-x)^2} \sqrt[3]{\frac{2-x}{2+x}}\, dx = 2\int \frac{dx}{(2-x)^{5/3}(2 + x/2 - x)^{1/3}}$

$= 2\displaystyle\int \frac{dx}{(2-x)^2\left(\frac{2+x}{2-x}\right)^{1/3}}$

Put $\left(\dfrac{2+x}{2-x}\right) = t$

$\Rightarrow \quad \dfrac{(2-x)\cdot 1 + (2+x)\cdot 1}{(2-x)^2}\, dx = dt$

$\Rightarrow \quad \dfrac{dx}{(2-x)^2} = \dfrac{dt}{4}$

$= \dfrac{1}{2}\displaystyle\int \frac{dt}{t^{1/3}}$

$= \dfrac{1}{3} t^{2/3} + c$

$= \dfrac{1}{3}\left(\dfrac{2+x}{2-x}\right)^{2/3} + c$

76. $\displaystyle\int \frac{dx}{\sqrt{x+1} + \sqrt[3]{x+1}}$

Let $x + 1 = t^6$

$\Rightarrow \quad dx = 6t^5\,dt$

$$= \int \frac{6t^5\,dt}{t^3 + t^2}$$

$$= 6\int \frac{t^3}{t + 1}\,dt$$

$$= 6\int \frac{(t^3 + 1) - 1}{t + 1}\,dt$$

$$= 6\int \frac{(t^3 + 1)}{t + 1}\,dt - 6\int \frac{dt}{t + 1}$$

$$= 6\int \frac{(t + 1)(t^2 - t + 1)}{t + 1}\,dt - 6\int \frac{dt}{t + 1}$$

$$= 6\int ((t^2 - t + 1) - \frac{1}{t + 1})\,dt$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t + 1|\right) + c$$

where $t = \sqrt[6]{x + 1}$

77. $\int \frac{dx}{\sqrt{x + 1} - (x + 1)^{1/4}}$

Let $(x + 1) = t^4$

$\Rightarrow \quad dx = 4t^3\,dt$

$$= \int \frac{4t^3\,dt}{t^2 - t}$$

$$= 4\int \frac{t^2\,dt}{t - 1}$$

$$= 4\int \left(t + 1 + \frac{1}{t - 1}\right)dt$$

$$= 4\left(\frac{t^2}{2} + t + \log|t - 1|\right) + c$$

$$= 4\left(\frac{(x + 1)^2}{2} + (x + 1) + \log|x|\right) + c$$

78. $\int \frac{1 + x^{1/2} - x^{1/3}}{1 + x^{1/3}}\,dx$

Let $x = t^6$

$\Rightarrow \quad dx = 6t^5\,dt$

$$= 6\int \left(\frac{1 + t^3 - t^2}{1 + t^2}\right)t^5\,dt$$

$$= 6\int \left(\frac{t^8 - t^7 + t^5}{1 + t^2}\right)dt$$

$$= 6\int \left(t^6 - t^5 - t^4 + t^3 - t^2 - t - 1 + \frac{t + 1}{t^2 + 1}\right)dt$$

$$= 6\left[\frac{t^7}{7} - \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} - \frac{t^2}{2} - t + \frac{1}{2}\log|t^2 + 1| + \tan^{-1}(t)\right] + c$$

where $t = \sqrt[6]{x}$

79. $\int \frac{dx}{x^{1/2} + x^{1/3}}$

Let $x = t^6$

$\Rightarrow \quad dx = 6t^5\,dt$

$$= 6\int \frac{t^5}{t^3 + t^2}\,dt$$

$$= 6\int \frac{t^3}{t + 1}\,dt$$

$$= 6\int \left(\frac{(t^3 + 1) - 1}{t + 1}\right)dt$$

$$= 6\int \left((t^2 - t + 1) - \frac{1}{t + 1}\right)dt$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t + 1|\right) + c$$

where $t = (x)^{1/6}$

80. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3} + 1}\,dx$

Put $x = t^4$

$\Rightarrow \quad dx = 4t^3\,dt$

$$= 4\int \frac{t^2 \times t^3}{t^3 + 1}\,dt$$

$$= 4\int \frac{t^5}{t^3 + 1}\,dt$$

$$= 4\int \left(t^2 - \frac{t^2}{t^3 + 1}\right)dt$$

$$= 4\int \left(\frac{t^3}{3} - \frac{1}{3}\log|t^3 + 1|\right) + c$$

where $t = x^{1/4}$

81. $\int \frac{\sqrt{x}}{\sqrt{x} + \sqrt[3]{\sqrt{x}}}\,dx$

Let $x = t^6$

$\Rightarrow \quad dx = 6t^5\,dt$

$$= 6\int \frac{t^3 \times t^5\,dt}{t^3 + t^2}$$

$$= 6\int \frac{t^6\,dt}{t + 1}$$

$$= 6\int \left(\frac{(t^6 - 1) + 1}{t + 1}\right)dt$$

$$= 6\int \left(1 + t + t^2 + t^3 + t^4 + t^5 + \frac{1}{t + 1}\right)dt$$

$$= 6\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \frac{t^5}{5} + \frac{t^6}{6} + \log|t + 1|\right) + c$$

where $t = x^{1/6}$

82. $\int \frac{dx}{(x-1)\sqrt{x+3}}$

Let $x - 3 = t^2$

$\Rightarrow dx = 2t\,dt$

Also, $x - 1 = t^2 - 3 - 1 = t^2 - 4$

$$= \int \frac{2t\,dt}{(t^2-4)t}$$

$$= 2\int \frac{dt}{(t^2 - 2^2)}$$

$$= \frac{1}{2}\log\left|\frac{t-2}{t+2}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x+3}-2}{\sqrt{x+3}+2}\right| + c$$

83. $\int \frac{dx}{(x+3)\sqrt{x+2}}$

Let $x + 2 = t^2$

$\Rightarrow dx = 2t\,dt$

$$= 2\int \frac{t\,dt}{(t^2+1)t}$$

$$= 2\int \frac{dt}{(t^2+1)}$$

$$= 2\tan^{-1}(t) + c$$

$$= 2\tan^{-1}(\sqrt{x+2}) + c$$

84. $\int \frac{\sqrt{x}}{x+1}\,dx$

Let $x^2 = t$

$\Rightarrow 2x dx = dt$

$$= 2\int \frac{t^2\,dt}{t+1}$$

$$= 2\int \left(\frac{(t^2-1)+1}{t+1}\right)dt$$

$$= 2\int \left((t-1) + \frac{1}{t+1}\right)dt$$

$$= 2\left(\frac{t^2}{2} - t + \log|t+1|\right) + c$$

$$= 2\left(\frac{x^4}{2} - x^2 + \log|x^2+1|\right) + c$$

85. $\int \frac{dx}{x\sqrt{x-2}}$

$$= 2\int \frac{t\,dt}{(t^2+2)t}$$

Let $(x - 2) = t^2$

$\Rightarrow 2x dx = dt$

$$= 2\int dt(t^2+2)$$

$$= \sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + c$$

$$= \sqrt{2}\tan^{-1}\left(\sqrt{\left(\frac{x-2}{2}\right)}\right) + c$$

86. $\int \frac{dx}{(x+3)\sqrt{x}}$

$$= 2\int \frac{t\,dt}{(t^2+3)t}$$

Let $x^2 = t$

$\Rightarrow 2x dx = dt$

$$= 2\int \frac{dt}{(t^2+3)}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right) + c$$

87. $\int \frac{dx}{(x+3)\sqrt{2x+1}}$

$$= \int \frac{t\,dt}{\left(\frac{t^2+5}{2}\right)} t$$

Let $2x + 1 = t^2$

$dx = t\,dt$

Also, $x = \frac{t^2-1}{2}$

$\Rightarrow x + 3 = \frac{t^2-1}{2} + 3 = \frac{t^2+5}{2}$

$$= 2\int \frac{dt}{(t^2+5)}$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + c$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{\frac{2x+1}{5}}\right) + c$$

88. $\int \frac{dx}{x^2\sqrt{x-1}}$

Let $x - 1 = t^2$

$\Rightarrow dx = 2t\,dt$

Also, $x = t^2 + 1$

$$= \int \frac{2t\,dt}{(t^2+1)2t}$$

$$= 2\int \frac{dt}{(t^2+1)^2}$$

Now, $\int \frac{dt}{t^2+1}$

$$= \frac{1}{t^2+1}\int dt - \int \frac{1}{(t^2+1)^2} \times (-2t)\cdot t\,dt$$

$$= \frac{t}{t^2+1} + 2\int \frac{(t^2+1)-1}{(t^2+1)^2}\,dt$$

$$= \frac{t}{t^2+1} + 2\int \frac{dt}{(t^2+1)} - 2\int \frac{dt}{(t^2+1)^2}$$

$$\Rightarrow \quad 2\int \frac{dt}{(t^2+1)^2} = \frac{t}{t^2+1} + 2\tan^{-1} t + c$$

$$= \frac{\sqrt{x-1}}{x} + 2\tan^{-1}(\sqrt{x-1}) + c$$

89. $\int \frac{dx}{(x^2-4)\sqrt{x+1}}$

Let $(x+1) = t^2$

$\Rightarrow \quad dx = 2t\,dt$

$$= 2\int \frac{t\,dt}{((x+1)^2-4)\,t}$$

$$= 2\int \frac{dt}{(t^2+2t-3)}$$

$$= 2\int \frac{dt}{(t+3)(t-1)}$$

$$= \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+3}\right)dt$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+3}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+3}\right| + c$$

90. $\int \frac{dx}{(x^2+1)\sqrt{x}}$

Let $x = t^2$

$\Rightarrow \quad dx = 2t\,dt$

$$= 2\int \frac{t\,dt}{(t^4+1)\,t}$$

$$= 2\int \frac{dt}{(t^3+1)}$$

$$= \int \frac{2\,dt}{(t^4+1)}$$

$$= \int \left(\frac{(t^2+1)-(t^2-1)}{(t^4+1)}\right)dt$$

$$= \int \left(\frac{(t^2+1)}{(t^4+1)} - \frac{(t^2-1)}{(t^4+1)}\right)dt$$

and then you do it.

91. $\int \frac{x\,dx}{(x^2+2x+2)\sqrt{x+1}} = \int \frac{x\,dx}{((x+1)^2+1\sqrt{x+1}}$

Let $(x+1) = t^2$

$\Rightarrow \quad dx = 2t\,dt$

$$= 2\int \frac{(t^2-1)t\,dt}{(t^4+1)t}$$

$$= 2\int \left(\frac{t^2-1}{t^4+1}\right)dt$$

$$= 2\int \left(\frac{1-(1/t^2)}{t^2+(1/t)^2}\right)dt$$

$$= 2\int \left(\frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2 - 2}\right)dt$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{\left(t+\frac{1}{t}-\sqrt{2}\right)}{\left(t+\frac{1}{t}+\sqrt{2}\right)}\right| + c$$

where $t = \sqrt{x+1}$

92. $\int \frac{dx}{(x^2-1)\sqrt{x}}$

Let $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$

$$= 2\int \frac{t\,dt}{(t^4-1)t}$$

$$= 2\int \frac{dt}{(t^4-1)}$$

$$= 2\int \frac{dt}{(t^2-1)(t^2+1)}$$

$$= \left(\frac{1}{t^2-1} - \frac{1}{t^2+1}\right)dt$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| - \tan^{-1}(t) + c$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| - \tan^{-1}(\sqrt{x}) + c$$

93. $\int \frac{x\,dx}{(x^2-2x+2)\sqrt{x-1}}$

$$= \int \frac{x\,dx}{\left((x-1)^2+1\right)\sqrt{x-1}}$$

Let $(x-1) = t^2$

$\Rightarrow \quad dx = 2t\,dt$

$$= \int \frac{(t^2+1)\,2t\,dt}{(t^4+1)\,t}$$

$$= 2\int \frac{(t^2 + 1)\,dt}{t^4 + 1)}$$

$$= 2\int \frac{\left(1 + \frac{1}{t^2}\right)dt}{\left(t^2 + \frac{1}{t^2}\right)}$$

$$= 2\int \frac{\left(1 + \frac{1}{t^2}\right)dt}{\left(t - \frac{1}{t}\right)^2 + 2}$$

$$= 2\int \frac{dz}{z^2 + 2} \quad \text{Let } z = \left(t - \frac{1}{t}\right)$$

$$= \sqrt{2}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$$

$$= \sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right)\right) + c$$

$$= \sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}(\sqrt{x-1} - \frac{1}{\sqrt{x-1}}\right) + c$$

94. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 1}}$

Let $(x + 1) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

Also, $x = \frac{1}{t} - 1 = \frac{1 - t}{t}$

$$= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(1 - \frac{1}{t}\right)^2 + 1}}$$

$$= -\int \frac{dt}{\sqrt{(t-1)^2 + t^2}}$$

$$= -\int \frac{dt}{\sqrt{(2t^2 - 2t + 1)}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t^2 - t + \frac{1}{2}\right)}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)}}$$

$$= -\frac{1}{\sqrt{2}}\log\left|\left(t - \frac{1}{2}\right) + \sqrt{t^2 - t + \frac{1}{2}}\right| + c$$

$$= -\frac{1}{\sqrt{2}}\log\left|\left(t - \frac{1}{2}\right) + \sqrt{t^2 - t + \frac{1}{2}}\right| + c$$

where $t = \frac{1}{x + 1}$

95. $\int \frac{dx}{(x + 2)\sqrt{x^2 + 2x + 2}}$

Put $(x + 1) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2} + 1}}$$

$$= -\int \frac{dt}{\sqrt{t^2 + 1}}$$

$$= -\log\left|t + \sqrt{t^2 + 1}\right| + c$$

$$= -\log\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right| + c$$

96. $\int \frac{dx}{x\sqrt{x^2 + 4}}$

Put $x = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2} + 4}}$$

$$= -\int \frac{dx}{\sqrt{4t^2 + 1}}$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t^2 + (1/2)^2}}$$

$$= -\frac{1}{2}\log\left|t + \sqrt{t^2 + \frac{1}{4}}\right| + c$$

$$= -\frac{1}{2}\log\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} + \frac{1}{4}}\right| + c$$

97. $\int \frac{dx}{(x - 1)\sqrt{x^2 + 4}}$

Put $(x - 1) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= \int - \frac{\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t} + 1\right)^2 + 4}}$$

$$= -\int \frac{dx}{\sqrt{(1+t)^2 + 4t^2}}$$

$$= -\int \frac{dt}{\sqrt{5t^2 + 2t + 1}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{t^2 + \frac{2}{5}t + \frac{1}{5}}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{\left(t + \frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2}}$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(t + \frac{1}{5}\right) + \sqrt{t^2 \frac{2}{5}t + \frac{1}{5}}\right| + c$$

where $t = \frac{1}{(x-1)}$

98. $\int \frac{dx}{(2x-1)\sqrt{x^2+1}}$

Put $(2x - 1) = \frac{1}{t}$

$\Rightarrow$ $dx = -\frac{dt}{2t^2}$

Also, $x = \frac{1}{2}\left(\frac{1}{t} + 1\right)$

$$= -\int \frac{\frac{dt}{2t^2}}{\frac{1}{t}\sqrt{\frac{1}{4}\left(\frac{1}{t} + 1\right)^2 + 1}}$$

$$= -2\int \frac{dt}{\sqrt{(1+t)^2 + 4t^2}}$$

$$= -2\int \frac{dt}{\sqrt{5t^2 + 2t + 1}}$$

$$= -\frac{2}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2 + \frac{2t}{5} + \frac{1}{5}}}$$

$$= -\frac{2}{\sqrt{5}}\int \frac{dt}{\sqrt{\left(t + \frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2}}$$

$$= -\frac{2}{\sqrt{5}}\log\left|\left(t + \frac{1}{5}\right) + \sqrt{t^2 + \frac{2}{5}t + \frac{1}{t}}\right| + c$$

where $t = \frac{1}{(2x-1)}$

99. $\int \frac{dx}{(3x+2)\sqrt{x^2-4}}$

Put $(3x + 2) = \frac{1}{t}$

$\Rightarrow$ $dx = -\frac{dt}{3t^2}$

Also, $x = \frac{1}{3}\left(\frac{1}{t} - 2\right)$

$$= -\int \frac{\frac{dt}{3t^2}}{\frac{1}{t}\sqrt{\frac{1}{9}\left(\frac{1}{t} - 2\right)^2 - 4}}$$

$$= -\int \frac{dt}{\sqrt{(1-2t)^2 - 36t^2}}$$

$$= -\int \frac{dt}{\sqrt{1 - 4t - 32t^2}}$$

$$= -\frac{1}{4\sqrt{2}}\int \frac{dt}{\sqrt{-\left\{t^2 - \frac{t}{8} - \frac{1}{32}\right\}}}$$

$$= -\frac{1}{4\sqrt{2}}\int \frac{dt}{\sqrt{-\left\{\left(t - \frac{1}{16}\right)^2 - \left(\frac{3}{16}\right)^2\right\}}}$$

$$= -\frac{1}{4\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{16}\right)^2 - \left(t - \frac{1}{16}\right)^2}}$$

$$= -\frac{1}{4\sqrt{2}}\sin^{-1}\left(\frac{\left(t - \frac{1}{16}\right)}{\frac{3}{16}}\right) + c$$

$$= -\frac{1}{4\sqrt{2}}\sin^{-1}\left(\frac{16t - 1}{3}\right) + c$$

100. $\int \frac{dx}{x^2\sqrt{x^2-1}}$

Put $x = \frac{1}{t}$

$\Rightarrow$ $dx - \frac{1}{t^2}dt$

Also, $x^2 - 1 = \frac{1}{t^2} - 1 = \frac{1-t^2}{t^2}$

Thus, $\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t^2}\sqrt{\frac{1-t^2}{t^2}}}$

$$= \int \frac{-t\,dt}{\sqrt{1-t^2}}$$

$$= \sqrt{1-t^2} + c$$

$$= \sqrt{1 - \left(\frac{1}{x}\right)^2} + c$$

$$= \frac{\sqrt{1 - x^2}}{x} + c$$

101. $\int \frac{dx}{(x^2 + 1\sqrt{x^2 + 2}}$

Put $x = \frac{1}{t}$

$\Rightarrow \quad dx = \frac{dt}{t^2}$

$$= \int \frac{-\frac{dt}{t^2}}{\left(\frac{1}{t^2} + 1\right)\sqrt{\frac{1}{t^2} + 2}}$$

$$= -\int \frac{t\,dt}{(t^2 + 1)\sqrt{1 + 2t^2}}$$

$$= -\frac{1}{2}\int \frac{dv}{(v + 1)\sqrt{1 + 2v}}, \text{ where } t^2 = v$$

$\Rightarrow \quad 2tdt = dx$

Let $(1 + 2v) = z^2$

$dv = z\,dz$

Also, $v = \frac{z^2 - 1}{2}$

$$= -\frac{1}{2}\int \frac{z\,dz}{\left(\frac{z^2 - 1}{2} + 1\right)z}$$

$$= -\int \frac{dz}{(z^2 + 1)}$$

$$= \cot^{-1}(z) + c$$

$$= \cot^{-1}(\sqrt{1 + 2v}) + c$$

$$= \cot^{-1}(\sqrt{1 + 2t^2}) + c$$

$$= \cot^{-1}\left(\sqrt{1 + \frac{2}{x^2}}\right) + c$$

102. $\int \frac{dx}{(x^2 - 1)\sqrt{x^2 + 2}}$

Let $x = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{dt}{t^2}$

$$= -\int \frac{\frac{dt}{t^2}}{\left(\frac{1}{t^2} - 1\right)\sqrt{\frac{1}{t^2} + 2}}$$

$$= -\int \frac{t\,dt}{(1 - t^2)\sqrt{1 + 2t^2}}$$

$$= \frac{1}{4}\int \frac{v\,dv}{\left(\frac{v^2 - 1}{2} - 1\right)v}$$

Let $(1 + 2t^2) = v^2$

$\Rightarrow \quad 4tdt = 2v\,dv$

$$= \frac{1}{2}\int \frac{dv}{(v^2 - 3)}$$

$$= \frac{1}{4\sqrt{3}}\log\left|\frac{v - \sqrt{3}}{v + \sqrt{3}}\right| + c$$

$$= \frac{1}{4\sqrt{3}}\log\left|\frac{\sqrt{1 + 2t^2} - \sqrt{3}}{\sqrt{1 + 2t^2} + \sqrt{3}}\right| + c$$

$$= \frac{1}{4\sqrt{3}}\log\left|\frac{\sqrt{\frac{1 + 2}{x^2}} - \sqrt{3}}{\sqrt{\frac{1 + 2}{x^2}} + \sqrt{3}}\right| + c$$

103. $\int \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$

Put $x = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{dt}{t^2}$

$$= -\int \frac{\frac{dt}{t^2}}{\left(1 + \frac{1}{t^2}\right)\sqrt{1 - \frac{1}{t^2}}}$$

$$= -\int \frac{t\,dt}{(1 + t^2)\sqrt{t^2 - 1}}$$

$$= -\int \frac{v\,dv}{(v^2 + 2)\,v},$$

Let $(t^2 - 1) = v$

$\Rightarrow \quad 2tdt = dx$

$$= -\int \frac{dv}{(v^2 + 2)}$$

$$= -\frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{v}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{\sqrt{t^2 - 1}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{x^2} - 1}\right) + c$$

104. $\int \frac{x\,dx}{(x^4 - 1)\sqrt{x^4 + 3}}$

Let $x^2 = t$

$\Rightarrow \quad 2xdx = dt$

$$= \frac{1}{2}\int \frac{dt}{(t^2-1)\sqrt{t^2+3}}$$

$$= -\frac{1}{2}\int \frac{\frac{dy}{y^2}}{\left(\frac{1}{y^2}-1\right)\sqrt{\frac{1}{y^2}+3}} \quad \text{Let } t = \frac{1}{y} \Rightarrow dt = -\frac{dy}{y^2}$$

$$= -\frac{1}{2}\int \frac{y\,dy}{(1-y^2)\sqrt{1+3y^2}}$$

$$= -\frac{1}{4}\int \frac{v\,dv}{(4-v^2)v} \quad \text{Let } (3y^2+1) = v^2 \Rightarrow 3ydy = 2vdu$$

$$= \frac{1}{4}\int \frac{dv}{(v^2-4)}$$

$$= \frac{1}{16}\log\left|\frac{v-2}{v+2}\right| + c$$

$$= \frac{1}{16}\log\left|\frac{\sqrt{3y^2+1}-2}{\sqrt{3y^2+1}+2}\right| + c$$

$$= \frac{1}{16}\log\left|\frac{\sqrt{\frac{3}{x^2}+1}-2}{\sqrt{\frac{3}{x^2}+1}+2}\right| + c$$

105. $\int \frac{dx}{(x^2-1)\sqrt{x^2+4x+5}}$

Let $x = \frac{1}{t}$

$\Rightarrow dx = -\frac{dt}{t^2}$

$$= \int \frac{-\frac{dt}{t^2}}{\left(\frac{1}{t^2}-1\right)\sqrt{\frac{1}{t^2}+\frac{4}{t}+5}}$$

$$= \int \frac{t\,dt}{(t^2-1)\sqrt{5t^2+4t+1}}$$

106. $\int (x+\sqrt{x^2+1})^{10}\,dx$

Let $(x+\sqrt{x^2+1}) = t$

$\Rightarrow \left(1+\frac{1\times 2x}{2\sqrt{x^2+1}}\right)dx = dt$

$\Rightarrow \left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)dx = dt$

$\Rightarrow \left(\frac{t}{\sqrt{x^2+1}}\right)dx = dt$

Also, $x^2+1 = \frac{1}{4}\left(t-\frac{1}{t}\right)^2+1$

$\Rightarrow \sqrt{x^2+1} = \frac{1}{2}\sqrt{\left(t-\frac{1}{t}\right)^2+4}$

$= \frac{1}{2}\sqrt{\left(t+\frac{1}{t}\right)^2}$

$= \frac{1}{2}\left(t+\frac{1}{t}\right)$

Thus, $dx = \left(\frac{\sqrt{x^2+1}}{t}\right)dt$

$\Rightarrow d\quad x = \left(\frac{\frac{1}{2}\left(t+\frac{1}{t}\right)}{t}\right)dt$

$\Rightarrow dx = \frac{1}{2}\left(\frac{t^2+1}{t^2}\right)dt$

Thus, the given integral reduces to

$$\frac{1}{2}\int t^{10}\left(\frac{t^2+1}{t^2}\right)dt = \frac{1}{2}\int t^8(t^2+1)\,dt$$

$$= \frac{1}{2}\int (t^{10}+t^8)\,dt$$

$$= \frac{1}{2}\left(\frac{t^{11}}{11}+\frac{t^9}{9}\right)+c,$$

where $(x+\sqrt{x^2+1}) = t$

107. $\int (x-\sqrt{x^2+4})^5\,dx$

Put $(x-\sqrt{x^2+4}) = t$

$\Rightarrow \left(1-\frac{1\times 2x}{2\sqrt{x^2+4}}\right)dx = dt$

$\Rightarrow \left(1-\frac{x}{\sqrt{x^2+4}}\right)dx = dt$

$\Rightarrow \left(\frac{x-\sqrt{x^2+4}}{\sqrt{x^2+4}}\right)dx = -dt$

$\Rightarrow \left(\frac{t}{\sqrt{x^2+4}}\right)dx = -dt$

$\Rightarrow \left(\frac{t}{-\left(\frac{t}{2}+\frac{2}{t}\right)}\right)dx = -dt$

$\Rightarrow dx = \frac{1}{t}\left(\frac{t}{2}+\frac{2}{t}\right)dt$

$$\Rightarrow \quad dx = \left(\frac{1}{2} + \frac{2}{t^2}\right)dt$$

$$= \int \left\{ t^5 \left(\frac{1}{2} + \frac{2}{t^2}\right) \right\} dt$$

$$= \int \left(\frac{t^5}{2} + 2t^3\right) dt$$

$$= \left(\frac{t^6}{12} + \frac{t^4}{2}\right) + c$$

where $t = (x - \sqrt{x^2 + 4})$

108. $\int (x + \sqrt{1 + x^2})^n \, dx$

Put $(x + \sqrt{x^2 + 1}) = t$

$$\Rightarrow \quad dx = \frac{1}{2}\left(\frac{t^2 + 1}{t^2}\right)dt$$

$$= \frac{1}{2}\int t^n \left(\frac{t^2 + 1}{t^2}\right) dt$$

$$= \frac{1}{2}\int t^{n-2}(t^2 + 1)\, dt$$

$$= \frac{1}{2}\int (t^n + t^{n-2})\, dt$$

$$= \frac{1}{2}\left(\frac{t^{n+1}}{n + 1} + \frac{t^{n-1}}{n - 1}\right) + c$$

where $t = (x + \sqrt{x^2 + 1})$

109. $\int \frac{dx}{(x + \sqrt{x^2 - 4})^{5/3}}$

Put $(x + \sqrt{x^2 - 4}) = t$

$$\Rightarrow \quad \left(1 + \frac{1 \times 2x}{2\sqrt{x^2 - 4}}\right)dx = dt$$

$$\Rightarrow \quad \left(1 + \frac{x}{\sqrt{x^2 - 4}}\right)dx = dt$$

$$\Rightarrow \quad \left(\frac{t}{\sqrt{x^2 - 4}}\right)dx = dt$$

$$\Rightarrow \quad dx = \frac{1}{t}\left(\frac{t}{2} - \frac{2}{t}\right)dt$$

$$\Rightarrow \quad dx = \left(\frac{1}{2} - \frac{2}{t^2}\right)dt$$

$$= \int \frac{\left(\frac{1}{2} - \frac{2}{t^2}\right)}{t^{5/3}} dt$$

$$= \int \left(\frac{t^{-5/3}}{2} - 2t^{-11/3}\right) dt$$

$$= \frac{3}{4}\left(t^{-8/3} - t^{-2/3}\right) + c$$

where $t = (x + \sqrt{x^2 - 4})$

110. Do yourself.

111. $\int \frac{dx}{x^2(x - \sqrt{x^2 + 9}} = \int \frac{dx}{x^3\left(1 - \sqrt{1 + \frac{9}{x^2}}\right)}$

Put $\left(1 + \frac{9}{x^2}\right) = t^2$

$$\frac{dx}{x^3} = -\frac{t}{9}dt$$

$$= -\frac{1}{9}\int \frac{t}{1 - t} dt$$

$$= \frac{1}{9}\int \frac{t}{t - 1}\, dt$$

$$= \frac{1}{9}\int \left(\frac{(t - 1) + 1}{t - 1}\right) dt$$

$$= \frac{1}{9}\int \left(1 + \frac{1}{t - 2}\right) dt$$

$$= \frac{1}{9}(t + \log|t - 1|) + c$$

where $t = \sqrt{1 + \frac{9}{x^2}}$

112. $\int \frac{dx}{x^{1/2}(2 + 3x)^{3/2}}$

Let $2 + 3x = tx$

$$\Rightarrow \quad x = \left(\frac{2}{t - 3}\right)$$

$$\Rightarrow \quad dx - \frac{-2\,dt}{(t - 3)^2}$$

$$= \int \frac{\frac{-2\,dt}{(t - 3)^2}}{\left(\frac{2}{t - 3}\right)^{1/2} t^{3/2} x^{3/2}}$$

$$= \int \frac{\frac{-2\,dt}{(t - 3)^2}}{\left(\frac{2}{t - 3}\right)^{1/2} t^{3/2} \left(\frac{2}{t - 3}\right)^{3/2}}$$

$$= -\int t^{-3/2}\, dt$$

$$= \frac{2}{\sqrt{t}} + c$$

$$= \frac{2}{\sqrt{\frac{2+3x}{x}}} + c$$

113. $\int \frac{dx}{x^{2/3}(2+3x)^{4/3}}$

Let $(2+3x) = tx$

$\Rightarrow x(t-3) = 2$

$\Rightarrow x = \frac{2}{(t-3)}$

$\Rightarrow dx = \frac{-2}{(t-3)^2}\,dt$

$$= -\int \frac{\frac{2}{(t-3)^2}}{\left(\frac{2}{t-3}\right)^{2/3} t^{4/3}\left(\frac{2}{t-3}\right)^{4/3}}\,dt$$

$$= -\int \frac{2}{4} \times t^{-4/3}\,dt$$

$$= -\frac{1}{2}\int t^{-4/3}\,dt$$

$$= \frac{3}{2}(t^{-4/3}) + c$$

$$= \frac{3}{2}\left(\frac{1}{\left(3+\frac{2}{x}\right)^{1/3}}\right) + c$$

114. $\int \frac{dx}{x^{3/4}(3x-1)^{5/4}}$

Put $(3x-1) = tx$

$\Rightarrow x = -\frac{1}{(t-3)}$

$\Rightarrow dx = \frac{1}{(t-3)^2}\,dt$

$$= \int \frac{\frac{1}{(t-3)^2}}{\frac{dt}{\left(\frac{1}{t-3}\right)^{3/4} t^{5/4}\left(\frac{1}{t-3}\right)^{5/4}}}$$

$$= \int \frac{dt}{t^{5/4}}$$

$$= \int t^{-5/4}\,dt$$

$$= -4t^{-1/4} + c$$

$$= -\frac{4}{\sqrt[4]{t}} + c$$

$$= -\frac{4}{\sqrt[4]{\left(3-\frac{1}{x}\right)}} + c$$

115. $\int dx/x^{1/3}(2x+1)^{5/3}$

Put $(2x+1) = tx$

$\Rightarrow x = \frac{1}{t-2}$

$\Rightarrow dx = -\frac{dt}{(t-2)^2}$

$$= -\int \frac{\frac{dt}{(t-2)^2}}{\left(\frac{1}{t-2}\right)^{1/3}(t)^{5/3}\left(\frac{1}{t-2}\right)^{5/3}}$$

$$= -\int t^{-5/3}\,dt$$

$$= \frac{3}{2}(t^{-2/3}) + c$$

$$= \frac{3}{2}\left(\frac{1}{\left(2+\frac{1}{x}\right)^{2/3}}\right) + c$$

116. Do yourself

117. Do yourself.

118. We have,

$$\int \frac{dx}{(x-1)^3(x+2)^4}$$

$$= \int \frac{dx}{\left(\frac{(x-1)}{(x+2)}\right)^3 (x+2)^7}$$

Put $\left(\frac{x-1}{x+2}\right) = t$

$\Rightarrow \frac{3}{(x+2)^2}dx = dt$

Also, $x = \frac{2t+1}{1-t}$

$\Rightarrow x + 2 = \frac{3}{1-t}$

$$= \frac{1}{3^6}\int \frac{dt}{t^3\left(\frac{1}{t-1}\right)^5}$$

$$= \frac{1}{3^6}\int \frac{(t-1)^5\,dt}{t^3}$$

$$= \frac{1}{3^6}\int \frac{t^5 - 5t^4 + 10t^3 - 10t^2 + 5t - 1)\,dt}{t^3}$$

$$= \frac{1}{3^6}\int \left(t^2 - 5t + 10 - \frac{10}{t} + \frac{5}{t^2} - \frac{1}{t^3}\right)dt$$

$$= \frac{1}{3^6}\left(\frac{t^3}{3} - \frac{5}{2}t^2 + 10t - 10\log|t| - \frac{5}{t} + \frac{1}{2t^2}\right) + c$$

where $t = \left(\frac{x-1}{x+2}\right)$

119. $$\int \frac{dx}{(x-1)^3(x-2)^2} = \int \frac{dx}{(x-1)^5\left(\frac{x-2}{x-1}\right)^2}$$

Put $\left(\frac{x-2}{x-1}\right) = t$

$$\frac{(x-1)\cdot 1 - (x-2)\cdot 1}{(x-1)^2}\,dx = dt$$

$$\frac{dx}{(x-1)^2} = dt$$

Also, $x = \frac{t-2}{t-1}$

$$(x-1) = \frac{t-2}{t-1} - 1 = -\frac{1}{(t-1)}$$

$$= \int \frac{dt}{\left(\frac{1}{t-1}\right)^3 t^2}$$

$$= \int \frac{(t-1)^3\,dt}{t^2}$$

$$= \int \frac{(t^3 - 3t^2 + 3t - 1)}{t^2}dt$$

$$= \int \left(t - 3 + \frac{3}{t} - \frac{1}{t^2}\right)dt$$

$$= \left(\frac{t^2}{2} - 3t + 3\log|t| + \frac{1}{t}\right) + c$$

where $t = \left(\frac{x-2}{x-1}\right)$

120. Do yourself.

121. $$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$$

$$= \int \frac{dx}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}$$

Put $\left(\frac{x-1}{x+2}\right) = t$

$$\Rightarrow \frac{(x+2)\cdot 1 - (x-1)\cdot 1}{(x+2)^2}\,dx = dt$$

$$\Rightarrow \frac{3}{(x+2)^2}dx = dt$$

$$\Rightarrow \frac{dx}{(x+2)^2} = \frac{dt}{3}$$

$$= \frac{1}{3}\int \frac{dt}{t^{3/4}}$$

$$= \frac{4}{3}\left(\frac{1}{\sqrt[4]{t}}\right) + c$$

$$= \frac{4}{3}\left(\frac{1}{\sqrt[4]{\left(\frac{x-1}{x-2}\right)}}\right) + c$$

122. $$\int \frac{dx}{x^2(x+5)^4} = \int \frac{dx}{\left(\frac{x}{x+5}\right)^2(x+5)^6}$$

Put $\left(\frac{x}{x+5}\right) = t$

$$\Rightarrow \frac{5\,dx}{(x+5)^2} = dt$$

$$\Rightarrow \frac{dx}{(x+5)^2} = \frac{dt}{5}$$

$$= \frac{1}{5^5}\int \frac{dt}{t^2\left(\frac{1}{1-t}\right)^4}$$

$$= \frac{1}{5^5}\int \frac{(t-1)^4\,dt}{t^2}$$

$$= \frac{1}{5^5}\int \frac{(t^4 - 4t^3 + 6t^2 - 4t + 1)\,dt}{t^2}$$

$$= \frac{1}{5^5}\int \left(t^2 - 4t + 6 - \frac{4}{t} + \frac{1}{t^2}\right)dt$$

$$= \frac{1}{5^5}\left(\frac{t^3}{3} - 2t^2 + 6t - 4\log|t| - \frac{1}{t}\right) + c$$

where $t = \left(\frac{x}{x+5}\right)$

123. $$\int \frac{dx}{(x-1)^{3/2}(x+1)^{5/2}}$$

$$= \int \frac{dx}{\left(\frac{x-2}{x+1}\right)^{3/2}(x+1)^2}$$

Let $\left(\frac{x-1}{x+1}\right) = t$

$$\Rightarrow \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}\,dx = dt$$

$$\Rightarrow \frac{2}{(x+1)^2}dx = dt$$

$$\Rightarrow \frac{dx}{(x+1)^2} = \frac{dt}{2}$$

$$= \frac{1}{2}\int \frac{dt}{t^{3/2}}$$

$$= \frac{1}{2}\int t^{-3/2}\, dt$$

$$= -\frac{1}{\sqrt{t}} + c$$

$$= -\frac{1}{\sqrt{\left(\frac{x-1}{x+1}\right)}} + c$$

$$= -\sqrt{\frac{x+1}{x-1}} + c$$

124. We have, $\int \frac{dx}{x\sqrt{3x^3+4}} = \int \frac{x^2\, dx}{x^3\sqrt{3x^3+4}}$

Let $x^3 = t$

$\Rightarrow \quad 3x^2\, dx = dt$

$$= \frac{1}{3}\int \frac{dt}{t\sqrt{3t+4}}$$

Again let $3t + 4 = v^2$

$\Rightarrow \quad 3\, dt = 2v\, dv$

Also, $t = \left(\frac{v^2-4}{3}\right)$

$$= \frac{2}{9}\int \frac{v\, dv}{\left(\frac{v^2-4}{3}\right)\cdot v}$$

$$= \frac{2}{3}\int \frac{dv}{(v^2-4)}$$

$$= \frac{2}{3}\times\frac{1}{2}\log\left|\frac{v-2}{v+2}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{\sqrt{3t+4}-2}{\sqrt{3t+4}+2}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{\sqrt{3x^3+4}-2}{\sqrt{3x^3+4}+2}\right| + c$$

125. $\int \frac{dx}{x\sqrt{5x^4+3}} = \int \frac{x^3}{x^4\sqrt{5x^4+3}}\, dx$

Let $5x^4 + 3 = t^2$

$\Rightarrow \quad 20x^3\, dx = 2t\, dt$

$\Rightarrow \quad x^3\, dx = \frac{t\, dt}{10}$

$$= \frac{1}{10}\int \frac{t\, dt}{\left(t^2 - \frac{3}{5}\right)t}$$

$$= \frac{1}{2}\int \frac{dt}{(t^2-3)}$$

$$= \frac{1}{4\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

where $t = \sqrt{5x^4+3}$

126. $\int \frac{dx}{x\sqrt{2-5x^6}} = \int \frac{x^5}{x^6\sqrt{2-5x^6}}\, dx$

Put $(2 - 5x^6) = t^2$

$\Rightarrow \quad -30x^5\, dx = 2t\, dt$

$\Rightarrow \quad x^5\, dx = -\frac{t\, dt}{15}$

$$= -\frac{1}{15}\int \frac{t\, dt}{\left(\frac{2-t^2}{5}\right)t}$$

$$= \frac{1}{3}\int dt(t^2-2)$$

$$= \frac{1}{3}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{\sqrt{2-5x^6}-\sqrt{2}}{\sqrt{2-5x^6}+\sqrt{2}}\right| + c$$

127. $\int \frac{dx}{x\sqrt{3x-2}}$

Let $3x - 2 = t^2$

$\Rightarrow \quad 3\, dx = 2t\, dt$

$$= \frac{2}{3}\int \frac{t\, dt}{\left(\frac{t^2+2}{3}\right)t}$$

$$= 2\int \frac{dt}{t^2+2}$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \sqrt{2}\tan^{-1}\left(\sqrt{\frac{3x-2}{2}}\right) + c$$

128. $\int \frac{dx}{x\sqrt{3x^9-2}} = \int \frac{x^8}{x^9\sqrt{3x^9-2}}\, dx$

Let $3x^9 - 2 = t^2$

$\Rightarrow \quad 27x^9\, dx = 2t\, dt$

$\Rightarrow \quad x^8\, dx = \frac{2t\, dt}{27}$

$$= \frac{2}{27}\int \frac{t\,dt}{\left(\frac{t^2+2}{3}\right)t}$$

$$= \frac{2}{9}\int \frac{dt}{(t^2+2)}$$

$$= \frac{\sqrt{2}}{9}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \frac{\sqrt{2}}{9}\tan^{-1}\left(\sqrt{\frac{3x^9-2}{2}}\right) + c$$

129. $\int \frac{dx}{x\sqrt{2x^{10}-3}} = \int \frac{x^9}{x^{10}\sqrt{2x^{10}-3}}\,dx$

Let $2x^{10} - 3 = t^2$

$\Rightarrow \quad 10x^9\,dx = 2t\,dt$

$\Rightarrow \quad x^9\,dx = \frac{t\,dt}{5}$

$$= \frac{1}{5}\int \frac{t\,dt}{\left(\frac{t^2+3}{10}\right)t}$$

$$= 2\int \frac{dt}{(t^2+3)}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{2x^{10}-3}{3}}\right) + c$$

130. $\int \frac{dx}{(2+3x^2)^{3/2}}$

Let $x = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= -\int \frac{t\,dt}{(2t^2+3)^{3/2}}$$

Again let $(2t^2 + 3) = v^2$

$\Rightarrow \quad 2t\,dt = v\,dv$

$$= -\frac{1}{2}\int \frac{v\,dv}{v^3}$$

$$= -\frac{1}{2}\int \frac{dv}{v^2}$$

$$= \frac{1}{2v} + c$$

$$= \frac{1}{2\sqrt{2t^2+3}} + c$$

$$= \frac{x}{2\sqrt{2+3x^2}} + c$$

131. $\int \frac{dx}{(c+dx^2)^{3/2}} = \int \frac{dx}{x^3\left(d+\frac{c}{x^2}\right)^{3/2}}$

Let $\left(d + \frac{c}{x^2}\right) = t^2$

$\Rightarrow \quad -\frac{2c}{x^3}dx = 2t\,dt$

$\Rightarrow \quad \frac{dx}{x^3} = -\frac{t\,dt}{c}$

$$= -\frac{1}{c}\int \frac{t\,dt}{t^3}$$

$$= -\frac{1}{c}\int \frac{dt}{t^2}$$

$$= \frac{1}{c} \times \frac{1}{t} + c$$

$$= \frac{1}{c} \times \frac{1}{\sqrt{\left(d+\frac{c}{x^2}\right)}} + c$$

132. Do yourself

133. Do yourself

134. $\int \frac{x\,dx}{(2-5x^4)^{3/2}} = \int \frac{x\,dx}{x^6\left(\frac{2}{x^4}-5\right)^{3/2}} = \int \frac{dx}{x^5\left(\frac{2}{x^4}-5\right)^{3/2}}$

Let $\left(\frac{2}{x^4} - 5\right) = t^2$

$\Rightarrow \quad -\frac{8}{x^5}dx = 2t\,dt$

$\Rightarrow \quad \frac{dx}{x^5} = -\frac{t\,dt}{4}$

$$= -\frac{1}{4}\int \frac{t\,dt}{t^3}$$

$$= -\frac{1}{4}\int \frac{dt}{t^2}$$

$$= \frac{1}{4t} + c$$

$$= \frac{1}{4}\sqrt{\left(\frac{2}{x^4}-5\right)} + c$$

135. $\int x^2 \frac{dx}{(1-4x^6)^{3/2}} = \int \frac{dx}{x^7\left(\frac{1}{x^6}-4\right)^{3/2}}$

Let $\left(\frac{1}{x^6} - 4\right) = t^2$

$\Rightarrow \quad -\frac{6}{x^7}dx = 2t\,dt$

$\Rightarrow \quad \frac{dx}{x^7} = -\frac{t\,dt}{3}$

$$= -\frac{1}{3}\int \frac{t\,dt}{t^3}$$

$$= -\frac{1}{3}\int \frac{dt}{t^2}$$

$$= \frac{1}{3t} + c$$

$$= \frac{1}{3}\sqrt{\frac{1}{x^6} - 4} + c$$

136. We have,

$$\int \frac{dx}{(x-2)^2\sqrt{x^2 - 4x + 7}}$$

$$= \int \frac{dx}{(x-2)^2\sqrt{(x-2)^2 + 3}}$$

Put $x - 2 = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

Thus, the given integral reduces to

$$\int \frac{-\frac{1}{t^2}}{\frac{1}{t^2}\sqrt{\frac{1}{t^2} + 3}}dt = -\int \frac{t}{\sqrt{1 + 3t^2}}\,dt$$

Again, let $1 + 3t^2 = v^2$

$\Rightarrow \quad 3t\,dt = v\,dv$

Now, the integral is $-\int \frac{v\,dv}{v}$

$$= -\int dv$$

$$= -v + c$$

$$= \sqrt{1 + 3t^2} + c$$

$$= \sqrt{1 + \frac{3}{(x-2)^2}} + c$$

137. $\displaystyle\int \frac{dx}{(x+1)^3\sqrt{x^2 + 2x + 4}} = \int \frac{dx}{(x+1)^3\sqrt{(x+1)^2 + 3}}$

Let $(x + 1) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= -\int \frac{\frac{1}{t^2}dt}{\frac{1}{t^3}\sqrt{\frac{1}{t^2} + 3}}$$

$$= -\int \frac{t^2\,dt}{\sqrt{1 + 3t^2}}$$

$$= -\frac{1}{3}\int \left(\frac{(1 + 3t^2) - 1}{\sqrt{1 + 3t^2}}\right)dt$$

$$= -\frac{1}{3}\int \left(\sqrt{(1 + 3t^2)} - \frac{1}{\sqrt{1 + 3t^2}}\right)dt$$

$$= -\sqrt{3}\int \sqrt{\left(t^2 + \left(\frac{1}{\sqrt{3}}\right)^2\right)}dt + \frac{1}{3\sqrt{3}}\int \frac{dt}{\sqrt{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2}}$$

$$= -\sqrt{3}\left(\frac{t^2}{2}\sqrt{t^2 + \frac{1}{3}} + \frac{1}{6}\log\left|t + \sqrt{t^2 + \frac{1}{3}}\right|\right)$$

$$+ \frac{1}{3\sqrt{3}}\log\left|t + \sqrt{t^2 + \frac{1}{3}}\right| + c$$

where $t = \frac{1}{(x+1)}$

138. $\displaystyle\int \frac{dx}{(x-2)^3\sqrt{4x^2 - 16x + 20}}$

$$= \frac{1}{2}\int \frac{dx}{(x-2)^3\sqrt{x^2 - 4x + 5}}$$

$$= \frac{1}{2}\int \frac{dx}{(x-2)^3\sqrt{(x-2)^2 + 1}}$$

Let $(x - 2) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= -\int \frac{\frac{dt}{t^2}}{\left(\frac{1}{t^3}\right)\sqrt{\frac{1}{t^2} + 1}}$$

$$= -\int \frac{t^2}{\sqrt{t^2 + 1}}\,dt$$

$$= -\int \left(\frac{(t^2 + 1) - 1}{\sqrt{t^2 + 1}}\right)dt$$

$$= -\int \left(\sqrt{t^2 + 1} - \frac{1}{\sqrt{t^2 + 1}}\right)dt$$

$$= -\left[\frac{t}{2}\sqrt{t^2 + 1} + \log\left|t + \sqrt{t^2 + 1}\right|\right.$$

$$\left. -\log\left|t + \sqrt{t^2 + 1}\right|\right] + c$$

where $t = \frac{1}{(x-2)}$

139. $\displaystyle\int \frac{dx}{(x^2 - 6x + 9)\sqrt{4x^2 - 24x + 20}}$

$$= \frac{1}{2}\int \frac{dx}{(x-3)^2\sqrt{x^2 - 6x + 5}}$$

$$= \frac{1}{2}\int \frac{dx}{(x-3)^2\sqrt{(x-3)^2-4}}$$

Let $(x-3) = \frac{1}{t}$

$\Rightarrow \quad dx = \frac{dt}{t^2}$

$$= -\int \frac{\frac{dt}{t^2}}{\left(\frac{1}{t}\right)^3\sqrt{\frac{1}{t^2}-4}}$$

$$= -\int \frac{t^2}{\sqrt{1-4t^2}}\,dt$$

$$= \frac{1}{4}\int\left((1-4t^2)-\frac{1}{\sqrt{1-4t^2}}\right)dt$$

$$= \frac{1}{4}\int\left(\sqrt{1-4t^2}-\frac{1}{\sqrt{1-4t^2}}\right)dt$$

$$= \frac{1}{2}\int\sqrt{\left(\frac{1}{2}\right)^2-t^2}\,dt - \frac{1}{4}\int\frac{dt}{\sqrt{\left(\frac{1}{2}\right)^2-t^2}}$$

$$= \frac{1}{2}\left[\frac{t}{2}\sqrt{\frac{1}{4}-t^2}+\frac{1}{8}\sin^{-1}(2t)\right]+\frac{1}{4}\sin^{-1}(2t)+c$$

where $t = \frac{1}{(x-3)}$

140. $\int \frac{dx}{(4x^2+4x+1)\sqrt{4x^2+4x+7}}$

$$= \int \frac{dx}{(2x+1)^2\sqrt{(2x+1)^2+6}}$$

Let $(2x+1) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{dt}{2t^2}$

$$= -\int \frac{\frac{dt}{2t^2}}{\left(\frac{1}{t}\right)^2\sqrt{\frac{1}{t^2}+6}}$$

$$= -\frac{1}{2}\int\frac{t\,dt}{\sqrt{6t^2+1}}$$

Again, let $6t^2+1 = v^2$

$\Rightarrow \quad 12t\,dt = 2v\,dv$

$\Rightarrow \quad t\,dt = \frac{1}{6}v\,dv$

$$= -\frac{1}{12}\int\frac{v\,dv}{v}$$

$$= -\frac{1}{12}\int dv$$

$$= -\frac{v}{12}+c$$

$$= -\frac{\sqrt{6t^2+1}}{12}+c$$

$$= \frac{\sqrt{\frac{6}{(2x+1)^2}+1}}{12}+c$$

141. $\int \frac{dx}{(x+1)^3\sqrt{x^2+2x-4}} = \int\frac{dx}{(x+1)^3\sqrt{(x+1)^2-5}}$

Let $(x+1) = \frac{1}{t}$

$\Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$= -\int\frac{\frac{dt}{t^2}}{\left(\frac{1}{t}\right)^3\sqrt{\frac{1}{t^2}-5}}$$

$$= -\int\frac{t^2\,dt}{\sqrt{1-5t^2}}$$

$$= \frac{1}{5}\int\left(\frac{1-5t^2)-1}{\sqrt{1-5t^2}}\right)dt$$

$$= \frac{1}{5}\int\left(\sqrt{1-5t^2}-\frac{1}{\sqrt{1-5t^2}}\right)dt$$

$$= \sqrt{5}\int\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2-t^2}\,dt - \frac{1}{5\sqrt{5}}\int\frac{dt}{\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2-t^2}}$$

$$= \sqrt{5}\left[\frac{t}{2}\sqrt{\frac{1}{5}-t^2}+\frac{1}{10}\sin^{-1}(t\sqrt{5})\right]$$

$$-\frac{1}{5\sqrt{5}}(\sin^{-1}t\sqrt{5})+c$$

where $t = \frac{1}{(x+1)}$

142. $\int\frac{(2x+3)}{(3x+4)\sqrt{x^2+2x+4}}\,dx$

Let $2x+3 = A(3x+4)+B$

Comparing the co-efficients of x and the constant term, we get

$\Rightarrow \quad A = 2/3,\ B = 1/3$

Thus, the given integral is reduces to

$$\frac{\frac{2}{3}(3x+4)+\frac{1}{3}}{(3x+4)\sqrt{x^2+2x+4}\,dx}$$

$$= \frac{2}{3}\int\frac{dx}{\sqrt{x^2+2x+4}} + \frac{1}{3}\int\frac{dx}{(3x+4)\sqrt{x^2+2x+4}}$$

$$= \frac{2}{3}\int\frac{dx}{\sqrt{(x+1)^2+3}} + \frac{1}{3}\int\frac{dx}{(3x+4)\sqrt{(x+1)^2+3}}$$

$$= \frac{2}{3}\log\left|(x+1)+\sqrt{x^2+2x+4}\right| + -\frac{3}{\sqrt{28}}\int\frac{dt}{\sqrt{28t^2-2t+1}},$$

where $3x + 4 = 1/t$

$$\Rightarrow \quad 3dx = \frac{dt}{t^2}$$

$$= \frac{2}{3}\log\left|(x+1)+\sqrt{x^2+2x+4}\right| + -\frac{3}{\sqrt{28}}\log\left|\left(t-\frac{1}{28}\right)+\sqrt{t^2-\frac{t}{14}+\frac{1}{28}}\right| + c$$

where $t = \frac{1}{3x+4}$

143. $\int \frac{(2x+3)}{(x+1)\sqrt{x^2+2x+9}}\,dx$

$$= \int\frac{2(x+1)+1}{(x+1)\sqrt{(x+1)^2+8}}\,dx$$

$$= 2\int\frac{dx}{\sqrt{(x+1)^2+8}} + \int\frac{dx}{(x+1)\sqrt{(x+1)^2+8}}$$

$$= 2\log\left|(x+1)+\sqrt{x^2+2x+9}\right| - \int\frac{\frac{dt}{t^2}}{\left(\frac{1}{t}\right)\sqrt{\frac{1}{t^2}+8}}, \text{ Let } (x+1)=\frac{1}{t}$$

$$dx = -\frac{dt}{t^2}$$

$$= 2\log\left|(x+1)+\sqrt{x^2+2x+9}\right| - \int\frac{dt}{\sqrt{8t^2+1}}$$

$$= 2\log\left|(x+1)+\sqrt{x^2+2x+9}\right| \frac{1}{2\sqrt{2}}\int\frac{dt}{\sqrt{t^2+\left(\frac{1}{2\sqrt{2}}\right)^2}}$$

$$= 2\log\left|(x+1)+\sqrt{x^2+2x+9}\right| - \frac{1}{2\sqrt{2}}\log\left|t+\sqrt{t^2+\frac{1}{8}}\right| + c$$

where $t = \frac{1}{(x+1)}$

144. $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+7}}\,dx$

$$= \int\frac{4(x+2)-1}{(x+2)\sqrt{(x+2)^2+3}}\,dx$$

$$= 4\int\frac{dx}{\sqrt{(x+2)^2+3}} - \int\frac{dx}{(x+2)\sqrt{(x+2)^2+3}}$$

$$= 4\log\left|(x+2)+\sqrt{x^2+3x+7}\right| - \int\frac{\frac{dt}{t^2}}{\left(\frac{1}{t}\right)\sqrt{\frac{1}{t^2}+3}}, \text{ Let } (x+2)=\frac{1}{t}$$

$$\Rightarrow \quad dx = \frac{dt}{t^2}$$

$$= 4\log\left|(x+2)+\sqrt{x^2+3x+7}\right| - \int\frac{dt}{\sqrt{3t^2+1}}$$

$$= 4\log\left|(x+2)+\sqrt{x^2+3x+7}\right| - \frac{1}{\sqrt{3}}\int\frac{dt}{\sqrt{t^2+\left(\frac{1}{\sqrt{3}}\right)^2}}$$

$$= 4\log\left|(x+2)+\sqrt{x^2+3x+7}\right| - \frac{1}{\sqrt{3}}\log\left|t+\sqrt{t^2+\frac{1}{3}}\right| + c$$

where $t = \frac{1}{(x+2)}$

145. $\int \frac{x^2+4x+2}{(x+1)\sqrt{x^2+2x+3}}\,dx$

Put $x^2 + 4x + 2$

$$= L(x+1)(2x+2) + M(x+1) + N$$
$$= L(2x^2+4x+2) + M(x+1) + N$$
$$= 2Lx^2 + (4L+M)x + (2L+M+N)$$

Comparing the coefficients of the like terms of both the sides, we get

$L = 1/2,\ M = 2,\ N = -1.$

Thus, the given integral reduces to

$$\frac{1}{2}\int\frac{(x+1)(2x+2)}{(x+1)\sqrt{x^2+2x+2}}\,dx$$

$$+ 2\int \frac{(x+1)}{(x+1)\sqrt{x^2+2x+2}}\,dx$$

$$-\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$$

$$= \frac{1}{2}\int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+2}} + 2\int \frac{dx}{\sqrt{x^2+2x+2}}$$

$$-\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$$

$$= \sqrt{x^2+2x+2}$$

$$+ 2\log\left|(x+1)+\sqrt{x^2+2x+2}\right|$$

$$-\sec^{-1}(x+1)+c$$

146. $\int \frac{x^2+5x+6}{(x+2)\sqrt{x^2+5x+4}}\,dx$

$$= \int \frac{\frac{1}{2}(x+2)(2x+5)+\frac{1}{2}(x+2)}{(x+2)\sqrt{(x+2)^2+1}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+5)\,dx}{\sqrt{x^2+2x+5}} + \frac{1}{2}\int \frac{dx}{\sqrt{(x+2)^2+1}}$$

$$= \sqrt{x^2+2x+5} + \frac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+5}\right| + c$$

147. $\int \frac{x^2+10x+6}{(x+2)\sqrt{x^2+4x+9}}\,dx$

$$= \int \left(\frac{\frac{1}{2}(x+2)(2x+4)+6(x+2)-10}{(x+2)\sqrt{x^2+4x+10)}}\right)dx$$

$$= \frac{1}{2}\int \frac{(2x+4)}{\sqrt{x^2+4x+10}}\,dx + 6\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$$-\int \frac{10\,dx}{(x+2)\sqrt{(x+2)^2+6}}$$

$$= \sqrt{x^2+4x+10} + 6\log\left|(x+2)+\sqrt{x^2+4x+10}\right|$$

$$-\frac{1}{2\sqrt{6}}\log\left|\frac{\sqrt{t^2+6}-\sqrt{6}}{\sqrt{t^2+6}+\sqrt{6}}\right| + c$$

where $t = (x+2)$

148. We have,

$$\int \sqrt[3]{x}\,(1+\sqrt{x})^3\,dx$$

$$= \int \sqrt[3]{x}\,(1+\sqrt[3]{x}+3x+x^{3/2})\,dx$$

$$= \int (x^{1/3}+3x^{5/6}+3x^{4/3}+x^{11/6})\,dx$$

$$= \left(\frac{3}{4}x^{4/3}+\frac{18}{11}x^{11/6}+\frac{9}{7}x^{7/3}+\frac{6}{17}x^{17/6}\right)+c$$

149. $\int \sqrt[3]{x^2}\,(3+x^{-2/3})^{-2}\,dx$

Here, $\alpha = -2 \Rightarrow$ a negative integer

Let $x = t^3$

$\Rightarrow \quad dx = 3t^2\,dt.$

$$= \int t^2(3+t^{-2})^{-2}\cdot 3t^2\,dt$$

$$= 3\int \frac{t^4}{(3+t^{-2})^2}\,dt$$

$$= 3\int \frac{dt}{\left(3t^2+\frac{1}{3}\right)^2}$$

$$= \frac{1}{3}\int \frac{dt}{\left(t^2+\frac{1}{3}\right)^2}$$

Now, $I = \int \frac{dx}{x^2+1/3}$

$$= \frac{1}{(x^2+1/3)}\cdot x - \int \frac{-2x\cdot x}{(x^2+1/3)^2}\,dx$$

$$= \frac{1}{(x^2+1/3)}\cdot x + 2\int \frac{(x^2+1.3)-1/3}{(x^2+1/3)^2}\,dx$$

$$= \frac{x}{(x^2+1/3)} + 2\int \frac{dx}{(x^2+1/3)} - \frac{2}{3}\int \frac{dx}{(x^2+1/3)^2} + c$$

$$= \frac{x}{(x^2+1/3)} + 6\tan^{-1}(3x) - \frac{2}{3}\int \frac{dx}{(x^2+1/3)^2} + c$$

$$= \frac{x}{(x^2+1/3)} + 6\tan^{-1}(3x) - \frac{2}{3}I + c$$

$$\Rightarrow \frac{5}{3}I = \frac{x}{(x^2+1/3)} + 6\tan^{-1}(3x) + c$$

$$\Rightarrow I = \frac{3x}{5(x^2+1/3)} + \frac{18}{5}\tan^{-1}(3x) + c$$

150. $\int \left(\frac{\sqrt{1+\sqrt[4]{x}}}{\sqrt[3]{x^4}}\right)dx$

Here, $\frac{\beta+1}{\gamma} = \frac{-\frac{3}{4}+1}{\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$

Let $(1 + x^{\frac{1}{4}}) = t^2$

$\Rightarrow \frac{1}{4}x^{-\frac{3}{4}} = 2t\,dt$

Thus, $\int x^{-3/4}(1 + x^{1/4})^{1/2}dx$

$= 8\int t^2 dt$

$= \frac{8t^3}{3} + c$

$= \frac{8}{3}\left(1 + x^{\frac{1}{4}}\right)^{3/2} + c$

151. $\int \frac{dx}{x^{7}\,{}^{2}\sqrt{1 + x^4}}$

Here, $\frac{\beta + 1}{\gamma} + P = \frac{-7 + 1}{4} - \frac{1}{2}$

$= \frac{-3}{2} - \frac{1}{2} = -2 =$ Integer.

Let $1 + x^4 = x^4 t^2$

$\Rightarrow x^4(1 - t^2) = 1$

$\Rightarrow x^4 = \frac{1}{(1 - t^2)}$

Thus, $\int \frac{dx}{x^7\sqrt{1 + x^4}}$

$= \int \frac{\frac{1}{2}\frac{t}{(t^2 - 1)^2}}{\left(\frac{1}{1 - t^2}\right)^3 t}\,dt$

$= \frac{1}{2}\int (1 - t^2)\,dt$

$= \frac{1}{2}\left(t - \frac{t^3}{3}\right) + c$

$= \frac{1}{2}\left(\sqrt{\frac{1 + x^4}{x^4}} - \frac{1}{3}\left(\frac{1 + x^4}{x^4}\right)^{3/2}\right) + c$

152. $\int \frac{dx}{\sqrt{x}(\sqrt[4]{x} + 1)^{10}} = \int x^{-1/2}(1 + x^{1/4})^{-10}dx$

Here, $\alpha = -10 \in I$

Let $x = t^4$

$\Rightarrow dx = 4t^3 dt$

$= \int \frac{4t^3\,dt}{t^2(1 + t)^{10}}$

$= 4\int \frac{t\,dt}{(t + 1)^{10}}$

$= 4\int \left(\frac{(t + 1) - 1}{(t + 1)^{10}}\right)dt$

$= 4\int \left(\frac{1}{(t + 1)^9} - \frac{1}{(t + 1)^{10}}\right)dt$

$= 4\left(\frac{9}{(t + 1)^9} - \frac{8}{(t + 1)^8}\right) + c$

$= 4\left(\frac{9}{(x^{1/4} + 1)^9} - \frac{8}{(x^{1/4} + 1)^8}\right) + c$

153. $\int x^{-1/2}(2 + 3x^{1/3})^{-2}dx$

Here, $\alpha = -2 \in I$

Let $x = t^6$

$\Rightarrow dx = 6t^5\,dt$

$= \int \frac{6t^5}{t^2(2 + 3t^2)}\,dt$

$= 2\int \left(\frac{(2 + 3t^2) - 2}{(2 + 3t^2)}\right)dt$

$= 2\int \left(1 - \frac{2}{(2 + 3t^2)}\right)dt$

$= 2\int dt - \frac{4}{3}\int \frac{dt}{t^2 + \left(\sqrt{\frac{2}{3}}\right)^2}$

$= 2t - \frac{4}{3} \times \sqrt{\frac{3}{2}}\tan^{-1}\left(\sqrt{\frac{3}{2}}t\right) + c$

$= 2(x^{1/6}) - \frac{2\sqrt{2}}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{3}{2}}x^{1/6}\right) + c$

154. $\int \sqrt[3]{x} \times \sqrt[7]{\left(1 + \sqrt[3]{x^4}\right)}\,dx$

Here, $\frac{\beta + 1}{\gamma} = \frac{\frac{1}{3} + 1}{\frac{4}{3}} = 1 \in I$

Let $(1 + x^{4/3}) = t^7$

$\Rightarrow \frac{4}{3}x^{1/3}\,dx = 7t^6\,dt$

$\Rightarrow x^{1/3}\,dx = \frac{21}{4}t^6\,dt$

$= \frac{21}{4}\int (t^6 \cdot t)\,dt$

$$= \frac{21}{4} \times \frac{t^8}{8} + c$$

$$= \frac{21}{4} \times \frac{(1 + x^{4/3})^{8/7}}{8} + c$$

155. $\int x^{-6}(1 + 2x^3)^{2/3}\,dx$

Here, $\frac{\beta + 1}{\gamma} + \alpha = \frac{-6 + 1}{3} + \frac{2}{3} = -1 \in I$

Let $1 + 2x^3 = x^3 t^3$

$\Rightarrow \quad x^3(t^2 - 2) = 1$

$\Rightarrow \quad x^3 = \frac{1}{(t^3 - 2)}$

$\Rightarrow \quad 3x^2\,dx = -\frac{3t^2\,dt}{(t^3 - 2)^2}$

$\Rightarrow \quad x^2\,dx = -\frac{t^2\,dt}{(t^3 - 2)^2}$

$$= \int \frac{(tx)^{3\times\frac{2}{3}}}{x^6}\,dx$$

$$= \int \frac{t^2 x^2\,dx}{x^6}$$

$$= \int \frac{t^2}{\left(\frac{1}{(t^3 - 2)^2}\right)} \times -\frac{t^2\,dt}{(t^3 - 2)^2}$$

$$= -\int t4\,dt$$

$$= -\left(\frac{t^5}{5}\right) + c$$

$$= -\frac{1}{5}\left(\frac{1 + 2x^3}{x^3}\right)^{5/3} + c$$

156. Do yourself.

157. $\int \frac{dx}{x^{11}\,{}^2\sqrt{1 + x^4}}$

$$= \int x^{-11}(1 + x^4)^{-1/2}\,dx$$

$$= \int x^{-11}(1 + x^4)^{-1/2}\,dx$$

Here, $\frac{\beta + 1}{\gamma} + \alpha = \frac{-11 + 1}{4} + \frac{2}{4} = -2 \in I$

Put $(1 + x^4) = x^4 t^2$

$\Rightarrow \quad x^4(t^2 - 1) = 1$

$\Rightarrow \quad x^4 = \frac{1}{(t^2 - 1)}$

$\Rightarrow \quad 4x^3\,dx = -\frac{2t}{(t^2 - 1)^2}\,dt$

$\Rightarrow \quad x^3\,dx = -\frac{t\,dt}{2(t^2 - 1)^2}$

$$= \int \frac{dx}{x^{11}\,{}^2\sqrt{1 + x^4}}$$

$$= -\frac{1}{2}\int (t^2 - 1)^2\,dt$$

$$= -\frac{1}{2}\int (t^4 - 2t^2 + 1)\,dt$$

$$= -\frac{1}{2}\left(\frac{t^5}{5} - \frac{2t^3}{3} + t\right) + c$$

where $t = \sqrt{\frac{1 + x^4}{x^4}}$

158. $\int \left(\frac{\sqrt[3]{1 + \sqrt[4]{x}}}{\sqrt[4]{x^3}}\right) dx$

Let $(1 + x^{1/4}) = t^3$

$\Rightarrow \quad \frac{1}{4}x^{-3/4}\,dx = 3t^2\,dt$

$\Rightarrow \quad \frac{dx}{x^{3/4}} = 12t^2\,dt$

$$= 12\int t \times t^2\,dt$$

$$= 12\int t^3\,dt$$

$$= 12\left(\frac{t^4}{4}\right) + c$$

$$= 3t^4 + c$$

$$= 3(1 + x^{1/4})^{4/3} + c$$

159. $\int \frac{dx}{(1 + \sqrt{x^2 + x + 1})}$

Here $a = 1 > 0$, therefore we put the Euler first substitution, i.e.

$\sqrt{x^2 + x + 1} = t - x$

$\Rightarrow \quad (x^2 + x + 1) = (t - x)^2$

$\Rightarrow \quad (x^2 + x + 1) = t^2 - 2tx + x^2$

$\Rightarrow \quad (1 + 2t)x = t^2 - 1$

$$\Rightarrow \quad x = \left(\frac{t^2 - 1}{2t + 1}\right)$$

$$\Rightarrow \quad dx = \frac{2(t^2 + t + 1}{(2t + 1)^2}\,dt$$

Now, $1 = \sqrt{x^2 + x + 1} = 1 + t - x$

$$= 1 + t - \left(\frac{t^2 - 1}{2t + 1}\right)$$

Thus, $\int \frac{dx}{\left(1 + \sqrt{x^2 + x + 1}\right)}$

$$= \int \frac{\frac{2(t^2 + t + 1)}{(2t + 1)^2}}{1 + t - \left(\frac{t^2 - 1}{2t + 1}\right)}\,dt$$

$$= \int \frac{2(t^2 + t + 1)}{(t^2 + 3t + 2)(2t + 1)}\,dt$$

$$= 2\left(-2\int \frac{dt}{t + 1} + \frac{8}{3}\int \frac{dt}{t + 2} + \frac{4}{3}\int \frac{dt}{2t + 1}\right)$$

$$-\frac{2}{3}\left(\int \frac{dt}{t + 2} - \int 2\frac{dt}{2t + 1}\right)$$

$$= 2\left(-2\log|t + 1| + \frac{8}{3}\log|t + 2| + \frac{2}{3}\log|2t + 1|\right)$$

$$-\frac{2}{3}(\log|t + 2| - \log|2t + 1|) + c$$

where $\quad t = x + \sqrt{x^2 + x + 1}$

160. $\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$

Here, $c = 1 > 0$, therefor we put the Euler second substitution, i.e.

$\sqrt{x^2 - x + 1} = tx - 1$

$$\Rightarrow \quad (x^2 - x + 1) = t^2x^2 - 2tx + 1$$

$$\Rightarrow \quad x = \frac{2t - 1}{t^2 - 1}$$

$$\Rightarrow \quad dx = \frac{-2(t^2 - t + 1)}{t^2 - 1)^2}\,dt$$

Thus, $\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$

$$= \int \frac{\frac{-2(t^2 - t + 1)}{(t^2 - 1)^2}}{\frac{t^2 - t + 1}{(t^2 - 1)}}\,dt$$

$$= -2\int \frac{dt}{t^2 - 1}$$

$$= -2 \times \frac{1}{2}\log\left|\frac{t - 1}{t + 1}\right| + c$$

$$= \log\left|\frac{t - 1}{t + 1}\right| + c$$

where, $t = \left(\frac{1 + \sqrt{x^2 - x + 1}}{x}\right)$

161. $\int \frac{x\,dx}{\sqrt{7x - 10 - x^2}}$

Here, $7x - 10 - x^2 = -(x^2 - 7x + 10)$

$= -(x - 2)(x - 5)$

$= (x - 2)(5 - x)$, so we can apply Euler third substitution, i.e.

$\sqrt{7x - 10 - x^2} = (x - 2)t$

$$\Rightarrow \quad \sqrt{-(x - 2)(x - 5)} = (x - 2)t$$

$$\Rightarrow \quad (5 - x) = (x - 2)t^2$$

$$\Rightarrow \quad x = \frac{5 + 2t^2}{1 + t^2}$$

$$\Rightarrow \quad dx = -\frac{6t\,dt}{(t^2 + 1)^2}$$

Now, $(x - 2)t = \left(\frac{5 + 2t^2}{t^2 + 1} - 2\right)t = \frac{3t}{t^2 + 1}$

Thus, $\int x\frac{dx}{\sqrt{7x - 10 - x^2}}$

$$= \int \frac{x}{(x - 2)t} \times \frac{-6t}{(1 + t^2)^2}\,dt$$

$$= \int \frac{\frac{5 + 2t^2}{(1 + t^2)}}{\frac{3t}{(1 + t^2)}} \times \frac{-6t}{(1 + t^2)^2}\,dt$$

$$= -2\int \frac{5 + 2t^2}{(1 + t^2)^2}\,dt$$

$$= -2\int \frac{2(1 + t^2) + 3}{(1 + t^2)^2}\,dt$$

$$= -4\int \frac{dt}{(1 + t^2)} - 6\int \frac{dt}{(1 + t^2)^2}$$

$$= -4\tan^{-1}(t) - 6\left(\frac{1}{2}\frac{t}{t^2+1} + \frac{1}{2}\tan^{-1}t\right) + c$$

$$= -7\tan^{-1}(t) - \frac{3t}{t^2+1} + c$$

where, $t = \frac{\sqrt{7x - 10 - x^2}}{(x-2)}$

162. $\int \frac{dx}{x - \sqrt{x^2 - x + 2}}$

Put $\sqrt{x^2 - x + 2} = t + x$

$\Rightarrow (x^2 - x + 2) = (t + x)^2$

$\Rightarrow (x^2 - x + 2) = t^2 + 2tx + x^2$

$\Rightarrow (2t + 1)x = 2 - t^2$

$\Rightarrow x = \left(\frac{2 - t^2}{2t + 1}\right)$

$\Rightarrow dx = \frac{(2x+1)(-2t) - (2 - t^2)2}{(2t+1)^2}dt$

$\Rightarrow dx = \frac{-4t^2 - 2t + 2t^2 - 4}{(2t+1)^2}dt$

$\Rightarrow dx = \frac{-2t^2 - 2t - 4}{(2t+1)^2}dt$

$$= \int \frac{1}{t} \times \frac{2t^2 + 2t + 4}{(2t+1)^2}dt$$

$$= 2\int \frac{1}{t} \times \frac{t^2 + t + 2}{(2t+1)^2}dt$$

$$= 2\left[2\int\frac{dt}{t} - \frac{7}{2}\int\frac{dt}{(2t+1)} - \frac{9}{2}\int\frac{dt}{(2t+1)^2}\right]$$

$$= 2\left(2\log|t| - \frac{7}{4}\log|2t+1| - \frac{9}{4(2t+1)}\right) + c$$

where $t = \left(\sqrt{x^2 - x + 2} - x\right)$

163. Do yourself.

164. $\int \frac{dx}{x\sqrt{x^2 - 3x + 2}}$

Here, $x^2 - 3x + 2 = (x-1)(x-2)$

Let $\sqrt{x^2 - 3x + 2} = (x-1)t$

$\Rightarrow (x^2 - 3x + 2) = (x-1)^2 t^2$

$\Rightarrow (x-1)(x-2) = (x-1)^2 t^2$

$\Rightarrow (x-2) = (x-1)t^2$

$\Rightarrow x(1 - t^2) = 2 - t^2$

$\Rightarrow x = \left(\frac{t^2 - 2}{t^2 - 1}\right)$

$\Rightarrow dx = \frac{(t^2 - 1)(2t) - (t^2 - 2)2t}{(t^2-1)^2}dt$

$\Rightarrow dx = \frac{2t(t^2 - 1 - t^2 + 2)}{(t^2-1)^2}dt$

$\Rightarrow dx = \frac{2t\,dt}{(t^2-1)^2}$

$$= \int \frac{\frac{2t\,dt}{(t^2-1)^2}}{\left(\frac{t^2-2}{t^2-1}\right) \times -\frac{1}{(t^2-1)}}dt$$

$$= -\int \frac{2t\,dt}{(t^2 - 2)}$$

$$= -\log|(t^2 - 2)| + c$$

where $t = \frac{\sqrt{x^2 - 3x + 2}}{(x-1)}$

165. $\int \frac{dx}{x + \sqrt{x^2 - 1}}$

$$= \int \frac{(x - \sqrt{x^2-1})}{(x + \sqrt{x^2-1})(x - \sqrt{x^2-1})}dx$$

$$= \int \frac{(x - \sqrt{x^2 - 1}}{(x^2 - x^2 + 1)}\,dx$$

$$= \int (x - \sqrt{x^2-1})\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x\sqrt{x^2-1}}{2} - \frac{1}{2}\log\left|x + \sqrt{x^2-1}\right|\right) + c$$

166. $\int \frac{x\,dx}{x + \sqrt{x^2-1}} = \int x(x - \sqrt{x^2-1})\,dx$

$$= \int (x^2 - x\sqrt{x^2-1})\,dx$$

$$= \frac{x^3}{3} - \int x\sqrt{x^2 - 1}\,dx$$

Let $x^2 - 1 = t^2$

$xdx = tdt$

$$= \frac{x^3}{3} - \int t^2\, dt$$

$$= \frac{x^3}{3} - \frac{t^3}{3} + c$$

$$= \frac{x^3}{3} - \frac{(x^2-1)^{3/2}}{3} + c$$

167. We have,

$$\int \sin^5 x\, dx$$

$$= -\frac{\cos x \sin^4 x}{5} + \frac{4}{5} I_3$$

$$= -\frac{\cos x \sin^4 x}{5} - \frac{4}{5}\left(-\frac{\cos x \sin^2 x}{3} + \frac{2}{3} I_1\right)$$

$$= -\frac{\cos x \sin^4 x}{5} - \frac{4}{15}(\cos x \sin^2 x + 2\cos x) + c$$

168. We have,

$$\int \sin^6 x\, dx$$

$$= -\frac{\cos x \sin^5 x}{6} + \frac{5}{6} I_4$$

$$= -\frac{\cos x \sin^5 x}{6} + \frac{5}{6}\left(-\frac{\cos x \cdot \sin^3 x}{4} + \frac{3}{4} I_2\right)$$

$$= \frac{\cos x \sin^5 x}{6} + \frac{5}{6}\left(-\frac{\cos x \cdot \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\, dx\right)$$

$$= -\frac{\cos x \sin^5 x}{6} - \frac{5}{24}\cos x \cdot \sin^3 x + \frac{3}{4}(x - \sin x \cos x) + c$$

169. We have,

$$\int \cos^7 x\, dx$$

$$= \frac{\cos^6 x \cdot \sin x}{9} + \frac{8}{9} I_5$$

$$= \frac{\cos^6 x \cdot \sin x}{9} + \frac{8}{9}\left(\frac{\cos^4 x \cdot \sin x}{7} + \frac{6}{7} I_3\right)$$

$$= \frac{\cos^6 x \cdot \sin x}{9} + \frac{8}{63}\cos^4 x \cdot \sin x + \frac{48}{63} I_3$$

$$= \frac{\cos^6 x \cdot \sin x}{9} + \frac{8}{63}\cos^4 x \cdot \sin x + \frac{48}{63}\left(\frac{\cos^2 x \cdot \sin x}{5} + \frac{4}{5} I_1\right)$$

$$= \frac{\cos^6 x \cdot \sin x}{9} + \frac{8}{63}\cos^4 x \cdot \sin x + \frac{48}{63}\left(\frac{\cos^2 x \cdot \sin x}{5} + \frac{4}{5}\sin x\right) + c$$

170. We have,

$$\int \cos^8 x\, dx$$

$$= \frac{\cos^7 x \cdot \sin x}{10} + \frac{9}{10} I_6$$

$$= \frac{\cos^7 x \cdot \sin x}{10} + \frac{9}{10}\left(\frac{\cos^5 x \cdot \sin x}{8} + \frac{7}{8} I_4\right)$$

$$= \frac{\cos^7 x \cdot \sin x}{10} + \frac{9}{80}\cos^5 x \cdot \sin x + \frac{63}{80} I_4$$

$$= \frac{\cos^7 x \cdot \sin x}{10} + \frac{9}{80}\cos^5 x \cdot \sin x + \frac{63}{80}\left(\frac{\cos^3 x \cdot \sin x}{6} + \frac{5}{6} I_2\right)$$

$$= \frac{\cos^7 x \cdot \sin x}{10} + \frac{9}{80}\cos^5 x \cdot \sin x + \frac{63}{80}\left(\frac{\cos^3 x \cdot \sin x}{6} + \frac{5}{12}(x + \sin x \cdot \cos x)\right) + c$$

171. We have,

$$\int \tan^5 x\, dx$$

$$= \left(\frac{\tan^4 x}{4} - I_3\right)$$

$$= \left(\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + I_1\right)$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \int \tan x\, dx$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$$

172. We have,

$$\int \tan^6 x\, dx$$

$$= \left(\frac{\tan^5 x}{5} - I_4\right)$$

$$= \left(\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + I_2\right)$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int \tan^2 x\, dx$$

$$= \left(\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int (\sec^2 x - 1)\, dx\right)$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

173. We have,

$$\int \cot^7 x\, dx.$$

$$= -\frac{\cot^6 x}{6} - I_5 + c$$

$$= -\frac{\cot^6 x}{6} - \left(-\frac{\cot^4 x}{4} - I_3\right) + c$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} + I_3 + c$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} + \left(-\frac{\cot^2 x}{2} - I_1\right) + c$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - I_1 + c$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log|\sin x| + c$$

174. We have,

$$-\frac{\cot^5 x}{5} - I_4 + C$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} + \int \cot^2 x\, dx + c$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} + \int (\operatorname{cosec}^2 x - 1)\, dx + c$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$$

175. We have,

$$\int \sec^2 x\, dx$$

$$= \frac{\sec x \cdot \tan x}{2} + \frac{1}{2} I_1$$

$$= \frac{\sec x \cdot \tan x}{2} + \frac{1}{2}\log|\sec x + \tan x| + C$$

176. We have,

$$\int \sec^4 x\, dx$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x + c$$

177. We have,

$$\int \sec^5 x\, dx$$

$$= \frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{4} I_3$$

$$= \frac{\sec^3 x \cdot \tan x}{4}$$

$$+ \frac{3}{4}\left(\sec x \cdot \tan\frac{x}{2} + \frac{1}{2}\log|\sec x + \tan x|\right) + c$$

178. We have,

$$\int \sec^7 x\, dx$$

$$= \frac{\sec^5 x \cdot \tan x}{6} + \frac{5}{6} I_5$$

$$= \frac{\sec^5 x \cdot \tan x}{6} + \frac{5}{6}\left(\frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{4} I_3\right)$$

$$= \frac{\sec^5 x \cdot \tan x}{6} + \frac{5}{24}\sec^3 x \cdot \tan x + \frac{15}{24} I_3$$

$$= \frac{\sec^5 x \cdot \tan x}{6} + \frac{5}{24}\sec^3 x \cdot \tan x$$

$$+ \frac{15}{24}\left(\frac{\sec x \cdot \tan x}{2} + \frac{1}{2}\log|\sec x + \tan x|\right) + c$$

179. We have,

$$\int \frac{\sin 3x}{\sin x} dx$$

$$= \frac{2\sin(3-1)x}{(3-1)} + I_{3-2}$$

$$= \sin 2x + I_1$$

$$= \sin 2x \int dx$$

$$= \sin 2x + x + c$$

180. We have,

$$\int \frac{\sin 5x}{\sin x} dx$$

$$= \frac{2\sin(5-1)x}{(5-1)} + I_{5-2}$$

$$= \frac{1}{2}\sin 4x + I_3$$

$$= \frac{1}{2}\sin 4x + \frac{2\sin(3-1)x}{(3-1)} + I_1$$

$$= \frac{2}{3}\sin 4x + \sin 2x + x + c$$

181. We have,

$$\int \frac{\sin 6x}{\sin x} dx$$

$$= \frac{2\sin(6-1)x}{(6-1)} + I_{6-2}$$

$$= \frac{2}{5}\sin 5x + I_4$$

$$= \frac{2}{5}\sin 5x + \frac{2}{3}\sin 3x + I_2$$

$$= \frac{2}{5}\sin 5x + \frac{2}{3}\sin 3x + \int \frac{\sin 2x}{\sin x} dx$$

$$= \frac{2}{5}\sin 5x + \frac{2}{3}\sin 3x + 2\int \cos x\, dx$$

$$= \frac{2}{5}\sin 5x + \frac{2}{3}\sin 3x + 2\sin x + c$$

182. We have,

$$\int \frac{\sin 8x}{\sin x}$$

$$= \frac{2\sin(8-1)x}{(8-1)} + I_6$$

$$= \frac{2}{7}\sin 7x + \frac{2}{5}\sin 5x + I_4$$

$$= \frac{2}{7}\sin 7x + \frac{2}{5}\sin 5x + \frac{2}{3}\sin 3x + I_2$$

$$= \frac{2}{5}\sin 7x + \frac{2}{5}\sin 5x + \frac{2}{3}\sin 3x + 2\sin x + c$$

183. We have $\int \frac{dx}{(x^2+2)^2}$

$$= \frac{x}{2.2(x^2+2)} + \frac{1}{2}\left(\frac{2.2-3}{2.2-2}\right)I_{2-1}$$

$$= \frac{x}{4}(x^2+2) + \frac{1}{4}\int \frac{dx}{(x^2+2)}$$

$$= \frac{x}{4}(x^2+2) + \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

184. We have,

$$\int \frac{dx}{(x^2+3)^2}$$

$$= \frac{x}{2(3-1)3(x^2+3)^2} + \frac{1}{3}\left(\frac{2.3-3}{22.3-2}\right)I_{3-1}$$

$$= \frac{x}{12(x^2+3)^2} + \frac{1}{4}I_2$$

$$= \frac{x}{12(x^2+3)^2} + \frac{1}{4}\left(\frac{x}{2.3(x^2+3)} + \frac{1}{3}\frac{1}{2}I_1\right)$$

$$= \frac{x}{12(x^2+3)^2} + \frac{x}{24(x^2+3)}$$

$$+ \frac{1}{24\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

185. We have,

$$\int \frac{x+1}{(x^2+3x+2)^2}dx$$

$$= \frac{1}{2}\int \frac{(2x+2)}{(x^2+3x+2)^2}dx$$

$$= \frac{1}{2}\int \frac{(2x+1)-1}{(x^2+3x+2)^2}dx$$

$$= \frac{1}{2}\int \frac{(2x+3)}{(x^2+3x+2)^2}dx - \frac{1}{2}\int \frac{dx}{(x^2+3x+2)^2}$$

$$= -\frac{1}{2(x^2+3x+2)} - \frac{1}{2}\int \frac{dx}{\left[\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^{22}\right]}$$

$$= -\frac{1}{2}(x^2+3x+2) - \frac{1}{2}\left[\frac{\left(x+\frac{3}{2}\right)}{\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}}\right.$$

$$\left. + \frac{1}{2\cdot\frac{1}{2}}\log\left(\frac{\left(x+\frac{3}{2}\right)-\frac{1}{2}}{\left(x+\frac{3}{2}\right)+\frac{1}{2}}\right)\right] + C$$

$$= -\frac{1}{2(x^2+3x+2)} - \frac{1}{2}\left[\frac{\left(x+\frac{3}{2}\right)}{\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}}\right.$$

$$\left. + \log\left(\frac{x+1}{x+2}\right)\right] + c$$

186. We have,

$$\int x^2 \log x\, dx$$

$$= (\log x)\cdot\frac{x^3}{3} - \frac{1}{3}I_{2,0}$$

$$= (\log x)\cdot\frac{x^3}{3} - \frac{1}{3}\int x^2 dx$$

$$= (\log x)\cdot\frac{x^3}{3} - \frac{x^3}{9} + C$$

187. We have,

$$\int x^2(\log x)^2 dx$$

$$= (\log x)^2\frac{x^3}{3} - \frac{2}{3}I_{2,1}$$

$$= (\log x)^2\frac{x^3}{3} - \frac{2}{3}\left(\int x^2\log x\right)dx$$

$$= (\log x)^2\frac{x^3}{3} - \frac{2}{3}\left(\log x)\frac{x^3}{3} - \frac{x^3}{9}\right) + c$$

188. We have,

$$\int x^3(\log x)^2 dx$$

$$= (\log x)^2\frac{x^4}{4} - \frac{2}{4}I_{3,1}$$

$$= (\log x)^2\frac{x^4}{4} - \frac{1}{2}\left((\log x)\frac{x^4}{4} - \frac{1}{4}I_{3,0}\right)$$

$$= (\log x)^2\frac{x^4}{4} - \frac{1}{2}\left((\log x)\frac{x^4}{4} - \frac{1}{4}\int x^3 dx\right)$$

$$= (\log)^2\frac{x^4}{4} - \frac{1}{2}\left((\log x)\frac{x^4}{4} - \frac{x^4}{16}\right) + c$$

189. We have,

$$\int x^2(1-x)^3\,dx$$

$$= \frac{x^3(1-x)^3}{(2+3+1)} + \frac{3}{(2+3+1)} I_{2,2}$$

$$= \frac{x^3(1-x)^3}{6} + \frac{1}{2} I_{2,2}$$

$$= \frac{x^3(1-x)^3}{6} + \frac{1}{2}\left(\frac{x^3(1-x)^2}{5} + \frac{2}{5} I_{2,1}\right)$$

$$= \frac{x^3(1-x)^3}{6} + \frac{1}{2}\left[\frac{x^3(1-x)^2}{5} + \frac{2}{5}\left(\frac{x^3}{4} + \frac{1}{4} I_{2,0}\right)\right]$$

$$= \frac{x^3(1-x)^3}{6} + \frac{1}{2}\left[\frac{x^3(1-x)^2}{5} + \frac{2}{5}\left(\frac{x^3}{4} + \frac{1}{4}\int x^2\,dx\right)\right]$$

$$= \frac{x^3(1-x)^3}{6} + \frac{1}{2}\left[\frac{x^3(1-x)^2}{5} + \frac{2}{5}\left(\frac{x^3}{4} + \frac{x^3}{12}\right)\right] + c$$

Note: Q-190 to 197, do yourself.

HINTS AND SOLUTIONS

Level III

(Problems for JEE-Advanced)

1. We have,

$$\int \frac{\cos 2x - \cos x}{1 - \cos x}\,dx$$

$$= \int \frac{2\cos^2 x - \cos x - 1}{1 - \cos x}\,dx$$

$$= \int \frac{(\cos x - 1)(2\cos x + 1)}{1 - \cos x}\,dx$$

$$= -\int (2\cos x + 1)\,dx$$

$$= -(2\sin x + x) + c$$

2. We have,

$$\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx$$

$$= \int \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x - \sin 6x}\,dx$$

$$= \int \frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)}{-2\cos\left(\frac{9x}{2}\right)\sin\left(\frac{3x}{2}\right)}\,dx$$

$$= -\int 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)dx$$

$$= -\int (\cos 2x + \cos x)\,dx$$

$$= -\left(\frac{\sin 2x}{2} + \sin x\right) + c$$

3. We have,

$$\int \left(\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x dx}\right)$$

$$= \int \left(\frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 3)\,dx$$

$$= (\tan x - \cot x - x) + c$$

4. We have,

$$\int \frac{\cos 2x}{\cos^2 x\sin^2 x}\,dx$$

$$= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x\sin^2 x}\,dx$$

$$= \int (\operatorname{cosec}^2 x - \sec^2 x)\,dx$$

$$= -(\cot x + \tan x) + c$$

5. We have,

$$\int \frac{\sin 2x}{\sin 5x\sin 3x}\,dx$$

$$= \int \frac{\sin(5x - 3x)}{\sin 5x\sin 3x}\,dx$$

$$= \int \frac{\sin 5x\cos 3x - \cos 5x\sin 3x}{\sin 5x\sin 3x}\,dx$$

$$= \int (\cot 3x - \cot 5x)\,dx$$

$$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$$

6. We have,

$$\int \frac{\sin x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \left(1 + \frac{\sin x - \cos x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}(x - \log|\sin x + \cos x|) + c$$

7. We have,

$$\int \frac{dx}{\sec x + \operatorname{cosec} x}$$

$$= \int \frac{\sin x \cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{2\sin x \cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(1 + \sin 2x) - 1}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)^2}{\sin x + \cos x} dx - \frac{1}{2}\int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{2}\int (\sin x + \cos x) dx - \frac{1}{2}\int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{2}\int (\sin x + \cos x) dx - \frac{1}{2\sqrt{2}}\int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)}$$

$$= \frac{1}{2}(\sin x - \cos x) - \frac{1}{2\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + c$$

8. We have,

$$\int \tan 3x \tan 2x \tan x \, dx$$

$$= \int (\tan 3x - \tan 2x - \tan x) dx$$

$$= \frac{1}{3}\log|\sec 3x| - \frac{1}{2}\log|\sec 2x| - \log|\sec x| + c$$

9. We have,

$$\int \frac{b}{a + ce^x} dx$$

$$= \int \frac{be^{-x}}{ce^{-x} + c} dx$$

$$= -\frac{b}{a}\int \frac{dt}{t}, \quad \text{Let } ae^{-x} + c = t$$

$$\Rightarrow \quad -ae^x dx = dt$$

$$= -\frac{b}{a}\log|t| + k$$

$$= -\frac{b}{a}\log|ae^{-x} + c| + k$$

10. We have,

$$\int \frac{dx}{1 + e^x}$$

$$= \int \frac{e^x dx}{e^x(e^x + 1)}$$

$$= \int \frac{dt}{t(t + 1)}, \quad \text{Let } e^x = t$$

$$\Rightarrow \quad e^x dx = dt$$

$$= \int \left(\frac{1}{t} - \frac{1}{t + 1}\right) dt$$

$$= \log\left|\frac{t}{t + 1}\right| + c$$

$$= \log\left|\frac{e^x}{e^x + 1}\right| + c$$

11. We have,

$$\int \frac{x + 9}{x^3 + 9x} dx$$

$$= \int \frac{x + 9 + x^2 - x^2}{x(x^2 + 9)} dx$$

$$= \int \frac{x + (9 + x^2) - x^2}{x(x^2 + 9)} dx$$

$$= \int \frac{dx}{x^2 + 9} + \int \frac{dx}{x} - \int \frac{x}{(x^2 + 9)} dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \log|x| - \frac{1}{2}\log|x^2 + 9| + c$$

12. We have

$$\int \sin x - \cos x / e^x + \sin x dx$$

$$= \int \frac{(e^x + \sin x) - (e^x + \cos x)}{e^x + \sin x)} dx$$

$$= \int \left(\frac{1 - (e^x + \cos x)}{(e^x + \sin x)}\right) dx$$

$$= (x - \log|e^x + \sin x|) + c$$

13. We have,

$$\int \frac{\cos x - \sin x + 1 - x}{|e^x + \sin x + x} dx$$

$$= \int \frac{(\cos x - 1) - (\sin x + x)}{e^x + \sin x + x} dx$$

$$= \int \frac{(e^x + \cos x - 1) - (e^x + \sin x + x)}{e^x + \sin x + x}\, dx$$

$$= \log|e^x + \sin x + x| - x + c$$

14. We have,

$$\int \frac{\sin(x+a)}{\sin(x+b)} dx$$

$$= \int \frac{\sin(t - b + a)}{\sin t} dt$$

$$= \int \frac{\sin(t + (b - a))}{\sin t} dt$$

$$= \int \frac{\sin t.\cos(a-b) + \cos t.\sin(a-b)}{\sin t} dt$$

$$= \cos(a-b) \int dt + \sin(a-b) \int \cot t\, dt$$

$$= t\cos(a-b) + \sin(a-b)\log|\sin t| + c$$

$$= (x+b)\cos(a-b) + \sin(a-b)\log|\sin(x+b)| + c$$

15. We have $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin((x-a)-(x-b))}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b)}{\sin(x-a)\sin(x-b)}\, dx - \frac{\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} dx \right]$$

$$= \frac{1}{\sin(b-a)} \int [\cot(x-b) - \cot(x-a)] dx$$

$$= \frac{1}{\sin(b-a)} [\log|\sin(x-b)| - \log|\sin(x-a)|] + c$$

$$= \frac{1}{\sin(b-a)} \left(\log\left| \frac{\sin(x-b)}{\sin(x-a)} \right| \right) + c$$

16. We have,

$$\int \frac{dx}{\sin(x-a)\cos(x-b)}$$

$$= \frac{1}{\cos(b-a)} \int \frac{\cos(b-a)}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(b-a)} \int \frac{\cos((x-a)-(x-b))}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(b-a)} \int \left[\frac{\cos(x-a)\cos(x-b)}{\sin(x-a)\cos(x-b)}\, dx + \frac{\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\, dx \right]$$

$$= \frac{1}{\cos(b-a)} \int (\cot(x-a) + \tan(x-b)) dx$$

$$= \frac{1}{\cos(b-a)} [\log|\sin(x-a)| - \log(\sec\{x-b\})] + c$$

$$= \frac{1}{\cos(b-a)} [\log|\sin(x-a)\cos(x-b)|] + c$$

17. We have,

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

$$= \int \frac{\sec^2 x \sqrt{\tan x}}{\tan x} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$$= \int \frac{2t\,dt}{t}, \qquad \text{Let } \tan x = t^2$$

$$\Rightarrow \quad \sec^2 x dx = 2t dt$$

$$= 2t + c$$

$$= 2\sqrt{\tan x} + c$$

18. We have,

$$\int \frac{dx}{x\sqrt{x^4 - 1}}$$

$$= \int \frac{x^3 dx}{x^4 \sqrt{x^4 - 1}}$$

$$= \frac{1}{2} \int t \frac{dt}{(t^2 + 1)t}$$

$$= \frac{1}{2} \int \frac{dt}{(t^2 + 1)}$$

$$= \frac{1}{2} \tan^{-1}(t) + c$$

$$= \frac{1}{2} \tan^{-1}\left(\sqrt{x^4 - 1}\right) + c$$

19. Let

$$\int \frac{dx}{(x^2 + 1)}$$

$$= \frac{1}{(x^2+1)} \int dx - \int \left(-\frac{2x \cdot x}{(x^2+1)^2} \right) dx$$

$$= \frac{1}{(x^2+1)} \int dx - 2\int \left(\frac{(x^2+1) - 1}{(x^2+1)^2} \right) dx$$

$$= \frac{x}{(x^2+1)} + 2\int \frac{dx}{(x^2+1)} - 2\int \frac{dx}{(x^2+1)^2}$$

$$\Rightarrow 2\int \frac{dx}{(x^2+1)^2} = \frac{x}{(x^2+1)} + 2\tan^{-1}x - \tan^{-1}x + c$$

$$\Rightarrow \int \frac{dx}{(x^2+1)^2} = \frac{x}{(x^2+1)} + \frac{1}{2}\tan^{-1}x + c$$

20. We have,

$$\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$$

$$= \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)} \times \frac{\sin(x-\alpha)}{\sin(x-\alpha)}}\,dx$$

$$= \int \frac{\sin(x-\alpha)}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= \int \frac{\sin x \cos\alpha - \cos x \sin\alpha}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= \cos\alpha \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx - \sin\alpha \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= \cos\alpha \int \frac{\sin x}{\sqrt{\cos^2\alpha - \cos^2 x}}\,dx - \sin\alpha \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= -\cos\alpha \sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha \log\left|\sin\alpha + \sqrt{\sin^2 x - \sin^2\alpha}\right| + c$$

21. We have,

$$\int \frac{x}{x^4 + x^2 + 1}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{t^2} + t + 1, \qquad \text{Let } x^2 = t \Rightarrow 2xdx = dt$$

$$= \frac{1}{2}\int \frac{dx}{\left(t+\frac{1}{2}\right)^2} + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$$

22. Let $I = \int \sec^3 x\,dx$

$$= \int (\sec^2 x \cdot \sec x)dx$$

$$= \sec x \int \sec^2 x\,dx - \int (\sec x \cdot \tan x \cdot \tan x)\,dx$$

$$= \sec x \int \sec^2 x\,dx - \int (\sec x \cdot \tan^2 x)dx$$

$$= \sec \cdot x \tan x - \int (\sec x(\sec^2 x - 1))dx$$

$$= \sec \cdot x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow 2I = \sec x \cdot \tan x + \int \sec x\,dx$$

$$\Rightarrow 2I = \sec x \cdot \tan x - \log|\sec x + \tan x| + c$$

$$\Rightarrow I = \frac{1}{2}(\sec x \cdot \tan x + \log|\sec x + \tan x|) + c$$

23. Let $I = \int \operatorname{cosec}^3 x\,dx$

$$= \int \operatorname{cosec}^3 x \cdot \operatorname{cosec} x\,dx$$

$$= \operatorname{cosec} x \int \operatorname{cosec}^2 x\,dx - \int (-\operatorname{cosec} x \cdot \cot x \cdot -\cot x)dx$$

$$= -\operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec} x \cdot \cot^2 x\,dx$$

$$= -\operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec} x(\operatorname{cosec}^2 x - 1)dx$$

$$= -\operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec}^3 x\,dx + \int \operatorname{cosec} x\,dx$$

$$\Rightarrow 2I = -\operatorname{cosec} x \cdot \cot x + \int \operatorname{cosec} x\,dx$$

$$\Rightarrow 2I = -\operatorname{cosec} x \cdot \cot x + \log|\operatorname{cosec} x - \cot x| + c$$

$$\Rightarrow I = \frac{1}{2}(-\operatorname{cosec} x \cdot \cot x + \log|\operatorname{cosec} x - \cot x| + c$$

24. We have,

$$\int \frac{x^2 - 1}{x\sqrt{x^4 + 3x^2 + 1}}\,dx$$

$$= \int \frac{x^2 - 1}{x^2\sqrt{\left(x^4 + \frac{1}{x^2}\right) + 3}}\,dx$$

$$= \int \left(1 - \frac{1}{x^2}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2 + 3}$$

$$= \int \frac{dt}{\sqrt{t^2 + 1}}, \quad \text{Let } x + \frac{1}{x} = t$$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$$

$$= \log\left|t + \sqrt{t^2 + 1}\right| + c$$

$$= \log\left|\left(x + \frac{1}{x}\right) + \sqrt{\left(x^2 + \frac{1}{x^2} + 3\right)}\right| + c$$

25. We have,

$$\int \frac{dx}{x^4 + 18x^2 + 81}$$

$$= \int \frac{dx}{(x^2 + 9)^2}$$

Now, $\int \frac{dx}{(x^2 + 9)^2}$

$$= \frac{1}{(x^2 + 9)}\int dx - \int\left(\frac{-2x \cdot x}{(x^2 + 9)^2}\right)dx$$

$$= \frac{x}{(x^2 + 9)} + 2\int\left(\frac{(x^2 + 9) - 9}{(x^2 + 9)^2}\right)dx$$

$$= \frac{x}{(x^2 + 9)} + 2\int \frac{dx}{(x^2 + 9)} - 18\int \frac{dx}{(x^2 + 9)^2}$$

$$\Rightarrow \quad 19\int \frac{dx}{(x^2 + 9)^2}$$

$$= \frac{x}{(x^2 + 9)} + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

$$= \frac{x}{(x^2 + 9)} + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

$$\Rightarrow \quad \int \frac{dx}{(x^2 + 9)^2} = \frac{x}{19(x^2 + 9)} + \frac{1}{57}\tan^{-1}\left(\frac{x}{3}\right) + c$$

26. We have,

$$\int\left(\frac{x + \sin x}{1 + \cos x}\right)dx$$

$$= \int\left(\frac{x}{1 + \cos x} + \frac{\sin x}{1 + \cos x}\right)dx$$

$$= \int\left(\frac{x}{2\cos^2(x/2)} + \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right)dx$$

$$= \int\left(\frac{x}{2}\sec^2\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right)\right)dx$$

$$= \frac{x}{2}\int \sec^2\left(\frac{x}{2}\right)dx - \int \tan\left(\frac{x}{2}\right)dx + \int \tan\left(\frac{x}{2}\right)dx$$

$$= \frac{x}{2}\left(2\tan\left(\frac{x}{2}\right)\right) + c$$

$$= x\tan\left(\frac{x}{2}\right) + c$$

27. We have,

$$\int \sin 4x \; e^{\tan 2x} dx$$

$$= \int 2\sin 2x . \cos 2x . e^{\tan 2x} dx$$

$$= \int 2\frac{2\tan x}{1 + \tan^2 x} \cdot \frac{1 - \tan^2 x}{1 + \tan^2 x} \cdot e^{\tan 2x} dx$$

$$= 2\int \frac{1 - \tan^2 x}{(1 + \tan^2 x)^3} \cdot 2\tan x \cdot \sec^2 x \cdot e^{\tan 2x} dx$$

Let $\tan^2 x = t$

$\Rightarrow \quad 2\tan x \sec^2 x \; dx = dt$

$$= 2\int \frac{e^t(1 - t)}{(1 + t)^3} dt$$

$$= -2\int e^t\left(\frac{(t + 1) - 2}{(t + 1)^3}\right)dt$$

$$= -2\int e^t\left(\frac{1}{(t + 1)^2} - \frac{2}{(t + 1)^3}\right)dt$$

$$= -2\int \frac{e^t}{(t + 1)^2} + c$$

28. We have,

$$\int \sin^{-1}\left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}}\right)dx$$

$$= \int \sin^{1}\left(\frac{2x + 2}{\sqrt{(2x + 8)^2 + 3^2}}\right)dx$$

Let $(2x + 2) = 3\tan\theta$

$\Rightarrow \quad 2dx = 3\sec^2\theta \; d\theta$

$$= \frac{3}{2}\int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\sec^2\theta d\theta$$

$$= \frac{3}{2}\int \theta \sec^2\theta d\theta$$

$$= \frac{3}{2}\left(\theta\int \sec^2\theta d\theta - \int \tan\theta d\theta\right)$$

$$= \frac{3}{2}\left(\theta \tan\theta - \log|\sec\theta|\right) + c$$

$$= \frac{3}{2}\left(\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right) + c$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + c$$

29. We have,

$$\int \frac{x^2}{(x\sin x + \cos x)^2} dx$$

$$= \int \frac{(x\cos x)(x\sec x)}{(x\sin x+\cos x)^2} dx$$

$$= x\sec x \int \frac{(x\cos x)}{(x\sin x+\cos x)^2} dx$$

$$- \int - \frac{(\sec x + x\sec x \cdot \tan x)}{x\sin x + \cos x} dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \sec^2 x\, dx$$

$$= -\frac{x\sec x}{x\sin x} + \cos x + \tan x + c$$

30. We have,

$$\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x+1)^2} dx$$

$$= \int x^2\left(\frac{x\sec^2 x+\tan x}{(x\tan x+1)^2}\right) dx$$

$$= x^2 \int \left(\frac{x\sec^2 x+\tan x}{(x\tan x+1)^2}\right) dx$$

$$- \int \left(2x. - \frac{1}{(x\tan x+1)}\right) dx$$

$$= -\frac{x^2}{(\tan x+1)} + 2\int \left(\frac{x\cos x}{(x\sin x+\cos x)}\right) dx$$

$$= -\frac{x^2}{(\tan x+1)} + 2\log|x\sin x + \cos x| + c$$

31. We have,

$$\int \frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2} dx$$

$$= \int \frac{2\cos x + 1}{(2+\cos x)^2} dx$$

$$= \int \frac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2} dx$$

$$= \int \frac{\cos x(2+\cos x) + \sin^2 x}{(2+\cos x)^2} dx$$

$$= \int \frac{\cos x}{(2+\cos x)} dx + \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

$$= \int \frac{1}{(2+\cos x)} \int \cos x\, dx - \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

$$+ \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

$$= \frac{\sin x}{(2+\cos x)} + c$$

32. We have,

$$\int \cos(2\theta) \times \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) d\theta$$

$$= \int \cos(2\theta)\log\left(\frac{1+\sin 2\theta}{\cos 2\theta}\right) d\theta$$

$$= \log\left(\frac{1+\sin 2\theta}{\cos 2\theta}\right)\int \cos(2\theta)\, d\theta$$

$$- \int \frac{\sin(2\theta)}{2} \cdot \frac{2}{\cos 2\theta} d\theta$$

$$= \log\left(\frac{1+\sin 2\theta}{\cos 2\theta}\right) \times \sin\frac{(2\theta)}{2} - \frac{1}{2}\log|\sec 2\theta| + c$$

33. We have,

$$\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right) dx$$

$$= \int e^x \frac{(x^4+1)+1}{(1+x^2)^{5/2}} dx$$

$$= \int e^x \frac{(x^2+1)^2 + (1-2x^2)}{(1+x^2)^{5/2}} dx$$

$$= \int e^x\left(\frac{(x^2+1)^2}{(1-x^2)^{5/2}} + \frac{(1-2x^2)}{(1+x^2)^{5/2}}\right) dx$$

$$= \int e^x\left(\frac{1}{(1+x^2)^{1/2}} + \frac{(1-2x^2)}{(1+x^2)^{5/2}}\right) dx$$

$$= \int e^x\left(\frac{1}{(1+x^2)^{1/2}} + \frac{x}{(1+x^2)^{5/2}}\right) dx$$

$$+ \int e^x\left(\frac{x}{(1+x^2)^{3/2}} + \frac{(1-2x^2)}{(1+x^2)^{5/2}}\right) dx$$

$$= e^x\left(\frac{1}{\sqrt{1+x^2}} + \frac{x}{(1+x^2)^{3/2}}\right) + c$$

34. We have,

$$\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$$

$$= \int e^{\sin x}(x\cos x - \sec x \cdot \tan x)\,dx$$

$$= \int e^{\sin x}\left(x\cos x - \frac{\sin x \cdot \cos x}{\cos^3 x}\right)dx$$

$$= \int e^{t}\left(\sin^{-1} t - \frac{t}{(1-t^2)^{3/2}}\right)dt, \quad \text{Let } t = \sin x$$

$$\Rightarrow \quad dt = \cos x\,dx$$

$$= \int e^{t}\left[\sin^{-1} t - \frac{1}{\sqrt{1-t^2}} + \left(\frac{1}{\sqrt{1-t^2}} - \frac{t}{(1-t^2)^{3/2}}\right)\right]dt$$

$$= e^{t}\left(\sin^{-1} t - \frac{1}{\sqrt{1-t^2}}\right) + c$$

$$= e^{\sin x}(x - \sec x) + c$$

35. We have,

$$\int \frac{dx}{\cos x + \operatorname{cosec} x}$$

$$= \int \frac{\sin x\,dx}{\sin x \cdot \cos x + 1}$$

$$= \int \frac{2\sin x\,dx}{2\sin x \cdot \cos x + 2}$$

$$= \int \frac{2\sin x\,dx}{\sin 2x + 2}$$

$$= \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin 2x + 2}\,dx$$

$$= \int \frac{(\sin x + \cos x)}{\sin 2x + 2}\,dx + \int \frac{(\sin x - \cos x)}{2 + \sin 2x + 2x}dx$$

Let $\sin x - \cos x = t$ and $\sin x + \cos x = v$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$ and $(\cos x - \sin x)dx = dy$

$$= \int \frac{dt}{1 + t^2 + 2} - \int \frac{dv}{v^2 - 1 + 2}$$

$$= \int \frac{dt}{t^2 - 3} - \int \frac{dv}{v^2 + 1}$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \tan^{-1}(v) + c$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{(\sin x - \cos x) - \sqrt{3}}{(\sin x - \cos x) + \sqrt{3}}\right|$$

$$-\tan^{-1}(\sin x + \cos x) + c$$

36. We have,

$$\int \cot^{-1}(1 + x + x^2)\,dx$$

$$= \int \tan^{-1}\left(\frac{1}{1 + x + x^2}\right)dx$$

$$= \int \tan^{-1}\left(\frac{(1+x) - x}{1 + x(1+x)}\right)dx$$

$$= \int \tan^{-1}(1+x)dx + \int \tan^{-1} x\,dx$$

Let $1 + x = t \Rightarrow dx = dt$

$$= \int \tan^{-1} t\,dx + \int \tan^{-1} x\,dx$$

$$= t\tan^{-1} t - \frac{1}{2}\log|t^2 + 1| +$$

$$x\tan^{-1} x - \frac{1}{2}\log|x^2 + 1| + c$$

$$= (x+1)\tan^{-1}(x+1) - \frac{1}{2}\log|(x+1)^2 + 1|$$

$$x\tan^{-1} x - \frac{1}{2}\log|x^2 + 1| + c$$

37. We have,

$$\int \tan^{-1}(1 + x + x^2)\,dx$$

$$= \int \left(\frac{\pi}{2} - \tan^{-1}(1 + x + x^2)\right)dx$$

$$= \frac{\pi}{2}x - (x+1)\tan^{-1}(x+1) - \frac{1}{2}\log|(x+1)^2 + 1|$$

$$-x\tan^{-1} x - \frac{1}{2}\log|x^2 + 1| + c$$

38. We have,

$$\int \frac{\sin x}{\sin 4x}\,dx$$

$$= \int \frac{\sin x}{2\sin 2x\cos 2x}\,dx$$

$$= \frac{1}{4}\int \frac{dx}{\cos x\,(2\cos^2 x - 1)}$$

$$= \frac{1}{4}\int \frac{\cos x\,dx}{(1 - \sin^2 x)(1 - 2\sin^2 x)}$$

$$= \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}, \quad \text{Let } t = \sin x$$

$$\Rightarrow \quad dt = \cos x\,dx$$

$$= \frac{1}{4}\int \frac{dt}{(t^2 - 1)(2t^2 - 1)}$$

$$= \frac{1}{4}\int\left(\frac{1}{t^2-1} - \frac{2}{2t^2-1}\right)dt$$

$$= \frac{1}{4}\int\frac{dt}{t^2-1} - \frac{1}{4}\int\frac{dt}{t^2 - \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{4}\times\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{t-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}}\right| + c$$

$$= \frac{1}{8}\log\left|\frac{t-1}{t+1}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{t\sqrt{2}-1}{t\sqrt{2}+1}\right| + c$$

where $t = \sin x$.

39. We have,

$$\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m}dx$$

$$= \int (x^{3m-1} + x^{2m-1} + x^{m-1}) \times x \times (2x^{2m} + 3x^m + 6)^{1/m}dx$$

$$= \int (x^{3m-1} + x^{2m-1} + x^{m-1})(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}dx$$

Let $2x^{3m} + 3x^{2m} + 6x^m = t$

$\Rightarrow\ 6m(x^{3m-1} + x^{2m-1} + x^{m-1})dx = dt$

$$= \frac{1}{6m}\int t^{1/m}\,dt$$

$$= \frac{1}{6(m+1)}t^{m+1} + c$$

$$= \frac{1}{6(m+1)}(2x^{3m} + 3x^{2m} + 6x^m)^{m-1} + c$$

40. We have,

$$\int \sqrt[3]{\tan x}\,dx$$

Let $\tan x = t^3$

$\Rightarrow\ \sec^2 x\,dx = 3t^2\,dt$

$\Rightarrow\ dx = \dfrac{3t^2}{\sec^2 x}dt = \dfrac{3t^2}{1+t^2}dt$

$$= \int\frac{3t^3}{t^6+1}dt$$

$$= \frac{3}{2}\int\frac{t^2.2t}{t^6+1}dt$$

$$= \frac{3}{2}\int\frac{v}{v^3+1}dv,$$

Let $t^2 = v$

$\Rightarrow\ 2t\,dt = dx$

$$= \frac{3}{2}\int\frac{(v+1)-1}{(v+1)(v^2-v+1)}dv$$

$$= \frac{3}{2}\int\frac{dv}{(v^2-v+1)} - \frac{3}{2}\int\frac{dv}{(v^3+1)}$$

$$= \frac{3}{2}\int\frac{dv}{\left(v-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{3}{2}\int\frac{v^2-(v^2-1)}{(v^3+1)}dv$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) - \frac{1}{2}\log|v^3+1| + \frac{3}{2}\int\frac{v-1}{v^2-v+1}dv$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) - \frac{1}{2}\log|v^3+1| + \frac{3}{2}\int\frac{(2v-1)-1}{v^2-v+1}dv$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(2v-\frac{1}{\sqrt{3}}\right) - \frac{1}{2}\log|v^3+1| + \frac{3}{2}\log|v^2-v+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) + c$$

$$= \frac{3}{2}\log|v^2-v+1| - \frac{1}{2}\log|v^3+1| + c$$

where $v = t^2 = (\tan x)^{2/3}$

41. We have $\displaystyle\int\frac{dx}{(e^x-1)^2}$

$$= \int\frac{e^x}{e^x(e^x-1)^2}dx$$

$$= \int\frac{dt}{t(t-1)^2} \quad \text{Let } e^x = t$$

$\Rightarrow e^x dx = dt$

$$= \int\left(\frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}\right)dt$$

$$= \log\left|\frac{t}{t-1}\right| - \frac{1}{t-1} + C$$

$$= \log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + C$$

42. We have

$$\int\frac{\tan^{-1}x}{x^4}dx$$

$$= \tan^{-1}x\int\frac{dx}{x^4} - \int\left[\frac{1}{1+x^2}\times\left(-\frac{1}{3x^3}\right)\right]dx$$

$$= \frac{\tan^{-1}x}{(-3x^3)} + \frac{1}{3}\int\frac{x\,dx}{x^4(1+x^2)}$$

$$= \frac{\tan^{-1}x}{(-3x^3)} + \frac{1}{6}\int \frac{dt}{t(t-1)^2}, \quad \text{Let } x^2 + 1 = t$$

$$\Rightarrow \quad 2x\,dx = dt$$

$$= \frac{\tan^{-1}x}{(-3x^3)} + \frac{1}{6}\int \left(\frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}\right)dt$$

$$= \frac{\tan^{-1}x}{(-3x^3)} + \frac{1}{6}\left(\log\left|\frac{t}{t-1}\right| - \frac{1}{t-1}\right) + c$$

$$= \frac{\tan^{-1}x}{(-3x^3)} + \frac{1}{6}\left(\log\left|\frac{x^2+1}{x^2}\right| - \frac{1}{x^2}\right) + c$$

43. We have,

$$\int \frac{dx}{(\sqrt{\cos x} + \sqrt{\sin x})^4}$$

$$= \int \frac{\sec^2 x}{(1 + \sqrt{\tan x})^4}\,dx$$

$$= \int \frac{2t\,dt}{(1+t)^4}, \quad \text{Let } \tan x = t^2$$

$$\Rightarrow \quad \sec^2 x\,dx = dt$$

$$= 2\int \frac{(t+1) - 1}{(1+t)^4}\,dt$$

$$= 2\int \left(\frac{1}{(1+t)^3} - \frac{1}{(t+1)^4}\right)dt$$

$$= -2\int \left(\frac{1}{3(1+t)^2} - \frac{1}{4(t+1)^3}\right) + c$$

$$= -2\int \left(\frac{1}{3(1+\sqrt{\tan x})^2} - \frac{1}{4(\sqrt{\tan x}+1)^3}\right) + c$$

44. $\int \frac{1 + x^{-2/3}}{1 + x}\,dx \; 3\int \left(\frac{1 + t^{-2}}{1 + t^3}\right)t^2\,dt,$

Let $x = t^3 \Rightarrow dx = 3t^2\,dt$

$$= 3\int \frac{t^2}{1+t^3}\,dt + 3\int \frac{dt}{1+t^3}$$

$$= \log|t^3 + 1| + 3\int \frac{t^2 - (t^2 - 1)}{t^3 + 1}\,dt$$

$$= 4\log|t^3 + 1| - 3\int \frac{(t-1)}{t^2 - t + 1}\,dt$$

$$= 4\log|t^3 + 1| - \frac{3}{2}\int \frac{(2t-1) - 1}{t^2 - t + 1}\,dt$$

$$= 4\log|t^3 + 1| - \frac{3}{2}\log|t^2 - t + 1| + \frac{3}{2}\int \frac{dt}{t^2 - t + 1}$$

$$= 4\log|t^3 + 1| - \frac{3}{2}\log|t^2 - t + 1|$$

$$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + c, \text{ Let } t = \sqrt[3]{x}$$

45. We have,

$$\int \frac{dx}{2\sin x + \sec x}$$

$$= \int \frac{\cos x}{\sin 2x + 1}\,dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin 2x + 1}\,dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\sin x - \cos x)}{\sin 2x + 1}\,dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)}{\sin 2x + 1}\,dx - \frac{1}{2}\int \frac{(\sin x - \cos x)}{\sin 2x + 1}\,dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)}{(\sin x + \cos x)^2}\,dx - \frac{1}{2}\int \frac{(\sin x - \cos x)}{(\sin x + \cos x)^2}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{(\sin x + \cos x)} - \frac{1}{2}\int \frac{(\sin x - \cos x)}{(\sin x + \cos x)^2}\,dx$$

$$= \frac{1}{2\sqrt{2}}\int \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx - \frac{1}{2}\int \frac{(\sin x - \cos x)}{(\sin x + \cos x)^2}\,dx$$

$$= \frac{1}{2\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| - \frac{1}{2}(\sin x + \cos x) + c$$

46. We have,

$$\int \frac{x^4 + 1}{x^6 + 1}\,dx$$

$$= \int \frac{(x^2+1)^2 - 2x^2}{(x^6 + 1)}\,dx$$

$$= \int \frac{(x^2+1)^2}{(x^6+1)}\,dx - 2\int \frac{x^2}{(x^6+1)}\,dx$$

$$= \int \frac{(x^2+1)}{(x^4 + x^2 + 1)}\,dx - 2\int \frac{x^2}{(x^6+1)}\,dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2} - 1\right)}\,dx - \frac{2}{3}\int \frac{3x^2}{((x^3)^2 + 1)}\,dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(\left(x + \frac{1}{x}\right)^2 + 1\right)}\,dx - \frac{2}{3}\int \frac{3x^2}{((x^3)^2 + 1)}\,dx$$

$$= \tan^{-1}\left(x - \frac{1}{x}\right) - \frac{2}{3}\tan^{-1}(x^3) + c$$

47. We have,

$$\int dx(e^x - 1)^2$$

$$= \int \frac{e^x dx}{e^x(e^x - 1)^2}$$

$$= \int \frac{dt}{t(t-1)^2}, \quad \text{Let } t = e^x$$

$$\Rightarrow \quad dt = ex\,dx$$

$$= \int \left(\frac{1}{t} - \frac{1}{(t-1)} + \frac{1}{(t-1)^2}\right) dt,$$

$$= \log\left|\frac{1}{t-1}\right| - \frac{1}{t-1} + C$$

$$= \log\left|\frac{e^x}{e^x - 1}\right| - \frac{1}{e^x - 1} + C$$

48. We have,

$$\int \frac{dx}{(x-1)\sqrt{x+2}}$$

$$= \int \frac{2t\,dt}{(t^2-3)t}, \quad \text{Let } x + 2 = t^2$$

$$\Rightarrow \quad dx = 2t\,dt$$

$$= 2\int \frac{dt}{\left(t^2 - (\sqrt{3})^2\right)}$$

$$= \frac{2}{2\sqrt{3}}\log\left|\frac{t - \sqrt{3}}{t + \sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{(x+2) - \sqrt{3}}{(x+2) + \sqrt{3}}\right| + c$$

49. We have,

$$\int \frac{dx}{x^2\sqrt{x+1}}$$

$$= \int \frac{2t\,dt}{(t^2-1)^2 t}, \quad \text{Let } x + 1 = t^2$$

$$\Rightarrow \quad dx = dtdt$$

$$= 2\int \frac{dt}{(t^2-1)^2}$$

Now, $\int \frac{dt}{(t^2-1)}$

$$= \frac{1}{(t^2-1)}\int dt - \int\left(-\frac{2t\cdot t}{(t^2-1)^2}\right)dt$$

$$= \frac{t}{(t^2-1)} + 2\int\left(\frac{(t^2-1)+1}{(t^2-1)^2}\right)dt$$

$$= \frac{t}{(t^2-1)} + 2\int\frac{dt}{(t^2-1)} + 2\int\frac{dt}{(t^2-1)^2}$$

$$\Rightarrow 2\int\frac{dt}{(t^2-1)^2} = -\frac{t}{(t^2-1)} - \int\frac{dt}{(t^2-1)}$$

$$= -\frac{t}{(t^2-1)} - \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c$$

where $t = \sqrt{x+1}$

50. We have,

$$\int \frac{x^4 - 1}{x^2\sqrt{x^4 + x^2 + 1}}\,dx$$

$$= \int \frac{x^4 - 1}{x^3\sqrt{\left(x^2 + \frac{1}{x^2}\right) + 1}}\,dx$$

$$= \int \frac{\left(x - \frac{1}{x^3}\right)dx}{\sqrt{\left(x^2 + \frac{1}{x^2}\right) + 1}}$$

$$= \int \frac{t\,dt}{t}, \quad \text{Let } x^2 + \frac{1}{x^2} + 1 = t^2$$

$$\Rightarrow \quad \left(2x - 2\frac{1}{x^3}\right)dx = 2tdt$$

$$= \int t\,dt$$

$$= t + c$$

$$= \sqrt{x^2 + \frac{1}{x^2} + 1} + c$$

51. We have,

$$\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}\,dx$$

$$= \int \frac{x^2 - 1}{x^5\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}\,dx$$

$$= \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)dx}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}$$

$$= \frac{1}{2}\int \frac{t\,dt}{t}, \quad \text{Let } 2 - \frac{2}{x^2} + \frac{1}{x^4} = t^2$$

$$\Rightarrow \quad \left(\frac{2}{25} - \frac{4}{25}\right)dx = 2tdt$$

$$= \frac{1}{2}t + c$$

$$= \frac{1}{2}\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + c$$

52. We have,

$$\int \frac{\sqrt{\cos 2x}}{\sin x} dx$$

$$= \int \sqrt{\frac{\cos 2x}{\sin^2 x}}\, dx$$

$$= \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}\, dx$$

$$= \int \sqrt{\cot^2 x - 1}\, dx$$

$$= \int \sqrt{\operatorname{cosec}^2 x - 2}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x - 2}{\sqrt{\operatorname{cosec}^2 x - 2}}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\operatorname{cosec}^2 x - 2}}\, dx - \int \frac{2}{\sqrt{\operatorname{cosec}^2 x - 2}}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}}\, dx - \int \frac{2\sin x}{\sqrt{1 - 2\sin^2 x}}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}}\, dx - \int \frac{2\sin x}{\sqrt{2\cos^2 x - 1}}\, dx$$

Let $\cot x = t$ and $\cos x = v$

$\Rightarrow \operatorname{cosec}^2 x dx = dt$ and $-\sin x dx = dx$.

$$= -\int \frac{dt}{\sqrt{t^2 - 1}} + 2\int \frac{dv}{\sqrt{2v^2 - 1}}$$

$$= -\int \frac{dt}{\sqrt{t^2 - 1}} + \sqrt{2}\int \frac{dv}{\sqrt{v^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -\log|t + \sqrt{t^2 - 1}| + \sqrt{2}\log\left|v + \sqrt{v^2 - \frac{1}{2}}\right| + c$$

$$= -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \sqrt{2}\log\left|\cos x + \sqrt{\cos^2 x - \frac{1}{2}}\right| + c$$

53. We have,

$$\int \frac{\cos\theta + \sin\theta}{\sqrt{5 + \sin(2\theta)}}\, d\theta$$

Let $\sin\theta - \cos\theta = 1$

$\Rightarrow (\cos\theta + \sin\theta)d\theta = dt$

Also, $t^2 = 1 - \sin 2\theta$

$\Rightarrow \sin 2\theta = 1 - t^2$

$$= \int \frac{dt}{\sqrt{5 + 1 + t^2}}$$

$$= \int \frac{dt}{\sqrt{(\sqrt{6})^2 - t^2}}$$

$$= \sin^{-1}\left(\frac{t}{\sqrt{6}}\right) + c$$

$$= \sin^{-1}\left(\frac{\sin\theta - \cos\theta}{\sqrt{6}}\right) + c$$

54. We have

$$\int \frac{\mathrm{se}^2 x}{(\sec x + \tan x)^{9/2}} dx$$

$$= \int \frac{\sec x . \sec x}{(\sec x + \tan x)^{9/2}} dx \quad \text{...(i)}$$

Let $\sec x + \tan x = t$

$\Rightarrow \sec x(\sec x + \tan x)\, dx = dt$

$\Rightarrow \sec x dx = \frac{dt}{t}$

Also $\sec x - \tan x = \frac{1}{(\sec x + \tan x)} = \frac{1}{t}$

Thus, $\sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)dt$

The given intergral (i) reduces to

$$\int \frac{\frac{1}{2}\left(t + \frac{1}{t}\right)\frac{dt}{t}}{t^{9/2}}$$

$$= \frac{1}{2}\int \frac{(t^2 + 1)}{t^{13/2}} dt$$

$$= \frac{1}{2}\int \left(\frac{1}{t^{9/2}} + \frac{1}{t^{13/2}}\right) dt$$

$$= \frac{1}{2}\int \left(\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right) + c$$

$$= \left(\frac{1}{7t^{7/2}} + \frac{1}{11t^{11/2}}\right) + c$$

$$= \frac{1}{7(\sec x - \tan x)^{7/2}} + \frac{1}{11(\sec x + \tan x)^{11/2}} + c$$

55. We have,

$$\int \frac{\tan 2\theta}{\sqrt{\sin^6\theta + \cos^6\theta}}\, d\theta$$

$$= \int \frac{\tan 2\theta}{\sqrt{1 - 3\sin^2\theta\cos^2\theta}}\, d\theta$$

$$= \int \frac{\sin 2\theta}{\cos 2\theta\sqrt{1 - \frac{3}{4}\sin^2 2\theta}}\, d\theta$$

$$= \int \frac{2\sin 2\theta}{\cos 2\theta\sqrt{4 - 3\sin^2 2\theta}}\, d\theta$$

$$= \int \frac{2\sin 2\theta}{\cos 2\theta\sqrt{3\cos^2 2\theta - 1}}\, d\theta$$

$$= \int \frac{dt}{t\sqrt{3t^2 - 1}}, \quad \text{Let } t = \cos 2\theta$$

$$\Rightarrow dt = 2\sin 2\theta\, d\theta$$

$$= -\int \frac{t\,dt}{t^2\sqrt{3t^2 - 1}}$$

$$= -\frac{1}{9}\int \frac{dv}{v^2 + 1}, \quad \text{Let } 3t^2 - 1 = v^2$$

$$6t\,dt = 2x\,dx$$

$$= -\frac{1}{9}\tan^{-1} v + c$$

$$= -\frac{1}{9}\tan^{-1}\left(\sqrt{3t^2 - 1}\right) + c$$

$$= -\frac{1}{9}\tan^{-1}\left(\sqrt{3\cos^2 2\theta - 1}\right) + c$$

Level IV

1. $\int \frac{x^2 + 6}{(x\sin x + 3\cos x)^6}\, dx$

$$= \int \frac{(x^2 + 6)x^4}{(x^3\sin x + 3x^2\cos x)^6}\, dx$$

$$= \int \frac{(x^2 + 6)x\cos x.(x^3\sec x)}{(x^3\sin x + 3x^2\cos x)^6}\, dx$$

$$= (x^3\sec x)\int \frac{x(x^2 + 6)\cos x}{(x^3\sin x + 3x^2\cos x)^6}\, dx + \int \sec^2 x\, dx$$

$$= \frac{-x\sec x}{x\sin x + 3\cos x} + \tan x + c$$

2. $\int \frac{\log(1 + \sin^2 x)}{\cos^2 x}\, dx$

$$= \int \sec^2 x \log(1 + \sin^2 x)\, dx$$

$$= \log(1 + \sin^2 x)\int \sec^2 x\, dx - \int \tan x \cdot \frac{2\sin x \cdot \cos x}{(1 + \sin^2 x)}\, dx$$

$$= \tan x \log(1 + \sin^2 x) - 2\int \frac{\sin^2 x}{1 + \sin^2 x}\, dx$$

$$= \tan x \log(1 + \sin^2 x) - 2\int\left(1 - \frac{1}{1 + \sin^2 x}\right)dx$$

$$= \tan x \log(1 + \sin^2 x) - 2(x) + 2\int \frac{\sec^2 x}{2\tan^2 x + 1}\, dx$$

$$= \tan x \log(1 + \sin^2 x) - 2(x) + 2\int \frac{dt}{2t^2 + 1}$$

Let $\tan x = t$

$\sec^2 x\, dx = dt$

$$= \tan x \log(1 + \sin^2 x) - 2(x) + 2\sqrt{2}\tan^{-1}(t\sqrt{2}) + c$$

where $t = \tan x$

3. $\int x^{-\frac{2}{3}}\left(1 + x^{\frac{2}{3}}\right)^{-1} dx = \int \frac{dx}{x^{2/3}(1 + x^{2/3})}$

Let $x^{1/3} = t$

$$\Rightarrow \frac{dx}{3x^{2/3}} = dt$$

$$\Rightarrow \frac{dx}{x^{2/3}} = 3dt$$

$$= 3\int \frac{dt}{1 + t^2}$$

$$= 3\tan^{-1}(t) + c$$

$$= 3\tan^{-1}(x^{1/3}) + c$$

4. $\int\left[1 + \tan\left(\frac{5\pi}{16} - x\right)\right]\left[1 + \tan\left(-\frac{\pi}{16} + x\right)\right]dx$

$$= \int 2\,dx$$

$$= 2x + c$$

$$\left(\because (1 + \tan A)(1 + \tan B) = 2, \text{ if } A + B = \frac{\pi}{4}\right)$$

5. $\int \tan(x - \alpha)\cdot\tan(x + \alpha)\cdot\tan 2x\, dx$

$$= \int(\tan(2x) - \tan(x - \alpha) - \tan(x + \alpha))dx$$

$$= -\frac{1}{2}\log|\cos(2x)| + \log|\cos(x - \alpha)|$$

$$+ \log|\cos(x - \alpha)| + c$$

6. $\int \cos\left(2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right)dx$

Let $x = \cos\theta$

$\Rightarrow \quad dx = -\sin\theta d\theta$

$$= \int \cos\left[2\cot^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\right]d\theta$$

$$= \int \cos\left[2\cot^{-1}\left\{\tan\left(\frac{\theta}{2}\right)\right\}\right]d\theta$$

$$= \int \cos\left[2\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}\right]d\theta$$

$$= \int \cos\left[2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]d\theta$$

$$= \int \cos(\pi-\theta)\,d\theta$$

$$= -\int \cos(\theta)\,d\theta$$

$$= -\sin\theta + c$$

$$= -\sqrt{1-x^2} + c$$

7. $\int \frac{(x-1)\sqrt{x^4+3x^3-x^2+2x+1}}{x^2(x+1)dx}$

$$= \int \frac{(x^2-1)\sqrt{x^2\left\{\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-1\right\}}}{x^2(x^2+2x+1)}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{\left\{\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-1\right\}}}{\left(x+\frac{1}{x}+2\right)}$$

Let $x + \frac{1}{x} + 2 = t$

$\Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{\sqrt{t^2+2t-3}}{(t+2)}\,dt$$

$$= \int \frac{t^2+2t-3}{(t+2)\sqrt{t^2+2t-3}}\,dt$$

$$= \int \frac{t(t+2)-3}{(t+2)\sqrt{t^2+2t-3}}\,dt$$

$$= \int \frac{t}{\sqrt{t^2+2t-3}}\,dt - \int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$$

$$= I_1 + I_2 \text{ (say)}$$

Now, $I_1 = \int \frac{dt\,t}{t^2+2t-3\,dt}$

$$= \frac{1}{2}\int\left(\frac{(2t+2)-2}{t^2+2t-3}\right)dt$$

$$= \frac{1}{2}\int\left(\frac{(2t+2)}{t^2+2t-3}\right)dt - \int\left(\frac{dt}{t^2+2t-3}\right)$$

$$= \frac{1}{2}\int\left(\frac{(2t+2)}{t^2+2t-3}\right)dt - \int\left(\frac{dt}{(t+1)^2-(2)^2}\right)$$

$$= \frac{1}{2}\log|t^2+2t-3| - \frac{1}{2\sqrt{2}}\log\left|\frac{(t+1)-\sqrt{2}}{(t+1)+\sqrt{2}}\right| + c$$

where $t = \left(x+\frac{1}{x}\right)$

Also, $I_2 = \int \frac{dx}{(x+2)\sqrt{x^2+2x-3}}$

$$= \int \frac{-\frac{dt}{t^2}}{\left(\frac{1}{t}\right)\sqrt{\left(\frac{1}{t}-2\right)^2+2\left(\frac{1}{t}-2\right)-3}}$$

$$= -\int \frac{dt}{\sqrt{(1-2t)^2+2t^2\left(\frac{1}{t}-2\right)-3t^2}}$$

$$= -\int \frac{dt}{\sqrt{(1-4t+4t^2)+2t-4t^2-3t^2}}$$

$$= -\int \frac{dt}{\sqrt{(1-2t+3t^2)}}$$

$$= -\int \frac{dt}{\sqrt{-\{3t^2+2t+1\}}}$$

$$= -\int \frac{dt}{\sqrt{-3\left\{t^2+\frac{2}{3}t-\frac{1}{3}\right\}}}$$

$$= -\int \frac{dt}{\sqrt{-3\left\{\left(t+\frac{1}{3}\right)^2-\frac{4}{9}\right\}}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(\frac{2}{3}\right)^2-\left(t+\frac{1}{3}\right)^2}}$$

$$= -\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3t+1}{2}\right) + c$$

where $t = \left(x+\frac{1}{x}\right)$

8. $\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan^2 x}}\,dx$

$$= \int \frac{(1-\tan x)\sec^2 x\,dx}{(1+\tan x)\sqrt{\tan^3 x+\tan^2 x+\tan x}}$$

$$= \int \frac{(1-t)dt}{(1+t)\sqrt{t^3+t^2+t}} \qquad \text{Let } \tan x = t$$

$$\Rightarrow \quad \sec^2 x dx = dt$$

$$= \int \frac{(1-t^2)dt}{(1+t)^2\sqrt{t^3+t^2+t}}$$

$$= \int \frac{(1-t^2)dx}{(t^2+2t+1)\sqrt{t^2\left(t+\frac{1}{t}+1\right)}}$$

$$= \int \frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}+2\right)\sqrt{\left(t+\frac{1}{t}+1\right)}}$$

Let $\left(t+\frac{1}{t}+1\right) = v^2 \Rightarrow \sec^2 xdx = dt$

$$= -2\int \frac{v\,dv}{(v^2+1)v}$$

$$= -2\int \frac{dv}{(v^2+1)}$$

$$= -2\tan^{-1}(v) + c$$

$$= -2\tan^{-1}\left(\sqrt{t+\frac{1}{t}+1}\right) + c$$

where $t = x + \frac{1}{x}$

9. $\int \frac{\tan^{-1}x}{x^4}dx$

Let $x = \tan\theta \Rightarrow dx = \sec^2 x d\theta$

$$= \int \frac{\tan^{-1}(\tan\theta)}{\tan^4\theta}\sec^2\theta d\theta$$

$$= \int \frac{\theta\sec^2\theta}{\tan^4\theta}d\theta$$

$$= \theta\int \frac{\sec^2\theta}{\tan^4\theta}d\theta + \frac{1}{3}\int \frac{d\theta}{\tan^3\theta}$$

$$= -\frac{\theta}{3\tan^3\theta} + \frac{1}{3}\int \frac{d\theta}{\tan^3\theta}$$

$$= -\frac{\theta}{3\tan^3\theta} + \frac{1}{3}\int \frac{\cos^3\theta}{\sin^3\theta}d\theta$$

$$= -\frac{\theta}{3\tan^3\theta} + \frac{1}{3}\int \frac{(1-\sin^2\theta)\cos\theta}{\sin^3\theta}d\theta$$

$$= -\frac{\theta}{3\tan^3\theta} + \frac{1}{3}\int \frac{(1-t^2)dt}{t^3}$$

$$= -\frac{\theta}{3\tan^3\theta} + \frac{1}{3}\int \left(\frac{1}{t^3}-\frac{1}{t}\right)dt$$

$$= -\frac{\theta}{3\tan^3\theta} - \frac{1}{3}\left(\frac{1}{2t^2}+\log|t|\right) + c$$

$$= -\frac{\theta}{3\tan^3\theta} - \frac{1}{3}\left(\frac{1}{2\sin^2\theta}+\log|\sin\theta|\right) + c$$

where $\tan\theta = x$

10. $\int \frac{(1-x\sin x)}{x(1-x^3e^{3\cos x})}dx$

$$= \int \frac{(1-x\sin x)e^{\cos x}}{xe^{\cos x}\left(1-(xe^{\cos x})^3\right)}dx$$

$$= \int \frac{dt}{t(1-t^3)}, \qquad \text{Let } t = xe^x$$

$$= \int \frac{dt}{t(1-t^3)}$$

$$= \int \frac{dt}{t(t^3-1)}$$

$$= \frac{1}{3}\int \frac{3t^2}{t^3(t^3-1)}dt$$

$$= \frac{1}{3}\int \frac{dv}{v(v-1)}, \qquad \text{where } v = t^3$$

$$= \frac{1}{3}\log\left|\frac{v-1}{v}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{t^3-1}{t^3}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{(xe^x)^3-1}{(xe^x)^3}\right| + c$$

11. $\int \frac{(1+x\cos x)}{x(1-x^2e^2\sin x)}dx$

$$= \int \frac{(1+x\cos x)e^{\sin x}}{xe^{\sin x}(1-(xe^{\sin x})^2)}dx$$

$$= \int \frac{dt}{t(1-t^2)}, \qquad \text{where } t = xe^{\sin x}$$

$$= \int \frac{dt}{t(1-t^2)}$$

$$= -\frac{1}{2}\int \frac{2t\,dt}{t^2(t^2-1)}$$

$$= -\frac{1}{2}\int \frac{dv}{v(v-1)}, \text{ where } v = t^2$$

$$= \frac{1}{2}\log\left|\frac{v}{v-1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{t^2}{t^2-1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{(xe^{\sin x})^2}{(xe^{\sin x})^2 - 1}\right| + c$$

12. $\int \frac{dx}{2\sin x + \sec x}$

$$= \int \frac{\cos x}{\sin 2x + 1}dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin 2x + 1}dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\cos x - \sin x)}{(\sin x + \cos x)^2}dx$$

$$= \frac{1}{2}\int \frac{dx}{(\sin x + \cos x)} + \frac{1}{2}\int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

$$= \frac{1}{2\sqrt{2}}\int \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx + \frac{1}{2}\int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}dx$$

$$= \frac{1}{2\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| - \frac{1}{(\sin x + \cos x)} + c$$

13. $\int \frac{dx}{\cos x\sqrt{\sin(2x+\alpha) + \sin\alpha}}$

$$= \int \frac{dx}{\cos x\sqrt{\sin(2x)\cos\alpha + \cos(2x)\sin\alpha + \sin\alpha}}$$

$$= \int \frac{dx}{\cos x\sqrt{\sin(2x)\cos\alpha + (1 + \cos(2x))\sin\alpha}}$$

$$= \int \frac{dx}{\cos x\sqrt{2\sin x\cos x\cos\alpha + 2\cos^2 x\sin\alpha}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\cos^2 x\sqrt{\tan x\cos\alpha + \sin\alpha}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2 x\,dx}{\sqrt{\tan x\cos\alpha + \sin\alpha}}$$

Let $\tan x\cos\alpha + \sin\alpha = t^2$

$$= \frac{1}{\sqrt{2}\cos\alpha}\int \frac{2t\,dt}{t}$$

$$= \frac{1}{\sqrt{2}\cos\alpha}\int dt$$

$$= \frac{\tan x\cos\alpha + \sin\alpha}{\sqrt{2}\cos\alpha} + c$$

14. $\int \left(3x^2\tan\left(\frac{1}{x}\right) - x\sec^2\left(\frac{1}{x}\right)\right)dx$

$$= \tan\left(\frac{1}{x}\right)\int (3x^2)dx - \int \sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)(x^3)\,dx$$

$$-\int x\sec^2\left(\frac{1}{x}\right)dx + c$$

$$= x^3\tan\left(\frac{1}{x}\right) + \int x\sec^2\left(\frac{1}{x}\right)dx - \int x\sec^2\left(\frac{1}{x}\right)dx + c$$

$$= x^3\tan\left(\frac{1}{x}\right) + c$$

15. $\int \frac{\sqrt{3\cos 2x - 1}}{\cos x}dx$

$$= \int \frac{3\cos 2x - 1}{\cos x\sqrt{\cos 2x - 1}}dx$$

$$= \int \frac{3(2\cos^2 x - 1)}{\cos x\sqrt{3\cos 2x - 1}}dx$$

$$= 6\int \frac{\cos x}{\sqrt{3(1 - 2\sin^2 x) - 1}}dx - 4\int \frac{dx}{\cos x\sqrt{3\cos 2x - 1}}dx$$

$$= 3\sqrt{2}\int \frac{\cos x}{\sqrt{(1 - 3\sin^2 x}}dx - 4\int \frac{\cos x\,dx}{(1 - \sin^2 x)\sqrt{1 - 3\sin^2 x}}$$

$$= \sqrt{6}\int \frac{\cos x}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - \sin^2 x}}dx - 4\int \frac{\cos x\,dx}{(1 - \sin^2 x)\sqrt{1 - 3\sin^2 x}}$$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) - 4\int \frac{\cos x\,dx}{(1 - \sin^2 x)\sqrt{1 - 3\sin^2 x}}$$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) - 4\int \frac{dt}{(1 - t^2)\sqrt{1 - 3t^2}}$$

where $t = \sin x$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 4\int \frac{y\,dy}{(y^2 - 1)\sqrt{y^2 - 3}}$$

where $t = 1/y$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 4\int \frac{v\,dv}{(v^2 + 2)v}, \; v^2 = y^2 - 3$$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 4\int \frac{dv}{(v^2 + 2)}$$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 2\sqrt{2}\tan^{-1}\left(\frac{v}{\sqrt{2}}\right) + c$$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 2\sqrt{2}\tan^{-1}\left(\sqrt{\frac{y^2 - 3}{2}}\right) + c$$

$$= \sqrt{6}\sin^{-1}(\sqrt{3}\sin x) + 2\sqrt{2}\tan^{-1}\left(\sqrt{\frac{\cos^2 x - 3}{2}}\right) + c$$

16. $$\int\left(\frac{1}{1-x^8}\right)\left[\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]dx$$

$$= \int\frac{dx}{(1-x^8)}\left(\frac{\pi}{2} - 2\tan^{-1}x + 2\tan^{-1}x\right)dx$$

$$= \frac{\pi}{2}\int\frac{dx}{(1-x^8)}$$

$$= \frac{\pi}{2}\int\frac{dx}{(1-x^4)(1+x^4)}$$

$$= \frac{\pi}{4}\left[\int\frac{dx}{(1-x^4)} + \int\frac{dx}{x^4+1}\right]$$

$$= \frac{\pi}{8}\left[\int\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right) + \int\frac{(x^2+1)-(x^2+1)}{x^4+1}\right]dx$$

$$= \frac{\pi}{8}\left[\int\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right) + \int\frac{(x^2+1)}{x^4+1} - \frac{(x^2+1)}{x^4+1}\right]dx$$

and then you do it.

17. $$\int\frac{x^2 - x^3}{(x+1)(x^3+x^2+x)^{3/2}}dx$$

$$= \int\frac{x^2(1-x)}{(x+1)x^3\left(x+\frac{1}{x}+1\right)^{3/2}}dx$$

$$= \int\frac{(1-x)}{x(x+1)\left(x+\frac{1}{x}+1\right)^{3/2}}dx$$

$$= \int\frac{(1-x^2)}{x(x+1)^2\left(x+\frac{1}{x}+1\right)^{3/2}}dx$$

$$= \int\frac{(1-x^2)}{x(x^2+2x+1)\left(x+\frac{1}{x}+1\right)^{3/2}}dx$$

$$= \int\frac{(1-x^2)}{x^2\left(x+\frac{1}{x}+2\right)\left(x+\frac{1}{x}+1\right)^{3/2}}dx$$

$$= \int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\left(x+\frac{1}{x}+1\right)^{3/2}}$$

Let $\left(x+\frac{1}{x}+2\right) = t^2$

$\Rightarrow \left(1-\frac{1}{x^2}\right)dx = 2t\,dt$

$$= -\int\frac{2t\,dt}{(t^2+1)t^3}$$

$$= -2\int\frac{dt}{(t^2+1)t^3}$$

$$= -2\int\left(\frac{1}{t^2} - \frac{1}{t^2+1}\right)dt$$

$$= -2\int\left(-\frac{1}{t} - \tan^{-1}t\right) + c$$

$$= 2\left(\frac{1}{\sqrt{x+\frac{1}{x}\ 1}} + \tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right)\right) + c$$

18. $$\int\frac{\cos^4 x}{\sin^3 x(\sin^5 x + \cos^5 x)^{3/5}}$$

$$= \int\frac{\cos^4 x}{\sin^6 x(1+\cot^5 x)^{3/5}}dx$$

$$= \int\frac{\cos^4 x\,\mathrm{cosec}^2 x\,dx}{(1+\cot^5 x)^{3/5}}$$

$$= \int\frac{t^4 dt}{(1+t^5)^{3/5}},$$ Let $\cot x = t$

$$= -\frac{1}{5}\int\frac{5t^4 dt}{(1+t^5)^{3/5}}$$

Let $(1+t^5) = y^5$

$\Rightarrow 5t^4 dt = 5y^4 dy$

$$= -\frac{1}{5}\int\frac{5y^4 dy}{y^3}$$

$$= -\int y\,dy$$

$$= -\left(\frac{y^2}{2}\right) + c$$

$$= -\frac{1}{2}(1+t^5)^{2/5} + c$$

$$= -\frac{1}{2}(1+\cot^5 x)^{2/5} + c$$

19. $$\int\frac{\sin^3 x\,dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$$

$$= \int\frac{(1-\cos^2 x)\sin x\,dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$$

$$= -\int\frac{(1-t^2)dt}{(t^4+3t^2+1)\tan^{=1}\left(t+\frac{1}{t}\right)},$$ where $\cos x = t$

$$= \int \frac{\left(1 - \frac{1}{t^2}\right)dt}{\left(t^2 + \frac{1}{t^2} + 3\right)\tan^{-1}\left(t + \frac{1}{t}\right)}$$

$$= \int \frac{\left(1 - \frac{1}{t^2}\right)dt}{\left(\left(t + \frac{1}{t}\right)^2 + 1\right)\tan^{-1}\left(t + \frac{1}{t}\right)}$$

$$= \int \frac{dy}{(y^2 + 1)\tan^{-1}y}, \text{ where } \left(t + \frac{1}{t}\right) = y$$

$$= \int \frac{dv}{v}, \text{ where } \tan^{-1}(y) = v$$

$$= \int \frac{dv}{v}$$

$$= \log|v| + c$$

$$= \log\left|\tan^{-1}\left(t + \frac{1}{t}\right)\right| + c$$

$$= \log|\tan^{-1}(\cos x + \sec x)| + c$$

20. $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{4 + 3\sin 2x}\,dx$

$$= \int\left(\frac{\tan x - 1}{\sqrt{\tan x}\left(4 + 3\left(\frac{2\tan x}{1 + \tan^2 x}\right)\right)}\right)dx$$

$$= \int\left(\frac{(\tan x - 1)\sec^2 x}{\sqrt{\tan x}\,(4 + 4\tan^2 x + 6\tan x)}\right)dx$$

$$= \int\left(\frac{(t^2 - 1)2t\,dt}{t(4t^4 + 6t + 4)}\right)dt, \qquad \text{where} \quad \tan x = t$$

$$= 2\int\left(\frac{(t^2 - 1)dt}{t(4t^4 + 6t^2 + 4)}\right)dt$$

$$= 2\int\left(\frac{\left(1 - \frac{1}{t^2}\right)dt}{\left(4\left(t^2 + \frac{1}{t^2}\right) + 6\right)}\right)dt$$

$$= 2\int\left(\frac{\left(1 - \frac{1}{t^2}\right)dt}{\left(4\left\{\left(t + \frac{1}{t}\right)^2 - 2\right\} + 6\right)}\right)dt$$

$$= 2\int\left(\frac{\left(1 - \frac{1}{t^2}\right)dt}{\left(4\left(t + \frac{1}{t}\right)^2 - 2\right)}\right)dt$$

$$= 2\int \frac{dy}{4y^2 - 2}, \; y = t + \frac{1}{t}$$

$$= \int \frac{dy}{2y^2 - 1}$$

$$= \frac{1}{2}\int \frac{dy}{y^2 - (1/\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{y\sqrt{2} - 1}{y\sqrt{2} + 1}\right| + c$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{\left(t + \frac{1}{t}\right)\sqrt{2} - 1}{\left(t + \frac{1}{t}\right)\sqrt{2} + 1}\right| + c$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{(\tan x + \cot x)\sqrt{2} - 1}{(\tan x + \cot x)\sqrt{2} + 1}\right| + c$$

21. $\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}\,dx$

$$= x^2\int \frac{(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}dx + 2\int \frac{x}{x\tan x + 1}dx$$

$$= -\frac{x^2}{(x\tan x + 1)} + 2\int \frac{x}{x\tan x + 1}dx$$

$$= -\frac{x^2}{(x\tan x + 1)} + 2\int \frac{x\cos x}{x\sin x + \cos x}\,dx$$

$$= -\frac{x^2}{(x\tan x + 1)} + 2\log|x\sin x + \cos x| + c$$

22. $\int \sqrt{\frac{\sin x - \sin^3 x}{1 - \sin^3 x}}\,dx$

$$= \int \sqrt{\frac{\sin x(1 - \sin^2 x)}{1 - \sin^3 x}}\,dx$$

$$= \int \frac{\cos x\sqrt{\sin x}}{\sqrt{1 - \sin^3 x}}\,dx$$

$$= \int \frac{\cos x\sqrt{\sin x}}{\sqrt{1 - (\sin^{3/2} x)^2}}\,dx$$

Let $(\sin^{3/2} x) = t$

$\Rightarrow \frac{3}{2}\sin^{1/2} x\cos x\,dx = dt$

$\Rightarrow \sin^{1/2} x\cos x\,dx = \frac{2}{3}dt$

$$= \frac{2}{3}\int \frac{dt}{\sqrt{1 - t^2}}$$

$$= \frac{2}{3}\sin^{-1}(t) + c$$

$$= \frac{2}{3}\sin^{-1}(\sin^{3/2}x) + c$$

23. $\int \frac{e^x(2-x)^2}{(1-x)\sqrt{1-x^2}}\, dx$

$$= \int e^x \left\{ \frac{\{1 + (1-x)\}^2}{(1-x)\sqrt{1-x^2}} \right\} dx$$

$$= \int e^x \left\{ \frac{1}{(1-x)\sqrt{1-x^2}} + \sqrt{\frac{1+x}{1-x}} \right\} dx$$

$$= \int e^x \left\{ \sqrt{\frac{1+x}{1-x}} + \frac{1}{(1-x)\sqrt{1-x^2}} \right\} dx$$

$$= \left(e^x \sqrt{\frac{1+x}{1-x}} \right) + c$$

24. $\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$

Put $(x - \beta) = t \; dx = dt$

$$= \int \frac{dt}{\sqrt{\sin(t + \beta + \alpha)\cos^3 t}}$$

$$= \int \frac{dt}{\sqrt{\sin(t+\theta)\cos^3 t}}, \text{ (given)}$$

$$= \int \frac{dt}{\sqrt{\cos^3 t(\sin t \cdot \cos\theta + \cos t \sin\theta)}}$$

$$= \int \frac{dt}{\sqrt{\cos^4 t(\tan t \cdot \cos\theta + \sin\theta)}}$$

$$= \int \frac{\sec^2 t\, dt}{\sqrt{(\tan t \cdot \cos\theta + \sin\theta)}}$$

Let $(\tan t \cdot \cos\theta + \sin\theta) = v^2$

$\Rightarrow \sec^2 t \cos\theta dt = 2v\, dv$

$\Rightarrow \sec^2 t\, dt = \frac{2v\, dv}{\cos\theta}$

$$= \frac{2}{\cos\theta}\int \frac{v\, dv}{v}$$

$$= \frac{2v}{\cos\theta} + c$$

$$= 2\sec\theta\sqrt{\cos\theta \tan t + \sin\theta} + c$$

$$= 2\sec(\alpha+\beta)$$

$$\sqrt{\cos(\alpha+\beta)\tan(x-\beta) + \sin(\alpha+\beta)} + c$$

25. $\int e^x \left(\frac{x+2}{x+4} \right)^2 dx$

$$= \int e^x \left(\frac{(x+2)^2}{(x+4)^2} \right) dx$$

$$= \int e^x \left(\frac{x^2 + 4x + 4}{(x+4)^2} \right) dx$$

$$= \int e^x \left(\frac{x^2 + 4x}{(x+4)^2} + \frac{4}{(x+4)^2} \right) dx$$

$$= \int e^x \left(\frac{x(x+4)}{(x+4)^2} + \frac{4}{(x+4)^2} \right) dx$$

$$= \int e^x \left(\frac{x}{(x+4)} + \frac{4}{(x+4)^2} \right) dx$$

$$= e^x \left(\frac{x}{(x+4)} \right) + c$$

26. $\int \frac{\sqrt{1+x^8}}{x^{13}} dx$

$$= \int \{ x^{-13}(1+x^8)^{1/2} \} dx$$

Here, $\frac{\beta + 1}{\gamma} + \alpha = \frac{-13+1}{8} + \frac{1}{2} = -1$

Let $(1 + x^8) = x^8 t^2$

$\Rightarrow x^8 = \frac{1}{(t^2-1)}$

$\Rightarrow 8x^7\, dx = -\frac{2t}{(t^2-1)} dt$

$\Rightarrow 4x^7\, dx = -\frac{t}{(t^2-1)} dt$

$$= \int \sqrt{\frac{1+x^8}{x^{13}}}\, dx$$

$$= \int \frac{(x^4 t)}{x^{20}} \times x^7 dx$$

$$= \int \frac{t x^7}{x^{16}} dx$$

$$= -\frac{1}{4}\int \frac{t}{\frac{1}{(t^2-1)^2}} \times -\frac{t}{(t^2-1)^2} dt$$

$$= -\frac{1}{4}\int t^2\, dt$$

$$= -\frac{t^3}{12} + c$$

$$= -\frac{1}{12}\left(\frac{1+x^8}{x^8}\right)^{3/2} + c$$

27. $\int \frac{dx}{\sin^3 x + \cos^3 x}$

$$= \int \frac{dx}{(\sin x + \cos x)(1 - \sin x \cos x)}$$

$$= \frac{1}{3}\int\left(\frac{2}{(\sin x + \cos x)} - \frac{\sin x + \cos x}{(1 - \sin x \cos x)}\right)dx$$

$$= \frac{2}{3}\int \frac{dx}{(\sin x + \cos x)} - \frac{1}{3}\int \frac{\sin x + \cos x}{(1 - \sin x \cos x)}dx$$

$$= \frac{2}{3\sqrt{2}}\int \frac{dx}{\left(\sin\left(x + \frac{\pi}{4}\right)\right)} - \frac{2}{3}\int \frac{\sin x + \cos x}{(1 + (\sin x - \cos x)^2)}dx$$

$$= \frac{2}{3\sqrt{2}}\int \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx - \frac{2}{3}\int \frac{\sin x + \cos x}{(1 + (\sin x - \cos x)^2)}dx$$

$$= \frac{2}{3\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| - \frac{2}{3}\tan^{-1}(\sin x - \cos x) + c$$

28. $\int \frac{(x + \sqrt{1+x^2})^3}{\sqrt{1+x^2}}dx$

Let $(x + \sqrt{1+x^2}) = t$

$$\Rightarrow \left(1 + \frac{x}{\sqrt{1+x^2}}\right)dx = dt$$

$$\Rightarrow \left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right)dx = dt$$

$$\Rightarrow \left(\frac{t}{\sqrt{1+x^2}}\right)dx = dt$$

$$\Rightarrow \left(\frac{dx}{\sqrt{1+x^2}}\right) = \frac{dx}{t}$$

$$= \int t^3 \times \frac{dt}{t}$$

$$= \int t^2 dt$$

$$= \frac{t^3}{3} + c$$

$$= \frac{(x + \sqrt{1+x^2})^3}{3} + c$$

29. $\int \frac{(x+1)}{x(1 + xe^x)^2}dx$

$$= \int \frac{e^x(x+1)}{xe^x(1+xe^x)^2}dx$$

$$= \int \frac{dt}{t(1+t)^2}, \quad \text{where } xe^x = t$$

$$= \int \frac{dt}{t} - \int \frac{dt}{(t+1)} - \int \frac{dt}{(1+t)^2}$$

$$= \log\left|\frac{t}{t+1}\right| + \frac{1}{(t+1)} + c$$

where $t = xe^x$

30. $\int \sqrt{\frac{\sin x}{2\sin x + 3\cos x}}\,dx$

$$= \int \frac{dx}{\sqrt{2 + 3\cot x}}$$

Let $2 + 3\cot x = t^2$

$$\Rightarrow -3\operatorname{cosec}^2 x\,dx = 2t\,dt$$

$$\Rightarrow dx = -\frac{2t\,dt}{3(t^4+1)}$$

$$= -\frac{2}{3}\int \frac{t\,dt}{(t^4+1)t}$$

$$= -\frac{2}{3}\int \frac{dt}{(t^4+1)}$$

$$= -\frac{1}{3}\int\left(\frac{(t^2+1) - (t^2-1)}{(t^4+1)}\right)dt$$

$$= -\frac{1}{3}\int\left(\frac{(t^2+1)}{(t^4+1)}\right)dt + \frac{1}{3}\int\left(\frac{(t^2-1)}{(t^4+1)}\right)dt$$

$$= -\frac{1}{3}\int\left(\frac{\left(1 + \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)}\right)dt + \frac{1}{3}\int\left(\frac{\left(1 - \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)}\right)dt$$

$$= -\frac{1}{3}\int\left(\frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2}\right)dt + \frac{1}{3}\int\left(\frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2} - 2\right)dt$$

$$= -\frac{1}{3\sqrt{2}}\tan^{-1}\left(t - \frac{1}{t}\right) + \frac{1}{6\sqrt{2}}\log\left|\frac{\left(t + \frac{1}{t}\right) - \sqrt{2}}{\left(t + \frac{1}{t}\right) + \sqrt{2}}\right| + c$$

where $t = \sqrt{2 + 3\cot x}$

31. $\int \frac{\cos 4x + 1}{\cot x - \tan x}dx$

$$= \int \frac{2\cos^2 x\, 2x\, dx}{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}$$

$$= \int \frac{2\cos^2 x\, 2x\, dx}{\frac{2\cos 2x}{2\sin x \cos x}}$$

$$= \int \cos 2x \sin 2x\, dx$$

$$= \frac{1}{2}\int 2\cos 2x \sin 2x\, dx$$

$$= \frac{1}{2}\int \sin(4x)\, dx$$

$$= -\frac{1}{8}\cos(4x) + c$$

32. $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}\, dx$

$$= \int \frac{\cos x(\cos^2 x + \cos^4 x)}{(\sin^2 x + \sin^4 x)} dx$$

$$= \int \frac{(1 - t^2) + (1 + t^4)^2}{t^4 + t^2} dt, \quad \text{where} \quad \sin x = t$$

$$= \int \frac{2 - 3t^2 + t^4}{t^4 + t^2} dt$$

$$= \int \left(1 + \frac{-4t^2 + 2}{t^4 + t^2}\right) dt$$

$$= \int \left(1 + \frac{-4t^2 - 4 + 6}{t^4 + t^2}\right) dt$$

$$= \int \left(1 + \frac{-4t^2 - 4 + 6}{t^2(t^2 + 1)}\right) dt$$

$$= \int \left(1 - \frac{4(t^2 - 1)}{t^2(t^2 + 1)} + \frac{6}{t^2(t^2 + 1)}\right) dt$$

$$= \int \left(1 - \frac{4}{t^2} + 6\left(\frac{1}{t^2} - \frac{1}{t^2 + 1}\right)\right) dt$$

$$= \int \left(1 + \frac{2}{t^2} - 6\left(\frac{1}{t^2 + 1}\right)\right) dt$$

$$= \left(1 - \frac{2}{t} - 6\tan^{-1}(t)\right) + c$$

$$= \sin x - 2\operatorname{cosec} x - 6\tan^{-1}(\sin x) + c$$

33. $\int \frac{dx}{\sin^6 x + \cos^6 x}$

$$= \int \frac{\sec^6 x\, dx}{\tan^6 x + 1}$$

$$= \int \frac{(1 + \tan^2 x)^2 \sec^2 x}{\tan^6 x + 1} dx$$

$$= \int \frac{(1 + t^2)^2}{t^6 + 1} dt, \quad \text{where} \quad t = \tan x$$

$$= \int \frac{(1 + t^2)^2}{(t^2 + 1)(t^4 - t^2 + 1)} dt$$

$$= \int \frac{(1 + t^2)}{(t^4 - t^2 + 1)} dt$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2} - 1\right)} dt$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(\left(t - \frac{1}{t}\right)^2 + 1\right)} dt$$

$$= \tan^{-1}\left(t - \frac{1}{t}\right) + c$$

$$= \tan^{-1}(\tan x - \cot x) + c$$

34. Let $f(x) = ax^2 + bx + c$

Given $f(0) = f(1) = -3 = 3 f(2)$

On solving, we get

$$a = 1,\ b = -1,\ c = -3$$

Thus, $I = \int \frac{x^2 - x - 3}{(x - 1)(x^2 + x + 1)} dx$

Now, $\frac{x^2 - x - 3}{(x - 1)(x^2 + x + 1)}$

$$= \frac{A}{(x - 1)} + \frac{Bx + C}{(x^2 + x + 1)}$$

Solving, we get

$$A = -1,\ B = 2,\ C = 2$$

The given integral reduces to

$$-\int \frac{dx}{(x - 1)} + \frac{2x + 2}{x^2 + x + 1} dx$$

$$= -\int \frac{dx}{(x - 1)} + \int \frac{2x + 1}{x^2 + x + 1} dx + \int \frac{dx}{x^2 + x + 1}$$

$$= -\log|(x-1)| + \log\left|(x^2+x+1)\right| + \int \frac{dx}{x^2+x+1}$$

$$= \log\left|\frac{(x^2+x+1)}{(x-1)}\right| + \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \log\left|\frac{(x^2+x+1)}{(x-1)}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Integer Type Questions

1. The given integral is

$$\int \frac{\cos x + \cos 2x}{2\cos x - 1}\, dx$$

$$= \int \frac{2\cos^2 x + \cos x - 1}{2\cos x - 1} dx$$

$$= \int \frac{(2\cos x - 1)(2\cos x + 1)}{2\cos x - 1}\, dx$$

$$= \int (\cos x + 1) dx$$

$$= \sin x + x + c$$

Clearly, $A = 1$, $B = 1$

Hence, the value of $A + B + 3 = 5$.

2. The given integral is

$$\int \frac{dx}{\sin^3 x + \cos^3 x}$$

$$= \int \frac{dx}{(\sin x + \cos x)(1 - \sin x \cos x)}$$

$$= \frac{1}{3}\int \left(\frac{2}{(\sin x + \cos x)} + \frac{(\sin x + \cos x)}{(1 - \sin x \cos x)}\right) dx$$

$$= \frac{2}{3}\int \left(\frac{dx}{(\sin x + \cos x)}\right) + \frac{2}{3}\int \frac{(\sin x + \cos x)}{(2 - 2\sin x \cos x)}\, dx$$

$$= \frac{\sqrt{2}}{3}\int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} + \frac{2}{3}\int \frac{(\sin x + \cos x)}{(1 + (\sin x - \cos x)^2} dx$$

$$= \frac{\sqrt{2}}{3}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + \frac{2}{3}\tan^{-1}(\sin x + \cos x) + c$$

Clearly, $L = 3$, $M = 8$, $N = 2$ and $P = 2$

Hence, the value of $L + M - N - P = 7$.

3. The given integral is

$$\int \frac{dx}{\sin^6 x + \cos^6 x}$$

$$= \int \frac{dx}{1 - 3\sin^2 x \cos^2 x}$$

$$= \int \frac{\sec^4 x\, dx}{\sec^4 x - 3\tan^2 x}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x}{(1+\tan^2 x)^2 - 3\tan^2 x} dx$$

$$= \int \frac{(1+t^2)dt}{(1+t^2)^2 - 3t^2}$$

$$= \int \left(\frac{t^2+1}{t^4 - t^2 + 1}\right) dt$$

$$= \int \left(\frac{1 + \frac{1}{t^2}}{\left(t^2 + \frac{1}{t^2}\right) - 1}\right) dt$$

$$= \int \left(\frac{1 + \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 + 1}\right) dt$$

$$= \tan^{-1}\left(t - \frac{1}{t}\right) + c$$

$$= \tan^{-1}(\tan x - \cot x) + c$$

Clearly, $L = 1$, $M = -1$

Hence, the value of $L + M + 4 = 4$.

4. The given integral is

$$\int \frac{x+1}{x^3+x} dx$$

$$= \int \frac{x+1}{x(x^2+1)} dx$$

$$= \int \frac{dx}{(x^2+1)} + \frac{dx}{x(x^2+1)}$$

$$= \tan^{-1} x + \frac{1}{2}\log\left|\frac{x^2}{x^2+1}\right| + c$$

Clearly, $L = 2$, $M = 2$ and $N = 2$

Hence, the value of $\left(\frac{L+M+N}{2}\right) = 2$

5. The given integral is

$$\int \left(\frac{\sin x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{2\sin x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\int\left(1 - \frac{(\cos x - \sin x)}{\sin x + \cos x}\right)dx$$

$$= \frac{1}{2}\int(x - \log|\sin x + \cos x|) + c$$

Clearly, $A = \frac{1}{2}$ and $B = -\frac{1}{2}$

Hence, the value of $A + B + 1$ is 1

6. The given integral is $\int\left(\frac{x^2}{(2x + 3)^2}\right)dx$

Let $2x + 3 = t$

$\Rightarrow \quad 2dx = dt$

$\Rightarrow \quad dx = 1/2dt$

$2x = 3 - t$

$\Rightarrow \quad x = \frac{3 - t}{2}$

$$= \int\left(\frac{x^2}{(2x + 3)^2}\right)dx$$

$$= \frac{1}{2}\int\frac{\left(\frac{3 - t}{2}\right)^2}{t^2}dt$$

$$= \frac{1}{8}\int\frac{(3 - t)^2}{t^2}dt$$

$$= \frac{1}{8}\int\frac{(t^2 - 6t + 9)}{t^2}dt$$

$$= \frac{1}{8}\int\left(1 - \frac{6}{t} + \frac{9}{t^2}\right)dt$$

$$= \frac{1}{8}\int\left(t - \log t - \frac{9}{t}\right) + c$$

$$= \frac{1}{8}\int\left((2x + 3) - \log(2x + 3) - \frac{9}{(2x + 3)}\right) + c$$

$$= \frac{(3 + 2x)}{L} + \frac{1}{M}\log|3 + 2x| - \frac{9}{8(2x + 3)} + c$$

Clearly, $L = 8$ and $M = -8$

Hence, the value of $L + M + 4 = 4$.

7. We have,

$$\int\left(\frac{3x^2 + 2x}{x^6 + 2x^5 + x^4 + 2x^3 + 2x^2 + 2}\right)dx$$

$$= \int\frac{3x^2 + 2x}{1 + \left(x^6 + x^4 + 1 + 2(x^5 + x^3 + x^2)\right)}dx$$

$$= \int\frac{3x^2 + 2x}{1 + (x^3 + x^2 + 1)^2}\,dx$$

$$= \tan^{-1}(x^3 + x^2 + 1) + c$$

Thus, $L = 1$, $M = 3$, $N = 2$, $P = 1$ and $Q = 1$

Hence, the value of $L + M + N + P + Q = 8$.

8. $\int\frac{\cos^4 x}{\sin^6 x(1 + \cot^5 x)^{3/5}}dx$

$$= \int\frac{\cos^4 x\,\text{cosec}^2 x\,dx}{(1 + \cot^5 x)^{3/5}}$$

$$= -\int\frac{t^4 dt}{(1 + t^5)^{3/5}}, \qquad \text{where } \cot x = t$$

$$= -\frac{1}{5}\int\frac{5t^4 dt}{(1 + t^5)^{3/5}}$$

Put $(1 + t^5) = y^5$

$\Rightarrow \quad 5t^4 dt = 5y^4 dy$

$$= -\frac{1}{5}\int\frac{5y^4 dy}{y^3}$$

$$= -\int y\,dy$$

$$= -\left(\frac{y^2}{2}\right) + c$$

$$= -\frac{1}{2}(1 + t^5)^{2/5} + c$$

$$= -\frac{1}{2}(1 + \cot^5 x)^{2/5} + c$$

$$= -\frac{1}{2}\left(\frac{\tan^5 x + 1}{\tan^5 x}\right)^{2/5} + c$$

Thus, $L = 2$, $M = 2$ and $N = 5$

Hence, the value of $L + M + N = 9$

9. Now, $\frac{\beta + 1}{\gamma} = \frac{\frac{13}{2} + 1}{\frac{5}{2}} = 3$

Put $(1 + x^{5/2}) = t^3$

$$\frac{5}{2}x^{3/2} = 3t^2 dt$$

$$x^{3/2} = \frac{6}{5}t^2 dt$$

Thus, the given integral reduces to

$$\frac{6}{5}\int(t^3 - 1)^2 \cdot t^{3/2} \cdot t^2 dt$$

$$= \frac{6}{5}\int(t^6 - 2t^3 + 1)t^{7/2} dt$$

$$= \frac{6}{5}\int(t^{19/2} - 2t^{13/2} + t^{7/2})\,dt$$

$$= \frac{6}{5}\left(\frac{t^{21/2}}{(21/2)} - \frac{2t^{15/2}}{(15/2)} + \frac{t^{9/2}}{(9/2)}\right) + c$$

$$= \frac{6}{5}\left(\frac{(1 + x^{5/2})^{21/2}}{(21/2)} - \frac{2(1 + x^{5/2})^{15/2}}{(15/2)} + \frac{(1 + x^{5/2})^{9/2}}{(9/2)}\right) + c$$

$$= \left(\frac{4}{35}(1 + x^{5/2})^{21/2} \frac{8}{25}(1 + x^{5/2})^{15/2} + \frac{4}{15}(1 + x^{5/2})^{9/2}\right) + c$$

Clearly, $L = 4$, $M = -1$ and $N = 4$

Hence, the value of $L + M + N = 7$

10. Do yourself.

Qustions asked in Past IIT-JEE Examinations

1. We have,

$$\int \frac{dx}{1 - \cot x}$$

$$= \int \left(\frac{\sin x}{\sin x - \cos x}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{2\sin x}{\sin x - \cos x}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{(\sin x - \cos x) + (\sin x + \cos x)}{\sin x - \cos x}\right) dx$$

$$= \frac{1}{2}\int \left(1 + \frac{(\sin x + \cos x)}{\sin x - \cos x}\right) dx$$

$$= \frac{1}{2}(x + \log|\sin x - \cos x|) + c$$

2. We have,

$$\int \frac{x\,dx}{1 + x^4}$$

$$= \frac{1}{2}\int \left(\frac{2x}{1 + (x^2)^2}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{dt}{1 + t^2}\right)$$

$$= \frac{1}{2}\tan^{-1}(t) + c = \frac{1}{2}\tan^{-1}(x^2) + c$$

3. We have,

$$\int \frac{x^2\,dx}{(a + bx)^2}$$

Put $a + bx = t$

$\Rightarrow \quad dx = \frac{1}{b}dt$

also $x = \frac{t - a}{b}$

Thus, the given integral reduces to

$$\int \frac{\left(\frac{t - a}{b}\right)^2}{t^2} \times \frac{1}{b}dt$$

$$= \frac{1}{b^3}\int \frac{(t - a)^2}{t^2} dt$$

$$= \frac{1}{b^3}\int \frac{t^2 - 2ta + a^2}{t^2} dt$$

$$= \frac{1}{b^3}\int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) dt$$

$$= \frac{1}{b^3}\left(t - 2a\log|t| - \frac{a^2}{t}\right) + c$$

$$= \frac{1}{b^3}\left((a + bx) - 2a\log|(a + bx)| - \frac{a^2}{(a + bx)}\right) + c$$

4. We have,

$$\int \sqrt{1 + \sin\left(\frac{x}{2}\right)}\,dx$$

$$= \int \sqrt{1 + 2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)}\,dx$$

$$= \int \left(\sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right)\right) dx$$

$$= 4\left(-\cos\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right)\right) + c$$

5. We have,

$$\int \frac{x^2}{\sqrt{1 - x}} dx$$

$$= \int \frac{(1 - t^2)^2}{t} \times -2t\,dt, \qquad \text{where} \quad 1 - x = t^2$$

$$= -2\int (1 - t^2)^2 dt$$

$$= -2\int (1 - 2t^2 + t^4) dt$$

$$= -2\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) + c$$

$$= -2\left((1 - x)^{1/2} - \frac{2}{3}(1 - x)^{3/2} + \frac{(1 - x)^{5/2}}{5}\right) + c$$

6. We have,

$$\int (e^{\log x} + \sin x)\cos x\,dx$$

$$= \int (e^{\log x}\cdot\sin x + \sin x\cos x)\,dx$$

$$= \int (x\cdot\sin x + \sin x\cos x)\,dx$$

$$= \int\left(x\cdot\sin x + \frac{\sin 2x}{2}\right)dx$$

$$= x\int \sin x\,dx + \int \cos x\,dx - \frac{\cos 2x}{4} + c$$

$$= -x\cos x + \sin x - \frac{\cos 2x}{4} + c$$

7. We have,

$$\int e^x\left(\frac{x-1}{(x+1)^3}\right)dx$$

$$= \int e^x\left(\frac{x+1-2}{(x+1)^2}\right)dx$$

$$= \int e^x\left(\frac{1}{(x+1)^2} + \left(\frac{-2}{(1+x)^3}\right)\right)dx$$

$$= \frac{e^x}{(x+1)^2} + c$$

8. We have,

$$\int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Let $\left(1+\frac{1}{x^4}\right) = t^4$

$$\Rightarrow \quad -\frac{4}{x^5}dx = 4t^3\,dt$$

$$\Rightarrow \quad \frac{1}{x^5}dx = -t^3\,dt$$

$$= -\int \frac{t^3}{t^3}dt$$

$$= -\int dt$$

$$= -t + c$$

$$= -\left(1+\frac{1}{x^4}\right)^{1/4} + c$$

9. We have $-\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Let $x = \cos^2\theta$

$\Rightarrow \quad dx = -\sin 2\theta\, d\theta$

$$= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \times -\sin(2\theta)\,d\theta$$

$$= \int \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \times 2\sin(\theta)\cos(\theta)\,d\theta$$

$$= 4\int \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \times \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos(\theta)\,d\theta$$

$$= 4\int \sin^2\left(\frac{\theta}{2}\right)\cos(\theta)\,d\theta$$

$$= 2\int\left(2\sin^2\left(\frac{\theta}{2}\right)\right)\cos(\theta)\,d\theta$$

$$= 2\int(1-\cos\theta)\cos(\theta)\,d\theta$$

$$= 2\int(\cos\theta - \cos 2\theta)\,d\theta$$

$$= 2\int\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2\int\left(\sin\theta - \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right) + c$$

$$= (2\sin\theta - (\theta + \sin 2\theta\cos\theta)) + c$$

$$= 2\sqrt{1-x} - \left[\left(\cos^{-1}(\sqrt{x}) + \sqrt{x}\sqrt{1-x}\right)\right] + c$$

10. We have,

$$\int\left(\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\right)dx$$

$$= \int\left(\frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\right)dx$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx$$

$$= \frac{4}{\pi}\int(\sin^{-1}\sqrt{x})dx - \int dx$$

$$= \frac{4}{\pi}\int(\theta\sin 2\theta)d\theta - \int dx$$

$$= \frac{4}{\pi}\int\left(\theta\int \sin(2\theta)\,d\theta + \int\left(\frac{\cos 2\theta}{2}\right)d\theta\right) - \int dx$$

where $x = \sin^2\theta$

$$= \frac{4}{\pi}\int\left(-\frac{\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right) - x + c$$

$$= \frac{4}{\pi}\left(-\frac{\sin^{-1}(\sqrt{x})(1-2x^2)}{2} + \frac{\sqrt{x}\sqrt{1-x}}{2}\right) - x + c$$

$$= \frac{2}{\pi}\left(-\sin^{-1}(\sqrt{x})(1-2x^2) + \sqrt{x}\sqrt{1-x}\right) - x + c$$

11. We have $\int \frac{\sqrt{\cos 2x}}{\sin x}dx$

$$= \int \frac{\sqrt{1-2\sin^2 x}}{\sin x}dx$$

$$= \int \sqrt{\frac{1}{\sin^2 x - 2}}\, dx$$

$$= \int \sqrt{\operatorname{cosec}^2 x - 2}\, dx$$

$$= \int \sqrt{\cot^2 x - 1}\, dx$$

Let $\cot x = \sec\theta$

$\Rightarrow$ $-\operatorname{cosec}^2 x\, dx = \sec\theta \tan\theta\, d\theta$

$$dx = -\frac{\sec\theta \tan\theta}{\operatorname{cosec}^2 x}\, d\theta$$

$$= -\frac{\sec\theta \tan\theta}{1 + \cot^2 x}\, d\theta$$

$$= -\frac{\sec\theta \tan\theta}{1 + \sec^2 x}\, d\theta$$

$$= -\frac{\sec\theta}{\cos^2\theta + 1}\, d\theta$$

$$= -\int \frac{\tan\theta \cdot \sin\theta}{\cos^2\theta + 1}\, d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos\theta(\cos^2\theta + 1)}\, d\theta$$

$$= -\int \frac{\sin^2\theta \cdot \cos\theta}{\cos^2\theta(\cos^2\theta + 1)}\, d\theta$$

$$= -\int \frac{t^2}{(1 - t^2)(2 - t^2)}\, dt, \quad \text{where} \quad \sin\theta = t$$

$$= -\int \frac{t^2}{(t^2 - 1)(t^2 - 2)}\, dt$$

$$= -\int \left(\frac{2}{t^2 - 2} - \frac{1}{t^2 - 1}\right) dt$$

$$= -\int \left(\frac{2}{t^2 - 2}\right) dt + \int \left(\frac{1}{t^2 - 1}\right) dt$$

$$= -\left(2 \cdot \frac{1}{2\sqrt{2}} \log\left|\frac{t - \sqrt{2}}{t + \sqrt{2}}\right|\right) + \frac{1}{2}\log\left|\frac{t - 1}{t + 1}\right| + c$$

where $t = \sin\theta$

12. We have,

$$\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$$

$$= \int \left(\frac{\tan x + 1}{\sqrt{\tan x}}\right) dx$$

$$= \int \left(\frac{t^2 + 1}{t}\right) \frac{2t}{t^4 + 1}\, dt$$

$$= 2\int \left(\frac{t^2 + 1}{t^4 + 1}\right) dt$$

$$= 2\int \left(\frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\right) dt$$

$$= 2\int \left(\frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2}\right) dt$$

$$= 2\int \frac{dz}{z^2 + 2}$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$$

$$= \sqrt{2} \tan^{-1}\left(\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right)\right) + c$$

$$= \sqrt{2} \tan^{-1}\left(\frac{1}{\sqrt{2}}(\tan x - \cot x)\right) + c$$

13. We have

$$I = \int \left(\frac{4e^x + 6e^{-x}}{9e^x + 4e^{-x}}\right) dx$$

$4e^x + 6e^{-x} = m(9e^x - 4e^{-x}) + n(9e^x + 4e^{-x})$

Comparing the co-efficients of e^x and e^{-x}, we get $m + n = \frac{4}{9}$ and $n - m = \frac{3}{2}$

Solving, we get,

$n = 35/36$ and $m = -19/36$.

The given integral reduces to

$$-\frac{19}{36}\int dx + \frac{35}{36}\int \left(\frac{9e^x + 4e^{-x}}{9e^x - 4e^{-x}}\right) dx$$

$$= -\frac{19}{36}x + \frac{35}{36}\log|9e^x - 4e^{-x}| + c$$

$$= -\frac{19}{36}x + \frac{35}{36}\log\left|\frac{9e^{2x} - 4}{e^x}\right| + c$$

$$= -\frac{19}{36}x + \frac{35}{36}\log|9e^{2x} - 4| - \frac{35}{36}\log|e^x| + c$$

$$= -\left(\frac{19}{36} + \frac{35}{36}\right)x + \frac{35}{36}\log|9e^{2x} - 4| + c$$

$$= -\frac{3}{2}x + \frac{35}{36}\log|9e^{2x} - 4| + c$$

14. $$\int \left(\frac{1}{x^{1/3} + x^{1/4}} + \frac{\log(1 + x^{1/6})}{\sqrt{x} + x^{1/3}}\right) dx$$

$= I_1 + I_2$ (say)

Now, $I_1 = \int \frac{dx}{x^{1/3} + x^{1/4}}$

$$= 12\int \frac{t^{11}\,dt}{t^4 + t^3}, \text{ where } x = t^{12}$$

$$= 12\int \frac{t^8\,dt}{t + 1}$$

$$= 12\int \left(\frac{(t^8 - 1) + 1}{t + 1}\right)dt$$

$$= 12\int \left(t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + \frac{1}{t + 1}\right)dt$$

$$= 12\int \left(\frac{t^8}{8} + \frac{t^7}{7} + \frac{t^6}{6} + \frac{t^5}{5} + \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} + \log|t + 1|\right) + c$$

Let $I_2 = \int \frac{\log(1 + x^{1/6})}{x^{1/2} + x^{1/3}}\,dx$

$$= 6\int \frac{\log(1 + t)}{t^3 + t^2}\,t^5 dt, \text{ where } x = t^6$$

$$= 6\int \frac{t^3 \log(1 + t)}{t + 1}\,dt$$

$$= 6\int \frac{(t^3 - 1 + 1)\log(1 + t)}{t + 1}\,dt$$

$$= 6\int \left(\frac{(t^3 - 1)\log(1 + t)}{t + 1} - \frac{\log(1 + t)}{(1 + t)}\right)dt$$

$$= 6\int \left((t^2 - t + 1)\log(1 + t) - \frac{\log(1 + t)}{(1 + t)}\right)dt$$

$$= 6\left[\log(1 + t)\left(\frac{t^3}{3} - \frac{t^2}{2} + t\right) - \int \frac{\frac{t^3}{3} - \frac{t^2}{2} + t}{1 + t}\,dt\right]$$

$$-6(\log(1 + t))^2 + c$$

$$= 6\log(1 + t)\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(1 + t)\right)$$

$$+ \int \frac{2t^3 - 3t^2 + 6}{t + 1}\,dt$$

$$= 6\log(1 + t)\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(1 + t)\right)$$

$$+ \int \left(2t^2 + 5t + 5 + \frac{1}{t + 1}\right)dt$$

$$= 6\log(1 + t)\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(1 + t)\right)$$

$$+ \left(\frac{2t^3}{3} + \frac{5t^2}{2} + 5t + \log(1 + t)\right) + c$$

where $t = x^{1/6}$.

15. We have,

$$I = \cos 2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta$$

$$= \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\int \cos(2\theta)\,d\theta$$

$$-\frac{1}{2}\int \sin(2\theta)\cdot\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right) + \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta$$

$$= \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\left(\frac{\sin 2\theta}{2}\right)$$

$$= -\frac{1}{2}\int \frac{\sin(2\theta)\cdot 2(\sin^2\theta + \cos^2\theta)}{\cos^2\theta - \sin^2\theta}\cdot d\theta$$

$$= \left(\frac{\sin 2\theta}{2}\right) \times \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \int \frac{\sin(2\theta)}{\cos(2\theta)}\cdot d\theta$$

$$= \left(\frac{\sin 2\theta}{2}\right) \times \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$\frac{1}{2}\log|\sec(2\theta)| + c$$

16. We have,

$$\int \left(\frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}\right)dx$$

$$= \left(\frac{\cos^2 x + \cos^4 x}{\sin^2 x + \sin^4 x}\right)\cos x\,dx$$

$$= \int \left(\frac{(1 - \sin^2 x) + (1 - \sin^2 x)^2}{\sin^2 x + \sin^4 x}\right)\cos x\,dx$$

$$= \int \left(\frac{(1 - t^2) + (1 - t^2)^2}{t^2 + t^4}\right)dt, \qquad \text{where} \quad t = \sin x$$

$$= \int \left(\frac{((t^4 + t^2) + (2 - 4t^2))}{t^2 + t^4}\right)dt$$

$$= \int \left(1 + \frac{(2 - 4t^2)}{t^2 + t^4}\right)dt$$

$$= \int \left(1 - \frac{2(2t^2 - 1)}{t^2(t^2 + 1)}\right)dt$$

$$= \int \left(1 - \frac{4}{t^2 + 1} + \frac{2}{t^2(t^2 + 1)}\right)dt$$

$$= \int \left(1 - \frac{4}{t^2 + 1} + 2\left(\frac{1}{t^2} - \frac{1}{(t^2 + 1)}\right)\right)dt$$

$$= \left(t - 4\tan^{-1}t - \frac{2}{t} - 2\tan^{-1}t\right) + c$$

$$= \left(t - 6\tan^{-1}t - \frac{2}{t}\right) + c$$

$$= (\sin x - 6\tan^{-1}(\sin x) - 2\operatorname{cosec} x) + c$$

17. We have,

$$\int \frac{x+1}{x(1+xe^x)^2}\,dx$$

$$= \int \frac{(x+1)e^x}{xe^x(1+xe^x)^2}\,dx$$

$$= \int \frac{dt}{t(1+t)^2}, \text{ Where } t = xe^x$$

$$= \int \left(\frac{1}{t} - \frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt$$

$$= \log\left|\frac{1}{t+1}\right| + \frac{1}{(t+1)} + c$$

$$= \log\left|\frac{xe^x}{xe^x+1}\right| + \frac{1}{(xe^x+1)} + c$$

18. We have,

$$\int \frac{dx}{\sqrt{(x-p)^3(x-q)}}$$

$$= \int \frac{dx}{(x-p)^{3/2}(x-q)^{1/2}}$$

$$= \int \frac{dx}{(x-p)^2\left(\dfrac{(x-q)}{(x-p)}\right)^{1/2}}$$

Let $\dfrac{(x-q)}{(x-p)} = t$

$$\Rightarrow \left(\frac{(x-p)\cdot 1 - (x-q)\cdot 1}{(x-p)^2}\right)dx = dt$$

$$\Rightarrow \frac{dx}{(x-p)^2} = \frac{dt}{(q-p)}$$

The given integral reduces to

$$= \frac{1}{(q-p)}\int \frac{dt}{t^{1/2}}$$

$$= \frac{1}{(q-p)} \times 2\sqrt{t} + c$$

$$= \frac{2}{(q-p)} \times 2\sqrt{\frac{x-p}{x-q}} + c$$

19. We have,

$$\int \frac{dx}{(\sin x + 4)(\sin x - 1)}$$

$$= \frac{1}{5}\left(\int \frac{1}{(\sin x - 1)} - \frac{1}{(\sin x - 4)}\right)dx$$

$$= \frac{1}{5}\int \frac{1}{(\sin x - 1)}\,dx - \frac{1}{5}\int \frac{1}{(\sin x + 4)}\,dx$$

$$= -\frac{1}{5}\int \frac{1}{(1-\sin x)}\,dx - \frac{1}{5}\int \frac{1}{(\sin x + 4)}\,dx$$

$$= -\frac{1}{5}\int \frac{1+\sin x}{\cos^2 x}\,dx - \frac{1}{5}\int \frac{1}{\left(\dfrac{2\tan(x/2)}{1+\tan^2(x/2)} + 4\right)}\,dx$$

$$= -\frac{1}{5}\int (\sec^2 x + \sec x \tan x)\,dx$$

$$= -\frac{1}{5}\int \frac{\sec^2(x/2)}{(4\tan^2(x/2) + 2\tan(x/2) + 4)}\,dx$$

$$= -\frac{1}{5}(\tan x + \sec x) - \frac{1}{5}\int \frac{dt}{2t^2 + t + 2}$$

where $\tan\left(\dfrac{x}{2}\right) = t$

and then you do it.

20. We have,

$$\int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)(2 + 2\sin 2x)dx$$

$$= 2\int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) \times (\cos x + \sin x)^2\,dx$$

$$= 2\int (\cos^2 x - \sin^2 x)dx$$

$$= 2\int \cos(2x)dx$$

$$= 2\left(\frac{\sin 2x}{2}\right) + c$$

$$= \sin 2x + c$$

21. $\displaystyle\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \times \frac{dx}{x}$

Let $x = \cos^2 2\theta$

$\Rightarrow dx = -2\cos x(2\theta)\sin(2\theta)$

$$= \int \sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}} \times \frac{-2\cos(2\theta)\sin(2\theta)}{\cos^2(2\theta)}d\theta$$

$$= -2\int \frac{\sin\theta}{\cos\theta} \times \frac{2\sin(\theta)\cos(\theta)}{\cos(2\theta)}d\theta$$

$$= -2\int \frac{2\sin^2(\theta)}{\cos(2\theta)}d\theta$$

$$= 2\int \frac{(\cos(2\theta) - 1)}{(\cos(2\theta))}d\theta$$

$$= 2\int (1 - \sec(2\theta))d\theta$$

$$= 2\left(\theta - \frac{1}{2}\log|\sec(2\theta) + \tan(2\theta)|\right) + c$$

$$= 2\left(\cos^{-1}(\sqrt{x}) - \frac{1}{2}\log\left|\frac{1}{\sqrt{x}} + \sqrt{1 + \frac{1}{x}}\right|\right) + c$$

22. We have,

$$\int \frac{dx}{(2x-7)\sqrt{x2 - 7x + 12}}$$

$$= 2\int \frac{dx}{(2x-7)\sqrt{4x^2 - 28x + 48}}$$

$$= \int \frac{2 \cdot dx}{(2x-7)\sqrt{(2x-7)^2 - 1}}$$

$$= \int \frac{dt}{t\sqrt{t^2 - 1}}, \text{ where } (2x - 7) = t$$

$$= \sec^{-1}(t) + c$$

$$= \sec^{-1}(2x - 7) + c$$

23. $\int \frac{x^2 + 3x + 2}{(x^2+1)^2(x+1)} dx$

$$= \int \frac{x^2 + 3x + 2)}{(x^2+1)^2(x+1)} dx$$

$$= \int \frac{(x+1)(x+2)}{(x^2+1)^2(x+1)} dx$$

$$= \int \frac{(x^2+2)}{(x^2+1)^2} dx$$

$$= \int \frac{(x+2)}{(x^2+1)^2} dx$$

$$= \frac{1}{2}\int \frac{2x}{(x^2+1)^2} dx + 2\int \frac{dx}{(x^2+1)^2}$$

$$= -\frac{1}{2(x^2+1)} + 2\int \frac{dx}{(x^2+1)^2}$$

$$= \left(-\frac{1}{2(x^2+1)} + \tan^{-1}x + \frac{x}{x^2+1}\right) + c$$

Let $I_1 = 2\int \frac{dx}{(x^2+1)^2}$

$$= 2\int \frac{\sec^2\theta d\theta}{\sec^4\theta}, \text{ where } x = \tan\theta$$

$$= 2\int \frac{d\theta}{\sec^2\theta}$$

$$= \int (2\cos^2\theta) d\theta$$

$$= \int (1 + \cos 2\theta) d\theta$$

$$= (\theta + \sin\theta + \cos\theta) + c$$

$$= \left(\tan^{-1}x + \frac{x}{x^2+1}\right) + c$$

24. We have,

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}}\right) dx$$

$$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2 + (3)^2}}\right)$$

Let $(2x + 2) = 3\tan\theta$

Thus, $\left(\frac{(2x+2)}{\sqrt{(2x+2)^2 + 3^2}}\right) = \sin\theta$

$$= \frac{3}{2}\int (\theta\sec^2\theta) d\theta$$

$$= \frac{3}{2}\int (\theta\tan\theta - \int \tan\theta d\theta)$$

$$= \frac{3}{2}(\theta\tan\theta - \log|\sec\theta|) + c$$

$$= \frac{3}{2}\left(\theta\tan\theta - \log\left|\sqrt{1 + \tan^2\theta}\right|\right) + c$$

$$= \frac{3}{2}(\theta\tan\theta) - \frac{3}{4}\log(1 + \tan^2\theta) + c$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2 + 8x + 13) + c$$

25. We have

$$\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx$$

$$= \int (x^{3m-1} + x^{2m-1} + x^{m-1})(2x^{2m} + 3x^m + 6)^{1/m} x\, dx$$

$$= \int (x^{3m-1} + x^{2m-1} + x^{m-1})(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$

Put $(2x^{3m} + 3x^{2m} + 6x^m) = t^m$

$$6m(x^{3m-1} + x^{2m-1} + x^{m-1})dx = m\,t^{m-1}dt$$

$$(x^{3m-1} + x^{2m-1} + x^{m-1})dx = \frac{t^{m-1}}{6}dt$$

Thus, the given integral reduces to $\frac{1}{6}\int t^m dt$

$$= \frac{1}{6}\left(\frac{t^{m+1}}{m+1}\right) + c$$

$$= \frac{1}{6}\left(\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{m+1}}{m+1}\right) + c$$

26. We have,

$$\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$$

$$= \int \frac{x^2 - 1}{x^3\sqrt{x^4\left(2 - \frac{2}{x^2} + \frac{1}{x^4}\right)}}\, dx$$

$$= \int \frac{x^2 - 1}{x^5\sqrt{\left(2 - \frac{2}{x^2} + \frac{1}{x^4}\right)}}\, dx$$

$$= \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)}{\sqrt{\left(2 - \frac{2}{x^2} + \frac{1}{x^4}\right)}}\, dx$$

Let $\left(2 - \frac{2}{x^2} + \frac{1}{x^4}\right) = t^2$

$\Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5}\right)dx = 2t\,dt$

$$= \frac{1}{4}\int \frac{2t\,dt}{t}$$

$$= \frac{1}{2} \times t + c$$

$$= \frac{1}{2} \times \left(2 - \frac{2}{x^2} + \frac{1}{x^4}\right)^{1/2} + c$$

27. We have,

$(f_0 f)(x) = f(f(x))$

$$= \frac{f(x)}{(1 + f(x))^{1/n}}$$

$$= \frac{x}{(1 + 2x^n)^{1/n}}$$

Also, $(f_0 f_0 f)(x)$

$$= f((f_0 f)(x))$$

$$= \frac{(f_0 f)(x)}{(1 + (f_0 f(x))^{1/n}}$$

$$= \frac{x}{(1 + 3x^n)^{1/n}}$$

Similarly, we can write, $(f_0 f_0 f_0 \cdots {}_0 f)(x)$

$$= \frac{x}{(1 + nx^n)^{1/n}}.$$

Thus, $I = \int x^{n-2} g(x)dx$

$$= \int x^{n-2}\left(\frac{x}{(1 + nx^n)^{1/n}}\right)dx$$

$$= \frac{1}{n}\int t^{n-2}dt, \text{ Where } t^n = (1 + nx^n)$$

$$= \frac{1}{n(n-1)} t^{n-1} + c$$

$$= \frac{1}{n(n-1)}(1 + nx^n)^{\frac{n-1}{n}} + c$$

$$= \frac{1}{n(n-1)}(1 + nx^n)^{\left(1 - \frac{1}{n}\right)} + c$$

28. We have

$J - I$

$$= \int \frac{e^x dx}{e^{4x} + e^{2x} + 1} - \int \frac{e^{-x}dx}{e^{-4x} + e^{-2x} + 1}$$

$$= \int \frac{(e^x - e^{3x})\,dx}{e^{4x} + e^{2x} + 1}$$

$$= \int \frac{(1 - e^{2x})\,e^x dx}{e^{4x} + e^{2x} + 1}$$

Let $e^x = t \Rightarrow e^x dx = dt$

$$= \int \left(\frac{1 - t^2}{t^4 + t^2 + 1}\right)dt$$

$$= -\int \left(\frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 1}\right)dt$$

$$= \int \left(\frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 1}\right)dt$$

$$= -\frac{1}{2}\log\left|\frac{\left(t + \frac{1}{t}\right) - 1}{\left(t + \frac{1}{t}\right) + 1}\right| + c$$

$$= -\frac{1}{2}\log\left|\frac{t^2 - t + 1}{t^2 + t + 1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{t^2 + t + 1}{t^2 - t + 1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right| + c$$

29. We have,

$$\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}\, dx$$

$$= \int \frac{\sec x \cdot \sec x}{(\sec x + \tan x)^{9/2}}\, dx \quad \text{...(i)}$$

Put $\sec x + \tan x = t$

$\Rightarrow \quad \sec x(\sec x + \tan x)\, dx = dt$

$\Rightarrow \quad \sec x\, dx = \frac{dt}{t}$

Also $\sec x - \tan x = \frac{1}{(\sec x + \tan x)} = \frac{1}{t}$

Thus, $\sec x = \frac{1}{2}\left(t + \frac{1}{t}\right) dt$

The given intergral (i) reduces to

$$\int \frac{\frac{1}{2}\frac{\left(t + \frac{1}{t}\right)dt}{t}}{t^{9/2}}$$

$$= \frac{1}{2}\int \frac{(t^2 + 1)}{t^{13/2}} dt$$

$$= \frac{1}{2}\int\left(\frac{1}{t^{9/2}} + \frac{1}{t^{13/2}}\right)dt$$

$$= -\frac{1}{2}\left(\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right) + c$$

$$= -\left(\frac{1}{7t^{7/2}} + \frac{1}{11t^{11/2}}\right) + c$$

$$= \left(\frac{1}{7(\sec x + \tan x)^{7/2}} + \frac{1}{11(\sec x + \tan x)^{11/2}}\right) + c$$

CHAPTER

2 Definite Integrals

CONCEPT BOOSTER

1. What is Definite Integral?

Let $f(x)$ be a function of x defined in the closed interval $[a, b]$ and $\varphi(x)$ be another function such that $\varphi'(x) = f(x)$ for every x in the domain of $f(x)$. Then

$$\int_a^b f(x)\,dx = [\varphi(x) + c]_a^b = \varphi(b) - \varphi(a)$$

is called the definite integral of the function $f(x)$ over the interval $[a, b]$.

Here, a is called the lower limit and b is called the upper limit.

Notes:

1. $\int_a^b$ is read as 'the integral of $f(x)$ from a to b'
2. To evaluate the definite integral, there is no need to keep the constant of integration.
3. Geometrically, $\int_a^b f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.
4. Area bounded means, we shall use, $\int_a^b |f(x)|\,dx$, when the area lies below x-axis.

2. Evaluation of Definite Integrals

Rules

1. Simply find the indefinite integrals of the given function.
2. There is no need to keep the constant of integration.
3. Use the limits of integration.

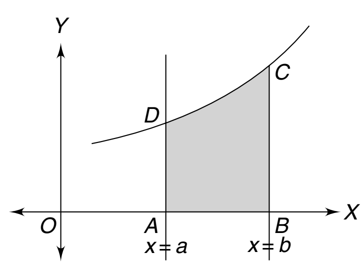
3. Evaluation of Definite Integrals by Substitution

Rules

1. When the variable in a definite integral is changed, the substitution in terms of new variable should be affected at three places.
   (a) in the integrand ($f(x)$ will be changed)
   (b) in the differentials (dx will be changed)
   (c) in the limits (old limits will be changed)

4. Geometrical Interpretation of Definite Integral

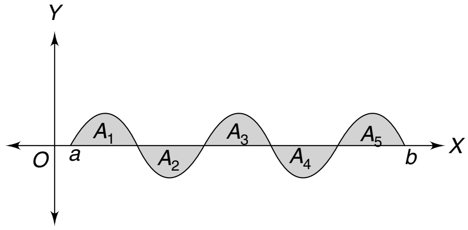
Consider the function $y = f(x)$, where $f(x) \geq 0$ for all x in $[a, b]$. The integral $\int_a^b$ is numerically equal to the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.



$$\text{Area of the region } ABCD = \int_a^b f(x)\,dx$$

In general, $\int_a^b f(x)\,dx$ represents an algebraic sum of the area bounded by the graph of the function $y = f(x)$, x-axis and the lines $x = a$ and $x = b$.

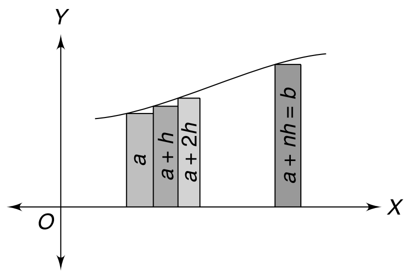
The area above the x-axis is taken as the sum with the positive sign, while the area which is below x-axis is taken as the sum with the negative sign. So, the value of the integral may be positive, negative or zero.



Thus, $\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$

5. Definite Integral as the Limit of Sum

Let f be a continuous real function on $[a, b]$. Assume that all the values taken by the function are non-negative, the graph of the function is a curve above the x-axis.



Divide the interval $[a, b]$ into n equal sub-intervals denoted by

$[a, a + h], [a + h, a + 2h], [a + 2h, a + 3h], ...,$

$[a + (n - 1)h, a + nh]$

Clearly, $b = a + nh$

$\Rightarrow \quad nh = b - a$

$\Rightarrow \quad h = \dfrac{b - a}{n}$

Thus, $\int_a^b f(x)\,dx$

= Area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$.

= Area of the region $PQRSP$ between the curve, x-axis and the ordinates $x = a$ and $x = b$.

$= \lim_{h\to 0} h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h]$

$= \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a + rh)$

$= \lim_{n\to\infty} \left(\dfrac{b - 1}{n}\right) \sum_{r=0}^{n-1} f(a + rh),$

where $nh = b - a$

which is known as the **first principle of integration.**

6. Evaluation of the Limit of the Sum using Newton-Leibnitz Formula

From the definition of the definite integral, we have

$$\int_a^b f(x)\,dx$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} f(a + rh)$$

$$= \lim_{n\to\infty} \left(\frac{b - a}{n}\right) \sum_{r=0}^{n} f\left[a + \left(\frac{b - a}{n}\right) r\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=\varphi(x)}^{\psi(x)} f\left(\frac{r}{n}\right),$$

where

(i) $\sum$ is replaced by $\int$ symbol

(ii) $\dfrac{1}{n}$ is replaced by dx

(iii) $\dfrac{r}{n}$ is replaced by x

(iv) $a = \lim_{n\to\infty} \left(\dfrac{\varphi(x)}{n}\right)$

and $b = \lim_{n\to\infty} \left(\dfrac{\psi(x)}{n}\right)$

7. Properties of Definite Integrals

Property I

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Proof: Let $\dfrac{d}{dx}[g(x) + c] = f(x)$

$\Rightarrow \quad \int f(x)\,dx = g(x) + c$

Now, $\int_a^b f(x)\,dx = g(x) + c|_a^b$

$= g(b) - g(a)$

Also, $\int_a^b f(t)dt = g(t) + c|_a^b$

$= g(b) - g(a)$

Hence, the result

Property II

$$\int_a^b f(x)dx = -\int_b^a f(x)\,dx$$

Proof: Let $\dfrac{d}{dx}[g(x) + c] = f(x)$

$\Rightarrow \quad \int f(x)dx = g(x) + c$

Now, $\int_a^b f(x)\,dx = g(x) + c|_a^b$

$= g(b) - g(a)$

Also, $-\int_b^a f(x)\,dx = -[g(x) + c]|_b^a$

$$= -[g(a) - g(b)]$$
$$= g(b) - g(a)$$

Hence, the result.

Property III

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x),$$

where $a < c < b$

Proof: Let $\frac{d}{dx}[g(x) + c] = f(x)$

$$\Rightarrow \quad \int f(x)\,dx = g(x) + c$$

Now, $\int_a^b f(x)\,dx = g(x) + c|_a^b$

$$= g(b) - g(a)$$

Also, $\int_a^c f(x)\,dx = g(x) + k|_a^c$

$$= g(c) - g(a)$$

and $\int_c^b f(x)dx = g(x) + k|_c^b$

$$= g(b) - g(a)$$

Thus, $\int_a^c f(x)dx + \int_c^b f(x)dx$

$$= g(c) - g(a) + g(b) - g(c)$$
$$= g(b) - g(a)$$
$$= \int_a^b f(x)\,dx$$

Hence, the result.

Note: In, general, the above property can be extended to any finite number of limits. Thus,

$$\int_a^b f(x)\,dx$$
$$= \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)dx + \ldots + \int_{c_n}^b f(x)dx$$

where $a < c_1 < c_2 < \ldots < c_n < b$

Property IV

$$\int_0^a f(x)dx = \int_0^a f(a - x)dx$$

Proof: Let $I = \int_0^a f(a - x)dx$

$$= -\int_a^0 f(t)\,dt, \quad \text{where } a - x = t$$
$$= +\int_0^a f(t)dt \quad \text{by Property II}$$
$$= \int_0^a f(x)\,dx \quad \text{by Property I}$$

Property V

$$\int_a^b f(x)\;dx = \int_a^b f(a + b - x)\,dx$$

Proof: We have

$$\int_a^b f(a + b - x)dx$$

Let $a + b - x = t$

$$\Rightarrow \quad dx = -dt$$
$$= -\int_b^a f(t)dt$$
$$= \int_a^b f(t)\,dt, \quad \text{by Property II}$$
$$= \int_a^b f(x)\,dx, \quad \text{by Property I}$$

Property VI

$$\int_{-a}^a f(x)\;dx$$
$$= \begin{cases} 2\int_0^a f(x)\,dx & : f(x) \text{ is an even function} \\ 0 & : f(x) \text{ is an odd function} \end{cases}$$

Proof: We have

$$\int_{-a}^a f(x)\,dx$$
$$= \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

Put $x = -t$ in the first integral only

$$= -\int_a^0 f(-t)dt + \int_0^a f(x)dx$$
$$= \int_0^a f(-t)\,dt + \int_0^a f(x)dx, \quad \text{by Property II}$$
$$= \int_0^a f(-x)\,dx + \int_0^a f(x)dx, \quad \text{by Property I}$$
$$= \begin{cases} \int_0^a f(x)\,dx + \int_0^a f(x)\,dx & : f(x) \text{ is even} \\ -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx & : f(x) \text{ is odd} \end{cases}$$
$$= \begin{cases} 2\int_0^a f(x)\,dx & : f(x) \text{ is even} \\ 0 & : f(x) \text{ is odd} \end{cases}$$

Property VII

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & : f(2a-x) = f(x) \\ 0 & : f(2a-x) = -f(x) \end{cases}$$

Proof: Let $I = \int_0^{2a} f(x)dx$

$= \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

$= I_1 + I_2$, (say)

Now, $I_2 = \int_0^a f(2a - t)\,dt$

Put $2a - t = x \Rightarrow dt = -dx$

$= -\int_a^0 f(2a - t)\,dt$

$= \int_0^a f(2a - t)$, by Property II

$= \int_0^a f(2a - x)dx$

Thus, $I = \int_0^{2a} f(x)dx$

$= \int_0^a f(x)dx + \int_a^{2a} f(2a - x)dx$

$$= \begin{cases} \int_0^a f(x)\,dx + \int_0^a f(x)\,dx & : f(2a-x) = f(x) \\ \int_0^a f(x)\,dx - \int_0^a f(x)\,dx & : f(2a-x) = -f(x) \end{cases}$$

$$= \begin{cases} 2\int_0^a f(x)\,dx & : f(2a-x) = f(x) \\ 0 & : f(2a-x) = -f(x) \end{cases}$$

Hence, the result.

Property VIII

If $f(x)$ be a periodic function with period T, then

(i) $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$

(ii) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$, $n \in Z$, $a \in R^+$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n - m)\int_0^T f(x)\,dx$, $m, n \in Z$

(iv) $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx$, $n \in Z$, $a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)dx$, $n \in Z$, $a, b \in R$

Proof:

(i) LHS $= \int_0^{nT} f(x)\,dx$

$= \sum_{r=1}^{n} \int_{(r-1)T}^{rT} f(x)\,dx$

$= \sum_{r=1}^{n} \int_0^T f[(r-1)T + y]\,dy$

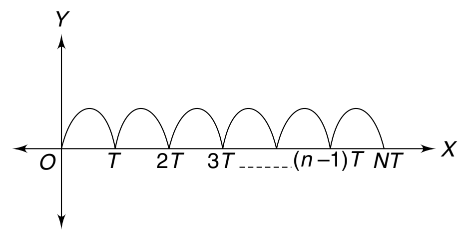
Put $x = (r - 1)T + y = \sum_{r=1}^{n} \int_0^T f(y)\,dy$,

since $f(x)$ is periodic, so $f[(r - 1)T + y] = f(y)$

$= n\int_0^T f(y)\,dy$

$= n\int_0^T f(x)\,dx$

Graphical Method



(ii) $\int_a^{a+nT} f(x)\,dx = \int_a^0 f(x)dx + \int_0^{nT} f(x)\,dx + \int_{nT}^{a+nT} f(x)\,dx$

Put $x = y + nT$ in the last integral only

RHS $= \int_a^0 f(x)\,dx + \int_0^{nT} f(x)\,dx + \int_0^a f(y + nT)\,dy$

$= \int_a^0 f(x)dx + \int_0^{nT} f(x)\,dx + \int_0^a f(x + nT)\,dx$

$= \int_a^0 f(x)\,dx + \int_0^{nT} f(x)\,dx + \int_0^a f(x)\,dx$

$= -\int_0^a f(x)\,dx + \int_0^{nT} f(x)\,dx + \int_0^a f(x)\,dx$

$= \int_0^{nT} f(x)\,dx$

$= n\int_0^T f(x)\,dx$ from (i)

(iii) LHS $= \int_{mT}^{nT} f(x)\,dx$

Pu $x = y + mT$

$= \int_0^{(n-m)T} f(y + mT)\,dy$

$= \int_0^{(n-m)T} f(x + mT)\,dx$

$$= \int_0^{(n-m)T} f(x)\,dx$$

$$= (n-m)\int_0^T f(x)\,dx \qquad \text{from (i)}$$

(iv) $\text{LHS} = \int_{nT}^{a+nT} f(x)\,dx$

$$= \int_{nT}^{0} f(x)\,dx + \int_0^a f(x)dx + \int_a^{a+nT} f(x)\,dx$$

$$= \int_{nT}^{0} f(x)\,dx + \int_0^a f(x)\,dx + \int_0^{nT} f(x)\,dx$$

$$= -\int_0^{nT} f(x)\,dx + \int_0^a f(x)\,dx + \int_0^{nT} f(x)\,dx$$

$$= \int_0^a f(x)\,dx$$

(v) $\text{LHS} = \int_{a+nT}^{b+nT} f(x)\,dx$

$$= \int_{a+nT}^{a} f(x)\,dx + \int_a^b f(x)\,dx + \int_b^{b+nT} f(x)\,dx$$

$$= \int_a^{a+nT} f(x)\,dx + \int_a^b f(x)\,dx + \int_b^{b+nT} f(x)\,dx$$

$$= -\int_0^{nT} f(x)\,dx + \int_a^b f(x)\,dx + \int_0^{nT} f(x)\,dx$$

$$= \int_a^b f(x)\,dx$$

Hence, the result.

Property IX

$$\frac{d}{dt}\left(\int_{\varphi(t)}^{\psi(t)} f(t)\,dt\right) = f[\psi(t)]\,\psi'(t) - f[\varphi(t)]\,\varphi'(t).$$

This is known as ***Newton–Leibnitz Rule.***

Example 1: Find $\frac{d}{dx}\left(\int_{x^3}^{x^4} e^t dt\right)$.

Solution: We have

$$\frac{d}{dx}\left(\int_{x^3}^{x^4} e^t dt\right) = (e^{x^4} \times 4x^3 - e^{x^3} \times 3x^2)$$

Example 2: Find $\frac{d}{dx}\left(\int_{x^4}^{x^5} \sin^2 t\,dt\right)$.

Solution: We have

$$\frac{d}{dx}\left(\int_{x^4}^{x^5} \sin^2 t\,dt\right) = \sin^2(x^5) \times 5x^4 - \sin^2(x^4) \times 4x^3$$

Example 3: Find $\frac{d}{dx}\left(\int_{x^3}^{x^6} \cos(t^2)\,dt\right)$.

Solution: We have

$$\frac{d}{dx}\left(\int_{x^3}^{x^6} \cos(t^2)\,dt\right)$$

$$= \cos(x^{12}) \times 6x^5 - \cos(x^6) \times 3x^2$$

Advance Properties of Definite Integral

Property I

If $f(t)$ is an odd function of x, then $\int_0^x f(t)dt$ is an even function of x.

Proof: Let $g(x) = \int_0^x f(t)\,dt.$

Then $g(-x) = \int_0^{-x} f(t)\,dt$

Now, $g(x) - g(-x)$

$$= \int_0^x f(t)\,dt - \int_0^{-x} f(t)\,dt$$

$$= \int_0^x f(t)\,dt + \int_{-x}^{0} f(t)\,dt$$

$$= \int_{-x}^{x} f(t)\,dt$$

$$= 0.$$

$\Rightarrow$ $g(x) = g(-x)$

$\Rightarrow$ $g(x)$ is an even function.

Property II

If $f(t)$ is an even function of x, then $\int_x^0 f(t)\,dt$ is an odd function of x.

Proof: Let $g(x) = \int_0^x f(t)\,dt.$

Then $g(-x) = \int_0^{-x} f(t)\,dt$

Now $g(x) + g(-x)$

$$= \int_0^x f(t)\,dt + \int_0^{-x} f(t)\,dt$$

$$= \int_0^x f(t)\,dt - \int_0^x f(t)\,dt$$

$$= 0$$

$\Rightarrow$ $g(x) = -g(x)$

$\Rightarrow$ $g(x)$ is an odd function of x.

Property III: Shifting Property

(i) $\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$

(ii) $\int_a^b f(x)\,dx = \int_{a+b}^{b+c} f(x-c)\,dx$

Property IV: Contraction Property

(i) $\int_a^b f(x)\,dx = k\int_{a/k}^{b/k} f(kx)\,dx$

(ii) $\int_a^b f(x)\,dx = \frac{1}{k}\int_{bk}^{ak} f\left(\frac{x}{k}\right)dx$

Property V: Reflection Property

$$\int_a^b f(x)\,dx = \int_{-a}^{-b} f(-x)\,dx.$$

Property VI

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)x+a]\,dx$$

Proof: Let $I = \int_a^b f(x)\,dx$

$= (b-a)\int_0^1 f[(b-a)x+a]\,dx$

put $(b-a)x + a = z$

$\Rightarrow$ $dx = \frac{dz}{b-a}$

When $x = 0$, then $z = a$

and when $x = 1$, then $z = b$

$= (b-a)\int_a^b f(z)\times\frac{dz}{b-a}$

$= \int_a^b f(z)\,dz = \int_a^b f(x)\,dx$

Hence, $\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)x+a].$

Property VII

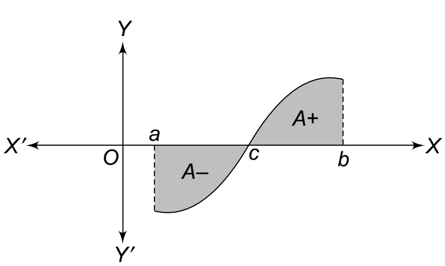
If $f(x)$ be a discontinuous function at $x = a$ then

$$\int_0^{2a} f(x)\,dx = \int_0^a \{f(a-x)+f(a+x)\}dx$$

Property VIII

If $f(x)$ be continuous in $[a, b]$ and $\int_a^b f(x)\,dx = 0$, the equation $f(x) = 0$ has at least one root in (a, b).

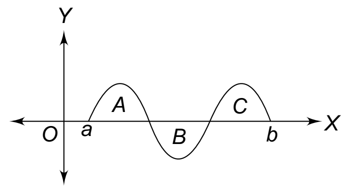
Graph:



Property IX

If $f(x)$ be defined in $[a, b]$, then $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Graph:



Proof: Clearly, $\left|\int_a^b f(x)\,dx\right| = |A - B + C|$

and $\int_a^b |f(x)|\,dx = (A + B + C):$

Hence, $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Property X

If m and M respectively be the least and the greatest value of $f(x)$ in $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$$

Proof: We have

$m \le f(x) \le M, \ \forall\ x \in [a, b]$

$\Rightarrow$ $\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$

$\Rightarrow$ $m(b-a) \le \int_a^b f(x)\,dx \le M(a-b)$

Property XI

Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. If

$I(\alpha) = \int_a^b f(x, \alpha)\,dx,$

then $\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\,dx.$

Property XII

If $f^2(x)$ and $g^2(x)$ are integrable in $[a, b]$, then

$$\left|\int_a^b f(x)\,g(x)\right|dx \le \sqrt{\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)}$$

Property XIII

If $f(x)$ be a periodic function with period T, then $\int_a^{a+T} f(x)\,dx$ is independent of a.

Proof: Let $\varphi(a) = \int_a^{a+T} f(x)\,dx$

$\Rightarrow$ $\frac{d}{da}(\varphi(a))$

$$= f(a+T)\frac{d}{da}(a+T) - f(a)\frac{d}{da}(a)$$
$$= f(a+T) - f(a)$$
$$= 0, \text{ since } f(a+T) = f(a)$$
$\Rightarrow$ $\varphi(a)$ is independent of a.

Property XIV

8. Mean Value of a Function over an Interval

Let $f(x)$ be a continuous function defined in $[a, b]$.

Then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Therefore, $f(c) = \frac{1}{(b-a)}\int_a^b f(x)\,dx$ is called the mean value of the function $f(x)$ over $[a, b]$.

9. Improper Integrals

If $f(x)$ be continuous in $[a, \infty)$, then $\int_a^\infty f(x)\,dx$ is called an improper integral and is defined as

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx. \qquad ...(1)$$

If there exists a finite limit on the RHS of (1), we say that the improper integral is convergent, otherwise it is divergent.

Notes:

1. Geometrically, for $f(x) > 0$, the improper integral $\int_a^\infty f(x)\,dx$, is the area bounded by the curve $y = f(x)$, the x-axis and the staright line $x = a$.
2. $\int_a^{-\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$
3. $\int_\infty^{-\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$

$$= \lim_{b\to-\infty}\int_a^b f(x)\,dx + \lim_{c\to\infty}\int_a^c f(x)\,dx$$

10. Gamma Function

If n be a positive rational number, the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is defined as Gamma Function and is denoted by $\Gamma(n)$, i.e.

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \text{ where } n \in Q^+.$$

Example 1. $\Gamma(1) = \int_0^\infty e^{-x}x^0\,dx$

$$= \lim_{b\to\infty}\int_0^b e^{-x}x^0\,dx$$
$$= \lim_{b\to\infty}(e^{-x})|_0^b$$
$$= \lim_{b\to\infty}(e^{-b}+1)$$
$$= 0 + 1 = 1.$$

Example 2. $\Gamma(2) = \int_0^\infty e^{-x}x\,dx$

$$= \lim_{b\to\infty}(-xe^{-x} - e^{-x})|_0^b$$
$$= \lim_{b\to\infty}(-be^{-b} - e^b + 1)$$
$$= 1.$$

Properties of Gamma Function

(i) $\Gamma(1) = 1$, $\Gamma(0) = \infty$ and $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(5) = 4\Gamma(4)$$
$$= 4\times 3\Gamma(3)$$
$$= 4\times 3\times 2\,\Gamma(2)$$
$$= 4\times 3\times 2\times 1$$
$$= 4!$$

(ii) If $n \in N$, then $\Gamma(n+1) = n\Gamma(n)$

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(iv) $\int_0^{\pi/2}\sin^m x\cdot\cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(v) $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, $0 < n < 1$

11. Beta Function

The Beta function is denoted by $B(m, n)$ and is defined as

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

where $m, n > 0$

Properties of Beta Function

1. $B(m, n) = B(n, m)$, where $m, n > 0$
2. $B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, where $m, n > 0$
3. $B(m, n), = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n-1}}dx$, where $m, n > 0$

12. Walli's Formula

If m and n be integers, then

$$\int_0^{\pi/2}\sin^m x\cdot\cos^n x\,dx,$$

$$= \begin{cases} \dfrac{(m-1)(m-2)\ldots(1 \text{ or } 2)\cdot(n-1)(n-2)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\dfrac{\pi}{2} \\ \text{where } m \text{ and } n \text{ are even integers} \\ \dfrac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)} \\ \text{where } m \text{ and } n \text{ are odd integers} \end{cases}$$

Notes:

1. (i) $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$

$$= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-5}\ldots\dfrac{2}{3} \\ \text{where } n \text{ is odd} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\ldots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} \\ \text{where } n \text{ is even} \end{cases}$$

(ii) $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\cdot\Gamma\left(\frac{n+2}{2}\right)} \times \sqrt{\pi}$$

2: Rule to evaluate

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx.$$

(i) For numerator

Start with as $(m - 1)$ as the first factor.

Subtract 2 from it and get the second factor $(m - 3)$ and continue till the last factor is either 2 or 1.

Similarly start with $(n - 1)$ as the first factor and subtract 2 from it and get the second factor $(n - 3)$ and continue till the last factor is either 2 or 1.

(ii) For denominator

Start with $(m + n)$ as the first factor and subtract 2 from it and get the second factor $(m + n - 2)$ and continue till the last factor is either 2 or 1.

Some Important Expansions to Remember

1. $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \ldots = \ln 2$
2. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$
3. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$
4. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$
5. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots = \frac{\pi^2}{24}$

EXERCISES

Level 1

(Problems Based on Fundamentals)

Evaluation of Definite Integrals

1. Evaluate: $\int_0^{\pi/2} \sin^2 x\, dx$
2. Evaluate: $\int_0^{\pi/4} \frac{dx}{1 + \sin x}$
3. Evaluate: $\int_0^{2} \frac{dx}{\sqrt{x+2} + \sqrt{x}}$
4. Evaluate: $\int_0^{\pi/2} \frac{\sin x}{\sin x - \cos x}\, dx$
5. Evaluate: $\int_1^{2} \frac{dx}{x(1 + x^4)}$
6. Evaluate: $\int_0^{1} \frac{dx}{x^2 + x + 1}$
7. Evaluate: $\int_0^{1} xe^x\, dx$
8. Evaluate: $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$
9. Evaluate: $\int_0^{\pi/2} \frac{dx}{(3\sin^2 x + 4\cos^2 x)}$
10. Evaluate: $\int_0^{1} (1 - x)x^n\, dx$
11. If $I_n = \int_0^{\pi/2} \tan^n x\, dx$, prove that
$$I_n + I_{n-2} = \frac{1}{n-1}$$
12. Prove that
$$\int_0^{\pi} \left(\frac{\sin nx}{\sin x}\right) dx = \begin{cases} 0, \text{ when } n \text{ is even} \\ \pi, \text{ when } n \text{ is odd} \end{cases}$$
13. Prove that $\int_0^{\pi/2} \left(\frac{\sin n\theta}{\sin\theta}\right)^2 d\theta = n\pi$,
where $n \in W$
14. $\int_0^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$
15. $\int_0^{1} x(1 - x)^5\, dx.$

16. $\int_0^{\pi/2} \left(\frac{\sin^2 x}{\sin^4 x + \cos^4 x}\right) dx$

17. $\int_1^2 \frac{dx}{x(x^3 + 1)}$

Evaluation of Definite Integrals by Substitution

18. Evaluate: $\int_2^4 \frac{x}{\sqrt{x-2}\,dx}$

19. Evaluate: $\int_0^1 x(1-x)^{2012}\, dx$

20. Evaluate: $\int_1^2 \frac{dx}{x(x^3+1)}$

21. Evaluate: $\int_0^{\pi/2} \frac{dx}{(2+\cos x)}$

22. Evaluate: $\int_0^{\pi/2} \frac{dx}{(2\sin^2 x + \cos^2 x)}$

23. Evaluate: $\int_0^{\pi/2} \left(\frac{\cos x}{6 - 5\sin x + \sin^2 x}\right) dx$

24. Evaluate: $\int_0^{\pi/2} \left(\frac{\sin^2 x}{\sin^4 x + \cos^4 x}\right) dx$

25. Evaluate: $\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{16 + \sin^2 x}\right) dx$

26. Evaluate: $\int_0^{\pi/2} \left(\frac{1}{\sqrt{\cos x} + \sqrt{\sin x}}\right)^4 dx$

27. Evaluate: $\int_0^2 \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}}$

28. Evaluate: $\int_0^{\pi/2} \left(\frac{\cos^2 x}{4\sin^2 x + \cos^2 x}\right) dx$

29. Evaluate: $\int_1^2 \frac{dx}{(2x+1)\sqrt{x^2+x}}$

30. Evaluate: $\int_0^4 \frac{dx}{x + \sqrt{x}}$

31. Evaluate: $\int_0^1 \frac{x\tan^{-1}}{(1+x^2)^{3/2}}\, dx$

32. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\, d\theta$

33. Evaluate: $\int_0^{\frac{\pi}{2}} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

34. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

35. Evaluate: $\int_0^{\frac{\pi}{4}} \left(\frac{\sin x + \cos x}{9 + 16\sin^2 x}\right) dx$

36. Evaluate: $\int_0^{\frac{\pi}{2}} \left(\frac{\sin^2 x}{\sin^4 x + \cos^4 x}\right) dx$

37. Evaluate: $\int_0^1 \left(\sqrt{\frac{1-x}{1+x}}\right) dx.$

Geometrical Interpretation of the Definite Integral

38. Evaluate: $\int_{-1}^4 (2x-3)\, dx$

39. Evaluate: $\int_{-2}^2 f(x)\, dx,$

where $f(x) = \min\{x - [x], -x - [-x]\}$

40. Evaluate: $\int_0^{2\pi} f(x)\, dx,$

where $f(x) = \sin^{-1}(\sin x).$

41. Evaluate: $\int_0^{2\pi} [\sin x]\, dx$

42. Evaluate: $\int_{-2}^2 f(x)\, dx,$

where $f(x) = \min\{|x-2|, |x|, |x+2|\}$

43. Evaluate: $\int_{-1}^1 f(x)\, dx,$

where $f(x) = \{|x| - 1, 1 - |x|\}$

44. Evaluate: $\int_{\pi/4}^{\pi/2} \left[\sin x + \left[\frac{2x}{\pi}\right]\right] dx$

45. Evaluate: $\int_0^1 [x d(x - [x])].$

46. Evaluate: $\int_0^2 [x^2]\, dx.$

47. Evaluate: $\int_0^{1.5} [x^2]\, dx$

48. Evaluate: $\int_0^{1.8} [x^2]\, dx$

49. Evaluate: $\int_0^2 [x^2 - x + 1]\, dx.$

50. Evaluate: $\int_1^2 [2x^2 - 3]dx$

51. Evaluate: $\int_1^2 [x^3 - 1]dx$

52. Evaluate: $\int_0^n [x]\, dx$

53. Evaluate: $\int_0^{10} [x]dx$

54. Evaluate: $\int_0^{10} [x + 2]dx$

55. Evaluate: $\int_0^2 [2x + 3]dx$

56. Evaluate: $\int_0^2 [3x - 2]dx$

57. Evaluate: $\int_0^n \{x\}\, dx$

58. Evaluate: $\int_2^0 (x[x])dx$

59. Evaluate: $\int_1^{e^6} \left[\frac{\log x}{3}\right]dx$

60. Evaluate: $\int_0^{\pi} [2\sin x]dx$

61. Evaluate: $\int_0^{\pi} [2\sin x]dx$

62. Evaluate: $\int_0^{\pi/3} [\sqrt{3}\tan x]\, dx$

63. Evaluate: $\int_1^{1.5} (\sin[x])dx$

64. Evaluate: $\int_0^{102} [\tan^{-1} x]dx$

65. Evaluate: $\int_0^{2n\pi} [\sin x + \cos x]dx$

66. Evaluate: $\int_0^{5\pi/12} [\tan x]dx$

67. Evaluate: $\int_{-2}^{1} \left[x\left\{1 + \cos\left(\frac{\pi x}{2}\right)\right\}\right]dx$

68. Evaluate: $\int_0^3 \left([x] + \left[x + \frac{1}{2}\right] + \left[x + \frac{2}{3}\right]\right)dx$

69. Evaluate: $\int_0^{10\pi} ([\sec^{-1} x] + [\tan^{-1} x])$

70. Evaluate: $\int_0^{\pi/4} [\sin x + \{\cos x + (\tan x + \sec x)\}]dx$

71. Evaluate: $\int_{\pi/4}^{\pi/2} \left[\sin x + \frac{2x}{\pi}\right]dx$

Definite Integral as the Limit of Sum

Q. Evaluate each of the following integrals from the first principle.

72. $\int_0^3 x\, dx$

73. $\int_0^2 x^2\, dx$

74. $\int_1^3 (x^2 + x + 2)dx$

75. $\int_0^1 e^{mx}\, dx$

76. $\int_0^{\pi/2} \sin x\, dx$

77. $\int_1^2 (2x + 3)dx$

78. $\int_0^2 (2x^2 + 1)dx$

79. $\int_0^2 (x^2 + 3x)dx$

80. $\int_0^3 (x^2 - 5x + 4)dx$

81. $\int_0^4 (x^3 - 4)dx$

82. $\int_0^4 (3x^3 + 2)dx$

83. $\int_0^4 (2x^3 - 3)dx$

84. $\int_0^1 e^x\, dx$

85. $\int_0^1 (e^3 x + e^x)dx$

86. $\int_a^b \cos x\, dx$

(Q.No. 77 to 86 for Board Exams Only)

Evaluation of the Limit of Sum using Newton–Leibnitz Formula

Q. Evaluate each of the following limit by N–L Formula.

87. $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n}+3+\ldots+\frac{1}{2n}\right]$

88. $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+n^2}+\frac{n}{n^2+3^2}+\ldots+\frac{n}{n^2+(n-1)^2}\right]$

89. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(2n-1)^2}\right]$

90. $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$

91. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\ldots+\frac{1}{\sqrt{3n^2}}\right)$

92. $\lim_{n\to\infty}\left(\frac{1m+2m+3m+\ldots+nm}{n^{m+1}}\right)$, $m>-1$

93. $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\ldots+\frac{1}{n}\right)$

94. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\right)$

95. $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$

96. $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

97. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\frac{1}{4n}\right)$

98. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{8n}\right)$

99. $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+\ldots+\frac{3}{5n}\right)$

100. $\lim_{n\to\infty}\left(\frac{n^2}{(n^2+1)^{3/2}}+\frac{n^2}{(n^2+2^2)^{3/2}}+\ldots+\frac{n^2}{(n^2+(n-1)^2)^{3/2}}\right)$

101. $\lim_{n\to\infty}\left(\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\ldots+\frac{1}{2n}\right)$

102. $\lim_{n\to\infty}\left(\frac{2^k+4^k+6^k+\ldots+(2n)^k}{n^k+1}\right)$, $k\neq-1$

103. $\lim_{n\to\infty}\left(\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+2\sqrt{n}}{n\sqrt{n}}\right)$

104. $\lim_{n\to\infty}\left(\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\ldots+\frac{1}{na+n}\right)$

105. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{n}\sqrt{2n}}\right)$

106. $\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{1}{n+r}\right)$

107. $\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{1}{\sqrt{n^2-r^2}}\right)$

108. $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{n^2+3rn}}\right)$

109. $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2}\right)$

110. $\lim_{n\to\infty}\sum_{r=1}^{10n}\left(\frac{1}{\sqrt{r}\sqrt{n}}\right)$

111. $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n}{(r+n)\sqrt{r^2+2nr}}\right)$

Properties of Definite Integrals

Property III

Q. 112 to 128, Evaluate each of the following integrals.

112. $\int_0^4 |x-2|\,dx$

113. $\int_0^4 (|x|+|x-2|)\,dx$

114. $\int_{-3}^3 (|x+2|+|x|+|x-2|)\,dx$

115. $\int_1^4 (|x-1|+|x-2|+|x-3|)\,dx$

116. $\int_0^3 |x^2-3x+2|\,dx$

117. $\int_{-4}^3 |x^2-4|\,dx$

118. $\int_0^\pi |\sin x|\,dx$

119. $\int_0^\pi ||\sin x|-|\cos x||\,dx$

120. $\int_{-1}^{3/2} |x\sin \pi x|\,dx$

121. $\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)\,dx$

122. $\int_{1/e}^{e} |\log x|\,dx$

123. $\int_0^{100} [\tan^{-1}x]\,dx$, where [,] = GIF

124. $\int_0^{1/2} [2\sin^{-1} x]\, dx$

125. $\int_{-2}^{1} \left[x\left[1 + \cos\left(\frac{\pi x}{2}\right)\right] + 1\right] dx$

126. $\int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1 - x^2}\right) dx$

127. $\int_{-\pi/2}^{\pi/2} (\cos|x| + \sin|x|)\, dx$

128. $\int_0^2 \left|\cos\left(\frac{\pi x}{2}\right)\right| dx$

129. Let $f(x) = \max\{x + |x|, x - [x]\}$ where [,] = GIF, find the value of $\int_{-2}^{2} f(x)\, dx$.

130. $\int_{-1}^{1} [x[1 + \sin \pi x] + 1]\, dx$, where [,] = GIF.

131. For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1 + t}\, dt$.

Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that

$f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

Here, $\ln t = \log_e t$.

132. Prove that $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})\, dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) = \frac{\pi}{4}$

133. Evaluate: $\int_0^2 |x^2 + 2x - 3|\, dx$

134. Evaluate: $\int_{1/e}^{e} |\log x|\, dx$

135. Evaluate: $\int_1^{10\pi} ([\sec^{-1} x] + [\cot^{-1} x])\, dx$

136. Evaluate: $\int_{-\pi/2}^{2\pi} ([\cot^{-1} x])\, dx$

137. Evaluate: $\int_1^2 [2x^2 - 3]\, dx$

138. Prove that $\int_0^{\pi} ||\sin x| - |\cos x||\, dx = 4(\sqrt{2} - 1)$

139. Prove that $\int_0^2 [x^2 - x + 1]\, dx = \left(\frac{5 - \sqrt{5}}{2}\right)$, where [,] = GIF

140. Prove that $\int_0^{\frac{5\pi}{12}} [\tan x]\, dx = \frac{\pi}{4}$

141. Prove that $\int_0^{102} [\tan^{-1} x]\, dx = 102 - \tan(1)$

142. Prove that $\frac{\int_0^n [x]\, dx}{\int_0^n \{x\}\, dx} = n - 1$

143. Prove that $\sum_{n=1}^{1000} \int_{n-1}^{n} e^{x-[x]}\, dx = 1000(e - 1)$

144. Prove that $\int_{1/e}^{\tan x} \frac{1}{(1 + t^2)}\, dt + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1 + t^2)} = 1.$

Property IV

145. Evaluate: $\int_0^{\pi/2} \left(\frac{\sin^n x}{\sin^n x + \cos^n x}\right) dx.$

146. Evaluate: $\int_0^{14} \left(\frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}\right) dx$

147. Evaluate: $\int_0^{\infty} \frac{dx}{(1 + x^a)(1 + x^2)}$

148. Evaluate: $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$

149. Evaluate: $\int_0^1 \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right) dx$

150. Evaluate: $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

151. Evaluate: $\int_0^{\pi} \left(\frac{dx}{1 + 2^{\tan x}}\right)$

152. Evaluate: $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

153. Evaluate: $\int_0^1 \frac{\log(1 + x)}{(1 + x^2)}\, dx$

154. Evaluate: $\int_0^1 x(1 - x)^{2013}\, dx$

155. Evaluate: $\int_0^{\pi} \left(\frac{x}{1 + \sin x}\right) dx$

156. Evaluate: $\int_0^{\pi/2} \cos(\pi \sin^2 x)\, dx$

157. Let $M = \int_0^1 \frac{e^t}{1+t}\,dt$ and $N = \int_0^1 e^t \log(1+t)dx$. Prove that $M + N = e \times \log 2$.

158. Let $a_n = \int_0^{\pi/2} \frac{\cos^2 nx}{\sin x}\,dx$. Prove that $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$ are in AP.

Q.No. 159 to 181 Evaluate each of the following integrals.

159. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

160. $\int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\,dx$

161. $\int_0^{\frac{\pi}{2}} \log(\tan x)\,dx$

162. $\int_0^{\theta} \log(1 + \tan\theta \tan x)dx,\ \theta \in \left(0, \frac{\pi}{2}\right)$

163. If $\int_0^{\pi} \log\sin x\ dx = k$, find the value of $\int_0^{\pi/4} \log(1 + \tan x)dx$.

164. $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right)dx$

165. $\int_0^{\frac{\pi}{2}} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x}\right)dx$

166. $\int_0^1 x(1-x)^n\,dx$

167. $\int_0^1 \left[\log\left(\frac{1}{x} - 1\right)\right]dx$

168. $\int_0^{\pi} \left(\frac{x\sin x}{1 + \cos^2 x}\right)dx$

169. $\int_0^{\pi} \left(\frac{x}{1 + \sin x}\right)dx.$

170. $\int_0^{\frac{\pi}{2}} \left(\frac{x\sin x \cos x}{\sin^4 x + \cos^4 x}\right)dx$

171. $\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$

172. $\int_0^1 \tan^{-1}(1 - x + x^2)\,dx$

173. $\int_0^{\infty} \frac{dx}{(1 + x^a)(1 + x^2)},\ (a > 0)$

174. $\int_0^{\pi/2} \frac{dx}{\sqrt{\tan x} - \sqrt{\cot x}}$

175. $\int_0^{\pi} \left(\frac{x}{a^2\cos^2 x + b^2\sin^2 x}\right)dx$

176. $\int_{1/2}^{2} \frac{1}{x}\,\mathrm{cosec}^{101}\left(x - \frac{1}{x}\right)dx$

177. $\int_0^{\pi} \frac{2\sin x \times \cos x}{2^{[\sin x]}}\,dx$

178. $\int_0^{\pi} x\log(\sin x)\,dx$

179. For $n > 0$, $\int_0^{2\pi} \frac{x\sin x^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

180. $\int_0^{\pi/2} \left(\frac{\cos x}{1 + \sin x + \cos x}\right)dx$

181. If $I = \int_0^{\pi/2} \frac{dx}{1 + \sin x}$, prove that

$$\int_0^{\pi} \frac{x^2\cos x}{(1 + \sin x)^2}\,dx = \pi I - \pi^2.$$

182. If $\lim_{t \to a} \frac{\int_a^t f(t)\,dt - \left(\frac{t-a}{2}\right)(f(t) + f(a))}{(t-a)^3} = 0$, prove that the maximum degree of $f(x)$ is 1.

Property V

183. Evaluate: $\int_a^b \left(\frac{f(x)}{f(x) + f(a + b - x)}\right)dx$

184. Evaluate: $\int_3^5 \left(\frac{[x^2]}{[x^2 - 16x + 64] + [x^2]}\right)dx$

185. Evaluate: $\int_{-2}^{2} \left(\frac{dx}{1 + 3^x}\right)$

186. Evaluate: $\int_{5\pi/6}^{\pi/6} \left(\frac{dx}{1 + e^{\tan x}}\right)$

187. Evaluate: $\int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1 + a^x}\right)dx$

188. If $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4}\cos^{-1}\left(\frac{2x}{1 + x^2}\right)\right)dx$

$$= k \int_0^{1\sqrt{3}} \left(\frac{x^4}{1 - x^4}\right) dx, \text{ find } k$$

189. Let f be a positive function and

$$I_1 = \int_{1-k}^{k} \cdot f(x(1 - x))\, dx,$$

$$I_2 = \int_{1-k}^{k} \cdot f(x(1 - x))\, dx, \text{ where } 2k - 1 > 0,$$

find $\frac{I_1}{I_2}$.

190. Evaluate: $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \left(\frac{x \sin(x^2)}{\sin x^2 + \sin(\ln(6 - x^2))}\right) dx$

191. Evaluate: If $I_n = \int_{-\pi}^{\pi} \left(\frac{\sin nx}{(1 + \pi^x)\sin x}\right) dx,\ n \in W$,

find the value of

(i) $\sum_{m=1}^{10} I_{2m+1}$ (ii) $\sum_{m=1}^{10} I_{2m}$

192. Evaluate: $\int_0^1 \left(\frac{x^4(1 - x)^4}{1 + x^2}\right) dx$

Q. Evaluate each of the following definite integrals.

193. If $f(a + b - x) = f(x)$, prove that

$$\int_a^b x\, f(x) = \frac{a + b}{2} \int_a^b f(x)\, dx$$

194. Prove that $\int_4^{10} \frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}\, dx = 3$

195. Prove that $\int_{\pi/4}^{3\pi/4} \left(\frac{dx}{1 + \cos x}\right) = 2.$

196. Prove that $\int_{-\pi/2}^{\pi/2} \left(\frac{\cos x}{1 + e^x}\right) dx = 1$

197. Prove that $\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1} = \frac{\pi}{2}$

198. Prove that $\int_{50}^{100} \frac{\ln x}{\ln x + \ln(150 - x)}\, dx = 25$

199. For any t in R and f be a continuous function.

Let $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x\, f(x(2 - x))\, dx$

and $I_2 = \int_{\sin^2 t}^{1 + \cos^2 t} f(x(2 - x))\, dx,$

prove that $\frac{I_1}{I_2} = 1.$

200. If $f(x) = \frac{e^x}{1 + e^x}$ and let

$$I_1 = \int_{f(-a)}^{f(a)} x\, g\{x(1 - x)\} dx \text{ and}$$

$$I_2 = \int_{f(-a)}^{f(a)} g\{x(1 - x)\} dx,$$

prove that $\frac{I_1}{I_2} = 2$

201. Prove that $\int_0^{4\pi} \sin^{2013} x\, dx = 0$

202. Prove that $\int_0^{2\pi} \cos^{2013} x\, dx = 0$

Property VI

203. Evaluate: $\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx$

204. Evaluate: $\int_{-\pi/4}^{\pi/4} x^3 \tan^4 x\, dx$

205. Evaluate: $\int_{-2}^{2} \log\left(\frac{2 - \sin x}{2 + \sin x}\right) dx$

206. Evaluate: $\int_0^2 x(x - 1)(x - 2)\, dx$

207. Evaluate:

$$\int_{2010}^{2014} [(x-2010)(x-2011)(x-2012)(x-2013)(x-2014)]\, dx$$

208. Find the value of

$$\int_{-\pi/4}^{\pi/4} \left(\frac{x^{2013} - 3x^{2011} + 5x^{2009} - 7x^{2007} + 1007}{\cos^2 x}\right) dx$$

209. Let $f(x) = \frac{\sin^{2013} x}{x^{2014} - x^{2012} + 1}$.

Find $\int_{-2010}^{2010} f(x)\, dx$

210. Find the value of

$$\int_{-10}^{10} \left(\tan^{2013} x + \left(\frac{2013\sqrt{x}}{1 + x^{2014}}\right) + 1007\right) dx$$

211. Let $f(x) = x^3\left(\frac{x^2}{e^{x^2} - 1} + \frac{x^2}{2} + 1\right)$

Find the value of $\int_{\sqrt{\ln(1/2)}}^{\sqrt{\ln(2)}} f(x) dx$

212. Find the value of the integral

$$\int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x + 1)\cos(x + 1)]\, dx$$

213. For any real number x, let $[x]$ denotes the largest integer less than or equal to x. Let f be a real function defined on [–10, 10] by

$$f(x) = \begin{cases} x-[x] & : \text{ if } [x] \text{ is odd} \\ 1+[x]-x & : \text{ if } [x] \text{ is even} \end{cases}.$$

Find the value of

$$\frac{\pi^2}{10} \times \int_{10}^{10} f(x)\cos(\pi x)\,dx$$

214. Find the value of

$$\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx.$$

215. Find the value of $\int_{-\pi}^{\pi} \left(\frac{2x(1+\sin x)}{1+\cos^2 x}\right) dx$

216. Find the value of

$$\int_{-\pi/3}^{\pi/3} \left(\frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right) dx$$

217. Find the value of

$$\int_{-1}^{1} \left(\tan^{-1}\left(\frac{x}{1+x^2}\right) + \tan^{-1}\left(\frac{1+x^2}{x}\right)\right) dx$$

Q. Evaluate each of the following integrals.

218. $\int_{-\pi/2}^{\pi/2} [\{f(x) + f(-x)\}] \times [\{g(x) - g(-x)\}]\,dx$

219. $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$

220. $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) \sin^2 x\,dx$

221. $\int_{\log(1/3)}^{\log 3} \tan\left(\frac{e^x - 1}{e^x + 1}\right) dx$

222. $\int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1+a^x}\right) dx$

223. $\int_{-1/2}^{1/2} \sqrt{\left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]}\,dx.$

224. $\int_{-\pi/2}^{\pi/2} \sin\left(\log\left(x + \sqrt{x^2+1}\right)\right) dx$

225. $\int_{-1/2}^{1/2} \left[[x] + \log\left(\frac{1-x}{1+x}\right)\right] dx$

226. $\int_{-1/2}^{1/2} \left(\cos x \times \log\left(\frac{1-x}{1+x}\right)\right) dx$

227. $\int_{-1}^{1} \sin^{10} x \cos^{11} x\,dx$

228. $\int_{-1}^{1} \sin^{2013} x\,dx$

229. $\int_{2}^{4} (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$

230. $\int_{2009}^{2011} [(x-2007)(x-2008)(x-2009)(x-2010)$
$(x-2011)(x-2012)(x-2013)]\,dx$

231. If $n \in N$, prove that $\int_{-n}^{n} (-1)^{[x]}\,dx = 0$

232. Prove that

$$\int_{-1}^{1} \frac{\sin x - x^2}{3 - |x|}\,dx = 2\int_0^1 \frac{-x^2}{3-[x]}\,dx$$

233. $\int_{-2}^{2} [\sqrt{2012 + 2013x + 2014x^2}$
$- \sqrt{2012 - 2013x + 2014x^2}]\,dx$

234. If $f(x) = \begin{cases} e^{\cos x}\sin x & : -2 \le x \le 2 \\ 2 & : \text{otherwise} \end{cases}$,

prove that $\int_{-2}^{3} f(x)\,dx = 2.$

235. Prove that $\int_{-1/2}^{1/2} \left|x\cos\left(\frac{\pi x}{2}\right)\right| dx$

$$= \frac{1}{\pi^2}\left(\pi\sqrt{2} + 4\sqrt{2} - 8\right)$$

236. Prove that

$$\int_0^{\pi} \left(\frac{x\sin 2x \cdot \sin\left(\frac{\pi}{2} \times \cos x\right)}{2x - \pi}\right) dx = \frac{8}{\pi^2}$$

Property VII

237. Evaluate: $\int_0^{\pi} \sin x\,dx$

238. Evaluate: $\int_0^{\pi} \cos x\,dx$

239. $\int_0^{\pi} \cos^{2013} x\,dx$

240. Evaluate: $\int_0^{\pi} (\sin^3 x + \cos^5 x + \tan^7 x)\,dx$

241. Evaluate: $\int_0^{\pi/2} \log \sin x\, dx$

242. Evaluate: $\int_0^{\pi} x \log(\sin x)\, dx$

243. Evaluate: $\int_0^{\pi/2} \sin 2x \log(\tan x)\, dx$

244. Evaluate: $\int_0^{\pi} \left(\frac{x\, dx}{1 + \cos\alpha \cdot \sin x}\right)$

245. Evaluate: $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$

246. Evaluate: $\int_0^{\pi} \frac{dx}{3\sin^2 x + 4\cos^2 x}$

247. Evaluate: $\int_0^{\pi} \left(\frac{x \sin x}{1 + \cos^2 x}\right) dx$

248. Evaluate: $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2}$

249. Evaluate: If $I_1 = \int_0^1 x^{1006}(1 - x)^{1006}\, dx$

and $I_2 = \int_0^1 x^{1006}(1 - x^{2014})^{1006}\, dx$

Find the value of $2^{2014} \times \left(\frac{I_2}{I_1}\right)$

Q. Evaluate each of the following integrals.

250. $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$

251. $\int_0^1 \frac{\log x}{\sqrt{1 - x^2}}\, dx$

252. $\int_0^{\pi} \log(1 - \cos x)\, dx$

253. $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2}$

254. $\int_0^{\pi} \frac{x}{1 + \cos^2 x}\, dx$

Property VIII

255. Evaluate: $\int_0^{400\pi} \sqrt{1 - \cos^2 x}\, dx$

256. Evaluate: $\int_0^{1000} e^{x-[x]}\, dx$

257. Evaluate: $\int_0^{2000\pi} \left(\frac{dx}{1 + 5^{\sin x}}\right)$

258. Evaluate: $\int_0^{1007} \{2x\}\, dx$.

259. Evaluate: $\int_0^{\pi} |\sin x + \cos x|\, dx$

260. Evaluate: $\int_0^{100\pi} (|\sin x| + |\cos x|)\, dx$

261. Evaluate: $\int_0^{100\pi} (|\sin x| - |\cos x|)\, dx$

262. Evaluate: $\int_{-503\pi}^{504\pi} |\cos x|\, dx$

263. Evaluate: $\int_0^{50\pi/3} \sqrt{\left(\frac{1 - \cos 2x}{2}\right)}\, dx$

264. Given $\int_0^1 \frac{\sin t}{1 + t}\, dt = \alpha$,

prove that

$$\int_{4\pi-2}^{4\pi} \left(\frac{\sin\left(\frac{t}{2}\right)}{4\pi + 2 - t}\right) dt = -\alpha$$

265. If $\int_0^1 \frac{e^t}{(t + 1)}\, dt = a$,

prove that

$$\int_{b-1}^{b} \left(\frac{e^{-t}}{(t - b - 1)}\right) dt = -ae^{-b}.$$

266. If $F(x)$ be a periodic function such that

$F(x) + F\left(\frac{1}{x} + 2\right) = 10$,

find the value of $\int_0^{201} F(x)\, dx$

267. If $\int_0^{50\pi} (\sin^4 x + \cos^4 x)\, dx = k \int_0^{\pi/2} \left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) dx$,

find k.

268. If $g(x) = \int_0^x \cos^4 t\, dt$, find the value of $g(x + \pi)$.

Q. Evaluate each of the following integrals

269. $\int_0^{2014\pi} \sqrt{1 - \cos 2x}\, dx$

270. $\int_0^{2012} e^{x-[x]}\, dx$

271. $\int_0^{4\pi} |\cos x|\, dx$

272. $\int_0^{10\pi} |\sin x|\,dx$

273. $\int_0^{[x]} dx$, where [,] = GIF

274. $\int_0^{4\pi} [\sin x + \cos x]\,dx$

275. $\int_1^{e^{37}} \frac{\pi\sin(\pi\log x)}{x}\,dx$

276. $\int_0^{\frac{32\pi}{3}} \sqrt{1+\cos 2x}\,dx$

278. $\int_0^{10\pi/3} |\sin x|\,dx$

280. $\int_{-\pi}^{199\pi} \sqrt{\left(\frac{1-\cos 2x}{2}\right)}\,dx$

281. $\int_{-20\pi}^{20\pi} |\cos x|\,dx$

282. $\int_0^{100} \sin(x-[x])\,dx$

283. $\int_0^{2014\pi} (|\sin x| + |\cos x|)\,dx$

284. $\int_0^{2014\pi} (|\sin x| - |\cos x|)\,dx$

285. Prove that $\int_0^{n\pi+V} |\sin x|\,dx = (2n+1) - \cos V$

286. Prove that $\int_0^{2n\pi} [\sin x + \cos x]\,dx = -n\pi$,.

287. Prove that $\int_0^{2\pi} \sin^2(100x)\,dx = \pi$, where [,] = GIF

288. Prove that $\int_{10\pi+\pi/6}^{10\pi+\pi/3} (\sin x + \cos x)dx = \sqrt{3} - 1$

289. Prove that $\int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\,dx = \frac{5\pi^2}{32}$,

where [,] = GIF

290. Let $F(x)$ be a non-negative continuous function defined on R such that $F(x) + F\left(x+\frac{1}{2}\right) = 3$, prove that $\int_0^{1500} F(x)\,dx = 2250$.

291. If for every integer n,

$\int_0^{n+1} f(x)\,dx = n^2$,

prove that $\int_{-2}^{4} f(x)\,dx = 19$

292. If $\int_0^{\pi/2} \sin^m x\cos^m x\,dx = k\int_0^{\pi/2} \sin^m x\,dx$,

prove that $k = 2^{-m}$.

293. If $M = \int_0^{\pi} f(\cos^2 x)\,dx$ and $N = \int_0^{3\pi} f(\cos^2 x)dx$, then prove that $N = 3M$.

Property IX

294. Let $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$, $x > 0$.

If $\int_1^4 \frac{2e^{\sin^2 x}}{x}\,dx = F(k) - F(1)$, find k.

295. Find the equation of the tangent to the curve

$$y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^2}} \text{ at } x = 1$$

296. Find the interval where the function

$f(x) = \int_0^x e^t(t-1)(t-2)\,dt$ is strictly increasing.

297. Find the point of maxima and minima for the function $f(x) = \int_0^x t(t+1)(t-2)dt$.

298. Find the point of inflection for the curve

$$f(x) = e^t(t-1)^3(t-2)^2\,dt$$

299. Prove that the equation

$y = \int_{1/10}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/10}^{\sin^2 x} \cos^{-1}\sqrt{t}\,dt$

is a striaght line parallel to x-axis, where $0 \le x \le \frac{\pi}{2}$

300. If $F(x) = \int_x^{2x} \sqrt{5-3\sin^2 t}\,dt + \int_0^y \sin t\,dt$, find $F'(x)$.

301. Let $f(x) = \int_1^x \sqrt{2-t^2}\,dt$. Find the real roots of the equation $x^2 - f'(x) = 0$.

302. If $f(x) = \int_{1/x^2}^{x^2} \cos\sqrt{t}\,dt$, the value of $f'(1)$ is ...

303. If $\int_{\sin x}^{1} t^2 f(t)\,dt = 1 - \sin x,\ x \in \left(0, \frac{\pi}{2}\right)$, find the value of $f\left(\frac{1}{\sqrt{3}}\right)$

304. Let $f:(0, \infty) \to R$ and $F(x) = \int_0^x f(t)\,dt$

If $F(x^2) = x^2(1 + x)$, find the value of $f(4)$.

305. If $\int_0^{t^2} x f(x)\,dx = \frac{2}{5}t^5$, find the value of $f\left(\frac{4}{25}\right)$.

306. Prove that the value of

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt \text{ is } \frac{\pi}{4}.$$

307. If $\int_0^x f(t)\,dt = x + \int_x^1 t f(t)\,dt$, prove that the value of $f(1)$ is 1/2.

308. Prove that the value of

$$\lim_{x \to 0} \frac{1}{x}\left[\int_y^a e^{\sin^2 t}\,dt - \int_{x+y}^a e^{\sin^2 t}\,dt\right] = e^{\sin^2 y}.$$

Advance Properties of Definite Integral

309. (i) Prove that $F(x) = \int_0^x \log\left(\frac{1-t}{1+t}\right)dt$ is an even function

(ii) Prove that $F(x) = \int_0^x \left(\frac{t}{e^t - 1} + \frac{t}{2} + 1\right)dt$ is an odd function.

(iii) Prove that the function

$f(t) = \int_0^t \sin\left(\log\left(\sqrt{x^2+1} + x\right)\right)dx$ is an even function.

(iv) Prove that the function

$f(x) = \int_0^x \left(\cos\left(\sin\left(\log\left(\sqrt{t^2+1} + t\right)\right)\right)\right)dt$ is an odd function.

(v) Prove that the function $g(x) = \int_0^x f(t)\,dt$ is an even function, where $f(x)$ satisfies the relation $f(x + y) = f(x) + f(y)$.

310. Evaluate: $\int_0^\pi \sin^{100}x \cos^{99}x\,dx$

311. If $f(x) = \int_0^x \frac{e^t}{t}dt,\ x > 0$, prove that

$$\int_1^x \frac{e^t}{(t+\alpha)}\,dt = e^{-a}\,[f(x + a) - f(1 + a)].$$

312. Prove that $\int_{-4}^{-5} e^{(x+5)^2}dx + \int_{1/3}^{2/3} e^{9(x-2/3)^2}dx = 0.$

313. Prove that $\int_0^\pi \sin^{2014}x \cos^{2013}x\,dx = 0.$

314. Evaluate: $\int_0^1 |\cos \pi x|\,dx$

315. Prove that $\int_0^1 |\sin 2\pi x|\,dx = \frac{2}{\pi}$

316. Prove that

$$\int_{-3/4}^{3/2} |x\cos \pi x|\,dx = \frac{1}{\pi^2}\left\{\frac{7\pi}{2} - \frac{3\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right\}$$

317. Evaluate: $\int_{-\pi/12}^{-5\pi/12} \left(\frac{\sin^{2014}x}{\sin^{2014}x + \cos^{2014}x}\right)dx$

318. Prove that $\int_{-\pi/3}^{-\pi/6} \frac{\sin^{2n}x}{\cos^{2n}x + \sin^{2n}x}\,dx = \frac{\pi}{12}.$

319. Prove that $\int_{-\pi/6}^{-\pi/3} \left(\frac{\sin^3 x + \cos^3 x}{\sin^{10}x + \cos^{10}x\,dx}\right) = 0$

320. Given $\int_0^1 \frac{\sin t}{1+t}dt = \alpha$,

prove that

$$\int_{4\pi-2}^{4\pi} \frac{\sin(t/2)}{4\pi + 2 - t\ dt} = -\alpha$$

321. Find the value of $\int_0^\pi \left(\frac{|\sin(2x)|}{|\sin x| + |\cos x|}\right)dx$

322. If $\int_a^b \left(\frac{\sin(x-a) - \cos(x-a)}{\sin(b-x) - \cos(b-x)}\right)dx$

$$= m\int_a^b \left(\frac{\sin(b-x) - \cos(b-x)}{\sin(x-a) - \cos(x-a)}\right)dx,$$

find the value of m.

323. If $\int_0^1 \frac{e^t}{(t+1)}\,dt = a$,

prove that

$$\int_{b-1}^{b} \frac{e^{-t}}{(t-b-1)}\,dt = -ae^{-b}.$$

324. Find the value of

$$\int_0^\pi \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi}\,dx = \frac{8}{\pi^2}$$

325. Let a, b and c be three non-zero real numbers such that $\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx$. Prove that the equation has at least one root in (1, 2).

326. Estimate the absolute value of the integral $\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx$.

327. Estimate the absolute value of the integral $\int_5^{10} \frac{\cos x}{1+x^4}\,dx$.

328. Prove that $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$.

329. Prove that the value of the integral $\int_0^1 e^{x^2}dx$ lies in between [1, e]

330. Prove that the value of the integral $\int_0^1 \frac{x}{x^3+16}\,dx$ lies in between [0, 1/17]

331. Prove that $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

332. Prove that $\frac{\pi}{6} < \int_0^1 \left(\frac{dx}{\sqrt{4-x^2-x^3}}\right) < \frac{\pi}{4\sqrt{2}}$

333. Prove that $1 \le \int_0^1 \sqrt{1+x^3}\,dx \le \frac{5}{4}$

334. Prove that $\frac{1}{17} \le \int_1^2 \frac{dx}{1+x^4} \le \frac{7}{24}$

335. Evaluate $\int_0^\infty \left(\frac{\tan^{-1}ax - \tan^{-1}x}{x}\right)dx$

336. Evaluate $\int_0^{\pi/2} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}$, $(|a| < 1)$

337. Prove that $I(b) = \int_0^1 \frac{x^b - 1}{\log x}\,dx = \log(b+1)$

338. Let $f(x)$ be a continuous function for all x such that $(f(x))^2 = \int_0^x f(t)\cdot\frac{2\sec^2 t}{4+\tan t}\,dt$ and $f(0) = 0$, prove that $f\left(\frac{\pi}{4}\right) = \log\left(\frac{5}{4}\right)$.

339. Prove that

$$f(x) = \int_0^{\pi/2} \frac{\log(1+x\sin^2\theta)}{\sin^2\theta}\,dx,\ x \ge 0 = \pi(\sqrt{1+x}-1)$$

340. Prove that

$$\int_0^1 \left(\frac{x^{\cos\alpha}-1}{\log_e x}\right)dx = \log|1+\cos\alpha|,$$

$[\alpha \ne (2n+1)\pi]$, where α is a parameter

341. Prove that

$$\int_0^\pi \ln(1+b\cos x)\,dx = \pi\ln\left(\frac{1+\sqrt{1-b^2}}{2}\right).$$

342. Find the maximum value of

$$\int_0^2 \sqrt{(1+x)(1+x^4)}\,dx.$$

343. Prove that the maximum value of

$$\int_0^1 \sqrt{(1+x)(1+x^3)}\,dx \text{ is } \sqrt{\frac{15}{8}}.$$

344. Prove that the value of

$$\int_a^{a+\frac{\pi}{2}} (\sin^4 x + \cos^4 x)\,dx \text{ is independent of } a.$$

345. Find the mean value of $f(x) = \sin^2 x$ over $[0, \pi]$.

Q. Find the mean value.

346. $f(x) = \frac{1}{e^x+1}$ on [0, 2]

347. $f(x) = \sqrt[3]{x}$ on [0, 1]

348. $f(x)\sin^3 x$ on $[0, 2\pi]$

349. $f(x) = \frac{1}{x+x^2}$ on [0, 1]

350. $f(x) = \frac{e^x}{1+e^x}$ on [0, 1]

Gamma and Beta Functions

351. Evaluate: $\int_0^1 \log x\,dx$

352. Evaluate: $\int_0^\infty e^{-x}x^6\,dx$

353. Evaluate: $\int_0^1 \left(\log\left(\frac{1}{x}\right)\right)^{10dx}$

Q. Evaluate each of the following integrals.

354. $\int_0^\infty e^{-x}x^3\,dx$

355. $\int_0^1 \left(\log\left(\frac{1}{x}\right)\right)^{n-1}dx$

356. Evaluate: $\int_0^1 x^4\sqrt{1-x^2}\,dx$

357. Evaluate: $\int_0^{\pi/2} \sin^8 x\cdot\cos^4 x\,dx$

Q.No. 358 to 362, Evaluate each of the following integrals.

358. $\int_0^1 x^6\sqrt{1-x^2}\,dx$

359. $\int_0^{\pi/2} \sin^4 x\cos^6 x\,dx$

360. $\int_0^1 x^{10}\sqrt{1-x^2}\,dx$

361. $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\,dx$

362. $\int_0^{\infty} e^{-a2x2}dx$

363. Evaluate: $\int_0^{\pi/2} \sin^7 x\cdot\cos^5 x\,dx$

364. Evaluate: $\int_0^{\pi/2} \sin^{10} x\,dx$

365. Evaluate: $\int_0^{\pi/2} \cos^7 x\,dx$

366. Evaluate: $\int_0^{\pi/4} 8\cos^4 x\cdot\sin^4 x\,dx$

367. Evaluate: $\int_0^{\pi/2} \sin^6 x\cdot\cos^5 x\,dx$

368. Evaluate: $\int_0^{\pi/2} \sin^5 x\cdot\cos^3 x\,dx$

Q.No. 369 to 373, Evaluate each of the following integrals.

369. $\int_0^{\pi/2} \sin^6 x\cdot\cos^4 x\,dx$

370. $\int_0^{\pi/2} \sin^7 x\cdot\cos x\,dx$

371. $\int_0^{\pi/2} \sin^7 x\cdot\cos^5 x\,dx$

372. $\int_0^{\pi/2} \sin^{11} x\,dx$

373. $\int_0^{\pi/2} \cos^8 x\,dx$

Level II

(Mixed Problems)

1. $\int_0^{\pi} \log(1+\cos x)\,dx =$
   (a) $\pi\log\frac{1}{2}$ (b) $\frac{\pi}{2}\log 2$
   (c) $-\pi\log 2$ (d) none

2. If $\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx = 200k$, then k is
   (a) $2\sqrt{2}$ (b) π
   (c) $\sqrt{3}$ (d) $\sqrt{2}$

3. $\int_0^{4\pi} |\cos x|\,dx =$
   (a) 4 (b) 8
   (c) 0 (d) none

4. $\int_0^{\frac{32\pi}{3}} (\sqrt{1+\cos 2x})\,dx =$
   (a) $\left(20\sqrt{2}-\sqrt{\frac{3}{2}}\right)$ (b) $\left(22\sqrt{2}+\sqrt{\frac{3}{2}}\right)$
   (c) $\left(22\sqrt{2}-\sqrt{\frac{3}{2}}\right)$ (d) none

5. If for every integer n,
   $\int_n^{n+1} f(x)dx = n^2,$
   the value of $\int_{-2}^{4} f(x)dx$ is
   (a) 16 (b) 14
   (c) 19 (d) none

6. If $\int_1^4 f(x)dx = 4$ and $\int_2^4 (3-f(x))dx = 7$, the value of $\int_2^{-1} [3-f(x)]dx$ is
   (a) 2 (b) -3
   (c) -5 (d) none

7. The value of $\int_0^{100} e^{x-[x]}$ is
   (a) $100e$ (b) $100\,(e-1)$
   (c) $100\,(e+1)$ (d) none

8. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} [f(x)]\,dx$ is
   (a) $\sin\theta + \operatorname{cosec}\theta$ (b) $\sin^2\theta$
   (c) $\operatorname{cosec}^2\theta$ (d) 0.

9. If $2f(x) + 3f\left(\frac{1}{x}\right) + \frac{1}{x} - 2$, $x \neq 0$, then $\int_1^2 f(x)dx$ is

(a) $-\frac{2}{5}\log 2 + \frac{1}{2}$ (b) $-\frac{2}{5}\log 2 - \frac{1}{2}$

(c) $\frac{2}{5}\log 2 + \frac{1}{2}$ (d) none

10. The value of the integral $\int_0^\infty \left(\frac{x\log x}{(1+x^2)^2}\right)dx$ is

(a) 1 (b) 0
(c) 2 (d) none

11. $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx =$

(a) –1 (b) 1
(c) 0 (d) none

12. The value of $\int_1^{e^{37}} \left(\frac{\pi\sin(\pi\log x)}{x}\right)dx$ is

(a) 1 (b) –1
(c) 2 (d) none

13. Let $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$, $x > 0$.

If $\int_1^4 \left(\frac{2e^{\sin x^2}}{x}\right)dx = F(k) - F(1)$, the value of K is

(a) 4 (b) 16
(c) 2 (d) none

14. Let $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$, $x > 0$.

If $\int_1^4 \left(\frac{3x^2}{x}e^{\sin x^3}\right)dx = F(k) - F(1)$, then k is

(a) 15 (b) 16
(c) 63 (d) 64

15. $\frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right)dx =$

(a) $\frac{1}{c}\int_a^b f(x)dx$ (b) $\int_a^b f(x)dx$

(c) $c\int_a^b f(x)dx$ (d) $\int_{ac^2}^{bc^2} f(x)dx$

16. If $g(x) = \int_0^\pi \cos^4 t\,dt$, then $g(x+\pi)$ is

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$
(c) $g(x)\cdot g(\pi)$ (d) $g(x)/g(\pi)$.

17. $\int_0^{\pi/2} \left(\frac{\sin x}{\sin x + \cos x}\right)dx =$

(a) $\frac{\pi}{2}$ (b) π

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

18. $\int_0^{\pi/2} \left(\frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x} + \sqrt{\cos^3 x}}\right)dx =$

(a) $\frac{\pi}{4}$ (b) π
(c) 0 (d) $\frac{\pi}{2}$

19. $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$

(a) 0 (b) 1
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

20. The value of $\int_0^{\pi/2} \left(\frac{\varphi(x)}{\varphi(x) + \varphi\left(\frac{\pi}{2} - x\right)}\right)dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) none

21. $\int_0^{\pi/4} \log(1+\tan x)dx =$

(a) $\frac{\pi}{2}\log 8$ (b) $\frac{\pi}{8}\log 2$

(c) $\frac{\pi}{4}\log 2$ (d) none

22. $\int_0^\infty \left(\log\left(x + \frac{1}{x}\right) \times \frac{dx}{1+x^2}\right) =$

(a) $\pi\log\frac{1}{2}$ (b) $\pi\log 2$

(c) $\frac{\pi}{2}\log 2$ (d) none

23. If $\int_0^\pi xf(\sin x)dx = k\int_0^{\pi/2} f(\sin x)dx$, the value of k is

(a) 2 (b) 1
(c) π (d) 0

24. For $n > 0$, $\int_0^{2\pi} \left(\frac{x\sin^{2n}\pi}{\sin^{2n}x + \cos^{2n}x}\right)dx =$

(a) π (b) π^2
(c) $\frac{\pi}{2}$ (d) 2π

25. The value of the integral $\int_0^1 x(1-x)^n dx$ is

(a) $\frac{1}{n+1} + \frac{1}{n+2}$ (b) $\frac{1}{(n+1)(n+2)}$

(c) $\frac{1}{n+2} - \frac{1}{n+1}$ (d) $2\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

26. If $x \int_0^1 x^m(1-x)^n dx = R \int_0^1 x^n(1-x)^m dx$, then
(a) $R = 1$ (b) $R = -1$
(c) $R = \frac{1}{2}$ (d) none

27. If $I = \int_0^1 \left(\frac{e^t}{1+t}\right) dt$, then $P = \int_0^1 e^t \log(1+t)\,dt$, where P is
(a) I (b) $2I$
(c) $e \log 2 - I$ (d) none

28. $\int_0^1 \tan^{-1}(1 - x + x^2)\,dx =$
(a) $\log 2$ (b) $\log\left(\frac{1}{2}\right)$
(c) $\pi \log 2$ (d) $\frac{\pi}{2}\log\left(\frac{1}{2}\right)$

29. If $a_n = \int_0^{\pi/2} \left(\frac{\cos^2 nx}{\sin x}\right) dx$, then $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$ are in
(a) AP (b) GP
(c) HP (d) none

30. If $f(x)$ and $g(x)$ be continuous functions satisfying $f(x) = f(a - x)$ and $g(x) + g(a - x) = 2$, then $\int_0^a f(x) g(x)\,dx$ is
(a) $\int_0^a g(x)\,dx$ (b) $\int_0^a f(x)\,dx$
(c) 0 (d) none

31. If $\lim_{t \to x} \left(\frac{\int_a^t f(t)\,dt - \left(\frac{t-1}{2}\right)(f(t) + f(a))}{(t-a)^3} \right) = 0$,
the maximum degree of $f(x)$ is
(a) 4 (b) 3
(c) 2 (d) 1

32. If $f(y) = e^y$, $g(y) = y$, $y > 0$ and $F(t) = \int_0^t f(t - y) g(y)\,dy$, then $F(t)$ is
(a) $1 - e^{-t}(1 + t)$ (b) $e^t - (1 + t)$
(c) te^t (d) te^{-t}

33. The value of $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2 - \sin\theta}{2 + \sin\theta}\right) d\theta$ is
(a) 0 (b) 1
(c) 2 (d) none

34. The value of $\int_{-1/2}^{1/2} \left([x] + \log\left(\frac{1+x}{1-x}\right)\right)$ is
(a) $-\frac{1}{2}$ (b) 0
(c) 1 (d) $2 \log\left(\frac{1}{2}\right)$

35. $f: R \to R$, $g: R \to R$ be continuous functions. The value of the integral
$\int_{-\pi/2}^{\pi/2} (f(x) + f(-x))(g(x) - g(-x))\,dx$ is
(a) π (b) 1
(c) -1 (d) 0

36. $\int_{-\pi/2}^{\pi/2} \left(\frac{2x(1 + \sin x)}{1 + \cos x}\right) dx =$
(a) π (b) π^2
(c) 0 (d) none

37. $\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi\pi)]\,dx =$
(a) $\frac{\pi^4}{32} + \frac{\pi}{2}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4} - 1$ (d) $\frac{\pi^4}{32}$.

38. The value of the integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cos x \log\left(\frac{1+x}{1-x}\right)\right) dx$ is
(a) 0 (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) none

39. The value of
$\int_{-2}^{0} [(x^3 + 3x^2 + 3x + 3) + (x + 1)\cos(x + 1)]\,dx$ is
(a) 2 (b) 4
(c) 0 (d) 8

40. The value of the integral $\int_{-1}^{1} \log\left(x + \sqrt{x^2 + 1}\right) dx$ is
(a) 0 (b) log 2
(c) $\log\left(\frac{1}{2}\right)$ (d) none

41. The function $f(x) = \int_a^x \log\left(t + \sqrt{1 + t^2}\right) dt$ is
(a) an even function (b) an odd functio
(c) a periodic function (d) none

42. The function $f(x) = \int_0^x \log\left(\frac{1-t}{1+t}\right) dt$ is
(a) even (b) odd
(c) periodic (d) none

43. $\int_{-1}^{1}\left(\frac{\sin x - x^2}{3 - |x|}\right)dx =$

(a) 0 (b) $\int_{1}^{0}\left(\frac{\sin x}{3 - |x|}\right)dx$

(c) $2\int_{0}^{1}\left(\frac{-x^2}{3 - |x|}\right)dx$ (d) $2\int_{0}^{1}\left(\frac{\sin x - x^2}{3 - |x|}\right)dx$

44. If $f(x) + f(y) = f(x + y)$ and $\int_{0}^{3} f(x)dx = \lambda$, then $\int_{-3}^{3} f(x)dx =$

(a) -2λ (b) 2λ

(c) 0 (d) $\frac{\lambda}{2}$

45. If $f(x) = \begin{cases} e^{\cos x} \times \sin x & : |x| \le 2 \\ 2 & : \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)dx =$

(a) 0 (b) 1

(c) 2 (d) 3

46. $\int_{3}^{6}\left(\frac{\sqrt{x}}{\sqrt{9 - x} + \sqrt{x}}\right)dx =$

(a) $\frac{3}{2}$ (b) 2

(c) 1 (d) $\frac{1}{2}$

47. If $f(a + b - x) = f(x)$, then $\int_{a}^{b} xf(x)dx$ is

(a) $\left(\frac{a + b}{2}\right)\int_{a}^{b} f(b - x)dx$ (b) $\left(\frac{a + b}{2}\right)\int_{a}^{b} f(x)dx$

(c) $\left(\frac{b - a}{2}\right)\int_{a}^{b} f(x)dx$ (d) none

48. Let f be a $+ve$ function If $I_1 = \int_{1-k}^{k} xf(x(1 - x))dx$ and $I_2 = \int_{1-k}^{k} f[x(1 - x)]dx$, where $2k - 1 > 0$, then I_1/I_2 is

(a) 2 (b) k

(c) $\frac{1}{2}$ (d) 1

49. Let $f(x) = \left(\frac{e^x}{1 + e^x}\right)$ If $I_1 = \int_{f(-a)}^{f(a)} x[g\{x(1 - x)\}]dx$ and $I_2 = \int_{f(-a)}^{f(a)} g[x(1 - x)]dx$, then the value of $\frac{I_1}{I_2}$ is

(a) 2 (b) -3

(c) -1 (d) 1

50. The value of $\int_{4}^{10}\left(\frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}\right)dx$ is

(a) 0 (b) 1

(c) 3 (d) none

51. $\int_{-\pi/2}^{\pi/2}\left(\frac{\cos x}{1 + e^x}\right)dx =$

(a) 1 (b) 2

(c) log2 (d) none

52. The value of $\int_{-\pi/2}^{\pi/2}\frac{dx}{e^{\sin x} + 1}$ is

(a) 0 (b) 1

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

53. $\int_{0}^{\pi}\frac{dx}{1 + 4\cos x} =$

(a) 0 (b) $\frac{\pi}{2}$

(c) π (d) 2π

54. The value of $\int_{-\pi}^{\pi}\left(\frac{\cos^2 x}{1 + a^x}\right)dx$, $a > 0$ is

(a) π (b) $a\pi$

(c) $\frac{\pi}{2}$ (d) 2π

55. If $F(x) = \frac{1}{x^2}\int_{4}^{x}(4t^2 - 2F'(t))$, then $F'(4)$ is

(a) 32 (b) $\frac{32}{3}$

(c) $\frac{32}{9}$ (d) none

56. The value of $\lim_{x\to 0}\left(\frac{\int_{0}^{x^2}\sec^2 t\,dt}{x\sin x}\right)$ is

(a) 1 (b) 2

(c) 3 (d) none

57. $\lim_{x\to 0}\left(\frac{\int_{0}^{x^2}\sin(\sqrt{t})dt}{x^3}\right) =$

(a) $\frac{1}{3}$ (b) 1

(c) $\frac{2}{3}$ (d) none.

58. Let $f(x) = \int_{1}^{x}\sqrt{2 - t^2}dt$, the real roots of $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$

(c) $\pm\frac{1}{2}$ (d) 0 and 1

59. If $\int_{\sin x}^{1} (t^2 f(t))dt = 1 - \sin x$, $x \in \left(0, \frac{\pi}{2}\right)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(a) 3 (b) $\frac{1}{3}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

60. If $\int_{0}^{t^2} xf(x)dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ is

(a) $\frac{2}{5}$ (b) $-\frac{5}{2}$

(c) 1 (d) $\frac{5}{2}$

61. The value of $\int_{0}^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_{0}^{\cos^2 x} \cos^{-1}(t)dt$ is

(a) $\frac{\pi}{2}$ (b) 1

(c) $\frac{\pi}{4}$ (d) none

62. If $f(x) = |2^x - 1| + |x - 1|$, then $\int_{-2}^{2} f(x)$ is

(a) $5 - \frac{9}{4}\log 2$ (b) $5 + \frac{9}{4}\log 2$

(c) $-5 - \frac{9}{4}\log 2$ (d) none

63. $\int_{-1}^{1} (x - [x])$ is

(a) 1 (b) $\frac{1}{2}$

(c) 0 (d) 2

64. $\int_{1}^{2} [2x^2 - 3]dx$, where [,] = GIF., then

(a) 4

(b) $\left(\frac{\sqrt{3}}{2} + \sqrt{2} + \sqrt{3} - 1\right)$

(c) $9 - \left\{\sqrt{\frac{3}{2}} + \sqrt{2} + \sqrt{\frac{5}{2}} + \sqrt{3} + \sqrt{\frac{7}{2}}\right\}$

(d) $15 - \sqrt{\frac{3}{2}} - \sqrt{2} - \sqrt{\frac{5}{2}} - \sqrt{3} - \sqrt{\frac{7}{2}} - \sqrt{\frac{9}{2}}$

65. $\int_{0}^{3/2} [x^2]\, dx =$

(a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$

(c) 3/2 (d) 3

66. $\int_{0}^{2} [x]dx =$

(a) $(2 - \sqrt{2})$ (b) $(2 + \sqrt{2})$

(c) $(\sqrt{2} - 1)$ (d) $(-\sqrt{2} - \sqrt{3} + 5)$

67. $\int_{0}^{[x]} (x - [x])dx =$

(a) $[x]$ (b) $2\,[x]$

(c) $\frac{1}{2[x]}$ (d) $\frac{1}{2}[x]$

68. $\int_{1}^{e^6} \left[\frac{\log x}{3}\right]dx$, where [,] = GIF, is

(a) 0 (b) $e^6 - e^3$

(c) $e^6 + e^3$ (d) $e^3 - e^6$

69. $\int_{1/e}^{e^2} \left|\frac{\log x}{x}\right|dx$, is

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$

(c) 3 (d) 5

70. $\sum_{n=1}^{1000} \int_{n-1}^{n} (e^{x-[x]})$, where [,] = GIF, is

(a) $\frac{e - 1}{1000}$ (b) $1000\,(e - 1)$

(c) $\frac{e^{1000} - 1}{e - 1}$ (d) $\frac{e^{1000} - 1}{1000}$

71. Let $f(x)$ be a continuous function such that $f(x) + f(2014 - x) = 0$ for all $x \in [0, 2014]$,

then $\int_{0}^{2014} \frac{dx}{1 + 5^{f(x)}}$ is

(a) 2014 (b) 1007

(c) $f(2014)$ (d) $f(1007)$

72. The value of $\int_{0}^{\pi} \frac{dx}{1 + 2\sin^2 x}$ is

(a) $\pi/\sqrt{3}$ (b) π

(c) 2π (d) $\sqrt{3}\pi$

73. The value of $\int_{0}^{4} \{\sqrt{x}\}dx$, where {,} is the fractional part of x, is

(a) 1/3 (b) 5/3

(c) 2 (d) 7/3

74. If $z \neq 0$, then $\int_{0}^{100} [arg|z|]dz$, where [,] = GIF, is

(a) 0 (b) not defined

(c) 100 (d) none

75. If $f(x) = \int_{0}^{x} \left(\frac{|\sin y|}{|\sin y| + |\cos y|}\right)dy$, then $f(2\pi)$ is

(a) $f(\pi/2)$ (b) $2f(\pi/2)$

(c) $4f(\pi/2)$ (d) none

76. The value of $\int_0^1 \left(\frac{\sin^{-1}x}{\sin^{-1}x + \cos^1 x}\right)dx$ is

(a) $\left(1 + \frac{2}{\pi}\right)$ (b) $\left(1 - \frac{2}{\pi}\right)$

(c) $-\left(1 + \frac{2}{\pi}\right)$ (d) $-\left(1 - \frac{2}{\pi}\right)$

77. The value of $\int_0^{\pi/2} [\sin 2x \cdot \tan^{-1}(\sin x)]dx$ is

(a) $\left(\frac{\pi}{2} + 1\right)$ (b) $\left(-\frac{\pi}{2} + 1\right)$

(c) $\left(\frac{\pi}{2} - 1\right)$ (d) $-\left(\frac{\pi}{2} + 1\right)$

78. The value of $\int_0^{\pi/2} \left(\frac{a\sin x + b\cos x}{\sin x + \cos x}\right)dx$ is

(a) $\pi/4$ (b) $(a + b)\pi/2$

(c) $(a + b)\pi$ (d) $(a + b)\pi/4$

79. The value of $\int_{-\pi}^{\pi} (1 - x^2)\sin x \cdot \cos^2 x\, dx$ is

(a) 0 (b) $\left(\pi - \frac{\pi^3}{3}\right)$

(c) $(2\pi - \pi^3)$ (d) $\left(\frac{7}{2} - \frac{2}{\pi^3}\right)$

80. If $\int_0^{\pi/2} \left(\frac{(x - \pi/4)^2 \sin x}{\sin x + \cos x}\right)dx = \frac{1}{3}\left(\frac{\pi}{a}\right)^3$, the value of a is

(a) 2 (b) 4

(c) 6 (d) 8

Objective Questions (More than one options are correct)

81. The value of $\int_{-\pi}^{\pi} \sin mx \cdot \cos nx\, dx$, where $m, n \in N$, is equal to

(a) 0 (b) $\int_{-\pi/8}^{\pi/8} x^8 \sin^9 x\, dx$

(c) $\int_{-\pi/2}^{\pi/2} \sin x f(\cos x)dx$ (d) $\int_{-1/2}^{1/2} e^{\cos x}dx$

82. The value of the integral $\int_0^{\pi} x f(\sin x)dx$ is

(a) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$ (b) $\frac{\pi}{4}\int_0^{\pi} f(\sin x)dx$

(c) $\pi \int_0^{\pi/2} f(\sin x)dx$ (d) $\pi \int_0^{\pi/2} f(\cos x)dx$

83. If $\int_a^b f(t)g(h(t))dt = \int_a^b f(h(t))g(t)dt$, where f, g, h are non negative continuous functions on $[a, b]$, then $h(t)$ is

(a) t (b) $a - b - t$

(c) $a + b - t$ (d) $b - t$

84. If $I_1 = \int_x^1 \left(\frac{dt}{1 + t^2}\right)$ and $I_2 = \int_1^{x/1} \left(\frac{dt}{1 + t^2}\right)$, $x > 0$, then

(a) $I_1 = I_2$ (b) $I_1 > I_2$

(c) $I_1 < I_2$ (d) $I_2 = \pi/4 - \tan^{-1}x$

85. The value of $\int_{1/e}^{\tan x} \left(\frac{tdt}{1 + t^2}\right) + \int_{1/e}^{\cot x} \left(\frac{dt}{t(1 + t^2)}\right)$ is

(a) $\frac{1}{2 + \tan^2 x}$ (b) 1

(c) $\pi/4$ (d) $\frac{2}{\pi}\int_{-1}^{1}\left(\frac{dt}{1 + t^2}\right)$

86. If $I = \int_0^1 \left(\frac{dx}{1 + x^{\pi/2}}\right)$, then

(a) $I > \ln 2$ (b) $I < \ln 2$

(c) $I < \pi/4$ (d) $I > \pi/4$

87. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, then

(a) $I_n + I_{n-2} = \frac{1}{n - 1}$

(b) $I_{n-1} + I_{n+1} = \frac{1}{n}$

(c) $\frac{1}{n + 1} < 2I_n < \frac{1}{n - 1}$, $n \in N(>1)$

(d) $(I_{n+2} + I_n)(n + 1) = 1$

88. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, then

(a) $I_7 + I_5 = 1/6$

(b) $I_{10} + I_8 = 1/9$

(c) $I_8 + I_{12} = 2/99$

(d) $I_8 + 2I_{10} + I_{12} = 20/99$

89. If $I_{n,k} = \int_0^1 x^{k-1}(\log x)^k dx$, its value can be

(a) $\frac{k!}{n^{k+1}}$, where k is an even number.

(b) $-\frac{k!}{n^{k+1}}$, k is an odd number

(c) $\dfrac{(-1)^k \times k!}{n^{k+1}}$

(d) $\dfrac{k!}{n^{k+1}}$

90. If $\int_{-2\pi}^{5\pi} \cot^{-1}(\tan x)dx$ is $\dfrac{l\pi^2}{2}$, then l is

(a) 3 (b) 4

(c) 6 (d) 7

91. The value of the integral $\int_0^{\pi/4} \left(\dfrac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right)$, is

(a) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}\right)$, $a > 0$, $b > 0$

(b) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}\right)$, $a < 0$, $b < 0$

(c) $\dfrac{\pi}{4}$, $(a = 1, b = 1)$

(d) none.

92. The value of $\int_{\pi}^{\pi/2} \left(\dfrac{\sin 2x}{\sin^4 x + \cos^4 x}\right)dx$ is

(a) $\pi/4$

(b) π

(c) $2\int_0^{\pi/4} \left(\dfrac{\tan x}{(\cos^2 x(1 + \tan^4 x)}\right)dx$

(d) $\dfrac{\pi}{2}$

Level III

(Problems for JEE-Advanced)

1. Let $f(x) = (|\sin x| - |\cos x|)$. Find the value of $\left[\int_0^{9\pi/4} f(x)dx\right]$, where [,] = GIF

2. Find the value of $\int_{-2}^{1} \left[x\left[1 + \cos\left(\dfrac{\pi x}{2}\right)\right] + 1\right]dx$.

3. Find the value of $\int_0^3 \left([x] + \left[x + \dfrac{1}{3}\right] + \left[x + \dfrac{2}{3}\right]\right)dx$.

4. Find the value of $\int_1^{10\pi} ([\sec^{-1}x] + [\tan^{-1}x])dx$ where [,] = GIF.

5. Find the value of $\int_0^{5\pi/12} [\tan x]dx$.

6. Find the value of $\int_0^{\pi/4} [\sin x + \{\cos x + \tan x + (\sec x)\}]dx$

7. Find the value of $\int_0^{\pi/2} [\sin^{-1}(\cos x) + \cos^{-1}(\sin x)]dx$.

8. Find the value of $\int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \left|\dfrac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\right|dx$.

9. If $m = \int_0^{100} \sec^{-1}(\sec \pi x)dx$ and $n = \int_0^{3/\pi} [\sec^{-1}(\sec \pi x)]dx$, find the value of $\left(\dfrac{m}{\pi} + n\pi + \dfrac{\pi}{4} + 4\right)$.

10. Find the value of
$$\int_{-3}^{-7} \tan(x^2 - 6)dx + \int_{-6}^{-2} \tan(x^2 + 18x + 75)dx.$$

11. Let $I_n = \int_{-2n\pi}^{2n\pi} |\sin x|[\sin x]dx$, where $n \in N$ and [,] = GIF, find the value of $\sum_{n=1}^{100} I_n$

12. If $m = \int_{-2}^{0} \left(\dfrac{|\sin x|}{\left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}}\right)dx$ and $n = \int_0^2 \left(\dfrac{|\sin x|}{\left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}}\right)dx$, where [,] = GIF, prove that $m + n = 0$.

13. Find the value of
$$\int_{-1}^{1} \left(\sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]\right)dx, \text{ where [,] = GIF.}$$

14. Find the value of
$$\int_{-\pi/3}^{0} (\cot^{-1}\left(\frac{2}{2\cos x - 1}\right) + \cot^{-1}\left(\cos x - \frac{1}{2}\right)dx$$

15. Find the value of $\int_0^{\pi/2} \left(\dfrac{8 + 7\cos x}{(7 + 8\cos x)^2}\right)dx$

16. If $f(x + y) = f(x) - f(y)$ for all x and y and $a = \int_1^3 (x - 1)^2 f(x - 1)dx$ and $b = \int_{-3}^{1} (x + 1)^2 f(x + 1)dx$, find the value of $2a - b + 4$.

17. Find the value of $\int_0^4 \left(\dfrac{(y^2 - 4y + 5)\sin(y - 2)}{2y^2 - 8y + 1}\right)dy$

18. Find the value of $\int_{-1}^{1} \left(\dfrac{\sin\alpha}{x^2 - 2x\cos\alpha + 1}\right)dx$

19. Find the value of $\int_0^1 \left(\dfrac{x^3\sin^{-1}x}{\sqrt{1 - x^2}}\right)dx$

20. Find the value of $\int_0^1 \left(\dfrac{1 - x^2}{1 + x^2} \times \dfrac{1}{\sqrt{1 + x^4}}\right)dx$

21. If $m = \int_0^1 \left(\frac{\log(1+x)}{1+x^2}\right)dx$ and $n = \int_0^\infty \left(\frac{\log(1+x^2)}{1+x^2}\right)dx$, prove that $8m = n$.

22. Find the value of
$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)}{e^x+1}\right)dx.$$

23. Find the value of $\int_{-3\pi/4}^{5\pi/4} \left(\frac{\sin x + \cos x}{e^{\left(x-\frac{\pi}{4}\right)}+1}\right)dx$.

24. Let $I_1 = \int_0^{\pi/2} \left(\frac{x}{\sin x}\right)dx$ and $I_2 = \int_0^1 \left(\frac{\tan^{-1}x}{x}\right)dx$, find $\frac{I_1}{I_2}$.

25. Let $I_1 = \int_0^1 \left(\frac{e^x}{x+1}\right)dx$ and $I_2 = \int_0^1 \left(\frac{x^2}{e^{x^3}(2-x^3)}\right)dx$, find $\frac{I_1}{I_2}$.

26. Find the value of $\int_0^\pi \left(\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx$.

27. Find the value of $\int_0^{\pi/2} \left(\frac{\sin 5x}{\sin x}\right)dx$.

28. If $m = \int_0^1 x^{50}(2-x)^{50}dx$ and $n = \int_0^1 x^{50}(1-x)^{50}dx$, prove that $\frac{m}{n} = 2^{100}$

29. If $\int_0^\infty \left(\frac{\sin x}{x}\right)dx = \frac{\pi}{2}$, find the value of $\int_0^\infty \left(\frac{\sin^3 x}{x}\right)dx$

30. If $A = \int_0^\pi \left(\frac{\cos x}{(x+2)^2}\right)dx$ and $B = \int_0^{\pi/2} \left(\frac{\sin 2x}{(x+1)}\right)dx$, prove that $A + B = \frac{1}{\pi+2} + \frac{1}{2}$.

31. If $\int_0^\infty \left(\frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\right)dx$
$= \frac{\pi}{2(a+b)(b+c)(c+a)}$,
find the value of
$\int_0^\infty \left(\frac{dx}{(x^2+4)(x^2+9)}\right)$.

32. Find the value of $\int_0^{\pi/2} \log(a^2\cos^2 x + b^2\sin^2 x)dx$.

33. Find the value of $\int_0^\infty \left(\frac{\ln x}{x^2+2x+4}\right)dx$.

34. Find the value of $\int_0^1 \left(\frac{\ln x}{1+x}\right)dx$.

35. Find the value of $\int_0^1 \left(\frac{\ln(1+x^4)}{x}\right)dx$.

36. If $I_n = \int_0^\pi \left(\frac{1 - x\cos nx}{1-\cos x}\right)dx$, prove that
$I_{n+2} + I_n = 2I_{n+1}$, $n \in I^+$
and deduce that $\int_0^{\pi/2} \left(\frac{\sin^2(nx)}{\sin^2(x)}\right) = \frac{n\pi}{2}$

37. Evaluate
$$\lim_{n\to\infty}\left(\frac{(1^2+2^2+3^2+\dots+n^2)(1^3+2^3+3^3+\dots+n^3)}{(1^6+2^6+3^6+\dots+n^6)}\right)$$

38. Evaluate $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n-r}{n^2}\cos\left(\frac{4r}{n}\right)\right)$.

39. Evaluate $\lim_{n\to\infty}\sum_{r=1}^{n}\int_0^1 (x^r - x^{r+1})dx$.

40. Find the value of $\int_0^\pi \left(\frac{\sin\theta}{a^2\cos^2\theta + b^2\sin^2\theta}\right)d\theta$, where $a, b \in R^+$, $a > b$

Level II

(Tougher Problems for JEE-Advanced)

1. Evaluate: $\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$

2. Let $I = \int_0^{\pi/2} \left(\frac{\cos x}{a\cos x + b\sin x}\right)dx$
and $J = \int_0^{\pi/2} \left(\frac{\sin x}{a\cos x + b\sin x}\right)dx$, where $a > 0$, $b > 0$, find I and J.

3. Evaluate: $\int_0^a \log(\cot a + \tan x)dx$, $a \in \left(0, \frac{\pi}{2}\right)$

4. Evaluate: $\int_0^{\pi/2} \left(\frac{\sin^6 x}{\sin x + \cos x}\right)dx$

5. Evaluate: $\int_0^1 \left(\frac{dx}{(5+2x-2x^2)(1+e^{(2-4x)})}\right)$

6. Evaluate: $\int_0^\infty \left(\frac{(x-1)^4}{x^8+x^{10}}\right)dx$

7. Let $I = \int_0^\pi \left(\frac{x}{1+x\sin x}\right)^2 dx$ and $J = \int_0^\pi \left(x^2\frac{(1+\cos x)}{(1+x\sin x)^2}\right)dx$, find $(J - I)$.

8. Evaluate: $\int_0^{\pi/2} (\sqrt{\sin x} + \sqrt{\cos x})^{-4}dx$

9. Prove that $\int_0^{\pi} \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}} \cdot a > 1$ and deduce that $\int_0^{\pi} \frac{dx}{(2-\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$

10. If $I_n = \int_0^{\pi/4} (\tan^n x)dx$, prove that

$\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5},$

$\frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7} \in$ A.P

11. Assuming $\int_0^{\pi} \log(\sin\theta)d\theta = -\pi\log 2$, prove that $\int_0^{\pi} \theta^3 \log(\sin\theta)d\theta = \frac{3\pi}{2}\int_0^{\pi} \theta^2 \log(\sqrt{2}\sin\theta)d\theta$

12. If $I_n = \int_0^{\infty} (e^{-x}x^{n-1}\log x)dx$, prove that

$I_{n+2} - (2n+1)I_{n+1} + n^2 I_n = 0$

13. If $af(x) + bf(-x) = \frac{1}{x} = \sin\left(x - \frac{1}{x}\right)$, find the values of $\int_{1/2}^{2} f(x)dx \ (a \neq b)$.

14. Evaluate: $\int_0^{\pi/6} \left(\frac{\sqrt{3\cos(2x) - 1}}{\cos(x)}\right)dx$

15. Evaluate:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{x^4 + x^3 + 2x^2 + x\tan^2 x + x^2\sin^3 x + \tan^{-1}(x) + 1}{x^6 + 3x^4 + 3x^2 + 1}\right)dx$$

Integer Type Questions

1. If the value of the integral is of the form $\int_0^{\pi/2} \sin^{10} x\,dx = \frac{63\pi}{A^B}$, find the value of $B - A$.

2. If the value of the integral $\int_0^{\pi/2} \cos^8 x\,dx = \frac{(LM)\pi}{P^Q}$, find the value of $L + M + P - Q$.

3. If $I_1 = \int_0^{\pi/2} \log(\sin x)dx$ and $I_2 = \int_0^{\pi/2} \log(\cos x)dx$, find the value of $I_1 - I_2 + 5$.

4. If $L = \int_0^{1} \frac{\log(1+x)}{1+x^2}dx$ and $M = \int_0^{\pi/4} \log(1+\tan\theta)d\theta$ such that $L + M = \frac{\pi}{A}\log B$, find the value of $A + B + 3$.

5. Let $I = \int_0^{\infty} e^{-x}x^{3/2}dx$ such that $I = \frac{3}{L} \times (\pi)^{\frac{M}{N}}$, find the value of $L + M + N$.

6. Let $M = \int_{-1}^{1} \min\{|x+1|, |x|, |x-1|\}dx$ and $N = \int_{-10}^{10} \min\{-x-[-x], x-[x]\}dx$. Find the value of $2M + N + 2$.

7. Let $M = \int_{-2}^{-2} (-1)^{[x]}\,dx$ and $N = \int_0^{4} [x+1]dx$, find the value of $M + N - 1$.

8. Let $L = \int_{-2}^{3} (|x-1| + 1)dx$ and $M = \int_0^{4} |x-2|dx$, find the value of $\frac{2}{3}(L - M)$

9. Let $L = \int_1^{3} [\log|x|]dx$ and $M = \int_1^{3} [\log x]dx$, find the value of $\frac{L}{M} + 5$.

10. Let $A = \int_0^{2\pi} [\sqrt{2}\sin x]dx$ and $B = \int_0^{10\pi} [\sin x + \cos x]dx$, find the value of $\frac{B}{A}$.

11. Evaluate:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{3n}} + \frac{1}{\sqrt{4n}} + \dots + \frac{1}{2n}\right)$$

12. Find the value of

$$\left(\frac{\int_0^{10} [x]dx}{\int_0^{10} \{x\}dx}\right), \text{ Where } [,] = \text{GIF},\ \{,\} = \text{FPF}$$

13. Find the value of $\int_2^{8} \left(\frac{[x^2]}{[x^2 - 20x + 100] + [x^2]}\right)dx$.

14. If $\int_{-1}^{1} x^2 d(\ln x) = \frac{e^m - 1}{2e^n}$, $m, n \in N$, find the value of $(m + n)$.

15. If $\int_0^{1} \frac{dx}{\sqrt{1+x} + \sqrt{1-x} + 2} = a\sqrt{2} - \frac{\pi}{2} + c$

where $a, b, c \in N$, find the value of $(a + b + c)$.

Comprehensive Link Passages

Passage I

For the integral,

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx & : f(-x) = f(x) \\ 0 & : f(-x) = -f(x) \end{cases}$$

From the above concepts, give the answer of the following questions.

1. the value of the integral $\int_{10}^{-10} x\left(\frac{x}{e^x - 1} + \frac{x}{2} + 1\right) dx$ is

(a) 2 (b) 0
(c) 1 (d) 4

2. The value of the integral $\int_{-5}^{5}\left(\frac{(-1)^{[x]}}{x^4 - x^2 + 1} \times \frac{(x^2 + 1)}{\tan(x^2)}\right)$ dx is

(a) 2/5 (b) 3/5
(c) 0 (d) 4/5

3. The value of the integral

$\int_{-2}^{0} [(x^3 + 3x^2 + 3x + 4) + (x + 1)^2 \sin(x + 1)]dx$ is

(a) 2 (b) 0
(c) 4 (d) 6.

4. The value of the integral

$\int_{-3}^{-1} [(x^3 + 6x^2 + 12x + 10) + (x + 2)\cos(x + 2)]dx$ is

(a) 0 (b) 2
(c) 4 (d) 10.

5. The value of the integral

$\int_{2}^{4} \{(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)\}dx$ is

(a) 0 (b) 2
(c) −2 (d) 1

6. The value of $\int_{49}^{51} \{(x - 2)(x - 4)(x - 6) \ldots (x - 48)$ $(x - 50) \ldots (x - 96)(x - 98)\}dx$ is

(a) *0 (b) 5050
(c) 650 (d) 420

7. The value of the integral

$$\int_{-10}^{10}\left(\frac{\sin\left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}\right)}{\log(x + \sqrt{x^2 + 1}) \times \left(\frac{10x - 1}{10x + 1}\right)}\right)dx \text{ is}$$

(a) 2 (b) 4
(c) 6 (d) 0.

8. The value of the integral

$$\int_{-2}^{0}\left(\frac{(x + 1)\tan(x + 1)}{\frac{1}{2} + [x + 1]} + (x + 1)\sin(x + 4 - [x])\right)dx \text{ is}$$

(a) −2 (b) 2
(c) 0 (d) 4.

9. The value of the integral

$\int_{-\pi/2}^{\pi/2} (\sin(\sin x) + x\cos(\sin x))dx$ is

(a) 0 (b) $-\pi/4$
(c) $\pi/4$ (d) $\pi/2$

10. the value of the integral

$$\int_{-\pi/2}^{\pi/2} \left(\sin\left(\frac{2^x - 1}{2^x + 1}\right) + x\left(\frac{x}{2012^x - 1} + \frac{x}{2}\right)dx \text{ is}$$

(a) $\frac{2^x}{\log 2}$ (b) 0
(c) $\frac{2012^x}{\log 2012}$ (d) 64.

Passage II

For the integral $\int_{0}^{2a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx & : f(2a - x) = f(x) \\ 0 & : f(2a - x) = -f(x) \end{cases}$

From the above concepts, give the answer of the following questions.

1. The value of the integral $\int_{0}^{2\pi} \sin^{2013} x\,dx$ is

(a) 0 (b) 256π
(c) 512π (d) 1024π

2. The value of the integral $\int_{0}^{\pi} \cos^{2013} x\,dx$ is

(a) 256π (b) 0
(c) 1024π (d) 2048π

3. The value of the integral $\int_{0}^{4\pi} \tan^{2013} x\,dx$ is

(a) 0 (b) 128π
(c) 512π (d) 1024π

4. The value of the integral $\int_{0}^{1} \frac{\log(2 + 2x)}{1 + x^2} dx$ is

(a) $\frac{3\pi}{8}\log 2$ (b) $\frac{\pi}{8}\log 2$
(c) $\frac{5\pi}{8}\log 2$ (d) $\frac{7\pi}{8}\log 2$

5. The value of the integral $\int_{0}^{\pi} \log(\sin x)dx$ is

(a) $-\frac{\pi}{2}\log 2$ (b) $\frac{\pi}{2}\log 2$

(c) $-\pi\log 2$ (d) $\pi\log 2$

6. The value of the integral $\int_0^{\pi} \log(\cos x)dx$ is

(a) 0 (b) $\frac{\pi}{4}\log 2$

(c) $\frac{3\pi}{4}\log 2$ (d) $\frac{5\pi}{4}\log 2$

7. The value of the integral $\int_0^{\pi} \frac{x}{1+\cos^2 x}dx$ is

(a) $\frac{\pi^2}{2\sqrt{2}}$ (b) $\frac{\pi^2}{3\sqrt{3}}$

(c) $\frac{\pi^2}{5\sqrt{2}}$ (d) $\frac{\pi^2}{7\sqrt{2}}$

8. The value of the integral $\int_0^1 \{\{f(x)+f(1-x)\} \times \{(g(x) - g(1-x)\}\}\, dx$ is

(a) 0 (b) 1

(c) 2 (d) 3.

9. The value of the integral $\int_0^2 \log\left(\frac{x}{2-x}\right)dx$ is

(a) 1 (b) –1

(c) 2 (d) 0.

10. The value of the integral $\int_0^1 ([x]-[1-x])d(x-[x])\, dx$ is

(a) 0 (b) –2

(c) 4 (d) 6

11. The value of the integral $\int_0^2 \{x(x+1)(2-x)(3-x)\}dx$ is

(a) 0 (b) 62/15

(c) 38/15 (d) 76/15.

12. The value of the integral $\int_0^4 \{2x^3 - 12x^2 + 78x - 124\}dx$ is

(a) 0 (b) 125

(c) 625 (d) 2525.

13. The value of the integral $\int_0^4 \{(|x^2+4x| + |x^2-16|) - (|x^2-8x| + |x^2-12x+32|)\}$ dx is

(a) 0 (b) 256

(c) 512 (d) 1024.

14. The value of the integral $\int_0^2 \left(\frac{e^{2-x}-e^x}{e^{2-x}+e^x}\right)dx$ is

(a) $e^2 - 1$ (b) 0

(c) $e^3 + 1$ (d) $e^2 - 4$.

15. The value of the integral $\int_0^2 (-1)^{[x]}dx$ is

(a) 2 (b) 4

(c) 8 (d) 0.

Passage III

For the integral $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

From the above concept, give the answer of the following questions.

1. The value of the integral $\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$ is

(a) $a/2$ (b) $a/4$

(c) $a/8$ (d) a.

2. The value of the integral $\int_0^1 \left(\frac{\log(2+4x+2x^2)}{(1+x^2)}\right)dx$ is

(a) $\frac{\pi}{2}\log 2$ (b) $-\pi\log 2$

(c) $\pi\log 2$ (d) $-\frac{\pi}{2}\log 2$

3. The value of the integral $\int_0^{\pi/2} \left(\frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\right)dx$ is

(a) $\frac{\pi^2}{16}$ (b) $\frac{\pi^2}{8}$

(c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{32}$

4. Let $I_1 = \int_0^2 (x\log(x(2-x)))dx$ and $I_2 = \int_0^2 (\log(x(2-x)))$ dx, then I_1/I_2 is

(a) 2 : 1 (b) 1 : 1

(c) 4 : 1 (d) 2 : 3

Matrix Match
(For JEE-Advanced Examination only)

1. **Match the following columns:**

Column I		Column II	
(A)	$\int_0^{\pi/2} \log(\tan x)dx$	(P)	$-\frac{\pi}{2}\log 2$
(B)	$\int_0^{\pi/2} \log(\sin x)dx$	(Q)	$-\pi\log 2$
(C)	$\int_0^{\pi/2} \log(\cos x)dx$	(R)	$\frac{\pi}{2}\log 2$
(D)	$\int_0^{\pi/2} \log(\operatorname{cosec} x)dx$	(S)	0

2. **Match the following columns:**

	Column I		Column II
(A)	$\int_0^{2\pi} \sin^5 x\,dx$	(P)	$\pi/2$
(B)	$\int_0^{\pi} \cos^{2013} x\,dx$	(Q)	0
(C)	$\int_0^{10\pi} \sin^{2013} x\,dx$	(R)	$\pi/4$
(D)	$\int_0^{2012\pi} \sin^{2013} x\,dx$	(S)	$\pi/2013$

3. **Match the following columns:**

	Column I		Column II
(A)	$\int_0^{1.5} [x^2]dx$	(P)	$5 - \sqrt{2} - \sqrt{3}$
(B)	$\int_0^{2} [x^2]dx$	(Q)	$2 - \sqrt{2}$
(C)	$\int_0^{2} [x^2 + 1]dx$	(R)	$6 - \sqrt{2} - \sqrt{3}$
(D)	$\int_0^{2} [2x^2 + 1]dx$	(S)	$\sqrt{2} - 1$

4. **Match the following columns:**

	Column I		Column II
(A)	$\int_0^{\pi/2} \sin^{10} x\,dx$	(P)	$\frac{63}{2^9} \times \mathrm{R}\pi$
(B)	$\int_0^{\pi/2} \sin^{11} x\,dx$	(Q)	$\frac{2^8}{693} \times \mathrm{R}\pi$
(C)	$\int_0^{\pi/2} \cos^{12} x\,dx$	(R)	$\frac{77}{5.2^4} \times \mathrm{R}\pi$
(D)	$\int_0^{\pi/2} \cos^{13} x\,dx$	(S)	$\frac{5 \times 2^7}{3003} \times \sqrt{\pi}$

5. **Match the following columns:**

	Column I		Column II
(A)	$\int_{-1}^{1} \log\left(\frac{1 - x}{1 + x}\right)dx$	(P)	0
(B)	$\int_{-2}^{2} \sin\left[\log\left(\frac{2 - x}{2 + x}\right)\right]dx$	(Q)	2
(C)	$\int_{-\pi/4}^{\pi/4} \tan[\log(x + \mathrm{R}x^2 + 1)]dx$	(R)	4
(D)	$\int_{-10}^{10} x\left(\frac{x}{e^x - 1} + \frac{x}{2 + 1}\right)dx$	(S)	1

6. **Match the following columns:**

	Column I		Column II
(A)	$\int_0^1 x^{m-1}(1 - x)^{n-1}dx,\quad m, n > 0$	(P)	$B(m, n)$
(B)	$\int_0^\infty \frac{x^{m-1}}{(1 + x)^{n-1}}dx,\quad m, n > 0$	(Q)	$\Gamma(n)$
(C)	$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m + n)},\quad m, n > 0$	(R)	$\sqrt{\pi}$
(D)	$\int_0^\infty e^{-x}x^{n-1}dx$	(S)	0

Questions asked in Previous Years' JEE-Advanced Examinations

1. Let $f: R \to R$ and $g: R \to R$ are continuous functions. The value of the integral

$$\int_{-\pi/2}^{\pi/2} [f(x) + f(-x))(g(-x) - g(-x)]dx \text{ is}$$

(a) π (b) 1

(c) 0 (d) None.

[IIT-JEE, 1990]

2. Show that $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos dx$

[IIT-JEE, 1990]

3. Prove that for any positive integer k,

$$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \dots + \cos(2k - 1)x]$$

Hence, prove that $\int_0^{\pi/2} \sin(2kx)\cot x\,dx = \frac{\pi}{2}$

[IIT-JEE, 1990]

4. If f be a continuous function with $\int_0^x f(t)dt \to \infty$ as $|x| \to \infty$, show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)dt = 2$.

[IIT-JEE, 1991]

5. Evaluate: $\int_0^{\pi} \left(\frac{x\sin(2x)\sin\left(\frac{x}{2}\cos x\right)}{2x - \pi}\right)dx$

[IIT-JEE, 1991]

6. The value of $\int_3^5 \left(\frac{x^2}{x^2-4}\right)dx$ is

(a) $2 - \log\left(\frac{15}{7}\right)$ (b) $2 + \log\left(\frac{15}{7}\right)$

(c) $2 + 4\log\left(\frac{15}{7}\right)$ (d) $2 - \tan^{-1}\left(\frac{15}{7}\right)$

[IIT-JEE, 1992]

7. The value of $\int_{-\pi}^{\pi} (1 - x^2) \sin x \cdot \cos^2 x\, dx$ is

(a) 0 (b) $\left(\pi - \frac{\pi^3}{3}\right)$

(c) $(2\pi - \pi^3)$ (d) $\left(\frac{7}{2} - 2\pi^3\right)$

[IIT-JEE, 1992]

8. The value of $\int_{-1}^{1} |1 - x| dx$ is

(a) −2 (b) 0

(c) 2 (d) 4

[IIT-JEE, 1992]

9. The value of $\int_0^1 |\sin(2\pi x)| dx$ is

(a) 0 (b) $\frac{3}{\pi}$

(c) $\frac{\pi}{4}$ (d) $\frac{2}{\pi}$.

[IIT-JEE, 1992]

10. Determine a positive integer $n \le 5$, such that

$$\int_0^1 e^x (x-1)^n dx = 16 - 6e$$

[IIT-JEE, 1992]

11. The value of $\int_0^{\pi/2} \left(\frac{1}{1 + \tan^3 \pi}\right)dx$ is

(a) 0 (b) 1

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

[IIT-JEE, 1993]

12. Find the value of $\int_2^3 \left(\frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)}\right)dx.$

[IIT-JEE, 1993]

13. Find the value of $\int_{\pi/4}^{3\pi/4} \left(\frac{\varphi}{1 + \sin\varphi}\right)d\varphi.$

[IIT-JEE, 1993]

14. The value of $\int_{-1}^{1} \left(\frac{\sin x - x^2}{3 - |x|}\right)dx$ is

(a) 0 (b) $2\int_0^1 \left(\frac{\sin x}{3 - |x|}\right)dx$

(c) $2\int_0^1 \left(\frac{-x^2}{3 - |x|}\right)dx$ (d) $2\int_0^1 \left(\frac{\sin x - x^2}{3 - |x|}\right)dx$

[IIT-JEE, 1994]

15. Prove that $\int_0^{n\pi + V} |\sin x| dx = 2n + 1 - \cos V$, where $n \in Z^+$ and $0 \le V \le \pi$.

[IIT-JEE, 1994]

16. Find the value of $\int_2^3 \left(\frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}}\right)dx$

[IIT-JEE, 1994]

17. The value of $\int_0^{3\pi} [2\sin x] dx$ is

(a) $-\frac{5\pi}{3}$ (b) $-\pi$

(c) $\frac{5\pi}{3}$ (d) -2π

[IIT-JEE, 1995]

18. If $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = 2\sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2A}{\pi}$, then the constants A and B are

(a) $\frac{\pi}{2}$ & $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ & $\frac{3}{\pi}$

(c) 0 & $\frac{-4}{\pi}$ (d) $\frac{8}{\pi}$ & $\frac{32}{\pi^2}$

[IIT-JEE, 1995]

19. Evaluate $\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2x}{1 + x^2}\right)\right)dx$

[IIT-JEE, 1995]

20. Let $U_n = \int_0^{\pi} \left(\frac{1 - \cos nx}{1 - \cos x}\right)dx$, where n is *a* positive integer, show that $U_{n+2} + U_n = 2U_{n+1}$.

Hence, deduce that $\int_0^{\pi/2} \left(\frac{\sin^2 n\theta}{\sin^2\theta}\right)d\theta = \frac{n\pi}{2}$

[IIT-JEE, 1995]

21. If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} = -5$, where $a \ne b$, then $\int_0^2 f(x)dx = ...$

[IIT-JEE, 1996]

22. For $n > 0$, evaluate $\int_0^{2\pi} \left(\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx.$

[IIT-JEE, 1996]

23. Find the value of $\int_{-\pi}^{\pi}\left(\frac{2x(1+\sin x)}{1+\cos^2 x}\right)dx$.

[IIT-JEE, 1997]

24. The value of $\int_{1}^{e^{37}}\left(\frac{\pi\sin(\pi\ln x)}{x}\right)dx$ is

[IIT-JEE, 1997]

25. Let $\frac{d}{dx}(f(x)) = \frac{e^{\sin x}}{x}$, $x > 0$ If $\int_{1}^{4}\left(\frac{2e^{\sin(x^2)}}{x}\right)dx = F(k) - F(1)$, the values of k is/are ...

[IIT-JEE, 1997]

26. Let $a + b = 4$, where $a < 2$ and let $g(x)$ be a differentiable function, If $\frac{dg}{dx} > 0$ for all x, prove that $\int_{0}^{a} g(x)dx + \int_{0}^{b} g(x)dx$ increases as $(b - a)$ increases.

[IIT-JEE, 1997]

27. If $g(x) = \int_{x}^{0}\cos^4 t\,dt$, then $g(x+\pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$
(c) $g(x)\cdot g(\pi)$ (d) $g(x)/g(\pi)$

[IIT-JEE, 1997]

28. If $\int_{0}^{x} f(t)dt = x + \int_{x}^{1} tf(t)dt$, the value of $f(1)$ is

(a) 1/2 (b) 0
(c) 1 (d) –1/2

[IIT-JEE, 1998]

29. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x, then $\int_{-1}^{1} f(x)dx$ is

(a) 1 (b) 2
(c) 0 (d) 1/2

[IIT-JEE, 1998]

30. Prove that $\int_{0}^{1}\tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_{0}^{1}\tan^{-1}x\,dx$

Hence or otherwise, evaluate the integral

$\int_{0}^{1}\tan^{-1}(1 - x + x^2)dx$.

[IIT-JEE, 1998]

31. $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to

(a) 2 (b) –2
(c) 1/2 (d) –1/2

[IIT-JEE, 1999]

32. If for a real number y, $[y]$ is the greatest integer is less than or equal to y, the value of the integral $\int_{\pi/2}^{3\pi/2}[2\sin x]dx$ is

(a) $-\pi$ (b) 0
(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$.

[IIT-JEE, 1999]

33. Evaluate the integral $\int_{0}^{\pi}\left(\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\right)$.

[IIT-JEE, 1999]

34. Let $g(x) = \int_{0}^{x} f(t)dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in [1, 2]$, then $g(2)$ satisfies the in equality

(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (b) $\frac{1}{2} \le g(2) \le \frac{3}{2}$
(c) $\frac{3}{2} < g(2) \le \frac{5}{2}$ (d) $2 < g(2) < 4$

[IIT-JEE, 2000]

35. If $f(x) = \begin{cases} e^{\cos x}\sin x & : |x| < 2 \\ 2 & : \text{otherwise} \end{cases}$, the value of $\int_{-2}^{3} f(x)dx$ is

(a) 0 (b) 1
(c) 2 (d) 3.

[IIT-JEE, 2000]

36. The value of the integral $\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$ is

(a) 3/2 (b) 5/2
(c) 3 (d) 5

[IIT-JEE, 2000]

37. For $x > 0$, let $f(x) = \int_{0}^{x}\left(\frac{\ln t}{1+t}\right)dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

Here, $\ln t = \log_e t$

[IIT-JEE, 2000]

38. The value of $\int_{-\pi}^{\pi}\left(\frac{\cos^2 x}{1+a^x}\right)$ is

(a) π (b) πa
(c) $\pi/2$ (d) 2π

[IIT-JEE, 2001]

39. Let $f: (0, \infty) \to R$ and $F(x) = \int_{0}^{x} f(t)dt$ and $F(x^2) = x^2(x + 1)$, then $f(4)$ equals

(a) 5/40 (b) 7
(c) 4 (d) 2

[IIT-JEE, 2001]

40. Let $f(x)$, $x \geq 0$ be a non-negative continuous function and let $F(x) = \int_0^x f(t)dt$, $x \geq 0$. If for some $c > 0$ $f(x) \leq cF(x)$ for all $x \geq 0$, show that $f(x) = 0$ for all $x \geq 0$.

[IIT-JEE, 2001]

41. Let $f(x)$, $\int_1^x \sqrt{2 - t^2}dt$. The real roots of the equation $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm 1/\sqrt{2}$
(c) $\pm 1/2$ (d) 0 and 1.

[IIT-JEE, 2002]

42. Let $T > 0$ be a fixed real number. Suppose f is continuous function such that for all x in R, $f(x + T) = f(x)$. If $I = \int_0^T f(x)dx$, the value of the integral $\int_3^{3+3T} f(2x)dx$ is

(a) $\frac{3}{2}I$ (b) $2I$
(c) $3I$ (d) $6I$

[IIT-JEE, 2002]

43. The value of $\int_{-1/2}^{1/2} \left([x] + \ln\left(\frac{1-x}{1+x}\right)\right)dx$ is

(a) –1/2 (b) 0
(c) 1 (d) $2\ln\left(\frac{1}{2}\right)$

[IIT-JEE, 2002]

44. If $I(m, n) = \int_0^1 t^m(1+t)^n dt$, the expression for $I(m, n)$ in terms of $I(m+1, n+1)$ is

(a) $\frac{2^n}{m+1} - \frac{n}{m+1}I(m+1, n-1)$

(b) $\frac{n}{m+1}\, I(m+1, n-1)$

(c) $\frac{2^n}{m+1} + \frac{n}{m+1}\, I(m+1, n-1)$

(d) $\frac{m}{n+1}\, I(m+1, n-1)$

[IIT-JEE, 2003]

45. Let $f(x)$ be an even function, prove that $\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \sqrt{2}\int_0^{\pi/2} f(\cos 2x)\cos dx$

[IIT-JEE, 2003]

46. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}dt$, $f(x)$ increases in

(a) $(-2, 2)$ (b) no value in x
(c) $(0, \infty)$ (d) $(-\infty, 0)$

[IIT-JEE, 2004]

47. If $f(x)$ is differentiable and $\int_0^{t^2} xf(x)dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals

(a) 2/5 (b) $-5/2$
(c) 1 (d) 5/2

[IIT-JEE, 2004]

48. The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ is

(a) $\left(\frac{\pi}{2} + 1\right)$ (b) $\left(\frac{\pi}{2} - 1\right)$
(c) –1 (d) 1

[IIT-JEE, 2004]

49. Let $f(x)$ be a differentiable function defined as $f: [0, 4] \to R$, show that

(i) $8f'(a)f(b) = \{f(4)\}^2 - \{f(0)\}^2$, when $a, b \in (0, 4)$

(ii) $\int_0^4 f(x)dx = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$, where $0 < \alpha, \beta < 2$

[IIT-JEE, 2004]

50. Find the value of $\int_{-\pi/3}^{\pi/3} \left(\frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right)dx$.

[IIT-JEE, 2004]

51. If $y(x) = \int_{\pi^2/16}^{x^2} \left(\frac{\cos x \cos(\sqrt{\theta})}{1 + \sin^2(\sqrt{\theta})}\right)d\theta$, find $y'(\pi)$.

[IIT-JEE, 2004]

52. $\int_{-2}^{0} [(x^3 + 3x^2 + 3x + 3) + (x+1)\cos(x+1)]dx$ is equal to

(a) –4 (b) 0
(c) 4 (d) 6.

[IIT-JEE, 2005]

53. If $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x$, then $f\left(\frac{1}{\sqrt{3}}\right)$ equals

(a) 1/3 (b) $1/\sqrt{3}$
(c) $\sqrt{3}$ (d) 3

[IIT-JEE, 2005]

54. Find the value of

$$\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\, dx$$

[IIT-JEE, 2005]

55. Find the value of $5050 \times \left(\dfrac{\int_0^1 (1 - x^{50})^{100} dx}{\int_0^1 (1 - x^{50})^{101} dx} \right)$.

[IIT-JEE, 2006]

56. $L = \lim_{x \to \frac{\pi}{4}} \left(\dfrac{\int_2^{\sec^2 x} f(t) dt}{x^2 - \left(\frac{\pi^2}{16}\right)} \right)$ equals

(a) $\frac{8}{\pi} f(2)$ (b) $\frac{2}{\pi} f(2)$

(c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (d) $4f(2)$

[IIT-JEE, 2007]

57. **Match the following columns:**

Column I		Column II	
(A)	$\int_{-1}^{1} \frac{dx}{1 + x^2}$	(P)	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B)	$\int_0^1 \frac{dx}{\sqrt{\sqrt{1 - x^2}}}$	(Q)	$2\log\left(\frac{2}{3}\right)$
(C)	$\int_2^3 \frac{dx}{1 - x^2}$	(R)	$\frac{\pi}{3}$
(D)	$\int_1^2 \frac{dx}{x\sqrt{x^2 - 1}}$	(S)	$\frac{\pi}{2}$

[IIT-JEE, 2007]

58. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'\left(\frac{1}{4}\right) = 0$, then

(a) $f'(x)$ vanishes at least twice on $[0, 1]$

(b) $f'\left(\frac{1}{2}\right) = 0$

(c) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$

(d) $\int_0^{1/2} f(t) e^{\sin(\pi t)} dt = \int_{1/2}^{1} f(1 - t) e^{\sin(\pi t)} dt$

[IIT-JEE, 2008]

59. Let $S_n = \sum_{k=1}^{n} \left(\frac{n}{n^2 + kn + k^2} \right)$ and

$T_n = \sum_{k=0}^{n-1} \left(\frac{n}{n^2 + kn + k^2} \right)$, for $n = 1, 2, 3, \ldots$, then

(a) $S_n < \frac{\pi}{3\sqrt{3}}$ (b) $S_n > \frac{\pi}{3\sqrt{3}}$

(c) $T_n < \frac{\pi}{3\sqrt{3}}$ (d) $T_n > \frac{\pi}{3\sqrt{3}}$

[IIT-JEE, 2008]

60. Let f be a non-negative function defined on [0, 1]. If $\int_0^x \sqrt{1 - (f'(t))^2} \, dt = \int_0^x f(t) dt$, $0 \le x \le 1$ and $f(0) = 0$, then

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(d) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

[IIT-JEE, 2009]

61. If $I_n = \int_{-\pi}^{\pi} \left(\frac{\sin nx}{(1 + \pi^c)\sin x} \right) dx$, $n = 0, 1, 2, \ldots$ then

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} (I_{2m+1}) = 10\pi$

(c) $\sum_{m=1}^{10} (I_{2m}) = 0$ (d) $I_n = I_{n+1}$

[IIT-JEE, 2009]

62. Let $f: R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t) dt$. Then the value of $f(\ln 5)$ is ...

[IIT-JEE, 2009]

63. Let f be a real valued function defined on $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} \, dt$ for all x in $(-1, 1)$ and let f^{-1} be the inverse function of f, then $(f^{-1})'(2)$ is equal to

(a) 1 (b) 1/3

(c) 1/2 (d) $1/e$

[IIT-JEE, 2010]

64. The value of the integral $\int_0^1 \left(\frac{x^4(1 - x)^4}{(1 - x^2)} \right) dx$ is (are)

(a) $\left(\frac{22}{7} - \pi\right)$ (b) $\frac{2}{105}$

(c) 0 (d) $\left(\frac{71}{15} - \frac{3\pi}{2}\right)$

[IIT-JEE, 2010]

65. The value of $\lim_{x \to 0} \left(\frac{1}{x^3} \int_0^x \frac{t \ln(1 + t)}{t^4 + 4} dt\right)$ is

(a) 0 (b) $\frac{1}{12}$

(c) $\frac{1}{24}$ (d) $\frac{1}{64}$

[IIT-JEE, 2010]

66. Let f be a function defined on $[-\pi, \pi]$ given by $f(0) = 9$ and $f(x) = \frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ for $x \neq 0$. Find the value of $\frac{2}{\pi} \int_{-\pi}^{\pi} f(x)dx$.

[IIT-JEE, 2010]

67. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real function defined on $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x] & : \text{ if } [x] \text{ is odd} \\ 1 + [x] - x & : \text{ if } [x] \text{ is even} \end{cases}.$$

Then find the value of $\frac{\pi^2}{10} \times \int_{10}^{10} f(x)\cos(\pi x)dx$

[IIT-JEE, 2010]

68. The value of $I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \left(\frac{x\sin(x^2)}{\sin(x^2) + \sin[\ln(6 - x^2)]}\right)dx$ is

(a) $\frac{1}{4} \ln\left(\frac{3}{2}\right)$ (b) $\frac{1}{2} \ln\left(\frac{3}{2}\right)$

(c) $\ln\left(\frac{3}{2}\right)$ (d) $\frac{1}{6} \ln\left(\frac{3}{2}\right)$

[IIT-JEE, 2011]

69. Let $f: [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$, if $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \geq 1$, the value of $f(2)$ is ...

[IIT-JEE, 2011]

70. The value of $\frac{\pi^2}{\ln 3} \int_{7/6}^{5/6} \sec(\pi x)dx$ is ...

[IIT-JEE, 2011]

71. If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, the value of $f\left(\frac{\pi}{6}\right)$ is ...

[IIT-JEE, 2011]

72. The value of the integral $\int_{-\pi/2}^{\pi/2} \left(x^2 + \ln\left(\frac{\pi + x}{\pi - x}\right)\right)\cos x\, dx$ is

(a) 0 (b) $\left(\frac{\pi^2}{2} - 4\right)$

(c) $\left(\frac{\pi^2}{2} + 4\right)$ (d) $\left(\frac{\pi^2}{2}\right)$

[IIT-JEE, 2012]

No questions asked in 2013.

73. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1 - x^2)^5\right\} dx$ is ...

[IIT-JEE, 2014]

74. The value of $\int_{-2}^{2} \left(\frac{3x^2}{1 + e^x}\right)$ is ...

[IIT-JEE, 2014]

75. The value of $\frac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1 + x}{1 - x}\right)dx}{\int_0^{1/2} \cos 2x \cdot \log\left(\frac{1 + x}{1 - x}\right)dx}$ is

[IIT-JEE, 2014]

77. The number of polynomials $f(x)$ with non-negative co-efficients of degree ≤ 2 satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is ...

[IIT-JEE, 2014]

78. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases} [x] & : x \leq 2 \\ 0 & : x > 2 \end{cases}$, where $[x]$ is the greatest integer less than and equal to x. If $I = \int_{-1}^{2} \left(\frac{xf(x^2)}{2 + f(x + 1)}\right)dx$, find the value of $(41 - 1)$

[IIT-JEE, 2015]

79. If $\alpha = \int_0^1 \left(e^{9x + \tan^{-1}x}\right)\left(\frac{12 + 9x^2}{1 + x^2}\right)dx$, where $\tan^{-1}x$ takes only principal values, find the value of $\left(\log_e|1 + \alpha| - \frac{3}{4}\right)$.

[IIT-JEE, 2015]

80. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the value of

(i) $\int_0^{\pi/4} f(x)dx$ (ii) $\int_0^{\pi/4} xf(x)dx$

[IIT-JEE, 2015]

81. Let $f'(x) = \dfrac{192x^3}{2 + \sin^4(\pi x)}$ for all x in R. with $f\left(\dfrac{1}{2}\right) = 0$.

If $m \le \int_{1/2}^{2} f(x)dx \le M$, find the possible values of m and M are

(a) $m = 13$, $M = 24$ (b) $m = 1/4$, $M = 1/2$

(c) $m = -11$, $M = 0$ (d) $m = 1$, $M = 12$

[IIT-JEE, 2015]

82. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} - L =$$

(a) $a = 2$, $L = \dfrac{e^{4\pi} - 1}{e^{\pi} - 1}$ (b) $a = 2$, $L = \dfrac{e^{4\pi} + 1}{e^{\pi} + 1}$

(c) $a = 4$, $L = \dfrac{e^{4\pi} - 1}{e^{\pi} - 1}$ (d) $a = 4$, $L = \dfrac{e^{4\pi} + 1}{e^{\pi} + 1}$

[IIT-JEE, 2015]

83. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \left(\dfrac{t^2}{1 + t^4}\right)dt = 2x - 1$ is ...

[IIT-JEE, 2016]

84. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\dfrac{x^2 \cos x}{1 + e^x}\right)dx$ is equal to

(a) $\left(\dfrac{\pi^2}{4} - 2\right)$ (b) $\left(\dfrac{\pi^2}{4} + 2\right)$

(c) $\left(\pi^2 - e^{\frac{\pi}{2}}\right)$ (d) $\left(\pi^2 + e^{\frac{\pi}{2}}\right)$

[IIT-JEE, 2016]

Answers

1. LEVEL II (OBJECTIVE QUESTIONS)

1. (a, c)	2. (d)	3. (b)	4. (c)	5. (c)
6. (c)	7. (c)	8. (d)	9. (a)	10. (c)
11. (c)	12. (c)	13. (b)	14. (d)	15. (b)
16. (a)	17. (c)	18. (a)	19. (d)	20. (a)
21. (b)	22. (b)	23. (c)	24. (b)	25. (b)
26. (a)	27. (c)	28. (a)	29. (c)	30. (b)
31. (d)	32. (b)	33. (a)	34. (a)	35. (d)
36. (b)	37. (b)	38. (a)	39. (b)	40. (a)
41. (a)	42. (a)	43. (c)	44. (c)	45. (c)
46. (a)	47. (b)	48. (c)	49. (d)	50. (c)
51. (a)	52. (d)	53. (b)	54. (c)	55. (c)
56. (a)	57. (d)	58. (a)	59. (a)	60. (a)
61. (c)	62. (b)	63. (a)	64. (c)	65. (b)
66. (d)	67. (d)	68. (b)	69. (a)	70. (b)
71. (a)	72. (a)	73. (d)	74. (a)	75. (c)
76. (b)	77. (c)	78. (d)	79. (a)	80. (b)
81. (a,b,c)	82. (a,c,d)	83. (a, c)	84. (a, d)	85. (b, d)
86. (a, c)	87. (a, b, c, d)		88. (a, b, c, d)	
89. (a,b,c)	90. (a, d)	91. (a, b, c)		92. (a, c)

INTEGER TYPE QUESTIONS

1. (7)	2. (2)	3. (5)	4. (7)	5. (5)
6. (8)	7. (9)	8. (5)	9. (7)	10. (5)
11. (4)	12. (9)	13. (3)	14. (6)	15. (3)

COMPREHENSIVE LINK PASSAGES

Passage I

1. (b)	2. (c)	3. (d)	4. (c)	5. (a)
6. (a)	7. (d)	8. (c)	9. (a)	10. (b)

Pssage II

1. (a)	2. (b)	3. (a)	4. (a)	5. (c)
6. (a)	7. (a)	8. (a)	9. (d)	10. (d)
11. (d)	12. (a)			

Passage III

1. (a)	2. (a)	3. (a)	4. (b)

MATRIX MATCH

1. (A)→(S), (B)→(P), (C)→(P), (D)→(R)
2. (A)→(Q), (B)→(Q), (C)→(Q), (D)→(Q)
3. (A)→(Q), (B)→(P), (C)→(R), (D)→(Q)
4. (A)→(P), (B)→(Q), (C)→(R), (D)→(S)
5. (A)→(P), (B)→(P), (C)→(P), (D)→(P)
6. (A)→(P), (B)→(P), (C)→(P), (D)→(Q)

Hints and Solutions

Level I

1. We have,

$$\int_0^{\pi/2} \sin^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

2. We have,

$$\int_0^{\pi/4} \frac{dx}{1 + \sin x}$$

$$= \int_0^{\pi/4} \frac{dx}{1 + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\pi/4} \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^2\left(\frac{\pi}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2} \times \left[-2\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]_0^{\pi/4}$$

$$= -\left[\tan\left(\frac{\pi}{4} - \frac{\pi}{8}\right) - 1\right]$$

$$= -\left[\tan\left(\frac{\pi}{8}\right) - 1\right]$$

$$= -[(\sqrt{2} - 1) - 1]$$

$$= 2 - \sqrt{2}$$

3. We have,

$$\int_0^2 \frac{dx}{\sqrt{x+2} + \sqrt{x}}$$

$$= \int_0^2 \frac{\sqrt{x-2} - \sqrt{x}}{(\sqrt{x+2} + \sqrt{x})(\sqrt{x+2} - \sqrt{x})}\,dx$$

$$= \int_0^2 \frac{(\sqrt{x+2} - \sqrt{x})}{x + 2 - x}\,dx$$

$$= \frac{1}{2}\int_0^2 (\sqrt{x+2} - \sqrt{x})\,dx$$

$$= \frac{1}{2}\left[\frac{2}{3}(x+2)^{3/2} - \frac{2}{3}x^{3/2}\right]_0^2$$

$$= \frac{1}{3}\left(8 - \frac{2}{3}2^{3/2}\right)$$

4. We have,

$$\int_0^{\pi/2} \frac{\sin x}{\sin x - \cos x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{2\sin x}{\sin x - \cos x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{(\sin x - \cos x) + (\sin x + \cos x)}{\sin x - \cos x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 + \frac{(\sin x + \cos x)}{(\sin x - \cos x)}\right)dx$$

$$= \frac{1}{2}\,(x + \log|\sin x - \cos x|)|_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

5. We have,

$$\int_1^2 \frac{dx}{x(1 + x^4)}$$

$$= \int_1^2 \frac{x^3\,dx}{x^4(1 + x^4)}$$

$$= \frac{1}{4}\log\left|\frac{x^4}{x^4 + 1}\right|_1^2$$

$$= \frac{1}{4}\left[\log\left(\frac{16}{17}\right) - \log\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\log\left(\frac{32}{17}\right)$$

6. We have,

$$\int_0^1 \frac{dx}{x^2 + x + 1}$$

$$= \int_0^1 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

7. We have,

$$\int_0^1 x e^x dx$$
$$= x \int e^x dx \Big|_0^1 - \int_0^1 e^x dx$$
$$= (xe^x - e^x)\Big|_0^1$$
$$= (e - e) - (0 - 1) = 1$$

8. We have,

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$
$$= \sqrt{2} \int_0^{\pi/2} \frac{dx}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}$$
$$= \sqrt{2} \int_0^{\pi/2} \frac{dx}{\sin x\left(x + \frac{\pi}{4}\right)}$$
$$= \sqrt{2} \int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$
$$= \sqrt{2} \log\left|\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|_0^{\pi/2}$$
$$= \sqrt{2} \log 3$$

9. We have

$$\int_0^{\pi/2} \frac{dx}{(3\sin^2 x + 4\cos^2 x)}$$
$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{3\tan^2 x + 4}$$
$$= \frac{1}{3}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{\tan^2 x + \left(\frac{2}{\sqrt{3}}\right)^2}$$
$$= \frac{1}{3} \times \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{2}\right)_0^{\pi/2}$$
$$= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{2} - 0\right)$$
$$= \frac{\pi}{4\sqrt{3}}$$

10. We have,

$$\int_0^1 (1 - x)x^n dx$$
$$= \int_0^1 (x^n - x^{n+1}) dx$$
$$= \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1$$
$$= \frac{1}{n+1} - \frac{1}{n+2}$$
$$= \frac{1}{(n+1)(n+2)}$$

11. We have,

$$I_n = \int_0^1 \tan^n x\, dx$$
$$= \int_0^1 \tan^{n-2} x \,.\, \tan^2 x\, dx$$
$$= \int_0^1 \tan^{n-2} x\, (\sec^2 x - 1)\, dx$$
$$= \int_0^1 \tan^{n-2} x \,.\, \sec^2 x\, dx - \int_0^1 \tan^{n-2} x\, dx$$
$$= \left(\frac{\tan^{n-1} x}{n-1}\right)_0^1 - I_{n-2}$$
$$\Rightarrow \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Hence, the result.

12. We have,

$\sin nx - \sin(n-2)x = 2\cos(n-1)x \sin x$

$$\Rightarrow \quad \frac{\sin nx}{\sin x} - \frac{\sin(n-2)x}{\sin x} = 2\cos(n-1)x$$
$$\Rightarrow \quad \frac{\sin nx}{\sin x} = 2\cos(n-1)\,x + \frac{\sin(n-2)x}{\sin x}$$
$$\Rightarrow \quad \int_0^{\pi} \frac{\sin nx}{\sin x} dx = \int_0^{\pi} 2\cos(n-1)x\, dx + \int_0^{\pi} \frac{\sin(n-2)x}{\sin x} dx$$
$$\Rightarrow \quad I_n = 0 + I_{n-2} = I_{n-4} = \ldots = I_2 \text{ or } I_1$$

according as n is even or odd.

Case I: When n is even

$$I_n = I_2 = \int_0^{\pi} \left(\frac{\sin 2x}{\sin x}\right) dx = 0$$

Case II: When n is odd

$$I_n = I_1 = \int_0^{\pi} \left(\frac{\sin 2x}{\sin x}\right) dx = x|_0^{\pi} = \pi$$

13. Let $\quad I_n = \int_0^{\pi/2} \left(\frac{\sin n\theta}{\sin\theta}\right)^2 d\theta$

Now,

$$I_n - I_{n-1} = \int_0^{\pi/2} \left(\frac{\sin^2 n\theta - \sin^2(n-1)\theta}{\sin^2\theta}\right) d\theta$$
$$= \int_0^{\pi/2} \left(\frac{\sin(2n-1)\sin\theta}{\sin^2\theta}\right) d\theta$$
$$= \int_0^{\pi/2} \left(\frac{\sin(2n-1)\theta}{\sin\theta}\right) d\theta$$
$$= \pi$$

14. $\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$

$= \frac{1}{(b^2 - a^2)} \int_0^\infty \left(\frac{1}{x^2 + a^2} - \frac{1}{x^2 + b^2}\right) dx$

$= \frac{1}{(b^2 - a^2)} \left(\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right)\right)_0^\infty$

$= \frac{1}{(b^2 - a^2)} \left(\frac{\pi}{2a} - \frac{\pi}{2b}\right)$

$= \frac{\pi}{2(b^2 - a^2)} \left(\frac{1}{a} - \frac{1}{b}\right)$

$= \frac{\pi}{2(b^2 - a^2)} \left(\frac{b - a}{ab}\right)$

$= \frac{\pi}{2(a + b)ab}$

15. $\int_0^1 x(1 - x)^5 dx$

$= \int_0^1 x(1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5) dx$

$= \int_0^1 (x - 5x^2 + 10x^3 - 10x^4 + 5x^5 - x^6) dx$

$= \left(\frac{x^2}{2} - \frac{5x^3}{3} + \frac{10x^4}{4} - \frac{10x^5}{5} + \frac{5x^6}{6} - \frac{x^7}{7}\right)_0^1$

$= \frac{1}{2} - \frac{5}{3} + \frac{10}{4} - \frac{10}{5} + \frac{5}{6} - \frac{1}{7}$

$= \frac{1}{50}.$

16. $\int_0^{\pi/2} \left(\frac{\sin 2x}{\sin^4 x + \cos^4 x}\right) dx$

$= \int_0^{\pi/2} \left(\frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x}\right) dx$

$= \int_0^{\pi/2} \left(\frac{2 \tan x \sec^2 x}{\tan^4 x + 1}\right) dx$

$= |\tan^{-1}(\tan x)|_0^{\pi/2}$

$= \left(\frac{\pi}{2} - 0\right)$

$= \frac{\pi}{2}$

17. $\int_1^2 \frac{dx}{x(x^3 + 1)}$

$= \int_1^2 \frac{x^2 dx}{x^3(x^3 + 1)}$

$= \frac{1}{2} \left|\log\left|\frac{x^3}{x^3 + 1}\right|\right|_1^2$

$= \frac{1}{2} \left|\log\left|\frac{16}{9}\right|\right|$

$= \log\left(\frac{4}{3}\right)$

18. We have,

$\int_2^4 \frac{x}{\sqrt{x - 2}} dx$

Let $x - 2 = t^2 \Rightarrow dx = 2t\,dt$

When $x = 0$, then $t = 0$

and $x = 4$, then $t = \sqrt{2}$

Thus, the given integral reduces to

$\int_0^{\sqrt{2}} \left(\frac{t^2 + 2}{t}\right) 2t\,dt$

$= 2\left(\frac{t^3}{3} + 2t\right)_0^{\sqrt{2}}$

$= 2\left(\frac{2\sqrt{2}}{3} + 2\sqrt{2}\right)$

$= \frac{16\sqrt{2}}{3}$

19. We have,

$\int_0^1 x(1 - x)^{2012} dx$

Let $1 - x = t \Rightarrow dx = -dt$

When $x = 0$, then $t = 1$

and $x = 1$, then $t = 0$

Thus, the given integral reduces to

$-\int_1^0 (1 - t) t^{2012} dt$

$= \int_1^0 (t^{2013} - t^{2012}) dt$

$= \left(\frac{t^{2014}}{2014} - \frac{t^{2013}}{2013}\right)_1^0$

$= \frac{1}{2014} - \frac{1}{2013}$

$= -\frac{1}{2013 \times 2014}$

20. We have,

$\int_1^2 \frac{dx}{x(x^3 + 1)}$

$= \int_1^2 \frac{dx}{x(x^3 + 1)}$

$= \int_1^2 \frac{x^2 dx}{x^3(x^3 + 1)}$

Let $x^3 = t \Rightarrow 3x^2 dx = dt$

When $x = 0$, then $t = 0$

and $x = 1$, then $t = 1$

$$= \frac{1}{3}\int_1^2 \frac{dt}{t(t+1)}$$

$$= \frac{1}{3}\log\left|\frac{t}{t+1}\right|\Big|_1^2$$

$$= \frac{1}{3}\log\left|\frac{4}{3}\right|$$

21. We have,

$$\int_0^{\pi/2} \frac{dx}{(2+\cos x)}$$

$$= \int_0^{\pi/2} \frac{dx}{2 + \dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}}$$

$$= \int_0^{\pi/2} \left(\frac{\sec^2\left(\frac{x}{2}\right)}{3+\tan^2\left(\frac{x}{2}\right)}\right)dx$$

Let $\tan\left(\frac{x}{2}\right) = t \Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2\,dt$

When $x = \pi/2$, then $t = 1$

and $x = 0$, then $t = 0$.

$$= 2\int_0^1 \left(\frac{dt}{t^2 + (\sqrt{3})^2}\right)$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\Big|_0^1$$

$$= \frac{\pi}{3\sqrt{3}}$$

22. We have,

$$\int_0^{\pi/2} \frac{dx}{(2\sin^2 x + \cos^2 x)}$$

$$= \int_0^{\pi/2} \frac{dx}{(2\sin^2 x + \cos^2 x)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{(2\tan^2 x + 1)}$$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

When $x = 0$, then $t = 0$

and $x = \frac{\pi}{2}$, then $t = \infty$

$$= \frac{1}{2}\int_0^\infty \frac{dt}{t^2 + (1/\sqrt{2})^2}$$

$$= \frac{1}{2}\times\frac{\sqrt{2}}{1}\tan^{-1}(t\sqrt{2})\Big|_0^\infty$$

$$= \frac{\pi}{2\sqrt{2}}$$

23. We have,

$$\int_0^{\pi/2} \left(\frac{\cos x}{6 - 5\sin x + \sin^2 x}\right)dx$$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

When $x = 0$, then $t = 0$

and $x = \pi/2$, then $t = 1$

$$= \int_0^1 \frac{dt}{t^2 - 5t + 6}$$

$$= \int_0^1 \frac{dt}{(t-2)(t-3)}$$

$$= \int_0^1 \left(\frac{1}{t-3} - \frac{1}{t-2}\right)dt$$

$$= \log\left|\frac{t-3}{t-2}\right|_0^1 = \log\left|\frac{4}{3}\right|$$

24. We have,

$$\int_0^{\pi/2} \left(\frac{\sin 2x}{\sin^4 x + \cos^4 x}\right)dx$$

$$= \int_0^{\pi/2} \left(\frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}\right)dx$$

$$= \int_0^{\pi/2} \left(\frac{2\tan x.\sec^2 x}{\tan^4 x + 1}\right)dx$$

Let $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$

When $x = 0$, then $t = 0$

and $x = \pi/2$, then $t = \infty$

$$= \int_0^\infty \frac{dt}{t^2+1}$$

$$= \tan^{-1}(t)\Big|_0^\infty$$

$$= \pi/2$$

25. We have, $\displaystyle\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{16 + \sin 2x}\right)dx$

Let $\sin x - \cos x = t$

$\Rightarrow$ $(\cos x + \sin x)dx = dt$

When $x = 0$, then $t = -1$

and $x = \pi/4$, then $t = 0$

Also, $t^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1 - t^2$

$$= \int_{-1}^{0} \frac{dt}{16 + 1 - t^2}$$

$$= -\int_{-1}^{0} \frac{dt}{t^2 - (\sqrt{17})^2}$$

$$= -\frac{1}{\sqrt{17}} \log\left|\frac{t - \sqrt{17}}{t + \sqrt{17}}\right|_{-1}^{0}$$

$$= \frac{1}{2\sqrt{17}} \log\left|\frac{9 - 2\sqrt{7}}{4}\right|$$

26. We have,

$$\int_0^{\pi/2} \left(\frac{1}{(\sqrt{\cos x} + \sqrt{\sin x})^4}\right) dx$$

$$= \int_0^{\pi/2} \frac{dx}{(\sqrt{\cos x} + \sqrt{\sin x})^4}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{(1 + \sqrt{\tan x})^4}$$

Let $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

When $x = 0$, then $t = 0$

and $x = \frac{\pi}{2}$, then $t = \infty$

$$= \int_0^{\infty} \frac{dt}{(t + 1)^4}$$

$$= -\frac{1}{3(t + 1)^3}\Bigg|_0^{\infty}$$

$$= \frac{1}{3}$$

27. We have,

$$\int_0^2 \frac{dx}{\sqrt{x + 1} + \sqrt{(x + 1)^3}}$$

Let $(x + 1) = t^2 \Rightarrow dx = 2t\, dt$

When $x = 0$, then $t = 0$

and $x = 2$, then $t = \sqrt{3}$

$$= \int_1^{\sqrt{3}} \frac{2t\, dt}{t + t^3}$$

$$= \int_1^{\sqrt{3}} \frac{2\, dt}{t^2 + 1}$$

$$= 2\tan^{-1}(t)\Big|_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

28. We have,

$$\int_0^{\pi/2} \left(\frac{\cos^2 x}{4\sin^2 x + \cos^2 x}\right) dx$$

$$= \int_0^{\pi/2} \frac{dx}{(4\tan^2 x + 1)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{(1 + \tan^2 x)(4\tan^2 x + 1)}$$

$$= \int_0^{\infty} \frac{dt}{(t^2 + 1)(4t^2 + 1)}$$

$$= \frac{1}{3}\int_0^{\infty} \left(\frac{4}{4t^2 + 1} - \frac{1}{t^2 + 1}\right) dt$$

$$= \frac{1}{3}\int_0^{\infty} \left(\frac{1}{t^2 + (1/2)^2} - \frac{1}{t^2 + 1}\right) dt$$

$$= \frac{1}{3}(2\tan^{-1}(2t) - \tan^{-1}(t))\Big|_0^{\infty}$$

$$= \frac{1}{3}\left(2\frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$= \frac{\pi}{6}$$

29. We have,

$$\int_1^2 \frac{dx}{(2x + 1)\sqrt{x^2 + x}}$$

Let $(2x + 1) = \frac{1}{t} \Rightarrow 2\, dx = = -\frac{1}{t^2} dt$

When $x = 1$, then $t = 1/3$

and $x = 2$, then $t = 1/5$

Also, $x^2 + x = \frac{1}{4}\left(\frac{1 - t}{t}\right)^2 + \frac{1}{2}\left(\frac{1 - t}{t}\right)$

$$= \frac{1}{4}\left(\frac{1 - t}{t}\right)\left(\frac{1 + t}{t}\right) = \frac{1}{4}\left(\frac{1 - t^2}{t^2}\right)$$

$$= -\int_{1/3}^{1/5} \frac{dt}{\sqrt{1 - t^2}}$$

$$= \cos^{-1}(t)\Big|_{1/3}^{1/5}$$

$$= \left(\cos^{-1}\left(\frac{1}{5}\right) - \cos^{-1}\left(\frac{1}{3}\right)\right)$$

30. We have,

$$\int_0^4 \frac{dx}{x + \sqrt{x}}$$

Let $x = t^2$

$\Rightarrow$ $dx = 2t\, dt$

When $x = 0$, $t = 0$

and $x = 4$, $t = 2$

$$= \int_0^2 \frac{2t\, dt}{t^2 + 1}$$

$$= \int_0^2 \frac{2\, dt}{t + 1}$$

$= 2(\log|t+1|)|_0^2$

$= 2\log 3$

31. We have,

$\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$

Put $\tan^{-1}(x) = t$

$\Rightarrow \frac{dx}{1+x^2} = dt$

$= \int_0^{\pi/4} \frac{t\tan t}{\sec t}\,dt$

$= \int_0^{\pi/4} (t\sin t)\,dt$

$= \left(t\int \sin t\,dt\right)_0^{\pi/4} + \int_0^{\pi/4} \cos t\,dt$

$= (-t\cos t + \sin t)|_0^{\pi/4}$

$= -\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$

32. We have,

$\int_0^{\frac{\pi}{2}} \frac{\cos}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$

$= \int_0^1 \frac{dt}{(t+1)(t+2)}$, Let $t = \sin\theta$

$\Rightarrow dt = \cos\theta\,d\theta$

$= \int_0^1 \left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$

$= \left(\log\left|\frac{t+1}{t+2}\right|\right)_0^1$

$= \log\left(\frac{4}{3}\right)$

33. We have,

$\int_0^{\frac{\pi}{2}} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$= \int_0^{\pi/2} \left(\frac{\tan x + 1}{\sqrt{\tan x}}\right)dx$

$= 2\int_0^{\infty} \left(\frac{t^2+1}{t^4+1}\right)dt$, Let $\tan x = t^2$

$\Rightarrow \sec^2 x\,dx = 2t\,dt$

$= 2\int_0^{\infty} \left(\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\right)dt$

$= 2\int_0^{\infty} \left(\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\right)dt$

$= \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)\right)\Big|_0^{\infty}$

$= \frac{2}{\sqrt{2}}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$

$= \pi\sqrt{2}$

34. We have,

$\int_0^{\frac{\pi}{2}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

$= \int_0^{\pi/2} \left(\frac{\sec^2 x\,dx}{4\tan^2 x + 5}\right)$

$= \int_0^{\infty} \frac{dt}{4t^2+5}$, Let $t = \tan x$

$= \frac{1}{4}\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2}$

$= \frac{1}{4} \times \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right)_0^{\infty}$

$= \frac{1}{2\sqrt{5}}\left(\frac{\pi}{2} - 0\right)$

$= \frac{\pi}{4\sqrt{5}}$

35. We have,

$\int_0^{\frac{\pi}{4}} \left(\frac{\sin x + \cos x}{9 + 16\sin 2x}\right)dx$

$= \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2}\right)dx$

$= \int_{-1}^{0} \frac{dt}{25 - 16t^2}$ Let $\sin x - \cos x = t$

$(\sin x + \cos x)\,dx = dt$

$= -\frac{1}{16}\int_{-1}^{0} \frac{dt}{t^2 - \left(\frac{5}{4}\right)^2}$

$= -\frac{1}{16} \times \frac{1}{2\left(\frac{5}{4}\right)}\log\left|\frac{5t-4}{5t+4}\right|_{-1}^{0}$

$= \frac{1}{40}\log 9$

36. We have,

$\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2x}{\sin^4 x + \cos^4 x}\right)dx$

$= \int_0^{\pi/2} \left(\frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}\right)dx$

$$= \int_0^{\pi/2} \left(\frac{2 \tan x \sec^2 x}{\tan^4 x + 1} \right) dx$$

$$= \int_0^{\infty} \frac{dt}{t^2 + 1}, \qquad \text{Let } \tan^2 x = t$$

$$= \tan^{-1}(t)\Big|_0^{\infty}$$

$$= \frac{\pi}{2}$$

37. We have, $\int_0^1 \left(\sqrt{\frac{1 - x}{1 + x}} \right) dx.$

$$= \int_0^1 \left(\frac{1 - x}{\sqrt{1 + x^2}} \right) dx$$

$$= \int_0^1 \left(\frac{dx}{\sqrt{1 - x^2}} \right) - \int_0^1 \left(\frac{x\,dx}{1 - x^2} \right)$$

$$= \int_0^1 \left(\frac{dx}{\sqrt{1 - x^2}} \right) + \int_0^1 dt, \quad \text{Let } (1 - x^2) = t^2$$

$$= \sin^{-1}(x)\Big|_0^1 + (t)_0^1$$

$$= \frac{\pi}{2} - 1$$

38. We have,

$$\int_{-1}^{4} (2x - 3)\,dx$$

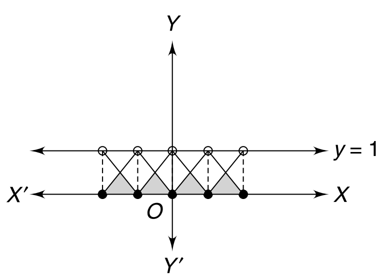
$$= (x^2 - 3x)_{-1}^{4}$$

$$= (16 - 12) - (1 + 3)$$

$$= 0$$

39. We have,

$$\int_{-2}^{2} f(x)\,dx$$

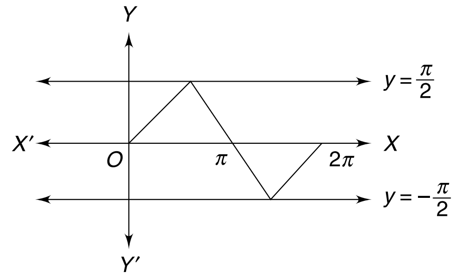


Area of $\Delta = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

Hence, the required area $= 4 \times \frac{1}{2} = 2$ sq.u.

40. We have,

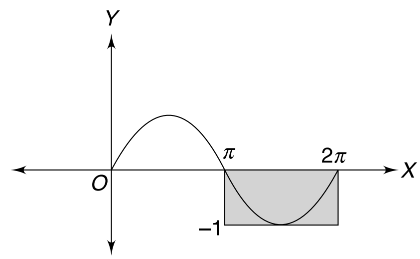
$$\int_0^{2\pi} f(x)\,dx$$



Hence, the required area $= 2 \times \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi^2}{2}$

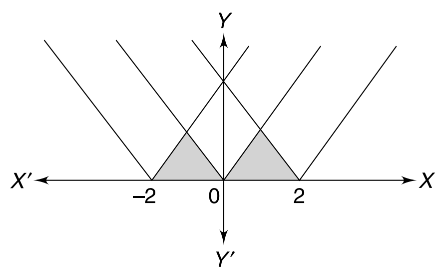
41. We have,

$$\int_0^{2\pi} [\sin x]\,dx$$



Hence, the required area $= \pi \times -1 = -\pi$

42. We have,

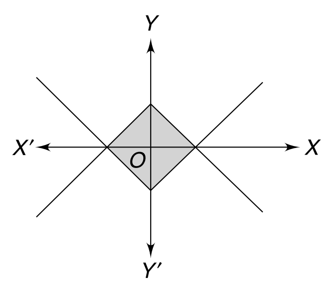


$$\int_{-2}^{2} f(x)\,dx$$

Hence, the required area $= 2 \times \frac{1}{2} \times 2 \times 1 = 2$ sq.u.

43. We have,

$$\int_{-1}^{1} g(x)\,dx$$



Hence, the required area

$$= 4 \times \Delta$$

$$= 4 \times \frac{1}{2} \times 1 \times 1$$

$$= 2 \text{ sq.u.}$$

44. Here, $\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow$ $\frac{1}{2} < \frac{2x}{\pi} < 1$

$\Rightarrow$ $\left[\frac{2x}{\pi}\right] = 0$

Thus, $\int_{\pi/4}^{\pi/2} \left[\sin x + \left[\frac{2x}{\pi}\right]\right] dx = 0$

45. Here, $0 < x < 1$

$\Rightarrow$ $[x] = 0$

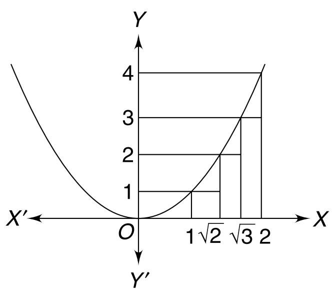
Thus,

$$\int_0^1 (x d(x - [x]))$$

$$= \int_0^1 x\, dx$$

$$= \left(\frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{2}$$

46.

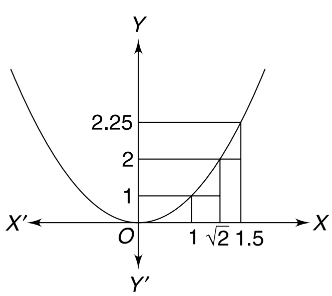


Hence, the required area

$$= 1 \cdot (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(\sqrt{3} - 2)$$

$$= (5 - \sqrt{2} - \sqrt{3}) \text{ sq.u.}$$

47.

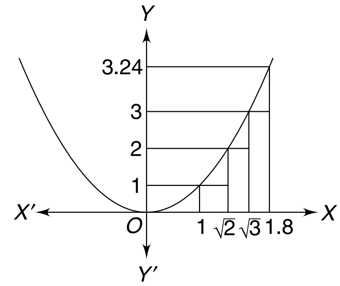


Hence, the required area

$$= 1 \cdot (\sqrt{2} - 1) + 2(1.5 - \sqrt{2})$$

$$= (2 - \sqrt{2}) \text{ sq.u.}$$

48.

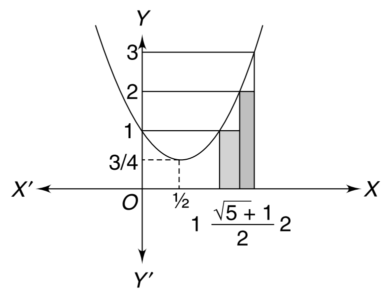


Hence, the required area

$$= 1 \cdot (\sqrt{2} - 1) + 2 \cdot (\sqrt{3} - \sqrt{2}) + 3(1.8 - \sqrt{3})$$

$$= (4.4 - \sqrt{2} - \sqrt{3}) \text{ sq. u.}$$

49.



Hence, the required area

$$= 0 + 1 \cdot \left(\frac{\sqrt{5} + 1}{2} - 1\right) + 2 \cdot \left(2 - \frac{\sqrt{5} + 1}{2}\right)$$

$$= 3 - \left(\frac{\sqrt{5} + 1}{2}\right)$$

$$= \left|\frac{5 - \sqrt{5}}{2}\right|$$

50. Now, $2x^2 - 3 = 0 \Rightarrow x = \sqrt{\frac{3}{2}}$

$$2x^2 - 3 = 1 \Rightarrow x = \sqrt{2}$$

$$2x^2 - 3 = 2 \Rightarrow x = \sqrt{\frac{5}{2}}$$

$$2x^2 - 3 = 3 \Rightarrow x = \sqrt{3}$$

$$2x^2 - 3 = 4 \Rightarrow x = \sqrt{\frac{7}{2}}$$

and $2x^2 - 3 = 5 \Rightarrow x = 2$

Hence, the required area

$$= (-1)\left(\sqrt{\frac{3}{2}} - 1\right) + 0 + 1 \cdot \left(\sqrt{\frac{5}{2}} - \sqrt{2}\right)$$

$$+ 2 \cdot \left(\sqrt{3} - \sqrt{\frac{5}{2}}\right) + 3 \cdot \left(\sqrt{\frac{7}{2}} - \sqrt{3}\right)$$

$$+ 4\cdot\left(2 - \sqrt{\frac{7}{2}}\right)$$

$$= \left(9 - \left(\sqrt{\frac{3}{2}} + \sqrt{2} + \sqrt{\frac{5}{2}} + \sqrt{3} + \sqrt{\frac{7}{2}}\right)\right) \text{ sq.u.}$$

51. Now, $x^3 - 1 = 0 \Rightarrow x = 1$

$x^3 - 1 = 1 \Rightarrow x = (2)^{1/3}$

$x^3 - 1 = 2 \Rightarrow x = (3)^{1/3}$

$x^3 - 1 = 3 \Rightarrow x = (4)^{1/3}$

$x^3 - 1 = 4 \Rightarrow x = (5)^{1/3}$

$x^3 - 1 = 5 \Rightarrow x = (6)^{1/3}$

$x^3 - 1 = 6 \Rightarrow x = (7)^{1/3}$

and $x^3 - 1 = 7 \Rightarrow x = 2$

Hence, the required area

$$= 0 + 1(3^{1/3} - 2^{1/3}) + 2\cdot(4^{1/3} - 3^{1/3}) + 3\cdot(5^{1/3} - 4^{1/3}) + 4\cdot(6^{1/3} - 5^{1/3}) + 5\cdot(6^{1/3} - 7^{1/3}) + 6\cdot(2^{1/3} - 7^{1/3})$$

$$= (12 - (2^{1/3} + 3^{1/3} + 4^{1/3} + 5^{1/3} + 6^{1/3} + 7^{1/3}))$$

52. We know that $\int_0^n [x]\,dx$

$$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \ldots + \int_{n-1}^n [x]\,dx$$

$$= \int_0^1 (0)\,dx + \int_1^2 (1)\,dx + \int_2^3 (2)\,dx + \ldots + \int_{n-1}^n (n-1)\,dx$$

$$= 0 + 1(2-1) + 2(3-2) + \ldots + (n-1)n-(n-1))$$

$$= 1 + 2 + 3 + 4 + \ldots + (n-1)$$

$$= \frac{n(n-1)}{2}$$

53. $\int_0^{10} [x]\,dx$

$$= \frac{10(10-1)}{2}$$

$$= 45$$

54. $\int_0^{10} [x+2]\,dx$

$$= \int_0^{10} [x]\,dx + \int_0^{10} 2\,dx$$

$$= \frac{10(10-1)}{2} + 2(10-0)$$

$$= 45 + 20$$

$$= 65.$$

55. $\int_0^2 [2x+3]\,dx$

$$= \int_0^2 [2x+3]\,dx$$

$$= \int_0^2 [2x]\,dx + \int_0^2 (3)\,dx$$

$$= \int_0^{1/2} [2x]\,dx + \int_{1/2}^1 [2x]\,dx + \int_1^{3/2} [2x]\,dx + \int_{3/2}^2 [2x]\,dx + \int_0^2 [3x]\,dx$$

$$= \int_0^{1/2} (0)\,dx + \int_{1/2}^1 (1)\,dx + \int_1^{3/2} (2)\,dx + \int_{3/2}^2 (3)\,dx + 6$$

$$= 0 + \left(1 - \frac{1}{2}\right) + 2\left(\frac{3}{2} - 1\right) + 3\left(2 - \frac{3}{2}\right) + 6$$

$$= \frac{1}{2} + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) + 6$$

$$= (1 + 2 + 6)$$

$$= 9$$

56. $\int_0^2 [3x-2]\,dx$

$$= \int_0^2 [3x]\,dx - \int_0^2 (2)\,dx$$

$$= \frac{1}{3}(1 + 2 + 3 + 4 + 5) - 4$$

$$= \left(\frac{15}{3} - 4\right)$$

$$= (5 - 4)$$

$$= 1$$

57. $\int_0^n \{x\}\,dx$

$$= \int_0^n \{x\}\,dx$$

$$= \int_0^n (x - [x])\,dx$$

$$= \int_0^n (x)\,dx - \int_0^n [x]\,dx$$

$$= \left(\frac{n^2}{2} - \frac{n(n-1)}{2}\right)$$

$$= \left(\frac{n}{2}\right)$$

58. $\int_0^2 (x[x])\,dx =$

$$\int_0^1 x[x]\,dx + \int_1^2 x[x]\,dx$$

$$= \int_0^1 x\cdot 0\,dx + \int_1^2 x\cdot 1\,dx$$

$$= \int_1^2 x\,dx$$

$$= \left(\frac{x^2}{2}\right)_1^2$$

$$= \left(2 - \frac{1}{2}\right) = \frac{3}{2}$$

59. $\int_1^{e^6} \left[\frac{\log x}{3}\right] dx$

$$= \int_1^{e^3} \left[\frac{\log x}{3}\right] dx + \int_{e^3}^{e^3} \left[\frac{\log x}{3}\right] dx$$

$$= \int_1^{e^3} 0\,.dx + \int_{e^3}^{e^3} 1\,.dx$$

$$= (e^6 - e^3).$$

60. $\int_0^{\pi} [2\sin x]\,dx$

$$= \int_0^{\pi/6} [2\sin x]\,dx + \int_{\pi/6}^{5\pi/6} [2\sin x]\,dx + \int_{5\pi/6}^{\pi} [2\sin x]\,dx$$

$$= \int_0^{\pi/6} (0)\,dx + \int_{\pi/6}^{5\pi/6} (1)\,dx + \int_{5\pi/6}^{\pi} (0)\,dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{6}\right).1$$

$$= \frac{2\pi}{3}$$

61. $\int_0^{2\pi} [2\sin x]\,dx$

$$= \int_0^{\pi/6} (0)\,dx + \int_{\pi/6}^{5\pi/6} (1)\,dx + \int_{5\pi/6}^{\pi} (0)\,dx$$

$$+ \int_{\pi}^{7\pi/6} (-1)\,dx + \int_{7\pi/6}^{11\pi/6} (-2)\,dx + \int_{11\pi/6}^{2\pi} (-1)\,dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{6}\right) + \left(\frac{7\pi}{6} - \pi\right)(-1)$$

$$+ \left(\frac{11\pi}{6} - \frac{7\pi}{6}\right)(-2) + \left(2\pi - \frac{11\pi}{6}\right)(-1)$$

$$= \frac{2\pi}{3} - \frac{\pi}{6} - \frac{8\pi}{6} - \frac{\pi}{6}$$

$$= \frac{2\pi}{3} - \frac{10\pi}{6}$$

$$= \frac{4\pi}{6} - \frac{10\pi}{6}$$

$$= -\frac{6\pi}{6} = -\pi$$

62. $\int_0^{\pi/3} [\sqrt{3}\tan x]\,dx$

$$= \int_0^{\pi/6} [\sqrt{3}\tan x]\,dx + \int_{\pi/6}^{\tan^{-1}(2\sqrt{3})} [\sqrt{3}\tan x]\,dx$$

$$+ \int_{\tan^{-1}(2\sqrt{3})}^{\pi/3} [\sqrt{3}\tan x]\,dx$$

$$= \int_0^{\pi/6} (0)\,dx + \int_{\pi/6}^{\tan^{-1}(2\sqrt{3})} (1)\,dx + \int_{\tan^{-1}(2\sqrt{3})}^{\pi/3} (2)\,dx$$

$$= \left(\frac{2\pi}{3} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - 1\right)$$

63. $\int_1^{1.5} (\text{sgn}[x])\,dx$

$$= \int_1^{3/2} (1)\,dx$$

$$= \left(\frac{3}{2} - 1\right)\cdot 1$$

$$= \frac{1}{2}$$

64. $\int_0^{102} [\tan^{-1} x]\,dx$

$$= \int_0^{\tan(1)} [\tan^{-1} x]\,dx + \int_{\tan(1)}^{102} [\tan^{-1} x]\,dx$$

$$= \int_0^{\tan(1)} (0)\,dx + \int_{\tan(1)}^{102} (1)\,dx$$

$$= (102 - \tan(1))$$

65. $\int_0^{2n\pi} [\sin x + \cos x]\,dx$

Since it is a periodic function with period 2π.

$$= n\int_0^{2\pi} [\sin x + \cos x]\,dx$$

$$= n\int_0^{2\pi} \left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right] dx$$

$$= n\int_{\pi/4}^{9\pi/4} [\sqrt{2}\sin x]\,dx$$

$$= n\left[\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\cdot 1 + \left(\frac{5\pi}{4} - \pi\right)(-1) + \left(\frac{7\pi}{4} - \frac{5\pi}{4}\right)(-2) + \left(2\pi - \frac{3\pi}{4}\right)\cdot(-1)\right]$$

$$= n\left(\frac{2\pi}{4} - \frac{\pi}{4} - \frac{\pi}{4} - \frac{4\pi}{4}\right)$$

$$= -n\pi$$

66. We have,

$$\int_0^{5\pi/12} [\tan^{-1} x]\,dx$$

Let $I = \int_0^{5\pi/12} [\tan x]\,dx$

$$= \int_0^1 [\tan^{-1} x]\,dx + \int_1^2 [\tan^{-1} x]\,dx + \int_2^3 [\tan^{-1} x]\,dx + \int_3^{5\pi/12} [\tan^{-1} x]\,dx$$

$$= 0 + (\tan^{-1} 2 - \tan^{-1} 1)\cdot 1 + (\tan^{-1} 3 - \tan^{-1} 2)\cdot 2 + \left(\frac{5\pi}{12} - \tan^{-1} 3\right)\cdot 3$$

$$= \frac{5\pi}{4} - \frac{\pi}{4} - (\tan^{-1} 3 + \tan^{-1} 2)$$

67. $\int_{-2}^{1} \left[\left(1 + \cos\left(\frac{\pi x}{2}\right)\right)\right] dx$

$$= \int_{-2}^{-1} (1)\,dx + \int_{-1}^{0} (0)\,dx + \int_0^1 (1)\,dx$$

$$= 1 + 0 + 1$$

$$= 2$$

68. We have,

$$\int_0^3 \left([x] + \left[x + \frac{1}{2}\right] + \left[x + \frac{2}{3}\right]\right) dx$$

Let $I = \int_0^3 \left([x] + \left[x + \frac{1}{2}\right] + \left[x + \frac{2}{3}\right]\right) dx$

$$= \int_0^{1/3} 0 . dx + \int_{1/3}^{2/3} 1\cdot dx + \int_{2/3}^{1} 2\cdot dx + \int_1^{4/3} 3\cdot dx + \int_{4/3}^{5/3} 4\cdot dx + \int_{5/3}^{2} 5\cdot dx + \int_2^{7/3} 6\cdot dx + \int_{7/3}^{8/3} 7\cdot dx + \int_{8/3}^{3} 8 . dx$$

$$= \frac{1}{3}(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$$

$$= \frac{1}{3} \times 8 \times \frac{9}{2}$$

$$= 12$$

69. $\int_0^{10\pi} ([\sec^{-1} x] + [\tan^{-1} x])dx$

$$= \int_1^{\sec 1} ([\sec^{-1} x] + [\cot^{-1} x])\,dx + \int_{\sec 1}^{10\pi} ([\sec^{-1} x] + [\cot^{-1} x])\,dx$$

$$= \int_1^{\sec 1} (0 + 0)\,dx + \int_{\sec 1}^{10\pi} (1 + 0)\,dx$$

$$= (10\pi - \sec 1)$$

70. $\int_0^{\pi/4} [\sin x + [\cos x + [\tan x + [\text{sex}\, x]]]]\,dx$

Let $I = \int_0^{\pi/4} [\sin x + [\cos x + \tan x + [\text{sex}\, x]]]\,dx$

Since $0 < x < \sqrt{2}$

$\Rightarrow$ $1 < \sec x < \sqrt{2}$

$\Rightarrow$ $[\sec x] = 1$

$\Rightarrow$ $[\tan x + [\sec x]] = [\tan x + 1]$

$= [\tan x] + 1$

$= 0 + 1 = 1$

$\Rightarrow$ $[\cos x + [\tan x + [\sec x]]]$

$= [\cos x + 1] = [\cos x] + 1 = 0 + 1 = 1$

$\Rightarrow$ $[\sin x + [\cos x + [\tan x + [\sec x]]]]$

$= [\sin x + 1] = [\sin x] + 1 = 0 + 1 = 1$

Thus, $I = \int_0^{\pi/4} [\sin x + [\cos x + \tan x + [\sec x]]]\,dx$

$$= \int_0^{\pi/4} 1\cdot dx$$

$$= \frac{\pi}{4}$$

71. We have,

$$\int_{\pi/4}^{\pi/2} [\sin x + \left[\frac{2x}{\pi}\right] dx$$

Here, $\frac{\pi}{4} < x < \frac{\pi}{2}$

$\Rightarrow$ $\frac{\pi}{2} < \pi\, 2x < \pi$

$\Rightarrow$ $\frac{1}{2} < \frac{2x}{\pi} < 1$

$\Rightarrow$ $\left[\frac{2x}{\pi}\right] = 0$

$\therefore$ The given integral reduces to

$$\int_{\pi/4}^{\pi/2} \left[\sin x + \left[\frac{2x}{\pi}\right]\right] dx$$

$$= \int_{\pi/4}^{\pi/2} [\sin x + 0]\, dx$$

$= 0$

72. Here, $a = 0,\ b = 3$

$\therefore$ $nh = b - a = 3 - 0 = 3,$

$f(x) = x$

$f(a + rh) = f(rh) = rh$

Now, $$\int_1^3 x\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(rh)$$

$$= \lim_{h\to 0} h\cdot\sum_{r=1}^{n} rh$$

$$= \lim_{h\to 0} h^2 \sum_{r=1}^{n} r$$

$$= \lim_{h\to 0} h^2\left(\frac{n(n+1)}{2}\right)$$

$$= \lim_{h\to 0}\left(\frac{nh(nh+h)}{2}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(3+h)}{2}\right)$$

$= 9/2$

73. Here, $a = 0,\ b = 2$

$nh = b - a = 2,\ f(x) = x^2$

Also, $f(a + rh) = f(rh) = (rh)^2$

Now, $$\int_0^2 x^2 dx$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} f(rh)$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} (rh)^2$$

$$= \lim_{h\to 0} h^3 \sum_{r=0}^{n} (r)^2$$

$$= \lim_{h\to 0} h^3\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{h\to 0} h^3\left(\frac{nh(nh+h)(2nh+1)}{6}\right)$$

$$= \lim_{h\to 0} h^3\left(\frac{2(2+h)(4+h)}{6}\right)$$

$$= \frac{8}{3}$$

74. Here, $a = 1,\ b = 3$

$nh = b - a = 2$

$f(x) = (x^2 + x + 2)$

$f(a + rh) = f(1 + rh)$

$(1 + rh)^2 + (1 + rh) + 2 = (rh)^2 + 3(rh) + 4$

Now, $$\int_1^3 (x^2 + x + 2)\,dx$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} f(a + rh)$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} ((rh)^2 + 3(rh) + 4)$$

$$= \lim_{h\to 0} h\left(\frac{h^2(n(n+1)(2n+1)}{6}\right)$$

$$+ 3h\left(\frac{n(n+1)}{2+4h}\right)$$

$$= \lim_{h\to 0}\left(\frac{nh(nh+h)(2nh+h)}{6}\right.$$

$$\left(+ 3\left(\frac{nh(nh+h)}{2}\right) + 4nh\right)$$

$$= \lim_{h\to 0}\left(\frac{2(2+h(4+h)}{6} + \left(\frac{2(2+h)}{2}\right) + 8\right)$$

$$= \frac{8}{3} + 6 + 8$$

$= 50/3$

75. Here $a = 0,\ b = 1$

$nh = b - a = 1 - 0 = 1.$

Also $f(x) = e^{mx},\ f(a + rh) = f(rh) = e^{mrh}$

Now, $$\int_0^1 e^{mx} dx$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} f(a + rh)$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} e^{mrh}$$

$$= \lim_{h\to 0} h(1 + e^{2mh} + \ldots + e^{nmh})$$

$$= \lim_{h\to 0} h\left(\frac{e^{mnh} - 1}{e^{mh} - 1}\right)$$

$$= \lim_{h\to 0} h\left(\frac{em - 1}{e^{mh} - 1}\right)$$

$$= \lim_{h\to 0}\left(\frac{mh}{e^{mh} - 1}\right) \times \left(\frac{e^m - 1}{m}\right)$$

$$= \left(\frac{e^m - 1}{m}\right)$$

76. Here, $a = 0,\ b = \frac{\pi}{2}$

$nh = \pi/2 - 0 = \pi/2,\ f(x) = \sin x$

Also $f(a + rh) = f(rh) = \sin(rh)$

Now, $\int_0^{\pi/2} \sin x\, dx$

$= \lim_{h\to 0} h \sum_{r=1}^{n} f(rh)$

$= \lim_{h\to 0} h \sum_{r=1}^{n} \sin(rh)$

$= \lim_{h\to 0} (\sin h + \sin 2h + \sin 3h + \ldots + \sin nh)$

$= \lim_{h\to 0} h\left(\frac{\sin(rh/2)}{\sin(h/2)} \times \frac{\sin(n+1)h}{2}\right)$

$= \lim_{h\to 0} h\left[\frac{\sin(\pi/4)}{\sin(h/2)} \times \sin\left(\frac{\frac{\pi}{2} + h}{2}\right)\right]$

$= \lim_{h\to 0} h\left[\frac{2}{\sqrt{2}} \times \frac{h/2}{\sin(h/2)} \times \sin\left(\frac{\pi}{4} + \frac{h}{2}\right)\right]$

$= \sqrt{2} \times \sin\left(\frac{\pi}{4}\right)$

$= 1$

77. Do yourself

87. $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}\right]$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{n+r}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{n}{n+r}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{1}{1 + (r/n)}\right)$

$= \int_0^1 \frac{dx}{1+x}$

$= \log|1 + x|\Big|_0^1$

$= \log 2$

88. $\lim_{n\to\infty}\left[\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots + \frac{n}{n^2+(n-1)^2}\right]$

$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{n}{n^2+r^2}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \left(\frac{n^2}{n^2+r^2}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \left(\frac{1}{1+(r/n^2)}\right)$

$= \int_0^1 \frac{dx}{1+x^2}$

$= \tan^{-1}(x)\Big|_0^1$

$= \frac{\pi}{4}$

89. $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(2n-1)^2}\right]$

$= \lim_{n\to\infty} \sum_{r=0}^{2n-1} \left(\frac{n}{n^2+r^2}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n-1} \left(\frac{n^2}{n^2+r^2}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n-1} \left(\frac{1}{1+(r/n)^2}\right)$

$= \int_0^2 \frac{dx}{1+x^2}$

$= \tan^{-1}(x)\Big|_0^2$

$= \tan^{-1} 2$

90. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \ldots + \sqrt{1+\frac{n}{n}}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{1} \left(\sqrt{1+\frac{r}{n}}\right)$

$= \int_0^1 \sqrt{1+x}\, dx$

$= \frac{2}{3}(1+x)^{3/2}\Big|_0^1$

$= \frac{2}{3}[(2)^{3/2} - 1]$

91. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \frac{1}{\sqrt{4n^2-9}} + \ldots + \frac{1}{\sqrt{3n^2}}\right)$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{\sqrt{4n^2-r^2}}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{n}{\sqrt{4n^2-r^2}}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{1}{\sqrt{4-(r/n)^2}}\right)$

$$= \int_0^1 \frac{dx}{\sqrt{4-x^2}}$$

$$= \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1$$

$$= \frac{\pi}{6}$$

92. $\lim_{n\to\infty}\left(\frac{1^m + 2^m + 3^m + \ldots + n^m}{n^{m+1}}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1^m + 2^m + 3^m + \ldots + n^m}{n^m}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^m + \left(\frac{2}{n}\right)^m + \left(\frac{3}{n}\right)^m + \ldots + \left(\frac{n}{n}\right)^m\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^m$$

$$= \int_0^1 x^m dx = \frac{x^{m+1}}{m+1}\Big|_0^1 = \frac{1}{m+1}$$

93. $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \ldots + \frac{1}{n}\right)$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \ldots + \frac{n+n}{n^2+n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n+r}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{n^2+nr}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1+(r/n)}{1+(r/n)^2}\right)$$

$$= \int_0^1\left(\frac{1+x}{1+x^2}\right)dx$$

$$= \tan^{-1}(x) + \frac{1}{2}\log|x^2+1|\Big|_0^1$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\log 2\right)$$

94. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}\right)$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{r}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{n}{\sqrt{n}\sqrt{r}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{r/n}}\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{x}}$$

$$= 2\sqrt{x}\,|_0^1$$

$$= 2(1-0) = 2$$

95. Let $A = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$

$$\Rightarrow \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{1.2.3\ldots n}{n^n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\log\left(\left(\frac{1}{n}\right)\cdot\left(\frac{2}{n}\right)\cdot\left(\frac{3}{n}\right)\ldots\left(\frac{n}{n}\right)\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \ldots + \log\left(\frac{n}{n}\right)\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\log\left(\frac{r}{n}\right)\right)$$

$$= \int_0^1 \log x\,dx$$

$$= (x\log x - x)_0^1$$

$$= (0-1) - \lim_{x\to 0^+}(x\log x) + 0$$

$$= -1 + 0 = -1$$

$$\Rightarrow \quad \log L = -1$$

$$\Rightarrow \quad L = e^{-1} = \frac{1}{e}$$

96. Let $L = \lim_{n\to\infty}\left[1 + \frac{1}{n^2}\left(1 + \frac{2^2}{n^2}\right)\left(1 + \frac{3^2}{n^2}\right)\ldots\left(1 + \frac{n^2}{n^2}\right)\right]^{1/n}$

$$\Rightarrow \quad \log A = \lim_{n\to\infty}\left(\frac{1}{n}\log\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\ldots\right.$$

$$\left.\log\left(1 + \frac{n^2}{n^2}\right) + \log\left(1 + \frac{n^2}{n^2}\right)\right)$$

$$\Rightarrow \quad \log A = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1 + \frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1 + \left(\frac{r}{n}\right)^2\right)$$

$$= \int_0^1 \log(1+x^2)\,dx$$

$$= \left(x\log|1+x^2| - 2x + 2\tan^{-1}x\right)_0^1$$

$$= \log 2 + \frac{\pi - 4}{2}$$

$$\Rightarrow \quad \log A - \log 2 = \frac{\pi - 4}{2}$$

$$\Rightarrow \quad \log\left(\frac{A}{2}\right) = \pi - \frac{4}{2}$$

$$\Rightarrow \quad A = 2e^{(\pi-4)/2}$$

97. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{4n}\right)$

$=\lim_{n\to\infty}\left(\frac{1}{n+0}+\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+3n}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{3n}\left(\frac{1}{n+r}\right)$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3n}\left(\frac{n}{n+r}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{3n}\left(\frac{1}{1+(r/n)}\right)$

$=\int_0^3\frac{dx}{1+x}$

$=\log(1+x)|_0^3$

$=\log 4$

$=2\log 2$

98. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{1}{8n}\right)$

$=\lim_{n\to\infty}\left(\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{(n+n)^3}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{n^2}{(n+r)^3}\right)$

$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{n}\left(\frac{n^3}{(n+r)^3}\right)\right]$

$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{n}\left(\frac{1}{(1+(r/n))^3}\right)\right]$

$=\int_0^1\left(\frac{dx}{(1+x)^3}\right)$

$=\left(-\frac{1}{2(x+1)^2}\right)_0^1$

$=\left|\frac{1}{2}-\frac{1}{8}\right|=\frac{3}{8}$

99. $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+\dots+\frac{3}{5n}\right)$

$=\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+\dots+\frac{n+2n}{n^2+(2n)^2}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{2n}\left(\frac{n+r}{n^2+r^2}\right)$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\left(\frac{n^2+rn}{n^2+r^2}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{1+\left(\frac{r}{h}\right)}{1+\left(\frac{r}{h}\right)^2}\right)$

$=\int_0^2\left(\frac{1+x}{1+x^2}\right)dx$

$=\left(\tan^{-1}x+\frac{1}{2}\log|1+x^2|\right)_0^2$

$=\left|\tan^{-1}|2|+\frac{1}{2}\log|5|\right|$

100. $\lim_{n\to\infty}\left(\frac{n^2}{(n^2+1)^{3/2}}+\frac{n^2}{(n^2+2^2)^{3/2}}+\dots+\frac{n^2}{(n^2+(n-1)^2)^{3/2}}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\left(\frac{n^2}{(n^2+r^2)^{3/2}}\right)$

$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{n-1}\left(\frac{n^3}{(n^2+r^2)^{3/2}}\right)\right]$

$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{n-1}\left(\frac{1}{\left(1+\left(\frac{r}{n}\right)^2\right)^{3/2}}\right)\right]$

$=\int_0^1\frac{dx}{(1+x^2)^{3/2}}$

$=\int_0^{\pi/4}\left(\frac{\sec^2\theta}{\sec^3\theta}\right)d\theta$

$=\int_0^{\pi/4}(\cos\theta)\,d\theta$

$=(\sin\theta)|_0^{\pi/4}$

$=\frac{1}{\sqrt{2}}$

101. $\lim_{n\to\infty}\left(\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\dots+\frac{1}{2n}\right)$

$=\lim_{n\to\infty}\left(\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\dots+\frac{n^2}{n^3+n^3}\right)$

$=\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{r^2}{n^3+r^3}\right)$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\left(\frac{nr^2}{n^3+r^3}\right)$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\left(\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}\right)$

$$= \int_0^1 \frac{x^2}{1 + x^3}\,dx$$

$$= \left(\frac{1}{3}\log|1 + x^3|\right)_0^1$$

$$= \frac{1}{3}\log|2|$$

102. $$\lim_{n\to\infty}\left(\frac{2^k + 4^k + 6^k + \ldots + (2n)^k}{n^{k+1}}\right),\ k \neq -1$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{2^k + 4^k + 6^k + \ldots + (2n)^k}{n^k}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\left(\frac{2}{n}\right)^k + \left(\frac{4}{n}\right)^k + \left(\frac{6}{n}\right)^k + \ldots + \left(\frac{2n}{n}\right)^k\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{2r}{n}\right)^k$$

$$= \int_0^1 2^k(x)^k\,dx$$

$$= 2k\left(\frac{x^{k+1}}{k+1}\right)_0^1$$

$$= \left(\frac{2^k}{k+1}\right)$$

103. $$\lim_{n\to\infty}\left(\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + 2\sqrt{n}}{n\sqrt{n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \ldots + \sqrt{\frac{4n}{n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{4n}\left(\sqrt{\frac{r}{n}}\right)$$

$$= \int_0^4 \sqrt{x}\,dx$$

$$= \left(\frac{2}{3}x^{3/2}\right)_0^4$$

$$= \frac{2}{3}\times 4^{3/2} = \frac{2}{3}\times 8 = \frac{16}{3}$$

104. $$\lim_{n\to\infty}\left(\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \ldots + \frac{1}{na+n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{na} + \frac{n}{na+1} + \frac{n}{na+2} + \ldots + \frac{n}{na+n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\left(\frac{n}{na+r}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\left(\frac{1}{a+(r/n)}\right)$$

$$= \int_0^1 \frac{dx}{a+x}$$

$$= (\log|a + x|)_0^1$$

$$= \log\left(\frac{a+1}{a}\right)$$

105. $$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n}\,\sqrt{2n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt{n}}{n\sqrt{n+1}} + \frac{\sqrt{n}}{n\sqrt{n+2}} + \frac{\sqrt{n}}{n\sqrt{n+3}} + \ldots + \frac{\sqrt{n}}{n\sqrt{n+n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\sqrt{\frac{n}{n+r}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\sqrt{\frac{1}{1+(r/n)}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{1+(r/n)}}\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{1+x}}$$

$$= \left(\frac{2}{3}(1 + x)^{3/2}\right)_0^1$$

$$= \left|\frac{2}{3}(2^{3/2} - 1)\right|$$

106. $$\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{1}{n+r}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{n}{n+r}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{1}{1+(r/n)}\right)$$

$$= \int_0^2 \frac{dx}{1+x}$$

$$= (\log|1 + x|)_0^2 = \log 3$$

107. $$\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{1}{\sqrt{n^2 - r^2}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{n}{\sqrt{n^2 - r^2}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{1}{\sqrt{1 - (r/n)^2}}\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$$

$$= [\sin^{-1}(x)]_0^1$$

$$= \frac{\pi}{2}$$

108. $$\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{n^2 + 3rn}}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{\sqrt{n^2 + 3rn}}\right)$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \sum_{r=1}^{n} \left(\frac{n}{\sqrt{n^2 + 3rn}}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \sum_{r=1}^{n} \left(\frac{1}{\sqrt{1 + 3(r/n)}}\right)\right]$$

$$= \int_0^1 \frac{dx}{\sqrt{1 + 3x}}$$

$$= \left(\frac{2}{9}(1 + 3x)^{3/2}\right)_0^1$$

$$= \frac{2}{9}(8 - 1) = \frac{14}{9}$$

109. $\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r} + 4\sqrt{n})^2}\right)$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{r=1}^{n} \left(\frac{n\sqrt{n}}{\sqrt{r}(\sqrt{r} + 4\sqrt{n})^2}\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{r=1}^{n} \left(\frac{1}{\sqrt{(r/n)}(\sqrt{(r/n)} + 4)^2}\right)\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{x}(\sqrt{x} + 4)^2}$$

$$= \int_4^5 \left(\frac{2\,dt}{t^2}\right)$$

$$= -\left(\frac{2}{t}\right)_4^5$$

$$= -2\left(\frac{1}{5} - \frac{1}{4}\right) = -2 \times -\frac{1}{20} = \frac{1}{10}$$

110. $\lim_{n\to\infty} \sum_{r=1}^{10n} \left(\frac{1}{\sqrt{r}\sqrt{n}}\right)$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{10n} \left(\frac{n}{\sqrt{r}\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{10n} \left(\frac{\sqrt{n}}{\sqrt{r}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{10n} \left(\frac{1}{\sqrt{(r/n)}}\right)$$

$$= \int_0^{10} \frac{dx}{\sqrt{x}}$$

$$= \left(\frac{2}{3}(x^{3/2})\right)_0^{10} = \frac{2}{3}(10)^{3/2}$$

111. $\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{n}{(r + n)\sqrt{r^2 + 2nr}}\right)$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{n^2}{(r + n)\sqrt{r^2 + 2nr}}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{(1 + (r/n))\sqrt{(r/n)^2 + 2(r/n)}}\right)$$

$$= \int_0^1 \frac{dx}{(1 + x)\sqrt{x^2 + 2x}}$$

$$= \int_0^1 \frac{dx}{(1 + x)\sqrt{(x + 1)^2 - 1}}$$

$$= \sec^{-1}(2) - \sec^{-1}(1)$$

$$= \left(\frac{\pi}{3} - 0\right) = \frac{\pi}{3}$$

112. Let $I = \int_0^4 |x - 2|\,dx$

$$= \int_0^2 |x - 2|\,dx + \int_2^4 |x - 2|\,dx$$

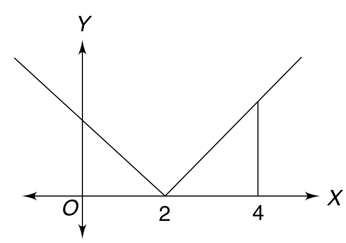
$$= -\int_0^2 (x - 2)\,dx + \int_2^4 (x - 2)\,dx$$

$$= \left(2x - \frac{x^2}{2}\right)\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^4$$

$$= (4 - 2) + (8 - 8) - (2 - 4)$$

$$= 2 + 2 = 4$$

Alternate Method

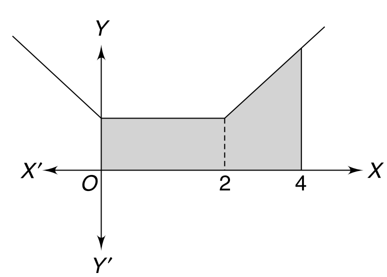


$$I = \int_0^4 |x - 2|\,dx$$

$$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 2 \times 2$$

$$= 4 \text{ sq.u.}$$

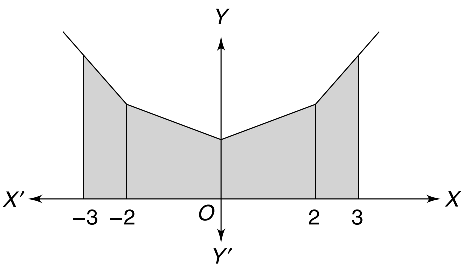
113.



Let $I = \int_0^4 (|x| + |x - 2|)\, dx$

$= 2 \times 2 + \frac{1}{2}(2 + 6) \times 2 = 4 + 8 = 12$

114. Let $I = \int_{-3}^{3} (|x + 2| + |x| + |x - 2|)\, dx$

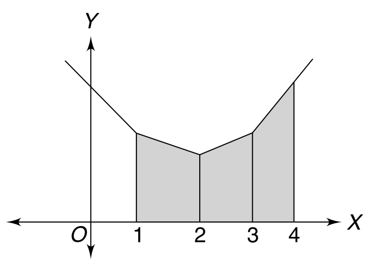


$= 2 \times \frac{1}{2} \times (9 + 6) + 2 \times \frac{1}{2} \times (4 + 6) \times 2$

$= 15 + 20$

$= 35$ sq.u.

115. Let $I = \int_1^4 (|x - 1| + |x - 2| + |x - 3|)\, dx$



$= 2 \times \frac{1}{2} \times (3 + 2) \times 1 + \frac{1}{2} \times (3 + 6) \times 1$

$= 5 + \frac{9}{2}$

$= 19/2$ sq.u.

116. Let $I = \int_0^3 |x^2 - 3x + 2|\, dx$

$= \int_0^1 |x^2 - 3x + 2|\, dx + \int_1^2 |x^2 - 3x + 2|\, dx$

$+ \int_2^3 |x^2 - 3x + 2|\, dx$

$= -\int_0^1 (x^2 - 3x + 2)\, dx + \int_1^2 (x^2 - 3x + 2)\, dx$

$-\int_2^3 (x^2 - 3x + 2)\, dx$

$= -\left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right)\Big|_0^1 + \left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right)\Big|_1^2$

$+ \left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right)_2^3$

$= -\frac{5}{6} + \frac{21}{6} - \frac{7}{6}$

$= \frac{9}{6}$

$= \frac{3}{2}$

117. Let $I = \int_{-4}^{3} |x^2 - 4|\, dx$

$= \int_{-4}^{-2} |x^2 - 4|\, dx + \int_{-2}^{2} |x^2 - 4|\, dx + \int_2^3 |x^2 - 4|\, dx$

$= \int_{-4}^{-2} (x^2 - 4)\, dx - \int_{-2}^{2} (x^2 - 4)\, dx + \int_2^3 (x^2 - 4)\, dx$

$= \left(\frac{x^3}{3} - 4x\right)_{-4}^{-2} - \left(\frac{x^3}{3} - 4x\right)_{-2}^{2} + \left(\frac{x^3}{3} - 4x\right)_2^3$

$= \left(8 - \frac{8}{3}\right) - \left(-\frac{64}{3} + 16\right) + 2\left(8 - \frac{8}{3}\right)$

$+ \left(\frac{27}{3} - 12\right) - \left(\frac{8}{3} - 8\right)$

$= 71/3$ sq.u.

118. Let $I = \int_0^{\pi} |\sin x|\, dx$

$= \int_0^{\pi} \sin x\, dx$

$= (-\cos x)_0^{\pi}$

$= -(\cos \pi - \cos 0)$

$= -(-1-1)$

$= 2$

Note: The values of

1. $\int_0^{2\pi} |\sin x|\, dx = 4$
2. $\int_0^{\pi} |\cos x|\, dx = 2$
3. $\int_0^{2\pi} |\cos x|\, dx = 4.$

119. We have,

$|\sin x| - |\cos x| = 0$

$\Rightarrow \quad |\sin x| = |\cos x|$

$\Rightarrow \quad |\tan x| = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$

Let $I = \int_0^{\pi} ||\sin x| - |\cos x|| \, dx$

$= 2\int_0^{\pi/2} |\sin x - \cos x| \, dx$

$= 2 \times \sqrt{2} \int_0^{\pi/2} \left|\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right| dx$

$= 2\sqrt{2} \int_0^{\pi/2} \left|\sin\left(x - \frac{\pi}{4}\right)\right| dx$

$= 2\sqrt{2} \int_{-\pi/4}^{\pi/4} |\sin x| \, dx$

$= 4\sqrt{2} \int_0^{\pi/4} |\sin x| \, dx$

$= 4\sqrt{2} \int_0^{\pi/4} \sin x \, dx$

$= 4\sqrt{2} (-\cos x)_0^{\pi/4}$

$= 4\sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right)$

$= 4(\sqrt{2} - 1)$

120. Let $I = \int_{-1}^{3/2} |x \sin \pi x| \, dx$

$= \int_{-1}^{1} |x \sin \pi x| \, dx + \int_{1}^{3/2} |x \sin \pi x| \, dx$

$= 2\int_0^1 ||x \sin \pi x \, dx + \frac{1}{\pi^2} \int_{\pi}^{3\pi/2} |x \sin x| \, dx$

$= 2\int_0^1 x \sin \pi x \, dx - \frac{1}{\pi^2} \int_{\pi}^{3\pi/2} x \sin x \, dx$

$= 2\left[x\left(\frac{-\cos \pi o}{\pi}\right) + \frac{\sin \pi x}{\pi^2}\right]_0^1$

$-\frac{1}{\pi^2} [x(-\cos x) + \sin x]_{\pi}^{3\pi/\pi}$

$= 2\left(\frac{1}{\pi}\right) - \frac{1}{\pi^2} \{(0 - 1) - \pi\}$

$= \frac{3}{\pi} + \frac{1}{\pi^2} = \left(\frac{3\pi + 1}{\pi^2}\right)$

121. $\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x) \, dx$

$= \int_{-\pi/2}^{0} \sin^{-1}(\sin x) \, dx$

$+ \int_0^{\pi} \sin^{-1}(\sin x) \, dx + \int_{\pi}^{2\pi} \sin^{-1}(\sin x) \, dx$

$= -\left(\frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2}\right) + \left(\frac{1}{2} \cdot \pi \cdot \frac{\pi}{2}\right) - \left(\frac{1}{2} \cdot \pi \cdot \frac{\pi}{2}\right)$

$= -\frac{\pi^2}{8} + \frac{\pi^2}{4} - \frac{\pi^2}{4}$

$= -\frac{\pi^2}{8}$

122. Let $I = \int_{1/e}^{e} |\log x| \, dx$

$= \int_{1/e}^{1} |\log x| \, dx + \int_1^e |\log x| \, dx$

$= \int_{1/e}^{1} (-\log x) \, dx + \int_1^e (\log x) \, dx$

$= -(x \log x - x)|_{1/e}^{1} + (x \log x - x)|_1^e$

$= \left(-1 + \frac{2}{e}\right) + 1$

$= \left(2 - \frac{2}{e}\right)$

123. Let $I = \int_0^{100} [\tan^{-1} x] \, dx$

$= \int_0^{\tan 1} [\tan^{-1} x] \, dx + \int_{\tan 1}^{100} [\tan^{-1} x] \, dx$

$= \int_0^{\tan 1} 0 \, . \, dx + \int_{\tan 1}^{100} 1 \, . \, dx$

$= 0 + 100 - \tan 1$

$= (100 - \tan 1)$

124. Let $I = \int_0^{1/2} [2 \sin^{-1} x] \, dx$

$= \int_0^{\sin(1/2)} [2\sin^{-1} x] \, dx + \int_{\sin(1/2)}^{1/2} [2\sin^{-1} x] \, dx$

$= \int_0^{\sin(1/2)} 0 \, . \, dx + \int_{\sin(1/2)}^{1/2} 1 \, . \, dx$

$= 0 + 1 . \left(\frac{1}{2} - \left(\frac{1}{2}\right)\right)$

$= \frac{1}{2} - \sin\left(\frac{1}{2}\right)$

125. Let $I = \int_{-2}^{1} \left[x\left[1 + \cos\left(\frac{\pi x}{2}\right)\right] + 1\right] dx$

$= \frac{2}{\pi} \int_{-\pi}^{\pi/2} \left[\frac{2x}{\pi} [1 + \cos x] + 1\right] dx$

$$= \frac{2}{\pi}\int_{-\pi}^{-\pi/2}\left[\frac{2x}{\pi}[1+\cos x]+1\right]dx + \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\left[\frac{2x}{\pi}[1+\cos x]+1\right]dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{-\pi/2}\left[\frac{2x}{\pi}\cdot 0+1\right]dx + \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\left[\frac{2x}{\pi}+1\right]dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{-\pi\pi//} 1\cdot dx + \frac{2}{\pi}\int_{-\pi/2}^{\pi//2}[x+1]\,dx$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2}+\pi\right) + \int_{-1}^{1}[x+1]\,dx$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2}+\pi\right) + \int_{-1}^{1}[x]\,dx + \int_{-1}^{1}1\,dx$$

$$= 1 + \int_{-1}^{0}[x]\,dx + \int_{0}^{1}1\,dx + 2$$

$$= 3 + (-1)\cdot 1 + 0$$

$$= 2$$

126. Let $I = \int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

$$= \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx + \int_1^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

$$= \int_0^1 2\tan^{-1}x\,dx + \int_1^{\sqrt{3}}(-\pi + 2\tan^{-1}x)\,dx$$

$$= 2\int_0^2 \tan^{-1}x\,dx - \int_1^{\sqrt{3}}(x)\,dx + \int_1^{\sqrt{3}}(2\tan^{-1}x)\,dx$$

$$= 2\int_0^{\sqrt{3}} \tan^{-1}x\,dx - \int_1^{\sqrt{3}}(\pi)\,dx$$

$$= 2\left[(x\tan^{-1}x)\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}}\left(\frac{x}{1+x^2}\right)dx\right]\pi - (x)\Big|_1^{\sqrt{3}}$$

$$= 2\left[(\sqrt{3}\tan^{-1}(\sqrt{3}) - 0) - \frac{1}{2}\log(1+x^2)\Big|_0^{\sqrt{3}}\right] - \pi(\sqrt{3}-1)$$

$$= \frac{2\pi\sqrt{3}}{3} - \log 4 - \pi(\sqrt{3}-1)$$

$$= \left[\pi\left(1-\frac{1}{\sqrt{3}}\right) - \log 4\right] \text{ sq.u}$$

Note: $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases}\pi + 2\tan^{-1}x, & x<-1\\ 2\tan^{-1}x, & -1<x<1\\ -\pi+2\tan^{-1}x, & x>1\end{cases}$

127. Let $I = \int_{-\pi/2}^{\pi/2}(\cos|x| + \sin|x|)\,dx$

$$= 2\int_0^{\pi/2}(\cos|x| + \sin|x|)\,dx$$

$$= 2\int_0^{\pi/2}(\cos x + \sin x)\,dx$$

$$= 2(\sin x - \cos x)\Big|_0^{\pi/2}$$

$$= 2((1+0)-(0-1))$$

$$= 4$$

128. Let $I = \int_0^2 \left|\cos\left(\frac{\pi x}{2}\right)\right|dx$

$$= \int_0^1 \left|\cos\left(\frac{\pi x}{2}\right)\right|dx + \int_1^2 \left|\cos\left(\frac{\pi x}{2}\right)\right|dx$$

$$= \int_0^1 \left|\cos\left(\frac{\pi x}{2}\right)\right|dx - \int_1^2 \left|\cos\left(\frac{\pi x}{2}\right)\right|dx$$

$$= \frac{2}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)\right)\Big|_0^1 - \frac{2}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)\right)\Big|_1^2$$

$$= \frac{2}{\pi}(1-0) - \frac{2}{\pi}(0-1)$$

$$= \frac{4}{\pi}$$

129. Let $I = \int_{-2}^{2} f(x)\,dx$

$$= \int_{-2}^{0} f(x)\,dx + \int_0^2 f(x)\,dx$$

$$= \int_{-2}^{0}(x - [x])\,dx + \int_0^2 (x + |x|)\,dx$$

$$= \int_{-2}^{0}(x)\,dx - \int_{-2}^{0}[x]\,dx + \int_0^2 (2x)\,dx$$

$$= \int_{-2}^{0}(x)\,dx - \int_{-2}^{-1}[x]\,dx - \int_{-1}^{0}[x]\,dx + \int_0^2(2x)\,dx$$

$$= -2 + 4 - \int_{-2}^{-1}[x]\,dx - \int_{-1}^{0}[x]\,dx$$

$$= 2 - [-2(-1+2) + (-1)(0+1)]$$

$$= -2 + 2 + 1 + 4$$

$$= 5$$

130. Let $I = \int_{-1}^{1}[x[1+\sin \pi x]+1]\,dx$

$$= \int_{-1}^{0}[x[1+\sin \pi x]+1]\,dx$$

$$+ \int_0^1 [x[1 + \sin \pi x] + 1]$$

$$= \int_{-1}^0 1\,dx + \int_0^1 [x + 1]\,dx$$

$$= \int_{-1}^0 1\,dx + \int_0^1 [x]\,dx + \int_0^1 1\,.dx$$

$$= \int_{-1}^1 1\,dx + \int_0^1 [x]\,dx$$

$$= \int_{-1}^1 1\,dx + \int_0^1 0\,.dx$$

$$= 1[1-(-1)]$$

$$= 2 \text{ sq.u.}$$

Notes: 1. If $-1 < x < 0 \Rightarrow [1 + \sin \pi x] = 0$

2. If $0 < x < 1 \Rightarrow [1 + \sin \pi x] = 1$

131. We have, $f(x) = \int_1^x \frac{\ln t}{1 + t}\,dt$ ...(i)

Now, $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1 + t}\,dt$

Let $t = \frac{1}{y} \Rightarrow dt = -\frac{1}{y^2}\,dy$

The given integral reduces to

$$= \int_1^y \frac{\ln\left(\frac{1}{y}\right)}{1 + \left(\frac{1}{y}\right)} \times -\frac{dy}{y^2}$$

$$= \int_1^y \frac{-\ln(y)}{1 + y} \times -\frac{dy}{y}$$

$$= \int_1^y \frac{\ln(y)}{y(1 + y)}\,dy$$

$$= \int_1^t \frac{\ln(t)}{t(1 + t)}\,dt \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln(t)}{(1 + t)}\,dt + \int_1^x \frac{\ln(t)}{t(1 + t)}\,dt$$

$$= \int_1^x \frac{(1 + t)\ln(t)}{t(1 + t)}\,dt$$

$$= \int_1^x \frac{\ln(t)}{t}\,dt$$

$$= \left.\frac{(\ln t)^2}{2}\right|_1^x = \frac{(\ln x)^2}{2}$$

Thus, $f(e) + f\left(\frac{1}{e}\right) = \left(\frac{\ln e}{2}\right)^2 = \frac{1}{2}.$

132. Let

$$f(x) = \int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})\,dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})\,dt$$

$$\Rightarrow \quad f'(x) = \sin^{-1}\left(\sqrt{\sin^2 x}\right)\sin 2x$$

$$+ \cos^{-1}\left(\sqrt{\cos^2 x}\right)(-\sin 2x)$$

$$= (\sin^{-1}(|\sin x|) - \cos^{-1}(|\cos x|))\sin 2x$$

$$= (\sin^{-1}(\sin x) - \cos^{-1}(\cos x))\sin 2x$$

$$= 0$$

$$\Rightarrow \quad f'(x) = 0$$

$$\Rightarrow \quad f(x) = c$$

$$\Rightarrow \quad c = \int_c^{\cos x} \sin^{-1}(\sqrt{t})\,dt + \int_0^{\sin^2 x} \cos^{-1}(\sqrt{t})\,dt$$

Let $x = \pi/4$, we get

$$c = \int_c^{1/2} \sin^{-1}(\sqrt{t})\,dt + \int_0^{1/2} \cos^{-1}(\sqrt{t})\,dt$$

$$= \int_c^{1/2} (\sin^{-1}(\sqrt{t}) + \cos^{-1}(\sqrt{t}))\,dt$$

$$= \int_c^{1/2} \left(\frac{\pi}{2}\right)dt$$

$$= \left(\frac{\pi}{4}\right)$$

133. We have,

$$\int_0^2 |x^2 + 2x - 3|\,dx$$

$$= \int_0^1 |x^2 + 2x - 3|\,dx + \int_1^0 |x^2 + 2x - 3|\,dx$$

$$= -\int_0^1 (x^2 + 2x - 3)\,dx + \int_1^2 (x^2 + 2x - 3)\,dx$$

$$= -\left(\frac{x^3}{3} + x^2 - 3x\right)_0^1 dx + \left(\frac{x^3}{3} + x^2 - 3x\right)_1^2$$

$$= -\left(\frac{1}{3} + 1 - 3\right) + \left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right)$$

$$= \left(\frac{8}{3} + 4 - 6\right) - 2\left(\frac{1}{3} + 1 - 3\right)$$

$$= \left(\frac{8}{3} - 2\right) - 2\left(\frac{1}{2} - 2\right)$$

$$= \left(\frac{8}{3} - \frac{2}{3}\right) + (4 - 2)$$
$$= 2 + 2$$
$$= 4$$

134. Do yourself

135. $\int_1^{10\pi} ([\sec^{-1}x] + [\cot^{-1}x])\,dx$

$$= \int_0^{\sec 1} ([\sec^{-1}x] + [\cot^{-1}x])\,dx + \int_{\sec 1}^{10\pi} ([\sec^{-1}x] + [\cot^{-1}x])\,dx$$
$$= \int_0^{\sec 1} (0 + 0)\,dx + \int_{\sec 1}^{10\pi} (1 + 0)\,dx$$
$$= (10\pi - \sec 1)$$

136. $\int_{-\pi/2}^{2\pi} ([\cot^{-1}x])\,dx$

$$= \int_{-\pi/2}^{\cot 2} ([\cot^{-1}x])\,dx + \int_{\cot 2}^{\cot 1} ([\cot^{-1}x])\,dx + \int_{\cot 1}^{2\pi} ([\cot^{-1}x])\,dx$$
$$= \int_{-\pi/2}^{\cot 2} (2)\,dx + \int_{\cot 2}^{\cot 1} (1)\,dx + \int_{\cot 1}^{2\pi} (0)\,dx$$
$$= (\pi + \cot 1 + \cot 2).$$

137. We have,

$$\int_1^2 [2x^2 - 3]\,dx$$

Now, $2x^2 - 3 = 0 \Rightarrow x = \sqrt{\frac{3}{2}}$

$2x^2 - 3 = 1 \Rightarrow x = \sqrt{2}$

$2x^2 - 3 = 2 \Rightarrow x = \sqrt{\frac{5}{2}}$

$2x^2 - 3 = 3 \Rightarrow x = \sqrt{3}$

$2x^2 - 3 = 4 \Rightarrow x = \sqrt{\frac{7}{2}}$

and $2x^2 - 3 = 5 \Rightarrow x = 2$

Thus, the given integral can be written as

$$\int_1^{\sqrt{3/2}} [2x^2 - 3]\,dx + \int_{\sqrt{3/2}}^{\sqrt{2}} [2x^2 - 3]\,dx + \int_{\sqrt{3/2}}^{\sqrt{5/2}} [2x^2 - 3]\,dx + \int_{\sqrt{5/2}}^{\sqrt{3}} [2x^2 - 3]\,dx + \int_{\sqrt{3}}^{\sqrt{7/2}} [2x^2 - 3]\,dx + \int_{\sqrt{7/2}}^{2} [2x^2 - 3]\,dx$$

$$= \int_1^{\sqrt{3/2}} (-1)\,dx + \int_{\sqrt{3/2}}^{\sqrt{2}} (0)\,dx + \int_{\sqrt{2}}^{\sqrt{5/2}} (1)\,dx$$
$$= \int_{\sqrt{5/2}}^{\sqrt{3}} (2)\,dx + \int_{\sqrt{3}}^{\sqrt{7/2}} (3)\,dx + \int_{\sqrt{7/2}}^{2} (4)\,dx$$
$$= (-1)\left(\sqrt{\frac{3}{2}} - 1\right) + 0 + 1\cdot\left(\sqrt{\frac{5}{2}} - \sqrt{2}\right) + 2\cdot\left(\sqrt{3} - \sqrt{\frac{5}{2}}\right) + 3\cdot\left(\sqrt{\frac{7}{2}} - \sqrt{3}\right) + 4\cdot\left(2 - \sqrt{\frac{7}{2}}\right)$$
$$= \left(9 - \left(\sqrt{\frac{3}{2}} + \sqrt{2} + \sqrt{\frac{5}{2}} + \sqrt{3} + \sqrt{\frac{7}{2}}\right)\right) \text{ sq. u.}$$

138. Do yourself

139. We have,

$$\int_0^1 [x^2 - x + 1]\,dx$$
$$= \int_0^1 [x^2 - x + 1]\,dx + \int_1^{\left(\frac{\sqrt{5}+1}{2}\right)} [x^2 - x + 1]\,dx + \int_{\left(\frac{\sqrt{5}+1}{2}\right)}^{2} [x^2 - x + 1]\,dx$$
$$= \int_0^1 (0)\,dx + \int_1^{\left(\frac{\sqrt{5}+1}{2}\right)} (1)\,dx + \int_{\left(\frac{\sqrt{5}+1}{2}\right)}^{2} (2)\,dx$$
$$= \left(\frac{\sqrt{5}+1}{2}\right) + 2\left(2 - \frac{\sqrt{5}+1}{2}\right)$$
$$= \left|3 - \frac{\sqrt{5}+1}{2}\right|$$
$$= \left|\frac{5 - \sqrt{5}}{2}\right|$$

140. Do yourself

141. Do yourself

142. We have,

$$\frac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx}$$
$$= \frac{\frac{n(n-1)}{2}}{\frac{n^2}{2} - \frac{n(n-1)}{2}}$$
$$= \frac{\frac{n(n-1)}{2}}{\frac{n}{2}}$$
$$= |n - 1|$$

143. We have,

$$\sum_{n=1}^{1000} \int_{n=1}^{n} (e^{x-[x]})\,dx$$

$$= \int_0^{1000} (e^{x-[x]})\,dx$$

$$= 1000 \int_0^1 (e^{x-[x]})\,dx$$

$$= 1000 \int_0^1 (e^{x-0})\,dx$$

$$= (1000(e^x))_0^1$$

$$= 1000(e-1)$$

144. We have,

$$\int_{\frac{1}{e}}^{\tan x} \frac{1}{(1+t^2)}\,dt + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{(1+t^2)}$$

$$= \int_{\frac{1}{e}}^{\tan x} \frac{1}{(1+t^2)}\,dt + \int_{e}^{\tan x} \frac{-\frac{dy}{y^2}}{\frac{1}{y}\left(1+\frac{1}{y^2}\right)},$$

Let $t = \frac{1}{y}$

$$= \int_{\frac{1}{e}}^{\tan x} \frac{1}{(1+t^2)}\,dt - \int_{e}^{\tan x} \frac{dy}{(y^2+1)}$$

$$= \int_{\frac{1}{e}}^{\tan x} \frac{1}{(1+t^2)}\,dt - \int_{e}^{\tan x} \frac{dt}{(t^2+1)}$$

$$= \int_{\frac{1}{e}}^{\tan x} \frac{1}{(1+t^2)}\,dt + \int_{\tan x}^{e} \frac{dt}{(t^2+1)}$$

$$= \int_{\frac{1}{e}}^{e} \frac{1}{(1+t^2)}\,dt$$

$$= (\tan^{-1}(t))_{1/e}^{e}$$

$$= \tan^{-1}(e) - \tan^{-1}\left(\frac{1}{e}\right)$$

145. Let $I = \int_0^{\pi/2} \left(\frac{\sin^n x}{\sin^n x + \cos^n x}\right) dx$ ...(i)

Also $I = \int_0^{\pi/2} \left(\frac{\cos^n x}{\cos^n x + \sin^n x}\right)$ ...(ii)

Adding Eqs (i) and (ii), we get

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\cos^n x + \sin^n x}\right) dx$$

$$= \int_0^{\pi/2} dx$$

$$= \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

146. Let $I = \int_4^{10} \left(\frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}\right) dx$ ...(i)

$$= \int_0^{14} \left(\frac{[x^2]}{[14-x]^2 + [x^2]}\right) dx$$

$$= \int_0^{14} \left(\frac{[(14-x)^2]}{[(x)^2] + [14-x]^2}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{14} \left(\frac{[x^2] + [(14-x)^2]}{[(x)^2] + [14-x]^2}\right) dx$$

$$= \int_0^{14} dx$$

$$= 14$$

$$\Rightarrow \quad I = 7$$

147. Let $I = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$

$$= \int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{(1+(\tan\theta)^a)(1+\tan^2\theta)}$$

Let $x = \tan\theta$

$$= \int_0^{\pi/2} \left(\frac{d\theta}{1+(\tan\theta)^a}\right)$$

$$= \int_0^{\pi/2} \left(\frac{\cos^a\theta}{\cos^a\theta + \sin^a\theta}\right) d\theta \quad \text{...(i)}$$

$$= \int_0^{\pi/2} \left(\frac{\sin^a\theta}{\sin^a\theta + \cos^a\theta}\right) d\theta \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left(\frac{\cos^a\theta + \sin^a\theta}{\sin^a\theta + \cos^a\theta}\right) d\theta$$

$$= \int_0^{\pi/2} d\theta$$

$$= \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

148. Let $I = \int_0^{\frac{\pi}{4}} (1 + \tan x)\,dx$

$$= \int_0^{\pi/4} \left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$$

$$= \int_0^{\pi/4} \left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log 2x\, dx - \int_0^{\pi/4} \log(1 + \tan x)\, dx$$

$$= \frac{\pi}{4} \log 2 - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4} \log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8} \log 2$$

149. Let $I = \int_0^1 \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right) dx$

$$= \int_0^1 \tan^{-1}\left(\frac{x + (x - 1)}{1 - x(x - 1)}\right) dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx$$

$$= 2\left(x \tan^{-1} x - \frac{1}{2} \log|1 + x^2|\right)_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{2} \text{ lgo } 2\right)$$

$$= \left(\frac{\pi}{2} - \log 2\right) \text{ sq.u.}$$

150. Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x - (x - 1)}{1 + x(x - 1)}\right) dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx$$

$$= 2\left(x \tan^{-1} x - \frac{1}{2} \log|1 + x^2|\right)_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{2} \log 2\right)$$

$$= \left(\frac{\pi}{2} - \log 2\right) \text{ sq.u.}$$

151. Let $I = \int_0^{\pi} \left(\frac{dx}{1 + 2^{\tan x}}\right)$ ...(i)

$$= \int_0^{\pi} \left(\frac{dx}{1 + 2^{\tan(\pi - x)}}\right)$$

$$= \int_0^{\pi} \left(\frac{dx}{1 + 2^{-\tan x}}\right)$$

$$= \int_0^{\pi} \left(\frac{2^{\tan x} dx}{1 + 2^{\tan x}}\right) \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi} \left(\frac{(1 + 2^{\tan x})}{(1 + 2^{\tan x})}\right) dx$$

$$= \int_0^{\pi} dx = \pi$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

152. Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

$$= \int_0^1 \log\left(\frac{1 - x}{x}\right) \quad \text{...(i)}$$

$$= \int_0^1 \log\left(\frac{1 - (1 - x)}{1 - x}\right) dx$$

$$= \int_0^1 \left(\frac{x}{1 - x}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{1 - x}{x}\right) dx + \int_0^1 \log\left(\frac{x}{1 - x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1 - x}{x} \times \frac{x}{1 - x}\right) dx$$

$$= \int_0^1 \log(1)\, dx$$

$$= 0$$

$$\Rightarrow \quad I = 0$$

153. Let $I = \int_0^1 \frac{\log(1 + x)}{(1 + x^2)}\, dx$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$= \int_0^{\pi/4} \frac{\log(1 + \tan\theta)}{(1 + \tan^2\theta)} \sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \log(1 + \tan\theta) d\theta \quad \text{...(i)}$$

$$= \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{2}{1 + \tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log(2) d\theta - \int_0^{\pi/4} \log(1 + \tan\theta) d\theta$$

$$= \int_0^{\pi/4} \log(2) d\theta - I$$

$$\Rightarrow \quad 2I = \int_0^{\pi/4} \log 2 d\theta$$

$$= \frac{\pi}{4} \log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8} \log 2$$

154. Let $I = \int_0^1 x(1 - x)^{2013} dx$

$$= \int_0^1 (1 - x)(1 - (1 - x))^{2013} dx$$

$$= \int_0^1 (1 - x)(x)^{2013} dx$$

$$= \int_0^1 (x^{2013} - x^{2014}) dx$$

$$= \left(\frac{x^{2014}}{2014} - \frac{x^{2015}}{2015}\right)_0^1$$

$$= \left(\frac{1}{2014} - \frac{1}{2015}\right)$$

$$= \left(\frac{2015 - 2014}{2014 \times 2015}\right)$$

$$= \left(\frac{1}{2014 \times 2015}\right)$$

155. Let $I = \int_0^{\pi} \left(\frac{x}{1 + \sin x}\right) dx \quad \text{...(i)}$

$$= \int_0^{\pi} \left(\frac{\pi - x}{1 + \sin(\pi - x)}\right) dx$$

$$= \int_0^{\pi} \left(\frac{\pi - x}{1 + \sin x}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi} \left(\frac{\pi}{1 + \sin x}\right) dx$$

$$= \pi \int_0^{\pi} \left(\frac{dx}{1 + \sin x}\right)$$

$$= \pi \int_0^{\pi} \left(\frac{(1 - \sin x) dx}{1 - \sin^2 x}\right)$$

$$= \pi \int_0^{\pi} \left(\frac{(1 - \sin x)}{\cos^2 x}\right) dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x) dx$$

$$= \pi (\tan x - \sec x)_0^{\pi}$$

$$= \pi(1 + 1) = 2\pi$$

$$\Rightarrow \quad I = \pi$$

156. Let $I = \int_0^{\pi/2} (\pi \sin^2 x) dx \quad \text{...(i)}$

$$= \int_0^{\pi/2} \cos\left(\pi \sin^2\left(\frac{\pi}{2} - x\right)\right) dx$$

$$= \int_0^{\pi/2} \cos(\pi \cos^2 x) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \cos(\pi \sin^2 x) dx + \int_0^{\pi/2} \cos(\pi \sin^2 x) dx$$

$$= \int_0^{\pi/2} (\cos(\pi \sin^2 x) + \cos(\pi \cos^2 x)) dx$$

$$= \int_0^{\pi/2} 2\cos\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2} \cos 2x\right) dx$$

$$= 0$$

$$\Rightarrow \quad I = 0$$

157. We have,

$$M = \int_0^1 \left(\frac{e^t}{1 + t}\right) dt$$

$$= \left(e^t \int \frac{dt}{1 + t}\right)_0^1 - \int_0^1 e^t \log(1 + t) dt$$

$$= (e^t \log|1 + t|)_0^1 - N$$

$$\Rightarrow \quad M + N = (e \times \log 2)$$

158. We have,

$$a_n = \int_0^{\pi/2} \frac{\cos^2 nx}{\sin x} dx$$

Then $a_{n+1} = \int_0^{\pi/2} \left(\frac{\cos^2(n+1)x}{\sin x}\right) dx$

Thus, $a_{n+1} - a_n$

$$= \int_0^{\pi/2} \left(\frac{\cos^2(n+1)x - \cos^2 nx}{\sin x}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{2\cos^2(n+1)x - 2\cos^2 nx}{\sin x}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{\cos(2n+2)x - \cos 2nx}{\sin x}\right) dx$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left(\frac{2\sin(2n+1)x \cdot \sin x}{\sin x}\right) dx$$

$$= -\int_0^{\pi/2} (\sin(2n+1)x)\, dx$$

$$= \left(\frac{\cos(2n+1)x}{2n+1}\right)_0^{\pi/2}$$

$$= -\frac{1}{2n+1}$$

159. Similiar to 145

160. Similiar to 145

161. Similiar to 145

162. Let $I = \int_0^{\theta} \log(1 + \tan\theta \cdot \tan x)\, dx,\ \theta \in \left(0, \frac{\pi}{2}\right)$

$$= \int_0^{\theta} \log(1 + \tan\theta \cdot \tan(\theta - x))\, dx$$

$$= \int_0^{\theta} \log\left(1 + \tan\theta\left(\frac{\tan\theta - \tan x}{1 + \tan\theta \cdot \tan x}\right)\right) dx$$

$$= \int_0^{\theta} \log\left(\frac{1 + \tan^2\theta}{1 + \tan\theta \cdot \tan x}\right) dx$$

$$= \int_0^{\theta} \log(1 + \tan^2\theta)\, dx - \int_0^{\theta} \log(1 + \tan\theta \cdot \tan x)\, dx$$

$$= \int_0^{\theta} \log(\sec^2\theta)\, dx - I$$

$\Rightarrow \quad 2I = 2\int_0^{\theta} \log(\sec\theta)\, dx$

$\Rightarrow \quad I = \theta \log(\sec\theta)$

163. Let $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$

$$= \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log(2)\, dx - \int_0^{\pi/4} \log(1 + \tan x)\, dx$$

$$= \frac{\pi}{4}\log(2) - I$$

$\Rightarrow \quad 2I = \frac{\pi}{4}\log(2)$

$\Rightarrow \quad I = \frac{\pi}{8}\log(2)$

But $k = \int_0^{\pi} \log(\sin x)\, dx = -\pi \log(2)$

Thus, $I = -\frac{k}{8}$

164. $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$

$$= \int_0^{\pi/2} \log(4 + 3\sin x)\, dx - \int_0^{\pi/2} \log(4 + 3\cos x)\, dx$$

$$= \int_0^{\pi/2} \log(4 + 3\cos x)\, dx - \int_0^{\pi/2} \log(4 + 3\cos x)\, dx$$

where $\int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx$

$= 0$

165. Let $I = \int_0^{\frac{\pi}{2}} \left(\frac{\sin x - \cos x}{1 + \sin x \cdot \cos x}\right) dx$ ...(i)

$$= \int_0^{\pi/2} \left(\frac{\cos x - \sin x}{1 + \sin x \cos x}\right) dx \quad ...(ii)$$

Adding (i) and (ii), we get,

$$2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x}\right) dx$$

$2I = 0$

$I = 0$

$$\int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x}\right) dx = 0$$

166. $\int_0^{1} x(1 - x)^n\, dx$

$$= \int_0^{1} x((1 - x)x^n)\, dx$$

$$= \int_0^{1} (x^n - x^{n+1})\, dx$$

$$= \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1$$

$$= \frac{1}{(n+1)(n+2)}$$

167. Let $I = \int_0^{1} \log\left(\frac{1}{x} - 1\right) dx$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right)dx \quad \text{...(i)}$$

$$= \int_0^1 \log\left(\frac{x}{1-x}\right)dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get,

$$2I = \int_0^1 \log\left(\frac{1-x}{x} \times \frac{x}{1-x}\right)dx$$

$$2I = \int_0^1 \log(1)\,dx$$

$$2I = 0$$

$$I = 0$$

$$\int_0^1 \log\left(\frac{1}{x} - 1\right)dx = 0$$

168. Let $I = \int_0^\pi \left(\frac{x\sin x}{1+\cos^2 x}\right)dx$

$$= \int_0^\pi \left(\frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\right)dx$$

$$= \int_0^\pi \left(\frac{(\pi - x)\sin(x)}{1+\cos^2(x)}\right)dx$$

$$= \pi\int_0^\pi \left(\frac{\sin(x)}{1+\cos^2(x)}\right)dx - \int_0^\pi \left(\frac{x\sin(x)}{1+\cos^2(x)}\right)dx$$

$$\pi\int_0^\pi \left(\frac{\sin(x)}{1+\cos^2(x)}\right)dx - 1$$

$$\Rightarrow \quad 2I = \pi\int_0^\pi \left(\frac{\sin(x)}{1+\cos^2(x)}\right)dx$$

$$= -\pi(\tan^{-1}(\cos x))_0^\pi$$

$$= -\pi(\tan^{-1}(-1) - \tan^{-1}(1))$$

$$= -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = -\pi\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}$$

169. Let $I = \int_0^\pi \left(\frac{x}{1+\sin x}\right)dx$

$$= \int_0^\pi \left(\frac{(\pi - x)}{1+\sin(\pi - x)}\right)dx$$

$$= \int_0^\pi \left(\frac{(\pi - x)}{1+\sin x}\right)dx$$

$$= \pi\int_0^\pi \left(\frac{dx}{1+\sin x}\right) - \int_0^\pi \left(\frac{x}{1+\sin x}\right)dx$$

$$= \pi\int_0^\pi \left(\frac{dx}{1+\sin x}\right) - I$$

$$\Rightarrow \quad 2I = \pi\int_0^\pi \left(\frac{dx}{1+\sin x}\right)$$

$$= \pi\int_0^\pi \left(\left(\frac{1-\sin x}{\cos^2 x}\right)dx\right)$$

$$= \pi\int_0^\pi ((\sec^2 x - \sec x \cdot \tan x)\,dx)$$

$$= \pi(\tan x - \sec x)_0^\pi$$

$$= \pi[0 - (-1) - (0 - 1)] = 2\pi$$

$$\Rightarrow \quad I = \pi$$

170. Let $I = \int_0^{\frac{\pi}{2}} \left(\frac{x\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\right)dx$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\left(\frac{\pi}{2} - x\right)\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\right)dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\right)dx - \int_0^{\frac{\pi}{2}} \left(\frac{x\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\right)dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\right)dx - 1$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\right)dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{2\tan x \cdot \sec^2 x}{\tan^4 x + 1}\right)dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{2\tan x \cdot \sec^2 x}{\tan^4 x + 1}\right)dx$$

$$\Rightarrow \quad I = \left(\frac{\pi}{8}\tan^{-1}(\tan^2 x)\right)_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad I = \frac{\pi}{8}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{16}$$

171. We have,

$$\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x + 1(1-x)}{1 - x\,(1 - x^2)}\right)dx$$

$$= \int_0^1 (\tan^{-1}(x) + \tan^{-1}(1-x))\,dx$$

$$= \int_0^1 (\tan^{-1}(x))\,dx + \int_0^1 (\tan^{-1}(1-x))\,dx$$

$$= \int_0^1 (\tan^{-1}(x))\,dx + \int_0^1 (\tan^{-1}(x))\,dx$$

$$= 2\int_0^1 \tan^{-1}(x))\,dx$$

$$= 2\left(x\tan^{-1}(x) - \frac{1}{2}\log|1 + x^2|\right)_0^1$$

$$= 2\left(x\tan^{-1}(1) - \frac{1}{2}\log(2)\right)$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{2}\log(2)\right)$$

$$= \left|\frac{\pi}{4} - \log(2)\right|$$

172. We have,

$$\int_0^1 \tan^{-1}(1 - x + x^2)\,dx$$

$$= \int_0^1 \left(\frac{\pi}{2} - \cot^{-1}(1 - x + x^2)\right)dx$$

$$= \int_0^1 \left(\frac{\pi}{2} - \cot^{-1}(1 - x + x^2)\right)dx$$

$$= \int_0^1 \left(\frac{\pi}{2}\right)dx - \int_0^1 (\cot^{-1}(1 - x + x^2))\,dx$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - \log 2\right)$$

$$= \log|2|$$

173. We have,

$$\int_0^\infty \frac{dx}{(1 + x^a)(1 + x^2)}, \; (a > 0)$$

Put $x = \tan\theta$

$$= \int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{(1 + \tan^a\theta)\sec^2\theta}$$

$$= \int_0^{\pi/2} \frac{d\theta}{(1 + \tan^a\theta)}$$

$$= \int_0^{\pi/2} \frac{\cos^a\theta}{(\cos^a\theta + \sin^a\theta)}\,d\theta$$

$$= \frac{\pi}{4}$$

174. Let $I = \int_0^{\pi/2} \frac{dx}{\sqrt{\tan x} - \sqrt{\cot x}}$

$$= \int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\tan x - 1)}\,dx \quad \text{...(i)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cot(x)}}{(\cot(x) - 1)}\,dx$$

$$= -\int_0^{\pi/2} \frac{\sqrt{\tan(x)}}{(\tan(x) - 1)}\,dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\tan(x)}}{(\tan(x) - 1)}\,dx - \int_0^{\pi/2} \frac{\sqrt{\tan(x)}}{(\tan(x) - 1)}\,dx$$

$\Rightarrow \quad 2I = 0$

$\Rightarrow \quad I = 0$

175. Let $I = \int_0^\pi \left(\frac{x}{a^2\cos^2 x + b^2\sin^2 x}\right)dx$

$$= \int_0^\pi \left(\frac{(\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}\right)dx$$

$$= \pi\int_0^\pi \left(\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right) - I$$

$$\Rightarrow \quad 2I = \pi\int_0^\pi \left(\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right)$$

$$= \pi\int_0^\pi \left(\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}\right)$$

$$= 2\pi\int_0^{\pi/2} \left(\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}\right)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \left(\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}\right)$$

$$= \pi\left(\frac{1}{ab}\tan^{-1}(b\tan x)\right)_0^{\pi/2}$$

$$= \frac{\pi}{ab} \times \frac{\pi}{2} = \frac{\pi^2}{2ab}$$

176. Do yourself

177. Let $I = \int_0^\pi \frac{2^{\sin x} \times \cos x}{2^{[\sin x]}}\,dx$

$$= \int_0^\pi \frac{2^{\sin x}}{2^{[\sin x]}}\cos(\pi - x)\,dx$$

$$= -\int_0^\pi \frac{2^{\sin x}}{2^{[\sin x]}}\cos(x)\,dx$$

$$= -I$$

$\Rightarrow \quad 2I = 0$

$\Rightarrow \quad I = 0$

178. Let $I = \int_0^\pi x\log(\sin x)\,dx$

$$= \int_0^\pi (\pi - x)\log[\sin(\pi - x)]\,dx$$

$$= \pi\int_0^\pi \log[\sin(x)]\,dx - \int_0^\pi x\log[\sin(x)]\,dx$$

$$= \pi\int_0^\pi \log[\sin(x)]\,dx - I$$

$$\Rightarrow \quad 2I = \pi\int_0^\pi \log(\sin(x))\,dx$$

$$= 2\pi \int_0^{\pi/2} \log(\sin(x))\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \log(\sin(x))\, dx$$

$$= \pi\left(\frac{1}{2}\log\left(\frac{1}{2}\right)\right)$$

$$= \left(\frac{\pi}{2}\log\left(\frac{1}{2}\right)\right)$$

179. For $n > 0$, $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}dx$

Let $I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx$

$$= \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx$$

$$= 2\pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx - \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx$$

$$= 2\pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx - I$$

$$\Rightarrow 2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx$$

$$\Rightarrow I = \pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx$$

$$= 4\pi \int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\, dx$$

$$= 4\pi \times \left(\frac{\pi}{4}\right)$$

$$= \pi^2$$

180. We have,

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x + \cos x}dx$$

$$= \int_0^{2\pi} \left(\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2) + 2\tan(x/2) + 1 - \tan^2(x/2)}\right)$$

$$= \int_0^1 \left(\frac{1 - t^2}{2(1 + t)} \times \frac{2dt}{1 + t^2}\right), \qquad \text{Let } \tan(x/2) = t$$

$$= -\int_0^1 \left(\frac{t - 1}{(1 + t^2)}\right)dt$$

$$= -\left(\frac{1}{2}\log|1 + t^2| - \tan^{-1}(t)\right)_0^1$$

$$= \left|\frac{\pi}{4} - \frac{1}{2}\log 2\right|$$

181. Given $I = \int_0^{\pi/2} \frac{dx}{1 + \sin x}$,

we have,

$$\int_0^{\pi} \left(\frac{x^2\cos x}{(1 + \sin x)^2}\right)dx$$

$$= \left[x^2 \int\left(\frac{\cos x}{(1 + \sin x)^2}\right)dx\right]_0^{\pi} + \int_0^{\pi} \frac{2x}{1 + \sin x}dx$$

$$= \left(-\frac{x^2}{(1 + \sin x)}\right)_0^{\pi} + \int_0^{\pi} \frac{x}{1 + \sin x}\, dx$$

$$= (-\pi^2) + 2\int_0^{\pi} \frac{x}{1 + \sin x}\, dx \qquad \text{...(i)}$$

$$= -\pi^2 + 2\int_0^{\pi} \frac{(\pi - x)}{1 + \sin x}\, dx \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = -2\pi^2 + 2\int_0^{\pi} \frac{\pi}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = -\pi^2 + \pi\int_0^{\pi} \frac{dx}{1 + \sin x}$$

$$= -\pi^2 + \pi I$$

$$= \pi I - \pi^2$$

182. We have,

$$\lim_{t \to a} \frac{\int_a^t f(t)\,dt - \left(\frac{t - a}{2}\right)(f(t) + f(a))}{(t - a)^3} = 0$$

$$\Rightarrow \lim_{t \to a} \frac{f(t) - \frac{1}{2}(f(t) + f(a)) - \frac{(t - a)}{2}f'(a)}{3(t - a)^2} = 0$$

$$\Rightarrow \lim_{t \to a} \frac{2f(t) - [f(t) + f(a)] - (t - a)f'(a)}{6(t - a)^2 = 0}$$

$$\Rightarrow \lim_{t \to a} \frac{[f(t) - f(a)] - (t - a)f'(a)}{6(t - a)^2} = 0$$

$$\Rightarrow \lim_{t \to a} \frac{f'(t) - f'(a)}{12(t - a)} = 0$$

$$\Rightarrow \lim_{t \to a} \frac{f''(t)}{12} = 0$$

$\Rightarrow \quad f''(a) = 0$

$\Rightarrow \quad f'(a) = c$

$\Rightarrow \quad f(a) = cx + k$

Clearly, $f(x)$ is of almost degree 1.

183. Let $\quad I = \int_a^b \left(\frac{f(x)}{f(x) + f(a+b-x)}\right) dx$ ...(i)

Also $\quad I = \int_a^b \left(\frac{f(a+b-x)}{f(a+b-x) + f(x)}\right) dx$ ...(ii)

Adding Eqs (i) and (ii), we get

$$2I = \int_a^b \left(\frac{f(x) + f(a+b-x)}{f(a+b-x) + f(x)}\right) dx$$

$$= \int_a^b dx$$

$$= (b - a)$$

$\Rightarrow \quad I = \left(\frac{b-a}{2}\right)$

184. Let $\quad I = \int_3^5 \left(\frac{[x^2]}{[x^2 - 16x + 64] + [x^2]}\right) dx$

$$= \int_3^5 \left(\frac{[x^2]}{[8-x]^2 + [x^2]}\right) dx \quad \text{...(i)}$$

$$= \int_3^5 \left(\frac{[(8-x)^2]}{[(x)]^2 + [(8-x)^2]}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_3^5 \left(\frac{[x^2] + [(8-x)^2]}{[(x)]^2 + [(8-x)^2]}\right) dx$$

$$= \int_3^5 dx$$

$$= (5 - 3) = 2$$

$\Rightarrow \quad I = 1$

185. Let $\quad I = \int_{-2}^2 \left(\frac{dx}{1 + 3^x}\right)$ ...(i)

Also $\quad I = \int_{-2}^2 \left(\frac{dx}{1 + 3^x}\right)$ ...(ii)

Adding Eqs (i) and (ii), we get,

$$2I = \int_{-2}^2 \left(\frac{1 + 3^x}{1 + 3^x}\right) dx$$

$$= \int_{-2}^2 dx$$

$$= |2 + 2| = 4$$

$\Rightarrow \quad I = 2$

186. Let $\quad I = \int_{\pi/6}^{5\pi/6} \left(\frac{dx}{1 + e^{\tan x}}\right)$ ...(i)

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{dx}{1 + e^{-\tan x}}\right)$$

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{e^{\tan x}}{e^{\tan x} + 1}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \left(\frac{e^{\tan x} + 1}{e^{\tan x} + 1}\right) dx = \int_{\pi/6}^{5\pi/6} dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \frac{2\pi}{3}$$

$\Rightarrow \quad I = \frac{\pi}{3}$

187. Let $\quad I = \int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1 + a^x}\right) dx$ ...(i)

$$= \int_{-\pi}^{\pi} \left(\frac{\cos^2(-x)}{1 + a^{-x}}\right) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{a^2 \cos(x)}{1 + a^x}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-\pi}^{\pi} \left(\frac{(1 + a^x)\cos^2(x)}{(1 + a^x)}\right) dx$$

$$= \int_{-\pi}^{\pi} \cos^2(x)\, dx$$

$$= 2 \int_{-\pi}^{\pi} \cos^2(x)\, dx$$

$$= \int_0^{\pi} (2\cos^2(x))\, dx$$

$$= \int_0^{\pi} (1 + \cos 2x)\, dx$$

$$= \left(x + \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= \pi$$

$\Rightarrow \quad I = \pi/2$

188. Let $\quad I = \int_{-1\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2x}{1 + x^2}\right)\right) dx$

$$= \int_{-1\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2(-x)}{1 + x^2}\right)\right) dx$$

$$= \int_{-1\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(-\frac{2(x)}{1 + x^2} \right) \right) dx$$

$$= \int_{-1\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\pi - \cos^{-1}\left(\frac{2(x)}{1 + x^2} \right) \right) \right) dx$$

$$= \int_{-1\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\pi x^4}{1 - x^4} - \frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2(x)}{1 + x^2} \right) \right) dx$$

$$= \int_{-1\sqrt{3}}^{1/\sqrt{3}} (\pi x^4/1 - x^4)\, dx$$

$$- \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2(x)}{1 + x^2} \right) \right) dx$$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\pi x^4}{1 - x^4} \right) dx - I$$

$$\Rightarrow \quad 2I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\pi x^4}{1 - x^4} \right) dx$$

$$= \int_{0}^{1/\sqrt{3}} \left(\frac{\pi x^4}{1 - x^4} \right) dx$$

$$\Rightarrow \quad I = \int_{0}^{1/\sqrt{3}} \left(\frac{\pi x^4}{1 - x^4} \right) dx$$

$$= \int_{0}^{1/\sqrt{3}} \left(\frac{x^4}{1 - x^4} \right) dx$$

$$\Rightarrow \quad k = \pi$$

189. Given $I_1 = \int_{1-k}^{k} x \cdot f(x(1 - x))\, dx$

$$= \int_{1}^{k} (1 - x) f[(1 - x)x]\, dx$$

$$= \int_{1}^{k} f[(1 - x)x]\, dx - \int_{1}^{k} x f[(1 - x)x]\, dx$$

$$= I_2 - I_1$$

$$\Rightarrow \quad I_1 = I_2 - I_1$$

$$\Rightarrow \quad 2I_1 = I_2$$

$$\Rightarrow \quad I_1/I_2 = 1/2$$

190. Let $I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \left(\frac{x \sin(x^2)}{\sin x^2 + \sin(\ln(6 - x^2))} \right) dx$

Let $x^2 = t \Rightarrow x\,dx = 1/2\, dt$

$$= \frac{1}{2} \int_{\ln 2}^{\ln 3} \left(\frac{\sin t}{\sin t + \sin(\ln 6 - t)} \right) dt \quad \text{...(i)}$$

$$= \frac{1}{2} \int_{\ln 2}^{\ln 3} \left(\frac{\sin(\ln - 6)}{\sin(\ln - 6) + \sin(t)} \right) dt \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{\ln 2}^{\ln 3} \left(\frac{\sin(t) + \sin(\ln - 6)}{\sin(\ln - 6) + \sin(t)} \right) dt$$

$$= \frac{1}{2} \int_{\ln 2}^{\ln 3} dt$$

$$= \frac{1}{2} (\ln 3 - \ln 2)$$

$$= \frac{1}{2} \left(\ln\left(\frac{3}{2} \right) \right)$$

$$\Rightarrow \quad I = \frac{1}{4} \left(\ln\left(\frac{3}{2} \right) \right)$$

191. Let $I_n = \int_{-\pi}^{\pi} \left(\frac{\sin nx}{(1 + \pi^x) \sin x} \right) dx$

$$= \int_{-\pi}^{\pi} \left(\frac{\sin nx}{(1 + \pi^x) \sin x} \right) dx \quad \text{...(i)}$$

$$= \int_{-\pi}^{\pi} \left(\frac{\sin n(-x)}{(1 + \pi^{-x}) \sin(-x)} \right) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{\sin n(x)}{(1 + \pi^{-x}) \sin(x)} \right) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{\pi x \sin n(x)}{(1 + \pi^x) \sin(x)} \right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \left(\frac{(1 + \pi^x) \sin n(x)}{(1 + \pi^x) \sin(x)} \right) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{\sin n(x)}{\sin(x)} \right) dx$$

$$= 2\int_{0}^{\pi} \left(\frac{\sin n(x)}{\sin(x)} \right) dx$$

$$\Rightarrow \quad I_n = \int_{0}^{\pi} \left(\frac{\sin n(x)}{\sin(x)} \right) dx$$

$$I_{n+2} - I_n = \int_{0}^{\pi} \left(\frac{\sin(n + 2)x - \sin nx}{\sin x} \right) dx$$

$$= \int_{0}^{\pi} \left(\frac{2\cos(n + 1)x \times \sin x}{\sin x} \right) dx$$

$$= \int_{0}^{\pi} 2\cos(n + 1)x\, dx$$

$$= 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$$

$\Rightarrow \quad I_{n+2} = I_n$

Now, $\quad I_0 = 0$ and $I_1 = \int_0^{\pi}\left(\frac{\sin x}{\sin x}\right)dx = \pi$

Since $\quad I_{n+2} = I_n$, so

$$I_1 = I_3 = I_5 = \ldots = I_{|2n-1|} = \pi$$

and $\quad I_0 = I_2 = I_4 = \ldots = I_{2n} = \pi$

Thus, $\quad \sum_{m=1}^{10} I_{2m+1} = I_3 + I_5 + \ldots + I_{21} = 10\pi$

and $\quad \sum_{m=1}^{10} I_{2m} = I_2 + I_4 + \ldots + I_{20} = 0$

192. Let $\quad I = \int_0^1 \left(\frac{x^4(1-x)^4}{1+x^2}\right)dx$

$$= \int_0^1 \left(\frac{x^4(x^4 - 4x^3 + 6x^2 - 4x + 1)}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4)}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{(x^8\ 6x^6 + x^4)}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{(x^6(x^2+1) + x^4(x^2+1) + 4x^6)}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{(x^2(x^2+1)(x^2+1) + 4x^6)}{x^2+1}\right)dx$$

$$= \int_0^1 \left((x^4 + x^2) + \frac{4(x^6+1-1)}{x^2+1}\right)dx$$

$$= \int_0^1 \left((x^4 + x^2) - \frac{4}{x^2+1} + 4(x^4 - x^2 + 1)\right)dx$$

$$= \int_0^1 \left((5x^4 - 3x^2 + 4) - \frac{4}{x^2+1}\right)dx$$

$$= \left(x^5 - x^3 + 4x - 4\tan^{-1}x\right)_0^1$$

$$= [4 - 4\tan^{-1}(1)] = (4 - \pi)$$

193. Given $\quad f(a+b-x) = f(x)$,
then prove that

$$\int_a^b xf(x) = \frac{a+b}{2}\int_a^b f(x)\,dx$$

Let $\quad I = \int_a^b xf(x)\,dx \quad$ ...(i)

$$= \int_a^b (a + \ b - c)f(a+b-x)\,dx$$

$$= \int_a^b (a+b-x)f(x)\,dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_a^b (a+b)f(x)\,dx$$

$$\Rightarrow \quad I = \int_a^b \left(\frac{a+b}{2}\right)f(x)\,dx$$

194. Let $\quad I = \int_4^{10} \frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}\,dx \quad$ ...(i)

$$= \int_4^{10} \left(\frac{[x^2]}{[(14-\ x)]^2 + [x^2]}\right)dx$$

$$= \int_4^{10} \left(\frac{[14-x])^2}{[(14-\ x)]^2 + [x^2]}\right)dx$$

$$= \int_4^{10} \left(\frac{[14-\ x)]^2 + [x^2]}{[(14-\ x)]^2 + [x^2]}\right)dx \quad \text{...(ii)}$$

$$\Rightarrow \quad 2I = \int_4^{10} \left(\frac{[14-x)^2] + [x^2]}{[(14-x)^2] + [x^2]}\right)dx$$

$$= \int_4^{10} dx$$

$$= (10 - 4) = 6$$

$\Rightarrow \quad I = 3$

195. Let $\quad I = \int_{\pi/4}^{3\pi/4} \left(\frac{dx}{1+\cos x}\right) \quad$ ...(i)

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{dx}{1+\cos(\pi - x)}\right)$$

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{dx}{1+\cos(\pi - x)}\right) \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos(x)} + \frac{1}{1-\cos(x)}\right)dx$$

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{2}{1+\cos^2(x)}\right)dx$$

$$\Rightarrow \quad I = \int_{\pi/4}^{3\pi/4} \left(\frac{dx}{1-\cos^2(x)}\right)$$

$$= \int_{\pi/4}^{3\pi/4} (\text{cosec}^2 x)\, dx$$

$$= (-\cot x)_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = -(-1-1) = 2$$

196. Let $I = \int_{-\pi/2}^{\pi/2} \left(\frac{\cos x}{1 + e^x}\right) dx$ ...(i)

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos(-x)}{1 + e^{-x}}\right) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos(x)}{1 + e^{-x}}\right) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{e^x \cos(x)}{1 + e^x}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$\Rightarrow \quad 2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{(1 + e^x)\cos(x)}{1 + e^x}\right) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(x))\, dx$$

$$= (\sin x)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\Rightarrow \quad I = 1$$

197. Let $I = \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$ ...(i)

$$= \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{-\sin x} + 1}$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{e^{\sin x}}{e^{\sin x} + 1}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-\pi/2}^{\pi/2} \left(\frac{e^{\sin x} + 1}{e^{\sin x} + 1}\right) dx$$

$$= \int_{-\pi/2}^{\pi/2} dx$$

$$= \left(\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\right) = \pi$$

$$\Rightarrow \quad I = \left(\frac{\pi}{2}\right)$$

198. Let $I = \int_{50}^{100} \frac{\ln x}{\ln x + \ln(150 - x)}\, dx$ ...(i)

$$= \int_{50}^{100} \frac{\ln(150 - x)}{\ln(150 - x)(x)}\, dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{50}^{100} \frac{\ln(150 - x) + \text{lon}(x)}{\ln(150 - x) + \ln(x)}\, dx$$

$$= \int_{50}^{100} dx$$

$$= (100 - 50)$$

$$= 50$$

$$\Rightarrow \quad I = 25$$

199. Given $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f(x(2 - x))\, dx$ ...(i)

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2 - x) f((2 - x)x))\, dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{\sin^2 t}^{1+\cos^2 t} (2 - x + x) f((2 - x)x))\, dx$$

$$= \int_{\sin^2 t}^{1+\cos^2 t} 2 f((2 - x)x))\, dx$$

$$\Rightarrow \quad I_1 = \int_{\sin^2 t}^{1+\cos^2 t} f((2 - x)x))\, dx = I_2$$

$$= I_2$$

$$\Rightarrow \quad \frac{I_1}{I_2} = 1$$

200. Given $I_1 = \int_{f(-a)}^{f(a)} x g\{x(1 - x)\} dx$ ...(i)

Also $= \int_{f(-a)}^{f(a)} (1 - x) g\{(1 - x)x\} dx$ ...(ii)

Adding Eqs (i) and (ii), we get,

$$2I = \int_{f(-a)}^{f(a)} g\{(1 - x)x\} dx$$

$$\Rightarrow \quad 2I_1 = I_2$$

$\Rightarrow \quad \frac{I_1}{I_2} = \frac{1}{2}$

201. Do yourself

202. Do yourself.

203. Let $f(x) = \sin^5 x$

$\Rightarrow \quad f(-x) = \sin^5(-5) = -\sin^5 x$

$\therefore \quad f(x)$ is an odd function.

Thus, the value of the given integral

$$= \int_{-\pi/2}^{\pi/2} \sin^5 x\, dx = 0$$

204. Let $f(x) = x^3 \tan^4 x$

$\Rightarrow \quad f(-x) = (-x)^3 \tan^4(-x)$

$= -x^3 \tan^4 x$

$\therefore \quad f(x)$ is an odd function.

Thus, the value of the given integral

$$= \int_{-\pi/4}^{\pi/4} x^3 \tan^4 x\, dx = 0$$

205. Let $f(x) = \log\left(\frac{2 - \sin x}{2 + \sin x}\right)$

$\Rightarrow \quad f(-x) = \log\left(\frac{2 + \sin x}{2 - \sin x}\right)$

$= -\log\left(\frac{2 - \sin x}{2 + \sin x}\right) = -f(x)$

$\therefore \quad f(x)$ is an odd function.

Thus, the value of the given integral is

$$= \int_{-2}^{2} \log\left(\frac{2 - \sin x}{2 + \sin x}\right) dx = 0$$

206. Let $I = \int_{0}^{2} x(x-1)(x-2)\, dx$

$= \int_{-1}^{1} (t-1)t(t+1)\, dt$, where $x - 1 = t$

$= \int_{-1}^{1} t(t^2 - 1)\, dt$

$= \int_{-1}^{1} x(x^2 - 1)\, dx$

$= 0$, since it is an odd function.

207. Let

$$I = \int_{2010}^{2014} [(x-2010)(x-2011)(x-2012)$$
$$(x-2013)(x-2014)]\, dx$$

Let $x - 2012 = t$

$= \int_{-2}^{2} (t+2)(t+1)(t-1)(t-2)\, dt$

$= \int_{-2}^{2} (t^2 - 4)(t^2 - 1)\, dt$

$= \int_{-2}^{2} x(x^2 - 4)(x^2 - 1)\, dx$

$= 0, \quad$ ($\because$ it is an odd function.)

208. Let

$$I = \int_{-\pi/4}^{\pi/4} \left(\frac{x^{2013} - 3x^{2011} + 5x^{2009} - 7x^{2007} + 1007}{\cos^2 x}\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \left(\frac{x^{2013} - 3x^{2011} + 5x^{2009} - 7x^{2007}}{\cos^2 x}\right) dx$$
$$+ \int_{-\pi/4}^{\pi/4} \left(\frac{1007}{\cos^2 x}\right) dx$$

$= 0 + \int_{-\pi/4}^{\pi/4} \left(\frac{1007}{\cos^2 x}\right) dx$

$= 2\int_{0}^{\pi/4} \left(\frac{1007}{\cos^2 x}\right) dx$

$= 2014 \int_{0}^{\pi/4} \sec^2 x\, dx$

$= 2014 \times (\tan x)_0^{\pi/4}$

$= 2014 \times (1 - 0) = 2014$

209. Given $f(x) = \frac{\sin^{2013} x}{x^{2014} - x^{2012} + 1}$

$\Rightarrow \quad f(-x) = \frac{-\sin^{2013} x}{x^{2014} - x^{2012} + 1}$

$= -f(x)$

$\therefore \quad f(x)$ is an odd function.

Hence, the value of the given integral

$$= \int_{-2010}^{2010} f(x)\, dx = 0$$

210. Let

$$I = \int_{-10}^{10} \left(\tan^{2013} x + \left(\frac{2013\sqrt{x}}{1 + x^{2014}}\right) + 1007\right) dx$$

$$= \int_{-10}^{10} (0 + 0 + 1007)\, dx$$

$$= 2\int_0^{10} (2007)\,dx$$

$$= 2014 \times (10 - 0)$$

$$= 20140$$

211. Let $I = \int_{\sqrt{\ln(1/2)}}^{\sqrt{\ln(2)}} x^3\left(\frac{x^2}{e^{x^2} - 1} + \frac{x^2}{2} + 1\right)dx$

$$= \frac{1}{2}\int_{\ln(1/2)}^{\ln(2)} t\left(\frac{t}{et - 1} + \frac{t}{2} + 1\right)dt$$

$$= \frac{1}{2}\int_{-\ln(2)}^{\ln 2} x\left(\frac{x}{e^x - 1} + \frac{x}{2 + 1}\right)dx$$

Let $\left[f(x) = x\left(\frac{x}{e^x - 1} + \frac{x}{2 + 1}\right)\right.$

$\Rightarrow$ $f(-x) = -x\left(\frac{-x}{e^{-x} - 1} - \frac{x}{2 + 1}\right)$

$$= -x\left(\frac{x}{1 - e^{-x}} - \frac{x}{2 + 1}\right)$$

$$= -x\left(\frac{xe^x}{e^x - 1} - \frac{x}{2} + 1\right)$$

$$= -x\left(\left(\frac{xe^x}{e^x - 1} - x\right) + \frac{x}{2} + 1\right)$$

$$= -x\left(\left(\frac{xe^x - xe^x + x}{e^x - 1}\right) + \frac{x}{2} + 1\right)$$

$$= -x\left(\frac{x}{e^x - 1} + \frac{x}{2} + 1\right)$$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function.

$\Rightarrow$ The value of the given Integral is zero.

$I = 0$, ($\because$ it is an odd function)

212. The given integral can be written as

$$\int_{-2}^{0} ((x + 1)^3 + 2 + (x + 1)\cos(x + 1)\,dx$$

Let $(x + 1) = t$

$$= \int_{-1}^{2} (t^3 + 2 + t\cos t)\,dt$$

$$= \int_{-1}^{1} (t^3 + t\cos t)\,dt + \int_{-1}^{1} 2\,dt$$

$$= 0 + \int_{-1}^{1} 2\,dt$$

$$= 2[1-(-1)] = 4$$

213. We have,

$$f(x) = \begin{cases} 1 - x & : 0 \le x < 1 \\ x - 1 & : 1 \le x < 2 \end{cases}$$

and f is periodic with period 2.

Also, since $[-x] = 1 - [x]$ for non integral values of x

So, f is an even function.

Thus, $\int_{-10}^{10} f(x)\cos(\pi x)\,dx$

$$= 2\int_0^{10} f(x)\cos(\pi x)\,dx$$

$$= 2.5\int_0^{2} f(x)\cos(\pi x)\,dx$$

$$= 10\int_0^{2} f(x)\cos(\pi x)\,dx$$

$$= 10(I + J),$$

Let $I = \int_0^1 f(x)\cos(\pi x)\,dx$

and $J = \int_1^2 f(x)\cos(\pi x)\,dx$

Now, $I = \int_0^1 f(x)\cos(\pi x)\,dx$

$$= \int_0^1 (1 - x)\cos(\pi x)\,dx$$

$$= \int_0^1 (1-(1 - x))\cos(\pi(1 - x))\,dx$$

$$= -\int_0^1 x\cos(\pi x)\,dx$$

Also, $J = \int_1^2 (x - 1)\cos(\pi x)\,dx$

$$= \int_0^1 t\cos(\pi(t + 1))\,dt, \text{ where } x - 1 = t$$

$$= -\int_0^1 t\cos(\pi t)\,dt$$

$$= -\int_0^1 x\cos(\pi x)\,dx$$

$$= I$$

Thus, $I = J = -\int_0^1 x\cos(\pi x)\,dx$

$$= \left(-\frac{x}{\pi}\sin(\pi x)\right)_0^1 + \frac{1}{\pi}\int_0^1 \sin(\pi x)\,dx$$

$$= -\frac{1}{\pi^2}(\cos\pi x)_0^1$$

$$= \frac{2}{\pi^2}$$

Therefore, $\int_{-10}^{10} f(x)\cos(\pi x)\,dx$

$$= 10\,(I + J)$$

$$= 20\,I$$

$$= 20 \times \frac{2}{\pi^2}$$

Now, $\frac{\pi^2}{10} \times \int_{10}^{10} f(x)\cos(\pi x)\,dx = \frac{\pi^2}{10} \times \frac{40}{\pi^2}$

$$= 4$$

214. Let

$$I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$$

$$= \int_{-2}^{2} (t+2)(t+1)\,t\,(t-1)(t-2)\,dt$$

$$= \int_{-2}^{2} t(t^2-4)(t^2-1)\,dt$$

$= 0,$ ($\because$ it is an odd function.)

215. Let $I = \int_{-\pi}^{\pi} \left(\frac{2x(1+\sin x)}{1+\cos^2 x}\right)dx$

$$= \int_{-\pi}^{\pi} \left(\frac{2x}{1+\cos^2 x}\right)dx + \int_{-\pi}^{\pi} \left(\frac{2x\sin x}{1+\cos^2 x}\right)dx$$

$$= 0 + 4\int_0^{\pi} \left(\frac{x\sin x}{1+\cos^2 x}\right)dx$$

$$= 4\int_0^{\pi} \left(\frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\right)dx$$

$$= 4\int_0^{\pi} \left(\frac{(\pi - x)\sin x}{1+\cos^2 x}\right)dx$$

$$= 4\pi\int_0^{\pi} \left(\frac{\sin x}{1+\cos^2 x}\right)dx - 4\int_0^{\pi} \left(\frac{x\sin x}{1+\cos^2 x}\right)dx$$

$$= 4\pi\int_0^{\pi} \left(\frac{\sin x}{1+\cos^2 x}\right)dx - I$$

$$\Rightarrow \quad 2I = 4\pi\int_0^{\pi} \left(\frac{\sin x}{1+\cos^2 x}\right)dx$$

$$\Rightarrow \quad I = 2\pi\int_0^{\pi} \left(\frac{\sin x}{1+\cos^2 x}\right)dx$$

$$= 4\pi\int_0^{\pi/2} \left(\frac{\sin x}{1+\cos^2 x}\right)dx$$

$$= -4\pi\int_1^0 \frac{dt}{1+t^2}$$

$$= -4\pi\left(\tan^{-1} t\right)_1^0$$

$$= -4\pi\left(-\frac{\pi}{4}\right)$$

$$= 4 \times \frac{\pi^2}{4}$$

$$= \pi^2$$

216. Let $I = \int_{-\pi/3}^{\pi/3} \left(\frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right)dx$

$$= \int_{-\pi/3}^{\pi/3} \left(\frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right)dx + \int_{-\pi/3}^{\pi/3} \left(\frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right)dx$$

$$= \int_{-\pi/3}^{\pi/3} \left(\frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right)dx + 0$$

$$= 2\int_0^{\pi/3} \left(\frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\right)dx$$

$$= 2\pi\int_0^{\pi/3} \left(\frac{dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}\right)$$

$$= 2\pi\int_{\pi/3}^{2\pi/3} \left(\frac{dt}{2 - \cos t}\right), \quad t = x + \pi/3$$

$$= 2\pi\int_{\pi/3}^{2\pi/3} \left(\frac{dt}{1 + (1 - \cos t)}\right)$$

$$= 2\pi\int_{\pi/3}^{2\pi/3} \left(\frac{dt}{1 + 2\sin^2(t/2)}\right)$$

$$= 4\pi\int_{\pi/6}^{\pi/3} \left(\frac{dy}{1 + 2\sin^2 y}\right), \qquad \text{Let } t/2 = y$$

$$= 4\pi\int_{\pi/6}^{\pi/3} \left(\frac{\sec^2 y\,dy}{\sec^2 y + 2\tan^2 y}\right)$$

$$= 4\pi\int_{\pi/6}^{\pi/3} \left(\frac{\sec^2 y}{1 + 3\tan^2 y}\right)dy$$

$$= \frac{4\pi}{3}\int_{\pi/6}^{\pi/3} \left(\frac{\sec^2 y}{(1/\sqrt{3})^2 + \tan^2 y}\right)dy$$

$$= \frac{4\pi}{3}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dv}{(1/\sqrt{3})^2 + v^2}$$

$$= \frac{4\pi}{3} \times \sqrt{3}\left(\tan^{-1}(v\sqrt{3})\right)_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}}\left(\tan^{-1}(3) - \frac{\pi}{4}\right)$$

217. Let

$$I = \int_{-1}^{1}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right) + \tan^{-1}\left(\frac{1+x^2}{x}\right)\right)dx$$

$$= \int_{-1}^{0}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right) + \tan^{-1}\left(\frac{1+x^2}{x}\right)\right)dx + \int_{0}^{1}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right) + \tan^{-1}\left(\frac{1+x^2}{x}\right)\right)dx$$

$$= \int_{-1}^{0}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right) + \cot^{-1}\left(\frac{x}{1+x^2}\right)\right)dx + \int_{0}^{1}(\tan^{-1}\left(\frac{x}{1+x^2}\right) + \cot^{-1}\left(\frac{x}{1+x^2}\right)dx$$

$$= \int_{-1}^{0}\left(-\pi + \frac{\pi}{2}\right)dx + \int_{0}^{1}\left(\frac{\pi}{2}\right)dx$$

$$= \int_{-1}^{0}\left(-\frac{\pi}{2}\right)dx + \int_{0}^{1}\left(\frac{\pi}{2}\right)dx$$

$$= \left(-\frac{\pi}{2}\right)(0-(-1)) + \left(\frac{\pi}{2}\right)(1-0)$$

$$= \left|-\frac{\pi}{2} + \frac{\pi}{2}\right|$$

$$= 0$$

237. Let $I = \int_0^{\pi} \sin x\, dx$

$$= 2\int_0^{\pi/2} \sin x\, dx \quad [\because \sin(\pi - x) = \sin x]$$

$$= 2(-\cos x)_0^{\pi/2}$$

$$= 2[-(0-1)]$$

$$= 2$$

238. Let $I = \int_0^{\pi} \cos x\, dx$

$$= 0 \quad [\because \cos(\pi - x) = -\cos x]$$

239. Let $I = \int_0^{\pi} \cos^{2103} x\, dx$

$$= 0 \quad [\because \cos(\pi - x) = -\cos x]$$

240. Let $I = \int_0^{\pi} (\sin^3 x + \cos^5 x + \tan^7 x)\, dx$

$$= \int_0^{\pi}(\sin^3 x)\,dx + \int_0^{\pi}(\cos^5 x)\,dx + \int_0^{\pi}(\tan^7 x)\,dx$$

$$= \int_0^{\pi}(\sin^3 x)\,dx + 0 + 0$$

$$= \int_0^{\pi}(\sin^3 x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi}(4\sin^3 x)\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}(3\sin x - \sin^3 x)\,dx$$

$$= \frac{1}{2}\left(-3\cos x + \frac{\cos^3 x}{3}\right)_0^{\pi/2}$$

$$= \frac{1}{2}\left(-3\left(0 - \left(1 + \frac{1}{3}\right)\right)\right)$$

$$= \frac{1}{2}\left(-3 \times -\frac{4}{3}\right)$$

$$= 2$$

241. Let $I = \int_0^{\pi/2} \log\sin x\, dx$ ...(i)

$$= \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} \log\cos x\, dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log\sin x\, dx + \int_0^{\pi/2} \log\cos x\, dx$$

$$= \int_0^{\pi/2} (\log\sin x + \log\cos x)\, dx$$

$$= \int_0^{\pi/2} (\log\sin x\cos x)\, dx$$

$$= \int_0^{\pi/2} \left(\log\left(\frac{2\sin x\cos x}{2}\right)\right)dx$$

$$= \int_0^{\pi/2} \left(\log\left(\frac{\sin 2x}{2}\right)\right)dx$$

$$= \int_0^{\pi/2} (\log(\sin 2x))dx - \int_0^{\pi/2} (\log 2)\, dx$$

Let $2x = t$

$$= \frac{1}{2}\int_0^{\pi} [\log(\sin t)]\,dt - \int_0^{\pi/2} [\log 2]\,dx$$

$$= \frac{1}{2}\int_0^{\pi} [\log(\sin x)]dx - \int_0^{\pi/2} [\log 2]\,dx$$

$$= \frac{1}{2}\int_0^{\pi} [\log(\sin x)]\,dx - \frac{\pi}{2}\log|2|$$

$$= \frac{2}{2}\int_0^{\pi/2} [\log(\sin x)]\,dx - \frac{\pi}{2}\log[2]$$

$$= I - \frac{\pi}{2}\log|2|$$

$$\Rightarrow \quad 2I = I - \frac{\pi}{2}\log 2$$

$$\Rightarrow \quad I = -\frac{\pi}{2}\log 2 = \frac{1}{2}\log\left(\frac{1}{2}\right)$$

Notes 1. $\int_0^{\pi/2} \log\cos dx = \frac{1}{2}\log\left(\frac{1}{2}\right)$

2. $\int_0^{\pi/2} \log\sec x\,dx = -\frac{1}{2}\log\left(\frac{1}{2}\right)$

3. $\int_0^{\pi/2} \log\operatorname{cosec} x\,dx = -\frac{1}{2}\log\left(\frac{1}{2}\right)$

4. $\int_0^{\pi/2} \log\left(\frac{\sin x\cos x}{\cot x}\right)dx = -\log\left(\frac{1}{2}\right)$

5. $\int_0^{\pi/2} \log\left(\frac{\sin x\cos x}{\tan x}\right)dx = -\log\left(\frac{1}{2}\right)$

242. Let $I = \int_0^{\pi} x\log(\sin x)\,dx$

$$= \int_0^{\pi} (\pi - x)\log(\sin(\pi - x))\,dx$$

$$= \int_0^{\pi} (\pi - x)\log(\sin x)\,dx$$

$$= \int_0^{\pi} \pi\log(\sin x)\,dx - \int_0^{\pi} x\log(\sin x)\,dx$$

$$\Rightarrow \quad I = \pi\int_0^{\pi} \log(\sin x)\,dx - I$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \log(\sin x)\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \log(\sin x)\,dx$$

$$\Rightarrow \quad I = 2 \times \frac{\pi}{2}\int_0^{\pi/2} \log(\sin x)\,dx$$

$$\Rightarrow \quad I = 2 \times \frac{\pi}{2}\left(\frac{\pi}{2}\log\left(\frac{1}{2}\right)\right)$$

$$\Rightarrow \quad I = \left|\frac{\pi^2}{2}\log\left(\frac{1}{2}\right)\right|$$

243. Let $I = \int_0^{\pi/2} \sin 2x\log(\tan x)\,dx$ ...(i)

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right)\log\left(\tan\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x)\log\left(\tan\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x)\log(\cot x)\,dx$$

$$= \int_0^{\pi/2} \sin 2x\log(\cot x)\,dx \quad \text{...(i)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \sin 2x\log(\tan x)\,dx + \int_0^{\pi/2} \sin 2x\log(\cot x)\,dx$$

$$= \int_0^{\pi/2} \sin 2x\,[\log(\tan x) + \log(\cot x)]\,dx$$

$$= \int_0^{\pi/2} \sin 2x\,[\log(\tan x\cdot\cot x)]\,dx$$

$$= \int_0^{\pi/2} \sin 2x(\log 1)\,dx$$

$$= 0$$

$$\Rightarrow \quad I = 0$$

244. Let $I = \int_0^{\pi} \left(\frac{x\,dx}{1 + \cos\alpha\cdot\sin x}\right)$

$$= \int_0^{\pi} \left(\frac{(\pi - x)\,dx}{1 + \cos\alpha\cdot\sin(\pi - x)}\right)$$

$$= \int_0^{\pi} \left(\frac{(\pi - x)dx}{1 + \cos\alpha\cdot\sin x}\right)$$

$$= \int_0^{\pi} \left(\frac{\pi\,dx}{1 + \cos\alpha\cdot\sin x}\right) - \int_0^{\pi}\left(\frac{x\,dx}{1 + \cos\alpha\cdot\sin x}\right)$$

$$= \int_0^{\pi} \left(\frac{\pi dx}{1 + \cos\alpha\cdot\sin x}\right) - I$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \left(\frac{\pi dx}{1 + \cos\alpha\cdot\sin x}\right)$$

$$= 2\int_0^{\pi/2} \left(\frac{\pi\,dx}{1 + \cos\alpha\cdot\sin x}\right)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \left(\frac{\pi dx}{1 + k\sin x}\right), \qquad \text{(where } k = \cos\alpha\text{)}$$

$$= \int_0^{\pi/2} \left(\frac{\pi dx}{1 + k\left(\frac{2\tan(x/2)}{1 + \tan^2(x/2)}\right)}\right)$$

$$= \int_0^{\pi/2} \left(\frac{\pi(1 + \tan^2(x/2))dx}{(1 + \tan^2(x/2)) + k(2\tan(x/2))}\right)$$

$$= \pi \int_0^{\pi/2} \left(\frac{\sec^2(x/2)dx}{(1 + \tan^2(x/2)) + k(2\tan(x/2))}\right)$$

$$= 2\pi \int_0^1 \left(\frac{dt}{(1 + t^2) + 2kt}\right)$$

$$= 2\pi \int_0^1 \left(\frac{dt}{(t + k)^2 + (1 - k)^2}\right)$$

$$= 2\pi \times \frac{1}{\sqrt{1 - k^2}} \tan^{-1}\left(\frac{t + k}{\sqrt{1 - k^2}}\right)_0^1$$

$$= 2\pi \times \frac{1}{\sqrt{1 - k^2}} \left(\tan^{-1}\left(\frac{1 + \cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right)$$

$$= 2\pi \times \frac{1}{\sqrt{1 - k^2}} \left[\tan^{-1}\left(\frac{2\cos^2(\alpha/2)}{2\sin(\alpha/2)\cos(\alpha/2)}\right) - \left(\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right)\right]$$

$$= \frac{2\pi}{\sqrt{1 - k^2}} \left[\tan^{-1}\left(\cot\left(\frac{\alpha}{2}\right) - \tan^{-1}(\cot\alpha)\right)\right]$$

$$= \frac{2\pi}{\sqrt{1 - k^2}} \left[\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - \left(\frac{\pi}{2} - \alpha\right)\right]$$

$$= \frac{2\pi}{\sqrt{1 - k^2}} \times \frac{\alpha}{2}$$

$$= \frac{\pi\alpha}{\sin\alpha}$$

245. Let $\quad I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$

$$= \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log\left(\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \log(\sqrt{2})\, dx + \int_{-\pi/4}^{\pi/4} \log\left(\sin\left(x + \frac{\pi}{4}\right)\right) dx$$

$$= 2\int_{-0}^{\pi/4} \log(\sqrt{2})\, dx + \int_{-\pi/4}^{\pi/4} \log\left(\sin\left(x + \frac{\pi}{4}\right)\right) dx$$

$$= 2 \times \frac{1}{2} \times \frac{\pi}{4} \times \log 2 + \int_{-\pi/4}^{\pi/4} \log\left(\sin\left(x + \frac{\pi}{4}\right)\right) dx$$

$$= \frac{\pi}{4} \times \log 2 + \int_{-\pi/4}^{\pi/4} \log\left(\sin\left(x + \frac{\pi}{4}\right)\right) dx$$

Let $\left(x + \frac{\pi}{4}\right) = t \Rightarrow dx = dt$

$$= \frac{\pi}{4} \times \log 2 + \int_0^{\pi/2} \log(\sin t)\, dt$$

$$= \frac{\pi}{4} \times \log 2 + \int_0^{\pi/2} \log(\sin x)\, dx$$

$$= \frac{\pi}{4} \times \log 2 - \frac{\pi}{2} \times \log 2$$

$$= -\frac{\pi}{2} \times \log 2$$

246. Let $\quad I = \int_0^{\pi} \frac{dx}{3\sin^2 x + 4\cos^2 x}$

$$= 2\int_0^{\pi/2} \frac{dx}{3\cos^2 x + 4\sin^2 x}$$

$$= 2\int_0^{\pi/2} \left(\frac{\sec^2 x\, dx}{3 + 4\tan^2 x}\right)$$

$$= 2\int_0^{\infty} \frac{dt}{3 + 4t^2}, \quad \text{Let } t = \tan x \Rightarrow dt = \sec^2 x dx$$

$$= \frac{1}{2}\int_0^{\infty} \left(\frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)_0^{\infty}$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{2\sqrt{3}}$$

247. Let $I = \int_0^{\pi} \left(\frac{x\sin x}{1 + \cos^2 x}\right) dx$

$$= \int_0^{\pi} \left(\frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\right) dx$$

$$= \int_0^{\pi} \left(\frac{(\pi - x)\sin x}{1 + \cos^2 x}\right) dx$$

$$= \int_0^{\pi} \left(\frac{\pi\sin x}{1 + \cos^2 x}\right) dx - \int_0^{\pi} \left(\frac{x\sin x}{1 + \cos^2 x}\right) dx$$

$$= \int_0^{\pi} \left(\frac{\pi\sin x}{1 + \cos^2 x}\right) dx - I$$

$\Rightarrow$ $$2I = \int_0^{\pi} \left(\frac{\pi\sin x}{1 + \cos^2 x}\right) dx$$

$$= 2\pi \int_0^{\pi/2} \left(\frac{\sin x}{1 + \cos^2 x}\right) dx$$

$\Rightarrow$ $$I = \pi \int_0^{\pi/2} \left(\frac{\sin x}{1 + \cos^2 x}\right) dx$$

$$= -\pi \int_0^{\infty} \left(\frac{dt}{1 + t^2}\right) \qquad \text{Let } \cos x = t$$

$$= -\pi(\tan^{-1} t)_0^{\infty}$$

$$= -\pi\left(\frac{\pi}{2} - 0\right)$$

$$= -\frac{\pi^2}{2}$$

248. Let $I = \int_0^{\infty} \log\left(x + \frac{1}{x}\right)\frac{dx}{1 + x^2}$

$$= \int_0^{\pi/2} \left(\frac{\log(\tan\theta + \cot\theta)\sec^2\theta}{(1 + \tan^2\theta)}\right) d\theta$$

$$= \int_0^{\pi/2} \log(\tan\theta + \cot\theta)\, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{2\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{\sin 2\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log 2\, d\theta - \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

$$= \frac{\pi}{2}\log 2 - \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt$$

$$= \frac{\pi}{2}\log 2 - \frac{1}{2}\int_0^{\pi} \log(\sin x)\, dx$$

$$= \frac{\pi}{2}\log 2 - \int_0^{\pi/2} \log(\sin x)\, dx$$

$$= \frac{\pi}{2}\log 2 - \left(-\frac{\pi}{2}\log 2\right)$$

$$= \pi \log 2$$

249. We have, $I_2 = \int_0^1 x^{1006}\left(1 - x^{2014}\right)^{1006} dx$

$$= \int_0^1 x^{1006}\left[1 - \left(x^{1007}\right)^2\right]^{1007} dx$$

Let $x^{1007} = t \Rightarrow 1007 \times x^{1006}\, dx = dt$

$$= \frac{1}{1007}\int_0^1 (1 - t^2)^{1007} dt$$

$$= \frac{1}{1007}\int_0^1 [1 - (1 - t)^2]^{1007} dt$$

$$= \frac{1}{1007}\int_0^1 [t(2 - t)]^{1007} dt$$

$$= \frac{1}{1007}\int_0^1 t^{1007}(2 - t)^{1007} dt$$

Let $t = 2y \Rightarrow 2\, dt = 2dy$

$$= \frac{2^{2014}}{1007}\int_0^1 (y)^{1007}(1 - y)^{1007} dy$$

$$= \frac{2^{2014}}{1007}\int_0^1 (x)^{1007}(1 - x)^{1007} dy$$

$$= \frac{2^{2014}}{1007} \times I_1$$

Now,

$$2^{2014} \times \left(\frac{I_2}{I_1}\right)$$

$$= 2^{2014} \times \left(\frac{2^{2014}}{1007} \times I_1 \times \frac{1}{I_1}\right)$$

$$= \frac{2^{4028}}{1007}$$

250. Do yourself.

251. Let $I = \int_0^1 \frac{\log x}{\sqrt{1 - x^2}}\, dx$

Let $x = \sin\theta$

$\Rightarrow dx = \cos\theta d\theta$

$$= \int_0^{\pi/2} \frac{\log(\sin\theta) \cdot \cos\theta d\theta}{\cos\theta}$$

$$= \int_0^{\pi/2} \log(\sin\theta)\, d\theta$$

$$= \left|-\frac{\pi}{2}\log 2\right|$$

$$= \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

252. Let $I = \int_0^{\pi} \log(1 - \cos x)\, dx$

$$= \int_0^{\pi} \log\left(2\sin^2\left(\frac{x}{2}\right)\right) dx$$

$$= \int_0^{\pi} \left(\log 2 + \log\left(\sin^2\left(\frac{x}{2}\right)\right)\right) dx$$

$$= |\pi \log(2)| + \int_0^{\pi} \left(\log\left(\sin^2\left(\frac{x}{2}\right)\right)\right) dx$$

Let $I_1 = \int_0^{\pi} \left(\log\left(\sin^2\left(\frac{x}{2}\right)\right)\right) dx$ ...(i)

$$= 2\int_0^{\pi} \left(\log\left(\sin\left(\frac{x}{2}\right)\right)\right) dx$$

$$= 2\int_0^{\pi} \left(\log\left(\sin\left(\frac{\pi - x}{2}\right)\right)\right) dx$$

$$= 2\int_0^{\pi} \left(\log\left(\cos\left(\frac{x}{2}\right)\right)\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I_1 = 2\int_0^{\pi} \left(\log\left(\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)\right) dx$$

$$\Rightarrow I_1 = \int_0^{\pi} \left(\log\left(\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)\right) dx$$

$$= \int_0^{\pi} \left(\log\left\{\frac{1}{2}\left(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)\right\}\right) dx$$

$$= \pi\log\left(\frac{1}{2}\right) + \int_0^{\pi} (\log(\sin x))\, dx$$

$$= \pi\log\left(\frac{1}{2}\right) + 2\int_0^{\pi} (\log(\sin x))\, dx$$

$$= \pi\log\left(\frac{1}{2}\right) + 2\left(-\frac{\pi}{2}\log(2)\right)$$

$$= \pi\log\left(\frac{1}{2}\right) + 2\left(\frac{\pi}{2}\log\left(\frac{1}{2}\right)\right)$$

$$= \pi\log\left(\frac{1}{2}\right) + \left(\pi\log\left(\frac{1}{2}\right)\right)$$

$$= 2\pi\log\left(\frac{1}{2}\right)$$

Thus, $I = \pi\log|2| + 2\pi\log\left(\frac{1}{2}\right)$

$$= \pi\log(2) - 2\pi\log|2|$$

$$= -\pi\log|2|$$

253. Let $I = \int_0^{\infty} \log\left(x + \frac{1}{x}\right)\frac{dx}{1 + x^2}$

$$= \int_0^{\pi/2} \log(\tan\theta + \cot\theta)\, d\theta, \text{ where } x = \tan\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{\sin 2\theta}\right) d\theta$$

$$= \left(\frac{\pi}{2}\log 2\right) - \int_0^{\pi/2} \log(\sin(2\theta))\, d\theta$$

$$= \left(\frac{\pi}{2}\log 2\right) - \frac{1}{2}\int_0^{\pi} \log(\sin(t))\, d\theta$$

$$= \left(\frac{\pi}{2}\log 2\right) - \int_0^{\pi} \log(\sin(t))\, d\theta$$

$$= \left(\frac{\pi}{2}\log 2\right) - \left(-\frac{\pi}{2}\log|2|\right)$$

$$= 2\left(\frac{\pi}{2}\log 2\right)$$

$$= |\pi\log(2)|$$

254. Let $I = \int_0^{\pi} \frac{x}{1 + \cos^2 x}\, dx$ ...(i)

$$= \int_0^{\pi} \frac{(\pi - x)}{1 + \cos^2 x}\, dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{dx}{1 + \cos^2 x}$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{1 + \cos^2 x}$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{dx}{1 + \cos^2 x}$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x\, dx}{\tan^2 x + 2}$$

$$= \frac{\pi}{\sqrt{2}} \left(\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)\right)_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{\sqrt{2}} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{2\sqrt{2}}$$

255. Let $\quad I = \int_0^{400\pi} \sqrt{1 - \cos 2x}\, dx$

$$= \int_0^{400\pi} \left(\sqrt{2\sin^2 x}\right) dx$$

$$= \int_0^{400\pi} (\sqrt{2}\, |\sin x|)\, dx$$

$$= \sqrt{2} \int_0^{400\pi} |\sin x|\, dx$$

$$= \sqrt{2} \times 400 \int_0^{\pi} |\sin x| dx$$

$$= 400 \times \sqrt{2} \int_0^{\pi} \sin x\, dx$$

$$= 400 \times \sqrt{2}\, (-\cos x)_0^{\pi}$$

$$= 400 \times \sqrt{2}\, (-(-1 - 1))$$

$$= 800\sqrt{2}$$

256. Let $\quad I = \int_0^{1000} e^{x-[x]} dx$

$$= 1000 \int_0^{1} e^{x-[x]} dx$$

$$= 1000 \int_0^{1} e^{x-0} dx$$

$$= 1000\, (e^x)_0^1$$

$$= 1000\, (e^1 - 1)$$

$$= 1000\, (e - 1)$$

257. Let $\quad I = \int_0^{2000\pi} \left(\frac{dx}{1 + 5^{\tan x}}\right)$

$$= 1000 \int_0^{2\pi} \left(\frac{dx}{1 + 5^{\sin x}}\right)$$

$$= 1000 \times \int_0^{2\pi} \frac{dx}{1 + 5^{\sin x}}$$

$$= 1000 \times \int_0^{2\pi} \frac{dx}{1 + 5^{\sin(2\pi - x)}}$$

$$= 1000 \times \int_0^{2\pi} \frac{dx}{1 + 5^{-\sin x}}$$

$$= 1000 \times \int_0^{2\pi} \frac{5^{\sin x} dx}{1 + 5^{\sin x}}$$

$$\Rightarrow \quad 2I = 1000 \times \int_0^{2\pi} \left(\frac{1 + 5^{\sin x}}{1 + 5^{\sin x}}\right) dx$$

$$= 1000 \times \int_0^{2\pi} dx = 1000 \times 2\pi$$

$$\Rightarrow \quad I = 1000 \times \pi$$

258. Let $\quad I = \int_0^{1007} \{2x\} dx$, where $\{,\}$ = FIP

$$= 2014 \int_0^{1/2} \{2x\} dx,$$

where the given function is periodic with period 1/2.

$$= 2014 \int_0^{1/2} (2x - [2x])\, dx$$

$$= 2014 \int_0^{1/2} (2x - 0)\, dx$$

$$= 2014 \times \left(\frac{2x^2}{2}\right)_0^{1/2}$$

$$= 2014 \times \frac{1}{4}$$

$$= \frac{1007}{2}$$

259. Let $I = \int_0^{\pi} |\sin x + \cos x|\, dx$

$= \sqrt{2}\int_0^{\pi} \left|\sin\left(x + \frac{\pi}{4}\right)\right| dx$

$= \sqrt{2}\int_{\pi/4}^{\pi+\pi/4} |\sin y|\, dy$, Let $x + \frac{\pi}{4} = y$

$= \sqrt{2}\int_0^{\pi} |\sin x|\, dx$

$= 2\sqrt{2}$, since $\int_0^{\pi} |\sin x|\, dx = 2$

260. Let $I = \int_0^{100\pi} (|\sin x| + |\cos x|)\, dx$

$= 200\int_0^{\pi/2} (|\sin x| + |\cos x|)\, dx$

$= 200\int_0^{\pi/2} (\sin x + \cos x)\, dx$

$= 200\sqrt{2}\int_0^{\pi/2} \left(\sin\left(x + \frac{\pi}{4}\right)\right) dx$

$= 200\sqrt{2}\int_{\pi/4}^{\pi/2+\pi/4} \sin y\, dy$, Let $x + \frac{\pi}{4} = y$

$= 200\sqrt{2}\int_0^{\pi/2} \sin x\, dx$

$= 200\sqrt{2}$

261. Let $I = \int_0^{100\pi} (|\sin x| - |\cos x|)\, dx$

$= 100\int_0^{\pi} (|\sin x| - |\cos x|) dx$

$= 100\left(\int_0^{\pi} (\sin x)\, dx - \int_0^{\pi/2} (\cos x) dx + \int_{\pi/2}^{\pi} (\cos x)\, dx\right)$

$= 100\left((-\cos x)_0^{\pi} - (\sin x)_0^{\pi/2} + (\sin x)_{\pi/2}^{\pi}\right)$

$= 100(2 - (-1) + (-1))$

$= 200$

262. Let $I = \int_{-503\pi}^{504\pi} |\cos x|\, dx$

$= (504 - (-503))\int_0^{\pi} |\cos x|\, dx$

$= 1007 \times \int_0^{\pi} |\cos x| dx$

$= 1007 \times \left(\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right)$

$= 1007 \times \left((\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right)$

$= 1007 \times (1 - (-1))$

$= 2014$

263. Let $I = \int_0^{50\pi/3} \sqrt{\left(\frac{1 - \cos 2x}{2}\right)}\, dx$

$= \int_0^{50\pi/3} \sqrt{\sin^2 x}\, dx$

$= \int_0^{50\pi/3} |\sin x|\, dx$

$= \int_0^{16\pi+2\pi/3} |\sin x| dx$

$= \int_0^{16\pi} |\sin x|\, dx + \int_{16\pi}^{16\pi+2\pi/3} |\sin x|\, dx$

$= \int_0^{16\pi} |\sin x|\, dx + \int_0^{2\pi/3} |\sin x|\, dx$

$= 16\int_0^{\pi} |\sin x|\, dx + \int_0^{2\pi/3} |\sin x|\, dx$

$= 16\int_0^{\pi} \sin x\, dx + \int_0^{2\pi/3} \sin x\, dx$

$= 16(-\cos x)_0^{\pi} + (-\cos x)_0^{2\pi/3}$

$= 16(-(-1 - 1)) + \left(-\left(-\frac{1}{2} - 1\right)\right)$

$= 16 \times 2 + \frac{3}{2}$

$= \frac{67}{2}$

264. Given $\alpha = \int_0^{1} \frac{\sin t}{1 + t}\, dt$

Now, $\int_{4\pi-2}^{4\pi} \left(\frac{\sin\left(\frac{t}{2}\right)}{4\pi + 2 - t}\right) dt$

$= \int_{4\pi-2}^{4\pi} \left(\frac{\sin\left(\frac{t}{2}\right)}{4\pi + 2 - t}\right) dt$

$= \int_{4\pi-2}^{4\pi} \left(\frac{\sin\left(\frac{t}{2}\right)}{2 + (4\pi - t)}\right) dt$

$$= \frac{1}{2}\int_{4\pi-2}^{4\pi}\left(\frac{\sin\left(\frac{t}{2}\right)}{1+\left(2\pi-\frac{t}{2}\right)}\right)dt$$

Let $\left(2\pi - \frac{t}{2}\right) = y \Rightarrow dt = -2\,dy$

$$= -\int_1^0\left(\frac{\sin(2\pi - y)}{1+y}\right)dy$$

$$= -\int_1^0\left(\frac{\sin y}{1+y}\right)dy$$

$$= -\int_0^1\left(\frac{\sin t}{1+t}\right)dt$$

$$= -\alpha$$

265. Given $\int_0^1\left(\frac{e^t}{t+1}\right)dt$

We have $\int_{b-1}^{b}\left(\frac{e^{-1}}{(t-b-1)}\right)dt$

Let $t - b = -y \Rightarrow dt = -dy$

$$= -\int_1^0\left(\frac{e^{-b}\cdot e^y}{-(y+1)}\right)dy$$

$$= -e^{-b}\int_0^1\left(\frac{e^y}{(y+1)}\right)dy$$

$$= -e^{-b}\int_0^1\left(\frac{e^t}{(t+1)}\right)dt$$

$$= -a^{e-b}$$

266. Given $F(x) + F\left(\frac{1}{x} + 2\right) = 10$ ...(i)

Replacing x by $\left(x + \frac{1}{2}\right)$, we get

$$F\left(x + \frac{1}{2}\right) + F(x+1) = 10 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$F(x) - F(x+1) = 0$$

$$\Rightarrow \quad F(x) = F(x+1)$$

$\Rightarrow$ Thus, the period of $F(x)$ is 1.

Now,

$$\int_0^{201} F(x)dx$$

$$= \int_0^{201} F(x)dx$$

$$= 201 \times \int_0^1 F(x)dx$$

$$= 201 \times \left(\int_0^{1/2} F(x)dx + \int_{1/2}^{1} F(x)dx\right)$$

Putting $x = \left(y + \frac{1}{2}\right) \Rightarrow dx = dy$ in the 2nd integral, we have

$$201 \times \left(\int_0^{1/2} F(x)dx + \int_0^{1/2} F\left(y + \frac{1}{2}\right)dy\right)$$

$$= 201 \times \left(\int_0^{1/2} F(x)dx + \int_0^{1/2} F\left(x + \frac{1}{2}\right)dx\right)$$

$$= 201 \times \int_0^{1/2}\left(F(x) + F\left(x + \frac{1}{2}\right)\right)dx$$

$$= 201 \times 10$$

$$= 2010$$

267. Let $I = \int_0^{50\pi}(\sin^4 x + \cos^4 x)\,dx$

$$= \int_0^{100\times\frac{\pi}{2}}(\sin^4 x + \cos^4 x)\,dx$$

$$= 100 \times \int_0^{\pi/2}(\sin^4 x + \cos^4 x)\,dx$$

$$= 100 \times \int_0^{\pi/2}(1 - 2\sin^2 x\cos^2 x)dx$$

$$= 50 \times \int_0^{\pi/2}(2 - 4\sin^2 x\cos^2 x)dx$$

$$= 50 \times \int_0^{\pi/2}(2 - \sin^2 2x)dx$$

$$= 25 \times \int_0^{\pi/2}(4 - 2\sin^2 2x)dx$$

$$= 25 \times \int_0^{\pi/2}[-4(1 - \cos^4 x)]dx$$

$$= 25 \times \int_0^{\pi/2}(3 + \cos^4 x)dx$$

$$= 100 \times \int_0^{\pi/2}\left(\frac{3}{4} + \frac{1}{4}\cos^4 x\right)dx$$

Thus, the value of k is 100.

268. We have,

$$g(x+\pi) = \int_0^{x+\pi}\cos^4 t\,dt$$

$$= \int_0^{\pi}\cos^4 t\,dt + \int_0^{x+\pi}\cos^4 t\,dt$$

$$= \int_0^{\pi}\cos^4 t\,dt + \int_0^{x}\cos^4 t\,dt$$

$$= g(\pi) + g(x)$$

$$= g(x) + g(\pi)$$

269. $\int_0^{2014\pi} \sqrt{1-\cos^2 x}\ dx$

$$= \int_0^{2014\pi} \sqrt{\sin^2 x}\ dx$$

$$= \int_0^{2014\pi} |\sin x| dx$$

$$= 2014 \int_0^{\pi} |\sin x| dx$$

$$= 2014 \times 2$$

$$= 4028.$$

270. $\int_0^{2012} e^{x-[x]} dx$

$$= 2012 \int_0^1 e^{x-[x]} dx$$

$$= 2012 \int_0^1 e^x dx$$

$$= 2012 (e^x)_0^1$$

$$= 2012(e-1)$$

271. $\int_0^{4\pi} |\cos x|\ dx$

$$= 4 \int_0^{\pi} |\cos(x)| dx$$

$$= 4 \times 2 = 8$$

272. $\int_0^{10\pi} |\sin x|\ dx$

$$= 10 \int_0^{\pi} |\sin(x)|\ dx$$

$$= 10 \times 2$$

$$= 20$$

273. $\int_0^{[x]} (x-[x]) dx$, where [,] = G.I.F

$$= [x] \int_0^1 (x-[x]) dx$$

$$= [x] \int_0^1 (x-0) dx$$

$$= [x] \int_0^1 (x) dx$$

$$= [x] \left(\frac{x^2}{2}\right)_0^1$$

$$= \frac{[x]}{2}$$

274. $\int_0^{4\pi} [\sin x + \cos x] dx$

$$= 2\left(\int_0^{2\pi} [\sin x + \cos x] dx\right)$$

$$= 2(-\pi)$$

$$= 2(-2\pi)$$

275. $\int_1^{e^{37}} \frac{\pi \sin(\pi \log x)}{x} dx$

$$= \int_0^{37\pi} \sin(t) dt,$$

where $\pi \log(x) = t$

$$= \int_0^{37\pi} \sin(t) dt$$

$$= (-\cos(t))_0^{37\pi}$$

$$= -(-1-1)$$

$$= 2$$

276. $\int_0^{\frac{32\pi}{3}} \sqrt{1+\cos 2x}\ dx$

$$= \int_0^{\frac{32\pi}{3}} (\sqrt{2}|\cos(x)|) dx$$

$$= \sqrt{2}\left[\int_0^{10\pi} (|\cos(x)|) dx + \int_{10\pi}^{\frac{32\pi}{3}} (|\cos(x)|) dx\right]$$

$$= \sqrt{2}\left[\int_0^{\pi} (|\cos(x)|) dx + \int_0^{\frac{2\pi}{3}} (|\cos(x)|) dx\right]$$

$$= \sqrt{2}\left[10 \times 2 + \int_0^{\frac{\pi}{2}} (|\cos(x))dx - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (|\cos(x))dx\right]$$

$$= \sqrt{2}\left[10 \times 2 + 1 - \left(\frac{\sqrt{3}}{2} - 1\right)\right]$$

$$= \sqrt{2}[\left[22 - \frac{\sqrt{3}}{2}\right]$$

$$= |22\sqrt{2} - \sqrt{6}|$$

278. $\int_0^{10\pi/3} |\sin x|\ dx$

$$= \int_0^{3\pi} |\sin x|\ dx + \int_{3\pi}^{10\pi/3} |\sin x|\ dx$$

$$= 3\int_0^{\pi} |\sin x|\ dx + \int_0^{\pi/3} |\sin x|\ dx$$

$$= 3 \times 2 - \left(\frac{1}{2} - 1\right)$$

$$= 6 + \frac{1}{2} = \frac{13}{2}$$

280. $\int_{-\pi}^{199\pi} \sqrt{\left(\frac{1-\cos^2 x}{2}\right)} dx$

$= (199 - (-)) \int_0^{\pi} |\sin(x)|\, dx$

$= 200 \int_0^{\pi} |\sin(x)|\, dx$

$= 200 \times 2 = 400$

281. $\int_{-20\pi}^{20\pi} |\cos x|\, dx$

$= [20 - (-20)] \int_0^{\pi} |\cos(x)|\, dx$

$= 40 \int_0^{\pi} |\cos(x)|\, dx$

$= 40 \times 2 = 80$

282. $\int_0^{100} \sin(x - [x]) dx$

$= 100 \int_0^{1} \sin(x - [x]) dx$

$= 100 \int_0^{1} \sin(x - [0]) dx$

$= 100 \int_0^{1} \sin(x) dx$

$= -100[\cos(x)]_0^1$

$= -100[\cos(1) - 1]$

$= 100[1 - \cos(1)]$

283. $\int_0^{2014\pi} (|\sin x| + |\cos x|) dx$

$= 4028 \int_0^{\pi/2} (|\sin x| + |\cos x|) dx$

$= 4028 \int_0^{\pi/2} (\sin x + \cos x) dx$

$= 4028(-\cos x + \sin x)_0^{\pi/2}$

$= 4028(1 + 1)$

$= 8056$

284. $\int_0^{2014\pi} (|\sin x| - |\cos x|) dx$

$= 2014 \left(\int_0^{\pi} (|\sin x| - |\cos x|) \right) dx$

$= 2014 \left(\int_0^{\pi} (|\sin x|) - \int_0^{\pi} (|\cos x|) \right) dx$

$= 2014 (2 - 2)$

$= 0.$

285. We have,

$\int_0^{n\pi + V} |\sin x|\, dx$

$= \int_0^{n\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi + V} |\sin x|\, dx$

$= \int_0^{n\pi} |\sin x|\, dx + \int_0^{V} |\sin x|\, dx$

$= n \int_0^{\pi} |\sin x|\, dx + \int_0^{V} (\sin x) dx$

$= 2n - (\cos(x))_0^V$

$= 2n - (\cos V - 1)$

$= (2n + 1) - \cos(V)$

286. We have,

$\int_0^{2n\pi} [\sin x + \cos x] dx$

$= n \int_0^{2\pi} [\sin x + \cos x] dx$

$= n \int_0^{2\pi} \left[\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) \right] dx$

$= n \int_{\pi/4}^{9\pi/4} [\sqrt{2} \sin(t)] dx, \quad \left[\text{Let} = \left(x + \frac{\pi}{4}\right) \right]$

$= n \int_{\pi/4}^{9\pi/4} [\sqrt{2} \sin(x)] dx$

$= n(-\pi)$

$= -|n\pi|$

287. We have,

$\int_0^{2\pi} \sin^2(100x) dx$

$= \frac{1}{100} \int_0^{200\pi} \sin^2(t) dt$

$= \frac{200}{100} \int_0^{\pi} \sin^2(x) dx$

$= \int_0^{\pi} [2 \sin^2(x)] dx$

$= \int_0^{\pi} [1 - \cos(2x)] dx$

$= \left(x - \frac{\sin(2x)}{2} \right)_0^{\pi}$

$= (\pi - 0)$

$= \pi$

288. We have,

$\int_{10\pi + \pi/6}^{10\pi + \pi/3} (\sin x + \cos x) dx$

$= \int_{\pi/6}^{\pi/3} (\sin x + \cos x) dx$

$= (\sin x - \cos x)_{\pi/6}^{\pi/3}$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$
$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$
$$= |\sqrt{3} - 1|$$

289. We have,
$$\int_0^5 \frac{\tan^{-1}(x - [x])}{1 + (x - [x])^2}\,dx$$
$$= 5\int_0^1 \frac{\tan^{-1}(x - [x])}{1 + (x - [x])^2}\,dx$$
$$= 5\int_0^1 \frac{\tan^{-1}[x - (0)]}{1 + [x - (0)]^2}\,dx$$
$$= 5\int_0^1 \frac{\tan^{-1}(x)}{1 + x^2}\,dx$$
$$= 5\left(\frac{(\tan^{-1}x)^2}{2}\right)_0^1$$
$$= 5\left(\frac{(\pi/4)^2}{2}\right)$$
$$= \frac{5\pi^2}{32}$$

290. Given $F(x) + F\left(x + \frac{1}{2}\right) = 3$

Replacing x by $\left(x + \frac{1}{2}\right)$, we get
$$F\left(x + \frac{1}{2}\right) + F(x + 1) = 3$$
Subtracting Eq. (ii) from Eq. (i), we get
$$F(x) - F(x + 1) = 0$$
$$\Rightarrow \quad F(x) = F(x + 1)$$
Thus, $F(x)$ is a periodic function with period 1.

Now, $\int_0^{1500} F(x)dx$
$$= 1500\int_0^1 F(x)dx$$
$$= 1500\left(\int_0^{1/2} F(x)dx + \int_{1/2}^1 F(x)dx\right)$$
$$= 1500\left(\int_0^{1/2} F(x)dx + \int_0^{1/2} F\left(t + \frac{1}{2}\right)dt\right),\ x = t + \frac{1}{2}$$
$$= 1500\left(\int_0^{1/2} F(x)dx + \int_0^{1/2} F\left(x + \frac{1}{2}\right)dx\right)$$
$$= 1500\left(\int_0^{1/2} F(x)dx + F\left(x + \frac{1}{2}\right)dx\right)$$
$$= 1500 \times 3 \times \frac{1}{2}$$
$$= 2250$$

291. It is given that $\int_0^{n+1} f(x)dx = n^2$, where $n \in 1$

Now, $\int_{-2}^{4} f(x)dx$
$$= \int_{-2}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx$$
$$= 4 + 1 + 0 + 1 + 4 + 9$$
$$= 19$$

292. We have $\int_0^{\pi/2} \sin^m x \cos^m x\,dx = k\int_0^{\pi/2} \sin^m x\,dx$
$$\Rightarrow \quad \frac{1}{2^m}\int_0^{\pi/2} (2\sin x\cos x)^m\,dx = k\int_0^{\pi/2} \sin^m x\,dx$$
$$\Rightarrow \quad \frac{1}{2^m}\int_0^{\pi/2} [\sin(2x)]^m dx = k\int_0^{\pi/2} \sin^m x\,dx$$
$$\Rightarrow \quad \frac{1}{2^{m+1}}\int_0^{\pi} [\sin(t)]^m dt = k\int_0^{\pi/2} \sin^m x\,dx$$
where $t = (2x)$
$$\Rightarrow \quad \frac{1}{2^{m+1}}\int_0^{\pi} (\sin(x)]^m dx = k\int_0^{\pi/2} \sin^m x\,dx$$
$$\Rightarrow \quad \frac{2}{2^{m+1}}\int_0^{\pi/2} (\sin(x)]^m dx = k\int_0^{\pi/2} \sin^m x\,dx,$$
$$\Rightarrow \quad \frac{1}{2^m}\int_0^{\pi/2} [\sin^m(x)]dx = k\int_0^{\pi/2} \sin^m x\,dx,$$
Thus, $k = \frac{1}{2m} = 2^{-m}$

293. Given $\quad M = \int_0^{\pi} f(\cos^2 x)\,dx$

Now, $\quad N = \int_0^{3\pi} f(\cos^2 x)dx$
$$= 3\int_0^{\pi} f(\cos^2 x)dx$$
$$= 3M$$
Hence, the result.

294. Given $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x},\ x > 0$
$$\Rightarrow \quad F(x) = \int\left(\frac{e^{\sin x}}{x}\right)dx$$
Now, $\quad \int_1^4 \frac{2e^{\sin^2}}{x}dx$
$$= \int_1^4 \frac{2xe^{\sin^2}}{x^2}dx$$

$$= \int_1^{16} \left(\frac{e^{\sin t}}{t}\right) dt$$

$$= \int_1^{16} \left(\frac{e^{\sin x}}{x}\right) dx$$

$$= F(16) - F(1)$$

Thus, $k = 16$.

295. At $x = 1$, $y = \int_1^1 \frac{dt}{\sqrt{1+t^2}} = 0$

So, the point is (1, 0).

Also $y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{\sqrt{1+x^6}} - \frac{2x}{\sqrt{1+x^4}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = \left(\frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

Hence, the equation of the tangent is

$$y - 0 = \frac{1}{\sqrt{2}}(x - 1)$$

$$\Rightarrow \quad y - y\sqrt{2} - 1 = 0$$

296. We have,

$$f(x) = \int_0^x e^t(t-1)(t-2)dt$$

$$\Rightarrow \quad f'(x) = e^x(x-1)(x-2)$$

By the sign scheme, we can say that $f(x)$ is strictly increasing in the interval $(-\infty, 1) \cup (2, \infty)$.

297. We have,

$$f(x) = \int_0^x t(t+1)(t-2)dt$$

$$\Rightarrow \quad f'(x) = x(x+1)(x-2)$$

By the sign scheme, we can say that the function $f(x)$ has points of minima at $x = -1$ and 2, and point of maxima at $x = 0$.

298. We have $f(x) = \int_0^x e^t(t-1)^3(t-2)^2 dt$

$$\Rightarrow \quad f'(x) = e^x(x-1)^3(x-2)^2$$

By the sign scheme, the function has a point of inflexion at $x = 2$

299. We have

$$y = \int_{1/10}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/10}^{\sin^2 x} \cos^{-1}\sqrt{t}\,dt$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin^{-1}(|\sin x|) - \cos^{-1}(|\cos x|))\sin^2 x$$

$$= (\sin^{-1}(\sin x) - \cos^{-1}(\cos x))\sin^2 x$$

$$= (x - x)\sin^2 x$$

$$= 0 \text{ for all } x$$

$\Rightarrow$ y = constant

$\Rightarrow$ Thus, the curve represents a straight line parallel to x-axis.

Since y is independent of x, so we can consider the value of $x = \pi/4$

$$\Rightarrow \quad y = \int_{1/10}^{1/2} \sin^{-1}\sqrt{t}\,dt + \int_{1/10}^{1/2} \cos^{-1}\sqrt{t}\,dt$$

$$= \int_{1/10}^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})dt$$

$$= \int_{1/10}^{1/2} \frac{\pi}{2} dt$$

$$= \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{10}\right) = \frac{\pi}{5}$$

300. Given $F(x) = \int_x^{2x} \sqrt{5 - 3\sin^2 t}\; dt + \int_0^y \sin t\; dt$

$$\Rightarrow \quad F'(x) = \sqrt{5 - 3\sin^2(2x)} - \sqrt{5 - 3\sin^2 x} + \sin y$$

301. Give,

$$f(x) = \int_1^x \sqrt{2 - t^2}\; dt$$

$$\Rightarrow \quad f'(x) = \sqrt{2 - x^2}$$

Also, $\quad x^2 - f'(x) = 0$

$$\Rightarrow \quad x^2 = \sqrt{2 - x^2}$$

$$\Rightarrow \quad x^4 = 2 - x^2$$

$$\Rightarrow \quad x^4 + x^2 - 2 = 0$$

$$\Rightarrow \quad (x^2 + 2)(x^2 - 1) = 0$$

$$\Rightarrow \quad (x^2 - 1) = 0, \qquad (\because x \text{ is real})$$

$$\Rightarrow \quad x = \pm 1$$

302. Given,

$$f(x) = \int_{1/x^2}^{x^2} \cos\sqrt{t}\; dt$$

$$\Rightarrow \quad f'(x) = 2x\left(\cos\sqrt{x^2}\right) + \left(\frac{2}{x^3}\right)\left(\cos\sqrt{\frac{1}{x^2}}\right)$$

$$= 2(\cos(1)) + 2(\cos(1))$$

$$= 4(\cos(1))$$

303. Given,

$$\int_{\sin x}^{1} t^2 f(t)\; dt = 1 - \sin x,\; x \in \left(0, \frac{\pi}{2}\right)$$

Now, $\quad -\sin^2 x\, f(\sin x)\cos x = -\cos x$

$$\Rightarrow \quad \sin^2 x\, f(\sin x)\cos x = \cos x$$

$$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x}$$

$$\Rightarrow \quad f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2} = 3$$

304. Given,

$$F(x) \int_0^x f(t)dt$$

$\Rightarrow \quad F'(x) = f(x)$

$\Rightarrow \quad f(x) = F'(x)$

Also, $\quad F(x^2) = x^2 + x^3$

$\Rightarrow \quad F'(x^2)(2x) = 2x + 3x^2$

$\Rightarrow \quad F'(x^2) = \frac{2x + 3x^2}{2x}$

$= \left(1 + \frac{3}{2}x\right)$

$\Rightarrow \quad F(x^2) = F'(x^2) = \left(1 + \frac{3}{2}x\right)$

$\Rightarrow \quad f(4) = \left(1 + \frac{3}{2} \times 2\right) = 1 + 3 = 4$

305. Given,

$\int_0^{t^2} x\, f(x)\, dx = \frac{2}{5}t^5$

$\Rightarrow \quad t^2 f(t^2) 2t = \frac{2}{5}(5t^4)$

$\Rightarrow \quad t^2 f(t^2) 2t = 2t^4$

$\Rightarrow \quad 2t^3 f(t^2) = 2t^4$

$\Rightarrow \quad f(t^2) = t$

Putting $t = \frac{2}{5}$, we get

$\Rightarrow \quad f\left(\frac{4}{25}\right) = \frac{2}{5}$

306. Let $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$

$\Rightarrow \quad f'(x) = \sin^{-1}(\sqrt{\sin^2 x})\sin(2x)$

$- \cos^{-1}(\sqrt{\cos^2 x})\sin(2x)$

Putting $x = \frac{\pi}{4}$, we get

$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cdot 1 - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\cdot 1$

$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi}{4} = 0$

$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow \quad f(x) = c$

Now,

$f\left(\frac{\pi}{4}\right) = \int_0^{1/2} \sin^{-1}(\sqrt{t})dt + \int_0^{1/2} \cos^{-1}(\sqrt{t})dt$

$= \int_0^{1/2} (\sin^{-1}(\sqrt{t}) + \cos^{-1}(\sqrt{t}))dt$

$= \int_0^{1/2} \left(\frac{\pi}{4}\right)dt = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$

307. Given,

$\int_0^x f(t)dt = x + \int_x^1 tf(t)dt$

$\Rightarrow \quad f(x) = 1 - xf(x)$

$\Rightarrow \quad (1 + x)f(x) = 1$

$\Rightarrow \quad f(x) = \frac{1}{(1 + x)}$

$\Rightarrow \quad f(1) = \frac{1}{(1 + 1)} = \frac{1}{2}$

308. We have,

$\lim_{x \to 0} \frac{1}{x}\left[\int_y^a e^{\sin^2 t}dt - \int_{x+y}^a e^{\sin^2 t}dt\right]$

$= \lim_{x \to 0} \frac{\left[\int_y^a e^{\sin^2 t}dt - \int_{x+y}^a e^{\sin^2 t}dt\right]}{x}$

$= \lim_{x \to 0} \left(\frac{0 + e^{\sin^2(x+y)}}{1}\right)$

$= e^{\sin^2 y}$

309. Do yourself.

310. Let $I = \int_0^{\pi} \sin^{100} x \cos^{99} x\, dx$

$= \int_0^{\pi} \cos^{100}\left(\frac{\pi}{2} - x\right)\sin^{99}\left(\frac{\pi}{2} - x\right)dx$

$= -\int_{\pi/2}^{-\pi/2} \cos^{100} y \sin^{99} y\, dy \qquad$ Let $\frac{\pi}{2} - x = y$

$= \int_{-\pi/2}^{\pi/2} \cos^{100} y \sin^{99} y\, dy$

$= \int_{-\pi/2}^{\pi/2} \cos^{100} x \sin^{99} x\, dx$

$= 0, \qquad$ ($\because$ it is an odd function.)

311. Do yourself.

312. Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx$

$= (-5 + 4)\int_0^1 e^{((-5+4)x-4+5)^2}dx$

$= -\int_0^1 e^{(-x+1)^2}dx$

$= -\int_0^1 e^{(x+1)^2}dx$

Also, let $I_2 = 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$

$= 3\left(\frac{2}{3} - \frac{1}{3}\right)\int_0^1 e^{9\left(\left(\frac{2}{3}-\frac{1}{3}\right)x+\frac{1}{3}-\frac{2}{3}\right)^2}dx$

$$= \int_0^1 e^{9\left(\left(\frac{1}{3}\right)x-\frac{1}{3}\right)^2} dx$$

$$= \int_0^1 e^{(x-1)^2} dx$$

Thus, $I_1 + I_2 = 0$

313. We have,

$$\int_0^{\pi} \sin^{2014}x\cos^{2013}x\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \sin^{2014}\left(\frac{\pi}{2}+x\right)\cos^{2013}\left(\frac{\pi}{2}+x\right)dx$$

$$= -\int_{-\pi/2}^{\pi/2} \sin^{2014}(x)\cos^{2013}(x)dx$$

$= 0,$ ($\because$ it is an odd function.)

314. Let $I = \int_0^1 |\cos \pi x|\,dx$

$$= \frac{1}{\pi}\int_0^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi} \qquad \left(\therefore \int_0^{\pi} |\cos x|\,dx = 2\right)$$

315. We have,

$$\int_0^1 |\sin 2\pi x|\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\sin(t)|\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\sin(x)|\,dx$$

$$= \frac{1}{2\pi} \times 4 = 2/\pi$$

316. We have,

$$\int_{-3/4}^{3/2} |x\cos \pi x|\,dx$$

$$= \frac{1}{\pi}\int_{-3\pi/4}^{3\pi/2} \left|\frac{t}{\pi}\cos(t)\right|dt$$

$$= \frac{1}{\pi^2}\int_{-3\pi/4}^{3\pi/2} |t\cos(t)|\,dt$$

$$= \frac{1}{\pi^2}\int_{-3\pi/4}^{3\pi/2} |x\cos(x)|\,dt$$

$$= \frac{1}{\pi^2}\int_{-3\pi/4}^{3\pi/4} |x\cos(x)|\,dt + \frac{1}{\pi^2}\int_{-3\pi/4}^{3\pi/2} |x\cos(x)|\,dt$$

$$= \frac{2}{\pi^2}\int_0^{3\pi/4} |x\cos(x)|\,dt + \frac{1}{\pi^2}\int_{-3\pi/4}^{3\pi/2} |x\cos(x)|\,dt$$

$$= \frac{2}{\pi^2}\int_0^{3\pi/4} |[x\cos(x)]|\,dx - \frac{1}{\pi^2}\int_{-3\pi/4}^{3\pi/2} [x\cos(x)]\,dt$$

$$= \frac{2}{\pi^2}\int_0^{\pi/2} [x\cos(x)]dx - \frac{2}{\pi^2}\int_{-\pi/2}^{3\pi/4} [x\cos(x)]dx$$

$$-\frac{1}{\pi^2}\int_{3\pi/4}^{3\pi/2} (x\cos(x))dx$$

$$= \frac{2}{\pi^2}(x\sin x + \cos x)^{\pi/2}{}_0 - \frac{2}{\pi^2}(x\sin x + \cos x)_{\pi/2}^{3\pi/4}$$

$$-\frac{1}{\pi^2}(x\sin x + \cos x)_{\pi/4}^{3\pi/2}$$

$$= \frac{2}{\pi^2}\left(\frac{\pi}{2}-1\right) - \frac{2}{\pi^2}\left(\frac{3\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{\pi}{2}\right)$$

$$-\frac{1}{\pi^2}\left(-\frac{3\pi}{2} - \frac{3\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\pi^2}\left(\frac{7\pi}{2} - \frac{3\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right)$$

317. Let $I = \int_{-\pi/12}^{-5\pi/12} \left(\frac{\sin^{2014}x}{\sin^{2014}x + \cos^{2014}x}\right)dx$

$$= \int_{\pi/12}^{5\pi/12} \left(\frac{\sin^{2014}(-x)}{\sin^{2014}(-x) + \cos^{2014}(-x)}\right)dx$$

$$= \int_{\pi/12}^{5\pi/12} \left(\frac{\sin^{2014}(x)}{\sin^{2014}(x) + \cos^{2014}(x)}\right)dx \qquad \text{...(i)}$$

$$= \int_{\pi/12}^{5\pi/12} \left(\frac{(\sin^{2014}\left(\frac{\pi}{2}-x\right)}{\sin^{2014}\left(\frac{\pi}{2}-x\right) + \cos^{2014}\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int_{\pi/12}^{5\pi/12} \left(\frac{\cos^{2014}(x)}{\cos^{2014}(x) + \sin^{2014}(x)}\right)dx \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{\pi/12}^{5\pi/12} \left(\frac{\sin^{2014}x + \cos^{2014}x}{\sin^{2014}x + \cos^{2014}x}\right)dx$$

$$= \int_{\pi/12}^{5\pi/12} dx$$

$$= \left(\frac{5\pi}{12} - \frac{\pi}{12}\right)$$

$$= \frac{\pi}{3}$$

$$\Rightarrow \quad I = \frac{\pi}{6}$$

318. Let $I = \int_{-\pi/3}^{-\pi/6} \frac{\sin^{2n}x}{\cos^{2n}x + \sin^{2n}x}dx$ ...(i)

$$= \int_{-\pi/3}^{-\pi/6} \frac{\sin^{2n}x\left(-\frac{\pi}{2}-x\right)}{\cos^{2n}\left(-\frac{\pi}{2}-x\right)+\sin^{2n}(\left(-\frac{\pi}{2}-x\right)}dx$$

$$= \int_{-\pi/3}^{-\pi/6} \frac{\cos^{2n}(x)}{\sin^{2n}(x)+\cos^{2n}(x)}dx \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-\pi/3}^{-\pi/6}\left(\frac{\sin^{2n}(x)+\cos^{2n}(x)}{\sin^{2n}(x)+\cos^{2n}(x)}\right)dx$$

$$= \int_{-\pi/3}^{-\pi/6} dx$$

$$= \left(-\frac{\pi}{6}+\frac{\pi}{3}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad I = \frac{\pi}{12}$$

319. Let $I = \int_{-\pi/6}^{-\pi/3}\left(\frac{\sin^3 x+\cos^3 x}{\sin^{10}x+\cos^{10}x}\right)dx$ ...(i)

$$= \int_{-\pi/6}^{-\pi/3}\left(\frac{\sin^3\left(-\frac{\pi}{2}-x\right)+\cos^3\left(-\frac{\pi}{2}-x\right)}{\sin^{10}\left(-\frac{\pi}{2}-x\right)+\cos^{10}\left(-\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int_{-\pi/6}^{-\pi/3}\left(\frac{-\cos^3(x)-\sin^3(x)}{\cos^{10}(x)+\sin^{10}(x)}\right)dx \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-\pi/6}^{-\pi/3}\left(\frac{\cos^3(x)+\sin^3(x)-\cos^3(x)-\sin^3(x)}{\cos^{10}(x)+\sin(x)}\right)dx$$

$$= 0$$

$$\Rightarrow \quad I = 0$$

320. Let $I = \int_{4\pi-2}^{4\pi} \frac{\sin\left(\frac{t}{2}\right)}{4\pi+2-t}dt$

$$= 2\int_0^1 \frac{\sin\left(\frac{4\pi-(4\pi-2)t+4\pi-2}{2}\right)dt}{(4\pi+2-(4\pi-(4\pi-2))t+(4\pi-2))}$$

$$= 2\int_0^1 \frac{\sin(t-1)dt}{(4-2t)}$$

$$= \int_0^1 \frac{\sin(t-1)dt}{(2-t)}$$

$$= \int_0^1 \left(\frac{\sin(1-t-1)}{(2-(1-t)}\right)dt$$

$$= 2\int_0^1 \left(\frac{\sin(-t)}{(1+t)}\right)dt$$

$$= -2\int_0^1 \left(\frac{\sin t}{(1+t)}\right)dt$$

$$= -\alpha$$

321. We have,

$$\int_0^{\pi}\left(\frac{|\sin(2x)|}{|\sin x|+|\cos x|}\right)dx$$

$$= 2\int_0^{\pi/2}\left(\frac{|\sin(2x)|}{|\sin x|+|\cos x|}\right)dx$$

$$= 2\int_0^{\pi/2}\left(\frac{\sin 2x}{\sin x+\cos x}\right)dx$$

$$= 2\int_0^{\pi/2}\left(\frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}\right)dx$$

$$= 2\int_0^{\pi/2}\left((\sin x+\cos x)-\frac{1}{\sin x+\cos x}\right)dx$$

$$= 2\int_0^{\pi/2}\left((\sin x+\cos x)-\frac{1}{\sqrt{2}}\operatorname{cosec}\left(x+\frac{\pi}{4}\right)\right)dx$$

$$= 2\left\{-\cos x+\sin x-\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|\right\}_0^{\frac{\pi}{2}}$$

$$= 2\left\{2-\frac{1}{\sqrt{2}}\left(\log\left|\tan\left(\frac{3\pi}{8}\right)\right|-\log\left|\tan\left(\frac{\pi}{8}\right)\right|\right)\right\}$$

$$= \left\{4-\frac{1}{\sqrt{2}}\left(\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right)\right\}$$

$$= \{4-\sqrt{2}\log(\sqrt{2}+1)\}$$

322. We have,

$$\int_a^b\left(\frac{\sin(x-a)-\cos(x-a)}{\sin(b-x)-\cos(b-x)}\right)dx$$

$$= m\int_a^b\left(\frac{\sin(b-x)-\cos(b-x)}{\sin(x-a)-\cos(x-a)}\right)dx$$

$$\Rightarrow \int_a^b\left(\frac{\sin(b-x)-\cos(b-x)}{\sin(x-a)-\cos(x-a)}\right)dx$$

$$= m\int_a^b\left(\frac{\sin(b-x)-\cos(b-x)}{\sin(x-a)-\cos(x-a)}\right)dx$$

$$\Rightarrow \quad m = 1$$

323. Given,

$$\int_0^1 \frac{e^t}{(t+1)}dt = a$$

Also, $\int_{b-1}^{b} \frac{e^{-t}}{(t-b-1)}dt = a$

$$= \int_0^1\left(\frac{e^{-((b-(b-1))t+b-1}}{(b-(b-1)t+(b-1)-b-1}\right)dt$$

$$= \int_0^1 \left(\frac{e^{-t-b+1}}{t-2}\right)dt$$

$$= e^{-b}\int_0^1 \left(\frac{e^{-(t-1)}}{t-2}\right)dt$$

$$= -e^{-b}\int_1^0 \left(\frac{e^{y}}{-(y+1)}\right)dy, \qquad \text{Let } (t-1) = -y$$

$$= -e^{-b}\int_0^1 \left(\frac{e^{t}}{-(t+1)}\right)dy$$

$$= -a e^{-b}$$

324. Let $I = \int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin 2\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}-x\right)\right)}{2\left(\frac{\pi}{2}-x\right)-\pi}dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin 2x \sin\left(\frac{\pi}{2}\sin x\right)}{-2x}dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}+x\right)(-\sin 2x)\sin\left(-\frac{\pi}{2}\sin x\right)}{-2x}dx$$

$$= \int_0^{\pi/2} \frac{2x(\sin 2x)\sin\left(\frac{\pi}{2}\sin x\right)}{2x}dx$$

$$= \int_0^{\pi/2} \frac{2\sin x\cos x\sin\left(\frac{\pi}{2}\sin x\right)}{x}dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} 2\left(\frac{2t}{\pi}\right)\sin t\,dt, \text{ where } t = \frac{\pi}{2}\sin x$$

$$= \frac{8}{\pi^2}\int_0^{\pi/2} t\sin t\,dt$$

$$= \frac{8}{\pi^2}(t(-\cos t) + \sin t)_0^{\pi/2}$$

$$= \frac{8}{\pi^2}(1-0) = \frac{8}{\pi^2}$$

325. Given,

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx$$

$$= \int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx$$

$$+ \int_1^2 (1+\cos^8 x)(ax^2+bx+c)dx$$

$$- \int_1^2 (1+\cos^8 x)(ax^2+bx+c)dx = 0$$

Thus, the equation has at least one root in (1, 2)

326. Since $|\sin x| \le 1$, for $x \ge 10$

$$\Rightarrow \quad \left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|} \qquad \text{...(i)}$$

For $10 \le x \le 19$,

$$1 + x^8 > x^8 \ge 10$$

$$\Rightarrow \quad \frac{1}{1+x^8} < \frac{1}{x^8} \le \frac{1}{10^8}$$

$$\Rightarrow \quad \frac{1}{|1+x^8|} \le \frac{1}{10^8} \qquad \text{...(ii)}$$

From Eqs (i) and (ii), we get

$$\left|\frac{\sin x}{1+x^8}\right| \le 10^{-8}$$

$$\Rightarrow \quad \int_{10}^{19} \left|\frac{\sin x}{1+x^8}\right| dx \le \int_{10}^{19} 10^{-8}dx$$

$$= (19-10)10^{-8} = 9 \times 10^{-8}$$

$$= 10^{-7} - 10^{-8} < 10^{-7}$$

Hence, $\int_{10}^{19} \left|\frac{\sin x}{1+x^8}\right| dx \le 10^{-7}$

327. Since $|\cos(x)| \le 1$, for $x \ge 5$

$$\Rightarrow \quad \left|\frac{\cos(x)}{1+x^4}\right| \le \frac{1}{|1+x^4|} \text{ for } x \ge 5 \qquad \text{...(i)}$$

Again, for $5 \le x \le 10$,

$$1 + x^4 > 5^4$$

$$\Rightarrow \quad \frac{1}{(1+x^4)} < \frac{1}{5^4}$$

$$\Rightarrow \quad \frac{1}{|(1+x^4)|} < 5^{-4} \qquad \text{...(ii)}$$

From Eqs (i) and (ii), we get

$$\left|\frac{\cos x}{(1+x^4)}\right| < 5^{-4}$$

$$\Rightarrow \quad \left|\int_5^{10} \frac{\cos x}{(1+x^4)}dx\right| < \int_5^{10} (5^{-4})dx$$

$$\Rightarrow \quad \left|\int_5^{10} \frac{\cos x}{(1+x^4)}dx\right| < 5^{-3}$$

328. We have,

$$1 \le x \le 3$$

$$\Rightarrow \quad 1 \le x^2 \le 9$$

$$\Rightarrow \quad 4 \le x^2 + 3 \le 12$$

$$\Rightarrow \quad 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$$

$$\Rightarrow \quad 2(3-1) \le \int_1^3 \sqrt{x^2+3}\, dx \le 2\sqrt{3}(3-1)$$

$$\Rightarrow \quad 4 \le \int_1^3 \sqrt{x^2+3}\, dx \le 4\sqrt{3}$$

Hence, the result.

329. We have,

$$0 \le x \le 1$$

$$\Rightarrow \quad 0 \le x^2 \le 1$$

$$\Rightarrow \quad e^0 \le e^{x^2} \le e^1$$

$$\Rightarrow \quad \int_0^1 (e^0)dx \le \int_0^1 (e^{x^2})dx \le \int_0^1 (e^1)dx$$

$$\Rightarrow \quad (1-0) \le \int_0^1 (e^{x^2})dx \le e(1-0)$$

$$\Rightarrow \quad 1 \le \int_0^1 (e^{x^2})dx \le e$$

330. Let $f(x) = \dfrac{x}{x^3+16}$.

$$\Rightarrow f'(x) = \frac{16-2x^3}{(x^3+16)^2} > 0, \ \forall\ x \in (0, 1)$$

$$\Rightarrow f''(x) = \frac{-(x^3+16)^2(6x^2) - (16-2x^3)2(x^3+16)\cdot 3x^2}{(x^3+16)^4} > 0, \quad \forall\ x \in (0, 1)$$

Thus, $f(x)$ is concave upaward.

$$\Rightarrow (1-0)f(0) \le \int_0^1 \left(\frac{x}{x^3+16}\right)dx \le \left(\frac{1-(0)(f(0)+f(1))}{2}\right)$$

$$\Rightarrow 0 \le \int_0^1 \left(\frac{x}{x^3+16}\right)dx \le \frac{17}{2}$$

Hence, the result.

331. Let $f(x) = \sqrt{3+x^3}$.

$$\Rightarrow \quad f'(x) = \frac{3x^2}{2\sqrt{3+x^3}}$$

$$\Rightarrow \quad f'(x) = \frac{3x^2}{2\sqrt{3+x^3}} > 0, \ \forall\ x \in (1, 3)$$

Thus, $f(x)$ increases in (1, 3).

So, $(3-1)f(1) \le \int_1^3 (\sqrt{3+x^3})dx \le (3-1)f(3)$

$$\Rightarrow \quad 4 \le \int_1^3 (\sqrt{3+x^3})dx \le 2\sqrt{30}$$

332. We have for every x in [0, 1],

$$\sqrt{4-2x^2} \le \sqrt{4-x^2-x^3} \le \sqrt{4-x^2}$$

$$\Rightarrow \frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\Rightarrow \left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^1 \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\left(\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right)_0^1$$

$$\Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$$

333. We have,

$$1 \le \sqrt{1+x^3} \le 1+x^3, \ x \ge 0$$

$$\Rightarrow \int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^3}\, dx \le \int_0^1 (1+x^3)dx$$

$$\Rightarrow 1 \le \int_0^1 \sqrt{1+x^3}\, dx \le \left(x + \frac{x^4}{4}\right)_0^1$$

$$\Rightarrow 1 \le \int_0^1 \sqrt{1+x^3}\, dx \le \frac{5}{4}$$

334. We have,

$$1 \le x \le 2$$

$$\Rightarrow \quad 1 \le x^4 \le 16$$

$$\Rightarrow \quad 2 \le (1+x^4) \le 17$$

$$\Rightarrow \quad \frac{1}{17} \le \frac{1}{(1+x^4)} \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{17} \le \int_0^1 \frac{dx}{1+x^4} \le \frac{7}{24}$$

335. Let $I = \int_0^\infty \left(\dfrac{\tan^{-1}ax - \tan^{-1}x}{x}\right)dx$

$$\Rightarrow \quad \frac{dI}{da} = \int_0^\infty \left(\frac{1.x}{(1+a^2x^2)x}\right)dx$$

$$= \int_0^\infty \left(\frac{dx}{(1+a^2x^2}\right)$$

$$= \frac{1}{a^2}\int_0^\infty \left(\frac{dx}{(1/a^2)+x^2}\right)$$

$$= \frac{1}{a^2}(a\tan^{-1}(ax))\Big|_0^\infty$$

$$= \frac{1}{a^2}\left(a\frac{\pi}{2}\right) = \frac{\pi}{2a}$$

$$\Rightarrow \quad I = \frac{\pi}{2}\ln a + c$$

When $a = 1$, $I = 0 \Rightarrow c = 0$

Hence, $I = \dfrac{\pi}{2}\ln a$

336. Let $I = \int_0^{\pi/2} \ln\left(\dfrac{1+a\sin x}{1-a\sin x}\right)\dfrac{dx}{\sin x}$, $(|a| < 1)$.

$$\Rightarrow \quad \frac{dI}{da} = \int_0^{\pi/2} \left(\frac{2\sin x}{1-a^2\sin^2 x}\right)\frac{dx}{\sin x}, \ (|a| < 1)$$

$$= \int_0^{\pi/2} \left(\frac{2\sin x}{1-a^2\sin^2 x}\right)\frac{dx}{\sin x}$$

$$= \int_0^{\pi/2} \left(\frac{2}{1-a^2\sin^2 x}\right)dx$$

$$= \int_0^{\pi/2} \left(\frac{2\sec^2 x}{\sec^2 x - a^2\tan^2 x}\right)dx$$

$$= \int_0^{\pi/2} \left(\frac{2\sec^2 x}{1+(1-a^2)\tan^2 x}\right)dx$$

$$= \int_0^{\infty} \left(\frac{2dt}{1+(1-a^2)t^2}\right), \quad \text{Let } t = \tan x$$

$$= \frac{1}{(1-a^2)} \int_0^{\infty} \left(\frac{2dt}{t^2 + \left(\frac{1}{\sqrt{(1-a^2)}}\right)^2}\right)$$

$$= \frac{2}{\sqrt{(1-a^2)}} \times \left(\tan^{-1}(t\sqrt{1-a^2})\right)_0^{\infty}$$

$$= \frac{2}{\sqrt{(1-a^2)}} \times \frac{\pi}{2}$$

$$= \frac{\pi}{\sqrt{(1-a^2)}}$$

$$\Rightarrow \quad I = \pi\sin^{-1}a + c$$

When $a = 0$, $I = 0$, then $c = 0$

Hence, $I = \pi\sin^{-1}a$

336. Given $I = \int_0^{\pi/2} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}, \ (|a| < 1)$

$$\Rightarrow \frac{dI}{da} = \int_0^{\pi/2} \left(\frac{1-a\sin x}{1+a\sin x}\right) \times \frac{2\sin x}{(1-a\sin x)^2} \times \frac{1}{\sin x}dx$$

$$= \int_0^{\pi/2} \left(\frac{2\,dx}{1-a^2\sin^2 x}\right)$$

$$= \int_0^{\pi/2} \left(\frac{2\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x - a^2}\right)dx$$

$$= \int_0^{\pi/2} \left(\frac{2\operatorname{cosec}^2 x}{\cot^2 x + (1-a^2)}\right)dx$$

$$= \frac{-2}{\sqrt{1-a^2}} \tan^{-1}\left(\frac{\cot x}{\sqrt{1-a^2}}\right)_0^{\pi/2}$$

$$= \frac{2}{\sqrt{1-a^2}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{1-a^2}}$$

Integrating, we get

$I = \pi\sin^{-1}(a) + c$

When $a = 0$, $I(a) = 0$, then $c = 0$

Thus, $I = \pi\sin^{-1}(a)$

337. Given $I(b) = \int_0^1 \frac{x^b - 1}{\log x}dx$

$$\Rightarrow \quad \frac{d[I(b)]}{db} = \int_0^1 \left(\frac{x^b \log x}{\log x}\right)dx$$

$$= \int_0^1 (x^b)dx$$

$$= \frac{x^{b+1}}{b+1}$$

Integrating both sides w.r.t. b, we get

$$I(b) = \log(b+1) + c$$

When $b = 0$, then $c = 0$

Thus, $I(b) = \log(b+1)$

338. We have,

$$(f(x))^2 = \int_0^x f(t).\frac{2\sec^2 t}{4+\tan t}dt$$

$$\Rightarrow \quad 2f(x)f'(x) = \frac{f(x).2\sec^2 x}{4+\tan x}$$

$$\Rightarrow \quad f'(x) = \frac{\sec^2 x}{4+\tan x}$$

Integrating, we get

$$f(x) = \log|(4+\tan(x))| + c$$

When $x = 0$, then $c = -\log(4)$

$$f(x) = \log|(4+\tan(x))| - \log(4)$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \log|(4+1)| - \log(4) = \log\left(\frac{5}{4}\right)$$

339. Given,

$$f(x) = \int_0^{\pi/2} \log(1 + x\sin^2\theta)/\sin^2\theta d\theta$$

$$\Rightarrow \quad f'(x) = \int_0^{\pi/2} \left(\frac{\sin^2\theta}{(1+x\sin^2\theta)} \times \frac{1}{\sin^2\theta}\right)d\theta$$

$$= \int_0^{\pi/2} \left(\frac{d\theta}{(1+x\sin^2\theta)}\right)$$

$$= \int_0^{\pi/2} \left(\frac{\operatorname{cosec}^2\theta d\theta}{(\cot^2\theta + (1+x))}\right)$$

$$= \left(-\frac{1}{\sqrt{1+x}}\tan^{-1}\left(\frac{\cot\theta}{\sqrt{1+x}}\right)\right)_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{1+x}}$$

Integrating, we get

$$f(x) \ \pi\sqrt{1+x+c}$$

When $x = 0$, $f(0) = 0$, then $c = -\pi$

$f(x)\ \pi\sqrt{1+x} - \pi = \pi(\sqrt{1+x} - 1)$

340. Let $I = \int_0^1 \left(\frac{x^{\cos\alpha} - 1}{\log_e x}\right) dx$

$$\Rightarrow \quad \frac{dI}{d\alpha} = \int_0^1 \left(\frac{x^{\cos\alpha}\log x}{\log x}\right) dx$$

$$= \int_0^1 (x^{\cos\alpha}) dx$$

$$= \left(\frac{x^{\cos\alpha+1}}{\cos\alpha + 1}\right)_0^1 = \frac{1}{1+\cos\alpha}$$

Integrating, we get

$$I(\alpha) = \log|(1+\cos\alpha)| + c$$

When $\alpha = \frac{\pi}{2}$, $I(\alpha) = 0$, then $c = 0$

Thus, $I(\alpha) = \log|1 + \cos\alpha|$

341. Do yourself

342. We have,

$$\int_0^2 \sqrt{(1+x)(1+x^4)}\, dx$$

$$\leq \sqrt{\int_0^2 (1+x)dx} \times \sqrt{\int_0^2 (1+x^4)dx}$$

$$= \sqrt{\left(x + \frac{x^2}{2}\right)_0^2} \times \sqrt{\left(x + \frac{x^5}{5}\right)_0^2}$$

$$= \sqrt{(2+2)} \times \sqrt{\left(2 + \frac{32}{5}\right)}$$

$$= 2 \times \sqrt{\frac{42}{5}}$$

Hence, the maximum value of the given integral is $2 \times \sqrt{\frac{42}{5}}$.

343. We have,

$$\int_0^1 \sqrt{(1+x)(1+x^3)}dx$$

$$\leq \sqrt{\int_0^1 (1+x)dx}\ \sqrt{\int_0^1 (1+x^3)dx}$$

$$= \sqrt{\left(x + \frac{x^2}{2}\right)_0^1}\ \sqrt{\left(x + \frac{x^4}{4}\right)_0^1}$$

$$= \sqrt{\frac{3}{2}}\ \sqrt{\frac{5}{4}} = \sqrt{\frac{15}{8}}$$

Hence, the maximum value is $\sqrt{\frac{15}{8}}$.

344. We have,

$$\int_a^{a+\frac{\pi}{2}} (\sin^4 x + \cos^4 x)dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^4 x + \cos^4 x)dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x\cos^2 x)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{\sin^2(2x)}{2}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\sin^2(2x)}{4}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1-\cos 4x}{4}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{3}{4} + \frac{1}{4}\cos|4x|\right)dx$$

$$= \left(\frac{3}{4}x + \frac{\sin|4x|}{16}\right)_0^{\pi/2}$$

$$= \left(\frac{3}{4} \times \frac{\pi}{2}\right)$$

$$= \frac{3\pi}{8}$$

345. Mean Value $= \frac{1}{(2\pi - 0)} \int_0^{2\pi} \sin^2 x\, dx$

$$= \frac{1}{4\pi}\int_0^{2\pi} 2\sin^2 x\, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (1 - \cos^2 x)\, dx$$

$$= \frac{1}{4\pi}\left(x - \frac{\sin^2 x}{2}\right)_0^{2\pi}$$

$$= \frac{1}{4\pi}(2\pi - 0)$$

$$= \frac{1}{2}$$

346. Mean value $= \frac{1}{(2-0)} \int_0^2 \frac{dx}{e^x + 1}$

$$= \frac{1}{2}\int_0^2 \frac{e^{-x}dx}{e^{-x} + 1}$$

$$= \left(-\frac{1}{2}\log|(e^{-x} + 1)|\right)_0^2$$

$$= \left(-\frac{1}{2}\log|2(e^{-2} + 1)|\right)$$

$$= \left(\frac{1}{2}\log\left|2\left(\frac{e^2}{e^2 + 1}\right)\right|\right)$$

347. Mean value $= \frac{1}{(1-0)}\int_0^1 (\sqrt[3]{x})dx$

$$= \frac{1}{(1-0)}\int_0^1 (\sqrt[3]{x})dx$$

$$= \frac{1}{(1-0)} \times \left(\frac{3}{4}x^{4/3}\right)_0^1$$
$$= \frac{3}{4}$$

348. Mean value $= \frac{1}{(2\pi - 0)} \int_0^{2\pi} (\sin^3 x)dx$
$$= \frac{1}{8\pi} \int_0^{2\pi} (3\sin x - \sin|3x|)dx$$
$$= \frac{1}{8\pi}\left(-3\cos x + \frac{\cos|3x|}{3}\right)_0^{2\pi}$$
$$= \frac{1}{8\pi}\left(-3 + \frac{1}{3} + 3 - \frac{1}{3}\right)$$
$$= 0$$

349. Mean value $= \frac{1}{(1-0)} \int_0^1 \frac{dx}{1+x^2}$
$$= \left(\tan^{-1}(x)\right)_0^1$$
$$= \frac{\pi}{4}$$

350. Mean value $= \frac{1}{(1-0)} \int_0^1 \left(\frac{e^x}{e^x+1}\right)dx$
$$= (\log|(e^x + 1)|)_0^1$$
$$= \log\left(\frac{e+1}{2}\right)$$

351. We have,
$$\lim_{\varphi \to 0}\left(\int_\varphi^1 \log(x)dx\right)$$
$$= \lim_{\varphi \to 0} (x\log(x) - x)_\varphi^1$$
$$= \lim_{\varphi \to 0} (\varphi - 1 - \varphi\log(\varphi))$$
$$= \lim_{\varphi \to 0} (\varphi - 1) - \lim_{\varphi \to 0} (\varphi\log(\varphi))$$
$$= \lim_{\varphi \to 0} (\varphi - 1) - \lim_{\varphi \to 0}\left(\frac{\log(\varphi)}{\frac{1}{\varphi}}\right)$$
$$= \lim_{\varphi \to 0} (\varphi - 1) - \lim_{\varphi \to 0}\left(\frac{\frac{1}{\varphi}}{-\frac{1}{\varphi^2}}\right)$$
$$= \lim_{\varphi \to 0} (\varphi - 1) + \lim_{\varphi \to 0} (\varphi)$$
$$= -1$$

352. We have,
$$\int_0^\infty e^{-x}x^6 dx$$
$$= \int_0^\infty e^{-x}x^{7-1} dx$$
$$= \Gamma(7)$$
$$= \Gamma(6+1)$$
$$= 6!$$
$$= 720$$

353. We have,
$$\int_0^1 \left(\log\left(\frac{1}{x}\right)\right)^{10} dx$$
$$= -\int_\infty^0 e^{-t}t^{10} dt$$
$$= \int_0^\infty e^{-x}x^{11-1} dx$$
$$= \Gamma(11)$$
$$= \Gamma(10+1)$$
$$= (10)!$$

354. We have,
$$= \int_0^\infty e^{-x}x^3 dx$$
$$= \int_0^\infty e^{-x}x^{4-1} dx$$
$$= \Gamma(7)$$
$$= (3)!$$
$$= 6$$

355. We have,
$$\int_0^1 \left(\log\left(\frac{1}{x}\right)\right)^{n-1} dx$$
$$= \int_0^1 (-\log|x|)^{n-1} dx$$
$$= \int_0^\infty (e^{-t}t^{n-1})\,dt, \qquad (\text{Let } t = -\log x)$$
$$= \int_0^\infty (e^{-x}t^{n-1})dx$$
$$= \Gamma(n)$$

356. Let $I = \int_0^1 x^4\sqrt{1-x^2}\, dx$
$$= \int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta \qquad (\text{Let } \sin\theta = x)$$
$$= \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$$
$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{2 \times 6}$$
$$= \frac{\pi}{32}$$

357. We have,

$$I = \int_0^{\pi/2} \sin^8 x \cos^4 x \, dx$$

$$= \frac{\Gamma\left(\frac{8+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{8+4+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{14}{2}\right)}$$

$$= \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \times \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot(6)!}$$

$$= \frac{7\pi}{2048}$$

358. We have,

$$\int_0^1 x^6\sqrt{1-x^2}\, dx$$

$$= \int_0^{\pi/2} \sin^6\theta \cos^2\theta \, d\theta \qquad (\text{Let } \sin\theta = x)$$

$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{6+2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(5)}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\pi}{2 \times 24}$$

$$= \frac{5\pi}{256}$$

359. We have,

$$\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$$

$$= \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{4+6+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(6)}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2 \times 120}$$

$$= \frac{3\pi}{512}$$

360. We have, $\int_0^1 x^{10}\sqrt{1-x^2}\, dx$

$$= \int_0^{\pi/2} \sin^{10}\theta \cos^2\theta \, d\theta \qquad (\text{Let } \sin\theta = x)$$

$$= \frac{\Gamma\left(\frac{10+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{10+2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(7)}$$

$$= \frac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2 \times (6!)}$$

$$= \frac{21\pi}{512}$$

361. We have,

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$$

$$= \int_0^{\pi/2} \sin^{-1/2} x \cos^0 x \, dx x \times \int_0^{\pi/2} \sin^{\frac{1}{2}} x \cos^0 x \, dx x$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \times \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \times \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\cdot\frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right)}$$

$$= \left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

$$= \pi$$

362. We have,

$$\int_0^{\infty} e^{-a^2x^2} dx$$

$$= \frac{1}{2a}\int_0^{\infty} e^{-t} t^{-1/2} dt, \qquad (\text{Let } (ax)^2 = t)$$

$$= \frac{1}{2a}\int_0^{\infty} e^{-t}\, t^{\left(1-\frac{1}{2}\right)} dt$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2a}$$

$$= \frac{\sqrt{\pi}}{2a}$$

363. We have,

$$\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$$

$$= \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{7+5+2}{2}\right)}$$

$$= \frac{\Gamma(2)\Gamma(3)}{2\Gamma(7)}$$

$$= \frac{3.2.1.2.1}{2.6.5.4.3.2.1}$$

$$= \frac{1}{120}$$

364. We have,

$$\int_0^{\pi/2} \sin^{10}x\, dx$$

$$= \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{63\pi}{512}$$

365. We have,

$$\int_0^{\pi/2} \cos^7 x\, dx$$

$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

$$= \frac{16}{35}$$

366. We have,

$$\int_0^{\pi/4} 8\cos^4 x \sin^4 x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} 16\cos^4 x\sin^4 x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} (\sin^2 x)^4 dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin^4 t) dx$$

$$= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi}{64}$$

367. We have,

$$\int_0^{\pi/2} \sin^6 x\cos^5 x\, dx$$

$$= \frac{(5\cdot 3\cdot 1)(4\cdot 2)}{(11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1)}$$

$$= \frac{8}{693}$$

368. We have, $\int_0^{\pi/2} \sin^5 x\cos^3 x\, dx$

$$= \frac{(4\cdot 2)(2)}{(8\cdot 6\cdot 4\cdot 2)}$$

$$= \frac{1}{24}$$

369. We have, $\int_0^{\pi/2} \sin^6 x\cos^4 x\, dx$

$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{6+4+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(6)}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\times(5!)}$$

$$= \frac{3\pi}{512}$$

370. We have,

$$\int_0^{\pi/2} \sin^7 x\cos x\, dx$$

$$= \int_0^1 t^7\, dt, \qquad \text{(Let } \sin x = t)$$

$$= \left(\frac{t^8}{8}\right)_0^1$$

$$= \frac{1}{8}$$

371. We have,

$$\int_0^{\pi/2} \sin^7 x\cos^5 x\, dx$$

$$= \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{7+5+2}{2}\right)}$$

$$= \frac{\Gamma(4)\Gamma(3)}{2\Gamma(7)}$$

$$= \frac{(3!)\times(2!)}{2\times(6!)}$$

$$= \frac{6\times 2}{2\times 720} = \frac{1}{120}$$

372. We have,

$$\int_0^{\pi/2} \sin^{11}x\, dx$$

$$= \left(\frac{11-1}{11}\right)\left(\frac{11-3}{11-2}\right)\left(\frac{11-5}{11-4}\right)\left(\frac{11-7}{11-6}\right)\left(\frac{11-9}{11-8}\right)$$

$$= \frac{10}{11}\cdot\frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

$$= \frac{256}{693}$$

373. We have, $\int_0^{\pi/2} \cos^8 x\, dx$

$$= \left(\frac{8-1}{8}\right)\left(\frac{8-3}{8-2}\right)\left(\frac{8-5}{8-4}\right)\left(\frac{8-7}{8-6}\right)\cdot\frac{\pi}{2}$$

$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{3}$$

$$= \frac{35\pi}{256}$$

Level 111

1. Let $I = \int_0^{9\pi/4} (|\sin x| - |\cos x|)dx$

$= \int_0^{2\pi} (|\sin x| - |\cos x|)dx + \int_{2\pi}^{2\pi+\pi/4} (|\sin x| - |\cos x|)dx$

$= 2\int_0^{\pi} (|\sin x| - |\cos x|)dx + \int_0^{\pi/4} (|\sin x| - |\cos x|)dx$

$= 4\int_0^{\pi/2} (\sin x - \cos x)dx + \int_0^{\pi/4} (\sin x - \cos x)dx$

$= 4(-\cos x - \sin x)_0^{\pi/2} + (-\cos x - \sin x)_0^{\pi/4}$

$= 4.0 + (1 + \sqrt{2})$

$= (1 + \sqrt{2})$

Hence, the value of the given integral

$= \lfloor(1 + \sqrt{2})\rfloor = -1$ sq.u.

2. Let $I = \int_{-2}^{1} \left[x\left[1 + \cos\left(\frac{\pi x}{2}\right)\right] + 1\right]dx$

$= \int_{-2}^{-1} \left[x\left[1 + \cos\left(\frac{\pi x}{2}\right)\right] + 1\right]dx$

$+ \int_{-1}^{0} \left[x\left[1 + \cos\left(\frac{\pi x}{2}\right)\right] + 1\right]dx$

$+ \int_{0}^{1} \left[x\left[1 + \cos\left(\frac{\pi x}{2}\right)\right] + 1\right]dx$

$= \int_{-2}^{-1} 1.dx + \int_{-1}^{0} 0.dx + \int_0^1 1.dx$

$= 1(-1-(-2)) + 0 + 1(1 - 0)$

$= 1 + 1$

$= 2$ sq.u.

3. Let $I = \int_0^3 \left([x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]\right)dx$

$= \int_0^{1/3} 0.dx + \int_{1/3}^{2/3} 1.dx + \int_{2/3}^{1} 2.dx$

$+ \int_1^{4/3} 3.dx + \int_{4/3}^{5/3} 4.dx + \int_{5/3}^{2} 5.dx$

$\int_2^{7/3} 6.dx + \int_{7/3}^{8/3} 7.dx + \int_{8/3}^{3} 5.dx$

$= \frac{1}{3}(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$

$= \frac{1}{3} \times \frac{8 \times 9}{2}$

$= 12$

4. Let $I = \int_1^{10\pi} ([\sec^{-1}x] + [\tan^{-1}x])dx$

$= \int_1^{\sec 1} ([\sec^{-1}x] + [\tan^{-1}x])dx$

$+ \int_{\sec 1}^{10\pi} ([\sec^{-1}x] + [\tan^{-1}x])dx$

$= \int_1^{\sec 1} (0 + 0)dx + \int_{\sec 1}^{10\pi} (1 + 0)dx$

$= (10\pi - \sec 1)$ sq. u.

5 Let $I = \int_0^{5\pi/12} [\tan x]dx$

$= \int_0^1 [\tan^{-1}x]dx + \int_1^2 [\tan^{-1}x]dx$

$+ \int_2^3 [\tan^{-1}x]dx + \int_3^{5\pi/12} [\tan^{-1}x]dx$

$= 0 + (\tan^{-1}2 - \tan^{-1}1)\cdot 1 + (\tan^{-1}3 - \tan^{-1}2)\cdot 2$

$+ \left(\frac{5\pi}{12} - \tan^{-1}3\right)\cdot 3$

$= \frac{5\pi}{4} - \frac{\pi}{4} - (\tan^{-1}3 + \tan^{-1}2)$

$= \pi + \pi - \tan^{-1}(-1)$

$= \frac{9\pi}{4}$ sq.u.

6. Let $I = \int_0^{\pi/4} [\sin x + \{\cos x + \tan x + (\sec x)\}]dx$

Since $0 < x < \frac{\pi}{4}$

$\therefore \quad 1 < \sec x < \sqrt{2}$

$\Rightarrow \quad [\sec x] = 1$

$\Rightarrow \quad [\tan x + [\sec x]] = [\tan x + 1]$

$= [\tan x] + 1 = 0 + 1 = 1$

$\Rightarrow \quad [\cos x + [\tan x + [\sec x]]]$

$= [\cos x + 1] = [\cos x] + 1 = 0 + 1 = 1$

$\Rightarrow \quad [\sin x + [\cos x + [\tan x + [\sec x]]]]$

$= [\sin x + 1] = [\sin x] + 1 = 0 + 1 = 1$

Thus, $I = \int_0^{\pi/4} [\sin x + [\cos x + [\tan x + [\sec x]]]dx$

$$= \int_0^{\pi/4} 1.dx$$

$$= \frac{\pi}{4}$$

7. Let $I = \int_0^{\pi/2} [\sin^{-1}(\cos x) + \cos^{-1}(\sin x)]dx$

Graph:

$$= \left(\frac{\pi - 1}{2}\right) + \left(\frac{\pi - 2}{2}\right) + \left(\frac{\pi - 3}{2}\right)$$

$$= \left(\frac{3\pi - 6}{2}\right) \text{ sq.u}$$

8. Let $I = \int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \left|\frac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\right| dx$

$$= \int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \left|\frac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\right| dx$$

$$= \int_\alpha^\beta 1dx$$

$$= x|_\alpha^\beta$$

$$= (\beta - \alpha)$$

9. We have,

$$m = \int_0^{100} \sec^{-1}(\sec \pi x)dx$$

$$= 50 \times \int_0^2 \sec^{-1}(\sec \pi x)dx$$

$$= 50 \times \frac{1}{2} \times 2 \times \pi$$

$$= 50\pi$$

Also, $n = \int_0^{3/\pi} [\sec^{-1}(\sec \pi x)]dx$

$$= \int_0^{1/2} [\sec^{-1}(\sec \pi x)]dx + \int_{1/2}^{1/\pi} [\sec^{-1}(\sec \pi x)]dx$$

$$+ \int_{1/\pi}^{2/\pi} [\sec^{-1}(\sec \pi x)]dx + \int_{2/\pi}^{3/\pi} [\sec^{-1}(\sec \pi x)]dx$$

$$= \int_0^{1/2} 0.dx + \int_{1/2}^{1/\pi} 1.dx + \int_{1/\pi}^{2/\pi} 2.dx + \int_{2/\pi}^{3/\pi} 3.dx$$

$$= 0 + 1.\left(\frac{1}{\pi} - \frac{1}{2}\right) + 2.\left(\frac{2}{\pi} - \frac{1}{\pi}\right) + 3.\left(\frac{3}{\pi} - \frac{2}{\pi}\right)$$

$$= \left(\frac{6}{\pi} - \frac{1}{2}\right)$$

Now, $\left(\frac{m}{\pi} + n\pi + \frac{\pi}{2} + 4\right)$

$$= 50 + 6 + 4$$

$$= 60$$

10. Let

$$I = \int_{-3}^{-7} \tan(x^2 - 6)dx + \int_{-6}^{-2} \tan(x^2 + 18x + 75)dx$$

$$= \int_{-3}^{-7} \tan(x^2 - 6)dx + \int_{-6}^{-2} \tan(x + 9)^2 - 6)dx$$

$$= \int_{-3}^{-7} \tan(x^2 - 6)dx + \int_3^7 \tan(y^2 - 6)dx$$

$$= \int_{-3}^{-7} \tan(x^2 - 6)dx + \int_3^7 \tan(x^2 - 6)dx$$

$$= \int_3^7 \tan(x^2 - 6)dx + \int_3^7 \tan(x^2 - 6)dx$$

$$= 0$$

11. We have,

$$I_n = \int_{-2n\pi}^{2n\pi} |\sin x|[\sin x]dx$$

$$= \int_{-2n\pi}^{2n\pi} |\sin x|[\sin x]dx$$

$$= \int_{-2n\pi}^{2n\pi} |\sin x|(-1 - [\sin x])dx$$

$$= -\int_{-2n\pi}^{2n\pi} |\sin x|dx - \int_{-2n\pi}^{2n\pi} |\sin x|[\sin x]dx$$

$$= -\int_{-2n\pi}^{2n\pi} |\sin x|dx - I_n$$

$$\Rightarrow 2I_n = -\int_{-2n\pi}^{2n\pi} |\sin x|dx$$

$$= -2\int_0^{2n\pi} |\sin x|dx$$

$$\Rightarrow I_n = -\int_0^{2n\pi} |\sin x|dx$$

$$= -2n\int_0^{\pi} |\sin x|dx$$

$$= -2n \times 2 = -4n$$

Now, $\sum_{n=1}^{100} (I_n)$

$$= \sum_{n=1}^{100} (-4n)$$

$$= -4\sum_{n=1}^{100}(n)$$

$$= -4(1 + 2 + 3 + ... + 100)$$

$$= -4 \times \frac{100 \times 101}{2}$$

$$= -20200$$

12. Given

$$m = \int_{-2}^{0}\left(\frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}\right)dx$$

Let $x = -y \Rightarrow dx = -dy$

Thus, $m = \int_{2}^{0}\left(\frac{|\sin(-y)|}{\left[\frac{-y}{\pi}\right] + \frac{1}{2}}\right) \times -dy$

$$= -\int_{2}^{0}\left(\frac{|\sin(y)|}{-1-\left[\frac{y}{\pi}\right]+\frac{1}{2}}\right) \times dy$$

$$= -\int_{0}^{2}\left(\frac{|\sin y|}{-\frac{1}{2}-\left[\frac{y}{\pi}\right]}\right)dy$$

$$= -\int_{0}^{2}\left(\frac{|\sin y|}{\left[\frac{y}{\pi}\right]+\frac{1}{2}}\right)dy$$

$$= -\int_{0}^{2}\left(\frac{|\sin x|}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\right)dx$$

$$= -n$$

$\Rightarrow \quad m + n = 0.$

Hence, the result.

13. Let

$$I = \int_{-1}^{1}\left(\sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]\right)dx$$

$$= 2\int_{0}^{1}\left(\sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]\right)dx$$

$$= 2\int_{0}^{1}\left(\sin^{-1}\left[x^2 + \frac{1}{2}\right]\right)dx + 2\int_{0}^{1}\left(\cos^{-1}\left[x^2 - \frac{1}{2}\right]dx\right)$$

$$= 2\left(\int_{0}^{1/\sqrt{2}}(\sin^{-1}(0))dx + \int_{1/\sqrt{2}}^{1}(\sin^{-1}(1))dx\right)$$

$$+2\left(\int_{0}^{1/\sqrt{2}}(\cos^{-1}(-1))dx + \int_{1/\sqrt{2}}^{1}\cos^{-1}(0)dx\right)$$

$$= 2\left(0 + \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right) + 2\left(\frac{\pi}{\sqrt{2}} + \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right)$$

$$= \pi\left(1 - \frac{1}{\sqrt{2}}\right) + 2\left(\frac{\pi}{2} + \frac{\pi}{2\sqrt{2}}\right)$$

$$= \pi\left(1 - \frac{1}{\sqrt{2}}\right) + \pi\left(1 + \frac{1}{2\sqrt{2}}\right)$$

$$= 2\pi$$

Notes: 1. $0 \le x \le 1 \Rightarrow 0 \le x^2 \le 1$

2. $\frac{1}{2} \le x^2 + \frac{1}{2} \le 1 + \frac{1}{2}$

3. $\frac{1}{2} \le x^2 + \frac{1}{2} \le 1 \Rightarrow 0 \le x^2 \le \frac{1}{2}$

$\Rightarrow \quad 0 \le x \le \frac{1}{\sqrt{2}}$

4. $1 \le x^2 + \frac{1}{2} \le \frac{3}{2} \Rightarrow \frac{1}{2} \le x^2 \le 1$

$\Rightarrow \quad \frac{1}{\sqrt{2}} \le x \le 1$

5. $\left[x^2 + \frac{1}{2}\right] = \begin{cases} 0 : & 0 \le x \le \frac{1}{\sqrt{2}} \\ 1 : & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$

14. Let

$$I = \int_{-\pi/3}^{0}\left(\cot^{-1}\left(\frac{2}{2\cos x - 1}\right) + \cot^{-1}\left(\cos x - \frac{1}{2}\right)\right)dx$$

$$= \int_{-\pi/3}^{0}\left(-\pi + \tan^{-1}\left(\cos x - \frac{1}{2}\right) + \cot^{-1}\left(\cos x - \frac{1}{2}\right)\right)dx$$

$$= \int_{-\pi/3}^{0}\left(-\pi + \frac{\pi}{2}\right)dx$$

$$= \int_{-\pi/3}^{0}\left(\frac{\pi}{2}\right)dx$$

$$= -\frac{\pi}{2}\left(0 - \left(-\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi^2}{6}$$

15. Let $I = \int_{0}^{\pi/2}\left(\frac{8 + 7\cos x}{(7 + 8\cos x)^2}\right)dx$

Dividing the numerator and the denominator by $\sin^2 x$, we get

$$I = \int_{0}^{\pi/2}\left(\frac{8\operatorname{cosec}^2 x + 7\operatorname{cosec} x \cot x}{(7\operatorname{cosec} x + 8\cot x)^2}\right)dx$$

Let $7\operatorname{cosec} x + 8\cot x = t$

$\Rightarrow \quad (8\operatorname{cosec}^2 x + 7\operatorname{cosec} x \cot x)dx = -dt$

Thus, $I = -\int_{\infty}^{7}\left(\frac{dt}{t^2}\right)$

$$= \left(\frac{1}{t}\right)_{\infty}^{7} = \frac{1}{7}$$

16. Given $f(x+y) = f(x) - f(y)$

Put $x = 0 = y$

$\Rightarrow$ $f(0) = 0$

Put $y = -x$, we get, $f(x) - f(-x) = 0$

$\Rightarrow$ $f(x) = f(-x)$

$\therefore$ $f(x)$ is an even function.

Now, $a = \int_1^3 (x-1)^2 f(x-1)dx$

$= \int_0^2 t^2 f(t)dt,$ (Let $x - 1 = t$)

$= \int_0^2 x^2 f(x)dx$

Also, $b = \int_{-3}^1 (x+1)^2 f(x+1)dx$

$= \int_{-2}^2 y^2 f(y)dy,$ (Let $x + 1 = y$)

$= \int_{-2}^2 y^2 f(x)dx$

$= 2\int_0^2 x^2 f(x)dx$

$= 2a$

Thus, the value of $2a - b + 4 = 0 + 4 = 4$

17. Let $I = \int_0^4 \left(\frac{(y^2 - 4y + 5)\sin(y-2)}{2y^2 - 8y + 1}\right)dy$

$= \int_0^4 \left(\frac{((y-2)^2 + 1)\sin(y-2)}{2(y-2)^2 - 7}\right)dy$

$= \int_{-2}^2 \left(\frac{(t^2+1)\sin t}{2t^2 - 7)}\right)dt,$ (Let $y - 2 = t$)

$= \int_{-2}^2 \left(\frac{(y^2+1)\sin y}{2y^2 - 7}\right)dy$

$= 0,$ ($\because$ it is an odd function.)

18. Let $I = \int_{-1}^1 \left(\frac{\sin\alpha}{x^2 - 2x\cos\alpha + 1}\right)dx$

$= \int_{-1}^1 \left(\frac{\sin\alpha}{(x-\cos\alpha)^2 + \sin^2\alpha}\right)dx$

Let $x - \cos\alpha = t \Rightarrow dx = dt$

$= \int_{-1-\cos\alpha}^{1-\cos\alpha} \left(\frac{\sin\alpha}{t^2 + \sin^2\alpha}\right)dt$

$= \left(\tan^{-1}\left(\frac{t}{\sin\alpha}\right)\right)_{-1-\cos\alpha}^{1-\cos\alpha}$

$= \left(\tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{-1-\cos\alpha}{\sin\alpha}\right)\right)$

$= \left(\tan^{-1}\left(\frac{2\sin^2(\alpha/2)}{\sin\alpha}\right) + \tan^{-1}\left(\frac{2\cos^2(\alpha/2)}{\sin\alpha}\right)\right)$

$= \left(\tan^{-1}\left(\tan\left(\frac{\alpha}{2}\right)\right) + \tan^{-1}\left(\cot\left(\frac{\alpha}{2}\right)\right)\right)$

$= \left(\frac{\alpha}{2} + \frac{\pi}{2} - \frac{\alpha}{2}\right)$

$= \frac{\alpha}{2}$

19. Let $I = \int_0^1 \left(\frac{x^3 \sin^{-1}x}{\sqrt{1-x^2}}\right)dx$

Let $\sin^{-1}x = t \Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

Thus, $I = \int_0^{\pi/2} (t\sin^3 t)dt$

$= \left(t\int \sin^3 t\,dt\right)_0^{\pi/2} - \int_0^{\pi/2}\left(\frac{\cos^3 t}{3} - \cos t\right)dt$

$= \left(t\left(\frac{\cos^3 t}{3} - \cos t\right)\right)_0^{\pi/2} - \int_0^{\pi/2}\left(\frac{\cos^3 t}{3} - \cos t\right)dt$

$= 0 - \int_0^{\pi/2}\left(\frac{\cos^3 t}{3} - \cos t\right)dt$

$= -\frac{1}{3}\int_0^{\pi/2}(\cos^3 t)dt + \int_0^{\pi/2}\cos t\,dt$

$= -\frac{1}{3}\left(\sin t - \frac{\sin^3 t}{3}\right)_0^{\pi/2} + (\sin t)_0^{\pi/2}$

$= -\frac{1}{3}\left(1 - \frac{1}{13}\right) + 1$

$= \left|1 - \frac{2}{9}\right|$

$= \frac{7}{9}$

20. Let $I = \int_0^1 \left(\frac{1-x^2}{1+x^2} \times \frac{1}{\sqrt{1+x^4}}\right)dx$

$= \int_0^1 \frac{1-x^2}{x^2\left(x+\frac{1}{x}\right)} \times \frac{dx}{\sqrt{x^2 + \frac{1}{x^2}}}$

$= \int_0^1 \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)} \times \frac{dx}{\sqrt{\left(x+\frac{1}{x}\right)^2 - 2}}$

Put $\left(x + \frac{1}{x}\right) = t \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$

$$= -\int_{\infty}^{2} \frac{dt}{t\sqrt{t^2 - 2}}$$

$$= -\int_{\infty}^{2} \frac{t\,dt}{t^2\sqrt{t^2 - 2}}$$

Let $t^2 - 2 = y^2 \Rightarrow t\,dt = y\,dy$

$$= \int_{\infty}^{\sqrt{2}} \frac{dy}{y^2 + 2}$$

$$= \left(-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)\right)\Bigg|_{\infty}^{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}\left(\frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$= \frac{\pi}{4\sqrt{2}}$$

21. Given,

$$m = \int_0^1 \left(\frac{\log(1+x)}{1 + x^2}\right)dx$$

$$= \int_0^{\pi/4} \log(1 + \tan\theta)d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1 + \tan\theta + 1 - \tan\theta}{1 + \tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log(2)d\theta - \int_0^{\pi/4} \log(1 + \tan\theta)d\theta$$

$$= \frac{\pi}{4}\log 2 - m$$

$$\Rightarrow 2m = \frac{\pi}{4}\log 2$$

$$\Rightarrow m = \frac{\pi}{8}\log 2$$

Also, $n = \int_0^{\infty} \left(\frac{\log(1 + x^2)}{1 + x^2}\right)dx$

$$= \int_0^{\pi/2} [\log(1 + \tan^2\theta)]d\theta$$

$$= \int_0^{\pi/2} (\log\sec^2\theta)d\theta$$

$$= 2\int_0^{\pi/2} (\log\sec\theta)d\theta$$

$$= -2\int_0^{\pi/2} (\log\cos\theta)d\theta$$

$$= -2\left(\frac{\pi}{2}\log\left(\frac{1}{2}\right)\right)$$

$$= \pi\log 2$$

$$= 8\left(\frac{\pi}{8}\log 2\right)$$

$$= 8m$$

Hence, the result.

22. Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\cos^{-1}\left(\frac{2x}{1 + x^2}\right) + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)}{e^x + 1}\right)dx$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\frac{\pi}{2} - \sin^{-1}\left(\frac{2x}{1 + x^2}\right) + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)}{e^x + 1}\right)dx$$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{\frac{\pi}{2} - 2\tan^{-1}x + 2\tan^{-1}x}{e^x + 1}\right)dx$$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{dx}{e^x + 1}\right) \quad \text{...(i)}$$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{dx}{e^{-x} + 1}\right)$$

$$= \pi/2\int_{-1/\sqrt{3}}^{1/\sqrt{3}} (e^x/e^x + 1)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$\Rightarrow 2I = \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{1 + e^x}{e^x + 1}\right)dx$$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} dx = \frac{\pi}{2} \times \frac{2}{\sqrt{3}}$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}}$$

23. Let $I = \int_{-3\pi/4}^{5\pi/4} \left(\frac{\sin x + \cos x}{e^{\left(x - \frac{\pi}{4}\right)} + 1}\right)dx \quad \text{...(i)}$

$$= \int_{-3\pi/4}^{5\pi/4} \left(\frac{\cos x + \sin x}{e^{\left(\frac{\pi}{2} - x - \frac{\pi}{4}\right)} + 1}\right)dx$$

$$= \int_{-3\pi/4}^{5\pi/4} \left(\frac{\cos x + \sin x}{e^{-\left(x - \frac{\pi}{4}\right)} + 1}\right)dx$$

$$= \int_{-3\pi/4}^{5\pi/4} \left(\frac{(\cos x + \sin x)e^{\left(x-\frac{\pi}{4}\right)}}{e^{\left(x-\frac{\pi}{4}\right)} + 1}\right)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-3\pi/4}^{5\pi/4} \frac{(\cos x + \sin x)\left(e^{\left(x-\frac{\pi}{4}\right)} + 1\right)}{\left(e^{\left(x-\frac{\pi}{4}\right)} + 1\right)} dx$$

$$= \int_{-3\pi/4}^{5\pi/4} (\cos x + \sin x)dx$$

$$= \sqrt{2}\int_{-3\pi/4}^{5\pi/4} \sin\left(x + \frac{\pi}{4}\right)dx$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{2}}\left(\cos\left(x + \frac{\pi}{4}\right)\right)_{-\frac{3\pi}{4}}^{\frac{5\pi}{4}} = 0$$

24. Given,

$$I_2 = \int_0^1 \left(\frac{\tan^{-1}x}{x}\right)dx$$

$$= \int_0^{\pi/4} \left(\frac{\theta}{\tan\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \left(\frac{\theta}{\sin\theta\cos\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \left(\frac{2\theta}{2\sin\theta\cos\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \left(\frac{2\theta}{\sin 2\theta}\right)d\theta$$

$$= \int_0^{\pi/2} \left(\frac{t}{\sin t}\right)dt, \quad (\text{Let } 2\theta = t)$$

$$= \int_0^{\pi/2} \left(\frac{x}{\sin x}\right)dt$$

$$= \frac{1}{2}I_1$$

Thus, $\frac{I_1}{I_2} = 2$

25. Given,

$$I_2 = \int_0^1 \left(\frac{x^2}{e^{x^3}(2 - x^3)}\right)dx$$

$$\left(\text{Let } 1 - x^3 = t \Rightarrow x^2 dx - \frac{1}{3}\, dt\right)$$

$$= -\frac{1}{3}\int_1^0 \left(\frac{dt}{e^{1-t}(t + 1)}\right)dx$$

$$= \frac{1}{3}\int_0^1 \left(\frac{dt}{e^{1-t}(t + 1)}\right)$$

$$= \frac{1}{3e}\int_0^1 \left(\frac{e^t dt}{(t + 1)}\right)$$

$$= \frac{1}{3e}\int_0^1 \left(\frac{e^x dx}{(x + 1)}\right)$$

$$= \frac{I_1}{3e}$$

Thus, $\frac{I_1}{I_2} = 3e$

26. Let $I_n = \int_0^{\pi} \left(\frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx.$

Now, $I_{n+1} - I_n$

$$= \int_0^x \left(\frac{\sin\left(n + 1 + \frac{1}{2}\right)x - \sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= 2\int_0^x \cos(n + 1)x\, dx$$

$$= \frac{2}{n + 1}(\sin(n + 1)x)_0^{\pi} = 0$$

Thus, $I_{n+1} - I_n$

$$\Rightarrow \quad I_{n+1} = I_n = I_{n-1} = I_{n-2} = \ldots = I_1 = I_0$$

Therefore, $I_0 = \int_0^{\pi} \sin x\, dx = (x)_0^{\pi} = \pi$

Hence, $I_n = \pi,\ n \geq 0.$

27. We have,

$\sin nx - \sin(n - 2)x = 2\cos(n - 1)x \sin x$

Now, $\int\left(\frac{\sin nx}{\sin x}\right)dx$

$$= \int 2\cos(n - 1)x\, dx + \int\left(\frac{\sin(n-2)x}{\sin x}\right)dx$$

$$\Rightarrow \quad \int\left(\frac{\sin 5x}{\sin x}\right)dx$$

$$= \int 2\cos 4x\, dx + \int\left(\frac{\sin^3 x}{\sin x}\right)dx$$

Thus,

$$\int_0^{\pi/2} \left(\frac{\sin 5x}{\sin x}\right)dx$$

$$= 2\int_0^{\pi/2} \cos^4 x\, dx + \int_0^{\pi/2}\left(\frac{\sin^3 x}{\sin x}\right)dx$$

$$= 0 + \int_0^{\pi/2} \left(\frac{3\sin x - 4\sin^3 x}{\sin x}\right)dx$$

$$= \int_0^{\pi/2} (3 - 4\sin^2 x)dx$$

$$= \int_0^{\pi/2} [3 - 2(1 - \cos^2 x)]dx$$

$$= \int_0^{\pi/2} (1 + 2\cos 2x)dx$$

$$= (x + \sin 2x)_0^{\pi/2}$$

$$= \left(\frac{\pi}{2}\right)$$

28. Given

$$m = \int_0^1 x^{50}(2 - x)^{50} dx$$

(Let $x = 2t \Rightarrow dx = 2dt$)

$$\Rightarrow \quad m = 2\int_0^{1/2} 2^{50} t^{50} 2^{50}(1 - t)^{50} dt$$

$$= 2^{100}\cdot 2\int_0^{1/2} t^{50}(1 - t)^{50} dt$$

$$= 2^{100}\int_0^1 t^{50}(1 - t)^{50} dt$$

$$= 2^{100}\int_0^1 x^{50}(1 - x)^{50} dx = 2^{100}.n$$

$$\Rightarrow \quad \frac{m}{n} = 2^{100}$$

Hence, the result.

29. Let $I = \int_0^\infty \left(\frac{\sin^3 x}{x}\right)dx$

$$= \frac{1}{4}\int_0^\infty \left(\frac{4\sin^3 x}{x}\right)dx$$

$$= \frac{1}{4}\int_0^\infty \left(\frac{3\sin x - \sin 3x}{x}\right)dx$$

$$= \frac{1}{4}\int_0^\infty \left(\frac{3\sin x}{x} - \frac{\sin 3x}{x}\right)dx$$

$$= \frac{3}{4}\int_0^\infty \left(\frac{\sin x}{x}\right)dx - \frac{3}{4}\int_0^\infty \left(\frac{\sin 3x}{3x}\right)dx$$

$$= \frac{3}{4}\int_0^\infty \left(\frac{\sin x}{x}\right)dx - \frac{1}{4}\int_0^\infty \left(\frac{\sin y}{y}\right)dy, \text{ (Let } 3x = y)$$

$$= \frac{3}{4}\int_0^\infty \left(\frac{\sin x}{x}\right)dx - \frac{1}{4}\int_0^\infty \left(\frac{\sin x}{x}\right)dx$$

$$= \left(\frac{3}{4} \times \frac{\pi}{2} - \frac{1}{4} \times \frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

30. We have,

$$B = \int_0^{\pi/2} \left(\frac{\sin 2x}{(x + 1)}\right)dx$$

$$\left(\text{Let } x = \frac{y}{2} \Rightarrow dx = \frac{dy}{2}\right)$$

$$\Rightarrow \quad B = \int_0^\pi \left(\frac{\sin y}{y + 2}\right)dy$$

$$= \int_0^\pi \left(\frac{\sin x}{x + 2}\right)dx$$

$$= \left(\frac{1}{x + 2}\int \sin x\, dx\right)_0^\pi - \int_0^\pi \left(\frac{\cos x}{(x + 2)^2}\right)dx$$

$$= \left(\frac{-\cos x}{x + 2}\right)_0^\pi - \int_0^\pi \left(\frac{\cos x}{(x + 2)^2}\right)dx$$

$$= \frac{1}{\pi + 2} + \frac{1}{2} - A$$

$$\Rightarrow \quad A + B = \frac{1}{\pi + 2} + \frac{1}{2}$$

31. We have,

$$\int_0^\infty \left(\frac{x^2}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)}\right)dx$$

$$= \left(\frac{\pi}{2(x + a)(x + b)(x + c)}\right)$$

Let $a = 2$, $b = 3$ and $c = 0$,

$$\int_0^\infty \left(\frac{dx}{(x^2 + 4)(x^2 + 9)}\right)$$

$$= \frac{\pi}{2(2 + 3)(3 + 0)(0 + 2)} = \frac{\pi}{60}$$

32. Let $I = \int_0^{\pi/2} \log(a^2\cos^2 x + b^2\sin^2 x)dx$

$$= \int_0^{\pi/2} \log[a^2(\cos^2 x + k^2\sin^2 x)]dx$$

(Let $k = b/a$)

$$= \pi\log a + \int_0^{\pi/2} \log(\cos^2 x + k^2\sin^2 x))dx$$

$$= \pi\log a + I_1,$$

Where $I_1 = \int_0^{\pi/2} \log(\cos^2 x + k^2\sin^2 x))dx$

$$\Rightarrow \frac{dI_1}{dk} = \int_0^{\pi/2} \left(\frac{2k\sin^2 x}{\cos^2 x + k^2\sin^2 x}\right)dx$$

$$= 2k\int_0^{\pi/2} \left(\frac{\tan^2 x\cdot\sec^2 x}{(1 + k^2\tan^2 x)(1 + \tan^2 x)}\right)dx$$

$$= 2k\int_0^{\pi/2} \left(\frac{t^2\, dt}{(1 + k^2t^2)(1 + t^2)}\right)dx,$$

(Let $t = \tan x$)

$$= \frac{2k}{k^2 - 1}\int_0^\infty \left(\frac{1}{1 + t^2} - \frac{1}{1 + k^2t^2}\right)dt$$

$$= \frac{2k}{k^2 - 1}\left(\tan^{-1}t - \frac{1}{k}\tan^{-1}(tk)\right)_0^\infty$$

$$= \frac{2k}{k^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2k}\right)$$

$$= \frac{2k}{k^2-1} \times \frac{\pi}{2}\left(\frac{k-1}{k}\right)$$

$$= \frac{2k}{k^2} - 1 \times \frac{\pi}{2}\left(\frac{k-1}{k}\right)$$

$$= \frac{\pi}{k+1}$$

Integrating, we get

$$I_1 = \pi \log|k+1| + c$$

When $k = 1$, $I_1 = 0$, then $c = -\pi\log 2$

Thus, $I_1 = \pi \log\left|\frac{1+k}{2}\right|$

Therefore, $I = \pi\log a + \pi\log\left|\frac{1+k}{2}\right|$

$$= \pi\log a + \pi\log\left|\frac{1+b/a}{2}\right|$$

$$= \pi\log a + \pi\log\left|\frac{a+b}{2a}\right|$$

$$= \pi\log\left|\frac{a+b}{2}\right|$$

33. Let $I = \int_0^\infty \left(\frac{\ln x}{x^2+2x+4}\right)dx$

(Let $x = 2t \Rightarrow dx = 2dt$)

$$= 2\int_0^\infty \left(\frac{\ln 2 + \ln t}{4(t^2+t+1)}\right)dt$$

$$= \frac{\ln 2}{2}\int_0^\infty \left(\frac{dt}{(t^2+t+1)}\right) + \frac{1}{2}\int_0^\infty \left(\frac{\ln t\, dt}{(t^2+t+1)}\right)$$

$$= I_1 + I_2 \text{ (say)}$$

Now, $I_2 = \frac{1}{2}\int_0^\infty \left(\frac{\ln t}{(t^2+t+1}\right)dt$

$$= -\frac{1}{2}\int_\infty^0 \left(\frac{\ln\left(\frac{1}{y}\right)}{\left(\frac{1}{y^2}+\frac{1}{y}+1\right)\frac{1}{y^2}}\right)dy, \quad \left(\text{Let } t = \frac{1}{y}\right)$$

$$= -\frac{1}{2}\int_\infty^0 \left(\frac{-\ln(y)}{(y^2+y+1)}\right)dy$$

$$= \frac{1}{2}\int_\infty^0 \left(\frac{\ln(y)}{(y^2+y+1)}\right)dy$$

$$= -\frac{1}{2}\int_0^\infty \left(\frac{\ln(y)}{(y^2+y+1)}\right)dy$$

$$= -\frac{1}{2}\int_0^\infty \left(\frac{\ln(t)}{(t^2+t+1)}\right)dt$$

$$= I_2$$

$$\Rightarrow 2I_2 = 0$$

$$\Rightarrow I_2 = 0$$

Also, $I_1 = \frac{\ln 2}{2}\int_0^\infty \left(\frac{dt}{(t^2+t+1)}\right)$

$$= \frac{\ln 2}{2}\int_0^\infty \left(\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\right)$$

$$= \frac{\ln 2}{2} \times \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}} \times \left(t+\frac{1}{2}\right)\right)_0^\infty$$

$$= \frac{\ln 2}{2} \times \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= \frac{\pi \ln 2}{3\sqrt{3}}$$

Thus, $I = 0 + \frac{\pi\ln 2}{3\sqrt{3}} = \frac{\pi\ln 2}{3\sqrt{3}}$

34. Let $I = \int_0^1 \left(\frac{\ln x}{1+x}\right)dx.$

$$= (\ln x \ln(1+x))_0^1 - \int_0^1 \left(\frac{\ln(1+x)}{x}\right)dx$$

$$= 0 - \int_0^1 \left(\frac{\ln(1+x)}{x}\right)dx$$

$$= -\int_0^1 \left(\frac{\ln(1+x)}{x}\right)dx$$

$$= -\int_0^1 \left(\frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots}{x}\right)dx$$

$$= -\int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)dx$$

$$= -\left(x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\right)_0^1$$

$$= -\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right)$$

$$= -\frac{\pi^2}{12}$$

35. Let $I = \int_0^1 \left(\frac{\ln(1+x^4)}{x}\right)dx$

$$= \int_0^1 \left(\frac{x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \frac{x^{16}}{4} + \frac{x^{20}}{5} - \ldots}{x}\right)dx$$

$$= \int_0^1 \left(x^3 - \frac{x^7}{2} + \frac{x^{11}}{3} - \frac{x^{15}}{4} + \frac{x^{19}}{5} - \ldots\right)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{36} - \frac{x^{16}}{64} + \ldots\right)_0^1$$

$$= \left(\frac{1}{4} - \frac{1}{16} + \frac{1}{36} - \frac{1}{64} + \ldots\right)$$

$$= \left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \ldots\right)$$

$$= \left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \ldots\right) - 2\left(\frac{1}{4^2} - \frac{1}{8^2} + \frac{1}{12^2} + \ldots\right)$$

$$= \left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \ldots\right) - \frac{2}{4^2}\left(1 + \frac{1}{2^2} + \frac{1}{12^2} + \ldots\right)$$

$$= \frac{\pi^2}{24} - \frac{1}{8}\cdot\frac{\pi^2}{6}$$

$$= \frac{\pi^2}{48}$$

36. Given,

$$I_n = \int_0^\pi \left(\frac{1 - x\cos nx}{1 - \cos x}\right)dx$$

Now,

$$I_{n+2} - I_{n+1}$$

$$= \int_0^\pi \left(\frac{1 - \cos(n+2)x}{1 - \cos x}\right)dx - \int_0^\pi \left(\frac{1 - \cos(n+1)x}{1 - \cos x}\right)dx$$

$$= \int_0^\pi \left(\frac{\cos(n+1)x - \cos(n+2)x}{1 - \cos x}\right)dx$$

$$= \int_0^\pi \left(\frac{2\sin\left(n + \frac{3}{2}\right)x\sin\left(\frac{x}{2}\right)}{1 - \cos x}\right)dx$$

$$= \int_0^\pi \left(\frac{\sin\left(n + \frac{3}{2}\right)x\sin\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_0^\pi \left(\frac{\sin\left(n + \frac{3}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx \quad \ldots\text{(i)}$$

Similarly,

$$I_{n+2} - I_n = \int_0^\pi \left(\frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx \quad \ldots\text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$I_{n+2} - 2I_{n+1} - I_n$$

$$= \int_0^\pi \left(\frac{\sin\left(n + \frac{3}{2}\right)x - \sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_0^\pi \left(\frac{2\cos(n+1)x\cdot\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_0^\pi 2\cos(n+1)x\,dx$$

$$= \left(\frac{2}{n+1}\right)\sin(n+1)x)_0^\pi$$

$$= 0$$

$\Rightarrow \quad I_{n+2} - 2I_{n+1} - I_n = 0$

$\Rightarrow \quad I_{n+2} - I_{n+1} = I_{n+1} - I_n$

Similarly, $I_{n+2} - I_{n+1} = I_{n+1} - I_n$

$= I_{n+2} - I_{n-1} = I_1 - I_0$.

Thus, $\quad I_n - I_{n-1} = I_1 - I_0 = \pi - 0 = \pi$

$\Rightarrow \quad I_n = \pi + I_{n-2} = \pi + \pi + I_{n-2} = 2\pi + I_{n-2}$

$= 3\pi + I_{n-3} = \ldots = n\pi + I_0 = n\pi + 0 = n\pi$

$$\Rightarrow \quad I_n = \int_0^\pi \left(\frac{1 - x\cos nx}{1 - \cos x}\right)dx = n\pi.$$

$$\text{Now, } I_0 = \int_0^\pi \left(\frac{1 - 1}{1 - \cos x}\right)dx = 0$$

$$\text{and} \quad I_1 = \int_0^\pi \left(\frac{1 - \cos x}{1 - \cos x}\right)dx = \pi$$

Subtracting, we get

$I_0 - I_1 = \pi$

$\Rightarrow \quad I_n = I_0 + n\pi = n\pi$

$I_n = n\pi$

$$\text{Now, } I_n = \int_0^{\pi/2} \left(\frac{\sin^2 n\theta}{\sin^2\theta}\right)d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1 - \cos(2n\theta)}{1 - \cos(2\theta)}\right)d\theta$$

$$= \frac{1}{2}\int_0^\pi \left(\frac{1 - \cos(nx)}{1 - \cos(x)}\right)d\theta = n\pi$$

37. We have

$$\lim_{n\to\infty}\left(\frac{(1^2 + 2^2 + 3^3 + \ldots + n^2)(1^3 + 2^3 + 3^3 + \ldots + n^3)}{(1^6 + 2^6 + 3^6 + \ldots + n^6)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\sum_{r=1}^{n}(r^2)\times\sum_{r=1}^{n}(r^3)}{\sum_{r=1}^{n}(r^6)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2\times\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^3}{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^6}\right)$$

$$= \frac{\left(\int_0^1 x^2\,dx\right)\times\left(\int_0^1 x^3\,dx\right)}{\left(\int_0^1 x^6\,dx\right)}$$

$$= \left(\frac{\left(\frac{x^3}{3}\right)_0^1\left(\frac{x^4}{4}\right)_0^1}{\left(\frac{x^7}{7}\right)_0^1}\right)$$

$$= \frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{7}} = \frac{7}{12}$$

38. We have,

$$\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n-r}{n^2}\cos\left(\frac{4r}{n}\right)\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\left(\frac{n-r}{n}\right)\cos\left(\frac{4r}{n}\right)\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\left(1-\frac{r}{n}\right)\cos\left(\frac{4r}{n}\right)\right)$$

$$= \int_0^1 (1-x)\cos(4x)dx$$

$$= \left((1-x)\frac{\sin(4x)}{4}\right)_0^1 + \frac{1}{4}\int_0^1 \sin(4x)dx$$

$$= 0 + \frac{1}{4}\int_0^1 \sin(4x)dx$$

$$= \left(-\frac{\cos(4x)}{16}\right)_0^1$$

$$= \frac{1}{16}[1-\cos(4x)]$$

39. We have,

$$\lim_{n\to\infty}\sum_{r=1}^{n}\int_0^1 (x^r - x^{r+1})dx$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{x^{r+1}}{r+1}-\frac{x^{r+2}}{r+2}\right)_0^1$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{n+2}\right)\right\}$$

$$= \left(\frac{1}{2}-0\right)$$

$$= \frac{1}{2}$$

40. We have

$$\int_0^{\pi}\left(\frac{\sin\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\right)d\theta,\ \forall\ a, b\in R^+,\ a>b$$

$$= 2\int_0^{\pi/2}\left(\frac{\sin\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\right)d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{\sin\theta}{a^2\cos^2\theta+b^2-b^2\cos^2\theta}\right)d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{\sin\theta}{b^2+(a^2-b^2)\cos^2\theta}\right)d\theta$$

Let $\cos\theta = t$

$$= -2\int_1^0\left(\frac{dt}{b^2+(a^2-b^2)t^2}\right)$$

$$= 2\int_0^1\left(\frac{dt}{b^2+(a^2-b^2)t^2}\right)$$

$$= \frac{2}{(a^2-b^2)}\int_0^1\left(\frac{dt}{\frac{b^2}{(a^2-b^2)}+t^2}\right)$$

$$= \frac{2}{(a^2-b^2)}\times\frac{1}{\frac{b}{\sqrt{a^2-b^2}}}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}t\right)_0^1$$

$$= \frac{2}{b\sqrt{(a^2-b^2)}}\times\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)$$

Level IV

1. We have,

$$\int_\alpha^\beta\sqrt{\frac{x-\alpha}{\beta-x}}\,dx$$

Let $x = \alpha\cos^2\theta+\beta\sin^2\theta$

$\Rightarrow\quad dx = (\beta-\alpha)\sin(2\theta)$

Now, $(x-\alpha) = \alpha\cos^2\theta+\beta\sin^2\theta-\alpha$

$= \beta\sin^2\theta-\alpha(1-\cos^2\theta)$

$= \beta\sin^2\theta-\alpha\sin^2\theta)$

$= (\beta-\alpha)\sin^2\theta$

and $(\beta-x) = \beta-\alpha\cos^2\theta-\beta\sin^2\theta$

$= \beta(1-\sin^2\theta)-\alpha\cos^2\theta$

$= (\beta-\alpha)\cos^2\theta$

The given integral reduces to

$$\int_0^{\pi/2} \sqrt{\frac{(\beta-\alpha)\sin^2\theta}{(\beta-\alpha)\cos^2\theta}} \times (\beta-\alpha)\sin(2\theta)d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \times (\beta-\alpha)\sin(2\theta)d\theta$$

$$= (\beta-\alpha)\int_0^{\pi/2} (2\sin^2\theta)d\theta$$

$$= (\beta-\alpha)\int_0^{\pi/2} (1-\cos(2\theta))d\theta$$

$$= (\beta-\alpha)\left(\theta - \frac{\sin(2\theta)}{2}\right)_0^{\pi/2}$$

$$= \frac{\pi}{2}(\beta-\alpha)$$

2. Now, $aI + bJ = \int_0^{\pi/2} dx$

$$\Rightarrow \quad aI + bJ = \frac{\pi}{2} \quad \text{...(i)}$$

Also, $$bI - aJ = \int_0^{\pi/2} \left(\frac{b\cos x - a\sin x}{a\cos x + b\sin x}\right)dx$$

$$= (\log|a\cos x + b\sin x|)_0^{\pi/2}$$

$$= \log\left(\frac{b}{a}\right) \quad \text{...(ii)}$$

Solving Eqs (i) and (ii), we get

$$I = \frac{1}{(a^2+b^2)}\left(\frac{a\pi}{2} + b\log\left(\frac{b}{a}\right)\right)$$

and

$$J = \frac{1}{(a^2+b^2)}\left(\frac{b\pi}{2} - a\log\left(\frac{b}{a}\right)\right).$$

3. We have,

$$\int_0^{a} \log(\cot a + \tan x)dx,\ a \in \left(0, \frac{\pi}{2}\right)$$

$$= \int_0^a \log\left(\frac{\cos a}{\sin a} + \frac{\sin x}{\cos x}\right)dx$$

$$= \int_0^a \log\left(\frac{\cos(a-x)}{\sin a\cos x}\right)dx$$

$$= \int_0^a \log[\cos(a-x)]dx - \int_0^a \log(\sin a)dx - \int_0^a \log(\cos x)dx$$

$$= \int_0^a \log(\cos x)dx - \int_0^a \log(\sin a)dx - \int_0^a \log(\cos x)dx$$

$$= -\int_0^a \log(\sin a)dx$$

$$= -a\log(\sin a)$$

$$= a\log(\operatorname{cosec} a)$$

4. Let $$\int_0^{\pi/2}\left(\frac{\sin^6 x}{\sin x + \cos x}\right)dx. \quad \text{...(i)}$$

$$= \int_0^{\pi/2}\left(\frac{\cos^6 x}{\sin x + \cos x}\right)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2}\left(\frac{\sin^6 x + \cos^6 x}{\sin x + \cos x}\right)dx$$

$$= 2\int_0^{\pi/4}\left(\frac{\sin^6 x + \cos^6 x}{\sin x + \cos x}\right)dx$$

$$I = \int_0^{\pi/4}\left(\frac{\sin^6 x + \cos^6 x}{\sin x + \cos x}\right)dx$$

$$= \int_0^{\pi/4}\left(\frac{1 - 3\sin^2 x\cos^2 x}{\sin x + \cos x}\right)dx$$

$$= \frac{1}{4}\int_0^{\pi/4}\left(\frac{4 - 3\sin^2 2x}{\sin x + \cos x}\right)dx$$

$$= \frac{1}{4}\int_0^{\pi/4}\left(\frac{4 - 3\sin^2\left\{2\left(\frac{\pi}{4} - x\right)\right\}}{\sin\left(\frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - x\right)}\right)dx$$

$$= \frac{1}{4\sqrt{2}}\int_0^{\pi/4}\left(\frac{4 - 3\cos^2 2x}{\cos x}\right)dx$$

$$= \frac{1}{4\sqrt{2}}\int_0^{\pi/4}\left(\frac{4 - 3(2\cos^2 2x - 1)^2}{\cos x}\right)dx$$

$$= \frac{1}{4\sqrt{2}}\int_0^{\pi/4}\left(\frac{4 - 3(4\cos^4 x - 4\cos^2 x + 1)}{\cos x}\right)dx$$

$$= \frac{1}{4\sqrt{2}}\int_0^{\pi/4}(\sec x + 12\cos x - 12\cos^3 x)dx$$

$$= \frac{1}{4\sqrt{2}}\left[\log|\sec x + \tan x| + 12\sin x - 12\left(\sin x - \frac{1}{3}\sin^3 x\right)\right]_0^{\pi/4}$$

$$= \frac{1}{4\sqrt{2}}\left[\log(\sqrt{2}+1) + 6\sqrt{2} - 12\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)\right]$$

$$= \frac{1}{4\sqrt{2}}[\log(\sqrt{2}+1) + \sqrt{2}]$$

5. We have

$$I = \int_0^1 \left(\frac{dx}{(5 + 2x - 2x^2)(1 + e^{(2-4x)})}\right) \quad ...(i)$$

$$= \int_0^1 \frac{dx}{\{5 + 2(1 - x) - 2(1 - x)^2\}(1 + e^{2-4(1-x)})}$$

$$= \int_0^1 \frac{dx}{(5 + 2x - 2x^2)(1 + e^{4x-2})} \quad ...(ii)$$

Adding Eqs (i) and (ii), we get

$$2i\int_0^1 \frac{dx}{(5+2x-2x^2)}\left\{\frac{1}{(1+e^{2-4x})} + \frac{1}{(1+e^{4x-2})}\right\}$$

$$= \int_0^1 \frac{dx}{(5 + 2x - 2x^2)}, \text{ since } \left\{\frac{1}{e^{2-4x}} + \frac{1}{e^{4x-2}}\right\} = 1$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \frac{dx}{-2\left(x^2 - x - \frac{5}{2}\right)}$$

$$= -\frac{1}{4}\int_0^1 \frac{dx}{\left(\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{11}}{2}\right)^2\right)}$$

$$= \left(-\frac{1}{4} \times \frac{1}{2 \cdot \frac{\sqrt{11}}{2}}\log\left|\frac{\left(x - \frac{1}{2}\right) - \left(\frac{\sqrt{11}}{2}\right)}{\left(x - \frac{1}{2}\right) + \left(\frac{\sqrt{11}}{2}\right)}\right|\right)_0^1$$

$$= \frac{1}{4} \times \frac{1}{\sqrt{11}}\log\left|\frac{(\sqrt{11} + 1)}{(\sqrt{11} - 1)}\right|$$

$$= \frac{1}{\sqrt{11}}\log\left(\frac{1 + \sqrt{11}}{\sqrt{10}}\right)$$

6. We have,

$$\int_1^\infty \left(\frac{(x - 1)^4}{x^8 + x^{10}}\right)dx$$

$$= \int_0^1 \left(\frac{t^4(t - 1)^4}{t^2 + 1}\right)dt, \qquad \left(\text{Let} \quad t = \frac{1}{x}\right)$$

$$= \int_0^1 \left(\frac{t^4(t^4 - 4t^3 + 6t^2 - 4t + 1)}{t^2 + 1}\right)dt$$

$$= \int_0^1 \left(\frac{(t^8 - 4t^7 + 6t^6 - 4t^5 + t^4)}{t^2 + 1}\right)dt$$

$$= \int_0^1 \left(t^6 - 4t^5 + 5t^4 - 4t^2 + 4 - \frac{4}{t^2 + 1}\right)dt$$

$$= \left(\frac{t^7}{7} - \frac{4t^6}{6} + t^5 - \frac{4t^3}{3} + 4t - 4\tan^{-1}t\right)_0^1$$

$$= \left(\frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\tan^{-1}(1)\right)$$

$$= \left(\frac{22}{7} - \pi\right)$$

7. Given $I = \int_0^\pi \left(\frac{x}{1 + x\sin x}\right)^2 dx$

and $J = \int_0^\pi \left(\frac{x^2(1 + \cos x)}{(1 + x\sin x)^2}\right)dx$

Now,

$$(J - I) = \int_0^\pi \frac{x^2\cos x}{(1 + \sin x)^2}dx$$

$$= \left(\frac{-x^2}{(1 + \sin x)}\right)_0^\pi + 2\int_0^\pi \left(\frac{x}{1 + \sin x}\right)dx$$

$$= -\pi^2 + 2k$$

where $k = \int_0^\pi \frac{x}{1 + \sin x}dx$

$$\Rightarrow \quad k = \int_0^\pi \frac{(\pi - x)}{1 + \sin x}dx$$

$$\Rightarrow \quad 2k = \int_0^\pi \frac{\pi}{1 + \sin x}dx$$

$$\Rightarrow \quad 2k = 2\pi\int_0^\pi \frac{dx}{1 + \sin x}$$

$$\Rightarrow \quad k = \pi\int_0^{\pi/2} \frac{dx}{1 + \cos\left(\frac{\pi}{2} - x\right)}$$

$$\Rightarrow \quad k = \pi\int_0^{\pi/2} \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$\Rightarrow \quad k = \frac{\pi}{2}\int_0^{\pi/2} \left(\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)dx$$

$$\Rightarrow \quad k = \left(\pi\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad k = \pi$$

Thus, $J - I = -\pi^2 + 2\pi = \pi(2 - \pi)$

8. We have,

$$\int_0^{\pi/2} (\sqrt{\sin x} + \sqrt{\cos x})^{-4}dx$$

$$= \int_0^{\pi/2} \frac{dx}{(\sqrt{\sin x} + \sqrt{\cos x})^4}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{(\sqrt{\tan x} + 1)^4}$$

$$= \int_0^\infty \frac{dt}{(t+1)^4}$$

$$= \left(-\frac{1}{3} \times \frac{1}{(t+1)}\right)_0^\infty$$

$$= \frac{1}{3}$$

9. We have,

$$\int_0^\pi \left(\frac{dx}{a - \cos x}\right)$$

$$= \int_0^\pi \left(\frac{dx}{a - \left(\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}\right)}\right)$$

$$= \int_0^\pi \left(\frac{\sec^2(x/2)dx}{a(1 + \tan^2(x/2)) - (1 - \tan^2(x/2))}\right)$$

$$= \int_0^\pi \left(\frac{\sec^2(x/2)dx}{(a-1) + (a+1)\tan^2(x/2)}\right)$$

$$= \frac{1}{(a+1)} \int_0^\pi \left(\frac{\sec^2(x/2)dx}{\frac{(a-1)}{(a+1)} + \tan^2(x/2)}\right)$$

$$= \frac{2}{(a+1)} \int_0^\infty \left(\frac{dt}{\frac{(a-1)}{(a+1)} + t^2}\right)$$

$$= \frac{2}{(a+1)} \times \sqrt{\frac{(a+1)}{(a-1)}} \tan^{-1}\left(t\sqrt{\frac{(a+1)}{(a-1)}}\right)_0^\infty$$

$$= \frac{2}{(a+1)} \times \sqrt{\frac{(a+1)}{(a-1)}} \times \frac{\pi}{2}$$

$$= \frac{\pi}{\sqrt{a^2 - 1}}$$

Thus, $\int_0^\pi \frac{dx}{(a - \cos x)} = \frac{\pi}{\sqrt{a^2 - 1}} \cdot a > 1$

Diffeerentiating w.r.t a, we get

$$-\int_0^\pi \frac{dx}{(a - \cos x)^2} = -\frac{\pi a}{(a^2 - 1)^{3/2}}$$

$$\Rightarrow \int_0^\pi \frac{dx}{(a - \cos x)^2} = \frac{\pi a}{(a^2 - 1)^{3/2}}$$

(Let $a = 2$)

$$\Rightarrow \int_0^\pi \frac{dx}{(2 - \cos x)^2} = \frac{2\pi}{3\sqrt{3}}$$

10. Given,

$$I_n = \int_0^{\pi/4} (\tan^n x)dx$$

$$= \int_0^{\Pi/4} (\tan^{n-2} x \cdot \tan^2 x)dx$$

$$= \int_0^{\Pi/4} (\tan^{n-2} x(\sec^2 x - 1))dx$$

$$= \int_0^{\Pi/4} (\tan^{n-2} x \cdot \sec^2 x)dx - \int_0^{\Pi/4} (\tan^{n-2} x\, dx$$

$$= \int_0^{\Pi/4} (\tan^{n-2} x \cdot \sec^2 x)dx - I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n-2} = \left(\frac{t^{n-1}}{n-1}\right)_0^1$$

$$\Rightarrow \quad I_n + I_{n-2} = \left(\frac{1}{n-1}\right)$$

$$\Rightarrow \quad \frac{1}{(I_n + I_{n-2})} = (n-1)$$

Thus, $\frac{1}{(I_2 + I_4)} = 3$

$$\frac{1}{(I_5 + I_3)} = 4$$

$$\frac{1}{(I_6 + I_4)} = 5$$

and $\frac{1}{(I_7 + I_5)} = 6$

Clearly, 3, 4, 5 and 6 are in AP.

11. Let $I = \int_0^\pi \theta^3 \log(\sin\theta)d\theta$ ...(i)

$$= \int_0^\pi (\pi - \theta)^3 \log(\sin(\pi - \theta))d\theta$$

$$= \int_0^\pi (\pi - \theta)^3 \log(\sin\theta)d\theta \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^\pi \{(\pi - \theta)^3 + \theta^3\}\log(\sin\theta)d\theta$$

$$= \int_0^\pi \{\pi^3 - 3\pi^2\theta + 3\pi\theta^2\}\log(\sin\theta)d\theta$$

$$= \pi^3 \int_0^\pi \log(\sin\theta)d\theta - 3\pi^2 \int_0^\pi (\theta)\log(\sin\theta)d\theta + 3\pi \int_0^\pi (\theta^2)\log(\sin\theta)d\theta$$

$$= -\pi^4 \log 2 - 3\pi^2 I_1 + 3\pi I_2$$

Now, $I_1 = \int_0^{\pi} \theta\log(\sin\theta)d\theta$

$= \int_0^{\pi} (\pi - \theta)\log(\sin\theta)d\theta$

$= \pi\int_0^{\pi} \log(\sin\theta)d\theta - I_1$

$\Rightarrow 2I_1 = \pi\int_0^{\pi} \log(\sin\theta)d\theta$

$= -\pi^2\log(2)$

$\Rightarrow I_1 = -\frac{\pi}{2}\log(2)$

Now, $2I = -\pi^4\log 2 - 3\pi^2\left(-\frac{\pi^2}{2}\log(2)\right) + 3\pi I_2$

$\Rightarrow I = -\frac{\pi^4}{2}\log 2 - \frac{3}{2}\pi^2\left(-\frac{\pi^2}{2}\log(2)\right) + \frac{3\pi}{2}I_2$

$= \frac{\pi^4}{4}\log 2 + \frac{3\pi}{2}I_2$

$= \frac{\pi^4}{4}\log 2 + \frac{3\pi}{2}\int_0^{\pi} [\theta^2\log\sin(\theta)]d\theta$

$= \frac{3\pi}{2}\cdot\frac{\pi^3}{6}\log 2 + \frac{3\pi}{2}\int_0^{\pi} [\theta^2\log\sin(\theta)]d\theta$

$= \frac{3\pi}{2}\cdot\frac{\pi^3}{6}\log(\sqrt{2}) + \frac{3\pi}{2}\int_0^{\pi} [\theta^2\log\sin(\theta)]d\theta$

$= \frac{3\pi}{2}\cdot\int_0^{\pi} \theta^2\log(\sqrt{2})d\theta + \frac{3\pi}{2}\int_0^{\pi} [\theta^2\log\sin(\theta)]d\theta$

$= \frac{3\pi}{2}\cdot\int_0^{\pi} \theta^2\log(\sqrt{2}\sin(\theta))d\theta$

Hence, the result.

12. Given,

$I_n = \int_0^{\infty} (e^{-x}x^{n-1}\log x)dx$

$= \int_0^{\infty} \{(e^{-x}\log x)\cdot x^{n-1}\}dx$

$= \left(e^{-x}\log x\cdot\frac{x^n}{n}\right)_0^{\infty} - \int_0^{\infty}\left\{\left(e^{-x}\cdot\frac{1}{x} - e^{-x}\log x\right)\frac{x^n}{n}\right\}dx$

$= 0 - \frac{1}{n}\int_0^{\infty} \{(e^{-x}\cdot x^{n-1} - e^{-x}\log x\cdot x^n)\}dx$

$= \frac{1}{n}\int_0^{\infty} e^{-x}\log x\cdot x^n\,dx - \frac{1}{n}\int_0^{\infty} e^{-x}\cdot x^{n-1}dx$

$= \frac{I_{n-1}}{n} - \frac{\Gamma(n)}{n}$

Thus, $nI_n = I_{n-1} - \Gamma(n)$

Replacing n by $(n + 1)$, we get

$(n + 1)I_{n+1} = I_{n+2} - \Gamma(n + 1)$

$(n + 1)I_{n+1} = I_{n+2} - \Gamma(n + 1)$

Also, $n^2I_n = nI_{n+1} - n\Gamma(n + 1)$

Thus, $I_{n+2} - (n + 1)I_{n+1} = nI_{n+1} - n^2I_n$

$\Rightarrow I_{n+2} - (n + 2)I_{n+1} + n^2I_n = 0$

13. Given,

$$af(x) = bf(-x) = \frac{1}{x}\sin\left(x - \frac{1}{2}\right) \quad \text{...(i)}$$

Replacing x by $-x$, we get

$af(-x) + bf(x) = -\frac{1}{x}\sin\left(\frac{1}{x} - x\right)$

$$\Rightarrow af(-x) + bf(x) = \frac{1}{x}\sin\left(x - \frac{1}{x}\right) \quad \text{...(ii)}$$

From Eqs (i) and (ii), we get

$(a^2 - b^2)f(x) = \left(a - \frac{b}{x}\right)\sin\left(x - \frac{1}{x}\right)$

$\Rightarrow f(x) = \left(\frac{1}{(a + b)x}\right)\sin\left(x - \frac{1}{x}\right)$

Let $I = \int_{1/2}^{2} f(x)dx, \ (a \neq b)$

$= \int_{1/2}^{2} \left(\left(\frac{1}{(a+b)x}\right)\sin\left(x - \frac{1}{x}\right)\right)dx$

$= \frac{1}{(a + b)}\int_{1/2}^{2}\left(\frac{1}{x}\sin\left(x - \frac{1}{x}\right)\right)dx$

$= \frac{1}{(a + b)}\int_{2}^{1/2}\left(\frac{1}{y}\sin\left(y - \frac{1}{y}\right)\right)dy, \ x = \frac{1}{y}$

$= -\frac{1}{(a + b)}\int_{1/2}^{2}\left(\frac{1}{y}\sin\left(y - \frac{1}{y}\right)\right)dy$

$= -I$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

$= \int_{1/2}^{2} f(x)dx = 0$

14. We have,

$\int_0^{\pi/6} \left(\frac{\sqrt{3\cos(2x) - 1}}{\cos(x)}\right)dx$

$= \int_0^{\pi/6} \left(\frac{\sqrt{6\cos^2 x - 4}}{\cos x}\right)dx$

$= \int_0^{\pi/6} \left(\sqrt{\frac{6\cos^2 x - 4}{\cos^2 x}}\right)dx$

$= \int_0^{\pi/6} (\sqrt{6 - 4\sec^2 x})dx$

$$= \int_0^{\pi/6} \left(\frac{6 - 4\sec^2 x}{\sqrt{6 - 4\cos^2 x}}\right) dx$$

$$= \int_0^{\pi/6} \left(\frac{6}{\sqrt{6 - 4\sec^2 x}}\right) dx + 4\int_0^{\pi/6} \left(\frac{\sec^2 x}{\sqrt{6 - 4\sec^2 x}}\right) dx$$

$$= \int_0^{\pi/6} \left(\frac{6\cos x}{\sqrt{6\cos^2 x - 4}}\right) dx + 4\int_0^{\pi/6} \left(\frac{\sec^2 x}{\sqrt{2 - 2\tan^2 x}}\right) dx$$

$$= \int_0^{\pi/6} \left(\frac{6\cos x}{\sqrt{2 - 6\sin^2 x}}\right) dx + 4\int_0^{\pi/6} \left(\frac{\sec^2 x}{\sqrt{2 - 2\tan^2 x}}\right) dx$$

Let $\sin x = t$ and $\tan x = v$

$$= \int_0^{1/2} \left(\frac{6t}{\sqrt{2 - 6t^2}}\right) dx + 4\int_0^{1/\sqrt{3}} \left(\frac{dv}{\sqrt{2 - 2v^2}}\right) dx$$

$$= \frac{6\pi}{3\sqrt{6}} - 4\cdot\frac{1}{2}\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

$$= \frac{2\pi}{\sqrt{6}} - 2\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

15. We have

$$\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{x^4 + x^3 + 2x^2 + x\tan^2 x + x^2\sin^3 x + \tan^{-1}(x) + 1}{x^6 + 3x^4 + 3x^2 + 1}\right) dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{x^4 + 2x^2 + 1}{(x^2 + 1)^3}\right) dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{(x^2 + 1)^2}{(x^2 + 1)^3}\right) dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{dx}{(x^2 + 1)}\right)$$

$$= 2\int_0^{\sqrt{3}} \frac{dx}{(x^2 + 1)}$$

$$= \left(2\tan^{-1}(x)\right)_0^{\sqrt{3}}$$

$$= 2[\tan^{-1}(\sqrt{3}) - 0]$$

$$= \frac{2\pi}{3}$$

Integer Type Questions

1. As we know that

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2.\Gamma\left(\frac{n+2}{2}\right)} \times \sqrt{\pi}$$

So, $$\int_0^{\pi/2} \sin^{10} x\, dx = \frac{\Gamma\left(\frac{11}{2}\right)}{2\Gamma(6)} \times \sqrt{\pi}$$

$$= \frac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{1}{2}\right)}{2\cdot(5!)} \times \sqrt{\pi}$$

$$= \frac{63 \times 15 \times \sqrt{\pi}}{2^5 \times 120} \times \sqrt{\pi}$$

$$= \frac{63 \times \pi}{2^5 \times 8}$$

$$= \frac{63 \times \pi}{2^8}$$

Thus, $A = 2$ and $B = 8$

Hence, the value of $B - A$ is 6.

2. As we know that

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2.\Gamma\left(\frac{n+2}{2}\right)} \times \sqrt{\pi}$$

So, $\int_0^{\pi/2} \cos^8 x\, dx$

$$= \frac{\Gamma\left(\frac{9}{2}\right)}{2\Gamma(5)} \times \sqrt{\pi}$$

$$= \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{2\cdot(4!)} \times \sqrt{\pi}$$

$$= \frac{105 \times \sqrt{\pi}}{2^4 \times 24} \times \sqrt{\pi}$$

$$= \frac{35 \times \pi}{2^7}$$

So, $L = 3$, $M = 5$, $P = 2$ and $Q = 7$

Hence, the value of

$$L + M + P - Q$$
$$= 3 + 5 + 2 - 7$$
$$= 3$$

3. We have,

$$I_1 = \int_0^{\pi/2} \log(\sin x) dx = \frac{\pi}{8}\log 2$$

and $$I_2 = \int_0^{\pi/2} \log(\cos x) dx = \frac{\pi}{8}\log 2$$

Hence, the value of

$I_1 - I_2 + 5 = 5$

4. Let $$L = \int_0^1 \frac{\log(1 + x)}{(1 + x^2)} dx$$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$= \int_0^{\pi/4} \frac{\log(1 + \tan\theta)}{(1 + \tan^2\theta)} \sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \log(1 + \tan\theta)d\theta \quad \text{...(i)}$$

$$= \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1 + 1 - \tan\theta}{1 + \tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log|2|d\theta - \int_0^{\pi/4} \log(1 + \tan\theta)d\theta$$

$$= \int_0^{\pi/4} \log|2|d\theta - I$$

$$\Rightarrow 2I = \int_0^{\pi/4} \log 2\, d\theta$$

$$= \frac{\pi}{4}\log 2$$

$$\Rightarrow I = \frac{\pi}{8}\log 2$$

and Let $M = \int_0^{\pi/4} \log(1 + \tan x)dx$

$$= \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right)dx$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right)dx$$

$$= \int_0^{\pi/4} \log 2\, dx - \int_0^{\pi/4} \log(1 + \tan x)dx$$

$$= \frac{\pi}{4}\log 2 - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

Now, $L + M = \frac{\pi}{8}\log 2 + \frac{\pi}{8}\log 2 = \frac{\pi}{4}\log 2$

So, $A = 4$ and $B = 2$

Hence, the value of

$A + B + 3 = 9$.

5. We have,

$$I = \int_0^{\infty} e^{-x}x^{3/2}\,dx$$

$$= \int_0^{\infty} e^{-x}x^{5/2 - 1}\,dx$$

$$= \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{3}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2} \times \sqrt{\pi}$$

$$= \frac{3}{2} \times (\pi)^{1/2}$$

Hence, the value of

$L + M + N$ is 5

6. Clearly, $M = \frac{1}{2} \times 1 \times \frac{1}{2} + \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{2}$

and $N = 20 \times \frac{1}{2} \times 1 \times \frac{1}{2} = 5$

Hence, the value of

$2M + N + 2 = 1 + 5 + 2 = 8$

7. We have,

$$M = \int_{-2}^{2} (-1)^{[x]}\,dx$$

$$= \int_{-2}^{-1} (-1)^{[x]}\,dx + \int_{-1}^{0} (-1)^{[x]}\,dx + \int_{0}^{1} (-1)^{[x]}\,dx + \int_{1}^{2} (-1)^{[x]}\,dx$$

$$= \int_{-2}^{-1} (-1)^{(-2)}\,dx + \int_{-1}^{0} (-1)^{(-1)}\,dx + \int_{0}^{1} (-1)^{(0)}\,dx + \int_{1}^{2} (-1)^{(1)}\,dx$$

$$= \int_{-2}^{-1} 1\cdot dx + \int_{-1}^{0} (-1)\cdot dx + \int_{0}^{1} (1)\,dx + \int_{1}^{2} (-1)dx$$

$$= (-1 + 2) - (0 + 1) + (1 - 0) - (2 - 1)$$

$$= 1 - 1 + 1 - 1 = 0$$

and $N = \int_0^4 [x + 1]dx$

$$= \int_0^4 ([x] + 1)dx$$

$$= \int_0^4 [x]dx + \int_0^4 1\cdot dx$$

$$= \frac{43}{2} + 4 = 10$$

Hence, the value of

$M + N - 1 = 9$.

8. We have,

$$L = \int_{-2}^{3} (|x - 1| + 1)dx$$

$$= \frac{1}{2}(4 + 1) \times 3 + \frac{1}{2}(1 + 3) \times 2$$

$$= \frac{23}{2}$$

Now, $M = \int_0^4 |x - 2| dx$

$$= 2 \times \frac{1}{2} \times 2 \times 2$$

$$= 4$$

Thus, $\frac{2}{3}(L - M) = \frac{2}{3}\left(\frac{23}{2} - 4\right) = 5$

9. Let $L = \int_1^3 [\log|x|] dx$ and $M = \int_1^3 [|\log x|] dx$ and, then find the value of $\frac{L}{M} + 5$.

10. We have,

$$A = \int_0^{2\pi} [\sqrt{2}\sin x] dx = \pi$$

and $B = \int_0^{10\pi} [\sin x + \cos x] dx = 5\pi$

Hence, the value of

$\frac{B}{A} = 5.$

11. We have,

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{3n}} + \frac{1}{\sqrt{4n}} + \ldots + \frac{1}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{r=1}^{4n} \frac{1}{\sqrt{r}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{4n} \frac{1}{\sqrt{\frac{r}{n}}}$$

$$= \int_0^4 \frac{dx}{\sqrt{x}}$$

$$= (2\sqrt{x})_0^4 = (2\sqrt{4} - 0) = 4$$

12. As we know that

$$\left(\frac{\int_0^n [x] dx}{\int_0^n \{x\} dx}\right) = \frac{\frac{n(n-1)}{2}}{\frac{n}{2}} = (n - 1)$$

Thus, $\left(\frac{\int_0^{10} [x] dx}{\int_0^{10} \{x\} dx}\right) = (10 - 1) = 9.$

13. As we know that

$$\int_a^b \left(\frac{f(x)}{f(a+b+x) + f(x)}\right) dx = \frac{b-a}{2}$$

Thus, the value of

$$\int_2^8 \left(\frac{[x^2]}{[x^2 - 20x + 100] + [x^2]}\right) dx$$

$$= \frac{8-2}{2} = 3$$

14. We have

$$\int_{-1}^{1} x^2 d(\ln x)$$

$$= \int_{1/e}^{e} x^2 \cdot \frac{1}{x} dx$$

$$= \int_{1/e}^{e} x\,dx$$

$$= \left(\frac{x^2}{2}\right)_{1/e}^{e}$$

$$= \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right)$$

$$= \frac{1}{2}\left(\frac{e^4 - 1}{e^2}\right)$$

Clearly, $m = 4$ and $n = 2$.

Hence, the value of

$(m + n) = 6.$

15. We have,

$$\int_0^1 \frac{dx}{\sqrt{1+x} + \sqrt{1-x} + 2}$$

Put $x = \sin(2\theta)$

$\Rightarrow$ $dx = 2\cos(2\theta) d\theta$

$$= \int_0^{\pi/4} \frac{2\cos(2\theta) d\theta}{\sqrt{1 + \sin(2\theta)} + \sqrt{1 - \sin(2\theta)} + 2}$$

$$= \int_0^{\pi/4} \frac{2\cos(2\theta) d\theta}{\cos\theta + \sin\theta + \cos\theta - \sin\theta + 2}$$

$$= \int_0^{\pi/4} \frac{2\cos(2\theta) d\theta}{2(\cos\theta + 1)}$$

$$= \int_0^{\pi/4} \frac{\cos(2\theta)}{(\cos\theta + 1)} d\theta$$

$$= \int_0^{\pi/4} \frac{1 - 2\sin^2\theta)}{(\cos\theta + 1)} d\theta$$

$$= \int_0^{\pi/4} \frac{1 - 2(1 - \cos^2\theta)}{(\cos\theta + 1)} d\theta$$

$$= \int_0^{\pi/4} \left(\frac{1}{1 + \cos\theta} - 2(1 - \cos\theta)\right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{1}{2\cos^2(\theta/2)} - 2(1 - \cos\theta)\right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) - 2(1 - \cos\theta)\right) d\theta$$

$$= \left(\tan\left(\frac{\theta}{2}\right) - 2(\theta - \sin\theta)\right)_0^{\frac{\pi}{4}}$$

$$= \left(\tan\left(\frac{\pi}{8}\right) - 2\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)\right)$$

$$= (\sqrt{2} - 1) - \frac{\pi}{2} + \sqrt{2}$$

$$= \left(2\sqrt{2} - \frac{\pi}{2} - 1\right)$$

Thus, $a = 2$, $b = 2$, $c = -1$

Hence, the value of

$(a + b + c) = 3$.

Questions asked in Past IIT-JEE Examinations

1. $\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)](g(x)) - g(-x))dx$

$$= \int_{-\pi/2}^{\pi/2} h(x)dx = 0$$

where $h(x) = [f(x) + f(-x)](g(x) - g(-x)$

$\Rightarrow h(x)$ is an odd function.

2. Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx$ ...(i)

$$= \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)\sin x + \cos x)dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\left(\sin\left(x + \frac{x}{4}\right)\right)dx$$

Let $\left(\frac{x}{4} - x\right) = t$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2t)(\cos t)dt$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2t)(\cos t)dt$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)(\cos x)dx$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)(\cos x)dx$$

$$\Rightarrow \quad I = 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)(\cos x)dx$$

3. Let $I = \int_0^{\pi/2} \sin(2kx)\cot x\,dx$

$$= \int_0^{\pi/2} \left(\frac{\sin 2kx}{\sin x}\right)\cos x\,dx$$

$$= \int_0^{\pi/2} 2((\cos x + \cos 3x + \dots + \cos(2k-1)x)\cos x\,dx$$

$$= \int_0^{\pi/2} [2\cos^2 x + 2\cos 3x\cos x + \dots + 2\cos(2k-1)x\cos x]dx$$

$$= \int_0^{\pi/2} [(1 + \cos 2x) + (\cos 4x + \cos 2x) + \dots + \cos(2k)x + \cos(2k-2)x]dx$$

$$= \left(1 + \frac{\sin 2x}{2}\right)_0^{\pi/2} + \left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right)_0^{\pi/2} + \frac{\sin(2k)x}{2k} + \frac{\cos(2k-2)x}{(2k-2)}\Bigg]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

4. If f is a continuous function with $\int_0^x f(t)dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)dt = 2$

5. Let $I = \int_0^{\pi} \left(\frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi}\right)dx$ ...(i)

$$= \int_0^{\pi} \left(\frac{(\pi - x)\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x\right)}{2(\pi - x) - \pi}\right)dx$$

$$= \int_0^{\pi} \left(\frac{(\pi - x)\sin(2x)\sin\left(\frac{\pi}{2}\cos(x)\right)}{\pi - 2x}\right)dx$$

$$= \int_0^{\pi} \left(\frac{(x - \pi)\sin(2x)\sin\left(\frac{\pi}{2}\cos(x)\right)}{(2x - \pi)}\right)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi} \left(\sin(2x)\sin\left(\frac{\pi}{2}\cos(x)\right)\right)dx$$

$$\Rightarrow 2I = 2\int_0^{\pi} \left(\sin(x)\cos x\sin\left(\frac{\pi}{2}\cos(x)\right)\right)dx$$

$$\Rightarrow I = \int_0^{\pi} \left(\sin(x)\cos x\sin\left(\frac{\pi}{2}\cos(x)\right)\right)dx$$

$$\Rightarrow I = -\frac{4}{\pi^2}\int_{\pi/2}^{-\pi/2} (t\sin t)dt, \qquad \left(\text{Let } \frac{\pi}{2}\cos x = t\right)$$

$$\Rightarrow \quad I = -\frac{4}{\pi^2}\int_{\pi/2}^{-\pi/2} (t\sin t)dt$$

$$= -\frac{8}{\pi^2}\int_0^{\pi/2} (x\sin x)dx$$

$$= -\frac{8}{\pi^2}(-x\cos x + \sin x)_0^{\pi/2}$$

$$= -\frac{8}{\pi^2}(1-0) = \frac{8}{\pi^2}$$

6. Given $\int_3^5 \left(\frac{x^2}{x^2-4}\right)dx$

$$= \int_3^5 \left(1 + \frac{4}{x^2-4}\right)dx$$

$$= \left(x + 4\cdot\frac{1}{2.2}\log\left|\frac{x-2}{x+2}\right|\right)_3^5$$

$$= \left(x + \log\left|\frac{x-2}{x+2}\right|\right)_3^5$$

$$= \left(5 + \log\left|\frac{3}{7}\right| - 3 - \log\left|\frac{1}{5}\right|\right)$$

$$= \left(2 + \log\left|\frac{15}{7}\right|\right)$$

7. Given, $\int_{-\pi}^{\pi} (1+x^2)\sin x\cos^2 x\,dx$

$$= \int_{-\pi}^{\pi} f(x)dx$$

$= 0$, since $(1-x^2)\sin x\cos^2 x$ is an odd function.

8. Given integral $= \int_{-1}^{1} |1-x|dx$

$$= \int_{-1}^{1} |x-1|dx$$

$$= \int_{-2}^{0} |t|dx, \qquad (\text{Let} \quad (x-1) = t)$$

$$= \int_{-2}^{0} |x|$$

$$= \int_{-2}^{0} (-x)dx$$

$$= \left(\frac{-x^2}{2}\right)_{-2}^{0}$$

$$= 2$$

9. Given integral $= \int_0^1 |\sin(2\pi x)|dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\sin t|dt, \qquad (\text{Let} \quad t = 2\pi x)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\sin x|dx$$

$$= \frac{1}{2\pi}\times 4 = \frac{2}{\pi}$$

10. Given integral $= \int_0^1 e^x(x-1)^n dx$

Let $I = \int_0^1 e^x(x-1)^n dx$

$$= ((x-1)^n e^x)_0^1 - \int_0^1 n(x-1)^{n-1}e^x dx$$

$$= (-1)n - n\int_0^1 (x-1)^{n-1}e^x dx$$

$$= -(-1)^n - nI_{n-1}$$

$$I_n = -(-1)^n - nI_{n-1} \qquad \text{...(i)}$$

Now, $I_1 = \int_0^1 e^x(x-1)dx$

$$= ((x-1)e^x)_0^1 - \int_0^1 e^x dx$$

$$= -(-1) - (e^x)_0^1$$

$$= 1 - (e-1) = 2 - e$$

From Eq. (i), we can write

$$I_2 = -1 - 2I_1$$
$$= -1 - 2(2-e) = 2e - 5$$

Again, from Eq. (i), we can write

$$I_3 = 1 - 3I_2$$
$$= 1 - 3(2e-5) = 16 - 6e$$

11. Given integral is $\int_0^{\pi/2} \left(\frac{1}{1+\tan^3 x}\right)dx$

$$= \int_0^{\pi/2} \left(\frac{\cos^3 x}{\cos^3 x + \sin^3 x}\right)dx \qquad \text{...(i)}$$

$$= \int_0^{\pi/2} \left(\frac{\sin^3 x}{\sin^3 x + \cos^3 x}\right) dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left(\frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x}\right) dx$$

$$= \int_0^{\pi/2} (1) dx = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

12. Given integral $= \int_2^3 \left(\frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2 + 1)(x^4 - 1)}\right) dx$

$$= \int_2^3 \left(\frac{(2x^5 - 2x^3) + (x^4 + 2x^2 + 1)}{(x^2 + 1)(x^4 - 1)}\right) dx$$

$$= \int_2^3 \left(\frac{(2x^3(x^2 - 1)) + (x^2 + 1)^2}{(x^2 + 1)^2(x^2 - 1)}\right) dx$$

$$= \int_2^3 \left(\frac{(2x^3)}{(x^2 + 1)^2}\right) dx + \int_2^3 \left(\frac{dx}{(x^2 - 1)}\right)$$

$$= \int_2^3 \left(\frac{x^2 . 2x\, dx}{(x^2 + 1)^2}\right) + \left(\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right)_2^3$$

$$= \int_5^{10} \left(\frac{t-1}{t^2}\right) dt + \left(\frac{1}{2}\log\left|\frac{1}{2}\right| - \frac{1}{2}\log\left|\frac{1}{3}\right|\right)$$

Let $x^2 + 1 = t$

$$= \int_5^{10} \left(\frac{1}{t} - \frac{1}{t^2}\right) dt + \left(\frac{1}{2}\log\left|\frac{3}{2}\right|\right)$$

$$= \left(\log|t| + \frac{1}{t}\right)_5^{10} + \left(\frac{1}{2}\log\left|\frac{3}{2}\right|\right)$$

$$= \left(\log|2| - \frac{1}{10}\right) + \left(\frac{1}{2}\log\left|\frac{3}{2}\right|\right)$$

$$= \left(\frac{1}{2}\log|6| - \frac{1}{10}\right)$$

13. Let $I = \int_{\pi/4}^{3\pi/4} \left(\frac{\varphi}{1 + \sin\varphi}\right) d\varphi$ ...(i)

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{\pi - \varphi}{1 + \sin(\pi - \varphi)}\right) d\varphi$$

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{\pi - \varphi}{1 + \sin(\varphi)}\right) d\varphi \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{\pi}{1 + \sin(\varphi)}\right) d\varphi$$

$$= \pi \int_{\pi/4}^{3\pi/4} \left(\frac{d\varphi}{1 + \sin(\varphi)}\right)$$

$$= \pi \int_{\pi/4}^{3\pi/4} \left(\frac{(1 - \sin\varphi) d\varphi}{1 - \sin^2(\varphi)}\right)$$

$$= \pi \int_{\pi/4}^{3\pi/4} \left(\frac{(1 - \sin\varphi)}{\cos^2\varphi}\right) d\varphi$$

$$= \pi \int_{\pi/4}^{3\pi/4} (\sec^2\varphi - \sec\varphi \tan\varphi) d\varphi$$

$$= \pi(\tan\varphi - \sec\varphi)_{\pi/4}^{3\pi/4}$$

$$= \pi[(-1 - \sqrt{2}) - (1 + \sqrt{2})]$$

$$= -2\pi(\sqrt{2} + 1)$$

$$\Rightarrow I = -\pi(\sqrt{2} + 1)$$

14. Given integral $= \int_{-1}^{1} \left(\frac{\sin x - x^2}{3 - |x|}\right) dx$

$$= \int_{-1}^{1} \left(\frac{\sin x}{3 - |x|}\right) dx - \int_{-1}^{1} \left(\frac{x^2}{3 - |x|}\right) dx$$

$$= 0 - \int_{-1}^{1} \left(\frac{x^2}{3 - |x|}\right) dx$$

$$= -2\int_0^1 \left(\frac{x^2}{3 - |x|}\right) dx$$

15. Given integral $= \int_0^{n\pi + V} |\sin x| dx$

$$= \int_0^V |\sin x| dx + \int_V^{n\pi + V} |\sin x| dx$$

$$= \int_0^V \sin x\, dx + \int_0^{n\pi} |\sin x| dx$$

$$= \int_0^V \sin x\, dx + n\int_0^{\pi} |\sin x| dx$$

$$= -(\cos x)\Big|_0^V + n \times 2$$

$$= 1 - \cos V + 2n$$

$$= (2n + 1 - \cos V)$$

16. The given integral =

$$\int_2^3 \left(\frac{\sqrt{x}}{\sqrt{5} - x + \sqrt{x}}\right) dx$$

$$= \frac{3 - 2}{2}$$

$= \frac{1}{2}$

17. Given integral $= \int_0^{2\pi} [2\sin x]dx$

$= \int_0^{\pi/4} [2\sin x]dx + \int_{\pi/4}^{3\pi/4} [2\sin x]dx + \int_{3\pi/4}^{\pi} [2\sin x]dx$

$+ \int_{\pi}^{5\pi/4} [2\sin x]dx + \int_{5\pi/4}^{7\pi/4} [2\sin x]dx + \int_{7\pi/4}^{2\pi} [2\sin x]dx$

$= 0 + \left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\cdot(1) + 0 + \left(\frac{5\pi}{4} - \pi\right)\cdot(-1)$

$+ \left(\frac{7\pi}{4} - \frac{5\pi}{4}\right)\cdot(-2) + \left(2\pi - \frac{7\pi}{4}\right)\cdot(-1)$

$= \frac{\pi}{2} - \frac{\pi}{4} - \pi - \frac{\pi}{4}$

$= -\pi$

18. Given $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B, f'\left(\frac{1}{2}\right) = 2\sqrt{2}$

Also, $\int_0^1 f(x)\,dx = \frac{2A}{\pi}$

$\Rightarrow \int_0^1 \left(A\sin\left(\frac{\pi x}{2}\right) + B\right) dx = \frac{2A}{\pi}$

$\Rightarrow \left(-\frac{A\sin(\pi x/2)}{(\pi/2)} + B\right)_0^1 = \frac{2A}{\pi}$

$\Rightarrow \left(-\frac{2A}{\pi} + B\right) = \frac{2A}{\pi}$

$\Rightarrow B = \frac{4A}{\pi}$

Also, $f'\left(\frac{1}{2}\right) = 2\sqrt{2}$

$\Rightarrow f'(x) = A\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$

$\Rightarrow f'\left(\frac{1}{2}\right) = A\frac{\pi}{2}\cos\left(\frac{\pi}{2} \times \frac{1}{2}\right)$

$= \frac{A\pi}{2}\cos\left(\frac{\pi}{4}\right) = \frac{A\pi}{2\sqrt{2}}$

$\Rightarrow \frac{A\pi}{2\sqrt{2}} = 2\sqrt{2}$

$\Rightarrow A = \frac{8}{\pi}$

$\Rightarrow B = \frac{4}{\pi} \times \frac{8}{\pi} = \frac{32}{\pi^2}$

19. Let $I = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2x}{1 + x^2}\right)\right)dx.$

$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{-2x}{1 + x^2}\right)\right)dx$

$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\left(\pi - \cos^{-1}\left(\frac{2x}{1 + x^2}\right)\right)\right)dx$

$= \pi\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\right)dx - \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\left(\cos^{-1}\left(\frac{2x}{1 + x^2}\right)\right)\right)dx$

$= \pi\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\right)dx - I$

$\Rightarrow 2I = \pi\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\right)dx$

$= 2\pi\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\right)dx$

$\Rightarrow I = \pi\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{1 - x^4}\right)dx$

$= -\pi\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^4}{x^4 - 1}\right)dx$

$= -\pi\int_0^{\frac{1}{\sqrt{3}}} \left(1 + \frac{1}{x^4 - 1}\right)dx$

$= -\pi\int_0^{\frac{1}{\sqrt{3}}} \left(1 + \frac{1}{(x^2 - 1)(x^2 + 1)}\right)dx$

$= -\pi\int_0^{\frac{1}{\sqrt{3}}} \left(1 + \frac{1}{2}\left(\frac{1}{(x^2 - 1)} - \frac{1}{(x^2 + 1)}\right)\right)dx$

$= -\pi\left(x + \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{x - 1}{x + 1}\right| - \tan^{-1}x\right)\right)_0^{\frac{1}{\sqrt{3}}}$

$= -\pi\left(\frac{1}{\sqrt{3}} + \frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1 - \sqrt{3}}{1 + \sqrt{3}}\right| - \frac{\pi}{6}\right)\right)$

20. We have,

$U_{n+2} - U_{n+1}$

$= \int_0^{\pi} \left(\frac{1 - \cos(n + 2)x}{1 - \cos x}\right)dx - \int_0^{\pi}\left(\frac{1 - \cos(n + 1)x}{1 - \cos x}\right)dx$

$$= \int_0^\pi \left(\frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}\right)dx$$

$$= \int_0^\pi \left(\frac{2\sin\left(n+\frac{3}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$\Rightarrow \quad U_{n+2} - U_{n+1} = \int_0^\pi \left(\frac{2\sin\left(n+\frac{3}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_0^\pi \left(\frac{2\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right)dx$$

Therefore,

$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$

$$= \int_0^\pi \left(\frac{2\sin\left(\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_0^\pi \left(\frac{2\cos(n+1)x\,\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_0^\pi (2\cos(n+1)x)\,dx$$

$$= 2\left(\frac{\sin(n+1)x}{(n+1)x}\right)_\pi^0$$

$$= 0$$

Thus, $\quad U_{n+1} + U_n = 2U_{n+1}$

Now, $\quad \left[U_0 = \int_0^\pi \left(\frac{1-1}{1-\cos x}\right)dx = 0\right.$

$$U_1 = \int_0^\pi \left(\frac{1-\cos x}{1-\cos x}\right)dx = \pi$$

Thus, $\quad U_1 - U_0 = \pi$

$\Rightarrow \quad U_1 = U_0 + \pi$

$\Rightarrow \quad U_n = U_0 + n\pi = n\pi]$

Also, $\quad \int_0^{\pi/2} \left(\frac{\sin^2 n\theta}{\sin^2\theta}\right)d\theta$

$$= \int_0^{\pi/2} \left(\frac{2\sin^2(n\theta)}{\sin^2(\theta)}\right)d\theta$$

$$= \int_0^{\pi/2} \left(\frac{(1-\cos^2 n\theta)}{(1-\cos^2\theta)}\right)d\theta$$

$$= \frac{1}{2}\int_0^\pi \left(\frac{(1-\cos nt)}{(1-\cos t)}\right)dt, \qquad (\text{Let } 2\theta = t)$$

$$= \frac{1}{2}\int_0^\pi \left(\frac{(1-\cos nx)}{(1-\cos x)}\right)dx$$

$$= \frac{1}{2}\times n\pi = \frac{n\pi}{2}$$

21. Given

$$a\,f(x) + b\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad \text{...(i)}$$

Replacing x by $1/x$, we get

$$a\,f\left(\frac{1}{x}\right) + bf(x) = x - 5 \qquad \text{...(ii)}$$

Multiplying (i) by a and (ii) by b, and subtracting, we get

$$a^2f(x) - b^2f(x) = a\left(\frac{1}{x} - 5\right) - b(x-5)$$

$$\Rightarrow \quad (a^2 - b^2)f(x) = a\left(\frac{1}{x} - 5\right) - b(x-5)$$

$$\Rightarrow \quad f(x) = \frac{a\left(\frac{1}{x} - 5\right) - b(x-5)}{(a^2 - b^2)}$$

Now,

$$\int_1^2 f(x)\,dx$$

$$= \int_1^2 \left(\frac{a\left(\frac{1}{x} - 5\right) - b(x-5)}{(a^2-b^2)}\right)dx$$

$$= \frac{1}{(a^2-b^2)}\int_1^2 \left(a\left(\frac{1}{x} - 5\right) - b(x-5)\right)dx$$

$$= \frac{1}{(a^2-b^2)}\left(a(\log|x| - 5x) - b\left(\frac{x^2}{3} - 5x\right)\right)_1^2$$

$$= \frac{1}{(a^2-b^2)}\left(\left(a(\log|2|) - \frac{22b}{3}\right) + 10b - 5a\right)$$

22. Let $\quad I = \int_0^{2\pi} \left(\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$

$$= \int_0^{2\pi} \left(\frac{(2\pi - x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$= 2\pi\int_0^{2\pi} \left(\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$- \int_0^{2\pi} \left(\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$= 2\pi\int_0^{2\pi} \left(\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx - I$$

$$\Rightarrow \quad 2I = 2\pi\int_0^{2\pi} \left(\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$\Rightarrow \quad I = \pi\int_0^{2\pi} \left(\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$\Rightarrow \quad I = 4\pi \int_0^{2\pi} \left(\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx \quad \text{...(i)}$$

$$= 4\pi \int_0^{2\pi} \left(\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = 4\pi \int_0^{\pi/2} \left(\frac{\sin^{2n}x + \cos^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$= 4\pi \int_0^{\pi/2} dx$$

$$= 4\pi\pi\frac{\pi}{2} = 4\pi^2$$

$$\Rightarrow \quad I = 2\pi^2$$

23. Let $I = \int_{-\pi}^{\pi} \left(\frac{2x(1 + \sin x)}{1 + \cos^2 x}\right)dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1 + \cos^2 x}\, dx$$

$$= 0 + \int_{-\pi}^{\pi} \frac{2x\sin x}{1 + \cos^2 x}\, dx$$

$$= 4\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}\, dx$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2 x}\, dx$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx$$

$$= 4\int_0^{\pi} \frac{\pi\sin x}{1 + \cos^2 x}\, dx - 4\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}\, dx$$

$$= 4\int_0^{\pi} \frac{\pi\sin x}{1 + \cos^2 x}\, dx - I$$

$$2I = 4\int_0^{\pi} \frac{\pi\sin x}{1 + \cos^2 x}\, dx$$

$$I = 2\int_0^{\pi} \frac{\pi\sin x}{1 + \cos^2 x}\, dx$$

$$I = 4\pi\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$$

$$= -4\pi\left(\tan^{-1}(\cos x)\right)_0^{\frac{\pi}{2}}$$

$$= -4\pi\left(0 - \frac{\pi}{4}\right)$$

$$= \pi^2$$

24. Given integral $= \int_1^{e^{37}} \left(\frac{\pi\sin(\pi\ln x)}{x}\right)dx$

$$= \int_0^{37\pi} \sin t\,dt, \quad (\text{Let } \pi \ln x = t)$$

$$= -(\cos t)|_0^{37\pi}$$

$$= -(\cos(37\pi) - 1)$$

$$= -(-1 - 1)$$

$$= 2$$

25. Given

$$\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x},\ x > 0$$

$$\Rightarrow \quad \int\left(\frac{e^{\sin x}}{x}\right)dx = F(x)$$

Now, $\int_1^4 \left(\frac{2e^{\sin(x^2)}}{x}\right) dx$

$$= \int_1^4 \left(\frac{2x\,e^{\sin(x^2)}}{2}\right)dx$$

$$= \int_1^{16} \left(\frac{e^{\sin t}}{t}\right)dt,\ x^2 = t$$

$$= \int_1^{16} \left(\frac{e^{\sin x}}{x}\right) dx$$

$$= (F(x))|_1^{16}$$

$$= F(16) - F(1)$$

$$\Rightarrow \quad k = 16$$

26. Given $a + b = 4$.

Let $f(a) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$$= \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx$$

$$\Rightarrow \quad \frac{df(a)}{da} = g(a) - g(4 - a)$$

$$\Rightarrow \quad \frac{df(a)}{d(b - a)} = g(a) - g(4 - a)$$

$$\Rightarrow \quad \frac{df(a)}{d(4 - 2a)} = g(a) - g(4 - a)$$

$$\Rightarrow \quad -\frac{df(a)}{2da} = g(a) - g(4 - a)$$

$$\Rightarrow \quad \frac{df(a)}{da} = \frac{g(a - 4) - g(a)}{2} > 0$$

Thus, $f(a) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

It increases as $(b - a)$ increases.

27. Given integral

$$g(x) = \int_0^x \cos^4 t\, dt$$

Now, $g(x + \pi)$

$$= \int_0^{x+\pi} \cos^4 t\ dt$$

$$= \int_0^x \cos^4 t\ dt + \int_x^{x+\pi} \cos^4 t\ dt$$

$$= \int_0^x \cos^4 t\ dt + \int_0^{\pi} \cos^4 t\ dt$$

$$= g(x) + g(\pi)$$

28. Given

$$\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$$

$\Rightarrow \quad f(x) = 1 - xf(x)$

$\Rightarrow \quad (1 + x)f(x) = 1$

$\Rightarrow \quad f(x) = \dfrac{1}{1 + x}$

$\Rightarrow \quad f(1) = \dfrac{1}{2}$

29. Given,

$$\int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{1} (x - [x])\,dx$$

$$= \int_{-1}^{1} (x)\,dx - \int_{-1}^{1} ([x])\,dx$$

$$= 0 - \int_{-1}^{1} ([x])\,dx$$

$$= -\int_{-1}^{0} ([x])\,dx - \int_{0}^{1} ([x])\,dx$$

$$= -\int_{-1}^{0} (-1)\,dx - \int_{0}^{1} (0)\,dx$$

$$= \int_{-1}^{0} dx$$

$$= (x)|_{-1}^{0}$$

$$= [0 - (-1)] = 1$$

30. Given,

$$\int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x - (x - 1)}{1 + x(x - 1)}\right) dx$$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \tan^{-1}(x - 1)\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \tan^{-1}[(1 - x) - 1]\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \tan^{-1}(-x)\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \tan^{-1}(x)\,dx$$

$$= 2\int_0^1 \tan^{-1} x\,dx$$

Also, $\displaystyle\int_0^1 \tan^{-1}(1 - x + x^2)\,dx$

$$= \frac{\pi}{2} - \int_0^1 \cot^{-1}(1 - x + x^2)\,dx$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx$$

$$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1} x\,dx$$

$$= \frac{\pi}{2} - 2\left(x\tan^{-1} x - \frac{1}{2}\log|x^2 + 1|\right)_0^1$$

$$= \frac{\pi}{2} - 2\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{2} + \log 2$$

$$= \log 2$$

31. Let $\displaystyle I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ ...(i)

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1 - \cos x} \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x}\right) dx$$

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{2}{\sin^2 x}\right) dx$$

$$= \int_{\pi/4}^{3\pi/4} (2\operatorname{cosec}^2 x)\,dx$$

$$= -(2\cot x)|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= -2(-1 - 1)$$

$$= 4$$

$\Rightarrow \quad I = 2$

32. Given,

$$\int_{\pi/2}^{3\pi/2} [2\sin x]dx$$

$$= \int_{\pi/2}^{5\pi/6} [2\sin x]\,dx + \int_{5\pi/6}^{\pi} [2\sin x]\,dx + \int_{\pi}^{7\pi/6} [2\sin x]dx + \int_{3\pi/2}^{7\pi/6} [2\sin x]\,dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right)\cdot 1 + (0) + \left(\frac{7\pi}{6} - \pi\right)\cdot(-1) + \left(\frac{3\pi}{2} - \frac{7\pi}{6}\right)\cdot(-2)$$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6}$$

$$= -\frac{\pi}{2}$$

33. Let $I = \int_0^{\pi} \left(\frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\right)$ ...(i)

$$= \int_0^{\pi} \left(\frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\right) \quad ...(ii)$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\pi} \left(\frac{e^{\cos x} + e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\right)dx$$

$$= \int_0^{\pi} dx$$

$$= (x)_0^{\pi}$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

34. Given $g(x) = \int_0^x f(t)\,dt$

$$\Rightarrow \quad g(2) = \int_0^2 f(t)\,\mathrm{dt}$$

$$= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

Now, $\frac{1}{2} \le f(t) \le 1$

$$\Rightarrow \quad \int_0^1 \frac{1}{2}dt \le \int_0^1 1\cdot dt$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt \le 1 \quad ...(i)$$

Also, $0 \le f(t) \le \frac{1}{2}$

$$\Rightarrow \quad \int_1^2 0\;dt \le \int_1^2 f(t)\,dt \le \int_1^2 \frac{1}{2}dt$$

$$\Rightarrow \quad 0 \le \int_1^2 f(t)\,dt \le \frac{1}{2} \quad ...(ii)$$

Adding Eqs. (i) and (ii), we get

$$\frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le 1 + \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le \frac{3}{2}$$

35. Given $\int_{-2}^{3} f(x)\,dx$

$$= \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= \int_{-2}^{2} e^{\cos x}\sin x\,dx + \int_2^3 2dx$$

$$= \underbrace{\int_{-2}^{2} e^{\cos x}\sin x\,dx}_{\text{odd function}} + \int_2^3 2dx$$

$$= 0 + 2(3 - 2)$$

$$= 2$$

36. Given $\int_{e-1}^{e^2} \left|\frac{\log_e x}{x}\right|dx$

$$= \int_{-1}^{2} t\,dt,\ \log_e x = 1$$

$$= \left(\frac{t^2}{2}\right)_{-1}^{2}$$

$$= \left(2 - \frac{1}{2}\right)$$

$$= \frac{3}{2}$$

37. We have,

$$f(x) = \int_1^x \frac{\ln t}{1 + t}\,dt \quad ...(i)$$

Now, $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1 + t}\,dt$

Let $t = \frac{1}{y} \Rightarrow dt = -\frac{1}{y^2}\,dy$

The given integral reduces to

$$= \int_1^y \frac{\ln\left(\frac{1}{y}\right)}{1 + \left(\frac{1}{y}\right)} \times -\frac{dy}{y^2}$$

$$= \int_1^y \frac{-\ln(y)}{1 + y} \times -\frac{dy}{y}$$

$$= \int_1^y \frac{\ln(y)}{y(1+y)}\,dy$$

$$= \int_1^t \frac{\ln(t)}{t(1+t)}\,dt \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln(t)}{(1+t)}\,dt + \int_1^x \frac{\ln(t)}{t(1+t)}\,dt$$

$$= \int_1^x \frac{(1+t)\ln(t)}{t(1+t)}\,dt$$

$$= \int_1^x \frac{\ln(t)}{t}\,dt$$

$$= \left.\frac{(\ln t)^2}{2}\right|_1^x = \frac{(\ln x)^2}{2}$$

Thus, $f(e) + f\left(\frac{1}{e}\right) = \left(\frac{\ln e}{2}\right)^2 = \frac{1}{2}.$

38. Let $I = \int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1+a^x}\right)dx$ ...(i)

$$= \int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1+a^{-x}}\right)dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{a^x\cos^2 x}{1+a^x}\right)dx \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$\Rightarrow \quad 2I = \int_{-\pi}^{\pi} \left(\frac{(1+a^x)\cos^2 x}{(1+a^x)}\right)dx$$

$$= \int_{-\pi}^{\pi} (\cos^2 x)dx$$

$$= 2\int_0^{\pi} (\cos^2 x)dx$$

$$= 2\int_0^{\pi} \left(\frac{1+\cos^2 x}{2}\right)dx$$

$$\Rightarrow \quad I = \left(\frac{1}{2}\left(x + \frac{\sin^2 x}{2}\right)\right)_0^{\pi}$$

$$= \frac{\pi}{2}$$

39. Given $F(x) = \int_0^x f(t)dt$

$\Rightarrow \quad F'(x) = f(x)$

Also, $F(x^2) = x^2(x+1) = x^3 + x^2$

$\Rightarrow \quad F'(x^2)\cdot 2x = 3x^2 + 2x$

$\Rightarrow \quad F'(x^2) = \frac{3x^2}{2x} + 1 = \frac{3}{2x} + 1$

$\Rightarrow \quad f(x^2) = \frac{3}{2x} + 1$

Putting $x = 2$, we get

$\Rightarrow \quad f(4) = \frac{3}{2}\cdot 2 + 1 = 4$

40. **

41. Given $f(x) = \int_1^x \sqrt{2-t^2}\,dt$

$\Rightarrow \quad f'(x) = \sqrt{2-x^2}$

Also, $x^2 - f'(x) = 0$

$\Rightarrow \quad x^2 = \sqrt{2-x^2}$

$\Rightarrow \quad x^4 = 2 - x^2$

$\Rightarrow \quad x^4 + x^2 - 2 = 0$

$\Rightarrow \quad (x^2+2)(x^2-1) = 0$

$\Rightarrow \quad x = \pm 1$

42. Given $\int_3^{3+3T} f(2x)dx$

$$= \frac{1}{2}\int_6^{6+6I} f(t)dt, \qquad \text{(Let } 2x = t\text{)}$$

$$= \frac{1}{2}\int_0^{6I} f(t)dt$$

$$= \frac{1}{2}\int_0^{6I} f(x)dx$$

$$= \frac{1}{2}\times 6\int_0^1 f(x)dx$$

$$= 3I$$

43. Given,

$$\int_{-1/2}^{1/2} \left([x] + \ln\left(\frac{1-x}{1+x}\right)\right)dx$$

$$= \int_{-1/2}^{1/2} ([x])dx + \int_{-1/2}^{1/2} \left(\ln\left(\frac{1-x}{1+x}\right)\right)dx$$

$$= \int_{-1/2}^{1/2} ([x])dx + 0$$

$$= \int_{-1/2}^{0} ([x])dx + \int_0^{1/2} ([x])dx$$

$$= \int_{-1/2}^{0} (-1)dx + \int_{0}^{1/2} (0)dx$$

$$= (-1)(x)|_{-1/2}^{0}$$

$$= (-1)\left(0 + \frac{1}{2}\right)\Big|$$

$$= -\frac{1}{2}$$

44. Given $I(m, n) = \int_0^1 t^m (1 + t)^n \, dt$

$$= \left((1+t)^n \frac{t^{m+1}}{m+1}\right)\Bigg|_0^1 - \frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1}\,dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} I(m+1, n-1)$$

45. Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x \, dx$ ...(i)

$$= \int_0^{\pi/2} f(\sin 2x)\cos x \, dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} f(\sin 2x)(\sin x + \cos x)dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} f(\sin 2x)\left(\sin\left(x + \frac{\pi}{4}\right)\right)dx$$

Let $\left(\frac{\pi}{4} - x\right) = t$

$$= \sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(\cos 2t)(\cos t)dt$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2t)(\cos t)dt$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2x)(\cos x)dx$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2x)(\cos x)dx$$

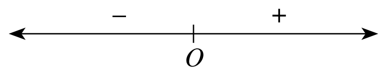
$$\Rightarrow \quad I = \sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2x)(\cos x)dx$$

46. Given $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\, dt$

$$\Rightarrow \quad f'(x) = e^{-(x^2+1)^2}\cdot 2x - e^{-x^4}\cdot 2x$$

$$\Rightarrow \quad f'(x) = 2x\left(e^{-(x^4+2x^4+1)} - e^{-x^4}\right)$$

$$\Rightarrow \quad f'(x) = 2xe^{-x^4}\left(e^{-(2x^2+1)} - 1\right)$$



Clearly $f(x)$ increases in $(0, \infty)$.

47. Given $\int_0^{t^2} xf(x)dx = \frac{2}{5}t^5$

$$t^2 f(t^2)(2t) = 2t^4$$

$$f(t^2) = \frac{1}{t}$$

Let $t = \frac{2}{5}$, then

$$f\left(\frac{4}{25}\right) = \frac{1}{\left(\frac{2}{5}\right)} = \frac{5}{2}$$

48. Given $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$

$$= -\int_{\pi/2}^{0}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\sin\theta\, d\theta,$$

Let $x = \cos\theta$

$$= \int_0^{\pi/2}\left(\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)\sin\theta\, d\theta$$

$$= \int_0^{\pi/2}\left(\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta$$

$$= \int_0^{\pi/2} 2\sin^2\left(\frac{\theta}{2}\right)d\theta$$

$$= \int_0^{\pi/2}(1 - \cos\theta)d\theta$$

$$= (\theta - \sin\theta)|_0^{\frac{\pi}{2}}$$

$$= \left|\frac{\pi}{2} - 1\right|$$

49. Let $f(x)$ be a differentiable function defined as $f: [0, 4] \to R$, show that

(i) $8f'(a)f(b) = \{f(4)\}^2 - \{f(0)\}^2$ when $a, b \in (0, 4)$

(ii) $\int_0^4 f(x)dx = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$

when $0 < \alpha, \beta < 2$

[IIT-JEE, 2004]

50. Let
$$I = \int_{-\pi/3}^{\pi/3} \left(\frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \right) dx$$
$$= \int_{-\pi/3}^{\pi/3} \left(\frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \right) dx + \int_{-\pi/3}^{\pi/3} \left(\frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \right) dx$$
$$= \int_{-\pi/3}^{\pi/3} \left(\frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \right) dx + 0$$
$$= 2\int_{0}^{\pi/3} \left(\frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \right) dx$$
$$= 2\pi \int_{0}^{\pi/3} \left(\frac{dx}{2 - \cos\left(x + \frac{\pi}{3}\right)} \right)$$
$$= 2\pi \int_{\pi/3}^{2\pi/3} \left(\frac{dt}{2 - \cos t} \right), \quad (\text{Let } t = x + \pi/3)$$
$$= 2\pi \int_{\pi/3}^{2\pi/3} \left(\frac{dt}{1 + (1 - \cos t)} \right)$$
$$= 2\pi \int_{\pi/3}^{2\pi/3} \left(\frac{dt}{1 + 2\sin^2(t/2)} \right)$$
$$= 4\pi \int_{\pi/6}^{\pi/3} \left(\frac{dy}{1 + 2\sin^2 y} \right), \quad (\text{Let } t/2 = y)$$
$$= 4\pi \int_{\pi/6}^{\pi/3} \left(\frac{\sec^2 y\, dy}{\sec^2 y + 2\tan^2 y} \right)$$
$$= 4\pi \int_{\pi/6}^{\pi/3} \left(\frac{\sec^2 y}{1 + 3\tan^2 y} \right) dy$$
$$= \frac{4\pi}{3} \int_{\pi/6}^{\pi/3} \left(\frac{\sec^2 y}{(1/\sqrt{3})^2 + \tan^2 y} \right) dy$$
$$= \frac{4\pi}{3} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dv}{(1/\sqrt{3})^2 + v^2}$$
$$= \frac{4\pi}{3} \times \sqrt{3} \left(\tan^{-1}(v\sqrt{3}) \right)_{1/\sqrt{3}}^{\sqrt{3}}$$

51. Given,
$$y(x) = \int_{\pi^2/16}^{x^2} \left(\frac{\cos x \cos(\sqrt{\theta})}{1 + \sin^2(\sqrt{\theta})} \right) d\theta$$
$$y = \cos x \int_{\frac{\pi^2}{16}}^{x^2} \left(\frac{\cos\sqrt{\theta}}{1 + \sin^2\sqrt{\theta}} \right) d\theta$$
Differentiating both sides w.r.t. x, we get
$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \left(\frac{\cos\sqrt{\theta}}{1 + \sin^2\sqrt{\theta}} \right) d\theta + \cos x \left(\frac{\cos x}{1 + \sin^2 x} \right)(2x)$$
Now, $$\left(\frac{dy}{dx} \right)_{x=\pi} = 0 + (-1)(2\pi) \cdot \frac{(-1)}{1 + 0} = 2\pi$$

52. The given integral can be written as
$$\int_{-2}^{0} ((x + 1)^3 + 2 + (x + 1)\cos(x + 1))dx$$
Let $(x + 1) = t$
$$= \int_{-1}^{1} (t^3 + 2 + t\cos t)\, dt$$
$$= \int_{-1}^{1} (t^3 + t\cos t)\, dt + \int_{-1}^{1} 2dt$$
$$= 0 + \int_{-1}^{1} 2dt$$
$$= 2(1 - (-1)) = 4$$

53. Given,
$$\int_{\sin x}^{1} t^2 f(t) dt = 1 - \sin x$$
$$\Rightarrow \quad 0 - \sin^2 x\, f(\sin x) \cdot \cos x = -\cos x$$
$$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x}$$
$$\Rightarrow \quad f(x) = \frac{1}{x^2}$$
$$\Rightarrow \quad f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2} = 3$$

54. Given integral is =
$$\int_0^{\pi} e^{|\cos x|} \left[2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right) \right] \sin x\, dx$$
$$= \int_0^{\pi} e^{|\cos x|} \left[2\sin\left(\frac{1}{2}\cos x\right) \right] \sin x\, dx + \int_0^{\pi} e^{|\cos x|} \left[3\cos\left(\frac{1}{2}\cos x\right) \right] \sin x\, dx$$
$$= 0 + \int_0^{\pi} e^{|\cos x|} \left[3\cos\left(\frac{1}{2}\cos x\right) \right] \sin x\, dx$$

$$\left(\because \int_0^{2a} f(x)\,dx = 0, f(2a - x) = -f(x)\right)$$

$$= \int_0^{\pi} e^{|\cos x|}\left[3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$$

$$= 2\int_0^{\pi/2} e^{|\cos x|}\left[3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$$

$$= 2\int_0^{\pi/2} e^{|\cos x|}\left[3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$$

$$= 6\int_0^{\pi/2} e^{\cos x}\left[\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$$

$$= -6\int_1^0 e^t\left[\cos\left(\frac{t}{2}\right)\right], \qquad (\text{Let } t = \cos x)$$

$$= 6\int_0^1 e^t\left[\cos\left(\frac{t}{2}\right)\right]dt$$

$$= 6\left(\frac{e^t}{1+\frac{1}{4}}\left[\frac{1}{2}\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right]\right)_0^1$$

$$= \frac{24}{5}\left[e\frac{1}{2}\sin\left(\frac{1}{2}\right) + \cos\left(\frac{1}{2}\right) - 1\right]$$

55. Let $I_n = \int_0^1 (1 - x^{50})^n\,dx$

Integrating by parts, we get

$$I_n = (1 - x^{50})^n \cdot x|_0^1 + \int_0^1 x \cdot 50n \cdot (1 - x^{50})^{n-1} \cdot x^{49} \cdot dx$$

$$= 0 + 50n\int_0^1 x^{50} \cdot (1 - x^{50})^{n-1} \cdot dx$$

$$= -50n\int_0^1 ((1 - x^{50}) - 1) \cdot (1 - x^{50})^{n-1} \cdot dx$$

$$= -50n\left(\int_0^1 (1 - x^{50})^n \cdot dx - \int_0^1 (1 - x^{50})^{n-1} \cdot dx\right)$$

$$= -50n\,(I_n - I_{n-1})$$

$$= -50n\,I_n + 50n\,I_{n-1}$$

$$\Rightarrow \quad I_n = -50n\,I_n + 50n\,I_{n-1}$$

$$\Rightarrow \quad (1 + 50n)I_n = 50n\,I_{n-1}$$

$$\Rightarrow \quad \frac{I_n}{I_{n-1}} = \frac{50n}{(1 + 50n)}$$

Putting $n = 101$, then we get

$$\Rightarrow \quad \frac{I_{101}}{I_{100}} = \frac{50 \times 101}{(1 + 50 \times 101)} = \frac{5050}{5051}$$

$$\Rightarrow \quad \frac{I_{100}}{I_{101}} = \frac{5051}{5050}$$

Thus, the value of

$$5050 \times \left(\frac{\int_0^1 (1 - x^{50})^{100}\,dx}{\int_0^1 (1 - x^{50})^{101}\,dx}\right) = 5051$$

56. Let

$$L = \lim_{x \to \frac{\pi}{4}}\left(\frac{\int_2^{\sec^2 x} f(t)dt}{x^2 - \left(\frac{\pi^2}{16}\right)}\right)$$

$$= \lim_{x \to \frac{\pi}{4}}\left(\frac{f(\sec^2 x)\,2\sec^2 x\,\tan x}{2x}\right)$$

$$= \frac{f\left(\sec^2\left(\frac{\pi}{4}\right)\right).\sec^2\left(\frac{\pi}{4}\right).1}{\frac{\pi}{4}}$$

$$= \frac{8f(2)}{\pi}$$

57.

(A) $\int_{-1}^{1} \frac{dx}{1 + x^2}$

$$= 2\int_0^1 \frac{dx}{x^2 + 1}$$

$$= 2(\tan^{-1} x)_0^1$$

$$= 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}$$

(B) $\int_0^1 \frac{dx}{\sqrt{\sqrt{1 - x^2}}}$

$$= (\sin^{-1} x)_0^1 = \frac{\pi}{2}$$

(C) $\int_2^3 \frac{dx}{1 - x^2}$

$$= \left(-\frac{1}{2}\log\left|\frac{x - 1}{x + 1}\right|\right)_2^3$$

$$= \frac{1}{2}\log\left(\frac{1}{3}\right) - \frac{1}{2}\log\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\log\left(\frac{2}{3}\right)$$

(D) $\int_1^2 \frac{dx}{x\sqrt{x^2 - 1}}$

$$= (\sec^{-1} x)_1^2$$

$$= (\sec^{=1} 2 - \sec^{-1} 1)$$

$$= \left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi}{3}$$

58. Given $f(x) = f(1 - x)$

$\Rightarrow$ $f'(x) = -f'(1 - x)$

Putting $x = 1/2$, we get

$$f'\left(\frac{1}{2}\right) = -f'\left(1 - \frac{1}{2}\right) = -f'\left(\frac{1}{2}\right)$$

$\Rightarrow$ $2f'\left(\frac{1}{2}\right) = 0$

$\Rightarrow$ $f'\left(\frac{1}{2}\right) = 0$

Also, it is given that $f'\left(\frac{1}{4}\right) = 0$

Putting $x = \frac{1}{4}$ in $f'(x) = -f'(1 - x)$, we get

$$f'\left(\frac{1}{4}\right) = -f'\left(\frac{3}{4}\right)$$

$\Rightarrow$ $f'\left(\frac{1}{4}\right) = -f'\left(\frac{3}{4}\right) = 0$

Thus, $f'(x) = 0$ vanishes at least twice.

Again, $f(x) = f(1 - x)$

Replacing x by $\left(x + \frac{1}{2}\right)$, we get

$\Rightarrow$ $f\left(x + \frac{1}{2}\right) = f\left(\frac{1}{2} - x\right)$

Thus, $f\left(x + \frac{1}{2}\right)$ is an even function.

$\Rightarrow$ $f\left(x + \frac{1}{2}\right)$ is an odd function.

Therefore, $\int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(x + \frac{1}{2}\right)\sin x\, dx = 0.$

Again, $\int_{1}^{1/2} f(1 - t)\, e^{\sin(\pi t)} dt$

$$= -\int_{1/2}^{0} f(y)\, e^{\sin(\pi(1 - y))} dy, \quad (\text{Let } 1 - t = y)$$

$$= \int_{0}^{1/2} f(y)\, e^{\sin(\pi(1 - \pi y))} dy$$

$$= \int_{0}^{1/2} f(y)\, e^{\sin(\pi(1 - \pi y))} dy$$

$$= \int_{0}^{1/2} f(t)\, e^{\sin(\pi y)} dt, \text{ by Property I.}$$

59. We have $S_n < \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2}\right)$

$$= \int_0^1 \frac{dx}{1 + x + x^2}$$

$$= \int_0^1 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \left(\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)\right)_0^1$$

$$= \frac{2}{\sqrt{3}} (\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \left(\frac{2}{\sqrt{3}} \times \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

Now, $T_n > \frac{\pi}{3\sqrt{3}}$

$$\left(\because\ h \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) > \int_0^1 f(x)\, dx > h \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)$$

60. Given $\int_0^x \sqrt{1 - [f'(t)]^2}\, dt = \int_0^x f(t)\, dt$

$\Rightarrow$ $\sqrt{1 - [f'(x)]^2} = f(x)$

$\Rightarrow$ $1 - [f'(x)]^2 = (f(x))^2$

$\Rightarrow$ $1 - \left(\frac{dy}{dx}\right)^2 = y^2$

$\Rightarrow$ $\left(\frac{dy}{dx}\right) = \pm\sqrt{1 - y^2}$

$\Rightarrow$ $\int \frac{dy}{\sqrt{1 - y^2}} = \pm dx$

$\Rightarrow$ $\sin^{-1} y = c \pm x$

$\Rightarrow$ $y = \sin(c \pm x)$

When $x = 0$, $y = 0$, then $c = 0$

Thus, the equation of the curve is

$$y = \sin(\pm x) = \pm \sin x$$

As $f(x) \geq 0$ for $0 \leq x \leq 1$, we get

$f(x) = \sin x$, for $0 \leq x \leq 1$

Since $\sin x < x$ for all $x > 0$, we get

$f(x) < x$ for $0 < x \leq 1$

Thus, $f\left(\frac{1}{2}\right) < \frac{1}{2}$

and $f\left(\frac{1}{3}\right) < \frac{1}{3}.$

61. Let $I_n = \int_{-\pi}^{\pi} \left(\frac{\sin nx}{(1+\pi^x)\sin x}\right) dx$

$= \int_{-\pi}^{\pi} \left(\frac{\sin nx}{(1+\pi^x)\sin x}\right) dx$ ...(i)

$= \int_{-\pi}^{\pi} \left(\frac{\sin n(-x)}{(1+\pi^{-x})\sin(-x)}\right) dx$

$= \int_{-\pi}^{\pi} \left(\frac{\sin n(x)}{(1+\pi^{x})\sin(x)}\right) dx$

$= \int_{-\pi}^{\pi} \left(\frac{\pi^x \sin n(x)}{(1+\pi^{x})\sin(x)}\right) dx$ ...(ii)

Adding Eqs (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \left(\frac{(1+\pi^x)\sin n(x)}{(1+\pi^{x})\sin(x)}\right) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{\sin n(x)}{\sin(x)}\right) dx$$

$$= 2\int_{0}^{\pi} \left(\frac{\sin n(x)}{\sin(x)}\right) dx$$

$\Rightarrow$ $I_n = \int_{0}^{\pi} \left(\frac{\sin n(x)}{\sin(x)}\right) dx$

$\therefore$ $I_{n+2} - I_n = \int_{0}^{\pi} \left(\frac{\sin(n+2)x - \sin nx}{\sin x}\right) dx$

$$= \int_{0}^{\pi} \left(\frac{2\cos(n+1)x \times \sin x}{\sin x}\right) dx$$

$$= \int_{0}^{\pi} 2\cos(n+1)x\, dx$$

$$= 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$$

$\Rightarrow$ $I_{n+2} = I_n$

Now, $I_0 = 0$ and $I_1 = \int_0^{\pi} \left(\frac{\sin x}{\sin x}\right) dx = \pi$

Since $I_{n+2} = I_n$, so

$I_1 = I_2 = I_5 = \ldots = I_{(2n-1)} = \pi$

and $I_0 = I_2 = I_4 = \ldots = I_{2n} = \pi$

Thus, $\sum_{m=1}^{10} I_{2m+1} = I_3 + I_5 + \ldots + I_{21} = 10\pi$

and $\sum_{m=1}^{10} I_{2m} = I_2 + I_4 + \ldots + I_{20} = 0$

62. Given,

$$f(x) = \int_0^x f(t)\,dt$$

$\Rightarrow$ $f'(x) = f(x)$

$\Rightarrow$ $\frac{dy}{dx} = y$

$\Rightarrow$ $\int \frac{dy}{y} = \int dx$

$\Rightarrow$ $\log|y| = x + C$

$\Rightarrow$ $y = e^x + C = Ae^x$

But $f(0) = \int_0^0 f(t)dt = 0$

When $x = 0$, $y = 0$, then $A = 0$

Thus, $y = Ae^x = 0$

$\Rightarrow$ $f(x) = 0$

$\Rightarrow$ $f(\ln 5) = 0$

63. Given $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$

$\Rightarrow$ $e^{-x} f'(x) - e^{-x} f(x) = \text{R}x^4 + 1$

Putting $x = 0$, we get

$\Rightarrow$ $f'(0) - f(0) = \sqrt{0+1} = 1$

$\Rightarrow$ $f'(0) = f(0) + 1$

$\Rightarrow$ $f'(0) = 2 + 1 = 3$

Let $f^{-1} = g$

$y = f(x) \Leftrightarrow x = g(y)$

when g is the inverse of f, then we shall use

$$g'(y) = \frac{1}{f'(x)}$$

Putting $y = 2$ and $x = 0$, then

$$g'(2) = \frac{1}{f'(0)} = \frac{1}{3}$$

64. Let $I = \int_0^1 \left(\frac{x^4(1-x)^4}{1+x^2}\right) dx$

$$= \int_0^1 \left(\frac{x^4(x^4 - 4x^3 + 6x^2 - 4x + 1)}{x^2+1}\right) dx$$

$$= \int_0^1 \left(\frac{(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4)}{x^2+1}\right) dx$$

$$= \int_0^1 \left(\frac{(x^8 + 6x^6 + x^4)}{x^2+1}\right) dx$$

$$= \int_0^1 \left(\frac{(x^6(x^2+1) + x^4(x^2+1) + 4x^6)}{x^2+1}\right) dx$$

$$= \int_0^1 \left(\frac{(x^2(x^2+1)(x^2+1) + 4x^6)}{x^2+1}\right) dx$$

$$= \int_0^1 \left((x^4 + x^2) + \frac{4(x^6 + 1 - 1)}{x^2+1}\right) dx$$

$$= \int_0^1 \left((x^4 + x^2) - \frac{4}{x^2+1} + 4(x^4 - x^2 + 1) \right) dx$$

$$= \int_0^1 \left((5x^4 - 3x^2 + 4) - \frac{4}{x^2+1} \right) dx$$

$$= (x^5 - x^3 + 4x - 4\tan^{-1} x)_0^1$$

$$= (4 - 4\tan^{-1}(1)) = (4 - \pi)$$

65. The given limit is

$$\lim_{x \to 0} \left(\frac{1}{x^3} \int_0^x \frac{t\ln(1+t)}{t^4+4} \right) dt$$

$$= \lim_{x \to 0} \left(\frac{\int_0^x \frac{t\ln(1+t)}{t^4+4}\, dt}{x^3} \right)$$

$$= \lim_{x \to 0} \left(\frac{\frac{x\ln(1+x)}{x^2+4}}{3x^2} \right)$$

$$= \lim_{x \to 0} \left(\frac{x\ln(1+x)}{3x^2(x^4+4)} \right)$$

$$= \lim_{x \to 0} \left(\frac{\ln(1+x)}{3x(x^4+4)} \right)$$

$$= \lim_{x \to 0} \left(\frac{\frac{\ln(1+x)}{x}}{3(x^4+4)} \right)$$

$$= \frac{1}{12}$$

66. The given integral is

$$\frac{2}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{4}{\pi} \int_0^{\pi} f(x)dx, \qquad (\because f(x) \text{ is an even function})$$

$$= \frac{4}{\pi} \int_0^{\pi} \left(\frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \right) dx$$

$$= \frac{8}{\pi} \int_0^{\pi/2} \left(\frac{\sin 9\theta}{\sin \theta} \right) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} \left(\frac{\sin 9\theta}{\sin 3\theta} \times \frac{\sin 3\theta}{\sin\theta} \right)$$

$$= \frac{8}{\pi} \int_0^{\pi/2} \left(\frac{\sin 3\cdot(3\theta)}{\sin 3\theta} \times \frac{\sin 3\theta}{\sin\theta} \right) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} \left(\frac{(3\sin 3\theta - 4\sin^3 3\theta)}{\sin 3\theta} \times \frac{(3\sin\theta - 4\sin^3\theta)}{\sin\theta} \right) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} ((3 - 4\sin^2 3\theta) \times (3 - 4\sin^2\theta)) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} [\{3 - 2(1 - \cos 6\theta) \times 3 - 2(1 - \cos 2\theta)\}] d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} [(1 + 2\cos 6\theta) \times (1 + 2\cos 2\theta)] d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} (1 + 2\cos 2\theta + 2\cos 6\theta + 4\cos 2\theta \cos 2\theta) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} (1 + 2\cos 2\theta + 2\cos 6\theta) d\theta + \frac{8}{\pi} \int_0^{\pi/2} (4\cos 6\theta \cos 2\theta) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} (1 + 2\cos 2\theta + 2\cos 6\theta) d\theta + \frac{16}{\pi} \int_0^{\pi/2} (2\cos 6\theta \cos 2\theta) d\theta$$

$$= \frac{8}{\pi} \int_0^{\pi/2} (1 + 2\cos 2\theta + 2\cos 6\theta) d\theta + \frac{16}{\pi} \int_0^{\pi/2} (\cos 8\theta + \cos 4\theta) d\theta$$

$$= \frac{8}{\pi} \left(\theta + \sin 2\theta + \frac{\sin 6\theta}{3} \right)_0^{\frac{\pi}{2}} + \frac{16}{\pi} \left(\frac{\sin 8\theta}{8} + \frac{\sin 4\theta}{12} \right)_0^{\frac{\pi}{2}}$$

$$= \frac{8}{\pi} \left(\frac{\pi}{2} + 0 \right) + \frac{16}{\pi}(0 + 0)$$

$$= 4$$

67. We have, $f(x) = \begin{cases} 1 - x & : 0 \le x < 1 \\ x - 1 & : 1 \le x < 2 \end{cases}$

and f is periodic with period 2.
Also, since $[-x] = 1 - [x]$ for non Integral values of x, so, f is an even function.

Thus, $\int_{-10}^{10} f(x)\cos(\pi x)dx$

$$= 2\int_0^{10} f(x)\cos(\pi x)dx$$

$$= 2\cdot 5\int_0^{2} f(x)\cos(\pi x)dx$$

$$= 10\int_0^{2} f(x)\cos(\pi x)dx$$

$$= 10(I + J),$$

where

$$I = \int_0^1 f(x)\cos(\pi x)dx$$

and $\quad J = \int_1^2 f(x)\cos(\pi x)dx$

Now, $\quad I = \int_0^1 f(x)\cos(\pi x)dx$

$$= \int_0^1 (1 - x)\cos(\pi x)dx$$

$$= \int_0^1 [1 - (1 - x))\cos(\pi(1 - x)]\,dx$$

$$= -\int_0^1 x\cos(\pi x)dx$$

Also, $\quad J = \int_1^2 (x - 1)\cos(\pi x)dx$

$$= \int_0^1 t\cos(\pi(t + 1))dt, \quad (\text{Let} \quad x - 1 = t)$$

$$= -\int_0^1 t\cos(\pi t)dt$$

$$= -\int_0^1 x\cos(\pi x)dx$$

$$= I$$

Thus, $\quad I = J$

$$= -\int_0^1 x\cos(\pi x)dx$$

$$= \left(-\frac{x}{\pi}\sin(\pi x)\right)_0^1 + \frac{1}{\pi}\int_0^1 \sin(\pi x)dx$$

$$= -\frac{1}{\pi^2}(\cos\pi x)_0^1$$

$$= \frac{2}{\pi^2}$$

Therefore,

$$\int_{-10}^{10} f(x)\cos(\pi x)dx$$

$$= 10(I + J)$$

$$= 20I$$

$$= 20 \times \frac{2}{\pi^2}$$

Now, $\quad \frac{\pi^2}{10} \times \int_{10}^{10} f(x)\cos(\pi x)$

$$= \frac{\pi^2}{10} \times \frac{40}{\pi^2}$$

$$= 4$$

68. Let $I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \left(\frac{x\sin(x^2)}{\sin x^2 + \sin(\ln(6 - x^2))}\right)dx$

Let $\quad x^2 = t \Rightarrow x\,dx = 1/2\ dt$

$$= \frac{1}{2}\int_{\ln 2}^{\ln 3} \left(\frac{\sin t}{\sin t + \sin(\ln 6 - t)}\right)dt \qquad \text{...(i)}$$

$$= \frac{1}{2}\int_{\ln 2}^{\ln 3} \left(\frac{\sin(\ln - 6)}{\sin(\ln - 6) + \sin(t)}\right)dt \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \left(\frac{\sin(t) + \sin(\ln - 6)}{\sin(\ln - 6) + \sin(t)}\right)dt$$

$$= \frac{1}{2}\int_{\ln 2}^{\ln 3} dt$$

$$= \frac{1}{2}(\ln 3 - \ln 2)$$

$$= \frac{1}{2}\left(\ln\left(\frac{3}{2}\right)\right)$$

$$\Rightarrow \quad I = \frac{1}{4}\left(\ln\left(\frac{3}{2}\right)\right)$$

69. Given,

$$6\int_1^x f(t)dt = 3xf(x) - x^3$$

$$\Rightarrow \quad 6f(x) = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow \quad 3f(x) = 3xf'(x) - 3x^2$$

$$\Rightarrow \quad f(x) = xf'(x) - x^2$$

$$\Rightarrow \quad y = x\frac{dy}{dx} - x^2$$

$$\Rightarrow \quad \frac{y}{x} = \frac{dy}{dx} - x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = x$$

which is a linear differential equation.

So, $\quad IF = e^{-\int\frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$

Hence, the solution is

$$\frac{y}{x} = \int \frac{1}{x^2}dx + c$$

$$\Rightarrow \quad \frac{y}{x} = -\frac{1}{x} + c$$

when $x = 1$, $y = 2$, then $c = 3$

Thus, the equation of the curve is

$$\frac{y}{x} = -\frac{1}{x} + 3$$

Now, when $x = 2$, then

$$y = -1 + 3x = -1 + 6 = 5$$

70. Given integral $= \frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)dx$

$$= \frac{\pi^2}{\ln 3} \times \frac{1}{\pi}\int_{7\pi/6}^{5\pi/6} \sec t\,dt$$

$$= \frac{\pi^2}{\ln 3} \times \frac{1}{\pi}\int_{7\pi/6}^{5\pi/6} \sec x\,dx$$

$$= \frac{\pi}{\ln 3} \times \int_{7\pi/6}^{5\pi/6} \sec x\, dx$$

$$= \frac{\pi}{\ln 3} (\log|\sec x + \tan x|)\Big|_{7\pi/6}^{5\pi/6}$$

$$= \frac{\pi}{\ln 3}\left(\log\left|-\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \left(-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\right|\right)$$

$$= \frac{\pi}{\ln 3}\left(\log\left|\frac{2}{\sqrt{3}}\right|\right)$$

71. Given,

$$\int_a^b (f(x) - 3x)dx = a^2 - b^2$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \frac{3}{2}(b^2 - a^2) - (b^2 - a^2)$$

$$\Rightarrow \quad \int_a^b f(x)dx = \frac{1}{2}(b^2 - a^2)$$

$$\Rightarrow \quad f(x) = x$$

$$\Rightarrow \quad f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

72. The given integral

$$= \int_{-\pi/2}^{\pi/2} \left[x^2 + \ln\left(\frac{\pi + x}{\pi - x}\right)\right]\cos x\, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[x^2 + \log\left(\frac{\pi + x}{\pi - x}\right)\right]\cos x\, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^2)\cos x\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\log\left(\frac{\pi + x}{\pi - x}\right)\right]\cos x\, dx$$

$$= 2\int_0^{\frac{\pi}{2}} (x^2)\cos x\, dx + 0$$

$$= 2(x^2\sin x + 2x\cos x - 2\sin x)_0^{\pi/2}$$

$$= 2\left(\frac{\pi^2}{4} - 2\right)$$

$$= \left(\frac{\pi^2}{2} - 4\right) \text{ sq.u.}$$

73. $$\int_0^1 4x^3 \frac{d^2}{dx^2}(1 - x^2)^5\, dx$$

$$= \left(4x^3 \frac{d}{dx}(1 - x^2)^5\right)_0^1 - \int_0^1 12x^2 \frac{d}{dx}(1 - x^2)^5\, dx$$

$$= (4x^3 \times 5(1 - x^2)^4 \times (-2x))_0^1$$

$$- 12\left[(x^2(1 - x^2)^5)_0^1 - \int_0^1 2x(1 - x^2)^5\, dx\right]$$

$$= 0 - 0 - 12[0 - 0] + 12\int_0^1 2x(1 - x^2)^5\, dx$$

$$= 12\left(-\frac{(1 - x^2)^6}{6}\right)_0^1 = 12\left(0 + \frac{1}{6}\right) = 2$$

74. Let $$I = \int_{-2}^{2} \left(\frac{3x^2}{1 + e^x}\right)dx \quad \text{...(i)}$$

$$= \int_{-2}^{2} \left(\frac{3x^2}{1 + e^{-x}}\right)dx$$

$$= \int_{-2}^{2} \left(\frac{3e^x x^2}{1 + e^x}\right)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-2}^{2} \left(\frac{3(e^x + 1)x^2}{(1 + e^x)}\right)dx$$

$$= \int_{-2}^{2} (3x^2)dx$$

$$= 2\int_0^2 (3x^2)dx$$

$$\Rightarrow \quad I = \int_0^2 (3x^2)dx$$

$$= (x^3)_0^2 = 8$$

75. The given integral

$$= \frac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1 + x}{1 - x}\right)dx}{\int_0^{1/2} \cos 2x \cdot \log\left(\frac{1 + x}{1 - x}\right)dx}$$

$= 0$, ($\because$ N_r = Odd × Even = Odd function)

76. Let $f(x) = ax^2 + bx$

Now, it is given that,

$$\int_0^1 f(x)dx = 1$$

$$\Rightarrow \quad 2a + 3b = 6$$

$$\Rightarrow \quad (a, b) = (3, 0) \text{ and } (0, 2)$$

Thus, the number of polynomials is 2.

77. $$I = \int_{-1}^{2} \left(\frac{x[x^2]}{2 + [x + 1]}\right)dx$$

$$= \int_{-1}^{2} \left(\frac{x[x^2]}{2 + [x + 1]}\right)dx$$

$$= \int_{-1}^{0} \left(\frac{x \cdot 0}{2 + (0)}\right)dx + \int_0^1 \left(\frac{x \cdot 0}{2 + (1)}\right)dx + \int_1^{\sqrt{2}} \left(\frac{x \cdot 1}{2 + (0)}\right)dx + 0$$

$$= \int_1^{\sqrt{2}} \left(\frac{x}{2}\right)dx$$

$$= \left(\frac{x^2}{4}\right)_1^{\sqrt{2}} = 2 - \frac{1}{4} = \frac{1}{4}$$

Thus, $4I - 1 = 0$

78. We have

$$\alpha = \int_0^1 \left(e^{9x + 3\tan^{-1}x}\right)\left(\frac{12 + 9x^2}{1 + x^2}\right)dx$$

Let $9x + 3\tan^{-1}x = t$

$$\Rightarrow \left(9 + \frac{3}{1 + x^2}\right)dx = dt$$

$$\Rightarrow \left(\frac{12 + 9x^2}{1 + x^2}\right)dx = dt$$

Thus, $\alpha = \int_0^{9+\frac{3\pi}{4}} e^t\,dt$

$$= \left(e^{9+\frac{3\pi}{4}} - 1\right)$$

$$\Rightarrow \alpha + 1 = e^{9+\frac{3\pi}{4}}$$

$$\Rightarrow \log(\alpha + 1) = 9 + \frac{3\pi}{4}$$

$$\Rightarrow \left(\log|(\alpha + 1)| - \frac{3\pi}{4}\right) = 9$$

79. We have

$$f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^2 x$$

$$= 7\tan^6 x(\tan^2 x + 1) - 3\tan^2 x(\tan^2 x + 1)$$

$$= (7\tan^6 x - 3\tan^2 x)(\tan^2 x + 1)$$

$$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$$

(i) $\int_0^{\pi/4} f(x)dx$

$$= \int_0^{\frac{\pi}{4}} (7\tan^6 x - 3\tan^2 x)\sec^2 x\,dx$$

$$= \int_0^1 (7t^6 - 3t^2)dt$$

$$= \left(\frac{7t^7}{7} - \frac{3t^3}{3}\right)_0^1 = 0$$

(ii) $\int_0^{\pi/4} x\,f(x)dx$

$$= \left(x\int f(x)\right)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left\{\int f(x)dx\right\} dx$$

$$= \left(x(\tan^7 x - \tan^3 x)\right)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\tan^7 x - \tan^3 x)dx$$

$$= 0 - \int_0^{\frac{\pi}{4}} (\tan^7 x - \tan^3 x)dx$$

$$= 0 - \int_0^{\frac{\pi}{4}} (\tan^4 x - 1)\tan^3 x\,dx$$

$$= 0 - \int_0^{\frac{\pi}{4}} (\tan^4 x - 1)\tan^3 x\sec^2 x\,dx$$

Let $\tan x = t$

$$= 0 - \int_0^1 (t^2 - 1)t^3 dt$$

$$= 0 - \left(\frac{t^6}{6} - \frac{t^4}{4}\right)_0^1$$

$$= 0 - \left(\frac{1}{6} - \frac{1}{4}\right) = -4 - \frac{6}{24} = \frac{2}{24} = \frac{1}{12}$$

81. We have

$$0 \le \sin^4(\pi x) \le 1$$

$$\Rightarrow 2 \le 2 + \sin^4(\pi x) \le 3$$

$$\Rightarrow \frac{1}{3} \le \frac{1}{2 + \sin^4(\pi x)} \le \frac{1}{2}$$

$$\Rightarrow \frac{192x^3}{3} \le \frac{192x^3}{2 + \sin^4(\pi x)} \le \frac{192x^3}{2}$$

$$\Rightarrow \frac{192x^3}{3} < f'(x) < \frac{192x^3}{2}$$

$$\Rightarrow \int_{1/2}^{x} \frac{192x^3}{3}\,dx \le \int_{1/2}^{x} f'(x)dx \le \int_{1/2}^{x} \frac{192x^3}{2}\,dx$$

$$\Rightarrow \int_{1/2}^{x} (64x^3)dx \le \int_{1/2}^{x} f'(x)dx \le \int_{1/2}^{x} (96x^3)dx$$

$$\Rightarrow 64\left(\frac{x^4}{4}\right)_{1/2}^{x} \le \int_{1/2}^{x} f'(x)dx \le 96\left(\frac{x^4}{4}\right)_{1/2}^{x}$$

$$\Rightarrow 16\left(x^4 - \frac{1}{16}\right) \le \int_{1/2}^{x} f'(x)dx \le 24\left(x^4 - \frac{1}{16}\right)$$

Integrating, we get

$$\Rightarrow \int_{1/2}^{1} 16\left(x^4 - \frac{1}{16}\right)dx \le \int_{1/2}^{1} f(x)dx$$

$$\le \int_{1/2}^{1} 24\left(x^4 - \frac{1}{16}\right)dx$$

$$\Rightarrow 16\left(\frac{x^5}{5} - \frac{x}{16}\right)_{1/2}^{1} \le \int_{1/2}^{1} f(x)dx$$

$$\le 24\left(\frac{x^5}{5} - \frac{x}{16}\right)_{1/2}^{1}$$

$$\Rightarrow 2.6 < \int_{1/2}^{1} f(x)dx < 3 \times 9$$

81. Ans. (*a*, *c*)

Given $$L = \frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}$$

Let $$I(k) = \int_0^{k\pi} e^t(\sin^6 at + \cos^4 at)dt$$

$$\Rightarrow \quad I'(k) = e^{k\pi}(\sin^6(ak\pi) + \cos^6(ak\pi))\cdot\pi$$

$$= \pi e^{k\pi} \text{ for } a = 2 \text{ as well as } a = 4$$

Integrating, we get

$$I(k) = e^{k\pi} + c$$

Since $I(0) = 0$, we get $c = -1$

$$\Rightarrow \quad I(k) = e^{k\pi} - 1$$

$$\Rightarrow \quad \frac{I(4)}{I(1)} = e^{4\pi} - 1/e^{\pi} - 1$$

82. Given,

$$\int_0^x \frac{t^2}{t^4 + 1}\, dt = 2x - 1$$

Let $$f(x) = \int_0^x \frac{t^2}{t^4} + 1dt - 2x + 1$$

$$\Rightarrow \quad f'(x) = \frac{x^2}{x^4} - 2 < 0 \ \forall\ x \in [0, 1]$$

$$\Rightarrow \quad f(0) = 1 \text{ and } f(t) = \int_0^1 \frac{t^2}{t^4} + 1dt$$

$$\Rightarrow \quad 0 \le \frac{t^2}{t^4 + 1} < \frac{1}{2} \ \forall\ t \in [0, 1]$$

$$\Rightarrow \quad \int_0^1 \frac{t^2}{t^4} + 1dt < \frac{1}{2}$$

$$\Rightarrow \quad f(1) < 0$$

$f(x) = 0$ exactly one root in [0, 1]

83. Let $$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{x^2\cos x}{1 + e^x}\right)dx \quad \text{...(i)}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{x^2\cos x}{1 + e^{-x}}\right)dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{e^x x^2\cos x}{1 + e^x}\right)dx \quad \text{...(ii)}$$

Adding Eqs (i) and (ii), we get

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ((e^x + 1)x^2\cos x/(1 + e^x))dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^2\cos x)dx$$

$$= 2\int_0^{\frac{\pi}{2}} (x^2\cos x)dx$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} (x^2\cos x)dx$$

$$= (x^2\sin x)_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (2x\sin x)dx$$

$$= \frac{\pi^2}{4} - 2\left[[x(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (-\cos x)dx\right]$$

$$= \frac{\pi^2}{4} - 2[-(0 - 0) + (\sin x)]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi^2}{4} - 2\right)$$

CHAPTER

3 Area Bounded by the Curves

CONCEPT BOOSTER

3.1 Rules to Draw Different Types of Curves

To find the area of the plane curves, first we need to draw the given curves.

So my dear friends, you should remember the basic rules to trace the given curve.

Rule I: Find the domain and range of the given function $y = f(x)$.

Rule II: Find the point of intersection on the co-ordinate axes.

Rule III: Find the intervals of the monotonocity of the function $y = f(x)$.

Rule IV: Find the local maximum and the local minimum values of the function $y = f(x)$.

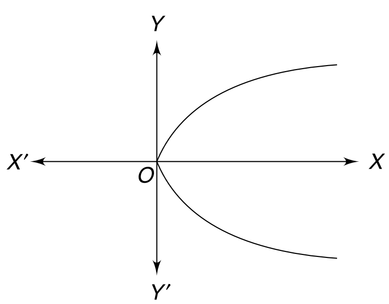
Rule V: Find the concavity and the points of inflection of the function $y = f(x)$.

Rule VI: Find out whether the function is periodic or not.

Rule VII: Find the symmetry as follows:

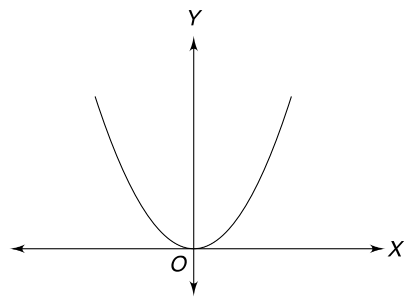
(a) A curve is symmetrical about x-axis, if the powers of y, which occur in its equation, are all even.

$y^2 = x$



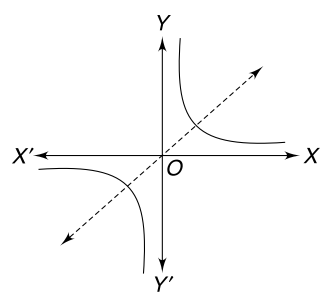
(b) A curve is symmetrical about y-axis, if the powers of x, which occur in its equation, are all even.

$y = x^2$

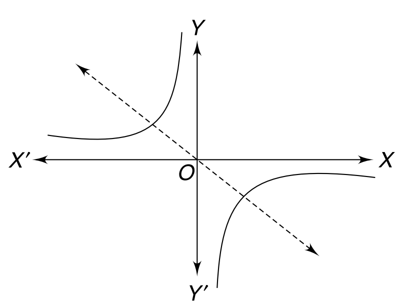


(c) A curve is symmetrical about the line $y = x$, if the curve remains unchanged by interchanging x and y in the equation.

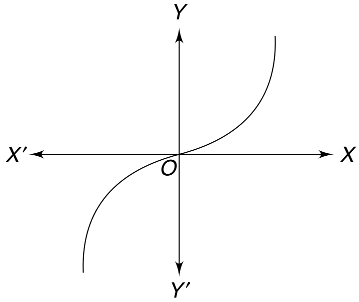
$y = \frac{1}{x}$



(d) A curve is symmetrical about the line $y = -x$ if the curve remains unchanged when x and y are replaced by $-x$ and $-y$, respectively.



(e) A curve is symmetrical in opposite quadrants, if the equation of the curve remains unchanged when x and y are replaced by $-x$ and $-y$, respectively.



Rule VIII: Find the asymptotes if any.

Asymptote

It is a straight line which touches the curve at infinity.

There are three kinds of asymptotes.

(i) **Vertical Asymptote:** The straight line $x = a$ is a vertical asymptote to the curve $y = f(x)$ if at least one of the values of $\lim_{x\to a^+} [f(x)]$ or $\lim_{x\to a^-} [f(x)]$ tends to $+\infty$ or $-\infty$.

(ii) **Horizontal Asymptote:** The straight line $y = b$ is a horizontal asymptote to the curve $y = f(x)$, where $\lim_{x\to+\infty} [f(x)] = b$ or $\lim_{x\to-\infty} [f(x)] = b$.

(iii) **Oblique Asymptote:** The staright line $y = mx + c$ is an oblique asymptote to the curve $y = f(x)$, where

$$m = \lim_{x\to\infty} \left(\frac{f(x)}{x}\right)$$

and $$c = \lim_{x\to\infty} (f(x) - mx)$$

For examples,

1. Let $y = \dfrac{x}{x-1}$.

The vertical asymptote to the given curve is $x = 1$. The horizontal asymptote to the curve is

$$y = \lim_{x\to\infty} \left(\frac{x}{x-1}\right) = 1$$

2. Let $y = \dfrac{1}{x^2-1}$.

The vertical asymptotes to the given curve are

$$x = \pm 1$$

The horizontal asymptote to the curve is

$$y = \lim_{x\to\infty} \left(\frac{1}{x^2-1}\right) = 0$$

3. Let $y = \left(\dfrac{3x}{x-1} + 2x\right)$.

The vertical asymptote to the given curve is $x = 1$

The oblique asymptote to the given curve is $y = 2x + 3$.

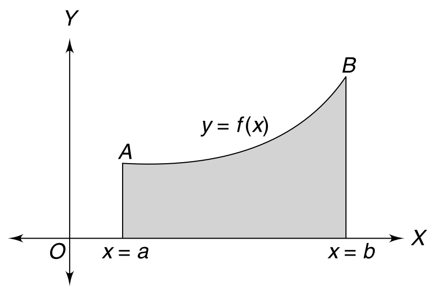
where as $m = \lim_{x\to\infty} \left(\frac{f(x)}{x}\right) = \lim_{x\to\infty} \left(\frac{y}{x}\right)$

$$= \lim_{x\to\infty} \left(\frac{3}{x-1} + 2\right) = 2$$

and $$c = \lim_{x\to\infty} (y - mx)$$

$$= \lim_{x\to\infty} \left(\frac{3x}{x-1} + 2x - 2x\right) = 3$$

3.2 Area of the Cartesian Curve



If $f(x)$ is a single valued and continuous function of x in the interval $[a, b]$. Let

$$f(x) \geq 0$$

for every x in $[a, b]$,

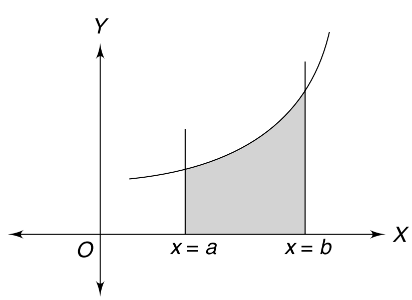
then the area bounded by the curve $t = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, $(a < b)$ is represented by

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$

3.3 Area of the Region Bounded by a Single Curve and the Co-ordinate Axes

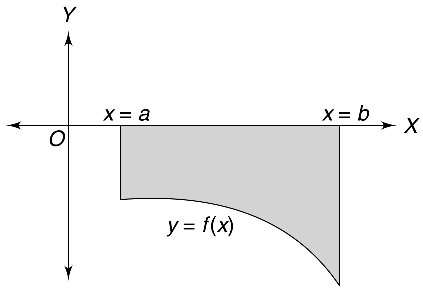
(i) The area bounded by the curve $y = f(x)$, x-axis and the straight lines $x = a$ and $x = b$ $(a < b)$ is given by

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

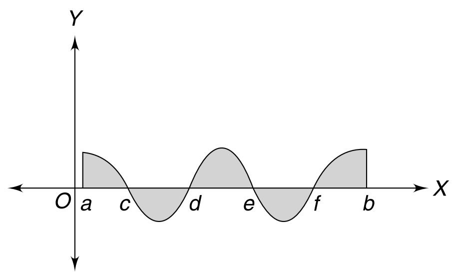


(ii) If $f(x) \le 0$ for all x in $[a, b]$, the area bounded by the curve $y = f(x)$, x-axis and the straight lines $x = a$ and $x = b$, $(a < b)$ is

$$A = \int_a^b (-y)\,dx = \left|\int_a^b f(x)\,dx\right|$$



(iii) If the graph of $y = f(x)$ crosses x-axis between $x = a$ and $x = b$ at $x = c, d, e, f$ respectively, its area (which lie below the x-axis is negative but the area is taken positive, so we take their modulus) is

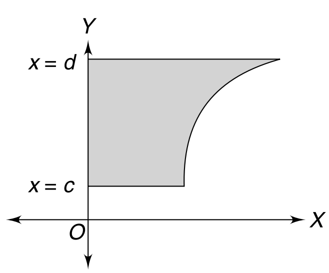


$$A = \int_a^b y\,dx$$

$$= \int_a^c f(x)\,dx + \left|\int_c^d f(x)\,dx\right| + \int_d^e f(x)\,dx + \left|\int_e^b f(x)\,dx\right| + \int_f^b f(x)\ dx$$

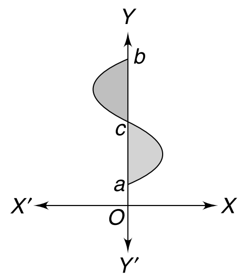
$$= \int_a^c f(x)\,dx - \int_c^d f(x)\,dx + \int_d^e f(x)\,dx - \int_e^b f(x)\,dx + \int_f^b f(x)\ dx$$

(iv) The area enclosed by the curve $x = f(y)$, y-axis and the abscissae $y = c$ and $y = d$, $(c < d)$ is given by



$$A = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

(v) When the curve $x = f(y)$ crosses y-axis, i.e. x changes its sign at $y = c$ between $y = a$ and $y = b$, its area is given by



$$A = \int_a^c x\,dy + \int_c^b (-x)\,dy$$

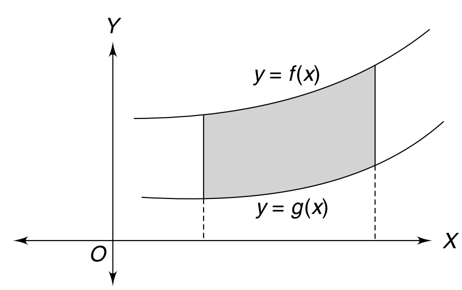
$$= \int_a^c x\,dy + \left|\int_c^b (x)\,dy\right|$$

$$= \int_a^c f(y)\,dy + \left|\int_c^b f(y)\,dy\right|$$

3.4 Area between Two Curves

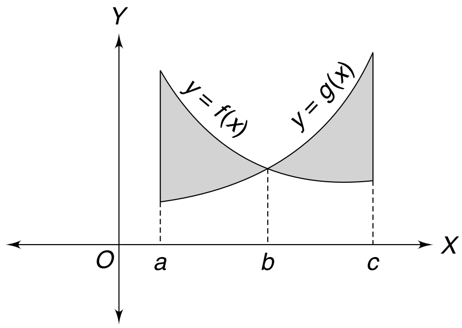
(i) Suppose we are given two curves reprsented by $y = f(x)$ and $y = g(x)$, where $f(x) \ge g(x)$ in $[a, b]$. Here the points of intersection of these two curves are given by $x = a$ and $x = b$ is obtained by taking common values of y from the given equation of two curves. Hence the required area is

$$A = \int_a^b (f(x) - g(x))\ dx$$



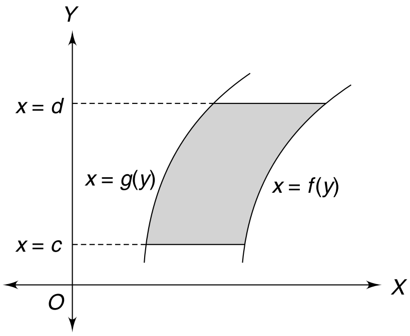
(ii) If the curves $y = f(x)$ and $y = g(x)$ intersect at $x = c$ in $[a, b]$, the required area is

$$A = \int_a^c (f(x) - g(x))\ dx + \int_c^b (g(x) - f(x))\ dx$$



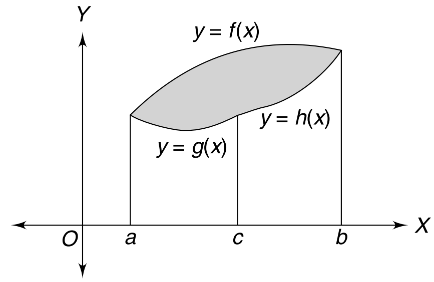
(iii) The area bounded by the curves $x = f(y)$ and $x = g(y)$ and the straight lines $y = c$ and $y = d$ where $f(y) \geq g(y)$ is given by

$$A = \int_c^d (f(y) - g(y)).$$



3.5 Area of the Region Bounded by the Several Curves

If $y = f(x)$ is a strictly monotonic function in (a, b) such that $f'(x) = 0$, the area bounded by the ordinates $x = a$ and $x = b$, $y = f(x)$ and $y = f(c)$, where $c \in (a, b)$, is minimum when $c = \frac{a+b}{2}$.



3.6 Least Value of a Variable Area

Example 1. If the area is bounded by the curve

$$f(x) = \frac{x^3}{3} - x^2 + b$$

and the straight lines $x = 0$ and $x = 2$ and x-axis is minimum, find the value of b.

Solution: Given

$$f(x) = \frac{x^3}{3} - x^2 + b.$$

$$\Rightarrow \quad f'(x) = x(x - 2)$$

Clearly $f(x)$ is monotonic in (0, 2).

Hence for minimum area, $f(x)$ must cross the x-axis at

$$x = \frac{0 + 2}{2} = 1.$$

Thus, $$f(1) = \frac{1}{3} - 1 + b = 0.$$

$$\Rightarrow \quad b = \frac{2}{3}.$$

Exercises

Level I

(Problems based on Fundamentals)

Area of a linear curve

1. Find the area of the region bounded by the curves $y = x^2 - 2x + 2$, $x = -1$ and $x = 2$.
2. Find the area bounded by the curves $y = \ln x + \tan^{-1} x$ and the ordinates = 1 and $x = 2$.
3. Find the area bounded by $y = x|\sin x|$, x-axis and the ordinates $x = 0$ and $x = 2\pi$.
4. Find the area bounded by $y = 1 \log_{\frac{1}{2}} x$ and the x-axis between $x = 1$ and $x = 2$.
5. Find the area of the region bounded by the curve $y = e^2 x - 3e^x + 2$ and the x-axis.

Area bounded by a function which changes sign

6. Find the area bounded by $y = x^3$ and x-axis between the ordinates $x = -1$ and $x = 1$.
7. Find the area bounded by the curve $y = \sin x$ and the x-axis, for $0 \leq x \leq 2\pi$.
8. Find the area bounded by the curve $y = \cos x$ and the x-axis, for $0 \leq x \leq 2\pi$.
9. Find the area bounded by the curve $y = x(x - 1)(x - 2)$ and the x-axis.
10. Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ and the x-axis.

Area of a region between two non-intersecting curves

11. Find the area of the region enclosed by the curves $y = x^2$ and $y = 2x - x^2$.

12. Find the area of the region bounded by the parabolas $y^2 = x$ and $x^2 = y$.
13. Find the area of the region bounded by the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.
14. Find the area of the region bounded by the curves $y = \frac{3x^2}{4}$ and the line $3x - 2y + 12 = 0$.
15. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.
16. Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
17. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.
18. Find the area of the region bounded by the curve $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.
19. Find the area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.
20. Find the area bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2x - x^2$, $y = 2^x$.
21. Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.
22. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?
23. Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.
24. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.
25. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} =1$.
26. Find the area of the region $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$.
27. Find the area of the region $\{(x, y): y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$.
28. Find the area of the region $\{(x, y): x^2 + y^2 \le 2ax, y^2 \ge ax, x \ge 0, y \ge 0\}$
29. Find the area of the region $\left\{(x, y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}\right\}$.
30. Find the area of the region bounded by $y = 1 + |x + 1|$, $x = -2$, $x = 3$, $y = 0$.
31. Find the area of the region bounded by $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$.
32. Find the area enclosed by the curves $y = |x - 1|$ and $y = -|x - 1| + 1$.
33. Find the area of the region bounded by the curves $f(x) = \log x$ and $g(x) = (\log x)^2$.
34. Find the area enclosed by $y^2 = x^2 - x^4$.
35. Find the area bounded by the curve $|y| + \frac{1}{2} \le e^{-|x|}$.
36. Find the area of the region $f(x, y) = [(x, y)$: $0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2]$.
37. Find the area bounded by the curves $y = x$ and $y = x^3$.
38. Find the area bounded by the curves $y = x(x - 1)^2$ the y-axis and the line $y = 2$.
39. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$.
40. Find the area of the region bounded by the curves $y = \log_e x$ and $y = 2^x$, $x = 1/2$ and $x = 2$.
41. Find ther area enclosed by the parabola $(y - 2)^2 = (x - 1)$, the tangent to the parabola at (2, 3) and x-axis.
42. Find the area enclosed by the loop of the curve $y^2 = (x - 1)(x - 2)^2$.
43. Find the area bounded by the curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y = 1/4$.
44. Find the area bounded by the curves $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$.
45. Find the area bounded by the curve $y = \log x$, $y = \log|x|$, $y = |\log x|$ and $y = |\log|x||$.
46. Find the area bounded by the curve $|x - 2| + |y + 1| = 1$.
47. Find the area of the region bounded by the curve $[x] + [y] = 3$ in first quadrant, where [,] = GIF.
48. Find the area of the region bounded by the curve $|x| + |y| = 1$ and $|x - 1| + |y| = 1$.
49. Find the area bounded by the curve $y = \cos^{-1}(\cos x)$ and $|x - \pi| + \left|y - \frac{\pi}{2}\right| = \frac{\pi}{2}$.
50. Find the area bounded by the curve $|y| = -(1 - |x|)^2 + 5$.
51. Find the area bounded by the curves $y = \sqrt{|x|}$ and $y = |x|$.
52. Find the area of the region bounded by the curves $|x + y| + |x - y| \le 2$.
53. Find the area bounded by the curve $y = x$ and $y = x + \sin x$, where $0 \le x \le \pi$.
54. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\left(\beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta\right)$, find $f\left(\frac{\pi}{2}\right)$.

Area of a region between two intersecting curves

55. Find the area bounded by the curves $y = \sin x$ and $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.
56. Find the area enclosed by the curves $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$.
57. The line $y = mx$ bisects the area enclosed by the lines $x = 0$, $y = 0$, $x = \frac{3}{2}$ and the curve $y = 1 + 4x - x^2$. Find m.
58. Find the area bounded by the curve $|y| + \frac{1}{2} \le e^{-|x|}$.

Area of a region by a horizontal strip

59. Find the area bounded between $y = \sin^{-1}x$ and y-axis between $y = 0$ and $y = \pi/2$.
60. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
61. Find the area enclosed by $x = -2y^2$ and $x = 1 - 3y^2$.
62. Find the area enclosed by the curves $y = \tan^{-1}x$ and $y = \cot^{-1}x$ and the y-axis.
63. Find the area bounded by the curve $x = -y^2 + y + 2$ and the y-axis.

Area of a region between several graphs

64. Find the area of the plane figure bounded by $y = \sqrt{x}$, $x \in [0, 1]$, $y = x^2$, $x \in [1, 2]$ and $y = -x^2 + 2x + 4$, $x \in [0, 2]$.
65. Find the area of the region bounded by the curves $4|y| = |4 - x^2|$ and $|y|(x^2 + 4) = 12$.
66. Find the area bounded by the ellipse $x^2 + 2y^2 = 2$ and the outside of the parabola $y = 1 - x^2$.
67. Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.
68. Find the area enclosed by the curves $y = \ln(x + e)$ and $x = \ln\left(\frac{1}{y}\right)$, and x-axis.
69. Find the area enclosed by the curves $x^2 + y^2 = 4$, parabola $y = x^2 + x + 1$ and the curve $y = \left[\sin^2\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right)\right]$ and x-axis.
70. Find the area enclosed by the curves $\frac{x^2}{4} + y^2 = 1$ and $\frac{x^2}{2} - y^2 = 1$.

Least value of a variable area

71. If the area bounded by $y = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0$ and $x = 2$ and the x-axis is minimum, find the value of a.
72. Find the value of a, for which the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight lines $y = 0$, $x = 0$ and $x = 1$ the least.
73. For what value of k, the area enclosed by the curves $y = x^2 - 3$ and $y = kx + 2$ is the least. Find also the least area.
74. If the area bounded by the curve $y = x^2 + 2x - 3$ and the line $y = mx + 1$ is least, find the value of m.
75. Find the value of a for which the area bounded by the curve $y = \frac{x}{6} + \frac{1}{x^2}$, and the lines $y = 0$, $x = a$ and $x = 2a$ is least.

Level II

(Mixed Problems)

1. The area bounded by the curve $y = 4x - x^2$ and the x-axis is

   (a) $\frac{30}{7}$ sq.u. (b) $\frac{31}{7}$ sq.u.

   (c) $\frac{32}{3}$ sq.u. (d) $\frac{34}{3}$ sq.u.

2. The area under the curve $y = \sqrt{3x + 4}$ between $x = 0$ and $x = 4$, is

   (a) $\frac{56}{9}$ sq.u. (b) $\frac{64}{9}$ sq.u.

   (c) 8 sq.u. (d) none of these

3. The area bounded by the curve $y = x^3$, x-axis and two ordinates $x = 1$ to $x = 2$ equal to

   (a) $\frac{15}{2}$ sq.u. (b) $\frac{15}{4}$ sq.u.

   (c) $\frac{17}{2}$ sq.u. (d) $\frac{17}{4}$ sq.u.

4. The measurement of the area bounded by the coordinate axes and the curve $y = \log_e x$ is

   (a) 1 (b) 2

   (c) 3 (d) ∞

5. If the area bounded by the curves $y^2 = 4ax$ and $y = mx$ is $\frac{a^2}{3}$, the value of m is

(a) 2 (b) −2
(c) 1/2 (d) none of these

6. The area bounded by the parabola $y = 4x^2$, y-axis and the lines $y = 1$, $y = 4$ is
(a) 3 sq.u. (b) $\frac{7}{5}$ sq.u.
(c) $\frac{7}{3}$ sq.u. (d) none of these

7. The area enclosed by the curves $y = \sin x$, $y = 0$, $x = 0$ and $\frac{\pi}{2}$ is
(a) π (b) 2π
(c) 1 (d) 2

8. The area of the region bounded by the x-axis and the curve defined by $y = \tan x \left(-\frac{\pi}{3} \le x \le \frac{\pi}{3}\right)$ is
(a) $\log \sqrt{2}$ (b) $-\log \sqrt{2}$
(c) $2\log 2$ (d) 0

9. The ratio of the areas bounded by the curves $y = \cos x$ and $y = \cos 2x$ between $x = 0$, $\frac{\pi}{3}$ and x-axis is
(a) $\sqrt{2} : 1$ (b) $1 : 1$
(c) $1 : 2$ (d) $2 : 1$

10. The area bounded by $y = [x]$ and the lines $x = 1$ and $x = 1.7$ is
(a) $\frac{17}{10}$ (b) 1
(c) $\frac{17}{5}$ (d) $\frac{7}{10}$

11. The area bounded by the x-axis and the curve $y = \sin x$ and $x = 0$, $x = \pi$ is
(a) 1 (b) 2
(c) 3 (d) 4

12. The area bounded by the parabola $y^2 = 2x$ and the ordinates $x = 1$, $x = 4$ is
(a) $\frac{4\sqrt{2}}{3}$ sq.u. (b) $\frac{28\sqrt{2}}{3}$ sq.u.
(c) $\frac{56}{3}$ sq.u. (d) none of these

13. The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
(a) πab sq.u. (b) $\frac{1}{2} \pi ab$ sq.u.
(c) $\frac{1}{4} \pi ab$ sq.u. (d) none of these

14. The area of the smaller segment cut off from the circle $x^2 + y^2 = 9$ by $x = 1$ is
(a) $\frac{1}{2}(9\sec^{-1}3 - \sqrt{8})$ (b) $9\sec^{-1}(3) - \sqrt{8}$
(c) $\sqrt{8} - 9\sec^{-1}3$ (d) none of these

15. The area of the upper half of the circle whose equation is $(x - 1)^2 + y^2 = 1$ is given by
(a) $\int_0^2 \sqrt{2x - x^2}\, dx$ (b) $\int_0^1 \sqrt{2x - x^2}\, dx$
(c) $\int_1^2 \sqrt{2x - x^2}\, dx$ (d) $\frac{\pi}{4}$

16. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the first quadrant is
(a) 9 (b) $\frac{27}{4}$
(c) 36 (d) 18

17. The area of the region $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$ is
(a) $\frac{\pi^2}{5}$ (b) $\frac{\pi^2}{2}$
(c) $\frac{\pi^2}{3}$ (d) $\left(\frac{\pi}{4} - \frac{1}{2}\right)$

18. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is
(a) 1 (b) 2
(c) $2\sqrt{2}$ (d) 4

19. The area of the figure bounded by $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$ is
(a) $e + \frac{1}{e}$ (b) $e - \frac{1}{e}$
(c) $e + \frac{1}{e} - 2$ (d) $e + \frac{1}{e} + 2$

20. The area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is
(a) $3 - e$ (b) $e - 3$
(c) $\frac{1}{2}(3 - e)$ (d) $\frac{1}{2}(e - 3)$

21. The area enclosed by the parabolas $y = x^2 - 1$ and $y = 1 - x^2$ is
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $\frac{4}{3}$ (d) $\frac{8}{3}$

22. The area bounded by the circle $x^2 + y^2 = 4$, line $x = \sqrt{3}y$ and x-axis lying in the first quadrant, is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) π

23. The area formed by triangular-shaped region bounded by the curves $y = \sin x$, $y = \cos x$ and $x = 0$ is
(a) $\sqrt{2} - 1$ (b) 1
(c) $\sqrt{2}$ (d) $1 + \sqrt{2}$

24. The area between the curve $y = \cos x$ and x-axis when $0 \le x \le 2\pi$ is
(a) 2 (b) 4
(c) 3 (d) 0

25. The area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
(a) $\sqrt{ab}$ (b) a/b
(c) $2ab$ (d) ab

26. The area bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and the x-axis is
(a) 4 (b) 2
(c) 3 (d) 1

27. The area bounded by lines $y = 2 + x$, $y = 2 - x$ and $x = 2$ is
(a) 3 (b) 4
(c) 8 (d) 16

28. The area enclosed by the parabola $y^2 = 4ax$ and the straight line $y = 2ax$, is
(a) $\frac{a^2}{3}$ sq.u. (b) $\frac{1}{3a^2}$ sq.u.
(c) $\frac{1}{3a}$ sq.u. (d) $\frac{2}{3a}$ sq.u.

29. The area bounded by the curves $x = at^2$, $y = 2at$ and the x-axis in $1 \le t \le 3$, is
(a) $26a^2$ (b) $8a^2$
(c) $\frac{26a^2}{3}$ (d) $\frac{104a^2}{3}$

30. The area bounded by the curves $y = \text{In}\, x$, $y = \text{In}\,|x|$, $y = |\text{In}\, x|$ and $y = |\log|x||$ is
(a) 4 sq.u. (b) 6 sq.u.
(c) 10 sq.u. (d) none of these

31. The area bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$, is
(a) 2 sq.u. (b) 3 sq.u.
(c) 4 sq.u. (d) 1 sq.u.

32. The area of the region lying inside $x^2 + (y - 1)^2 = 1$ and outside $c^2x^2 + y^2 = c^2$, where $c = (\sqrt{2} - 1)$ is
(a) $(4 - \sqrt{2})\frac{\pi}{4} + \frac{1}{\sqrt{2}}$ (b) $(4 + \sqrt{2})\frac{\pi}{4} - \frac{1}{\sqrt{2}}$
(c) $(4 + \sqrt{2})\frac{\pi}{4} + \frac{1}{\sqrt{2}}$ (d) none of these

33. The area enclosed between the curves $y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis, is
(a) 2 (b) 1
(c) 4 (d) none of these

34. The area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $x \le \frac{5}{2}$ is
(a) $\frac{\pi}{6} - \frac{\sqrt{3}+1}{8}$ (b) $\frac{\pi}{4} + \frac{\sqrt{3}-1}{8}$
(c) $\frac{\pi}{6} - \frac{\sqrt{3}-1}{8}$ (d) none of these

35. Let $f(x) = \text{maximum } [x^2, (1 - x)^2, 2x(1 - x)]$ where $0 \le x \le 1$. The area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$ is
(a) $\frac{17}{27}$ (b) $\frac{14}{27}$
(c) $\frac{19}{27}$ (d) none of these

36. The area enclosed by the parabola $ay = 3(a^2 - x^2)$ and x-axis is
(a) $4a^2$ sq.u. (b) $12a^2$ sq.u.
(c) $4a^3$ sq.u. (d) none of these

37. The area bounded by the curve $y = k \sin x$ between $x = \pi$ and $x = 2\pi$, is
(a) 2k sq.u. (b) 0
(c) $\frac{k^2}{2}$ (d) k sq.u.

38. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\left(\beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\,\beta\right)$, then $f\left(\frac{\pi}{2}\right)$ is
(a) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$ (b) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$
(c) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$ (d) $\left(\frac{\pi}{4} - \sqrt{2} + 1\right)$

39. The area in the first quadrant between $x^2 + y^2 = \pi^2$ and $y = \sin x$ is
(a) $\frac{(\pi^3 - 8)}{4}$ (b) $\frac{\pi^3}{3}$
(c) $\frac{(\pi^3 - 16)}{4}$ (d) $\frac{(\pi^3 - 8)}{2}$

40. For $0 \le x \le \pi$, the area bounded by $y = x$ and $y = x + \sin x$, is

(a) 2 (b) 4
(c) 2π (d) 4π

41. The area bounded by $y = x \sin x$ and x-axis between $x = 0$ and $x = 2\pi$, is
(a) 0 (b) 2π sq.u.
(c) π sq.u. (d) 4π sq.u.

42. The area bounded by the lines $y = |x - 2|$, $|x| = 3$ and $y = 0$ is
(a) 13 unit^2 (b) 5 unit^2
(c) 9 unit^2 (d) 7 unit^2

43. The area bounded by the curve $x^2 = ky$, $k > 0$ and the line $y = 3$ is 12 unit^2. Then k is
(a) 3 (b) $3\sqrt{3}$
(c) $\frac{3}{4}$ (d) none of these

44. The area of the portion enclosed by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the axes of reference is
(a) $\frac{a^2}{6}$ (b) a^2
(c) $\frac{a^2}{2}$ (d) $\frac{a^2}{4}$

45. The area bounded by the curve $x = \cos^{-1} y$ and the lines $|x| = 1$ is
(a) sin 1 (b) cos 1
(c) 2 sin 1 (d) 2cos 1

46. The area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$ is
(a) $\frac{3\pi - 8}{3}$ (b) $\frac{\pi - 8}{3}$
(c) $\frac{2\pi - 8}{3}$ (d) none

47. A point P moves inside a triangle formed by $A(0, 0)$, $B\left(1, \frac{1}{\sqrt{3}}\right)$, $C(2, 0)$ such that $\min\{PA, PB, PC\} = 1$, the area bounded by the curve traced by the point P is
(a) $\left(3\sqrt{3} - \frac{3\pi}{2}\right)$ (b) $\sqrt{3} + \frac{\pi}{2}$
(c) $\left(\sqrt{3} - \frac{\pi}{2}\right)$ (d) $\left(3\sqrt{3} + \frac{3\pi}{2}\right)$.

48. The area bounded by the curves $y = \cos^{-1}(\cos x)$ and $y = |x - \pi|$, is
(a) π^2 (b) $2\pi^2$
(c) $\pi^2/2$ (d) none

49. The area bounded by the curves
$$y = \begin{cases} x^{1/\ln x} & : x \neq 1 \\ e & : x = 1 \end{cases} \text{ and } y = |x - e| \text{ is}$$
(a) $\frac{e^2}{2}$ (b) e^2
(c) $2e^2$ (d) 1.

50. The area bounded by $e\ln(x + 1)$, $|x| \le 1$, is
(a) 1 (b) 2
(c) 4 (d) none

51. The area common to the region determined by $y \ge \sqrt{x}$ and $x^2 + y^2 < 2$ has the value
(a) π (b) $(2\pi - 1)$
(c) $\left(\frac{\pi}{4} - \frac{1}{6}\right)$ (d) none

52. The area of the region bounded by $||x| - |y|| \le 1$ and $x^2 + y^2$ in the xy-plane is
(a) π (b) 2π
(c) 3π (d) 1.

53. The area bounded by $1 \le |x - 2| + |y + 1| \le 2$ is
(a) 2 (b) 4
(c) 6 (d) none

54. The area bounded by the curve $y = \tan x + \cot x| - |\tan x - \cot x|$ and between the lines $x = 0$ and $x = \frac{\pi}{2}$ and the x-axis is
(a) ln 4 (b) ln $\sqrt{2}$
(c) 2 ln 2 (d) $\sqrt{2}$ ln 2

55. The area is enclosed by the graph of the curve $y = \ln^2 x - 1$ lying in the fourth quadrant is
(a) $\frac{2}{e}$ (b) $\frac{4}{e}$
(c) $2\left(e + \frac{1}{e}\right)$ (d) $4\left(e - \frac{1}{e}\right)$.

56. The area bounded by the curve $y = \frac{3}{|x|}$ and $y = 2 - |x - 2|$ is
(a) $\frac{4 - \ln 27}{3}$ (b) $2 - \ln 3$
(c) $2 + \ln 3$ (d) none

57. The area enclosed by the curve $|x + y - 1| + |2x + y + 1| = 1$ is
(a) 2 (b) 1
(c) 4 (d) 3/2

58. The area of the region $\{f(x, y) : x^2 + y^2 \le 1\ |x| + |y|\}$ is
(a) π (b) $\pi - 1$
(c) $\pi - 2$ (d) $\pi - 3$

59. The area of the region of the xy-plane is bounded by $|x| + |y| + |x + y| \le 1$ is
(a) 1/2 (b) 3/4
(c) 1 (d) 4

60. The area of the region bounded by the curves $y = [x]$ and $y = \{x\}$, where [,] = GIF and {,} = LIF, is

(a) 2 (b) 1
(c) 1/2 (d) 4

61. The smaller area is bounded by the curves $|x| + |y| = 1$ and $\sqrt{|x|} + \sqrt{|y|} = 1$, is
(a) 1/3 (b) 2/3
(c) 4/3 (d) 5/3.

62. The area bounded by the curves $|x| + |y| = 1$ and $|x - 1| + |y| = 1$ is
(a) 1/2 (b) 1/4
(c) 3/2 (d) 3/4

63. The area of the region bounded by the curves $x = -2y^2$ and $x = 1 - 3y^2$ is
(a) 4/3 (b) 3/4
(c) 5/3 (d) 3/5

64. The area enclosed by the parabola $(y - 2)^2 = x - 1$ and the tangent to it at (2, 3) and x-axis is
(a) 10 (b) 7
(c) 9 (d) 5

65. The area enclosed by the curves $(y - x)^2 = x^3$ and the straight line $x = 1$ is
(a) 5/4 (b) 4/5
(c) 2/3 (d) 3/2

(More than one options are correct)

66. The area bounded by the curve $y = |e^{|x|} - e^{-x}|$, the x-axis and $x = 1$ is $\left(e^m + \frac{1}{e^n} + p\right)$ sq unit, then
(a) $m = 1$ (b) $n = 1$
(c) $m + n + p = 3$ (d) $m + n + p = 0$

67. Let the curves $y^2 = 4ax$, $y = ax$ and $y = \frac{x}{a}$ where $1 \le a \le 2$. Then
(a) Area $= \frac{8}{5}\left(a^5 - \frac{1}{a}\right)$
(b) Max area = 84
(c) Max area = 75
(d) Points of intersections are $\left(\frac{a}{4}, 4\right)$ and $(4a^3, 4a^3)$

68. Let $f(x) = \min\left\{\tan x, \cot x, \frac{1}{\sqrt{3}}\right\} \forall x \in \left[0, \frac{\pi}{2}\right]$. Then the are bounded by the curve $y = f(x)$ and the x-axis is $\left(\log\left(\frac{m}{n}\right) + \frac{\pi}{p\sqrt{3}}\right)$ sq units, where $m, n, p \in N$ then
(a) $m = 4$ (b) $n = 4$
(c) $p = 6$ (d) $m + n + p = 14$

69. The area bounded by the curve $y = x - x^2$ and the line $y = m\,x$ is $a/2$ then the value of m is/are
(a) 1 (b) -2
(c) 2 (d) 4

70. A normal to the curve $x^2 + kx - y + 2 = 0$ at the point, whose abscissa '1' is paral let to the line $y = x$. Then
(a) the value of k is -3
(b) Equation of the normal is $x - y = 1$
(c) The area bounded by the curve the normal and the x-axis is 7/6 sq u.
(d) Equation of tangent is $x + y + 1 = 0$

71. If the area bounded by the curve $y = 3x^3 + 2x$ and the lines $x = a$ and $y = 0$ is unity, then the value of a is/are
(a) $\frac{-1}{\sqrt{3}}$ (b) $\sqrt{\frac{2}{3}}$
(c) $-\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{3}}$

72. Let the parabola by $y^2 = 4x$ and the solpe of the normal by -1. Then
(a) Equation of the normal is $x + y = 3$
(b) The points of intersections are (1, 2) and (9, -6)
(c) The area bounded by the curve and its normal is 64/3 sq. units
(d) None of these

Level III

(Problems for JEE-Advanced)

1. Find the area bounded by the curve $a^2y^2 = x^2(a^2 - x^2)$ in xy-plane by the method of integration. **[Roorkee, 1985]**
2. Find the area bounded by the curve $y = (x - 1)(x - 3)$ and x-axis lying between the ordinates $x = 0$ and $x = 3$. **[Roorkee, 1986]**
3. Find the area of the region bounded by the curves $y = \log e_x$ and y = $\sin 4(\pi x)$ and $x = 0$. **[Roorkee, 1987]**
4. Find the area between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of the two minima of the curve. **[Roorkee, 1988]**
5. Find the area of the portion of the circle $x^2 + y^2$ which is exterrior to the parabola $y^2 = 12x$. **[Roorkee, 1989]**
6. Find the area enclosed between the curves $y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis. **[Roorkee, 1990]**
7. The line $y = mx$ bisects the area enclosed by the lines $x = 0$, $y = 0$, $x = 3/2$ and the curve $y = 1 + 4x - x^2$. Find the value of m. **[Roorkee, 1991]**

8. Find the area bounded by the curve $y = 2x - x^2$ and the line $y = -x$. **[Roorkee, 1992]**
9. Find the area bounded by the curve $y = (x - 1)(x - 2)$ and the axis of x. **[Roorkee, 1993]**
10. Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$. **[Roorkee, 1994]**
11. Find the area given by the curves $x + y \le 6$. $x^2 + y^2 \le 6y$ and $y^2 \le 8x$. **[Roorkee, 1995]**
12. Find the area of the region bounded by the curves $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$, $x \le 5/2$. **[Roorkee, 1996]**
13. Indicate the region bounded by the curves $x^2 = y$, $y = x + 2$ and x-axis and obtain the area enclosed by them. **[Roorkee, 1997]**
14. Indicate the region bounded by the curves $y = x\log x$ and $y = 2x - x^2$ and obtain the area enclosed by them. **[Roorkee, 1998]**
15. Find the area of the region lying inside $x^2 + (y + 1)^2$ and outside $c^2x^2 + y^2 = c^2$, where $c = \sqrt{2} - 1$. **[Roorkee, 1999]**
16. Find the area enclosed by the parabola $(y - 2)^3 = x - 1$, the tangent to the parabola at (2, 3) and the x-axis. **[Roorkee, 2000]**
17. A point P moves inside a triangle formed by $A(0, 0)$, $B\left(1, \frac{1}{\sqrt{3}}\right)$, $C(2, 0)$ such that $\{PA, PB, PC\} = 1$, find the area bounded by the curve traced by P.
18. Find the area bounded by the curve $f(x) = \cos^{-1}(\cos x)$, $0 \le x \le 2\pi$ with the tangent to the curve $f(x) = |\cos x|$ at $x = \pi$.
19. Find the area enclosed between the curves $|y| = 1 - x^2$, $x^2 + y^2 = 1$.
20. Find the area bounded by the curves $|y| = e^{-|x|} - \frac{1}{2}$ and $|x| + |y| = \ln 2$.
21. Find the area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$.
22. Find the area enclosed between the smaller arc of the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + 1 - 2\sqrt{3}$.
23. Find the smaller of the two areas enclosed between the ellipse $9x^2 + 4y^2 - 36x + 8y + 4 = 0$ and the line $3x + 2y = 10$.
24. Let $f(x) = x^2 + 3x + 2$ and $g(x)$ be its inverse. Find the area bounded by $g(x)$, the x-axis and the ordinates $x = 1$ and $x = 4$.
25. Find the area bounded by the curve $g(x)$, the x-axis, the ordinates $x = -1$ and $x = 4$, where $g(x)$ is the inverse of the function $f(x) = \frac{x^3}{24} + \frac{x^2}{8} + \frac{13}{12x} + 1$.
26. Find the area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$.
27. Find the area of the region bounded by $|x + 2y| + |x + 2y| \ge 8$ and $xy \ge 2$.
28. Find the area enclosed by $|x| + |y| \le 3$ and $xy \ge 2$.
29. Find the area of the region bounded by $|x - y| + |x + y| \le 8$ and $xy \ge 2$.
30. Find the area of the curve enclosed by $|x + y| \le 2$ and $x^2 + y^2 \ge 2$.

Level IV

(Tougher Problems for JEE-Advanced)

1. Find the area enclosed by $2x^2 + 6xy + 5y^2 = 1$.
2. Find the area enclosed by the curve $x^{2/3} + y^{2/3} = a^{2/3}$.
3. A polynomial function $f(x)$ satisfies the condition $f(x + 1) = f(x) + 2x + 1$. Find $f(x)$, if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.
4. Find the area bounded by the curve $y = xe^{-x^2}$, the x-axis and the line $x = c$, where $y(x)$ is maximum.
5. For what value of a, the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight lines $x = 0$, $y = 0$ and $x = 1$ is the least?
6. Find the possible value of a for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, a^2 + 1)$, $(a, a^2 + 1)$.
7. Find the area bounded by the curve $y = xe^{-x}$, $xy = 0$ and $x = c$ where c is the x-coordinate of the curves inflection point.
8. Find the area of the region bounded by $|x| + |y| \ge 1$ and $x^2 - 2x + 1 \le 1 - y^2$.
9. Find the area bounded by the curves $|x + y| \le 1$, $|y - x| \le 1$ and $3x^2 + 3y^2 \ge 1$.
10. Find the area bounded by the curves $|x - 2| + |y - 2| \le 3$, $x^2 - 4x + y + 3 \le 0$ and $x^2 + y^2 + 2x - 9 \le 0$.
11. Find the area enclosed by the curves $(x^2 + y^2) \le 4 \le 2(|x| + |y|)$
12. Find the area bounded by the curves $y = \frac{4 - x^2}{4}$ and $y = 7 - |x|$.
13. Find the area bounded by the curves $y = x^2 + x - 2$, $y = 2x$ for which $|x^2 + x - 2| + |2x| = |x^2 + 3x - 2|$ is satisfied.
14. Find the area bounded by the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$.
15. Find the area bounded by the parabola $y = x^2 - 2x + 3$, the line tangent to it at the point $P(2, -5)$ and the y-axis.

16. Find the area enclosed between the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + (1 - 2\sqrt{3})$.
17. Find the area enclosed by the curves $a^2x^2 + b^2y^2 = 1$ and $b^2x^2 + a^2y^2 = 1$.
18. Find the area bounded by the curves $|y| + \frac{1}{2} \le e^{-|x|}$ and $\max\{|x|, |y|\} \le 2$.
19. Find the area enclosed by the curves $y = \min\{x^3, |x - 2|, e^{3-x}\}$, x-axis, y-axis and $x = 4$.
20. Find the area of the region bounded by the set $S = S_3 - S_1 - S_2$, where

$$S_1:\{(x, y): x^2 + 2y^2 \le 2\}$$

$$S_2:\{(x,y): 2x^2 + y^2 \le 2\}$$

and $S_3:\{(x, y) : x^2 + y^2 \le 2$

Integer Type Questions

1. Find the area bounded by the curves $y = \left[\sin^2 x + \cos\left(\frac{x}{4}\right)\right]$, $x = -2$ and $x = 2$ and x-axis.
2. Find the area bounded by the curves $|x| + |y| = 1$.
3. Find the area bounded by the curves $|x - 2| + |y - 2| = 1$.
4. Find the area bounded by the curve $e^{\ln(x+1)} \ge |y|$, $|x| \le 1$.
5. Find the area bounded by the curves $y = \ln|x|$ and $y = 1 - |x|$.
6. Find the area bounded by the curves $y = 3 - |x|$ and $y = |x - 1|$.
7. Find the area bounded by the curves $y = \frac{|x|}{x}$, $x \ne 0$ and $y = x(x - 1)(x - 3)$.
8. Find the area bounded by the curves $y = e^x$, $x = 0$, $y = e$.
9. Find the area bounded by $1 \le |x - 2| + |y + 1| \le 2$.
10. Find the area bounded by the curve $|y| = \sin(2x)$, where $0 < x < 2\pi$.
11. Find the area enclosed by the curves $|x + y| + |x - y|, \le 4$, $|x| \le 1$ and $y \le \sqrt{x^2 - 2x + 1}$.
12. Find the area bounded by the curves $4 \le x^2 + y^2 \le 2(|x| + |y|)$.
13. If the area bounded by the curves $y = \frac{1}{x}$, $y = \frac{1}{2x - 1}$, $x = 2$, $x = a$ is $\frac{4}{\sqrt{5}}$, find a.

Comprehensive Link Passages (For JEE-Advanced Examinations)

Passage I

Consider the functions defined implicity $y^3 - 3y + x = 0$ on various interval in the real line.

If $x \in (-\infty, 2) \cup (2, \infty)$, the equation implicity defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicity define a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

On the basis of the above information, answer the following questions.

1. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is
   (a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $\frac{4\sqrt{2}}{7^2 3^2}$
   (c) $\frac{4\sqrt{2}}{7^2 3}$ (d) $\frac{4\sqrt{2}}{7^3 3}$
2. The area of the region bounded by the curves $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is
   (a) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$
   (b) $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$
   (c) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$
   (d) $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$.
3. The value of $\int_{-1}^{1} g'(x)$ is
   (a) $2g(-1)$ (b) 0
   (c) $-2g(1)$ (d) $2g(1)$.

Passage II

If $x = f(y)$ and $x = g(y)$ be two functional curves, the area bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$ is given by

$$\int_c^d |f(y) - g(y)|\, dy,$$

where $d > c$.

In case of $g(y)$ always left of $f(y)$, required area $= \int_c^d [f(y) - g(y)]$.

On the basis of the above information, answer the following questions.

1. The area bounded by $y^2 = 2x + 1$ and $x - y - 1 = 0$, will be
   (a) 16/9 (b) 16/3
   (c) 8/3 (d) none.

2. The area bounded by $y = \log_e x$, x-axis and y-axis is given by

(a) $\int_1^{\infty} \log e\, y\, dy$ (b) $\int_{-\infty}^{0} e^{-y} dy$

(c) $\int_{-\infty}^{0} e^{y} dy$ (d) none

3. The area bounded by $y = \tan^{-1}x$, $y = \cot^{-1}x$ and y-axis is equal to

(a) $\log\sqrt{2}$ (b) log 4

(c) log 2 (d) none

Passage III

Conseider the function

$$f(x) = \begin{cases} x - [x] - \frac{1}{2} & : x \notin I \\ 0 & : x \in I \end{cases},$$

where [,] = GIF

and if $g(x) = \max\{x^2, f(x), [x]\}$,

where $-2 \le x \le 2$

On the basis of the above information, answer the following questions.

1. The area bounded by the curves $y = f(x)$, x-axis and the ordinates $x = 1/2$ and $x = 1$ is

(a) 1/4 (b) 1/8

(c) 1/2 (d) none.

2. The area bounded by $y = g(x)$, x-axis and the straight line $x = 1$ is

(a) 1/4 (b) 1/2

(c) 1/8 (d) none

3. The area bounded by $y = g(x)$, where $-2 \le x \le 2$, is

(a) 175/48 (b) 275/48

(c) 175/24 (d) 275/24.

Passage IV

A normal to the curve $x^2 + kx - y + 2 = 0$ at the point whose abscissa 1 is parallel to the line $y = x$.

On the basis of the above information, answer the following questions.

1. The value of k is

(a) –3 (b) 1

(c) 0 (d) 2

2. The equation of the normal is given by

(a) $y = x + 1$ (b) $y = x - 1$

(c) $y = -x + 1$ (d) $y = -x - 1$.

3. The area in the first quadrant bounded by the curves, the normal and the x-axis is

(a) 13/6 (b) 7/6

(c) 10/3 (d) 43/6.

Passage V

Consider the functions $f(x) = \sqrt{x^2 - 4}$, $g(x) = |x - 2|$ and $h(x) = \sqrt{x - 2}$ for $x \in R$, a function is defined as $F(x)$ = max or min $\{f(x), g(x), h(x)\}$.

On the basis of the above information, answer the following questions.

1. The area of $F(x) = \min\{f(x), g(x), h(x)\}$ between the co-odinates axes for $x < 0$ is

(a) 2π (b) π

(c) 4π (d) none

2. The area enclosed by $F(x) = \min\{f(x), g(x), h(x)\}$ and $F(x) = \max\{f(x), g(x), h(x)\}$ for $x \in [0, 2]$ is

(a) π (b) $\pi + 2$

(c) $\pi - 2$ (d) $\pi - 1$.

3. The area bounded by the curves $F(x) = \min\{f(x), g(x), h(x)\}$, $x = 2$, $y = 0$ and $x = 3$, is

(a) 3/2 (b) 1/2

(c) 1 (d) none

Matrix Match (For JEE-Advanced Examination Only)

1. **Match the following Columns:**

	Column I	Column II	
(A)	The area bounded by $y = 3x$ and $y = x^2$ is	(P)	2/3
(B)	The area bounded by $x = 4 - y^2$ and the y-axis is	(Q)	17/12
(C)	The area in squares units of the region bounded by the curve $x^2 = 4y$ and the line $x = 2$ and the x-axis is	(R)	32/3
(D)	The area bounded by $y = x^3$, $y = x^2$ and $x = 1$, $x = 2$ is	(S)	32/9
		(T)	9/2

2. **Match the following Columns:**

	Column I	Column II	
(A)	The area enclosed by the curve $(y - \sin^{-1}x)^2 = x - x^2$ is	(P)	0
(B)	The area of the region represented by $\sqrt{2} \le \lvert x + y\rvert + \lvert x - y\rvert \le 2$ is	(Q)	1
(C)	If the area bounded by $y = x^2 - 3$ and the line $y = ax + 2$ attains its maximum value, the value of a is	(R)	π
(D)	If k is a positive number and the area of the region bounded by the curves $y = x - kx^2$ and $ky = x^2$ attains its maximum value, the value of k is	(S)	6
		(T)	$\pi/4$

3. **Match the following Columns:**

	Column I		Column II
(A)	The area of the figure bounded by $y = \tan x$, $y = \tan^2 x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is	(P)	4π
(B)	The area bounded by the curve $y = x \sin x$ and the x-axis, $x = 0$ and $x = 2\pi$ is	(Q)	$1 - \frac{\pi}{4}$
(C)	The area bounded by the curves $x = \tan^{-1}\sqrt{y}$, $x = 0$, $x = \pi/4$ and $y = 0$ is	(R)	$2(\pi + 1) + \sqrt{2}$
(D)	The area bounded by the curves $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}$, between the co-ordinates $x = -\pi$ and $x = \pi$ is	(S)	$\left(\log_e 2 - 2 + \frac{\pi}{2}\right)$

4. **Match the following Columns:**

	Column I		Column II
(A)	The area bounded by the curves $y = \lvert\sin x\rvert$, x-axis and the lines $\lvert x\rvert = \pi$.	(P)	2 s.u.
(B)	The area bounded by the curves $y = \sin^{-1} x$, y-axis, $\lvert y\rvert = \pi/2$	(Q)	1 s.u.
(C)	The area bounded by the curves $y = \left[\frac{x^2}{64} + 2\right]$, [,] = GIF, $y = x - 1$, $x = 0$ is	(R)	8 s.u.
(D)	The area bounded by the curves $y = \lvert x - 1\rvert$ and $y = 3 - \lvert x\rvert$, is	(S)	4 s.u.

Assertion and Reason

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is not the correct explanation of A

(C) A is true and R is false

(D) A is false and R is true.

1. *Assertion (A)*: The area bounded by the curve $\lvert x\rvert + \lvert y\rvert = 1$ is 1 s.u.

   Reason (R): The area bounded by the curve $\lvert x - 2\rvert + \lvert y - 2\rvert = 1$ is also 1 s.u.

2. *Assertion (A):* The area bounded by the curve $y = [x]$ and the lines $x = 0$ and $x = n$ is $\left(\frac{n(n-1)}{2}\right)$ s.u.

   Reason (R): The area bounded by the curve $y = [x]$ and the lines $x = 0$ and $x = 10$ is 45 s.u.

3. *Assertion (A):* The area bounded by the curves $f(x, y) = \{(x, y) : x^2 + y^2 \le a^2, x + y \ge a\}$ is $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ s.u.

   Reason (R): The area bounded by the curves $f(x, y) = \{(x, y) : x^2 + y^2 \le 1, x + y \ge 1,\}$ is $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ s.u.

4. *Assertion (A):* The area bouned by the curves $y = \min\{\lvert x + 2\rvert, \lvert x\rvert, \lvert x - 2\rvert\}$ is 2 s.u.

   Reason (R): The area bounded by the curves $y = 1 - \lvert x\rvert$ and $y = \lvert x\rvert - 1$ is 2 s.u.

5. *Assertion (A):* The area of the region bounded by the curves $y = \lvert x - 1\rvert$ and $y = 3 - \lvert x\rvert$ is 4 s.u.

   Reason (R): The area of the region bounded by the curves $y = \ln x$, $y = \ln\lvert x\rvert$, $y = \lvert\ln x\rvert$ and $y = \lvert\ln\lvert x\rvert\rvert$ is 4 s.u.

Questions asked in past IIT-JEE Examinations

1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. **[IIT-JEE, 1981]**
2. Find the area enclosed by $\lvert x\rvert + \lvert y\rvert = 1$. **[IIT-JEE, 1981]**
3. Find the area bounded by the curve $y = f(x)$, x-axis and the co-ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$. Then $f(x)$ is
   (a) $(x - 1)\cos(3x + 4)$
   (b) $\sin(3x + 4)$
   (c) $\sin(3x + 4)(x - 1)\cos(3x + 4)$
   (d) none. **[IIT-JEE, 1982]**
4. Find the area bounded by the x-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the co-ordinates at $x = a$ divides the area into two equal parts, find a. **[IIT-JEE, 1983]**
5. Find the area of the region bounded by the x-axis and the curves defned by
$$\begin{cases} y = \tan x & : -\frac{\pi}{3} \le x \le \frac{\pi}{3} \\ y = \cot x & : \frac{\pi}{6} \le x \le \frac{\pi}{2} \end{cases}$$
[IIT-JEE, 1984]
6. Find the area bounded by the curves $y = \sqrt{5 - x^2}$ and $y = \lvert x - 1\rvert$ **[IIT-JEE, 1985]**
7. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}y$ and $y = x$. **[IIT-JEE, 1986]**

8. Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ above the x-axis. **[IIT-JEE, 1987]**

9. Find the area of the region bounded by the curve $y = \tan x$, tangent drawn to the curve at $x = \frac{\pi}{4}$ and the x-axis. **[IIT-JEE, 1988]**

10. Find all maxima and minima of the function $y = x(x - 1)^2 : 0 \le x \le 2$. Also, determine the area bounded by the curve $y = x(x - 1)^2$, the y-axis and the line $y = 2$. **[IIT-JEE, 1989]**

11. Find the area of the region bounded by the curves $y = ex \ln x$, $y = \frac{\ln x}{ex}$ and $x = 1$. **[IIT-JEE, 1990]**

12. Sketch the curves and identify the region bounded by $y = \ln x$, $y = 2^x$, $x = \frac{1}{2}$ and $x = 2$. Find the area of the region. **[IIT-JEE, 1991]**

13. Find the area bounded by the curves $y = x^2$ and $y = \frac{2}{1 + x^2}$. **[IIT-JEE, 1992]**

14. The area of the region bounded by the curves $y = |x - 1|$ and $y = 1$ is
(a) 1 (b) 2
(c) 1/2 (d) none
[IIT-JEE, 1993]

15. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$? **[IIT-JEE, 1994]**

16. Consider a square with vertices at (1, 1), (–1, 1) (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are near to the origin than to any edge. Sketch the region S and find its area. **[IIT-JEE, 1995]**

17. The slope of the tangent to the curve $y = f(x)$ at a point (x, y) is $(2x + 1)$ and the curve passes through (1, 2). The area of the region bounded by the curve, the x-axis and the line $x = 1$ is
(a) 5/3 (b) 5/6
(c) 6/5 (d) 6
[IIT-JEE, 1996]

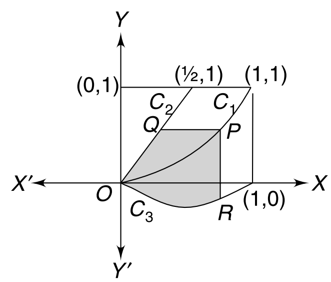
18. Let A_n be the area bounded by the curve $y = \tan^n x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$. If $n \ge 2$, then $(A_n + A_{n+2})$ is
(a) $\frac{1}{(n + 1)}$ (b) $\frac{1}{n}$
(c) $\frac{1}{n - 1}$ (d) none
[IIT-JEE, 1997]

19. Find all possible values of $b > 0$ so that the area of the region bounded between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum. **[IIT-JEE, 1997]**

20. Let $O(0, 0)$, $A(2, 0)$, $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB, which satisfy $d(P, OA) \ge \min\{(P, OB), d(P, AB)\}$ where d denotes the distance from the point to the corresponding line. Sketech the region R and find its area. **[IIT-JEE, 1997]**

21. Let $f(x) = \max\{x^2, (1 - x)^2, 2x(1 - x)\}$ where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$, $x = 1$. **[IIT-JEE, 1997]**

22. Let C_1 and C_2 be the graphs of the functions $y = x^2$, $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$.



For a point P on C_1, let the lines through P parallel to the axes meet C_2 and C_3 at Q and R, respectively. If for every position of P on C_1, the area of the shaded regions OPQ and ORP are equal, determine the function $f(x)$. **[IIT-JEE, 1998]**

23. For what values of m, the area of the region bounded by the curves $y = x - x^2$ and the line $y = mx$ equals 9/2?
(a) –4 (b) –2
(c) 2 (d) 4
[IIT-JEE, 1999]

24. Let $f(x)$ be a continuous function given by
$$f(x) = \begin{cases} 2x & : |x| \le 1 \\ x^2 + ax + b & : |x| > 1 \end{cases}.$$
Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying in the left of the line $8x + 1 = 0$.
[IIT-JEE, 1999]

No questions asked in 2000.

25. Let $b \ne 0$ and for $j = 0, 1, 2, ..., n$, let S_1 be the area of the region bounded by the y-axis and the curve $xe^{xy} = \sin by$, $\frac{j\pi}{b} \le y \le \frac{(j + 1)\pi}{b}$. Show that S_0, S_1, ..., S_n are in GP. Also, find their sum for $a = -1$ and $b = -1$ and $b = \pi$.
[IIT-JEE, 2001]

26. The area bounded by the curves $y = |x| - 1$ and $y = 1 - |x|$ is
    (a) 1 (b) 2
    (c) $2\sqrt{2}$ (d) 4
    [IIT-JEE, 2002]

27. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the first quadrant is
    (a) 9 (b) 27/4
    (c) 36 (d) 18
    [IIT-JEE, 2003]

28. Find the curve passing through (2, 0) and having slope of tangent at any point $P(x, y)$ as $\left(\frac{(x+1)^2 + (y-3)}{(x+1)}\right)$.
    Find also the area enclosed by the curve and x-axis in the fourth quadrant **[IIT-JEE, 2004]**

29. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq.u., the value of a is
    (a) $1/\sqrt{3}$ (b) 1/2
    (c) 1 (d) 1/3
    [IIT-JEE, 2004]

30. The area bounded by the curves $y = (x + 1)^2$ and $y = (x - 1)^2$ and the line $y = 1/4$ is
    (a) 4 s.u. (b) 1/6 s.u.
    (c) 4/3 s.u. (d) 1/3 s.u.
    [IIT-JEE, 2005]

31. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$, and $y^2 = 4x - 3$ **[IIT-JEE, 2005]**

32. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$
    and $f(x)$ is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with x-axis and point B is such that the chord AB subtends a right angle at V. Find the area enclosed by $f(x)$ and chord AB. **[IIT-JEE, 2005]**
    No questions asked in between 2006 to 2007.

33. The area of the region between the curves
    $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is
    (a) $\int_0^{\sqrt{2}-1}\left(\frac{t}{(1+t^2)\sqrt{1-t^2}}\right)dt$
    (b) $\int_0^{\sqrt{2}-1}\left(\frac{dt}{(1+t^2)\sqrt{1-t^2}}\right)$
    (c) $\int_0^{\sqrt{2}+1}\left(\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}\right)$
    (d) $\int_0^{\sqrt{2}+1}\left(\frac{t\,dt}{(1+t^2)\sqrt{1-t^2}}\right)$ **[IIT-JEE, 2008]**

Comprehension

Consider the functions defined implicitly by the equation $y^2 - 3y + x = 0$ on varoius intervals in the real line.

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real-valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation uniquely implicity defines a unique real-valued differentiable function $y = g(x)$ satisfying $g(0) = 0$

34. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$
    (a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$
    (c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

35. The area of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, where $- < \infty < a < b < -2$, is
    (a) $\int_a^b \left(\frac{x}{3(f(x)^2 - 1}\right)dx + bf(b) - af(a)$
    (b) $-\int_a^b \left(\frac{x}{3(f(x)^2 - 1)}\right)dx + bf(b) - af(a)$
    (c) $\int_a^b \left(\frac{x}{3(f(x)^2 - 1)}\right)dx - bf(b) + af(a)$
    (d) $-\int_a^b \left(\frac{x}{3(f(x)^2 - 1)}\right)dx - bf(b) + af(a)$

36. $\int_{-1}^{1} g'(x)\,dx =$
    (a) $2g(-1)$ (b) 0
    (c) $-2g(1)$ (d) $2g(1)$
    [IIT-JEE, 2008]

37. Area of the region bounded by $y = e^x$ and $x = 0$ and $x = e$ is
    (a) $e - 1$ (b) $\int_1^e \log(1 + e - y)\,dy$
    (c) $e - \int_0^1 e^x\,dx$ (d) $\int_1^e \ln y\,dy$
    [IIT-JEE, 2009]

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$.

Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

38. The real number s lies in the interval
    (a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-1, \frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

39. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$

(c) (9, 10) (d) $\left(0, \frac{21}{64}\right)$

40. The function $f'(x)$ is

(a) inc in $\left(-t, -\frac{1}{4}\right)$ and dec. in $\left(-\frac{1}{4}, t\right)$

(b) dec. in $\left(-t, -\frac{1}{4}\right)$ and inc. in $\left(-\frac{1}{4}, t\right)$

(c) inc in $(-t, t)$

(d) dec in $(-t, t)$

[IIT-JEE, 2010]

41. Let $f: [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = (1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)\,dx$ and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$ and $x = 2$ and the x-axis, Then

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$

(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

[IIT-JEE, 2011]

42. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts R_1 $(0 \le x \le b)$ and $R_2 (b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$, Then b is

(a) 3/4 (b) 1/2

(c) 1/3 (d) 1/4

[IIT-JEE, 2011]

43. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then

(a) $S \ge \frac{1}{e}$ (b) $S \ge 1 - \frac{1}{e}$

(c) $S \ge \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$

(d) $S \ge \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$ **[IIT-JEE, 2012]**

44. The area of the region enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

(a) $4(\sqrt{2} - 1)$ (b) $2\sqrt{2}(\sqrt{2} - 1)$

(c) $2(\sqrt{2} + 1)$ (d) $2\sqrt{2}(\sqrt{2} + 1)$

[IIT-JEE, 2013]

No questions asked in 2014, 2015.

45. Area of the region $\{(x, y) : y \le \sqrt{|x + 3|}, 5y \le x + 9 \le 15\}$ is

(a) 1/6 (b) 4/3

(c) 3/2 (d) 5/3

[IIT-JEE, 2016]

ANSWERS

LEVEL-II

1. (c)	2. (d)	3. (b)	4. (d)	5. (a)
6. (c)	7. (c)	8. (c)	9. (d)	10. (d)
11. (b)	12. (b)	13. (a)	14. (b)	15. (a)
16. (a)	17. (d)	18. (b)	19. (c)	20. (a)
21. (d)	22. (c)	23. (a)	24. (b)	25. (c)
26. (d)	27. (b)	28. (c)	29. (d)	30. (a)
31. (c)	32. (a)	33. (a)	34. (c)	35. (a)
36. (a)	37. (a)	38. (b)	39. (a)	40. (a)
41. (d)	42. (a)	43. (a)	44. (a)	45. (c)
46. (a)	47. (c)	48. (c)	49. (b)	50. (c)
51. (c)	52. (a)	53. (c)	54. (a)	55. (b)
56. (a)	57. (a)	58. (c)	59. (b)	60. (b)
61. (b)	62. (a)	63. (a)	64. (c)	65. (d)
66. (a, b, d)	67. (a, b, d)		68. (a, b, c, d)	
69. (b, d)	70. (a, b, c, d)		72. (b, c)	72. (a, b, c)

INTEGER TYPE QUESTIONS

1. 4	2. 2	3. 2	4. 4	5. 3
6. 4	7. 2	8. 1	9. 6	10. 4
11. 2	12. 8	13. a = 8		

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (b)	2. (a)	3. (d)
Passage II :	1. (b)	2. (c)	3. (c)
Passage III :	1. (b)	2. (a)	3. (b)
Passage IV :	1. (a)	2. (b)	3. (b)
Passage V :	1. (b)	2. (c)	3. (b)

MATRIX MATCH

1. (A)→(T), (B)→(R), (C)→(P), (D)→(Q)
2. (A)→(T), (B)→(S), (C)→(P), (D)→(Q)
3. (A)→(S), (B)→(P), (C)→(Q), (D)→(R)
4. (A)→(S), (B)→(P), (C)→(S), (D)→(S)

ASSERTION AND REASON

1. B	2. A	3. A	4. C	5. B

Hints and Solutions

Level I

1. Hence, the required area

$$= \int_{-1}^{2} y\,dx$$

$$= \int_{-1}^{2} (x^2 - 2x + 2)\,dx$$

$$= \left(\frac{x^3}{3} - x^2 + 2x\right)\Bigg|_{-1}^{2}$$

$$= \left(\frac{8}{3} - 4 + 4 + \frac{1}{3} + 1 - 2\right)$$

$$= 3 + 1 - 2 = 2 \text{ sq.u.}$$

2. Hence, the required area

$$= \int_{1}^{2} (\ln x + \tan^{-1} x)\,dx$$

$$= (x \log x - x)|_1^2 + \left(x \tan^{-1} x - \frac{1}{2} \log|1 + x^2|\right)\Bigg|_1^2$$

$$= (2 \log 2 - 1) + \left(2 \tan^{-1} 2 - \frac{1}{2} \log 5 - \frac{\pi}{4}\right)$$

$$= \left(2(\log 2 + \tan^{-1} 2) - \frac{1}{2} \log 5 - \left(\frac{\pi}{4} + 1\right)\right) \text{ s.u.}$$

3. Hence, the required area

$$= \int_{0}^{2\pi} (x|\sin x|)\,dx$$

$$= \int_{0}^{\pi} (x \sin x)dx - \int_{\pi}^{2\pi} (x \sin x)dx$$

$$= (-x \cos x + \sin x)|_0^{\pi} - (-x \cos x + \sin x)|_{\pi}^{2\pi}$$

$$= \pi - (-2\pi - \pi)$$

$$= 4\pi. \text{ sq.u.}$$

4. Hence, the required area

$$= \int_{1}^{2} (\log_{1/2} x)\,dx$$

$$= -\frac{1}{\log 2} (x \log x - x)|_1^2$$

$$= -\frac{1}{\log 2} (2\log\ 2 - 3)$$

$$= \frac{1}{\log 2} (3 - 2\log 2) \text{ sq.u.}$$

5. Hence, the required area

$$= \int_{0}^{\log 2} (e^{2x} - 3e^x + 2)dx$$

$$= \left(\frac{e^{2x}}{2} - 3e^x + 2x\right)\Bigg|_0^{\log 2}$$

$$= \left(4 - 6 + 2\log 2 + 3 - \frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + 2\log 2\right) \text{ sq.u.}$$

6. Hence, the required area

$$= \int_{-1}^{1} x^3 dx$$

$$= 0$$

7. Hence, the required area

$$= \int_{0}^{2\pi} \sin x\,dx$$

$$= \int_{0}^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

$$= (-\cos x)|_0^{\pi} - (-\cos x)|_{\pi}^{2\pi}$$

$$= -(\cos x)|_0^{\pi} + (\cos x)|_{\pi}^{2\pi}$$

$$= -(-1 - 1) + (1 - (-1)) = 4 \text{ sq.u.}$$

8. Hence, the required area

$$= \int_{0}^{2\pi} \cos x\,dx$$

$$= \int_{0}^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$

$$= (\sin x)|_0^{\pi/2} - (\sin x)|_{\pi/2}^{3\pi/2} + (\sin x)|_{3\pi/2}^{2\pi}$$

$$= (1 - 0) - (-1 - 1) + (0 - (-1))$$

$$= 4 \text{ sq.u.}$$

9. Hence, the required area

$$= \int_{0}^{2} \{x(x-1)(x-2)\}dx$$

$$= \int_{0}^{1} \{x(x-1)(x-2)\}dx - \int_{1}^{2} \{x(x-1)(x-2)\}dx$$

$$= \int_{0}^{1} (x^3 - 3x^2 + 2x)dx - \int_{1}^{2} (x^3 - 3x^2 + 2x)dx$$

$$= \left(\frac{x^4}{4} - x^3 + x^2\right)\Bigg|_0^1 - \left(\frac{x^4}{4} - x^3 + x^2\right)\Bigg|_1^2$$

$$= \left(\frac{1}{4} - 1 + 1\right) - \left((4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1\right)\right)$$

$$= 2\left(\frac{1}{4} - 1 + 1\right)$$

$$= \frac{1}{2} \text{ sq.u.}$$

10. Hence, the required area

$$= \int_{0}^{3} \{(x-1)(x-2)(x-3)\}dx$$

$$= -\int_0^1 \{(x-1)(x-2)(x-3)\}dx$$

$$+\int_1^2 \{(x-1)(x-2)(x-3)\}dx$$

$$-\int_2^3 \{(x-1)(x-2)(x-3)\}dx$$

$$= -\left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)\Bigg|_0^1$$

$$+\left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_1^2$$

$$-\left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_2^3$$

$$= -\left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right) + (4 - 8 + 22 - 12)$$

$$-\left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right)$$

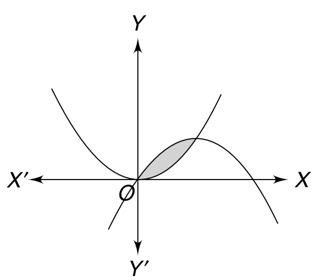
$$-\left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right) + (4 - 8 + 22 - 12)$$

$$= -2\left(\frac{23}{4} - 8\right) + 12 - \left(\frac{279}{4} - 72\right)$$

$$= \frac{46}{4} - \frac{279}{4} + 100$$

$$= 100 - \frac{233}{4} = \frac{400 - 233}{4} = \frac{167}{4} \text{ sq.u.}$$

11. Given curves are $y = x^2$ and $y = 2x - x^2$



Hence, the required area

$$= \int_0^1 (y_2 - y_1)dx$$

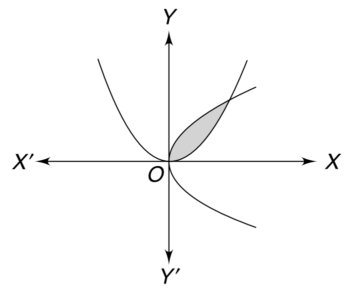
$$= \int_0^1 (2x - x^2 - x^2)dx$$

$$= \int_0^1 (2x - 2x^2)dx$$

$$= \left(x^2 - \frac{2}{3}x^3\right)\Bigg|_0^1$$

$$= \left(1 - \frac{1}{3}\right) = \frac{2}{3} \text{ sq.u.}$$

12. Given curves are $y^2 = x$ and $x^2 = y$.



Hence, the required area

$$= \int_0^1 (y_2 - y_1)\,dx$$

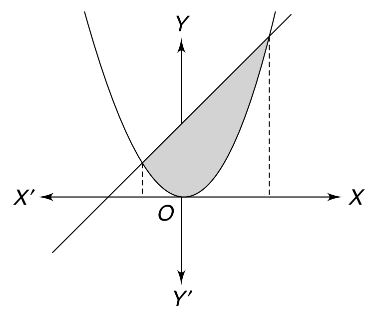
$$= \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Bigg|_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3} \text{ sq.u.}$$

13. Do yourself.

14. Given curves are $y = \frac{3x^2}{4}$ and $3x - 2y + 12 = 0$



Hence, the required area

$$= \int_{-2}^{4} (y_2 - y_1)\,dx$$

$$= \int_{-2}^{4} \left(\frac{3x + 12}{2} - \frac{3x^3}{4}\right)dx$$

$$= \frac{3}{4}\int_{-2}^{4} (2x + 8 - x^2)\,dx$$

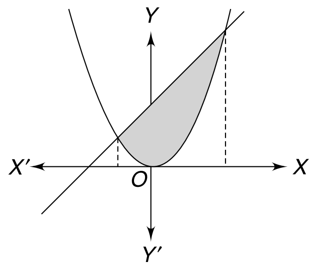
$$= \frac{3}{4}\left(x^2 + 8x - \frac{x^3}{3}\right)_{-2}^{4}$$

$$= \frac{3}{4}\left(\left(16 + 32 - \frac{54}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)\right)$$

$$= \frac{3}{4}\left(36 - \frac{62}{3}\right)$$

$$= \frac{3}{4}\left(\frac{108 - 62}{3}\right) = \frac{46}{4} = \frac{23}{2} \text{ sq.u.}$$

15. Given curves are $x^2 = 4y$ and $x = 4y - 2$



Hence, the required area

$$= \int_{-1}^{2} (y_2 - y_1)\, dx$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right) dx$$

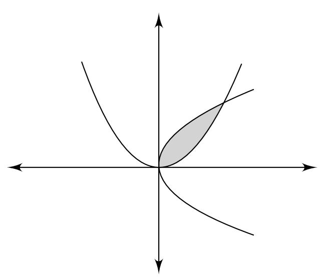
$$= \frac{1}{4}\int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)_{-1}^{2}$$

$$= \frac{1}{4}\left(\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right)$$

$$= \frac{1}{4}\left(\frac{10}{3} + \frac{7}{6}\right) = \frac{1}{4}\left(\frac{27}{6}\right) = \frac{9}{8} \text{ sq.u.}$$

16. Given curves are $y^2 = 4ax$ and $x^2 = 4ay$.



Hence, the required area

$$= \int_{0}^{4a} (y_2 - y_1)\, dx$$

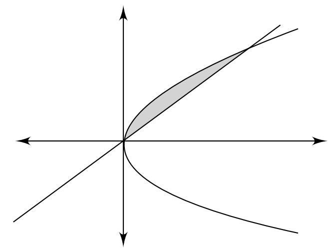
$$= \int_{0}^{4a} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a}\left(\frac{2}{3}x^{3/2}\right)_0^{4a} - \left(\frac{x^3}{12a}\right)_0^{4a}$$

$$= 2\sqrt{a}\left(\frac{2}{3}(4a)^{3/2}\right) - \left(\frac{64a^3}{12a}\right)$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq.u.}$$

17. Given curves are $y^2 = 4ax$ and $y = mx$.



Hence, the required area

$$= \int_{0}^{4a} (y_2 - y_1)\, dx$$

$$= \int_{0}^{4a} (2\sqrt{a}\sqrt{x} - mx)\, dx$$

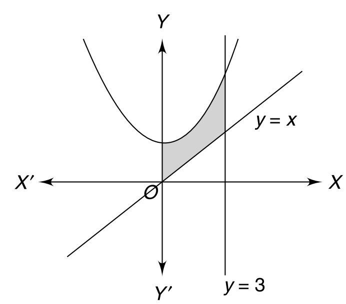
$$= \left(2\sqrt{a}\left(\frac{2}{3}x^{3/2}\right) - m\left(\frac{x^2}{2}\right)\right)_0^{4a}$$

$$= \left(2\sqrt{a}\left(\frac{2}{3}(4a)^{3/2}\right) - m\left(\frac{16a^2}{2}\right)\right)$$

$$= \left(\frac{32}{3} - 8m\right)a^2 \text{ sq.u.}$$

18. Given curves are

$$y = x^2 + 2,\ y = x,\ x = 0 \text{ and } x = 3.$$



Hence, the required area

$$= \int_{0}^{3} (y_2 - y_1)\, dx$$

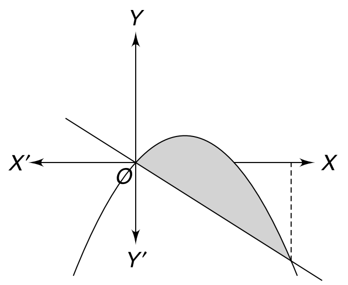
$$= \int_{0}^{3} (x^2 + 2 - 2x)\, dx$$

$$= \left(\frac{x^3}{3} + 2x - x^2\right)_0^3$$

$$= \left(\frac{27}{3} + 6 - 9\right) = 6 \text{ sq.u.}$$

19. Given curves are

$$y = 2x - x^2 \text{ and } y = -x.$$



Hence, the required area

$$= \int_0^3 (y_2 - y_1)\,dx$$

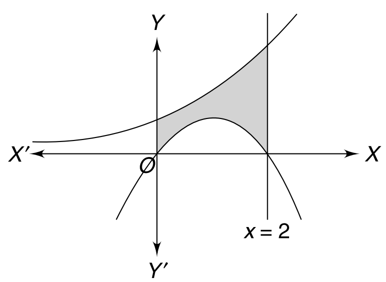
$$= \int_0^3 (2x - x^2 + x)\,dx$$

$$= \int_0^3 (3x - x^2)\,dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)_0^3$$

$$= \left(\frac{27}{2} - 9\right) = \frac{9}{2} \text{ sq.u.}$$

20. Given curves are $y = 2x - x^2$, $y = 2^x$ and $x = 0$, $x = 2$



Hence, the required area

$$= \int_0^2 (y_2 - y_1)\,dx$$

$$= \int_0^2 (2x - x^2 - 2^x)\,dx$$

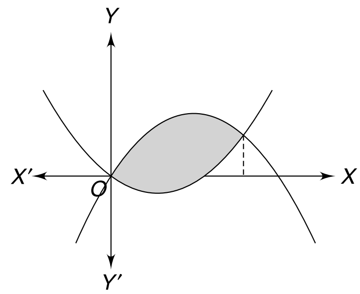
$$= \left(x^2 - \frac{x^3}{3} - \frac{2^x}{\log 2}\right)_0^2$$

$$= \left(4 - \frac{8}{3} - \frac{4}{\log 2} + \frac{1}{\log 2}\right)$$

$$= \left(\frac{4}{3} - \frac{3}{\log 2}\right) \text{ sq.u.}$$

21. Given curves are

$$y = 6x - x^2 \text{ and } y = x^2 - 2x.$$



We have $6x - x^2 = x^2 - 2x$.

$2x^2 = 8x$

$x = 0, 4$

Hence, the required area

$$= \int_0^4 (y_2 - y_1)\,dx$$

$$= \int_0^4 (6x - x^2 - x^2 + 2x)\,dx$$

$$= \int_0^4 (8x - 2x^2)\,dx$$

$$= \left(4x^2 - \frac{2}{3}x^3\right)_0^4$$

$$= \left(64 - \frac{128}{3}\right) = \frac{64}{3} \text{ sq.u.}$$

22. Let A_1 be the area of the curve $y = 4x - x^2$ with x-axis and A_2 be the area of the curve $y = x^2 - x$ with x-axis

$$A_1 = \int_0^4 y\,dx$$

$$= \int_0^4 (4x - x^2)\,dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^4$$

$$= \left(32 - \frac{64}{3}\right) = \frac{32}{3}$$

and

$$A_2 = \int_0^1 y\,dx$$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

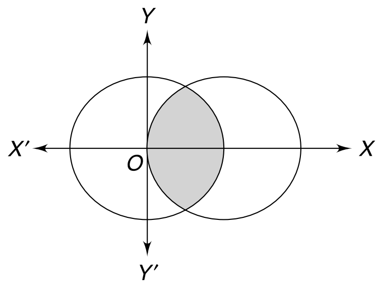
$$= \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}$$

Hence, the required ratio,

$$\frac{A_1}{A_2} = \frac{32}{3} : \frac{1}{6} = \frac{32}{3} \times \frac{6}{1} = 64 : 1$$

23. Given curves are

$$x^2 + y^2 = 4 \text{ and } (x-2)^2 + y^2 = 4.$$



Hence, the required area

$$= 2\left(\int_0^1 \sqrt{4-(x-2)^2} + \int_1^2 \sqrt{4-x^2}\right)dx$$

$$= 2\left(\frac{(x-2)}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right)_0^1$$

$$+2\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_1^2$$

$$= 2\left(-\frac{\sqrt{3}}{2} + 2\sin^{-1}\left(-\frac{1}{2}\right) - 2\sin^{-1}(-1)\right)$$

$$+2\left(\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\left(\frac{1}{2}\right)\right)$$

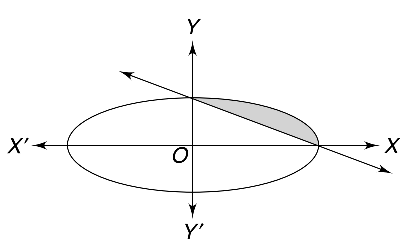
$$= 2\left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3} + \pi + \frac{\pi}{2} - \frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)$$

$$= 2\left(\frac{3\pi}{2} - \frac{2\pi}{3} - \sqrt{3}\right)$$

$$= \left(\frac{5\pi}{3} - 2\sqrt{3}\right) \text{ sq.u.}$$

24. Do yourself.

25. Given curves are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x}{a} + \frac{y}{b} = 1$.

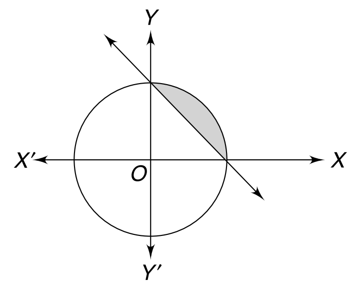


Hence, the required area

$$= \frac{\pi ab}{4} - \frac{1}{2}ab$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 1\right) \text{ sq.u.}$$

26. Given curves are $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$.



Hence, the required area

$$= \int_0^1 (C - L)\,dx$$

$$= \int_0^1 (\sqrt{1-x^2} - (1-x))\,dx$$

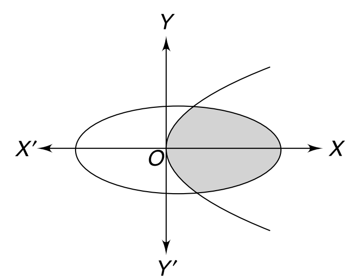
$$= \left(\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) - x + \frac{x^2}{2}\right)_0^1$$

$$= \left(\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - 1 + \frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - 1\right) \text{ sq.u.}$$

27. Given curves are $\{x: y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$.



Solving $y^2 = 4x$ and $4x^2 + 4y^2 = 9$ we get.

$$4x^2 + 16x - 9 = 0$$

$\Rightarrow$ $4x^2 - 2x + 18x - 9 = 0$

$\Rightarrow$ $2x(2x-1) + 9(2x-1) = 0$

$\Rightarrow$ $(2x+9)(2x-1) = 0$

$\Rightarrow$ $x = \frac{1}{2}, -\frac{9}{2}$

Hence, the required area

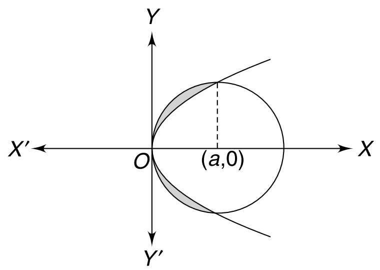
$$= \int_0^{1/2} P\,dx + \int_{1/2}^{3/2} E\,dx$$

$$= \int_0^{1/2} 2\sqrt{x}\,dx + \frac{1}{2}\int_{1/2}^{3/2} \sqrt{9-4x^2}\,dx$$

$$= \int_0^{1/2} 2\sqrt{x}\,dx + \frac{1}{2}\int_{1/2}^{3/2} \sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,dx$$

$$= \left(2 \times \frac{2}{3} \times x^{2/3}\right)_0^{1/2} + \frac{1}{2}\left(\frac{x}{2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\right.$$
$$\left. + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right)_{1/2}^{3/2}$$
$$= \frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{9}{8} \times \frac{1}{2}\sin^{-1}(1)$$
$$- \frac{1}{2}\left(\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right)$$
$$= \frac{2}{3\sqrt{2}} + \frac{9\pi}{32} - \frac{1}{4\sqrt{2}} - \frac{9}{16}\sin^{-1}\left(\frac{1}{3}\right)$$
$$= \left(\frac{5}{12\sqrt{2}} + \frac{9\pi}{32} - \frac{9}{16}\sin^{-1}\left(\frac{1}{3}\right)\right) \text{ sq.u.}$$

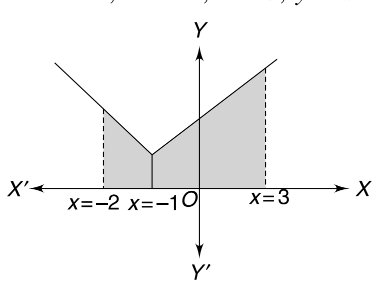
28. Given curves are
$\{x : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x \ge 0,\ y \ge 0\}$



Hence, the required area
$$= \int_0^a (y_2 - y_1)\,dx$$
$$= \int_0^a \left(\sqrt{a^2 - (x-a)^2} - a\sqrt{x}\right)dx$$
$$= \left(\frac{x-a}{2}\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) - \frac{2a}{3}x^{3/2}\right)_0^a$$
$$= \frac{2a}{3}a\sqrt{a} - \frac{a^2}{2}\sin^{-1}(-1)$$
$$= \left(\frac{2a^{5/2}}{3} + \frac{\pi a^2}{4}\right) \text{ sq.u.}$$

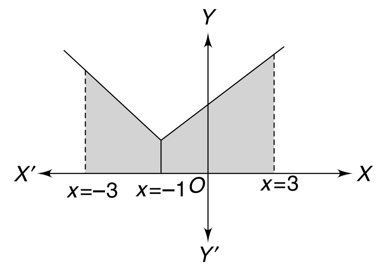
29. Do yourself.

30. Given curves are
$y = 1 + |x + 1|,\ x = -2,\ x = 3,\ y = 0.$



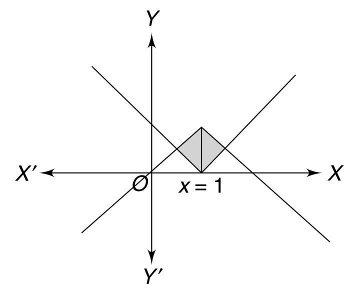
Hence, the required area
$$= \frac{1}{2} \times (2+1) \times 1 + \frac{1}{2} \times (1+2) \times 1 + \frac{1}{2} \times (2+5) \times 3$$
$$= \frac{1}{2}(3 + 2 + 21) = \frac{27}{2} \text{ sq.u.}$$

31. Given curves are
$y = 1 + |x + 1|,\ x = -3,\ x = 3,\ y = 0.$



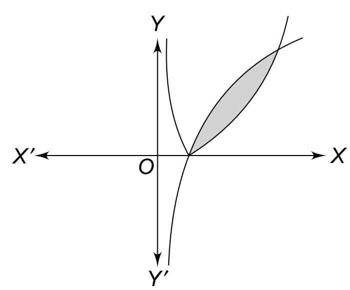
Hence, the required area
$$= \frac{1}{2} \times (3+1) \times 2 + \frac{1}{2} \times (1+2) \times 1 + \frac{1}{2} \times (2+5) \times 3$$
$$= \frac{1}{2}(4 + 3 + 21) = \frac{28}{2} = 14$$

32. Given curves are
$y = |x - 1|$ and $y = -|x - 1| + 1.$



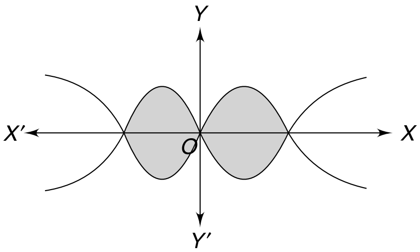
Hence, the required area
$$= 2\left(\frac{1}{2} \times 1 \times \frac{1}{2}\right) = \frac{1}{2} \text{ sq.u.}$$

33. Given curves are
$f(x) = \log x$ and $g(x) = (\log x)^2$



Hence, the required area
$$= \int_1^e (\log x - (\log x)^2)dx$$
$$= (x\log x - x - (x(\log x)^2 - 2(x\log x - x)))_1^e$$
$$= ((3(x\log x - x) - x(\log x)^2))_1^e$$
$$= (3 - e) \text{ sq.u.}$$

34. Given curve is $y^2 = x^2 - x^4$.



Hence, the required area

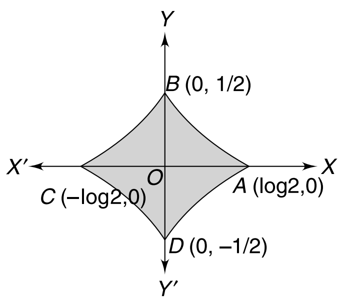
$$= 4\int_0^1 (x^2 - x^4)\,dx$$

$$= 4\left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1$$

$$= 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15} \text{ sq.u.}$$

35. Given curve is

$$|y| + \frac{1}{2} \le e^{-|x|}$$



Hence, the required area

$$= 4\left(\int_0^{\log 2} e^{-x}dx\right)$$

$$= 4(-e^{-x})_0^{\log 2}$$

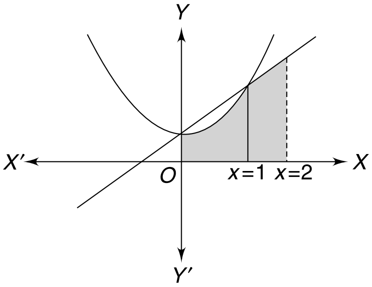
$$= 4(1 - e^{-\log 2})$$

$$= 4\left(1 - \frac{1}{2}\right) = 4 \times \frac{1}{2} = 2 \text{ sq.u.}$$

36. Given curve is

$$f(x, y) = [(x, y): 0 \le y \le x^2 + 1,$$

$$0 \le y \le x + 1,\ 0 \le x \le 2]$$



Hence, the required area

$$= \int_0^1 (x^2 + 1)dx + \int_1^2 (x + 1)\,dx$$

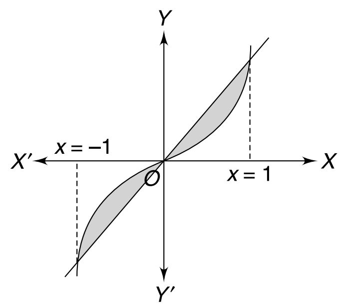
$$= \left(\frac{x^3}{3} + x\right)_0^1 + \left(\frac{x^2}{2} + x\right)_1^2$$

$$= \left(\frac{1}{3} + 1\right) + \left(4 - \frac{3}{2}\right)$$

$$= \frac{4}{3} + \frac{5}{3} = 3 \text{ sq.u.}$$

37. Given curves are

$$y = x \text{ and } y = x^3.$$



Hence, the required area

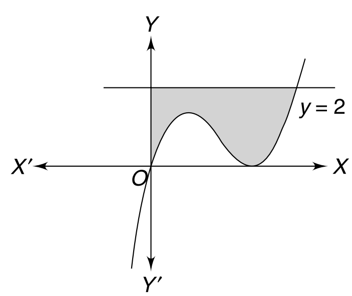
$$= 2\int_0^1 (x - x^3)\,dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = 2 \times \frac{2}{4} = 1 \text{ sq.u.}$$

38. Given curve is

$$y = x(x - 1)$$



Hence, the required area

$$= \int_0^2 (2 - x(x - 1)^2)\,dx$$

$$= \int_0^2 (2 - x(x^2 - 2x + 1))\,dx$$

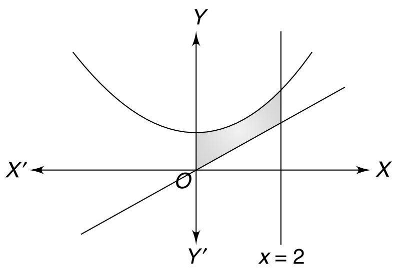
$$= \int_0^2 (2 - (x^3 - 2x^2 + x))\,dx$$

$$= \left(2x - \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{x^2}{2}\right)_0^2$$

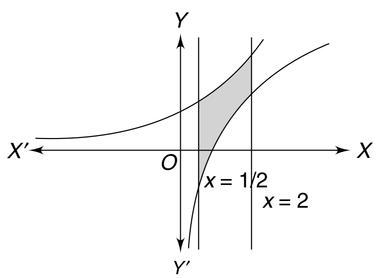
$$= \left(4 - 4 + \frac{16}{3} - 2\right)$$

$$= \frac{10}{3} \text{ sq.u.}$$

39. Given curves are
$y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$.



40. Given curves are
$y = \log_e x$ and $y = 2^x$, $x = 1/2$ and $x = 2$.



Hence, the required area

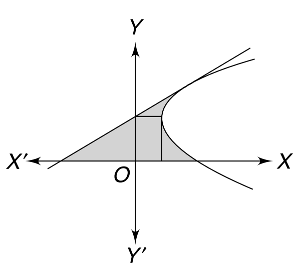
$$= \int_{1/2}^{2} (2x - \log x)\,dx$$

$$= \left(\frac{2^x}{\log 2} - (x \log x - x)\right)_{1/2}^{2}$$

$$= \left(\frac{4}{\log 2} - 21\log 2 + 2 - \frac{\sqrt{2}}{\log 2} + \frac{1}{2}\log\left(\frac{1}{2}\right) - \frac{1}{2}\right)$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right) \text{ sq.u.}$$

41. Given curve is $(y - 2)^2 = (x - 1)$ and the equation of the tangent to the curve at (2, 3) is $x - 2y + 4 = 0$.



Hence, the required area

$$= \int_0^3 (x_2 - x_1)\,dy$$

$$= \int_0^3 ((y - 2)^2 + 1 - (2y - 4))\,dy$$

$$= \int_0^3 (y^2 - 4y + 4 + 1 - 2y + 4)\,dy$$

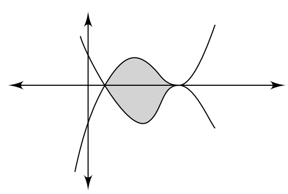
$$= \int_0^3 (y^2 - 6y + 9)dy$$

$$= \left(\frac{y^3}{3} - 3y^2 + 9y\right)_0^3$$

$$= 9 - 27 + 27$$

$$= 9 \text{ sq.u.}$$

42. Given curve is
$y^2 = (x - 1)(x - 2)^2$



Hence, the required area

$$= 2\int_1^2 (x - 2)\sqrt{x - 1}\,dx$$

$$= 4\int_0^1 (t^2 - 1)t^2 dt$$

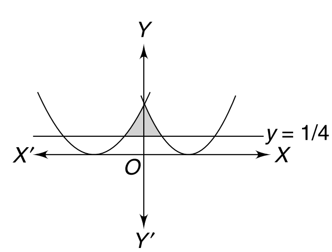
$$= \left|4\int_0^1 (t^4 - t^2)dt\right|$$

$$= \left|4\left(\frac{t^5}{5} - \frac{t^3}{3}\right)_0^1\right|$$

$$= \left|4\left(\frac{1}{5} - \frac{1}{3}\right)\right|$$

$$= \frac{8}{15} \text{ sq.u.}$$

43. Given curves are
$y = (x - 1)^2$, $y = (x + 1)^2$ and $y = 1/4$.



Hence, the required shaded area

$$= \int_{-1/2}^{0} \left((x+1)^2 - \frac{1}{4}\right) dx + \int_{0}^{1/2} \left((x-1)^2 - \frac{1}{4}\right) dx$$

$$= \left(\frac{(x+1)^3}{3} - \frac{1}{4}x\right)_{-1/2}^{0} + \left(\frac{(x-1)^3}{3} - \frac{1}{4}x\right)_{0}^{1/2}$$

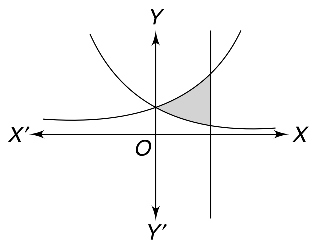
$$= \frac{1}{3} - \left(\frac{1}{24} + \frac{1}{8}\right) - \frac{1}{24} - \frac{1}{8} + \frac{1}{3}$$

$$= \frac{1}{3} - 2\left(\frac{1}{24} + \frac{1}{8}\right) + \frac{1}{3}$$

$$= \frac{2}{3} - \frac{2}{6} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq.u.}$$

44. Given curves are

$y = e^x$, $y = e^{-x}$ and the straight line $x = 1$



Hence, the required area

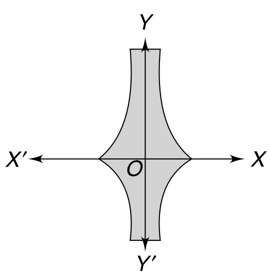
$$= \int_{0}^{1} (e^x - e^{-x})$$

$$= (e^x + e^{-x})_0^1$$

$$= \left(e + \frac{1}{e} - 2\right) \text{ sq.u.}$$

45. Given curves are

$y = \log x$, $y = \log|x|$, $y = |\log x|$ and $y = |\log|x||$



Hence, the required area

$$= 4\int_{-\infty}^{0} x\,dy$$

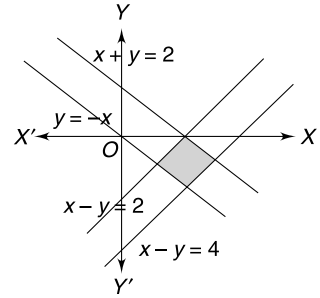
$$= 4\int_{-\infty}^{0} e^y dy$$

$$= 4(e^y)_{-\infty}^{0}$$

$$= 4(e^0 - 0) = 4(1 - 0) = 4 \text{ sq.u.}$$

46. Given curve is

$$|x - 2| + |y + 1| = 1$$



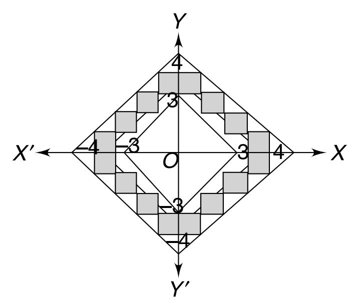
Thus, the length of each side

$$= \sqrt{(2-1)^2 + (0+1)^2} = \sqrt{2}$$

Hence, the required area

$$= (\sqrt{2})^2 \text{ sq.u.}$$

47. Given curve is $[x] + [y] = 3$



If $[x] = 0, 1, 2, 3$ then $[y] = 3, 2, 1, 0$.

Hence, the required area

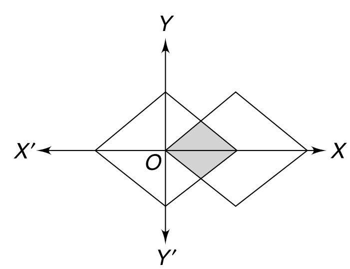
$$= 4(1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 1)$$

$$= 4 \times 4$$

$$= 16 \text{ sq.u.}$$

48. Given curves are

$|x| + |y| = 1$ and $|x + 1| + |y| = 1$.



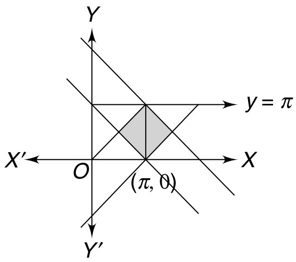
Hence, the required area

$$= 2 \times \frac{1}{2} \times 1 \times \frac{1}{2}$$

$$= \frac{1}{2} \text{ sq.u.}$$

49. Given curves are

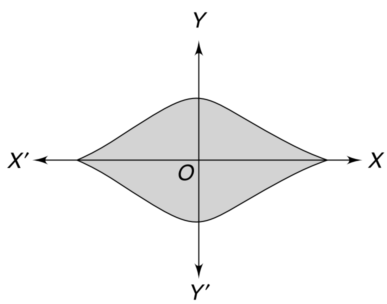
$$y = \cos^{-1}(\cos x) \text{ and } |x - \pi| + \left|y - \frac{\pi}{2}\right| = \frac{\pi}{2}$$



Hence, the required area

$$= 2 \times \frac{1}{2} \times \pi \times \frac{\pi}{2} = \frac{\pi^2}{2} \text{ sq.u.}$$

50. Since the given curve is symmetrical about x-axis as well as y-axis.



Hence, the required area

$$= 4\int_0^{\sqrt{5}+1} y\,dx$$

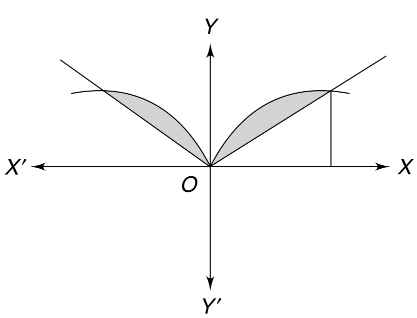
$$= 4\int_0^{\sqrt{5}+1} (5 - (x - 1)^2)dx$$

$$= 4\left(5x - \frac{(x-1)^3}{3}\right)_0^{\sqrt{5}+1}$$

$$= \frac{8}{3}(7 + 5\sqrt{5}) \text{ sq.u.}$$

51. Given curves are

$$y = \sqrt{|x|} \text{ and } y = |x|$$



Hence, the required area

$$= 2\int_0^1 (\sqrt{x} - x)\,dx$$

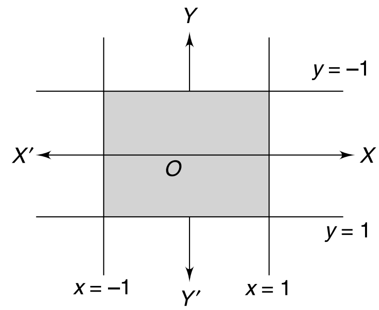
$$= 2\left(\frac{2}{3}x^{2/3} - \frac{x^2}{2}\right)_0^1$$

$$= 2\left(\frac{2}{3} - \frac{1}{2}\right)$$

$$= 2\left(\frac{4-3}{6}\right) = 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq.u.}$$

52. Given curve is $|x + y| + |x - y| \le 2$

$$= \begin{cases} x \le 1, x \ge -1 \\ y \le 1, y \ge -1 \end{cases}$$



Hence, the required area

$$= (2)^2 = 4 \text{ sq.u.}$$

53. Given curves are

$$y = x,\ y = x + \sin x,\ 0 \le x \le \pi$$

Hence, the required area

$$= \int_0^{\pi} (x\sin x - x)\,dx$$

$$= \int_0^{\pi} (\sin x)\,dx$$

$$= -(\cos x)_0^{\pi} = -(-1 - 1) = 2 \text{ sq.u.}$$

54. Hence, the required area

$$= \int_{\frac{\pi}{4}}^{\beta} f(x)dx = \left(\beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta\right)$$

Applying, Newton and Leibnitz rule, we get

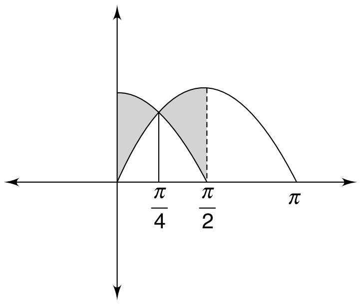
$$f(\beta) = \beta\cos\beta + \sin\beta - \frac{\pi}{4}\sin\beta + \sqrt{2}$$

$$f\left(\frac{\pi}{2}\right) = 0 + 1 - \frac{\pi}{4} + \sqrt{2}$$

$$f\left(\frac{\pi}{2}\right) = \left(1 - \frac{\pi}{4}\right) + \sqrt{2}$$

55. Given curves are $y = \sin x$ and $y = \cos x$

$$x = 0 \text{ and } x = \frac{\pi}{2}.$$



Hence, the required area

$$\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)_0^{\frac{\pi}{4}} - (\cos x + \sin x)_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

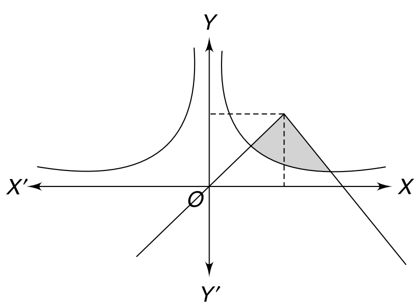
$$= \left(\frac{2}{\sqrt{2}} - 1\right) - \left(1 - \frac{2}{\sqrt{2}}\right)$$

$$= 2\left(\frac{2}{\sqrt{2}} - 1\right)$$

$$= 2(\sqrt{2} - 1) \text{ sq.u.}$$

56. Given curves are

$y = 2 - |2 - x|$ and $y = \dfrac{3}{|x|}$.



Hence, the required area

$$= \int_{\sqrt{3}}^{2} \left(x - \frac{3}{x}\right)dx + \int_{2}^{2+\sqrt{7}} \left(4 - x - \frac{3}{x}\right)dx$$

$$= \left(\frac{x^2}{2} - 3\log x\right)_{\sqrt{3}}^{2} + \left(4x - \frac{x^2}{2} - 3\log x\right)_{2}^{2+\sqrt{7}}$$

$$= \left(2 - 3\log 2 - \frac{3}{2} + \frac{3}{2}\log 2\right)$$

$$+ \left(4(2 + \sqrt{7}) - \frac{(2 + \sqrt{7})^2}{2} - 3\log(2 + \sqrt{7}) - 8 + 2 + 3\log 2\right)$$

$$= \left(\frac{1}{2} + \frac{3}{2}\log 2\right) + 3\log\left(\frac{2}{2 + \sqrt{7}}\right) + \left(2\sqrt{7} - \frac{11}{2}\right)$$

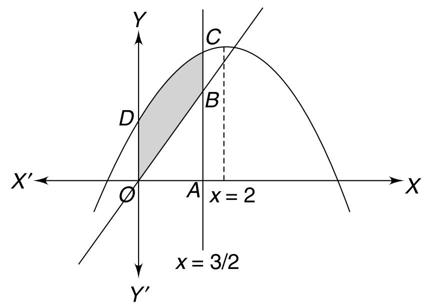
$$= \left(2\sqrt{7} + \frac{3}{2}\log 2 - 5\right) + 3\log\left(\frac{2}{2 + \sqrt{7}}\right)$$

$$= (2\sqrt{7} - 5) + 3\log\left(\frac{2}{2 + \sqrt{7}}\right) + 3\log(\sqrt{2})$$

$$= (2\sqrt{7} - 5) + 3\log\left(\frac{2\sqrt{2}}{2 + \sqrt{7}}\right)$$

57. Given curve is $y = 4x - x^2$

$\Rightarrow \quad y = 5 - (x - 2)^2$



Now, the area *OABO*

$$= \int_0^{3/2} (mx)\,dx$$

$$= \left(\frac{mx^2}{2}\right)_0^{3/2} = \left(\frac{9m}{8}\right)$$

Hence, the area *OABCDO*

$$= \int_0^{3/2} (1 + 4x - x^2)\,dx$$

$$= \left(x + 2x^2 - \frac{x^3}{x}\right)_0^{3/2}$$

$$= \left(\frac{3}{2} + 2\cdot\frac{9}{4} - \frac{27}{24}\right)$$

$$= \left(6 - \frac{9}{8}\right) = \frac{39}{8} \text{ sq.u}$$

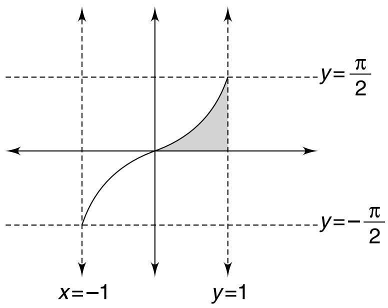
According to the question, $\dfrac{9m}{8} = \dfrac{1}{2}\cdot\dfrac{39}{8}$

$\Rightarrow \quad m = \dfrac{13}{6}$

Hence, the value of m is $\dfrac{13}{6}$.

58. Do yourself.

59. Given curve is $y = \sin^{-1}x$ and y-axis between $y = 0$ and $y = \pi/2$.



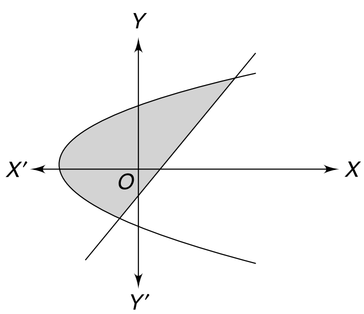
Hence, the required area

$$= \int_0^{\frac{\pi}{2}} (1 - \sin x)\, dx$$

$$= (y + \cos y)_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - 1\right) \text{ sq.u.}$$

60. Given curves are $y^2 = 2x + 6$ and $y = x - 1$



On solving, we get, $(x - 1)^2 = 2x + 6$

$$y^2 = 2(y + 1) + 6$$

$$y^2 - 2y - 8 = 0$$

$$(y - 4)(y + 2) = 0$$

$$y = -2, 4$$

Hence, the required area

$$= \int_{-2}^{4} (x_2 - x_1)\, dy$$

$$= \int_{-2}^{4} \left(\frac{y^2 - 6}{2} - (y + 1)\right) dy$$

$$= \frac{1}{2} \int_{-2}^{4} (y^2 - 6 - 2y - 2)\, dy$$

$$= \frac{1}{2} \int_{-2}^{4} (y^2 - 2y - 8)\, dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3} - y^2 - 8y\right)_{-2}^{4}$$

$$= \left|\frac{1}{2}\left(\frac{64}{3} - 48 + \frac{8}{3} + 4 - 16\right)\right|$$

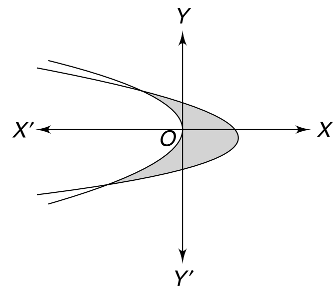
$$= \left|\frac{1}{2}(24 - 48 + 4 - 16)\right|$$

$$= \left|\frac{1}{2}(28 - 64)\right|$$

$$= \frac{1}{2}|(28 - 64)| = \frac{36}{2} = 18 \text{ sq.u.}$$

61. Given curves are

$$x = -2y^2 \text{ and } x = 1 - 3y^2.$$



On solving, we get,

$$-2y^2 = 1 - 3y^2$$

$$y^2 = 1$$

$$y = \pm 1$$

Hence, the required area

$$= \int_{-1}^{1} (x_2 - x_1)\, dy$$

$$= \int_{-1}^{1} ((-2y^2) - (1 - 3y^2))\, dy$$

$$= \int_{-1}^{1} (y^2 - 1)\, dy$$

$$= 2\int_{0}^{1} (y^2 - 1)\, dy$$

$$= 2\left(\frac{y^3}{3} - y\right)_0^1$$

$$= 2\left|\left(\frac{1}{3} - 1\right)\right| = \frac{4}{3} \text{ sq.u.}$$

62. Given curves are $y = \tan^{-1}x$ and $y = \cot^{-1}x$ and the y-axis.

Hence, the required area

$$= \int_0^{\frac{\pi}{4}} \tan y\, dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot y\, dy$$

$$= (\log(\sec y))_0^{\frac{\pi}{4}} + (\log(\sin y))_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

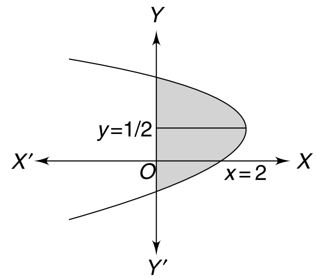
$$= \log(\sqrt{2}) + \left(0 - \log\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= 2\log(\sqrt{2})$$

$$= \log(2) \text{ sq.u.}$$

63. Given curve is

$x = y^2 + y + 2$ and the y-axis.



Hence, the required area

$$= \int_{-1}^{2} x\,dy$$

$$= \int_{-1}^{2} (-y^2 + y + 2)dy$$

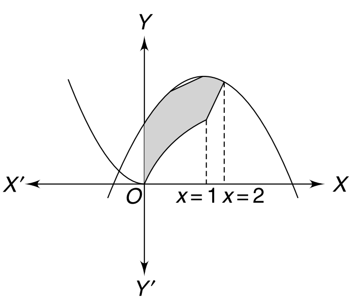
$$= \left(-\frac{y^3}{3} + \frac{y^2}{2} + 2y\right)_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 2 + 4 + \frac{1}{3} - \frac{1}{2} - 2\right)$$

$$= \left(4 - \frac{7}{3} - \frac{1}{2}\right) = \frac{7}{6} \text{ sq.u.}$$

64. Given curves are $y = \sqrt{x}$, $y = x^2$

$y = -(x^2 - 2x - 4) = 5 - (x - 1)^2$



Hence, the required area

$$= \int_{0}^{2} (-x^2 + 2x + 4)dx - \int_{0}^{1} \sqrt{x}\,dx - \int_{1}^{2} x^2 dx$$

$$= \left(-\frac{x^3}{3} + x^2 + 4x\right)_0^2 - \left(\frac{2}{3}x^{3/2}\right)_0^1 - \left(\frac{x^3}{3}\right)_1^2$$

$$= \left(-\frac{18}{3} + 12\right) - \left(\frac{2}{3}\right) - \left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \frac{28}{3} - \frac{2}{3} - \frac{7}{3}$$

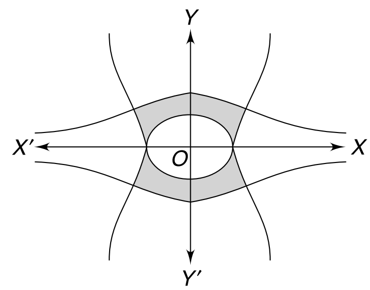
$$= \frac{28 - 9}{3}$$

$$= \frac{19}{3} \text{ sq.u.}$$

65. Find the area of the region bounded by the curves $4|y| = |4 - x^2|$ and $|y|(x^2 + 4) = 12$.

65. Given curves are

$4|y| = |4 - x^2|$ and $|y|(x^2 + 4) = 12$.



Hence, the required area

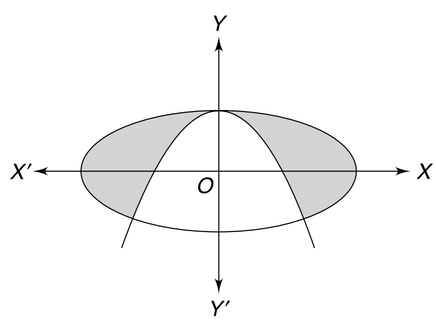
$$= 4\left[\int_0^{2\sqrt{2}} \left(\frac{12}{x^2 + 4}\right)dx - \left(\int_0^2 \left(\frac{4 - x^2}{4}dx\right) + \left(\int_2^{2\sqrt{2}} \left(\frac{x^2 - 4}{4}\right)dx\right)\right)\right]$$

$$= 4\left[\left(6\tan^{-1}\left(\frac{x}{2}\right)\right)_0^{2\sqrt{2}} - \left(x - \frac{x^3}{12}\right)_0^2 - \left(\frac{x^3}{12} - x\right)_2^{2\sqrt{2}}\right]$$

$$= 4\left[6\tan^{-1}(\sqrt{2}) - \frac{4}{3} + \frac{4 - 2\sqrt{2}}{3}\right]$$

$$= \frac{4}{3}\left(18\tan^{-1}(\sqrt{2}) - 2\sqrt{2}\right) \text{ sq.u.}$$

66. Given curves are $x^2 + 2y^2 = 2$ and $y = 1 - x^2$



Hence, the required area

$$= 2\int_{-1/2}^{1} \left(\sqrt{2 - 2y^2} - \sqrt{1 - y}\right)dy$$

$$= 2\sqrt{2}\int_{-1/2}^{1}\sqrt{1-y^2}\,dy - 2\int_{-1/2}^{1}\sqrt{1-y}\,dy$$

$$= 2\sqrt{2}\left(\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}(y)\right)_{-1/2}^{1}$$

$$-2\left(\frac{2}{-3}(1-y)^{3/2}\right)_{-1/2}^{1}$$

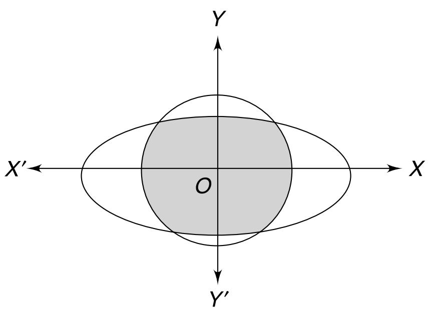
$$= 2\sqrt{2}\left(\frac{\pi}{4} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right) - \frac{4}{3}\left(\frac{3}{2}\right)^{3/2}$$

$$= 2\sqrt{2}\left(\frac{\pi}{4} + \frac{\sqrt{3}}{8} + \frac{\pi}{12}\right) - \frac{4}{3}\left(\frac{3\sqrt{3}}{2\sqrt{2}}\right)$$

$$= 2\sqrt{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) - 2\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$$

$$= \left(2\sqrt{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) - \sqrt{6}\right) \text{ sq.u.}$$

67. Given curves are $x^2 + y^2 = 4$ and $x^2 + 4y^2 = 9$



Solving the above equations, we get,

$$x^2 + 4(4 - x^2) = 9$$

$$\Rightarrow \quad 3x^2 = 7$$

$$\Rightarrow \quad x = \sqrt{\frac{7}{3}}$$

Hence, the required area

$$= 4\left(\int_{0}^{\sqrt{7/3}} \frac{1}{2}\sqrt{9-x^2}\,dx + \int_{\sqrt{7/3}}^{2}\sqrt{4-x^2}\,dx\right)$$

$$= 4\left[\frac{1}{2}\left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^{\sqrt{7/3}}\right.$$

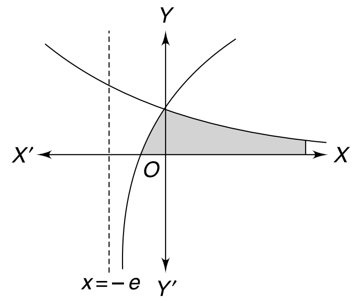
$$\left. + \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{4}\right)\right)\Bigg]_{\sqrt{7/3}}^{2}$$

$$= \left\{4\pi + 9\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{7}{3}}\right) - 8\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{7}{3}}\right)\right\}$$

68. Given curves are

$$y = \ln(x+e) \text{ and } x = \ln\left(\frac{1}{y}\right)$$

$$\Rightarrow \quad y = \ln(x+e),\ y = e^{-x},\ y = 0$$



Hence, the required area

$$= \int_{1-e}^{0}\log(x+e)\,dx + \int_{0}^{\infty}e^{-x}dx$$

$$= ((x+e)\log(x+e) - (x+e))_{1-e}^{0} - (e^{-x})_0^{\infty}$$

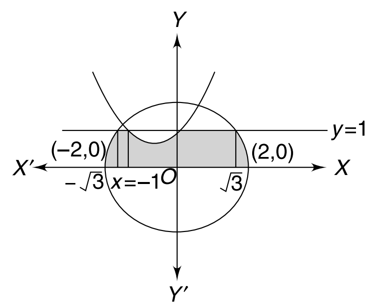
$$= (e\log e - e) - (1\log 1 - (1)) - (e^{-x})_0^{\infty}$$

$$= (1 - (0 - 0))$$

$$= 2 \text{ sq.u.}$$

69. Given curves are $x^2 + y^2 = 4$, $y = x^2 + x + 1$ and

$$y = \left[\sin^2\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right)\right] \text{ and } y = 0.$$



Hence, the required area

$$= (\sqrt{3}\times 1) + (\sqrt{3} - 1)$$

$$+\int_{-1}^{0}(x^2 + x + 1)\,dx + 2\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$$

$$= (2\sqrt{3} - 1) + \left(\frac{x^3}{3} + \frac{x^2}{2} + x\right)_{-1}^{0}$$

$$+2\left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right)_{\sqrt{3}}^{2}$$

$$= (2\sqrt{3} - 1) + \left(0 - \left(-\frac{1}{3} + \frac{1}{2} - 1\right)\right)$$

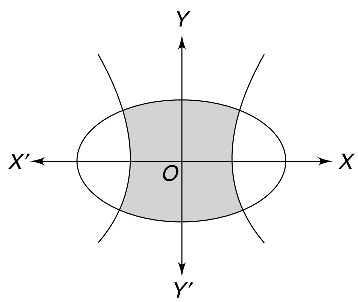
$$+2\left\{(0 + \pi) - \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)\right\}$$

$$= \left\{(2\sqrt{3} - 1) + \frac{5}{6} + \frac{2\pi}{3} - \sqrt{3}\right\}$$

$$= \left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right) \text{ sq.u.}$$

70. Given curves are

$$\frac{x^2}{4} + y^2 = 1 \text{ and } \frac{x^2}{2} - y^2 = 1$$



Solving the above equations, we get

$$\frac{x^2}{4} + \frac{x^2}{2} = 2$$

$$\Rightarrow \quad \frac{3x^2}{4} = 2$$

$$\Rightarrow \quad x^2 = \frac{8}{3}$$

Now, $\quad y^2 = 1 - \frac{x^2}{4}$

$$\Rightarrow \quad y^2 = 1 - \frac{8}{4 \times 3} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}$$

Hence, the required area

$$= 4\left(\int_0^{1/\sqrt{3}} (\text{Hyperbola})\,dy + \int_{1/\sqrt{3}}^1 (\text{Ellipse})\,dy\right)$$

$$= 4\left(\int_0^{1/\sqrt{3}} \sqrt{2}\left(\sqrt{1 + y^2}\right)dy + \int_{1/\sqrt{3}}^1 2\sqrt{1 + y^2}dy\right)$$

$$= 4\left[\sqrt{2}\left(\frac{y}{2}\sqrt{1 + y^2} + \frac{1}{2}\log\left|y + \sqrt{y^2 + 1}\right|\right)_0^{1/\sqrt{3}}\right.$$

$$\left. + 2\left(\frac{y}{2}\sqrt{1 + y^2} + \frac{1}{2}\log\left|y + \sqrt{y^2 + 1}\right|\right)_{1/\sqrt{3}}^1\right]$$

$$= 4\left[\sqrt{2}\left(\frac{1}{2\sqrt{3}} \times \frac{2}{\sqrt{3}} + \frac{1}{2}\log\left|\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right|\right)\right.$$

$$\left. + 2\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\log(\sqrt{2}+1) - \frac{1}{2\sqrt{3}} \times \frac{2}{\sqrt{3}} - \frac{1}{4}\log 3\right)\right]$$

$$= 4\left[\sqrt{2}\left(\frac{1}{3} + \frac{1}{4}\log(3)\right)\right.$$

$$\left. + 2\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\log(\sqrt{2} + 1) - \frac{1}{3} - \frac{1}{4}\log 3\right)\right] \text{ sq.u.}$$

71. Let the area be A.

Then $\quad A = \int_0^2 \left(\frac{x^3}{3} - x^2 + a\right)dx$

$$= \left(\frac{x^4}{12} - \frac{x^3}{3} + ax\right)_0^2$$

$$= \left(\frac{4}{3} - \frac{8}{3} + 2a\right)$$

$$= \left(2a - \frac{4}{3}\right)$$

$$\frac{dA}{da} = 2 > 0$$

So, the area is minimum.

It will be minimum when $A = 0$.

Thus, $\quad 2a - \frac{4}{3} = 0$

$$\Rightarrow \quad a = \frac{2}{3}$$

Hence, the value of a is $\frac{2}{3}$.

72. Hence, the required area

$$= A = \int_0^1 (a^2x^2 + ax + 1)\,dx$$

$$= \left(\frac{a^2x^3}{3} + \frac{ax^2}{2} + x\right)_0^1$$

$$= \left(\frac{a^2}{3} + \frac{a}{2} + 1\right)$$

$$\Rightarrow \quad \frac{dA}{da} = \frac{2a}{3} + \frac{1}{2}$$

$$\Rightarrow \quad \frac{d^2A}{da^2} = \frac{2}{3} > 0$$

So, the area is minimum.

For maximum or minimum, $\frac{2a}{3} + \frac{1}{2} = 0$

$$\Rightarrow \quad \frac{2a}{3} = -\frac{1}{2}$$

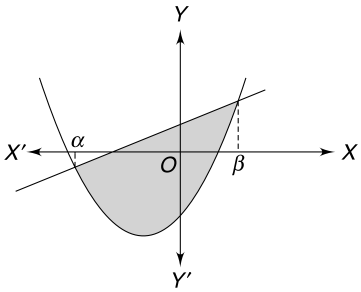
$$\Rightarrow \quad a = -\frac{3}{4}$$

Hence, the value of a is $-3/4$

73. Do yourself.

74. Given curves are

$$y = x^2 + 2x - 3 \text{ and } y = mx + 1$$



Solving, we get

$$x^2 + 2x - 3 = mx + 1$$

$$x^2 + (2 - m)x - 4 = 0$$

Let α, β be the roots such that

$$\alpha + \beta = m - 2, \ \alpha\beta = -4$$

Hence, the required area

$$A = \int_\alpha^\beta (\text{Line} - \text{Parabola})\, dx$$

$$= \int_\alpha^\beta ((mx + 1) - (x^2 + 2x - 3))\, dx$$

$$= \left(\frac{mx^2}{2} + x - \frac{x^3}{3} - x^2 + 3x\right)_\alpha^\beta$$

$$= \left(\left(\frac{m\beta^2}{2} + 4\beta - \frac{\beta^3}{3} - \beta^2\right) - \left(\frac{m\alpha^2}{2} + 4\alpha - \frac{\alpha^3}{3} - \alpha^2\right)\right)$$

$$= \left(\left(\frac{m}{2}(\beta^2 - \alpha^2) + 4(\beta - \alpha) - \frac{(\beta^3 - \alpha^3)}{3} - (\beta^2 - \alpha^2)\right)\right)$$

$$= \left(\left(\left(\frac{m}{2} - 1\right)(\beta^2 - \alpha^2) + 4(\beta - \alpha) - \frac{(\beta^3 - \alpha^3)}{3}\right)\right)$$

$$= (\beta - \alpha)\left(\left(\frac{m}{2} - 1\right)(\beta + \alpha) + 4 - \frac{(\beta^2 + \alpha\beta + \alpha^2)}{3}\right)$$

$$= (\beta - \alpha)\left(\frac{(m-2)^2}{2} + 4 - \frac{(m-2)^2 + 4}{3}\right)$$

$$= (\beta - \alpha)\left(\frac{(m-2)^2}{6} + \frac{16}{3}\right)$$

$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\left(\frac{(m-2)^2}{6} + \frac{16}{3}\right)$$

$$= \sqrt{(m-2)^2 + 16}\left(\frac{(m-2)^2}{6} + \frac{16}{3}\right)$$

The area is minimum, if $m = 2$ and the least area is $= \frac{64}{3}$.

75. Hence, the required area

$$= A = \int_a^{2a} \left(\frac{x}{6} + \frac{1}{x^2}\right) dx$$

$$= \left(\frac{x^2}{12} - \frac{1}{x}\right)_a^{2a}$$

$$= \left(\frac{a^2}{3} - \frac{1}{2a} - \frac{a^2}{12} + \frac{1}{a}\right)$$

$$= \left(\frac{a^2}{4} + \frac{1}{2a}\right)$$

$$\frac{dA}{da} = \frac{a}{2} - \frac{1}{2a^2}$$

$$\Rightarrow \quad \frac{d^2A}{da^2} = \frac{1}{2} + \frac{1}{a^3}$$

Now for maximum or minimum,

$$\frac{dA}{da} = 0$$

$$\Rightarrow \quad \frac{a}{2} - \frac{1}{2a^2} = 0$$

$$\Rightarrow \quad \frac{a}{2} = \frac{1}{2a^2}$$

$$\Rightarrow \quad a^3 = 1$$

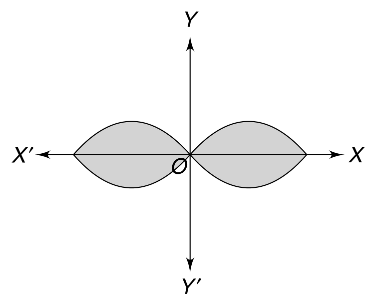
$$\Rightarrow \quad a = 1$$

Hence, the value of a is 1, when the area is least.

Level III

(Problems for JEE-Advanced)

1. Given curve is $a^2 y^2 = x^2(a^2 - x^2)$.



Hence, the required area

$$= 4\int_0^a y\, dx$$

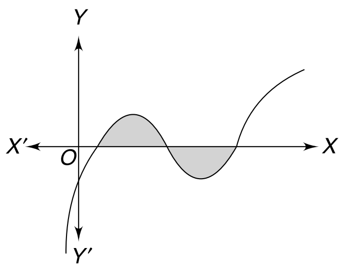
$$= 4\int_0^a \left(\frac{x}{a}\sqrt{a^2 - x^2}\right) dx$$

$$= 4\int_0^{\pi/2} a^2 \sin^2\theta \cos\theta\, d\theta, \quad (\text{Let } x = a\sin\theta)$$

$$= 4a^2 \int_0^1 t^2\, dt, \quad (\text{Let } t = \sin\theta)$$

$$= \frac{4a^2}{3} \text{ sq.u.}$$

2. Given curve is $y = (x - 1)(x - 2)(x - 3)$.



Hence, the required area

$$= \int_0^3 \{(x-1)(x-2)(x-3)\}dx$$

$$= -\int_0^1 \{(x-1)(x-2)(x-3)\}dx + \int_1^2 \{(x-1)(x-2)(x-3)\}dx - \int_2^3 \{(x-1)(x-2)(x-3)\}dx$$

$$= -\left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)\Bigg|_0^1 + \left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_1^2 - \left(\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right)_2^3$$

$$= -\left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right) + (4 - 8 + 22 - 12) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right)$$

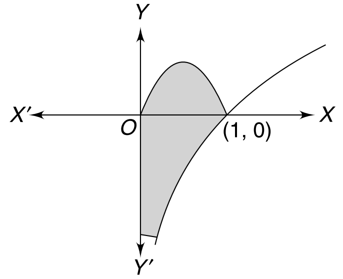
$$-\left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right) + (4 - 8 + 22 - 12)$$

$$= -2\left(\frac{23}{4} - 8\right) + 12 - \left(\frac{279}{4} - 72\right)$$

$$= \frac{46}{4} - \frac{279}{4} + 100$$

$$= 100 - \frac{233}{4} = \frac{400 - 233}{4} = \frac{167}{4} \text{ sq.u.}$$

3. Given curves are $y = \log_e x$, $y = \sin^4(\pi x)$ and $x = 0$



Hence, the required area

$$= \left|\int_0^1 \log x\, dx\right| + \int_0^1 \sin^4 x\, dx$$

$$= |x(\log x - 1)|_0^1 + \int_0^1 \left(\frac{1 - \cos 2\pi x}{2}\right)^2 dx$$

$$= 1 + \int_0^1 \left(\frac{1 - \cos 2\pi x}{2}\right)^2 dx$$

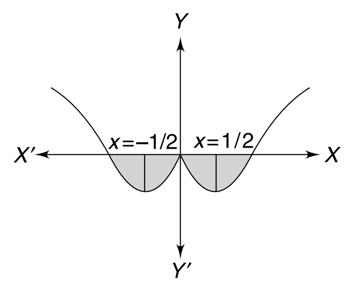
$$= 1 + \frac{1}{4}\int_0^1 (1 - 2\cos(2\pi x) + \cos^2(2\pi x))\, dx$$

$$= 1 + \frac{1}{4}\int_0^1 \left(1 - 2\cos(2\pi x) + \left(\frac{1 + \cos(4\pi x)}{2}\right)\right) dx$$

$$= 1 + \frac{1}{4}\left(\frac{3x}{2} - \frac{2\sin(2\pi x)}{2\pi} + \frac{\sin(4\pi x)}{8\pi}\right)_0^1$$

$$= 1 + \frac{1}{4}\left(\frac{3}{2} - 0\right) = 1 + \frac{3}{8} = \frac{11}{8}$$

4. Given curve is $y = 2x^4 - x^2$.



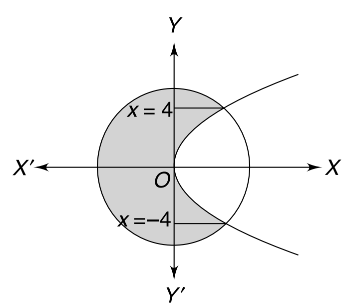
Hence, the required area

$$= \left|2\int_0^{1/2} (2x^4 - x^2)\, dx\right|$$

$$= \left|2\left(\frac{2x^5}{5} - \frac{x^3}{3}\right)_0^{1/2}\right|$$

$$= \left|2\left(\frac{1}{80} - \frac{1}{24}\right)\right| = \frac{7}{120} \text{ sq.u.}$$

5. Given curves are $x^2 + y^2 = 64$ and $y^2 = 12x$.



Hence, the required area

$= \text{area of a semi-circle} - 2\left(\int_0^4 P\,dx + \int_4^8 C\,dx\right)$

$= 64\pi - 2\left(\int_0^4 \sqrt{12}\sqrt{x}\,dx + \int_4^8 \sqrt{64 - x^2}\,dx\right)$

$= 64\pi - 2\left(2\sqrt{3} \times \frac{3}{2}x^{3/2}\right)_0^4$

$\quad -2\left(\frac{x}{2}\sqrt{64 - x^2} + 32\sin^{-1}\left(\frac{x}{2}\right)\right)_4^8$

$= 64\pi - 24\sqrt{3} - 2\left(\frac{32\pi}{2} - 8\sqrt{3} - 32\sin^{-1}\left(\frac{1}{2}\right)\right)$

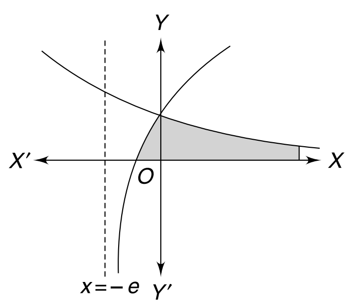
$= 32\pi - 8\sqrt{3} + \frac{64\pi}{3}$

$= \left(\frac{16\pi}{3} - 8\sqrt{3}\right)$ sq.u.

6. Given curves are

$y = \ln(x + e)$ and $x = \ln\left(\frac{1}{y}\right)$

$\Rightarrow y = \ln(x + e),\ y = e^{-x},\ y = 0$



Hence, the required area

$= \int_{1-e}^{0} \log(x + e)dx + \int_0^{\infty} e^{-x}\,dx$

$= ((x + e)\log(x + e) - (x + e))_{1-e}^{0} - (e^{-x})_0^{\infty}$

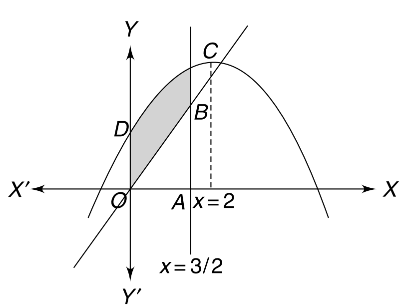
$= (e\log e - e) - (1\log 1 - (1)) - (e^{-x})_0^{\infty}$

$= (1 - (0 - 1))$

$= 2$ sq.u.

7. Given curve is $y = 1 + 4x - x^2$

$\Rightarrow y = 5 - (x - 2)^2$



Now, the area *OABO*

$= \int_0^{3/2} (mx)\,dx$

$= \left(\frac{mx^2}{2}\right)_0^{3/2} = \left(\frac{9m}{8}\right)$

Hence, the area *OABCDO*

$= \int_0^{3/2} (1 + 4x - x^2)\,dx$

$= \left(x + 2x^2 - \frac{x^3}{3}\right)_0^{3/2}$

$= \left(\frac{3}{2} + 2\cdot\frac{9}{4} - \frac{27}{24}\right)$

$= \left(6 - \frac{9}{8}\right) = \frac{39}{8}$ sq.u.

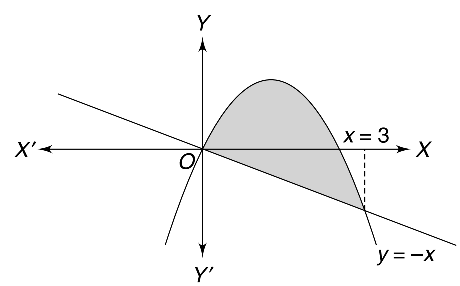
According to the question,

$\frac{9m}{8} = \frac{1}{2}\cdot\frac{39}{8}$

$\Rightarrow \quad m = \frac{13}{6}$

Hence, the value of m is $\frac{13}{6}$.

8. Given curve is



Hence, the required area

$= \int_0^3 (y_2 - y_1)\,dx$

$= \int_0^3 (2x - x^2 + x)\,dx$

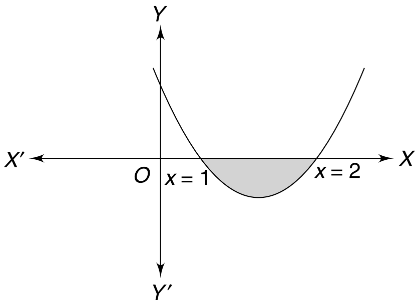
$= \int_0^3 (3x - x^2)\,dx$

$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)_0^3$

$= \left(\frac{27}{2} - \frac{27}{3}\right)$

$= \left(\frac{27 - 18}{2}\right) = \frac{9}{2}$ sq.u.

9. Given curve is



Hence, the required area

$$= \left|\int_1^2 \{(x-1)(x-2)\}dx\right|$$

$$= \left|\int_1^2 (x^2 - 3x + 2)\,dx\right|$$

$$= \left|\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)_1^2\right|$$

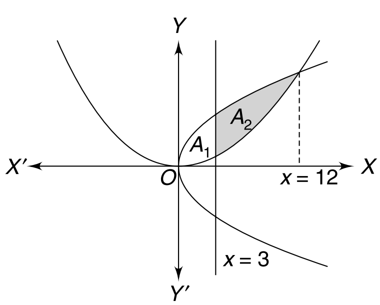
$$= \left|\left(\frac{8}{3} - 6 + 4 - \frac{1}{3} + \frac{3}{2} - 2\right)\right|$$

$$= \left|\left(\frac{7}{3} + \frac{3}{2} - 4\right)\right|$$

$$= \left|\left(\frac{14+9}{6} - 4\right)\right|$$

$$= \left(\frac{23}{6} - 4\right) = \frac{1}{6} \text{ sq.u.}$$

10. Given curves are $y^2 = 12x$ and $x^2 = 12y$ and $x = 3$.



Here, $A_1 = \int_0^3 \left(2\sqrt{3}\sqrt{x} - \frac{x^2}{12}\right)dx$

$$= \left(2\sqrt{3}\left(\frac{2}{3}x^{3/2}\right) - \frac{x^3}{36}\right)_0^3$$

$$= 2\sqrt{3}\left(\frac{2}{3} \times 3\sqrt{3}\right) - \frac{27}{36}$$

$$= 12 - \frac{3}{4} = \frac{45}{4} \text{ sq.u.}$$

Also, $A_2 = \int_3^{12} \left(2\sqrt{3}\sqrt{x} - \frac{x^2}{12}\right)dx$

$$= \left(2\sqrt{3}\left(\frac{2}{3}x^{3/2}\right) - \frac{x^3}{36}\right)_3^{12}$$

$$= \left(2\sqrt{3} \times \frac{2}{3} \times 12\sqrt{12}\,\frac{12}{36}\right) - \left(\frac{2}{3} \times 2\sqrt{3} \times 3\sqrt{3}\,\frac{27}{36}\right)$$

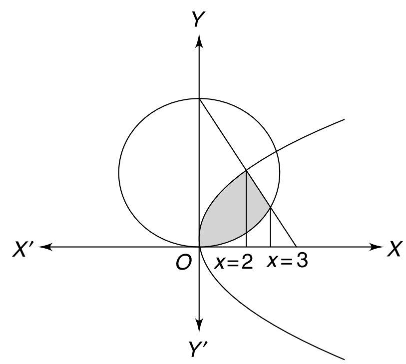
$$= (96 - 48) - \left(12 - \frac{3}{4}\right)$$

$$= 48 - 12 + \frac{3}{4} = 36 + \frac{3}{4} = \frac{147}{4}$$

Thus, $A_1 : A_2 = 45 : 147$

$= 15 : 49$

11. Given curves are
$x + y \le 6$. $x^2 + y^2 \le 6y$ and $y^2 \le 8x$



Hence, the required area

$$= \int_0^2 \left\{(2\sqrt{2x}) - \left(3 - \sqrt{9 - x^2}\right)\right\}dx$$

$$+ \int_2^3 \left\{(6 - x) - \left(3 - \sqrt{9 - x^2}\right)\right\}dx$$

$$= 2\sqrt{2}\cdot\frac{2}{3}(x^{3/2})_0^2 - \int_0^3 (3 - \sqrt{9 - x^2})dx + \left(6x - \frac{x^2}{2}\right)_2^3$$

$$= \frac{16}{3} + \left(18 - \frac{9}{2} - 12 + 2\right)$$

$$- \left(3x - \frac{x\sqrt{9-x^2}}{2} - \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^3$$

$$= \frac{16}{3} - 9 + \frac{9\pi}{4} + \frac{27}{2} - 10$$

$$= \frac{9\pi}{4} + \left(\frac{16}{3} + \frac{27}{2} - 19\right)$$

$$= \left(\frac{9\pi}{4} - \frac{1}{6}\right) \text{ sq.u.}$$

12. Given curve is

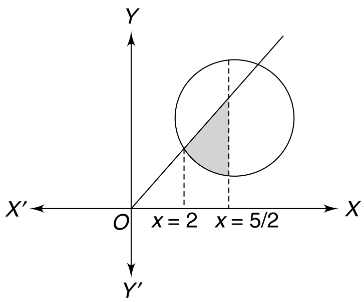
$x^2 + y^2 - 6x - 4y + 12 \le 0,\ y \le x,\ x \le 5/2$

$\Rightarrow (x-3)^2 + (y-2)^2 \le 1,\ y \le x,\ x \le 5/2$

$\Rightarrow (y-2)^2 = \{1 - (x-3)^2\}$

$\Rightarrow y = 2 \pm \sqrt{1-(x-3)^2}$

$\Rightarrow y = 2 - \sqrt{1-(x-3)^2}$ as $x \le \frac{5}{2}$



Hence, the required area

$= \frac{1}{2}\left(2 + \frac{5}{2}\right)\frac{1}{2} - \int_2^{5/2} \left(2 - \sqrt{1-(x-3)^2}\right)dx$

$= \frac{9}{8} - \int_2^{5/2} \left(2 - \sqrt{1-(x-3)^2}\right)dx$

$= \frac{9}{8} - \left(2x - \frac{(x-3)}{2}\sqrt{1-(x-3)^2} + \frac{1}{2}\sin^{-1}(x-3)\right)_2^{5/2}$

$= \frac{9}{8} - \left\{2\cdot\frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\right) \times \frac{\sqrt{3}}{2}\right.$

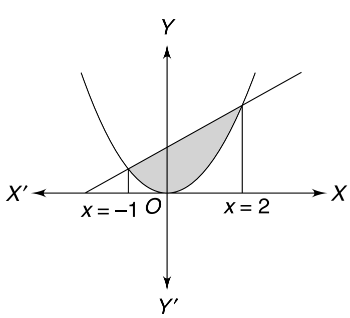
$\left. + \frac{1}{2}\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right)\right\}$

$= \frac{9}{8} - \left(1 + \frac{\sqrt{3}}{8} + \frac{1}{2}\left(-\frac{\pi}{6} + \frac{\pi}{2}\right)\right)$

$= \frac{9}{8} - \left(1 + \frac{\sqrt{3}}{8} + \frac{\pi}{6}\right)$

$= \left(\frac{1}{8} - \frac{\sqrt{3}}{8} - \frac{\pi}{6}\right)$ sq.u.

13. Given curves are $x^2 = y,\ y = x + 2$



Hence, the required area

$= \int_{-1}^{2} (x + 2 - x^2)dx$

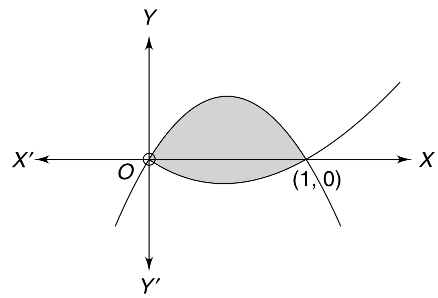
$= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 + \frac{1}{3}\right)$

$= \left(8 - \frac{1}{2} - \frac{7}{3}\right)$

$= \frac{48 - 3 - 14}{6} = \frac{31}{6}$ sq.u.

14. Given curve is $y = x\log x$ and $y = 2x - x^2$



Hence, the required area

$= \int_0^1 ((2x - 2x^2) - x\log x)\,dx$

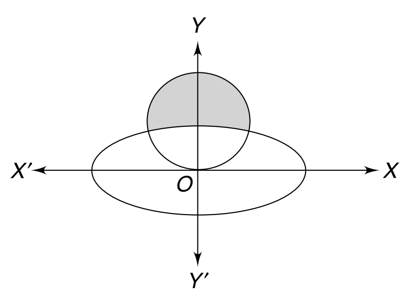
$= \left(x^2 - \frac{2}{3}x^3\right)_0^1 - \left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right)_0^1$

$= \left(1 - \frac{2}{3}\right) + \frac{1}{4}$

$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ sq.u.

15. Given curves are $x^2 + (y-1)^2 = 1$

and $c^2x^2 + y^2 = c^2$ where $c = \sqrt{2} - 1$



Hence, the required area

$= \pi\cdot(1)^2 - 2\int_0^{1/\sqrt{2}} \left(c\sqrt{1-x^2} - 1 - \sqrt{1-x^2}\right)$

$= \pi\cdot(1)^2 - 2\int_0^{1/\sqrt{2}} \left((c+1)\sqrt{1-x^2} - 1\right)$

$$= \pi \cdot (1)^2 - 2\left((c+1)\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right\} - x\right)_0^{1/\sqrt{2}}$$

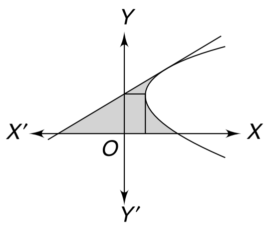
$$= \pi \cdot (1)^2 - 2\left(\sqrt{2}\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right\} - x\right)_0^{1/\sqrt{2}}$$

$$= \pi - 2\left(\sqrt{2}\left\{\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{\pi}{4}\right\} - \frac{1}{\sqrt{2}}\right)$$

$$= \pi - \frac{(\pi - 2)\sqrt{2}}{4}$$

$$= \frac{1}{4}\{(4-\sqrt{2})\pi + 2\sqrt{2}\} \text{ sq.u.}$$

16. Given curve is $(y-2)^2 = (x-1)$ and the equation of the tangent to the curve at (2, 3) is $x - 2y + 4 = 0$.



Hence, the required area

$$= \int_0^3 (x_2 - x_1)\,dy$$

$$= \int_0^3 ((y-2)^2 + 1 - (2y-4))\,dy$$

$$= \int_0^3 (y^2 - 4y + 4 + 1 - 2y + 4)\,dy$$

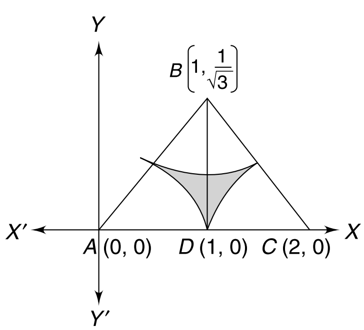
$$= \int_0^3 (y^2 - 6y + 9)\,dy$$

$$= \left(\frac{y^3}{3} - 3y^2 + 9y\right)_0^3$$

$$= 9 - 27 + 27$$

$$= 9 \text{ sq.u.}$$

17. Here D, E and F are the mid-points of the sides of the trangle ABC respectively.



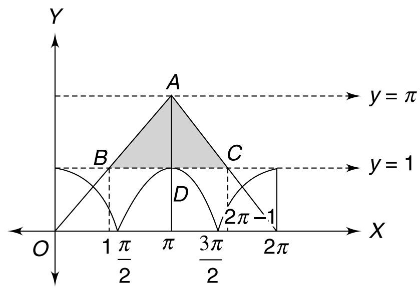
Hence, the required area

$$= 3\left(2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{3}} - \frac{1}{6}\pi\cdot 1\right)$$

$$= 3\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$= \left(\sqrt{3} - \frac{\pi}{2}\right) \text{ sq.u.}$$

18. The tangent at $x = \pi$ to the curve $f(x) = |\cos x|$ will be parallel to x-axis and cuts the curve $f(x) = \cos^{-1}(\cos x)$ at B and C.



Thus, $AD = \pi - 1$

Hence, the area of ΔABC

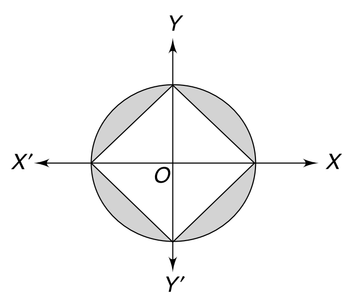
$$= \frac{1}{2} \times (2\pi - 2) \times (\pi - 1)$$

$$= \frac{1}{2} \times 2(\pi - 1) \times (\pi - 1)$$

$$= (\pi - 1)^2 \text{ sq.u.}$$

19. Given curves are

$|y| = 1 - x^2$, $x^2 + y^2 = 1$.



Hence, the required area

$$= \pi - 4\left(\int_0^1 (1 - x^2)\,dx\right)$$

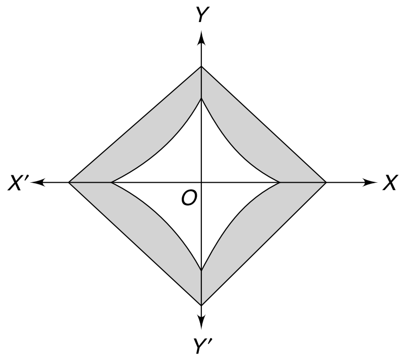
$$= \pi - 4\left(x - \frac{x^3}{3}\right)_0^1$$

$$= \pi - 4\left(1 - \frac{1}{3}\right)$$

$$= \left(\pi - \frac{8}{3}\right) \text{ sq.u.}$$

20. Given curves are

$|y| = e^{-|x|} - \frac{1}{2}$ and $|x| + |y| = \ln 2$



Hence, the required area

$$= 4\left(\frac{1}{2}(\ln 2)^2 - \int_0^{\ln 2}\left(e^{-x} - \frac{1}{2}\right)dx\right)$$

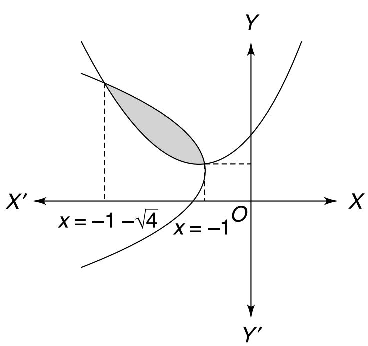
$$= 4\left(\frac{1}{2}(\ln 2)^2 - \left(\frac{e^{-x}}{-1} - \frac{1}{2}\right)_0^{\ln 2}\right)$$

$$= 2(\ln 2)^2 + 2\ln\left(\frac{2}{e}\right).$$

21. Given curves are

$$y^2 - 2y + 4x + 5 = 0 \text{ and } x^2 + 2x - y + 2 = 0$$

$$\Rightarrow \quad (y-1)^2 = -4(x+1) \text{ and } (x+1)^2 = (y-1).$$



Hence, the required area

$$= \int_{-1-3\sqrt{4}}^{-1}\left(1 + \sqrt{-4(x+1)}\right) - (1 + (x+1)^2)\,dx$$

$$= \left(x + 2\cdot\frac{2}{3}(-(x+1)^{3/2}) - \left(x + \frac{(x+1)^3}{3}\right)\right)_{-1-\sqrt[3]{4}}^{-1}$$

$$= \left(\frac{4}{3}(-(x+1)^{3/2}) - \left(\frac{(x+1)^3}{3}\right)\right)_{-1-\sqrt[3]{4}}^{-1}$$

$$= \left(\frac{4}{3}\left((-\sqrt[3]{4})^{3/2}\right) + \left(\frac{(-\sqrt[3]{4})^3}{3}\right)\right)$$

$$= \left|\left(\frac{4}{3}(-2) - \frac{4}{3}\right)\right| = \frac{12}{3} = 4 \text{ sq.u.}$$

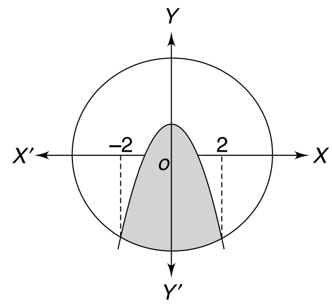
22. Given curves are

$$x^2 + y^2 - 2x + 4y - 11 = 0$$

and $\quad y = -x^2 + 2x + 1 - 2\sqrt{3}$

$\Rightarrow \quad (x-1)^2 + (y+2)^2 = 16$

and $\quad y + 2(\sqrt{3} - 1) = -(x-1)^2$



Hence, the required area

$$= 2\left(\int_0^2 (\text{Parabola} - \text{Circle})\,dx\right)$$

$$= 2\left(\int_0^2 ((-x^2 + 2x + 1 - 2\sqrt{3}) - \left(-2 + \sqrt{16 - (x-1)^2}\right))\,dx\right)$$

$$= 2\left[\left(-\frac{x^3}{3} + x^2 + x - 2\sqrt{3}\,x\right)_0^2 - \left(-2x + \frac{(x-1)}{2}\sqrt{16-(x-1)^2} + 8\sin^{-1}\left(\frac{x-1}{4}\right)\right)_0^2\right]$$

$$= 2\left(-\frac{8}{3} + 6 - 4\sqrt{3} - \left((-4) + \frac{\sqrt{15}}{2} + 8\sin^{-1}\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{15}}{2} - 8\sin^{-1}\left(\frac{1}{4}\right)\right)\right)\right)$$

$$= 2\left(10 - \frac{8}{3} - 4\sqrt{3}\right)$$

$$= 2\left(\frac{22}{3} - 4\sqrt{3}\right) \text{ sq.u.}$$

23. Given curve is

$$9x^2 + 4y^2 - 36x + 8y + 4 = 0$$

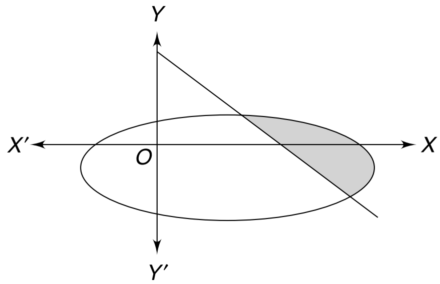
$\Rightarrow \quad (9x^2 - 36x) + (4y^2 + 8y) + 4 = 0$

$\Rightarrow \quad 9(x^2 - 4x) + 4(y^2 + 2y) + 4 = 0$

$\Rightarrow \quad 9(x-2)^2 + 4(y+1)^2 = 36 + 4 - 4 = 36$

$\Rightarrow \quad 9(x-2)^2 + 4(y+1)^2 = 36$

$\Rightarrow \quad \frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} = 1$



Solving, we get

$x = 2$ and $x = 4$

Hence, the required area

$$= \int_2^4 (E - L)\,dx$$

$$= \int_2^4 \left(-1 + 3\sqrt{1 - \left(\frac{(x-2)^2}{4}\right)} - \left(\frac{10 - 3x}{2}\right)\right)dx$$

$$= \int_2^4 \left(-1 + 3\sqrt{1 - \left(\frac{(x-2)^2}{4}\right)} - 5 + \frac{3x}{2}\right)dx$$

$$= \int_2^4 \left(-6 + 3\sqrt{1 - \left(\frac{(x-2)^2}{4}\right)} + \frac{3x}{2}\right)dx$$

$$= \left(-6x + 3\left(\frac{x-2}{4}\sqrt{1 - \left(\frac{x-2}{2}\right)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right) + \frac{3x^2}{4}\right)_2^4$$

$$= \left(-24 + 3\left(\frac{\pi}{4}\right) + 12 + 12 - 3\right)$$

$$= 3\left(\frac{\pi}{4} - 1\right) \text{ sq.u.}$$

24. The required area will be equal to area enclosed by $y = f(x)$ and the y-axis between the abscissae $y = -2$ and $y = 6$.

$\therefore \quad f(0) = 2, f(-1) = -2, f(1) = 6$

Clearly, $f(x)$ is monotonic in $[-1, 1]$.

Hence, the required area

$$= \int_0^1 (6 - f(x))\,dx + \int_0^{-1} (f(x) - (-2))\,dx$$

$$= \int_0^1 (4 - x^3 - x)\,dx + \int_0^{-1} (x^3 + 3x + 4)\,dx$$

$$= \left(4x - \frac{x^4}{4} - \frac{x^2}{2}\right)_0^1 + \left(\frac{x^4}{4} + \frac{3x^2}{2} + 4x\right)_{-1}^0$$

$$= \left|\left(4 - \frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} + \frac{3}{2} - 4\right)\right|$$

$$= \left|8 - \frac{2}{4} - 2\right|$$

$$= 6 - \frac{1}{2} = \frac{11}{2} \text{ sq.u.}$$

25. The required area will be equal to area enclosed by $y = f(x)$ and the y-axis between the abscissae $y = -1$ and $y = 4$

$\therefore \quad f(0) = 1, f(-2) = -1, f(2) = 4$

Clearly, $f(x)$ is monotonic in $[-2, 2]$.

Hence, the required area

$$= \int_0^2 (4 - f(x))\,dx + \int_{-2}^0 (f(x) - (-1))\,dx$$

$$= \int_0^2 \left(3 - \frac{x^3}{24} - \frac{x^2}{8} - \frac{13}{12}x\right)dx$$

$$+ \int_{-2}^0 \left(\frac{x^3}{24} + \frac{x^2}{8} + \frac{13}{12}x + 2\right)dx$$

$$= \left(3x - \frac{x^4}{96} - \frac{x^3}{24} - \frac{13x^2}{24}\right)_0^2$$

$$+ \left(\frac{x^4}{96} + \frac{x^3}{24} + \frac{13x^2}{24} + 2x\right)_{-2}^0$$

$$= \left(6 - \frac{16}{96} - \frac{8}{24} - \frac{52}{24}\right) - \left(\frac{16}{96} - \frac{8}{24} + \frac{52}{24} - 4\right)$$

$$= \left|10 - \frac{32}{96} - \frac{104}{24}\right|$$

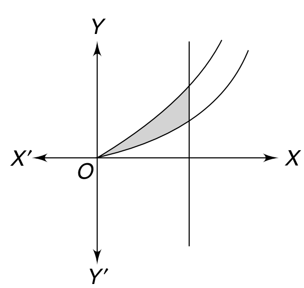
$$= \left|10 - \frac{1}{3} - \frac{13}{3}\right|$$

$$= \left|10 - \frac{14}{3}\right)$$

$$= \frac{16}{3} \text{ sq.u.}$$

26. Given curves are

$y = xe^x,\ y = xe^{-x}$



Hence, the required area

$$= \int_0^1 (xe^x - xe^{-x})\,dx$$

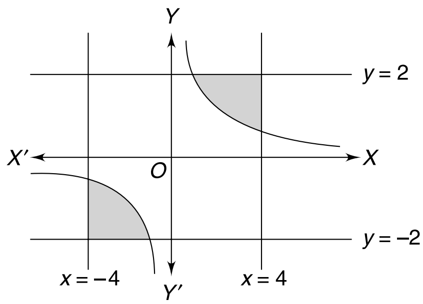
$$= (x(e^x + e^{-x}))_0^1 - \int_0^1 (ex + e^{-x})\,dx$$

$$= \left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right)$$

$$= \frac{2}{e} \text{ sq.u.}$$

27. Given curves are

$$|x - 2y| + |x + 2y| \le 8 \text{ and } xy \ge 2.$$



Hence, the required area

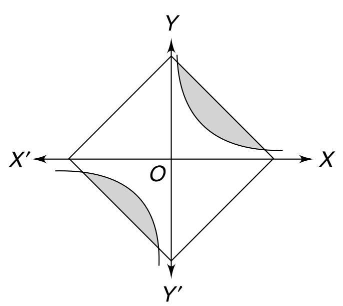
$$= 2\int_1^4 \left(2 - \frac{2}{x}\right) dx$$

$$= 2(2x - 2\log x)_1^4$$

$$= 2(8 - 2\log 2 - 2)$$

$$= 2(6 - 2\log 2) \text{ sq.u.}$$

28. Given curves are $|x| + |y| \ge 3$ and $xy \ge 2$.



Hence, the required area

$$= \int_1^2 \left(3 - x - \frac{2}{x}\right) dx$$

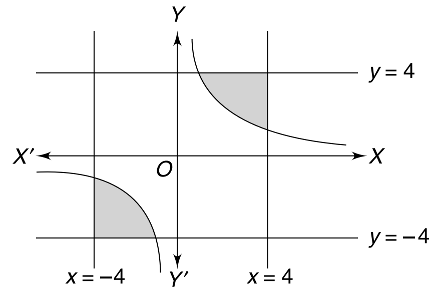
$$= \left(3x - \frac{x^2}{2} - 2\log x\right)_1^2$$

$$= \left(6 - 2 - 2\log 2 - 3 + \frac{1}{2}\right)$$

$$= \left(\frac{3}{2} - 2\log 2\right) \text{ sq.u.}$$

29. Given curves are

$$|x - y| + |x + y| \le 8 \text{ and } xy \ge 2.$$



Hence, the required area

$$= 2\int_{1/2}^4 \left(4 - \frac{2}{x}\right) dx$$

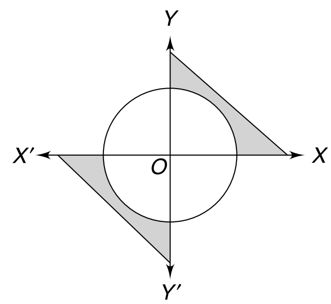
$$= 2(4x - 2\log x)_{1/2}^2$$

$$= 2\left(8 - 2\log 2 - 2 + 2\log\left(\frac{1}{2}\right)\right)$$

$$= 2(6 - 4\log 2) \text{ sq.u.}$$

30. Given curves are

$$|x + y| \le 2 \text{ and } x^2 + y^2 \ge 2.$$



Hence, the required area

$$= 2\left(\frac{1}{2} \times 2 \times 2 - \frac{\pi \cdot (\sqrt{2})^2}{4}\right)$$

$$= \left(4 - \frac{\pi}{2}\right) \text{ sq.u.}$$

Level IV

(Tougher Problems for JEE-Advanced)

1. Given curve is

$$2x^2 + 6xy + 5y^2 = 1$$

Let $x = r\cos\theta,\ y = r\sin\theta$

$$\Rightarrow \quad r^2 = \frac{1}{2\cos^2\theta + 6\sin\theta\cos\theta + 5\sin^2\theta}$$

Hence, the required area

$$= \int_{-\pi/2}^{\pi/2} r^2 \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2\cos^2\theta + 6\sin\theta\cos\theta + 5\sin^2\theta} \right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{\sec^2\theta}{2 + 6\tan\theta + 5\tan^2\theta} \right) d\theta$$

$$= \int_{-\infty}^{\infty} \left(\frac{dt}{5t^2 + 6t + 2} \right), \quad (\text{Let } t = \tan\theta)$$

$$= \frac{1}{5} \int_{-\infty}^{\infty} \left(\frac{dt}{t^2 + \frac{6}{5}t + \frac{2}{5}} \right)$$

$$= 1/5 \int_{-\infty}^{\infty} \left(\frac{dt}{\left(t + \frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)} \right)$$

$$= \frac{1}{5} \times \left(5\tan^{-1}\left(\frac{5t+3}{1}\right) \right)_{-\infty}^{\infty}$$

$$= \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right)$$

$$= \pi \text{ sq.u.}$$

2. Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$.

Hence, the required area

$$= 4 \int_0^a y\,dx$$

$$= 4 \int_0^a (a^{2/3} - x^{2/3})^{3/2} dx$$

$$= 4 \int_{\frac{\pi}{2}}^{0} (a \sin 3\theta)(-3a \cos^2\theta \sin\theta)\, d\theta$$

(Let $x = a\cos^2\theta$)

$$= 12a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta \, d\theta$$

$$= 12a^2 \times \frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \left(\frac{3\pi a^2}{8} \right) \text{sq.u.}$$

3. Clearly, $f(x) = x^2 + 1$

$\Rightarrow \quad f'(x) = 2x$

Now, $m = (f'(x))_{x=0} = 0$

Equation of the tangent at (α, β) is

$$y - \beta = m(x - a)$$

$$\Rightarrow \quad y - \beta = 2\alpha(x - \alpha) \quad \text{...(i)}$$

Since the point (α, β) lies on the curve, so

$$\beta = \alpha^2 + 1 \quad \text{...(ii)}$$

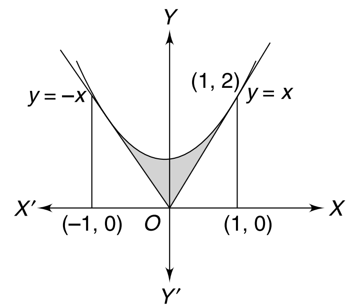
From Eq. (i) and (ii), we get

$$\alpha = \pm 1, \ \beta = 2$$

Hence, the equation of the tangent is

$$y - 2 = \pm 2(x - 1)$$

$$y = \pm 2x$$



Hence, the required area

$$= 2\int_0^1 (x^2 + 1 - x)\,dx$$

$$= 2\left(\frac{x^3}{3} - \frac{x^2}{2} + x \right)_0^1$$

$$= 2\left(\frac{1}{3} - \frac{1}{2} + 1 \right)$$

$$= 2\left(\frac{4}{3} - \frac{1}{2} \right)$$

$$= 2 \times \frac{5}{6}$$

$$= \frac{5}{3} \text{ sq. u.}$$

4. Given curve is

$$y = xe^{-x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{-x^2} - 2x^2 e^{-x^2}$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad e^{-x^2} - 2x^2 e^{-x^2} = 0$$

$$\Rightarrow \quad 1 - 2x^2 = 0$$

$$\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

By the sign scheme, it is maximum at $x = \frac{1}{\sqrt{2}}$

Thus, $c = \frac{1}{\sqrt{2}}$

Hence, the required area

$$= \int_0^c x e^{-x^2} dx$$

$$= -\frac{1}{2} \int_0^{-c^2} e^t\,dt, \quad (\text{Let } -x^2 = t)$$

$$= -\frac{1}{2}(e^t)_0^{-c^2}$$

$$= -\frac{1}{2}(e^{-c^2} - 1)$$

$$= \frac{1}{2}(1 - e^{-c^2}) \text{ sq. u.}$$

The area is maximum, when $c = \frac{1}{\sqrt{2}}$

Hence, the maximum area

$$= \frac{1}{2}\left(1 - e^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{e}}\right)$$

5. Let the required area be A.

Thus, $A = \int_0^1 (a^2x^2 + ax + 1)\,dx$

$$= \left(\frac{a^2x^3}{3} + \frac{ax^2}{2} + x\right)_0^1$$

$$= \left(\frac{a^2}{3} + \frac{a}{2} + 1\right)$$

$\Rightarrow \quad \frac{dA}{da} = \left(\frac{2a}{3} + \frac{1}{2}\right)$

$\Rightarrow \quad \frac{d^2A}{da^2} = \frac{2}{3} > 0$

So, the area is the least.

For maximum or minimum,

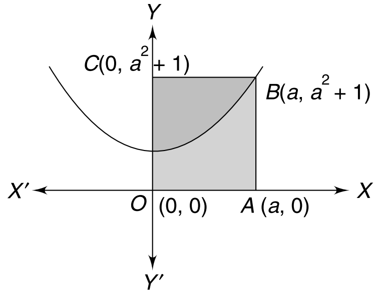
$$\frac{dA}{da} = 0$$

$\Rightarrow \quad \frac{2a}{3} + \frac{1}{2} = 0$

$\Rightarrow \quad a = -\frac{3}{4}$

Hence, the value of a is $-3/4$.

6. Given curve is $y = x^2 + 1$



Hence, the required area

$$= \int_0^a (x^2 + 1)\,dx$$

$$= \left(\frac{x^3}{3} + x\right)_0^a$$

$$= \left(\frac{a^3}{3} + a\right)$$

It is given that,

$$2\left(\frac{a^3}{3} + a\right) = (a^3 + a)$$

$\Rightarrow \quad 2a^3 + 6a = 3a^3 + 3a$

$\Rightarrow \quad a^3 = 3a$

$\Rightarrow \quad a^2 = 3$

$\Rightarrow \quad a = \sqrt{3}$

Hence, the value of a is $\sqrt{3}$.

7. Given curve is

$$y = xe^{-x}$$

$\Rightarrow \quad \frac{dy}{dx} = e^{-x} - xe^{-x}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -e^{-x} - e^{-x} + xe^{-x}$

For point of inflection,

$$\frac{d^2y}{dx^2} = 0$$

$\Rightarrow \quad -e^{-x} - e^{-x} + xe^{-x} = 0$

$\Rightarrow \quad x - 2 = 0$

$\Rightarrow \quad x = 2$

So $\quad c = 2$

Hence, the required area is

$$= \int_0^2 xe^{-x}\,dx$$

$$= (-xe^{-x})_0^2 + \int_0^2 e^{-x}\,dx$$

$$= -(xe^{-x} + e^{-x})_0^2$$

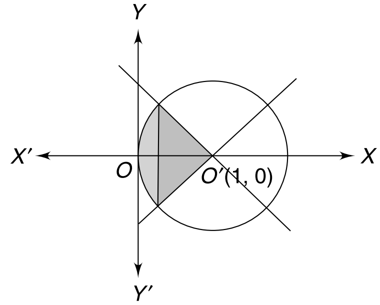
$$= (1 - 3e^{-2}) \text{ sq.u.}$$

8. Given curves are

$$|x| + |y| \geq 1 \text{ and } x^2 - 2x + 1 \leq 1 - y^2$$

$$|x| + |y| \geq 1 \text{ and } (x - 1)^2 + y^2 \leq 1$$

$\Rightarrow \quad \begin{cases} x + y \geq 1, x + y \leq -1 \\ x - y \geq 1, x - y \leq -1 \end{cases}$ and $(x - 1)^2 + y^2 \leq 1$



Hence, the required area

$$= 2\left(\int_0^1 \sqrt{1-(x-1)^2}\, dx + \frac{1}{2} \times 1 \times 1\right)$$

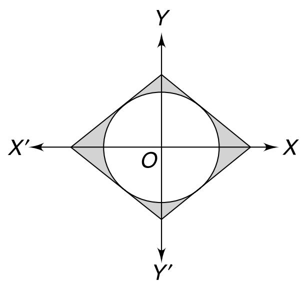
$$= 2\left(\frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{2}\right)\right)_0^1 + 1$$

$$= 2\left(\frac{1}{2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right) + 1$$

$$= \left(2 + \frac{\pi}{6}\right) \text{ sq.u.}$$

9. Given curves are

$|x + y| \le 1$, $|y - x| \le 1$ and $3x^2 + 3y^2 \ge 1$.



Hence, the required area

= area of the square – area of the circle

$$= \left(2 - \frac{\pi}{2}\right) \text{ sq.u.}$$

10. Given curves are

$$|x - 2| + |y - 2| \le 3,\ x^2 - 4x + y + 3 \le 0$$

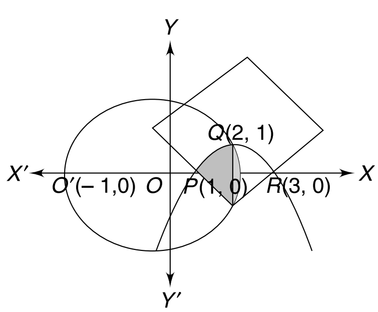
and $\quad x^2 + y^2 + 2x - 9 \le 0$

$\Rightarrow \quad |x - 2| + |y - 2| \le 3,\ (x - 2)^2 \le -(y + 1)$

and $\quad (x + 1)^2 + y^2 \le 10$

$\Rightarrow \quad (x - 2)^2 \le -(y + 1),\ (x + 1)^2 + y^2 \le 10$

$$\begin{cases} x + y \le 7, x + y \ge 1 \\ x - y \le 3, x - y \ge -3 \end{cases}$$



Hence, the required area

$$= \int_1^2 (-x^2 + 4x - 3)\, dx$$

$$+ \left|2\int_2^{\sqrt{10}-1} \sqrt{10-(x+1)^2}\, dx\right| + \left|\int_1^1 (1-x)\, dx\right|$$

$$= \left(-\frac{x^3}{3} + 2x^2 - 3x\right)_1^2$$

$$+ 2\left[\frac{(x+1)\sqrt{10-(x+1)^2}}{2} + 5\sin^{-1}\left(\frac{x+1}{\sqrt{10}}\right)\right]_2^{\sqrt{10}-1}$$

$$- \left(x - \frac{x^2}{2}\right)_1^2$$

$$= \left(-\frac{8}{3} + 8 - 6 + \frac{1}{3} - 2 + 3\right)$$

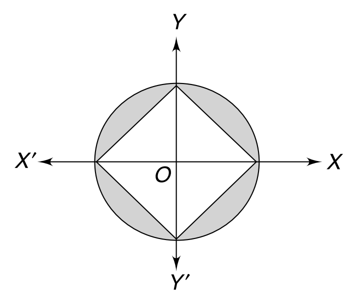
$$+ 2\left(0 + 5\sin^{-1}(1) - \frac{3}{2} - 5\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)\right) + \frac{1}{2}$$

$$= \frac{2}{3} + \frac{1}{2} + 5\pi - 3 - 10\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

$$= \left(5\pi - \frac{11}{6} - 10\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)\right) \text{ sq.u.}$$

11. Given curves are

$(x^2 + y^2) \le 4 \le 2(|x| + |y|)$



Hence, the required area

= area of a circle-area of a square

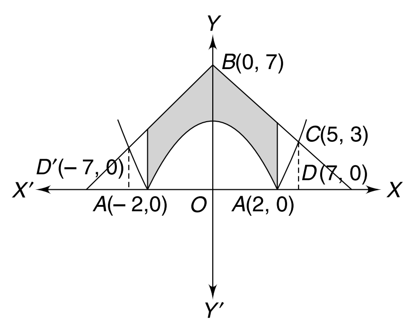
$= 4\pi - 4$

$= 4(\pi - 1)$ sq.u.

12. Given curves are

$$y = \frac{|4 - x^2|}{4} \text{ and } y = 7 - |x|$$

$$\Rightarrow \quad y = \frac{|x^2 - 4|}{4} \text{ and } y = 7 - |x|$$



Hence, the required area

$$= 2\left[\frac{1}{2}(7 + 3) \times 4 - \int_0^2 \left(\frac{4 - x^2}{4}\right)dx - \int_2^4 \left(\frac{x^2 - 4}{2}\right)dx\right]$$

$$= 2\left[40 - \left(x - \frac{x^3}{12}\right)_0^2 - \left(\frac{x^2}{12} - x\right)_2^4\right]$$

$$= 2\left[20 - \left(2 - \frac{8}{12}\right) - \left(\frac{64}{12} - 4 - \frac{8}{12} + 2\right)\right]$$

$$= 2\left[18 + \frac{2}{3} - \frac{16}{3} + 4 + \frac{2}{3} - 2\right]]$$

$= 32$ sq.u.

13. Given curves are

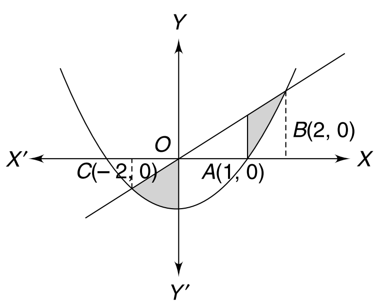
$$y = x^2 + x - 2,\ y = 2x \text{ where}$$

$$x(x^2 + x - 2) \geq 0$$

$$\Rightarrow \quad x(x - 1)(x + 1) \geq 0$$

Solving, $y = x^2 + x - 2$, $y = 2x$, we get,

$$x = -1,\ x = 2$$



Hence, the required area

$$= \int_{-1}^{0} [2x - (x^2 + x - 2)]\,dx + \int_{1}^{2} [2x - (x^2 + x - 2)]dx$$

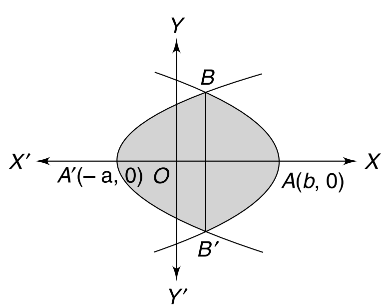
$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_1^2$$

$$= \left[\frac{7}{6} + \left(\frac{10}{3} - \frac{13}{6}\right)\right]$$

$$= \frac{14}{6} = \frac{7}{3} \text{ sq.u.}$$

14. Given curves are

$y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$



On solving, we get,

$$B(b - a, \sqrt{4ab}),\ B'(b - a, -\sqrt{4ab})$$

Hence, the required area

$$= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} \left[\left(b - \frac{y^2}{4b}\right) - \left(\frac{y^2}{4a} - a\right)\right]dy$$

$$= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} \left[(a + b) - \frac{1}{4}\left(\frac{y^2}{a} + \frac{y^2}{b}\right)\right]dy$$

$$= 2(a + b)\cdot 2\sqrt{ab} - \int_0^{\sqrt{ab}} \left[\frac{1}{4}\left(\frac{y^2}{a} + \frac{y^2}{b}\right)\right]dy$$

$$= 4(a + b)\sqrt{ab} - \frac{1}{2}\int_0^{\sqrt{ab}} \left[\left(\frac{y^2}{a} + \frac{y^2}{b}\right)\right]dy$$

$$= 4(a + b)\sqrt{ab} - \frac{1}{2}\left(\frac{y^3}{3a} + \frac{y^3}{3b}\right)_0^{2\sqrt{ab}}$$

$$= 4(a + b)\sqrt{ab} - \frac{1}{2}\left(\frac{8ab\sqrt{ab}}{3a} + \frac{8ab\sqrt{ab}}{3b}\right)$$

$$= 4(a + b)\sqrt{ab} - \frac{4}{3}(a + b)\sqrt{ab}$$

$$= \frac{8}{3}(a + b)\sqrt{ab} \text{ sq.u.}$$

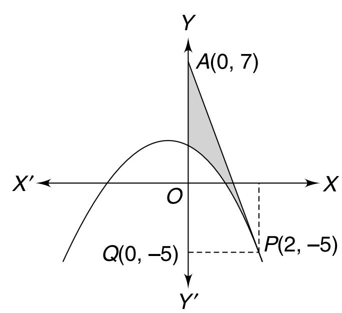
15. Given curve is

$$y = -x^2 - 2x + 3$$

$$\Rightarrow \quad \frac{dy}{dx} = -2x - 2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=2} = -6$$

The equation of the tangent to the given curve at $P(2, -5)$ is $6x + y = 7$



Hence, the required area

$$= \frac{1}{2}\cdot 2\cdot 12 - \left|\int_{-5}^{3} x\,dy\right|$$

$$= 12 - \left|\int_{-5}^{3} (-1 + \sqrt{4 - y})\,dy\right|$$

$$= 12 - \left|\left[-y - \frac{2}{3}(4 - y)^{3/2}\right]_{-5}^{3}\right|$$

$$= 12 - \left[10 - \frac{2}{3}\right]$$

$$= \frac{8}{3} \text{ sq.u.}$$

16. Given curves are

$$x^2 + y^2 - 2x + 4y - 11 = 0$$

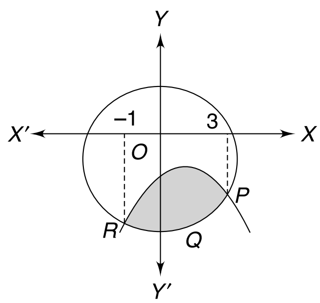
and $y = -x^2 + 2x + (1 - 2\sqrt{3})$

$\Rightarrow$ $(x - 1)^2 + (y + 2)^2 = 16$

and $y = -(x - 1)^2 + 2(1 - \sqrt{3})$

Solving, we get

$$x = -1, 3$$



Hence, the required area

$$= \int_{-1}^{3} (y_p - y_c)\, dx$$

$$= \int_{-1}^{3} [(-x^2 + 2x + 1 - 2\sqrt{3}) - (-2 - \sqrt{16 - (x-1)^2})]\, dx$$

$$= \left[-\frac{x^3}{3} + x^2 + (3 - 2\sqrt{3})x + \frac{(x-1)}{2}\sqrt{16 - (x-1)^2} + \frac{16}{2}\sin^{-1}\left(\frac{x-1}{4}\right)\right]_{-1}^{3}$$

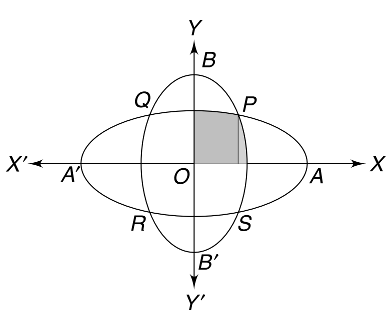
$$= \left(\frac{32}{3} - 4\sqrt{3} + \frac{8\pi}{3}\right) \text{ sq.u.}$$

17. Given curves are

$$a^2x^2 + b^2y^2 = 1 \text{ and } b^2x^2 + a^2y^2 = 1$$

On solving, we get,

$$x = \pm\frac{1}{\sqrt{a^2 + b^2}} = y$$



Hence, the required area

$$= 4\left[\int_0^{\alpha} \frac{\sqrt{1 - a^2x^2}}{b} + \int_{\alpha}^{\frac{1}{b}} \frac{\sqrt{1 - b^2x^2}}{a}\, dx\right]$$

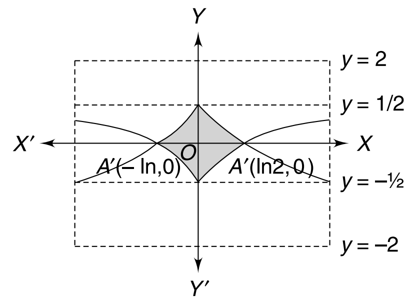
$$\left(\text{Let } \alpha = \frac{1}{\sqrt{a^2 + b^2}}\right)$$

$$= 4\left[\frac{a}{b}\int_0^{\alpha}\left(\sqrt{\frac{1}{a^2} - x^2}\right)dx + \frac{b}{a}\int_{\alpha}^{\frac{1}{b}}\left(\sqrt{\frac{1}{b^2} - x^2}\right)dx\right]$$

$$= \left(\frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)\right) \text{ sq.u.}$$

18. Given curves are

$$|y| + \frac{1}{2} \le e^{-|x|} \text{ and } \{|x|, |y|\} \le 2$$



Hence, the required area

$$= 4\int_0^{\ln 2}\left(e^{-x} - \frac{1}{2}\right)dx$$

$$= 4\left(-e^{-x} - \frac{x}{2}\right)_0^{\ln 2}$$

$$= 4\left(-e^{-\ln 2} - \frac{\ln 2}{2} + 1\right)$$

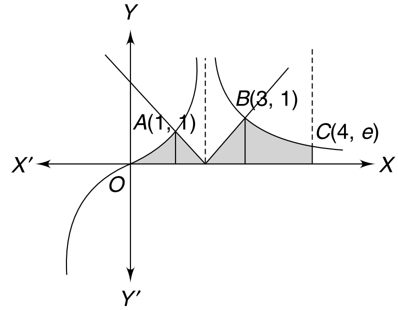
$$= 4\left(-\frac{1}{2} - \frac{\ln 2}{2} + 1\right)$$

$$= 4\left(\frac{1}{2} - \frac{\ln 2}{2}\right)$$

$$= (2 - 2\ln 2) \text{ sq.u.}$$

19. Given curves are

$y = \min\{x^3, |x - 2|, e^{3-x}\}$, x-axis, y-axis and $x = 4$.



On solving, we get,

$A = (1, 1)$, $B = (3, 1)$, $C = (4, e)$

Hence, the required area

$$= \int_0^1 x^3\,dx + \int_1^2 (2 - x)\,dx + \int_2^3 (x - 2)\,dx + \int_3^4 (e^{3-x})\,dx$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2 + \left(\frac{x^2}{2} - 2x\right)_2^3 + \left(-e^{3-x}\right)_3^4$$

$$= \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{e}\right)$$

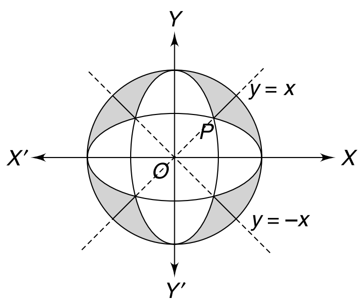
$$= \left(\frac{9}{4} - \frac{1}{e}\right) \text{ sq.u.}$$

20. Given curves are

$S_1: \{(x, y): x^2 + 2y^2 \le 2\}$

$S_2: \{(x, y): 2x^2 + y^2 \le 2\}$

and $S_3: \{(x, y): x^2 + y^2 \le 2\}$



The area of the first quadrant

$$= \int_0^{\sqrt{2/3}} \left(\sqrt{2}\sqrt{1 - x^2} - x\right) dx$$

$$= \left(\sqrt{2}\left(\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right)\right) - \frac{x^2}{2}\right)_0^{\sqrt{2/3}}$$

$$= \left|\frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) + \frac{1}{3} - \frac{1}{3}\right|$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

Hence, the required area

$$= 8\left(\frac{2\pi}{8} - \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right)$$

$$= \left(2\pi - 4\sqrt{2}\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right) \text{ sq.u.}$$

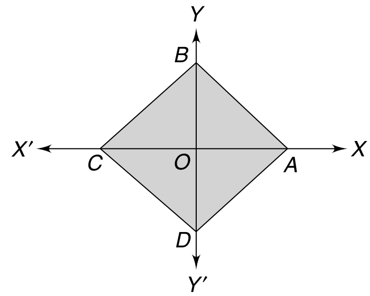
Integer Type Questions

1. For $x \in [-2, 2]$, $y = \left[\sin^2 x + \cos\left(\frac{x}{4}\right)\right] = 1$

Hence, the required area

$$= \int_{-2}^{2} 1\,dx = (x)_{-2}^{2} = 4$$

2. Given curve is $|x| + |y| = 1$



Hence, the required area

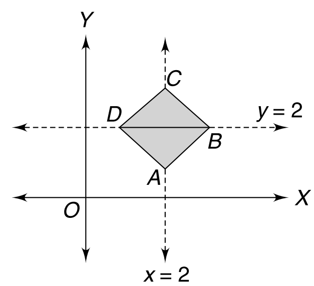
$= \text{ar}(ABCD)$

$= 4(\text{ar } \Delta OAB)$

$= 4 \times \frac{1}{2} \times 1 \times 1$

$= 2$ sq.u.

3. Given curve is $|x - 2| + |y - 2| = 1$



Hence, the required area

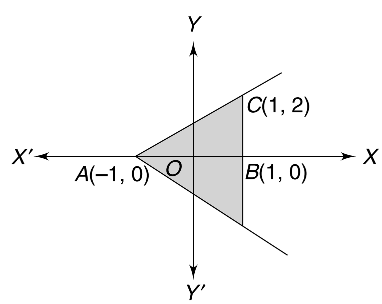
$= 4 \times \frac{1}{2} \times 1 \times 1$

$= 2$ sq.u.

4. Given curve are

$e^{\ln(x+1)} \ge |y|, |x| \le 1$

$x + 1 \ge |y|, |x| \le 1$



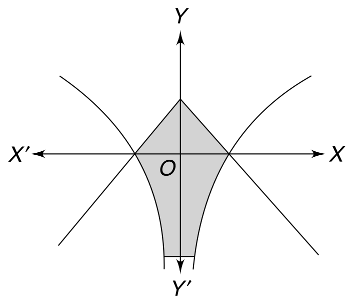
Hence, the required area

$$= 2 \times \frac{1}{2} \times 2 \times 2$$

$$= \text{sq.u.}$$

5. Given curves are

$$y = \ln|x| \text{ and } y = 1 - |x|.$$



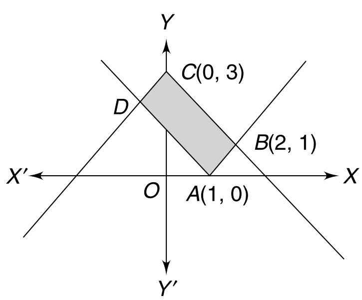
Hence, the required area

$$= 2\left[\frac{1}{2} \times 1 \times 1 + \int_{-\infty}^{0} e^x\,dx\right]$$

$$= 2\left(\frac{1}{2} + 1\right) = 2 \times \frac{3}{2} = 3 \text{ sq.u.}$$

6. Given curves are

$$y = 3 - |x| \text{ and } y = |x - 1|$$



Here, $AB = \sqrt{1^2 + 1^2} = \sqrt{2}$

and $BC = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Hence, the required area

$$= AB \times BC$$

$$= \sqrt{2} \times 2\sqrt{2} = 4 \text{ sq.u.}$$

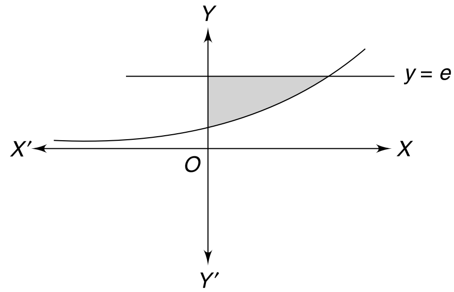
7. Given curves are

$$y = \frac{|x|}{x}, x \neq 0 \text{ and } y = x(x - 1)(x - 3)$$

$$y = \begin{cases} 1 & : x > 0 \\ -1 & : x < 0 \end{cases} \text{ and } y = x(x - 1)(x - 3)$$

8. Given curves are

$$y = e^x, x = 0, y = e$$



Hence, the required area

$$= \int_0^1 (e - e^x)\,dx$$

$$= (e - e^x)_0^1$$

$$= (e - e + 1)$$

$$= 1.$$

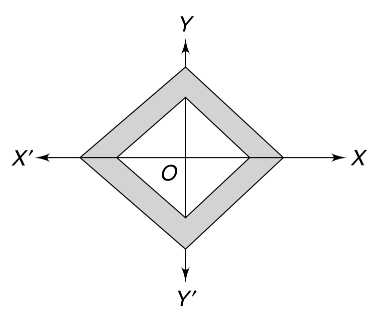
9. Given curves are

$$1 \leq |x - 2| + |y + 1| \leq 2$$

Putting $X = x - 2, Y = y + 1$

Thus, the equation redices to

$$1 \leq |X| + |Y| \leq 2$$



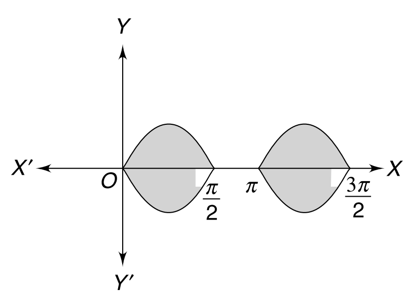
Hence, the required area

$$= 4\left(\frac{1}{2} \times 2 \times 2 - 2 \times \frac{1}{2} \times 1 \times 1\right)$$

$$= 4 \times \frac{3}{2} = 6 \text{ sq.u.}$$

10. Given curve is

$$|y| = \sin(2x), \text{ where } 0 < x < 2\pi$$



Hence, the required area

$$= 4\int_0^{\pi/2} (\sin(2x))\,dx$$

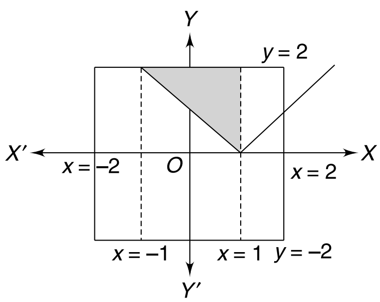
$$= 4\left(-\frac{\cos(2x)}{2}\right)_0^{\pi/2}$$

$$= -2(-1 - 1)$$

$$= 4 \text{ sq.u.}$$

11. Given curve is

$$|x + y| + |x - y| \le 4,\ |x| \le 1,\ y \ge |x - 1|$$



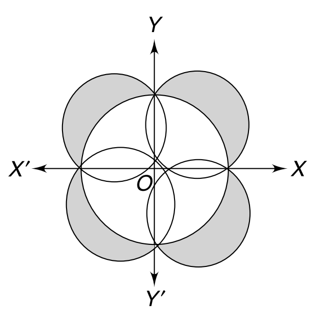
Hence, the required area

$$= \frac{1}{2} \times 2 \times 2$$

$$= 2 \text{ sq.u.}$$

12. Given curves are

$$4 \le x^2 + y^2 \le 2\,(|x| + |y|)$$



Hence, the required area

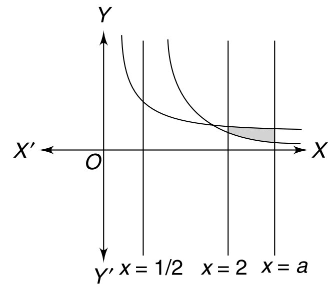
$$= 4\left(\frac{\pi(\sqrt{2})^2}{2} - (\pi - 2)\right)$$

$$= 4(\pi - (\pi - 2))$$

$$= 8 \text{ sq.u.}$$

13. Given curves are

$$y = \frac{1}{x},\ y = \frac{1}{2x - 1},\ x = 2,\ x = a, \text{ where } a > 2$$



Hence, the required area,

$$\int_2^a \left(\frac{1}{x} - \frac{1}{2x - 1}\right)dx = \log\left(\frac{4}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \left(\frac{1}{2}\log\left(\frac{x^2}{2x - 1}\right)\right)_2^a = \log\left(\frac{4}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \left(\log\left(\frac{a^2}{2a - 1}\right) - \log\left(\frac{4}{3}\right)\right) = \log\left(\frac{16}{5}\right)$$

$$\Rightarrow \quad \log\left(\frac{a^2}{2a - 1}\right) = \log\left(\frac{64}{15}\right)$$

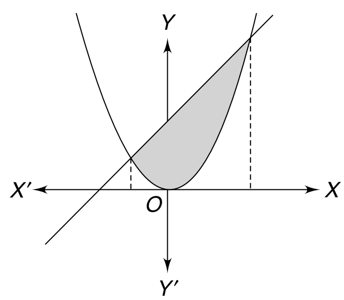
$$\Rightarrow \quad \left(\frac{a^2}{2a - 1}\right)$$

$$= \left|\frac{64}{15}\right|$$

$$\Rightarrow \quad a = 8$$

Hence the value of a is 8.

1. Given curves are $x^2 = 4y$ and $x = 4y - 2$



Hence, the required area

$$= \int_{-1}^{2} (y_2 - y_1)\,dx$$

$$= \int_{-1}^{2} \left(\frac{x + 2}{4} - \frac{x^2}{4}\right)dx$$

$$= \frac{1}{4}\int_{-1}^{2} (x + 2 - x^2)\,dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)_{-1}^{2}$$

$$= \frac{1}{4}\left(\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right)$$

$$= \frac{1}{4}\left(\frac{10}{3} + \frac{7}{6}\right) = \frac{1}{4}\left(\frac{27}{6}\right) = \frac{9}{8} \text{ sq.u.}$$

2. The required area

$$= 4 \times \frac{1}{2} \times 1 \times 1 = 2 \text{ sq.u.}$$

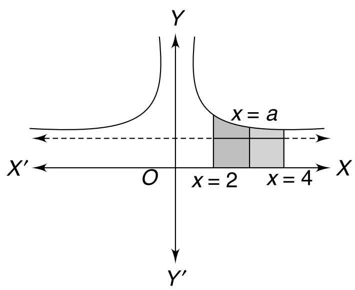
3. Given $\int_1^b f(x)\,dx = (b - 1)\sin(3b + 4)$

Differentiating both sides w.r.t. b, we get

$$f(b) = \sin(3b + 4) + 3(b - 1)\cos(3b + 4)$$

$$\Rightarrow f(x) = \sin(3x + 4) + 3(x - 1)\cos(3x + 4)$$

4. Given curve is



It is given that

$$\int_1^a \left(1 + \frac{8}{x^2}\right)dx = \int_a^4 \left(1 + \frac{8}{x^2}\right)dx$$

$$\Rightarrow \left(x - \frac{8}{x}\right)_2^a = \left(x - \frac{8}{x}\right)_a^4$$

$$\Rightarrow \left(a - \frac{8}{a} - 2 + 4\right) = \left(4 - 2 - a + \frac{8}{a}\right)$$

$$\Rightarrow 2a - \frac{16}{a} = 0$$

$$\Rightarrow 2a^2 = 16$$

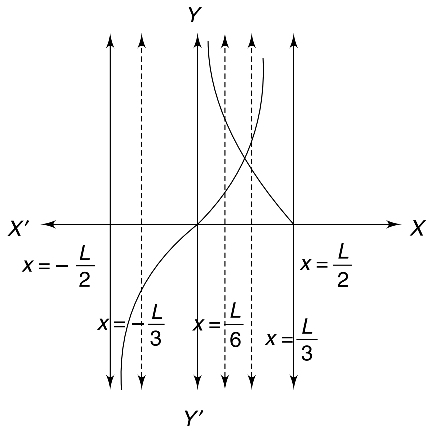
$$\Rightarrow a^2 = 8$$

$$\Rightarrow a = 2\sqrt{2}$$

Hence, the value of a is $2\sqrt{2}$.

5. Given curves are

$$\begin{cases} y = \tan x & : -\frac{\pi}{3} \le x \le \frac{\pi}{3} \\ y = \cot x & : \frac{\pi}{6} \le x \le \frac{\pi}{2} \end{cases}$$



Hence, the required area

$$= \left|\int_{-\frac{\pi}{3}}^{0} \tan x\,dx\right| + \int_0^{\frac{\pi}{4}} \tan x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$$

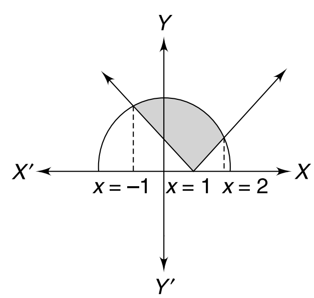
$$= \left|(\log \sec x)_{-\frac{\pi}{3}}^{0}\right| + (\log \sec)_0^{\frac{\pi}{4}}$$

$$+ (\log(\sin x))_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \log(2) + \log\left(\frac{1}{\sqrt{2}}\right) - \log\left(\frac{1}{\sqrt{2}}\right)$$

$$= \log(2) \text{ sq.u.}$$

6. Given curves are



Hence, the required area

$$= \int_{-1}^{2} \sqrt{5 - x^2}\,dx - \int_{-1}^{2} |x - 1|\,dx$$

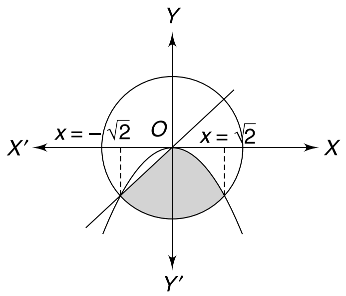
$$= \left(\frac{3}{2} + \frac{5\pi}{4}\right) - \frac{5}{2}$$

$$= \left(\frac{3}{2} + \frac{5\pi}{4}\right) - 2$$

$$= \left(\frac{5\pi}{4} - \frac{1}{2}\right) \text{ sq.u.}$$

7. Given curves are

$$x^2 + y^2 = 4,\ x^2 = -\sqrt{2}y \text{ and } y = x$$



Hence, the required area

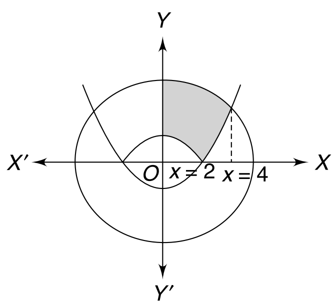
$$= \left(\int_{-\sqrt{2}}^{0} x\,dx + \int_{0}^{\sqrt{2}} \left(-\frac{x^2}{R2}\right)dx\right) - \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{5 - x^2}\,dx$$

$$= \left(-\frac{5}{3}\right) + (2 + 2\pi)$$

$$= \left(2\pi + \frac{1}{3}\right) \text{ sq.u.}$$

8. Given curves are

$$x^2 + y^2 = 25,\ 4y = |4 - x^2| \text{ and } x = 0$$



Hence, the required area

$$= \frac{\pi \cdot 5^2}{4} - \left(\int_0^2 \left(1 - \frac{x^2}{4}\right)dx + \int_2^4 \left(\frac{x^2}{4} - 1\right)dx + \int_4^5 \sqrt{25 - x^2}\,dx\right)$$

$$= \frac{25\pi}{4} - \left(\int_0^2 \left(1 - \frac{x^2}{4}\right)dx + \int_2^4 \left(\frac{x^2}{4} - 1\right)dx + \int_4^5 \sqrt{25 - x^2}\,dx\right)$$

$$= \frac{25\pi}{4} - \left(\frac{4}{3} + \frac{8}{3} + \frac{25\pi}{4} - \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right) + 6\right)$$

$$= \left(2 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)\right) \text{ sq.u.}$$

9. Given curve is $y = \tan x$

$$\Rightarrow \quad \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \sec^2\left(\frac{\pi}{4}\right) = 2$$

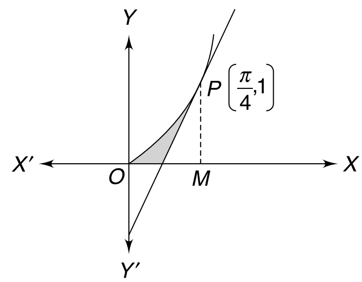
When $x = \frac{\pi}{4}$, then $y = 1$

Hence, the equation of the tangent is

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$y = 2x + \left(1 - \frac{\pi}{2}\right)$$



Hence, the required area

$$\int_0^{\frac{\pi}{4}} \tan x\,dx - \frac{1}{2}\left(\frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right) \times 1$$

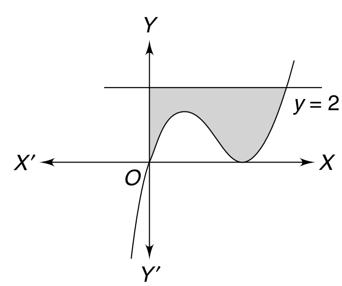
$$(\log(\sec x))_0^{\frac{\pi}{4}} - \frac{1}{2}\left(\frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right)$$

$$\log(\sqrt{2}) - \frac{1}{4}$$

$$\frac{1}{2}\log(2) - \frac{1}{4}$$

10. Given curve is

$$y = x(x - 1)^2$$



Hence, the required area

$$= \int_0^2 (2 - x(x - 1)^2)dx$$

$$= \int_0^2 (2 - x(x^2 - 2x + 1))dx$$

$$= \int_0^2 (2 - (x^3 - 2x^2 + x))dx$$

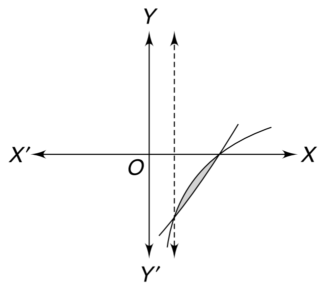
$$= \left(2x - \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{x^2}{2}\right)_0^2$$

$$= \left(4 - 4 + \frac{16}{3} - 3\right)$$

$$= \frac{10}{3} \text{ sq.u.}$$

11. Given curves are

$$y = ex\ln x,\ y = \frac{\ln x}{ex} \text{ and } x = 1$$



Hence, the required area

$$= \int_{1/e}^{1}\left(\frac{\log x}{ex} - ex\log x\right)dx$$

$$= \frac{1}{e}\int_{1/e}^{1}\left(\frac{\log x}{x}\right)dx - e\int_{1/e}^{1} x\log x\,dx$$

$$= \frac{1}{e}\left(\frac{(\log x)^2}{2}\right)_{1}^{1/e} + \left(\frac{x^2}{2}\log x\right)_{1/e}^{1} - \left(\frac{x^2}{4}\right)_{1/e}^{1}$$

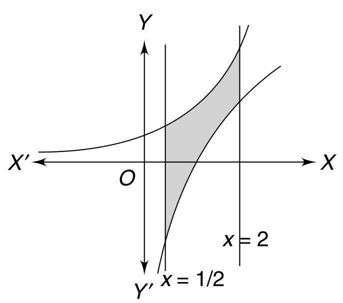
$$= -\frac{1}{2e} - e\left\{\frac{1}{2e^2} - \frac{1}{4} + \frac{1}{4e^2}\right\}$$

$$= -\frac{1}{2e} - \frac{1}{2e} + \frac{e}{4} - \frac{1}{4e}$$

$$= \left|\frac{e^2 - 5}{4e}\right|$$

12. Given curves are

$$y = \ln x,\ y = 2^x,\ x = \frac{1}{2} \text{ and } x = 2$$



Hence, the required area

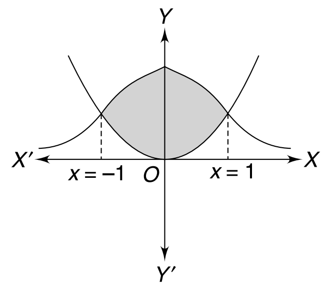
$$= \int_{1/2}^{2} (2x - \log x)\,dx$$

$$= \left(\frac{2^x}{\log 2}\right)_{1/2}^{2} - (x\log x - x)_{1/2}^{2}$$

$$= \left(\frac{4}{\log 2} - \frac{\sqrt{2}}{\log 2}\right) - \left(2\log 2 - 2 - \frac{1}{2}\log\left(\frac{1}{2}\right)\right)$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right) \text{ sq.u.}$$

13. Given curves are $y = x^2$ and $y = \dfrac{2}{1 + x^2}$



Hence, the required area

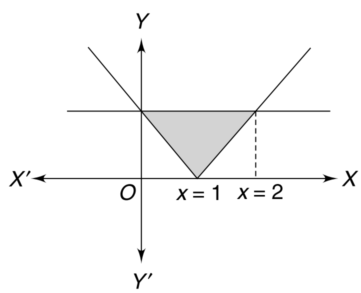
$$= 2\int_{0}^{1}\left(\frac{2}{1 + x^2} - x^2\right)dx$$

$$= 2\left(2\tan^{-1}x - \frac{x^3}{3}\right)_0^1$$

$$= 2\left(2\cdot\frac{\pi}{4} - \frac{1}{3}\right)$$

$$= \left(\pi - \frac{2}{3}\right) \text{ sq.u.}$$

14. Given curves are $y = |x - 1|$ and $y = 1$

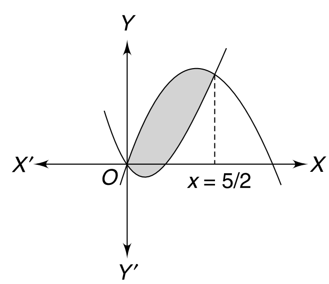


Hence, the required area

$$= \frac{1}{2} \times 2 \times 1 = 1 \text{ sq.u.}$$

15. Given curves are

$$y = 4x - x^2 \text{ and } y = x^2 - x$$



Now, $A_1 = \int_0^{5/2} ((4x - x^2) - (x^2 - x))\,dx$

$= \int_0^{5/2} (5x - 2x^2)\,dx$

$= \left(\frac{5x^2}{2} - \frac{2x^3}{3}\right)_0^{5/2}$

$= \left(\frac{125}{8} - \frac{250}{24}\right) = \frac{125}{24}$ sq.u.

and $A_2 = \int_0^{1} (0 - (x^2 - x))\,dx$

$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^{1}$

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq.u.

Now, Area of the above x-axis

$= A_1 - A_2$

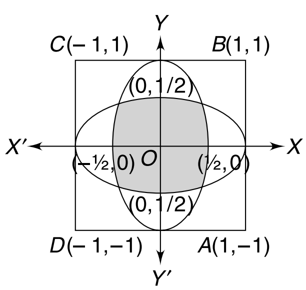
$= \frac{125}{24} - \frac{1}{6} = \frac{121}{24}$

Hence, the required ratio

$= (A_1 - A_2) : A_2$

$= \frac{121}{24} : \frac{1}{6} = 121 : 4$

16.



Equation of the sides of a square are

$x = 1,\ x = -1,\ y = 1$ and $y = -1$

Let (x, y) be any point inside the region S.

Thus, according to the given conditions

$\sqrt{x^2 + y^2} < |1 + x|, |1 - x|, |1 + y|, |1 - y|$

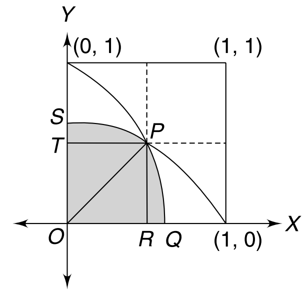
$\Rightarrow \quad (x^2 + y^2) < (1 + x)^2, (1 - x)^2, (1 + y)^2, (1 - y)^2$

$y < 1 - 2x,\ y^2 < 1 + 2x$

and $\quad x^2 < 1 - 2y,\ x^2 < 1 + 2y$

The region S is the region lying inside the four parabolas $y^2 = 1 - 2x,\ y^2 = 1 + 2x$

$x^2 = 1 + 2y,\ x^2 = 1 - 2y$



The parabola $y^2 = 1 - 2x,\ x^2 = 1 - 2y$ intersect the line $y = x$ and the point of intersection is (a, a), where $a = \sqrt{2} - 1$.

Let $\quad A$ = Area of the region $OPQO$

= Area of OPR + area $RPQR$

$= \frac{1}{2}a^2 + \int_a^{1/2} \sqrt{(1 - 2x)}\,dx$

$= \frac{1}{2}a^2 - \left(\frac{2}{3} \times \frac{1}{2} \times (1 - 2x)^{3/2}\right)_a^{1/2}$

$= \frac{1}{2}a^2 + \frac{1}{3}(1 - 2a)^{3/2}$

$= \frac{a^2}{2} + \frac{a^3}{3}$

$= \frac{a^2}{6}(3 + 2a)$

$= \frac{1}{6}(3 - 2\sqrt{2})(3 + 2\sqrt{2} - 2)$

$= \frac{1}{6}(4\sqrt{2} - 5)$ sq.u.

Hence, the required area

= Area of the shaded region

$= 4(2A)$

$= 8A$

$= \frac{8}{6}(4\sqrt{2} - 5)$

$= \frac{4}{3}(4\sqrt{2} - 5)$ sq.u.

17. It is given that,

$\frac{dy}{dx} = (2x + 1)$

$dy = (2x + 1)dx$

Integrating, we get

$y = x^2 + x + c$

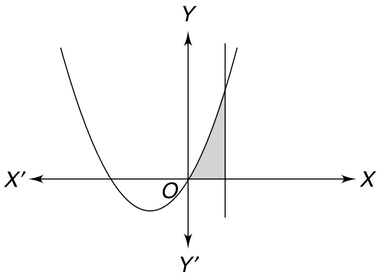
which is passing through the point (1, 2), so

$2 = 1 + 1 + c$

$\Rightarrow \quad c = 0$

Hence, the equation of the curve is

$$y = x^2 + x$$



Hence, the required area

$$= \int_0^1 (x^2 + x)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \text{ sq.u.}$$

18. Given curves are

$$y = \tan^n x,\ y = 0,\ x = 0,\ x = \frac{\pi}{4}.$$

Now, $A_n = \int_0^{\frac{\pi}{4}} (\tan^n x)\,dx$

Thus, $A_n + A_{n-2}$

$$= \int_0^{\frac{\pi}{4}} (\tan^n x)dx + \int_0^{\frac{\pi}{4}} (\tan^{n-2} x)dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan^{n-2} x)(\tan^2 x + 1)dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan^{n-2} x)\sec^2 x\,dx$$

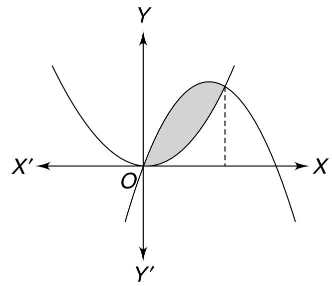
$$= \int_0^1 t^{n-2} dt,\ t = \tan x$$

$$= \left(\frac{t^{n-1}}{n-1}\right)_0^1$$

$$= \left|\frac{1}{n-1}\right|$$

19. Given curves are

$$y = x - bx^2 \text{ and } y = \frac{x^2}{b}$$



Hence, the required area = A

$$= \int_0^t \left(x - bx^2 - \frac{x^2}{b}\right)dx, \quad \left(\text{Let } t = \frac{b}{b^2+1}\right)$$

$$= \int_0^t \left(x - \frac{x^2}{t}\right)dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3t}\right)_{01}^{t}$$

$$= \left(\frac{t^2}{2} - \frac{t^2}{3}\right)$$

$$= \frac{3t^2 - 2t^2}{6}$$

$$= \frac{t^2}{6}$$

$$= \frac{1}{6}\left(\frac{b}{b^2+1}\right)^2$$

Taking logarith of both sides, we get

$$\log A = 2\log b - 2\log(b^2 + 1) - \log 6$$

$$\Rightarrow \quad \frac{1}{A}\frac{dA}{db} = \frac{2}{b} - \frac{4b}{(b^2+1)}$$

$$\Rightarrow \quad \frac{dA}{db} = A\left(\frac{2(b^2 - 2b + 1)}{b(b^2+1)}\right)$$

For maximum or minimum, $\frac{dA}{db} = 0$ gives

$b = 1$

Since $b > 0$, so $b = 1$

Thus A is maximum, when $b = 1$

20. Let the co-ordinates of P be (x, y)

Equation of the line $OA : y = 0$

Equation of the line $OB : \sqrt{3}y = x$

Equation of the line $AB : \sqrt{3}y = 2 - x$

$d(P, OA)$ = Distance of P from the line $OA = y$

$d(P, OB)$ = Distance of P from the line OB

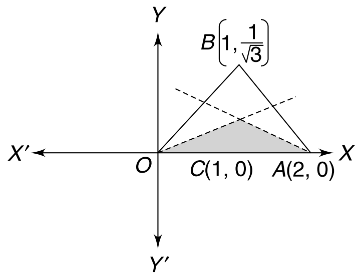
$$= \frac{|R3y - x|}{2}$$

$d(P, AB)$ = Distance of P from the line AB

$$= \frac{|R3y - x - 2|}{2}$$

It is given that,

$d(P, OA) \le \min\{d(P, OB), d(P, OA)\}$



$$y \le \min\left\{\frac{|R3y - x|}{2}, \frac{|\sqrt{3}y - x + 2|}{2}\right\}$$

$$y \le \frac{|R3y - x|}{2} \text{ and } y \le \frac{|R3y - x + 2|}{2}$$

Case I: When $y \le \frac{|R3y - x|}{2}$, then

$$y \le \frac{(x - \sqrt{3}y)}{2}$$

$\Rightarrow \quad (2 + \sqrt{3})\, y \le x$

$\Rightarrow \quad y \le (2 - \sqrt{3})x$

$\Rightarrow \quad y \le (\tan 15°)x$

Case II: When $y \le \frac{|\sqrt{3}y - x + 2|}{2}$, then

$2y \le 2 - x - \sqrt{3}y$

$\Rightarrow \quad (2 + \sqrt{3})\, y \le 2 - x$

$\Rightarrow \quad y \le (2 - \sqrt{3})(2 - x)$

$\Rightarrow \quad y \le -(\tan(15°))(x - 2)$

Hence, the area of the shaded region

$$= \frac{1}{2} \times \text{base} \times \text{height}$$

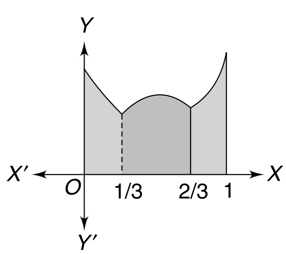
$$= \frac{1}{2} \times 2 \times (1 \cdot \tan 15°)$$

$$= \tan 15°$$

$$= (2 - \sqrt{3}) \text{ sq.u.}$$

21. Given curve is

$$f(x) = \max\{x^2, (1 - x)^2, 2x(1 - x)\}$$



From the graph, it is clear that

$$f(x) = \begin{cases} (1 - x)^2 & : 0 \le x \le \frac{1}{3} \\ 2x(1 - x) & : \frac{1}{3} \le x \le 1 \\ x^2 & : \frac{1}{3} \le x \le 1 \end{cases}$$

Hence, the required area

$$= \int_0^{1/3} (1 - x)^2 dx + \int_{1/3}^{2/3} 2x(1 - x)dx + \int_{2/3}^{1} x^2 dx$$

$$= \left(-\frac{(1 - x)^3}{3}\right)_0^{1/3} + \left(x^2 - \frac{2x^3}{3}\right)_{1/3}^{2/3} + \left(\frac{x^3}{3}\right)_{2/3}^{1}$$

$$= \left(-\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\right)$$

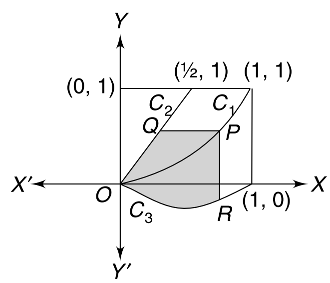
$$+\left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3$$

$$+\left(\frac{1}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^3\right)$$

$$= \frac{17}{27} \text{ sq.u.}$$

22. Let C_1 and C_2 be the graphs of dunctions $y = x^2$, $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function

$$y = f(x),\ 0 \le x \le 1,\ f(0) = 0.$$



Let the co-ordinates P be (x, x^2), where $0 \le x \le 1$.

Area $(ORPO)$ $= \int_0^x t^2 dt - \int_0^x f(t)dt$

Area $(ORPO)$ $= \int_0^x \sqrt{t}\, dt - \int_0^{x^2} \left(\frac{t^2}{2}\right)dt$

$$= \left(\frac{2}{3}t^{3/2}\right)_0^x - \left(\frac{t^2}{4}\right)_0^{x^2}$$

$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)$$

According to the question,

$$\frac{x^3}{3} - \int_0^x f(t)dt = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)$$

Differentiating both sides w.r.t. x, we get

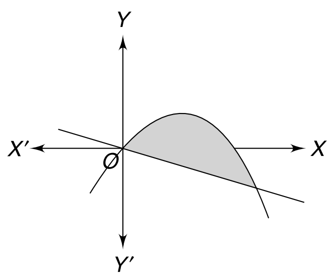
$$x^2 - f(x) = 2x^2 - x^3$$

$$\Rightarrow \quad f(x) = x^3 - x^2$$

Hence, the required curve is

$$f(x) = x^3 - x^2, \text{ where } 0 \le x \le 1.$$

23. **Case-I:** When $m < 0$



Area of the region

$$= \int_0^{1-m} ((x - x^2) - mx)\,dx$$

$$= \left((1 - m)\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^{1-m}$$

$$= \frac{1}{2}(1 - m)^3 - \frac{1}{3}(1 - m)^3$$

$$= \frac{1}{6}(1 - m)$$

Also, it is given that,

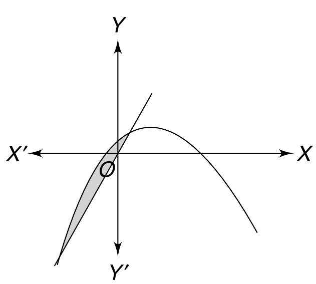
$$\frac{1}{6}(1 - m)^3 = \frac{9}{2}$$

$$\Rightarrow \quad (1 - m)^3 = 27$$

$$\Rightarrow \quad (1 - m) = 3$$

$$\Rightarrow \quad m = -2.$$

Case II: When $m > 0$



Area of the region

$$= \int_{1-m}^{0} (x - x^2 - mx)\,dx$$

$$= \left((1 - m)\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{1-m}^{0}$$

$$= -\frac{1}{6}(1 - m)^3$$

It is also given that

$$-\frac{1}{6}(1 - m)^3 = \frac{9}{2}$$

$$\Rightarrow \quad (1 - m)^3 = -27$$

$$\Rightarrow \quad 1 - m = -3$$

$$\Rightarrow \quad m = 4.$$

24. We have,

$$f(x) = \begin{cases} x^2 + ax + b & : x > -1 \\ 2x & : -1 \le x \le 1 \\ x^2 + ax + b & : x > 1 \end{cases}$$

As f is continuous on R, f is continuous at –1 and 1,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x) = f(-1)$$

and $$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

Thus, $1 - a + b = -2$ and $1 + a + b = 2$

$\Rightarrow$ $a - b = 3$ and $a + b = 1$

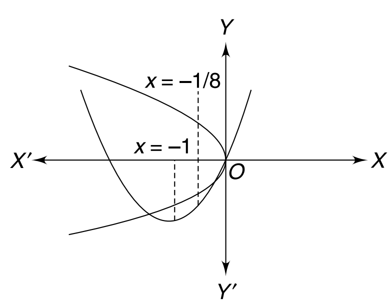
$\Rightarrow$ $a = 2$ and $b = -1$.

Hence, $$f(x) = \begin{cases} x^2 + 2x - 1 & : x > -1 \\ 2x & : -1 \le x \le 1 \\ x^2 + 2x - 1 & : x > 1 \end{cases}$$

Let us find the point of intersection of

$$x = -2y^2 \text{ and } y = f(x).$$

These two curves meet at (–2, –1)



The required area

$$= \int_{-2}^{-1/8} \left(\sqrt{\frac{-x}{2}} - f(x)\right)dx$$

$$= \int_{-2}^{-1/8} \left(\sqrt{\frac{-x}{2}}\right) dx - \int_{-2}^{-1/8} f(x)\, dx$$

$$= \int_{-2}^{-1/8} \left(\sqrt{\frac{-x}{2}}\right) dx - \int_{-2}^{-1} (x^2 + 2x - 1)\, dx$$

$$- \int_{-1}^{-1/8} f(x)\, dx$$

$$= \left(\frac{-2}{3\sqrt{2}} (-x)^{3/2}\right)_{-2}^{-1/8} - \left(\frac{x^3}{3} + x^2 - x\right)_{-2}^{-1}$$

$$-(-x^2)_{-1}^{-1/8}$$

$$= -\frac{2}{3\sqrt{2}} \left(\left(\frac{1}{8}\right)^{3/2} - 2^{3/2}\right) - \left(-\frac{1}{3} + 1 + 1\right)$$

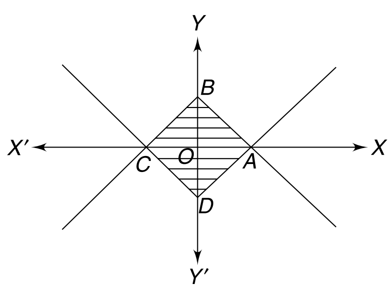
$$+\left(-\frac{8}{3} + 4 + 2\right) - \left(\frac{1}{64} - 1\right)$$

$$= \frac{\sqrt{2}}{3} \left(\sqrt{2} - 2^{-9/2}\right) + \frac{5}{3} + \frac{63}{64}$$

$$= \frac{881}{192}$$

25. Let $b \neq 0$ and for $j = 0, 1, 2, ..., n$, let S_1 be the area of the region bounded by the y-axis and the curve xe^{xy}, $\frac{j\pi}{b} \leq y \leq \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, ..., S_n$ are in G.P. Also, find their sum for $a = -1$ and $b = \pi$ **[IIT-JEE, 2001]**

26. The given curves are $y = |x| - 1$ and $y = 1 - |x|$



Hence, the required area

$$= ar(\text{Quad } ABCD)$$

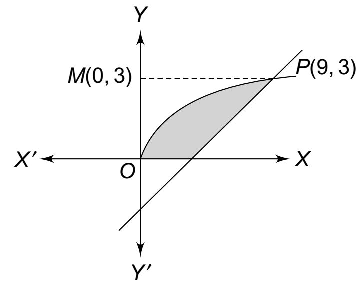
$$= 4ar(\Delta OAB)$$

$$= 4 \times \frac{1}{2} \times 1 \times 1$$

$$= 2 \text{ sq.u.}$$

27. The given curves are

$$y = \sqrt{x},\ 2y + 3 = x \text{ and } y = 0$$



Hence, the required area

$$= \int_0^3 (2y + 3 - y^2)\, dy$$

$$= \left(y^2 + 3y - \frac{y^3}{3}\right)_0^3$$

$$= 9 + 9 - 9$$

$$= 9 \text{ sq.u.}$$

28. Given $\dfrac{dy}{dx} = \dfrac{(x+1)^2 + (y-3)}{(x+1)}$

$$\Rightarrow \quad \frac{dy}{dx} = (x+1) + \frac{y}{(x+1)} - \frac{3}{(x+1)}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{(x+1)} = (x+1) - \frac{3}{(x+1)}$$

which is a linear differential equation.

Thus, $\text{IF} = e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = \dfrac{1}{(x+1)}$

Therefore, the solution is

$$\frac{y}{x+1} = \int\left(1 - \frac{3}{(x+1)^2}\right) dx$$

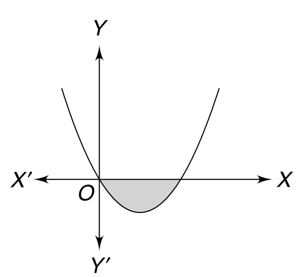
$$\Rightarrow \quad \frac{y}{x+1} = x + \frac{3}{x+1} + c$$

Put $x = 2$ and $y = 0$, then $c = -3$

Hence, the equation of the curve is

$$\frac{y}{x+1} = x + \frac{3}{x+1} - 3$$

$$\Rightarrow \quad y = x^2 + x + 3 - 3x - 3 = x^2 - 2x.$$



Hence, the required area

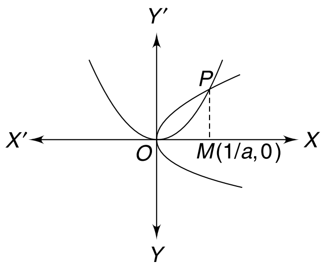
$$= \int_0^2 (0 - (x^2 - 2x))dx$$

$$= \left(x^2 - \frac{x^3}{3}\right)_0^2$$

$$= \left(4 - \frac{8}{3}\right) = \frac{4}{3} \text{ sq.u.}$$

29. The given curves are

$$y = ax^2 \text{ and } x = ay^2, \text{ where } a > 0$$



Hence, the required area

$$= \int_0^{1/a} \left(\sqrt{\frac{x}{a}} - ax^2\right)dx$$

$$= \left(\frac{1}{\sqrt{a}} \cdot \frac{2}{3} x^{3/2} - \frac{1}{3}ax^3\right)_0^{1/a}$$

$$= \left(\frac{1}{\sqrt{a}} \cdot \frac{2}{3} a^{3/2} - \frac{1}{3}a \cdot \frac{1}{a^3}\right)$$

$$= \frac{1}{3a^2}$$

It is also given that,

$$\frac{1}{3a^2} = 1$$

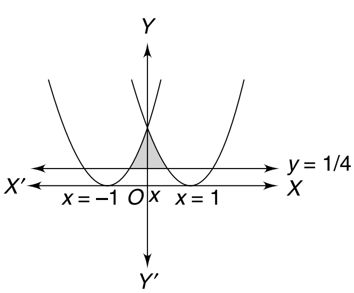
$$\Rightarrow \quad a^2 = \frac{1}{3}$$

$$\Rightarrow \quad a = \pm\sqrt{\frac{1}{3}}$$

$$\Rightarrow \quad a = \frac{1}{\sqrt{3}}, \text{ since } a > 0.$$

30. The given curves are

$$y = (x + 1)^2, \; y = (x - 1)^2 \text{ and } y = 1/4$$



Hence, the required shaded area

$$= \int_{-1/2}^{0} \left((x + 1)^2 - \frac{1}{4}\right)dx + \int_0^{1/2} \left((x - 1)^2 - \frac{1}{4}\right)dx$$

$$= \left(\frac{(x + 1)^3}{3} - \frac{1}{4}x\right)_{-1/2}^{0} + \left((x - 1)^3/3 - \frac{1}{4}x\right)_0^{1/2}$$

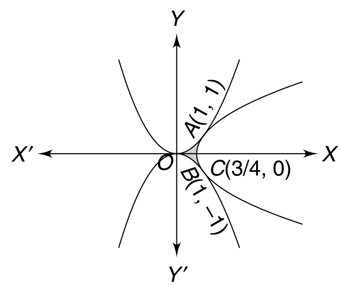
$$= \frac{1}{3} - \left(\frac{1}{24} + \frac{1}{8}\right) - \frac{1}{24} - \frac{1}{8} + \frac{1}{3}$$

$$= \frac{1}{3} - 2\left(\frac{1}{24} + \frac{1}{8}\right) + \frac{1}{3}$$

$$= \frac{2}{3} - \frac{2}{6} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq.u.}$$

31. The given curves are

$$x^2 = y, \; x^2 = -y \text{ and } y^2 = 4x - 3$$



Hence, the required area

$$= 2\left(\int_0^1 x^2 dx - \int_{3/4}^{1} \sqrt{4x - 3}\,dx\right)$$

$$= 2\left(\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2}{3} \cdot \frac{(4x - 3)^{3/2}}{4}\right)_{3/4}^{1}\right)$$

$$= 2\left(\left(\frac{1}{3}\right) - \left(\frac{1}{6} - 0\right)\right)$$

$$= 2 \cdot \frac{1}{6} = \frac{1}{3}$$

32. The given system of equations

$4x^2 f(-1) + 4xf(1) + f(2) = 3x^2 + 3x$

is satisfied for 3 distinct real numbers a, b and c.

Comparing the co-efficients of x^2, x and constant terms, we get

$$4f(-1) = 3, \; 4f(1) = 3, \; f(2) = 0$$

Let $\quad f(x) = ax^2 + bx + c$

Given $\quad f(-1) = \dfrac{3}{4}$

$$\Rightarrow \quad a - b + c = \frac{3}{4}$$

$$f(1) = \frac{3}{4}$$

$$\Rightarrow \quad a + b + c = \frac{3}{4}$$

Thus, $b = 0$

And $f(2) = 0$

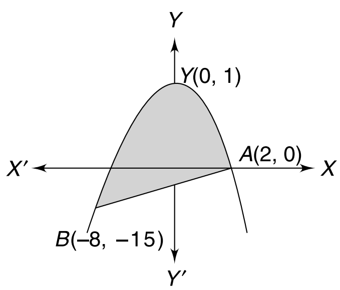
$$\Rightarrow \quad 4a + 2b + c = 0$$

$$\Rightarrow \quad c = -4a$$

Solving, we get

$$a = -\frac{1}{4}, b = 0, c = 1$$

Thus, $f(1) = -\frac{1}{4}x^2 + 1$



Let the co-ordinates of B be $B\left(t, 1 - \frac{t^2}{4}\right)$

Let $m_1 = m(AV) = \frac{1-0}{0-2} = -\frac{1}{2}$

$$m_2 = m(BV) = \frac{1 - \frac{t^2}{4} - 1}{t - 0} = -\frac{t}{4}$$

$\because \angle AVB = \frac{\pi}{2}$, so

$$m_1 \times m_2 = -1$$

$$\Rightarrow \quad \left(-\frac{1}{2}\right)\left(-\frac{t}{4}\right) = -1$$

$$\Rightarrow \quad t = -8$$

Therefore, the co-ordinates of $B = (-8, -15)$.

Hence, the required area

$$= \int_{-8}^{2} \left(\left(1 - \frac{x^2}{4}\right) - \frac{3}{2}(x-2)\right) dx$$

$$= \int_{-8}^{2} \left(4 - \frac{x^2}{4} - \frac{3}{2}x\right) dx$$

$$= \left(4x - \frac{x^3}{12} - \frac{3x^2}{4}\right)_{-8}^{2}$$

$$= \left(\left(8 - \frac{8}{12} - 3\right) - \left(-32 + \frac{128}{3} - 48\right)\right)$$

$$= \left|80 + 8 - 3 - \frac{130}{3}\right|$$

$$= \left|85 - \frac{130}{3}\right|$$

$$= \frac{125}{3}$$

33. We have $C_1 : y = \sqrt{\frac{1 + \sin x}{\cos x}}$

$$= \sqrt{\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}}$$

$$= \sqrt{\frac{2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}}$$

$$= \sqrt{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

Similarly, $C_2 : y = \sqrt{\frac{1 - \sin x}{\cos x}} = \sqrt{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}$

Hence, the area of the shaded region

$$= \int_0^{\pi/4} \left(\sqrt{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)} - \sqrt{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right) dx$$

$$= \int_0^{\pi/4} \left(\frac{\left(1 + \tan\left(\frac{x}{2}\right)\right) - \left(1 - \tan\left(\frac{x}{2}\right)\right)}{\sqrt{\left(1 - \tan^2\left(\frac{x}{2}\right)\right)}}\right) dx$$

$$= \int_0^{\pi/4} \left(\frac{2\tan\left(\frac{x}{2}\right)}{\sqrt{\left(1 - \tan^2\left(\frac{x}{2}\right)\right)}}\right) dx$$

$$= \int_0^{\sqrt{2}-1} \left(\frac{2t(2dt)}{(1 + t^2)\sqrt{1 - t^2}}\right)$$

$$= \int_0^{\sqrt{2}-1} \left(\frac{4t\,dt}{(1 + t^2)\sqrt{1 - t^2}}\right), \left(\text{Let } \tan\left(\frac{x}{2}\right) = t\right)$$

34. Given curve is

$$y^3 - 3y + x = 0$$

$$\Rightarrow \quad 3y^2 y' - 3y' + 1 = 0$$

$$\Rightarrow \quad (3y^2 - 3)y' + 1 = 0$$

$$\Rightarrow \quad y' = -\frac{1}{(3y^2 - 3)} = -\frac{1}{3(y^2 - 1)} \quad \text{...(i)}$$

$$\Rightarrow \quad y'' = -\frac{2yy'}{3(y^2 - 1)^2} \quad \text{...(ii)}$$

From Eq. (i), we get

$$y'(-10\sqrt{2}) = \frac{1}{3(1-8)} = -\frac{1}{21}$$

From Eq. (ii), we get

$$y''(-10\sqrt{2}) = \frac{2(2\sqrt{2})(-1)}{3(1-8)^2(21)} = -\frac{4\sqrt{2}}{(3)^2(7)^3}$$

35. Hence, the required area

$$= \int_a^b f(x)dx$$

$$= (xf(x))_a^b - \int_a^b (xf'(x))dx$$

$$= (bf(b) - af(a)) - \frac{1}{3}\int_a^b \left(\frac{x}{1-y^2}\right)dx$$

$$= (bf(b) - af(a)) + \frac{1}{3}\int_a^b \left(\frac{x}{y^2-1}\right)dx$$

$$= (bf(b) - af(a)) + \frac{1}{3}\int_a^b \left(\frac{x}{(f(x))^2-1}\right)dx$$

36. For $x \in (-2, 2)$,

$$(g(x))^3 - 3g(x) + x = 0 \quad \text{...(i)}$$

$$(g(-x))^3 - 3g(-x) - x = 0 \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$\Rightarrow \{(g(x))^3 + (g(x))^3\} - 3\{g(x) + g(-x)\} = 0$

$\Rightarrow \{g(x)+g(-x)\}\{(g(x))^2+(g(-x))^2-g(x)g(-x))-3\}=0$

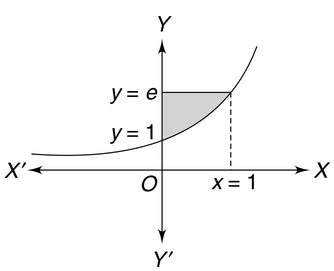
$\Rightarrow \{g(x) + g(-x)\} = 0$

Now, $\int_{-1}^{1} g'(x)dx$

$= (g(x))_{-1}^{1}$

$= g(1) - g(-1) = g(1) + g(1) = 2g(1)$

37. We have



Hence, the required area

$$= \int_1^e x\,dy$$

$$= \int_1^e \ln y\,dy$$

$$= \int_0^e \ln(1 + e - y)\,dy$$

Also, the shaded area

$$= \int_0^1 (e - e^x)\,dx$$

$$= \int_0^1 e\,dx - \int_0^1 e^x dx$$

$$= e - \int_0^1 e^x dx$$

38. Given $f(x) = 1 + 2x + 3x^2 + 4x^3$

$f'(x) = 2 + 6x + 12x^2$

> 0 for all x in R

Thus, $f(x)$ is an increasing function on R. So, $f(x)$ can have atmost one root. It is clear that $f(x)$ cannot have a poistive real root.

We have

$$f\left(-\frac{3}{4}\right) = 1 - \frac{3}{2} + \frac{27}{16} - \frac{27}{16} = -\frac{1}{2} < 0$$

and $$f\left(-\frac{1}{2}\right) = 1 - 1 + \frac{3}{4} - \frac{1}{2} = \frac{1}{4} > 0$$

Hence, $f(x)$ has a root in $\left(-\frac{3}{4}, -\frac{1}{2}\right)$.

39. Let $\alpha \in \left(-\frac{3}{4}, -\frac{1}{2}\right)$ and $t = |x| = |\alpha|$

As $-\frac{3}{4} < \alpha < -\frac{1}{2}$

$\Rightarrow -\frac{3}{4} < t < -\frac{1}{2}$

$\Rightarrow \int_0^{1/2} f(x)dx < \int_0^t f(x)dx < \int_0^{3/4} f(x)\,dx$

$\Rightarrow (x + x^2 + x^3 + x^4)_0^{1/2}$

$< \int_0^t f(x)dx < (x + x^2 + x^3 + x^4)_0^{3/4}$

$\Rightarrow \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) < \int_0^t f(x)dx$

$< \left(\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256}\right)$

$\Rightarrow \frac{15}{16} < \int_0^t f(x)dx < \frac{525}{256}$

$\Rightarrow \frac{3}{4} < \frac{15}{16} < \int_0^t f(x)dx < \frac{525}{256} < 3$

40. We have,

$f(x) = 1 + 2x + 3x^2 + 4x^3$

$\Rightarrow f'(x) = 2 + 6x + 12x^2$

$\Rightarrow \quad f''(x) = 6 + 24x$

By the sign scheme, we get that

$f'(x)$ decreases in $\left(-\infty, -\frac{1}{4}\right)$ and increases in $\left(-\frac{1}{4}, \infty\right)$

In particular $f'(x)$ decreases in $\left(-t, -\frac{1}{4}\right)$ and increases in $\left(-\frac{1}{4}, t\right)$.

41. We have $R_1 = \int_{-1}^{2} x f(x)\, dx$

$$= \int_{-1}^{2} (1 - x) f(1 - x)\, dx$$

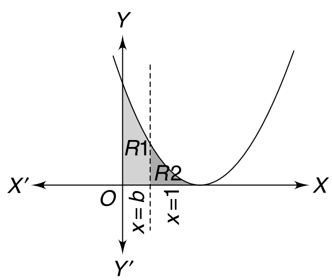
$$= \int_{-1}^{2} (1 - x) f(x)\, dx$$

$$= \int_{-1}^{2} f(x) dx - \int_{-1}^{2} x f(x)\, dx$$

$$= \int_{-1}^{2} f(x) dx - R_1$$

$$\Rightarrow \quad 2R_1 = \int_{-1}^{2} f(x) dx = R_2$$

42. The given curves are $y = (1 - x)^2$, $y = 0$ and $x = 0$



We have, $R_1 = \int_{0}^{b} (1 - x)^2 dx$

$$= \left(-\frac{(1 - x)^3}{3}\right)_0^b = \frac{1}{3}(1 - (1 - b)^3)$$

Also, $\quad R_2 = \int_{b}^{1} (1 - x^2) dx$

$$= \left(-\frac{(1 - x)^3}{3}\right)_b^1 = \frac{1}{3}((1 - b)^3)$$

Now, $\quad R_1 - R_2 = \frac{1}{4}$

$$\Rightarrow \quad \frac{1}{3}(1 - (1 - b)^3) - \frac{1}{3}(1 - b)^3 = \frac{1}{4}$$

$$\Rightarrow \quad \frac{2}{3}(1 - b)^3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

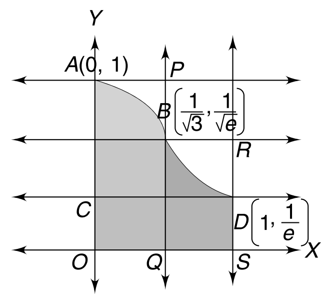
$$\Rightarrow \quad \frac{2}{3}(1 - b)^3 = \frac{1}{12}$$

$$\Rightarrow \quad (1 - b)^3 = \frac{1}{8}$$

$$\Rightarrow \quad (1 - b) = \frac{1}{2}$$

$$\Rightarrow \quad b = \frac{1}{2}$$

43.



$S \ge \frac{1}{e}$

(Since area of a rectangle $OCDS = 1/e$),

Since $\quad e^{-x^2} \ge e^{-x}, \ \forall\ x \in [0, 1]$

$$\Rightarrow \quad S \ge \int_0^1 e^{-x} dx = \left(1 - \frac{1}{e}\right)$$

Area of a rectangle $OAPQ$ +Area of a rectangle $QBRS > S$

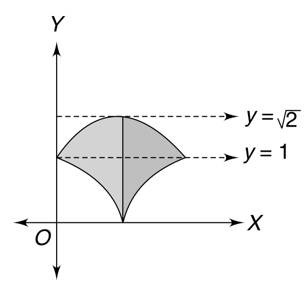
$$\Rightarrow \quad S \le \frac{1}{\sqrt{2}}(1) + \left(\frac{1 - 1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$$

Since $\quad \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) \le 1 - \frac{1}{e}$

Thus, option (c) is incorrect.

44. The given curves are

$$y = \sin x + \cos x \text{ and } y = |\cos x - \sin x|$$



Hence, the required area

$$= \int_0^{\pi/2} (\sin x + \cos x)\, dx$$

$$- \left(\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx\right)$$

$$= -(\cos x)_0^{\pi/2} + (\sin x)_0^{\pi/2}$$

$$-\left((\sin x)_0^{\pi/4} + (\cos x)_0^{\pi/4} - (\cos x)_{\pi/4}^{\pi/2}\right.$$

$$\left.(\sin x)_{\pi/4}^{\pi/2}\right)$$

$$= -(0 - 1) + (1 - 0) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)$$

$$-\left(0 - \frac{1}{\sqrt{2}}\right) - \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= 2 - \left(\sqrt{2} - 1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\right)$$

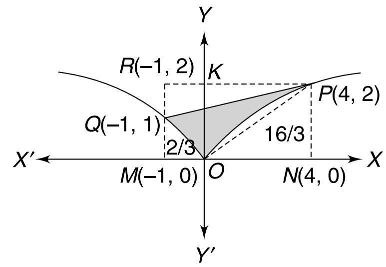
$$= 2 - (2\sqrt{2} - 2)$$

$$= 4 - 2\sqrt{2}$$

$$= 2\sqrt{2}(\sqrt{2} - 1). \text{ Sq.u.}$$

45. Shifting the origin to (–3, 0), we get the

Area $= \{(x, y) \in R^2 : y \geq \sqrt{|x|},\ 5y \leq x + 6 \leq 15\}$



Hence, the required area

= Region (*OPK*) + Region (*QLKR*)

+Region (*OLQ*) – Triangle (*PQR*)

$$= \frac{8}{3} + 1 + \frac{1}{3} - \frac{5}{2} = 4 - \frac{5}{2} = \frac{3}{2} \text{ sq.u.}$$

CHAPTER

4 Differential Equation

CONCEPT BOOSTER

1. Introduction

A differential equation is a mathematical equation for an unknown function of one or more variables that relates the values of the function itself and its derivatives of various orders. It plays a prominent role in engineering, physics, economics, biology, and other disciplines.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modelled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and the velocity as the time value varies.

Newton's laws allow us (given the position, velocity, acceleration and various forces acting on the body) to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly.

An example of modelling a real world problem using differential equations is the determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the acceleration due to air resistance. The gravity is considered constant, and the air resistance may be modelled as proportional to the ball's velocity. It means that the ball's acceleration, which is a derivative of its velocity, depends on the velocity (and the velocity depends on time).

Finding the velocity as a function of time involves solving a differential equation.

Differential equations are mathematically, studied from different perspectives, mostly concerned with their solutions—the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulae; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

2. Definition

An equation that involves the dependent variables, independent variables and the derivatives of dependent variables is called a differential equation, i.e.

$$\left\{x,\ y,\ \frac{dy}{dx}\right\} = 0 \text{ or } \left\{x,\ y,\ \frac{dx}{dy}\right\} = 0$$

For examples,

1. $\frac{dy}{dx} = \sin x$
2. $\frac{dy^3}{dx^3} + 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 10y = 0$
3. $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{2 + \left(\frac{d^2y}{dx^2}\right)}$
4. $\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = u + v + w$

each are differential equations.

3. Ordinary Differential Equation

A differential equation that involves the derivative with respect to a single independent variable is called an ordinary differential equation.

It can also be expressed as a function of variables x and y, and the derivatives of y w.r.t. x, i.e.

$$f\left(x,\ y,\ \frac{dy}{dx}\right) = 0.$$

Thus,

1. $\frac{dy}{dx} + 10y = 0$

2. $\dfrac{dy}{dx} + 2y = \sin x$

3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$

4. $\dfrac{d^ny}{dx^n} + ny = 0$

are ordinary differential equations.

4. Partial Differential Equation

A differential equation which contains two or more independent variables and partial derivatives w.r.t. them is called a partial differential equation.

For examples,

1. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

2. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = x$

5. Order of a Differential Equation

The order of a differential equation is the highest order derivative of a differential equation.

Example 1. The order of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

is 2.

Example 2. The order of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^{100} + \frac{d^4y}{dx^4} + \frac{dy}{dx} + 50y = 0$$

is 4.

Example 3. The order of the differential equation

$$\sin\left(\frac{dy}{dx}\right) = x$$

is 1.

Example 4. The order of the differential equation of the curve

$$y = (a + b)e^{3x} + (c + d)e^{-2x}$$

$$a, b, c, d, \in R$$

is 2.

Example 5. The order of the differential equation of the curve

$$y = (a + b)\cos((c + d)x + e),$$

where $a, b, c, d, e, \in R$

is 3.

Example 6. The order of the differential equation whose genral solution is given by

$$y = (c_1 + c_2)\cos(x + c^3) - c^4 e^{x + c5},$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants is 3.

6. Degree of a Differential Equation

The index power of the highest order derivative of a differential equation is called the degree of a differential equation, and is free from any radical sign.

Example 1. The degree of the differential equation

$$\left(\frac{d^3y}{dx^3}\right)^{2102} + \frac{d^4y}{dx^4} + \left(\frac{dy}{dx}\right)^{2013} + 5005y = 0$$

is 1.

Example 2. The degree of the differential equation

$$\sqrt[4]{\left(1 + \frac{d^2y}{dx^2}\right)} = \sqrt[5]{\left(10 + \frac{d^2y}{dx^2}\right)}$$

is 5.

Example 3. The degree of the differential equation

$$\frac{dy}{dx} + \frac{dx}{dy} = y$$

is 2.

Example 4. The degree of the differential equation

$$e^{\frac{dy}{dx}} = x + 1$$

is not defined.

Example 5. The degree of the differential equation

$$\log\left(1 + \frac{dy}{dx}\right) = x$$

is not defined.

Example 6. The order and the degree of the differential equation

$$\left(1 + 3\frac{dy}{dx}\right)^{2/3} = 4\left(\frac{d^3y}{dx^3}\right)$$

are 3 and 3, respesctively

Example 7. The order and the degree of the differential equation

$$\sqrt[5]{1 + \frac{d^2y}{dx^2}} = \sqrt[4]{\left(y + \left(\frac{dy}{dx}\right)^5\right)}$$

are 2 and 4, respectively.

7. Linear and Non-linear Differential Equation

A differential equation in which the dependent variables and all its derivatives present occur in the first degree only and no products of dependent variables and/or derivatives occur is known as linear differential equation.

A differential equation which is not linear is called a non-linear differential equation.

For examples,

1. $\frac{dy}{dx} + 5y = x$ is linear.
2. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ is linear.
3. $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 10x = 0$ is not linear.
4. $\frac{d^{2013}y}{dx^{2013}} + 5\frac{dy}{dx} + 10y = 0$ is linear.

8. Formation of a Differential Equation

Let the equation of the curve be

$$f(x, y, c_1, c_2 ..., c_n) = 0, \quad ...(i)$$

where $c_1, c_2, ..., c_n$ are n arbitrary constants apart from x and y.

To obtain the differential equation of the curve (i), we have to differentiate (i) up to n times w.r.t. x and eliminate those arbitrary constants from $(n + 1)$ equations, which yield the required differential equation.

9. Differential Equation of First Order and First Degree

A differential equation of the first order and the first degree can be written as the form of

$\frac{dy}{dx} = f(x, y)$ or $M dx \pm N dy = 0$, where M and N are functions of x and y or constants.

All differential equations of the first order cannot be always solvable. However they can be solved by suitable methods if they belong to any of the following standard forms.

(i) A differential equation is of the form $\frac{dy}{dx} = f(x)$.
(ii) Variable separable form.
(iii) Reducible to variable separable form.
(iv) Homegeneous differential equation.
(v) Reducible to homegeneous differential equation.
(vi) Linear differential equation.
(vii) Bernoulli's differential equation.
(viii) Exact differential equation.
(ix) Reducible to exact differential equation.
(x) Clairauts differential equation.

9.1 Differential Equation of the Form

$$\frac{dy}{dx} = f(x)$$

$$\Rightarrow \quad dy = f(x)\,dx$$

Integrating, we get

$$\Rightarrow \quad \int dy = \int f(x)\,dx$$

Thus, $y = \varphi(x) + c$ is the required solution.

9.2 Variable Separable Form

Let the differential equation be

$$\frac{dy}{dx} = f(x) \cdot f(y)$$

$$\Rightarrow \quad \frac{dy}{f(y)} = f(x)\,dx$$

Integrating, we get

$$\int \frac{dy}{f(y)} = \int f(x)\,dx$$

$$\Rightarrow \quad \varphi(y) = \psi(x) + c,$$

which is the required solution.

9.3 Reducible to Variable Separable Form

Let a differential equation is of the form

$$\frac{dy}{dx} = f(ax + by + c) \quad ...(i)$$

Let $(ax + by + c = v)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

The Eq. (i) reduces to

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v)$$

$$\Rightarrow \quad \frac{dv}{bf(v) + a} = dx$$

Integrating, we get

$$\Rightarrow \quad \int \frac{dv}{bf(v) + a} = \int dx$$

$$\Rightarrow \quad \varphi(v) = x + c$$

$$\Rightarrow \quad \varphi(ax + by + c) = x + c,$$

which is the required solution.

9.4 Homogeneous Differential Equation

Homogeneous Equation

If the degree of each term throughout the equation is the same, it is called a homogeneous equation.

For examples,

1. $x^2 + y^2 = 0$, $x^2 + xy + y^2 = 0$
   are the homogeneous equations of 2nd degree.
2. $x^3 + x^2y + y^3 = 0$,
   $x^3 + xy^2 + x^2y + y^3 = 0$
   are the homogeneous equation of 3rd degree.

A homogeneous differential equation is of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} \quad \text{...(i)}$$

where the degree of $f(x, y)$ is the same as the degree of $g(x, y)$.

In this case, divide the numerator and the denominator by the highest power of x.

Thus the given differential equation reduces to

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{...(ii)}$$

Let $\frac{y}{x} = v \Rightarrow y = vx.$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

The Eq. (ii) reduces to

$$v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow \quad \frac{dv}{f(v) - v} = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \varphi(v) = \log|x| + c$$

$$\Rightarrow \quad \varphi\left(\frac{y}{x}\right) = \log|x| + C$$

which is the required solution.

9.5 Reducible to Homogeneous Differential Equation

Let a differential equation is of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a^2x + b^2y + c_2},$$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, ...(i)

The Eq. (i) can be reduced into homogeneous equation form by means of suitable substitutions.

$x = X + \alpha$, $y = Y + \beta$, where α, β are constants.

$$\Rightarrow \quad dx = dX,\ dy = dY$$

Thus, $\frac{dy}{dx} = \frac{dY}{dX}.$

Now, Eq. (i) can be reduced to

$$\frac{dY}{dX} = \frac{a_1(X + \alpha) + b_1(Y + \beta) + c_1}{a_2(X + \alpha) + b^2(Y + \beta) + c_2}$$

$$= \frac{(a_1X + b_1Y) + (a_1\alpha + b_1\beta + c_1)}{(a_2X + b_2Y) + (a_2\alpha + b_2\beta + c_2)} \quad \text{...(ii)}$$

Let us choose α and β in such a way that

$a_1\alpha + b_1\beta + c_1 = 0$

and $a_2\alpha + b_2\beta + c_2 = 0.$

Solving, we get

$$\alpha = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } \beta = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}.$$

Then the Eq. (ii) reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y} \quad \text{...(iii)}$$

which is a homogeneous equation of the first degree.

Let $\frac{Y}{X} = V$

$$\Rightarrow \quad \frac{dY}{dX} = V + \frac{XV}{dX}$$

Then the Eq. (iii) reduces to

$$V + X\frac{dV}{dX} = \frac{a_1 + b_1V}{a_2 + b_2V}.$$

Integrating, we get

$$\varphi(V) = \log|X| + k$$

$$\Rightarrow \quad \varphi\left(\frac{Y}{X}\right) = \log(X) + c$$

$$\Rightarrow \quad \varphi\left(\frac{y - \beta}{x - \alpha}\right) = \log(x - \alpha) + c$$

which is the required solution of the given differential equation.

9.6 Linear Differential Equation

A differential equation is said to be linear differential eqation of the first order, when the dependent variable and its derivatives appear only in the first degree.

Let a differential equation is of the form

$$\frac{dy}{dx} + Py = Q \qquad ...(i)$$

where P and Q are either functions of x or constants.

Here IF $= e^{\int p\,dx} = e^{px}$.

Multiplying both sides of Eq. (i) by IF, we get

$$e^{px}\frac{dy}{dx} + Pe^{px}y = Q\cdot e^{px}$$

$$\Rightarrow \quad \frac{d}{dx}(ye^{px}) = Q\cdot e^{px}$$

Integrating, we get

$$y\cdot e^{px} = \int Q\cdot e^{px}dx + c$$

$$\Rightarrow \quad y\cdot(\text{I.F}) = \int Q\cdot(\text{I.F})\,dx + c$$

$$\Rightarrow \quad y\cdot(\text{I.F}) = \varphi(x) + c$$

$$\Rightarrow \quad y\cdot e^{px} = \varphi(x) + c$$

$$\Rightarrow \quad y = (\varphi(x) + c)e^{-px}$$

which is the required solution of the given differential equation.

Note: Sometimes it is convenient to express the differential equation in the form

$$\frac{dx}{dy} + Px = Q$$

where P and Q are either functions of y or constants.

Then IF $= e^{\int P\,dy} = e^{Py}$

Thus the solution is $x(\text{IF}) = \int Q(\text{I.F})\,dx + c$

9.7 Bernouli's Differential Equation

A differential equation is of the form

$$\frac{dy}{dx} = Py = Qy^n \qquad ...(i)$$

where P and Q are either functions of x alone or constants.

Dividing both sides of Eq. (i) by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q,$$

Let $\quad y^{1-n} = v \Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}.$

Thus, $\quad \frac{dv}{dx} + P(1-n)v = Q(1-n)$

which is a linear differential equation.

$$\text{IF} = e^{\int P(1-n)\,dx} = e^{(1-n)\int p\,dx}$$

Thus the required solution is

$$v(\text{IF}) = \int Q\cdot(\text{IF})\,dx + c$$

$$\Rightarrow \quad y^{(1-n)}\cdot e^{(1-n)\int p\,dx} = \int Q\left(e^{(1-n)\int p\,dx}\right)dx + c$$

is the required solution of the given differential equation.

9.8 Exact Differential Equation

A differential equation is said to be an exact differential equation, if it is formed by the equation, an exact differential to zero.

Thus a differential equation is of the form

$M(x, y)dx + N(x, y)dy = 0$

said to be exact, if the expression

$M(x, y)dx + N(x, y)dy$

is the exact differential of some function $u(x, y)$ i.e.

$$du = Mdx + Ndy = 0$$

Its solution is given by $u(x, y) = c,\ \forall\ c \in R$.

For examples,

1. $e^x\cdot\cos y\,dy + e^x\cdot\sin y\,dx = 0$
   is an exact differential equation.
   Since, it is derived by
   $d(e^x\sin y) = 0$
   $\Rightarrow \quad e^x\sin y = c$
   which is the required solution.
2. $x\,dy + y\,dx = 0$
   is an exact differential equation.
   $\Rightarrow \quad d(xy) = 0$
   $\Rightarrow \quad xy = c$
   which is the required solution.

General Form of Variable Seperation (By Inspection)

Dear friends you should remember the follwoing exact differentials. These help us to find the integrating factors.

1. $d(x \pm y) = dx \pm dy$
2. $d(xy) = x\,dy + y\,dx$
3. $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
4. $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$
5. $d[\log(x+y)] = \frac{dx + dy}{x + y}$
6. $d[\log(xy)] = \frac{x\,dy + y\,dx}{xy}$
7. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$
8. $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$
9. $d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right] = \frac{x\,dx + y\,dy}{x^2 + y^2}$

10. $d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right] = \frac{x\,dy - y\,dx}{x^2 - y^2}$

11. $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$

12. $d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{x^2 + y^2}$

13. $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2 + y^2}$

14. $d\left(\sqrt{x^2 + y^2}\right) = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$.

15. When the expression is in the form of $\rightarrow (x^2 + y^2)$

Let $x = r\cos\theta,\ y = r\sin\theta,$

where $x^2 + y^2 = r^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow\quad x\,dx + y\,dy = r\,dr,\ x\,dy - y\,dx = r^2 d\theta$

16. When the expression is in the form of: $\rightarrow (x^2 - y^2)$

Let $x = r\sec\theta,\ y = r\tan\theta,$

where $x^2 - y^2 = r^2,\ \theta = \sin^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow\quad x\,dx - y\,dy = r\,dr,\ x\,dy - y\,dx = r^2\sec\theta\,d\theta$

10. Orthogonal Trajectories

If a curve intersects a family of curves at right angles, then previous one is the orthogonal trajectory to the later one.

The differential equation of the orthogonal trajectories of the curves $f\left(x, y, \frac{dy}{dx}\right) = 0$ is the family of the curves, whose differential equation is $f\left(x, y, -\frac{dx}{dy}\right) = 0$.

Rule to find out the orthogonal trajectory:

(i) Let the equation of the family of curves is

$f(x, y, c) = 0$, where $c \in R$.

(ii) Form a differential equation of the given curve.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

(iv) Solve the equation $f\left(x, y, -\frac{dx}{dy}\right) = 0,$

We will get the required orthogonal trajectories.

11. First Order Higher Degree Differential Equation

The most general form of a differential equation of the first order and of higher degrees (say nth degree) is

$$\left(\frac{dy}{dx}\right)^n + f_1(x, y)\left(\frac{dy}{dx}\right)^{n-1} + f_2(x, y)\left(\frac{dy}{dx}\right)^{n-2} + \ldots + f_{n-1}(x, y)\left(\frac{dy}{dx}\right) + f_n(x, y) = 0 \quad \ldots\text{(i)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^n + f_1\left(\frac{dy}{dx}\right)^{n-1} + f_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots + f_{n-1}\left(\frac{dy}{dx}\right) + f_n = 0$$

$$\Rightarrow p^n + f_1 p^{n-1} + f_2 p^{n-2} + \ldots + f_{n-1}p + f_n = 0 \quad \ldots\text{(i)}$$

where $p = \frac{dy}{dx}$

and $f_1, f_2, \ldots, f_n$ are functions of x and y.

The Eq. (i) can aslo be written as

$$f(x, y, p) = 0 \quad \ldots\text{(ii)}$$

For Examples,

1. The equation

$$y\left(\frac{dy}{dx}\right)^5 + 2013x\left(\frac{dy}{dx}\right) - y = 0$$

is a differential equation of 5th degree.

2. The equation

$$\left(\frac{dy}{dx}\right)^{2014} + x\left(\frac{dy}{dx}\right) - 2015y = 0$$

is a differential equation of 2014th degree.

The differential equation $f(x, y, p) = 0$ can be solvable

(i) for p

(ii) for x

(iii) for y

(iv) for the first degree in x and y.

Now, we shall discuss the varoius methods to solve the above types of equations.

(i) Equations Solvable for p

Consider a differential equation of the form

$(p - f_1(x, y))(p - f_2(x, y)) \ldots (p - f_n(x, y)) = 0,$

which is solvable for p.

$\Rightarrow\quad [p - f_1(x, y)] = 0,$

$(p - f_2(x, y)) = 0 \vdots,$

$(p - f_n(x, y)) = 0$

Each of these equations is of it first order and of first degree.

Let the solutions of these equations are given by $\varphi(x, y, c_1) = 0,$

$\varphi_2(x, y, c_2) = 0 \vdots,$

$\varphi_n(x, y, c_n) = 0,$

where $c_1, c_2,$

c_n are arbitrary constants.

Hence, the general solution is

$\varphi_1(x, y, c_1)\cdot\varphi_2(x, y, c_2) \ldots \theta_n(x, y, c_n) = 0$

which contains n arbitrary constants, whereas an equation of the first order and of first degree should contain only one arbitrary constant.

So, we can take, without loss of generality,

$c_1 = c_2 = c_n = c.$

Hence, the genral solution of Eq. (i) is

$$\varphi_1(x, y, c)\cdot\varphi_2(x, y, c)$$

$$\varphi_n(x, y, c)\cdot\varphi_2(x, y, c) = 0.$$

(ii) **Equations Solvable for x**

When the differential equation

$f(x, y, p) = 0$

is solvable for x, it can be expressed in the form of $x = f(y, p)$ ...(i)

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = f\left(y, p, \frac{dp}{dy}\right)$$

$$\Rightarrow \quad \frac{1}{p} = f\left(y, p, \frac{dp}{dy}\right) \qquad \text{...(ii)}$$

which is a differential equation of two variables y and p and hence it can be solved.

Let us consider its solution be $\varphi(y, p, c) = 0$ ...(iii)

Eliminating p between (i) and (iii) gives the required solution.

(iii) **Equations Solvable for y**

If the given equation is solvable for y, it can be expressed in the form

$$y = f(x, p) \qquad \text{...(i)}$$

Differentiaing w.r.t. x, we get

$$\frac{dy}{dx} = p = \varphi\left(x, p, \frac{dp}{dx}\right) = 0$$

which is a differential equation in two variables x and p and hence its solution is possible.

Let us consider its solution be

$$\psi(x, p, c) = 0 \qquad \text{...(ii)}$$

Eliminating p between (i) and (ii) we get the required solution of Eq. (i).

(iv) **Clairaut's Differential Equation**

A differential equation of the first order in the form

$$y = Px + f(P),$$

where $P = \dfrac{dy}{dx}$,

is called the Clairaut's differential equation.

Rule to find the Clairaut's Differential Equation

1. Given $\quad y = Px + f(P)$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$\Rightarrow \quad P = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$\Rightarrow \quad (x + f'(P))\frac{dP}{dx} = 0$$

$$\Rightarrow \quad \frac{dP}{dx} = 0 \text{ or } (x + f'(P)) = 0$$

$$\Rightarrow \quad P = c \qquad \text{...(ii)}$$

$$\text{or,} \quad (x + f'(P)) = 0 \qquad \text{...(iii)}$$

Eliminating P between Eq. (i) and (ii), we get,

$$y = cx + f(c)$$

where is the general solution of Eq. (i).

Eliminating P between (i) and (iii), we get an equation, which has no constant.

This is the *singular solution* of (i).

12. Higher Order Differential Equation

(i) Let a differential equation is of the form

$$\frac{d^2y}{dx^2} = f(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$$

Integrating, we get

$$\frac{dy}{dx} = f(x) + c_1$$

Integrating again, we get

$$y = \int f(x)\,dx + c_1x + c_2$$

$$\Rightarrow \quad y = \varphi(x) + c_1x + c_2$$

which is the required solution.

(ii) Let a differential equation is of the form

$$\frac{d^2y}{dx^2} = f(y) \qquad \text{...(i)}$$

Let $\quad \dfrac{dy}{dx} = p$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\cdot\frac{dp}{dy}$$

From Eq. (i), we get,

$$p\frac{dp}{dy} = f(y)$$

$$\Rightarrow \quad p\,dp = f(y)\,dy$$

Integrating, we get

$$\frac{p^2}{2} = \int f(y)\,dy + c_1$$

$$\Rightarrow \quad p^2 = 2\int f(y)\,dy + 2c_1$$

$$= \varphi(y) + 2c_1$$

$$\Rightarrow \quad p = \pm\sqrt{\varphi(y) + 2c_1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\varphi(y) + 2c_1}$$

$$\Rightarrow \quad dy = \pm\sqrt{\varphi(y) + 2c_1}\,dx$$

Integrating, we get

$$y = \int \pm\sqrt{\varphi(y) + 2c_1}\,dx + c_2$$

$$\Rightarrow \quad y = \psi(x) + c_2$$

which is the required solution.

(iii) Let a differential equation is of the form

$$\varphi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}\right) = 0 \quad \text{...(i)}$$

Let $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dp}{dx}$

The Eq. (i) reduces to

$$\varphi\left(\frac{dp}{dx}, p\right) = 0 \quad \text{...(ii)}$$

which is an equation of the first order.

Then solve Eq. (ii) and get the required solution.

Also, Eq. (i) can be written in the form

$$\varphi\left(p\frac{dp}{dy}, p\right) = 0 \quad \text{...(iii)}$$

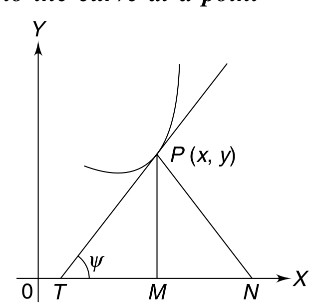
which is also an equation of the first order.

Solve it and get the required solution.

13. Applications of Differential Equation

Geometrical Application

(i) *Lengths of tangent, sub-tangent, normal and sub-normal to the curve at a point*



Let the curve be $y = f(x)$ and the point be $P(x, y)$.

Then $\tan(\psi) = \frac{dy}{dx}$.

(i) *Length of the tangent* (PT)

$$\sin\psi = \frac{y}{PT}$$

$$\Rightarrow \quad PT = y\operatorname{cosec}\psi$$

$$= y\sqrt{1 + \cot^2\psi}$$

$$= y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

(ii) *Length of the sub-tangent* (TM):

$$\tan\psi = \frac{y}{TM}$$

$$\Rightarrow \quad TM = y\cot\psi$$

$$= y\cdot\left(\frac{dx}{dy}\right)$$

(iii) *Length of the normal* (PN):

$$\cos\psi = \frac{y}{PN}$$

$$\Rightarrow \quad PN = y\sec\psi$$

$$= y\sqrt{1 + \tan^2\psi}$$

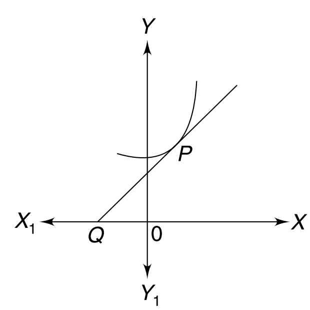
$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(iv) *Length of the sub-normal* (MN):

$$\tan\psi = \frac{y}{MN}$$

$$\Rightarrow \quad MN = y\left(\frac{dy}{dx}\right).$$

(ii) *Length of Intercepts of the Tangent by the Axes:*



(i) The *equation of the tangent* to a curve at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{\text{at}(x_1, y_1)}(x - x_1)$$

(ii) The *length of the x-intercept* is

$$x_1 - y_1\left(\frac{dx}{dy}\right)_{\text{at}(x_1, y_1)}$$

(iii) The *length of the y-intercept* is

$$= x - y\left(\frac{dx}{dy}\right)_{\text{at}(x_1, y_1)}$$

EXERCISES

Level I

(Problems based on Fundamentals)

Find the order and the degree of each of the following differential equations:

1. $\frac{d^2y}{dx^2} + 4y = 0$

2. $\left(\frac{dy}{dx}\right)^2 + \frac{dx}{dy} = 2$

3. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + xy = 0$

4. $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt[3]{c \cdot \frac{d^2y}{dx^2}}$

5. $x + \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

6. $y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

7. $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$

8. $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^6 = x\sin\left(\frac{d^2y}{dx^2}\right)$

9. $e^{\frac{dy}{dx}} = (x + 1)$

10. $\sin\left(\frac{dy}{dx}\right) + \cos\left(\frac{dy}{dx}\right) = x$

Formation of a Differential Equation

11. Find the differential equation of all non-vertical lines in a plane.
12. Find the differential equation of the family of parabolas having vertex at the origin and the axis along positive x-axis.
13. Find the differential equation of the family of all circles, whose centre lies on x-axis and touches the y-axis at the origin.
14. Find the differential equation of all parabolas whose axes are parallel to the x-axis and have latus rectum $4a$.
15. Find the differential equation corresponding to the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.
16. Find the differential equation of the system of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are arbitrary constants.
17. $y = mx$, where m is a parameter.
18. $y = A\cos x + B\sin x$, where A and B are parameters.
19. $y = ae^x + be^{-x}$, where a and b are parameters.
20. $y = \sin(bx + c)$, b and c being parameters.
21. $y = ae^{2x} + be^{3x}$, where a and b are parameters.
22. Find the differential equation of all circles touching the
    (a) x-axis at the origin.
    (b) y-axis at the origin.
23. Obtain the differential equation of all circles of the radius r.
24. Find the differential equation of all the circles in the first quadrant which touch the co-ordinate axes.
25. Find the differential equation of all the circles which pass through the origin and the centre lies on x-axis.
26. Find the differential equation of all non-vertical lines in a plane.
27. Find the differential equation of all the parabolas with the latus rectum $4a$ and whose axes are parallel to the x-axis.
28. Find the differential equation of all the ellipses having foci on the x-axis and the centre at the origin.
29. Find the differential equation of the family of parabolas having vertex at the origin and the axis along positive y-axis.
30. Find the differential equation of the family of curves $y^2 = 2c(x + \sqrt{c})$.
31. Find the order and the degree of the differential equation of all parabolas whose axis of symmetry is parallel to x-axis.

Differential equation of first order and first degres

32. Solve: $\frac{dy}{dx} = \frac{x^2 - 1}{x^2 + 1}$.

33. Solve: $\frac{dy}{dx} = \frac{1}{e^x + 1}$.

34. Solve: $\frac{dy}{dx} = \frac{\cos 2x - \cos x}{1 - \cos x}$.

35. Solve: $\frac{dy}{dx} = \frac{1}{x(x^4 + 1)}$.

36. Solve: $\frac{dy}{dx} = \frac{\sqrt{\sin x}}{\sin x \cos x}$.

37. Solve: $\frac{dy}{dx} = \frac{x^2}{(3 + 2x)^2}$.

38. Solve: $\frac{dy}{dx} = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$.

39. Solve: $\frac{dy}{dx} = (\sqrt{\tan x} + \sqrt{\cot x})$.

40. Solve: $\frac{dy}{dx} = \frac{1}{\sin^3 x + \cos^3 x}$.

41. Solve: $\frac{dy}{dx} = \frac{1}{(\sqrt{\sin x} + \sqrt{\cos x})^4}$.

Variable Separable Form

42. Solve: $\frac{dy}{dx} = 1 + x + y + xy$.

43. Solve: $\frac{dy}{dx} = e^{x-y} + x^2 \cdot e^{-y}$.

44. Solve: $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$.

45. Solve: $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$.

46. Solve: $\frac{dy}{dx} = \frac{\sin y + \cos y}{x(2\log x + 1)}$.

47. Solve: $x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.

48. Solve: $xy^2\frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$.

49. Solve: $(x+1)\frac{dy}{dx} = 2xy$.

50. Solve: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

51. Solve: $(1 + e^{2x})dy + (1 + y^2)e^x dx = 0$.

52. Solve: $\frac{dy}{dx} = e^{x+y} + x^2e^y$.

53. Solve: $y\sqrt{1+x^2} + x\sqrt{1+x^2}\frac{dy}{dx} = 0$.

54. Solve: $\sqrt{1+x^2}\, dy + \sqrt{1+y^2}\, dx = 0$.

55. Solve: $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy = 0$.

56. Solve: $xy\frac{dy}{dx} + 1 + x + y + xy = 0$.

57. Solve: $(1+x)(1+y^2)dx + (1+y)(1+x^2)dy = 0$.

58. Solve: $x\sqrt{1-y^2}\, dx + y\sqrt{1-x^2}\, dy = 0$.

59. Solve: $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$.

60. Solve: $xy\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$.

61. Solve: $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$.

Reducible to Variable Separable Form

62. Solve: $\frac{dy}{dx} = (x + y + 1)^2$.

63. Solve: $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

64. Solve: $(x+y)(dx - dy) = (dx + dy)$.

65. Solve: $\tan y\frac{dy}{dx} = \sin(x+y) + \sin(x-y)$.

66. Solve: $\frac{dy}{dx} - x\tan y\,(y - x) = 1$.

67. Solve: $\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$.

68. Solve: $(x+y)(dx - dy) = dx + dy$.

69. Solve: $\frac{dy}{dx} = \sec(x+y)$.

70. Solve: $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$.

71. Solve: $\frac{dy}{dx} = \cos(x + y + 1)$.

72. Solve: $(x^2 + 2xy + y^2 + 1)\frac{dy}{dx} = 2(x+y)$.

73. Solve: $\frac{dy}{dx} = \sin(10x + 6y)$.

74. Solve: $\frac{y}{x}\frac{dy}{dx} + \frac{2(x^2+y^2)-1}{x^2+y^2+1} = 0$.

Homogeneous Differential Equation

75. Solve: $(x^3 + y^3)dy - x^2y\,dx = 0$.

76. Solve: $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$.

77. Solve: $(x^2 - y^2)dx = 2xy\,dy$.

78. Solve: $(1 + 2e^{x/y})dx + 2e^{x/y}(1 - x/y)dy = 0$.

79. Solve: $x\left(\frac{dy}{dx} + 1\right) = y\left(1 - \frac{dy}{dx}\right)$.

80. Solve: $x\frac{dy}{dx} = x + y$.

81. Solve: $\frac{dy}{dx} = \frac{y-x}{y+x}$.

82. Solve: $2xy\frac{dy}{dx} = x^2 + y^2$.

83. Solve: $(y^2 - 2xy)dx = (x^2 - 2xy)dy$.

84. Solve: $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

85. Solve: $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$.

86. Solve: $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$.

87. Solve: $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$.

88. Solve: $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$.

89. Solve: $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$.

Reducible to Homogeneous Differential Equation

90. Solve: $\frac{dy}{dx} = \frac{x + 2y + 3}{2x + 3y + 4}$.

91. Solve: $\frac{dy}{dx} = \frac{(x + y)^2}{(x + 2)(y - 2)}$.

93. Solve: $(x + 2y + 3)\,dx = (2x + 3y + 4)\,dy$.

94. Solve: $\frac{dy}{dx} = \frac{x + 2y - 5}{2x + y - 4}$.

95. Solve: $\frac{dy}{dx} = \frac{4x + 6y - 5}{6x + 9y + 7}$.

96. Solve: $\frac{dy}{dx} = \frac{6x - 2y - 7}{3x - y + 4}$.

Linear Differential Equation

97. Solve: $y\,dx - x\,dy + \ln x\,dx = 0$.

98. Solve: $x^2\frac{dy}{dx} - 3xy = 4x^4 + 2x^2$.

99. Solve: $\frac{dy}{dx} = y\tan x - \sin x$.

100. Solve: $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$.

101. Solve: $x\,dx = \left(\frac{x}{y^2} - y\right)dy$.

102. Solve: $(x + 3y + 2)\frac{dy}{dx} = 1$.

103. Solve: $\frac{dy}{dx} + 2y = e^{3x}$.

104. Solve: $x\frac{dy}{dx} = x + y$.

105. Solve: $x\frac{dy}{dx} + y = xe^x$.

106. Solve: $x\frac{dy}{dx} + y = x\log x$.

107. Solve: $\frac{dy}{dx} + \frac{y}{x} = x^3$.

108. Solve: $\frac{dy}{dx} = y\tan x - 2\sin x$.

109. Solve: $\frac{dy}{dx} - \frac{y}{x} = 2x^2$.

110. Solve: $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$.

111. Solve: $y\,dx - (x + 2y^2)\,dy = 0$.

112. Solve: $y\,dx + (x - y^3)\,dy = 0$.

113. Solve: $x\frac{dy}{dx} - ay = x + 1$.

114. Solve: $(x + 1)\frac{dy}{dx} - ny = e^x(x + 1)^{n+1}$.

115. Solve: $(1 + y^2)\,dx = (\tan^{-1}y - x)\,dx$.

116. Solve: $(x + 3y + 2)\frac{dy}{dx} = 1$.

117. Solve: $(1 + y^2)\,dx = (xy + y^3 + y)\,dy$.

Bernoulli's Differential Equation

118. Solve: $\frac{dy}{dx} + \frac{y}{x}\log x = \frac{y}{x^2}(\log y)^2$.

119. Solve: $\frac{dy}{dx} + xy = x^2y^6$.

120. Solve: $\frac{dy}{dx} + \frac{\sin^2 y}{x} = x^3\cos^2 y$.

121. Solve: $\frac{dy}{dx} - x^3y^2 + xy = 0$.

122. Solve: $(1 - x^2)\frac{dy}{dx} + xy = xy^2$.

123. Solve: $x\frac{dy}{dx} + y = x^3y^6$.

124. Solve: $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$.

125. Solve: $\frac{dy}{dx}(x^2y^3 + xy) = 1$.

126. Solve: $(y\log x - 1)\,y\,dx = x\,dy$

127. Solve: $\frac{dy}{dx} = x^2y^3 - xy$

128. Solve: $\frac{dy}{dx} + \frac{xy}{1 - x^2} = x\sqrt{y}$.

129. Solve: $\frac{dy}{dx} + \frac{y}{x}\cdot\log y = \frac{y}{x^2}(\log y)^2$

130. Solve: $\frac{dy}{dx} + \frac{1}{x}\cdot\sin 2y = x^3\cdot\cos 2y$.

131. Solve: $(xy^2 - e^{1/x^2})\,dx - xy^2\,dy = 0$.

132. Solve: $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$.

Exact Differential Equation

133. Solve: $x\,dx + y\,dy = x\,dy - y\,dx$.

134. Solve: $x\,dy - y\,dx = x^4 dx$.

135. Solve: $x\,dy + y\,dx = \sin y\,dy$.

136. Solve: $x\,dy + y\,dx + y^2(x\,dy - y\,dx) = 0$.

137. Solve: $x\,dy + y\,dx + xy^2\,dx - x^2y\,dy = 0$.

138. Solve: $(4x - 3y)\,dx + (2y - 3x)\,dy = 0$.

139. Solve: $\left(\sin y + y\sin x + \frac{1}{x}\right)dx.$

$+ \left(x\cos y - \cos x + \frac{1}{y}\right)dy = 0$.

140. Solve: $\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \frac{y\,dx - x\,dy}{x^2}$.

141. Solve: $\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0$.

142. Solve: $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$.

143. Solve: $x + y\frac{dy}{dx} = \frac{a^2\left(x\frac{dy}{dx} - y\right)}{x^2 + y^2}$.

144. Solve: $x\,dx + y\,dy = (x^2 + y^2)\,y\,dy$.

145. Solve: $(x + y)\,dx + (x - y)\,dy = 0$.

146. Solve: $y\,dx + x(x - 1)\,dy = 0$.

147. Solve: $y\,dx + x(1 - xy)\,dy = 0$.

148. Solve: $(x + y)(dx - dy) = (dx + dy)$.

149. Solve: $dx + dy = x\,dy + y\,dx$.

150. Solve: $x\,dy - y\,dx = (x^2 + y^2)\,dx$.

151. Solve: $\left(\frac{1}{x} + \frac{1}{y}\right)(x\,dy + y\,dx) = dx + dy$.

152. Solve: $(x\,dy + y\,dx)\sqrt{x^2 + y^2} = x^2y\,dx + xy^2\,dy$.

153. Solve: $(x - x^3)\,dy = y(dx + x^2\,dy)$.

154. Solve: $(x + 2y)\,dy + y\,dx = 0$.

155. Solve: $(x + \sin y)\,dy + y\,dx = 0$.

156. Solve: $e^y\,dx + (xe^y - 2y)\,dy = 0$.

157. Solve: $2y\,dy + (\cos x\cdot\cot y - y^2)\,dx$.

$= (2xy + \sin x\cdot\mathrm{cosec}^2 y)\,dy$

158. Solve: $\frac{dy}{dx} = \left(\frac{2x - 3y + 1}{3x - 2y - 2}\right)$.

159. Solve: $x(dy + dx) = y(dx - dy)$.

160. Solve: $x(\tan^{-1}x - \log x)\,dy + y\,dx = \left(\frac{xy}{1 + x^2}\right)dx$.

161. Solve: $x(dy - x\,dx) + y\,dx = 0$.

162. Solve: $\frac{y - x\frac{dy}{dx}}{y + x\frac{dy}{dx}} = \frac{1}{x^2} + \frac{1}{y^2}$.

163. Solve: $x^2y\,dy - xy^2\,dx = x^4\,dy + x^3y\,dx$.

164. Solve: $dx + x(y\,dx + x\,dy) = e^{-ey}\,dx$.

165. Solve: $y(y^2\,dx + (x^2\,dy - xy\,dx)) = e^{-\frac{x}{y}}\,dy$.

166. Solve: $y\,dx - x\,dy + xy^2\,dx = 0$.

167. Solve: $2\left(x - y\frac{dy}{dx}\right)(x^2 + y^2) = (x^2 - y^2)\left(y - x\frac{dy}{dx}\right)$.

168. Solve: $(x^2 + y^2 + a^2)\,y\,dy = (x^2 + y^2 - a^2)\,x\,dx$.

169. Solve: $(1 + xy)\,y\,dx + x(1 - xy)\,dy = 0$.

170. Solve: $\frac{x\,dy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$.

171. Solve: $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$.

172. Solve: $x\,dx + y\,dy = m(x\,dy - y\,dx)$.

173. Solve: $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\sin^2(x^2 + y^2)}{y^3}$.

174. Solve: $x + y\frac{dy}{dx} = \frac{a^2\left(x\frac{dy}{dx} - y\right)}{x^2 + y^2}$.

175. Solve: $\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$.

176. Solve: $\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 + x^2 - y^2}{x^2 - y^2}}$.

177. Solve: $\sin\left(\frac{x}{y}\right)(y\,dx - x\,dy) = xy^2(x\,dy + y\,dx)$.

178. Solve: $x\,dy + y\,dx + y^2(x\,dy - y\,dx) = 0$.

179. Solve: $(4x^3 + e^x\sin y)\,dx + e^x\cos y\,dy = 0$.

Orthogonal Trajectories

180. Find the orthogonal trajectories of the family of the straight lines which are passing through the origin.
181. Determine the 45° trajectories of the family of concentric circles $x^2 + y^2 = a^2$.
182. Find the orthogonal trajectories of the family of the curves $ax^2 + y^2 = 1$.
183. Find the orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$, where a is a parameter.
184. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$, where a is a parameter.
185. Find the orthogonal trajectories of the family of rectangular hyperbola $xy = c^2$.
186. Find the orthogonal trajectories of the family of curves $x^2 - y^2 = c^2$.

First Order Higher Degree Differential Equation. Equation Solvable for p

187. Solve: $(x + y + p)(2x + p) = 0$.
188. Solve: $p^2 - p(e^x + e^{-x}) + 1 = 0$.
189. Solve: $p^2 + 2py\cot x = y^2$.
190. Solve: $p^2 - px - xy - y^2 = 0$.
191. Solve: $p^2y + (x - y)p - x = 0$.
192. Solve: $(p^2 - 1)xy = (x^2 - y^2)p$.
193. Solve: $xyp^2 - (x^2 + y^2)p + xy = 0$.

Equation Solvable for *x*

194. Solve: $px - yp^2 = ap$.
195. Solve: $y = 2px + y^2p^3$.
196. Solve: $y^2 \log y = xyp + p^2$.
197. Solve: $p^2y + 2px = y$.
198. Solve: $p^3 - 4xyp + 8p^2 = 0$.
199. Solve: $yp = 2p^2x + y^2p^4$.
200. Solve: $xp^2 - yp - p + 1 = 0$.

Equations Solvable for *y*

201. Solve: $y = (1 + p)x + ap^2$.
202. Solve: $y = yp^2 + 2px$.
203. $y = 2px + \tan^{-1}(xp^2)$.
204. $x^2p^4 + 2xp = y$.
205. $y + px = x^4p^2$.
206. $y = p\sin p + \cos p$.

Clairaut Differential Equation

207. Solve: $(y + 1)P - xP^2 + 2 = 0$.
208. Solve: $P^3x - P^2y - 1 = 0$.
209. Solve: $(y + 1)P - xP^2 + 2 = 0$.
210. Solve: $\sin y \cdot \cos Px - \cos y \cdot \sin Px - P = 0$.
211. Solve: $(x - a)P^2 + (x - y)P - y = 0$.
212. Solve: $y = px + a\sqrt{1 + p^2}$.
213. Solve: $y = P(x - b) + \dfrac{a}{P}$.

Higher Order Differential Equation

Differential equation is the form $\dfrac{d^2y}{dx^2} = f(x)$

214. Solve: $\dfrac{d^2y}{dx^2} = x + \sin x$.
215. Solve: $\dfrac{d^2y}{dx^2} = e^{2x} + e^x + 2014$.
216. Solve: $\dfrac{d^2y}{dx^2} = \sin^2 x$.
217. Solve: $\dfrac{d^2y}{dx^2} = \cos^3 x$.
218. Solve: $\dfrac{d^2y}{dx^2} = \dfrac{1}{\sin^2 x \cos^2 x}$.
219. Solve: $\dfrac{d^2y}{dx^2} = \sin^4 x + \cos^4 x$.
220. Solve: $\dfrac{d^2y}{dx^2} = xe^x$.

Differential equation of the form $\dfrac{d^2y}{dx^2} = f(y)$

221. Solve: $\dfrac{d^2y}{dx^2} + y = 0$.
222. Solve: $\dfrac{d^2y}{dx^2} = \dfrac{1}{y^3}$.
223. Solve: $\dfrac{d^2y}{dx^2} = \dfrac{1}{4\sqrt{y}}$.
224. Solve: $a^2\dfrac{d^2y}{dx^2} - y = 0$.
225. Solve: $\dfrac{d^2y}{dx^2} = e^{2y}$.
226. Solve: $2\dfrac{d^2y}{dx^2} = 3y^2$,

$$y(-2) = -1, \left(\frac{dy}{dx}\right)_{x=-2} = -1,$$

227. Solve: $y^3\left(\frac{d^2y}{dx^2}\right) = -1,$

$$y(1) = -1, \left(\frac{dy}{dx}\right)_{x=-1} = 0.$$

Differential equation is the form $\varphi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}\right) = 0$

228. Solve: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0.$

229. Solve: $\frac{d^2y}{dx^2} = \frac{1}{x}\frac{dy}{dx} + x.$

230. Solve: $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$

231. Solve: $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{dy}{dx}\right).$

232. Solve: $(x + a)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right).$

233. Solve: $x\frac{d^2y}{dx^2} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right).$

234. Solve: $y\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2.$

235. Solve: $\frac{d^2y}{dx^2} = 2y\left(\frac{dy}{dx}\right).$

236. Solve: $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{dy}{dx}\right).$

237. Solve: $\frac{d^2y}{dx^2} + \frac{1}{x}\left(\frac{dy}{dx}\right) - \frac{a}{x^2} = 0.$

238. Solve: $y\frac{d^2y}{dx^2} - y\left(\frac{dy}{dx}\right)\ln y = \left(\frac{dy}{dx}\right)^2.$

Geometrical Applications

239. The slope of the curve at any point is the reciprocal of twice the ordinate at the point. The curve passes through the point (4, 3).
Find the equation of the curve.

240. Find the equation of the curve whose slope at any point is $y + 2x$ and which passes through the origin.

241. Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to

$$\frac{x^4 + 2xy - 1}{x^2 + 1}.$$

242. A curve passes through (2, 0) and the slope of the tangent at $P(x, y)$ is $\frac{(x + 1)^2 + y - 3}{(x + 1)}$.
Find the equation of the curve.

243. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinate axes at A and B, P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve.

244. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve.

245. If the length of the tangent at any point on the curve $y = f(x)$ intercepted between the point and the x-axis is of length 1. Find the equation of the curve.

246. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve, given that it passes through $(-2, 1)$.

247. The ordinate and the normal at any point P on the curve meet the x-axis at points A and B respestively. Find the equation of the family of the curves satisfying the condition AB is the AM of abscissa and the ordinate of P.

248. Find the nature of the curve for which the length of the normal at any point P is equal to the radius vector.

Level II

(Mixed Problems)

1. The order and the degree of the differential equation

$$\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7y = 0$$

are

(a) 1 and 1/2 (b) 2 and 1
(c) 1 and 1 (d) 1 and 2

2. The order of the differential equation whose general solution is given by

$$y = c_1e^{2x+c_2} + c_3e^x + c_4\sin(x + c_5)$$

is

(a) 5 (b) 4
(c) 3 (d) 2

3. The differential equation of all straight lines passing through the origin is

(a) $y = \sqrt{x\frac{dy}{dx}}$ (b) $\frac{dy}{dx} = y + x$

(c) $\frac{dy}{dx} = \frac{y}{x}$ (d) none of these

4. A differential equation associated by the promitive $y = a + be^{5x} + ce^{-7x}$ is
(a) $y_3 + 2y_2 + y_1 = 0$
(b) $4y_3 + 5y_2 - 20y_1 = 0$
(c) $y_3 + 2y_2 - 35y_1 = 0$
(d) none of these

5. The differential equation of the family of curves $y = a\cos(x + b)$ is
(a) $\frac{d^2y}{dx^2} - y = 0$ (b) $\frac{d^2y}{dx^2} + y = 0$
(c) $\frac{d^2y}{dx^2} + 2y = 0$ (d) none of these

6. The differential equation, whose general solution is $y = A\sin x + B\cos x$, is
(a) $\frac{d^2y}{dx^2} + y = 0$ (b) $\frac{d^2y}{dx^2} - y = 0$
(c) $\frac{dy}{dx} + y = 0$ (d) none of these

7. The differential equation of all straight lines passing through the point (1, –1) is
(a) $y = (x + 1)\frac{dy}{dx} + 1$ (b) $y = (x + 1)\frac{dy}{dx} - 1$
(c) $y = (x - 1)\frac{dy}{dx} + 1$ (d) $y = (x - 1)\frac{dy}{dx} - 1$

8. The differential equation of the family of parabolas with focus at the origin and the x-axis as axis is
(a) $y\left(\frac{dy}{dx}\right)^2 + 4x\frac{dy}{dx} = 4y$
(b) $-y\left(\frac{dy}{dx}\right)^2 = 2x\frac{dy}{dx} - y$
(c) $y\left(\frac{dy}{dx}\right)^2 + y = 2xy\frac{dy}{dx}$
(d) $y\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + y = 0$

9. The differential equation of all lines in the xy-plane is
(a) $\frac{dy}{dx} - x = 0$ (b) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$
(c) $\frac{d^2y}{dx^2} = 0$ (d) $\frac{d^2y}{dx^2} + x = 0$

10. The solution of the differential equation
$$\frac{dy}{dx} + \frac{1 + x^2}{x} = 0$$
is
(a) $y = -\frac{1}{2}\tan^{-1}x + c$
(b) $y + \ln x + \frac{x^2}{2} + c = 0$
(c) $y = \frac{1}{2}\tan^{-1}x + c$
(d) $y - \ln x - \frac{x^2}{2} = c$

11. The solution of the differential equation
$$x\cos y\, dy = (xe^x\ln x + e^x)\,dx$$
is
(a) $\sin y = \frac{1}{x}e^x + c$
(b) $\sin y + e^x\ln x + c = 0$
(c) $\sin y = e^x\ln x + c$
(d) none of these

12. The solution of the differential equation
$$x(e^{2y} - 1)\,dy + (x^2 - 1)\,e^y\,dx = 0$$
is
(a) $e^y + e^{-y} = \ln x - \frac{x^2}{2} + c$
(b) $e^y - e^{-y} = \ln x - \frac{x^2}{2} + c$
(c) $e^y + e^{-y} = \ln x + \frac{x^2}{2} + c$
(d) none of these

13. Solution of the equation
$$(1 - x^2)\,dy + xy\,dx = xy^2dx$$
is
(a) $(y - 1)^2(1 - x^2) = 0$
(b) $(y - 1)^2(1 - x^2) = c^2y^2$
(c) $(y - 1)^2(1 + x^2) = c^2y^2$
(d) none of these

14. The general solution of the differential equation
$y\,dx + (1 + x^2)\tan^{-1}x\,dy = 0$,
is
(a) $y\tan^{-1}x = \text{c}$ (b) $x\tan^{-1}y = \text{c}$
(c) $y + \tan^{-1}x = c$ (d) $x + \tan^{-1}y = c$

15. For solving $\frac{dy}{dx} = (4x + y + 1)$, suitable substitution is
(a) $y = vx$ (b) $y = 4x + v$

(c) $y = 4x$ (d) $y + 4x + 1 = v$

16. The solution of the differential equation
$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$
is
(a) $ay^2 = e^{x^2/y^2}$ (b) $ay = e^{x/y}$
(c) $y = ex^2 + ey^2 + c$ (d) $y = ex^2 + y^2 + c$

17. $(x^2 + y^2)dy = xy\,dx$, if $y(x_0) = e$, $y(1) = 1$, the value of $x_0 =$
(a) $\sqrt{3}e$ (b) $\sqrt{e^2 - \frac{1}{2}}$
(c) $\sqrt{\frac{e^2 - 1}{2}}$ (d) $\sqrt{\frac{e^2 + 1}{2}}$

18. The solution of $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$ is
(a) $\frac{x}{y} + \frac{1}{xy} = k$ (b) $\log\left(\frac{x}{y}\right) = \frac{1}{xy} + k$
(c) $\frac{x}{y} = e^{xy} + k$ (d) $\log\left(\frac{x}{y}\right) = xy + k$

19. The solution of the equation $(x + \log y)dy + y\,dx = 0$ is
(a) $xy + y\log y = c$
(b) $xy + y\log y - y = c$
(c) $xy + \log y - x = c$
(d) none of these

20. The solution of $(x - y^3)dx + 3xy^2dy = 0$ is
(a) $\log x + \frac{x}{y^3} = k$
(b) $\log x + \frac{y}{x^3} = k$
(c) $\log x - \frac{x}{y^3} = k$
(d) $\log xy - y^3 = k$

21. The solution of the differential equation
$$\frac{dy}{dx} + \frac{3x^2}{1 + x^3}y = \frac{\sin^2 x}{1 + x^3}$$
is
(a) $y(1 + x^3) = x + \frac{1}{2}\sin 2x + c$
(b) $y(1 + x^3) = cx + \frac{1}{2}\sin 2x + c$
(c) $y(1 + x^3) = cx - \frac{1}{2}\sin 2x + c$
(d) $y(1 + x^3) = \frac{x}{2} - \frac{1}{2}\sin 2x + c$

22. The solution of the equation
$$x\frac{dy}{dx} + 3y = x$$
is
(a) $x^3y + \frac{x^4}{4} + c = 0$ (b) $x^3y = \frac{x^4}{4} + c$
(c) $x^3y + \frac{x^4}{4} + c$ (d) none of these

23. The solution of the differential equation
$$\frac{dy}{dx} + 2y\cot x = 3x^3\operatorname{cosec}^2 x$$
is
(a) $y\sin^2 x = x^3 + c$ (b) $y\sin x = c$
(c) $y\cos x^2 = c$ (d) $y\sin x^2 = c$

24. The integrating factor of the differential equation
$$x\,dy - y\,dx = xy^2dx$$
is
(a) $\frac{1}{x^2}$ (b) $\frac{1}{y^2}$
(c) $\frac{1}{xy}$ (d) $\frac{1}{x^2y^2}$

25. The equation of the curve through the point (1, 0) which satisfies the differential equation
$$(1 + y^2)dx - xy\,dy = 0,$$
is
(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$
(c) $2x^2 + y^2 = 2$ (d) none of these

26. The equation of a curve passing through $\left(2, \frac{7}{2}\right)$ and having gradient $\left(1 - \frac{1}{x^2}\right)$ at (x, y) is
(a) $y = x^2 + x + 1$ (b) $xy = x^2 + x + 1$
(c) $xy = x + 1$ (d) none of these

27. The differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$ is
(a) $y_1(y^2 - x^2) + 2xy + a^2 = 0$
(b) $y_1y^2 + xy + a^2x^2 = 0$
(c) $y_1(y^2 - x^2 + a^2) + 2xy = 0$
(d) none of these

28. The number of solutions of $y = \frac{y + 1}{x - 1}$, $y(1) = 2$ is
(a) none (b) one
(c) two (d) infinite

29. The solution of

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2,\ y(0) = 0$$

is

(a) $y^2 = e^{\left(x + \frac{x^2}{2}\right) - 1}$

(b) $y^2 = 1 + ce^{\left(x + \frac{x^2}{2}\right)}$

(c) $y = \tan(c + x + x^2)$

(d) $y = \tan\left(x + \frac{x^2}{2}\right)$

30. If $\frac{d^2y}{dx^2} = 0$, then

(a) $y = ax + b$ (b) $y^2 = ax + b$

(c) $y = \log x$ (d) $y = e^x + C$

31. The order of the differential equation whose general solution is given by

$$y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x + c_5}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is

(a) 5 (b) 4

(c) 3 (d) 2

32. A solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$$

is

(a) $y = 2$ (b) $y = 2x$

(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

33. If $y(t)$ is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$ then $y(1)$ is equal to

(a) $-1/2$ (b) $e + 1/2$

(c) $e - 1/2$ (d) $1/2$

34. If $y = y(x)$ and $\frac{2 + \sin x}{y + 1}\left(\frac{dy}{dx}\right) = -\cos x\ y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ equals

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $-\frac{1}{3}$ (d) 1

35. If $x\,dy = y(dx + y\,dy)$, $y(1)$ and $y(x) > 0$, then $y(-3) =$

(a) 3 (b) 2

(c) 1 (d) 0

36. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is of

(a) order 1 (b) order 2

(c) degree 3 (d) degree 4

37. If $f'(x) + f(x) = x$, when $f(0) = 2$, then $f(x)$ is

(a) $x + 2$ (b) $(x + 1) + e^x$

(c) $(x - 1) + 3e^{-x}$ (d) none.

38. The degree of the differential equation

$$\left(\sqrt{1 + x^2} + \sqrt{1 + y^2}\right) = A\left(x\sqrt{1 + y^2} - y\sqrt{1 + x^2}\right)$$

is

(a) 2 (b) 3

(c) 4 (d) none

39. The degree of the differential equation

$$\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$$

is

(a) 1 (b) 2

(c) 3 (d) none.

(More than one options are correct)

40. If m and n be the order and the degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^5 + 4\left(\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\left(\frac{d^3y}{dx^3}\right)}\right) + \frac{d^3y}{dx^3} = x^2 - 1,$$

then

(a) $m = 3$ and $n = 5$ (b) $m = 3$ and $n = 1$

(c) $m = 3$ and $n = 3$ (d) $m = 3$ and $n = 2$

41. If $y = e^{4x} + 2e^{-x}$ satisfies the relation $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$, then

(a) $A = -3$ (b) $B = -4$

(c) $A - B = 1$ (d) $A + B = -7$

42. The solution of $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying $y(1) = 1$ is given by

(a) a system of hyperbola

(b) a system of circles

(c) $y^2 = x(1 + x) - 1$

(d) $(x - 2) + (y - 3)^2 = 5$.

43. The function $f(x)$ satisfying the relation
$(\{f(x)\}^2 + 4 \cdot f(x) \cdot f'(x) + \{f'(x)\}^2) = 0$
is given by
(a) $ke^{(2+\sqrt{3})x}$ (b) $ke^{(2-\sqrt{3})x}$
(c) $ke^{(4-\sqrt{3})x}$ (d) $ke^{(4+\sqrt{3})x}$

44. If the order and degree of the differential equation $y = c_1 \sin^{-1}x + c_2 \cos^{-1}x + c_3 \tan^{-1}x + c_4 \cot^{-1}x$ where c_1, c_2, c_3 and c_4 are arbitrary constants, are m and n respectively, then
(a) $m = 3$ (b) $n = 2$
(c) $m + n = 5$ (d) $m - n = 1$

45. The value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses
$x^2 + 2y^2 - 2y = c$ is
(a) 1/2 (b) 1/4
(c) 2 (d) 4.

46. A differential function f satisfies the relation $f\left(\dfrac{x+y}{1+xy}\right) = f(x)\, f(y)\ \forall\ x, y \in R - \{-1\}$,
where $f(0) \neq 0$ and $f'(0) = 1$. Then
(a) $f(0) = 1$ (b) $\dfrac{dy}{dx} = \dfrac{y}{1-x^2}$
(c) $f(x) = \dfrac{1-x}{1+x}$ (d) None of these

47. Let $\dfrac{d}{dx}(x^2 y) = x - 1$, where $x \neq 0$ and $x = 1$, $y = 0$. Then
(a) The solution of the curve is $y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{1}{2x^2}$
(b) The set of values of x, where x is increasing is $(-\infty, 0) \cup (1, \infty)$
(c) The set of values of x where x is decreasing is $(0, 1)$
(d) None of these.

48. Let $y = x\left(\dfrac{dy}{dx}\right) + \left(\dfrac{dy}{dx}\right) - \left(\dfrac{dy}{dx}\right)^2$. then
(a) The genral solution is $y = cx + c - c^2$
(b) The particular solution is $y = 2x - 4$
(c) The singular solution is $4y = (x + 1)^2$
(d) None of these

49. Let $\dfrac{dy}{dx} = \dfrac{1-3x-3y}{1+x+y}$.
If the solution of the above differential equation is $mx + y + n \log_e|1 - x - y| + C$ then
(a) $m = 2$ (b) $n = 3$
(c) $m + n = 5$ (d) $n - m = 1$

50. Let $\dfrac{dy}{dx} = \dfrac{ax+b}{cy+d}$. If the solution of the given differential equation is a parabola, then
(a) $a = 2$, $b = 0$
(b) $c = 0$, $d = 4$
(c) $a = 0$, $b = 2$, $c = 4$
(d) $c = 0$, $a = 2$, $b = 2$, $d = 4$

Level III

(Problems for JEE-Advanced)

1. Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to $\dfrac{x^4 + 2xy - 1}{1 + x^2}$.
[Roorkee-JEE, 2001]

2. Solve the differential equation
$$y\cos\left(\frac{y}{x}\right)(x\,dy - y\,dx) + x\sin\left(\frac{y}{x}\right)(x\,dy + y\,dx) = 0$$
when $y(1) = \dfrac{\pi}{2}$. **[Roorkee-JEE, 1997]**

3. Solve the differential equation
$$\cos^2 x\frac{dy}{dx} - (\tan 2x)y = \cos^4 x$$
where $-\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$ and $y\left(\dfrac{\pi}{6}\right) = \dfrac{3\sqrt{3}}{8}$
[Roorkee-JEE, 1996]

4. If $y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$, find $y(x)$.
[Roorkee-JEE, 1995]

5. Solve the differential equation
$$x(1 - x^2)\,dy + (2x^2 y - y - 5x^3)\,dx = 0$$
[Roorkee-JEE,1994]

6. A tangent to the curve $y = f(x)$ cuts the line $y = x$ at a point which is at a distance of one unit from y-axis. Find the equation of the curve.

7. Solve: $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$.

8. Solve: $\left(xe^{y/x} - y\sin\left(\dfrac{y}{x}\right)\right)dx + x\sin\left(\dfrac{y}{x}\right)dy = 0$.

9. Solve: $(x + 2y)(dx - dy) = dx + dy$.

10. Solve: $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$.

11. Solve: $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - 2y - 1}$.

12. Solve: $x\,dy - \{y + xy^3(1 + \log x)\}\,dx = 0$.

13. Solve: $\sec^2 x \cdot \tan y\,dx + \sec^2 y \cdot \tan x\,dy = 0$.

14. Solve: $\sqrt{x + y + 1} \cdot \dfrac{dy}{dx} = x + y - 1$.

15. Solve: $(1 + e^{x/y})\,dy + \left(1 - \dfrac{x}{y}\right)e^{x/y}dy = 0$.

16. Solve: $(4x - y + 3)\,dy + (2x - 3y - 1)\,dx = 0$.
17. Solve: $(1 + y^2) + (x - e^{\tan^{-1}y})\dfrac{dy}{dx} = 0$.
18. Solve: $y\,dx - x(1 + xy)\,dy = 0$.
19. Prove that the equation of a curve whose slope at (x, y) is $\dfrac{-(x + y)}{x}$ and which passes through the point (2, 1), is $x^2 + 2xy = 8$.
20. If the square of the intercept cut by any tangent on the y-axis is equal to the product of the coordinate of the point of contact, find the equation of such curves.
21. Find the curves for which the length of the normal is equal to the radius vector.
22. Solve: $\left(\dfrac{dy}{dx}\right)^2 - (e^x + e^{-x})\dfrac{dy}{dx} + 1 = 0$.
23. Solve: $\left(\dfrac{dy}{dx}\right)^2 + 2x\,\dfrac{dy}{dx} = 3x^2$.
24. Solve: $xy\left\{\left(\dfrac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\dfrac{dy}{dx}$.
25. Find the equation of the curve passing through the point (0, −2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.
26. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (−4, −3). Find the equation of the curve, given that it passing through (−2, 1).
27. Find the equation of the curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate and the product of the x-coordinate and y-coordinate of that point.
28. Find the equation of the curve which touches the line $y = 1$ and passes through the point (0, 1) and satisfies the differential equation $y^3\left(\dfrac{d^2y}{dx^2} - y\right) = 1$.
29. The ordinate and the normal at any point P on the curve meet the x-axis at points A and B respectively. Find the equation of the family of curves satisfying the condition AB = arithmetic mean of abscissa and ordinate of P.
30. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. PQ is of the constant length k. Find the coordinates of Q. Also find the differential equation of the given curve.
31. Find the equation of the curve, the slope of whose tangent at any point (x, y) is $\dfrac{2y}{x}$, for all $x, y > 0$ and which pass through the point (1, 1).
32. Let C be a curve such that the normal at any point P on it meets x-axis and y-axis at A and B, respectively. If $PB : PA = 1 : 2$ (internally) and the curve passes through the point (0, 4), prove that the curve is a hyperbola and it passes through the point $(\sqrt{10}, 6)$.
33. A normal is drawn at any point $P(x, y)$ of a curve, it meets the x-axis and the y-axis in points A and B respectively, such that $\dfrac{1}{OA} + \dfrac{1}{OB} = 1$, where O is the origin. Find the equation of such a curve passing through (5, 4).

Level III

(Tougher Problems for JEE-Advanced)

1. Find the order of the differential equation whose general solution is given by
$y = C_1\cos(2x + C_2) - (C_3 + C_4)3^{x+C_5} + C_6\sin(x - C_7)$
2. Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to x-axis.
3. The slope of the tangent to a curve at a point $P(x, y)$ is $\dfrac{2y}{x}$, $x, y > 0$ and which passes through the point (1, 1), find the equation of the curve.
4. A conic C passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the coordinate axes is bisected at the point of tangentcy. Find the equation of the conic.
5. Solve the differential equation
$$2y\frac{dy}{dx} = e^{\left(\frac{x^2 + y^2}{x}\right)} + \left(\frac{x^2 + y^2}{x} - 2x\right).$$
6. Find the integral curve of the differential equation
$x(1 - x\ln y)\dfrac{dy}{dx} + y = 0,$
which passes through the the point $\left(e, \dfrac{1}{e}\right)$.
7. Solve the differential equation
$2\left(x - y\dfrac{dy}{dx}\right)(x^2 + y^2) = (x^2 - y^2)\left(y - x\dfrac{dy}{dx}\right)$
8. If $y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$, find y as a function of x.
9. Solve the differential equation
$(x\,dy + y\,dx)\,\sin(xy) + (x^2y\,dx + xy^2dx)\,\cos(xy) = 0$.
10. Solve the differential equation
$\dfrac{dy}{dx} = \dfrac{2y^2\cos x + y\sin 2x + 2\cos x \cdot \sin^2 x}{\sin^2 x}$.
11. Solve the differential equation

$$y\cos\left(\frac{y}{x}\right)(x\,dy - y\,dx) + x\sin\left(\frac{y}{x}\right)(x\,dy + y\,dx) = 0,$$

where $y(1) = \frac{\pi}{2}$.

12. Solve the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{x^4y^2 - x^6 + 2x^4y - x^6y^2 - 2x^6y + x^4}{y^2 - x^2y^2 + x^3y^2 - x^5y^2}}$$

13. If $f(x) = \int_1^x \frac{\log t}{1 + t + t^2}\,dt$, $x \geq 1$, prove that $f(x) = f\left(\frac{1}{x}\right)$.

14. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2 + 3x + 3y + 2xy + 1}{x^2 + y^2 - 3x - 3y + 2xy + 2}.$$

15. A differentiable function f satisfies the relation $f(x + y) + f(x)\cdot f(y) = f(xy + 1)$.
Given $f(0) = -1$, $f'(0) = f'(1) = 1$, find the function f.

16. A differentiable function f satisfies the relation $f(xy) = xf(y) + yf(x)$ for every x, y in R.
If $f'(1) = 1$, the find the function f.

17. Solve: $\left(\frac{1}{y}\sin\left(\frac{x}{y}\right) - \frac{y}{x^2}\cos\left(\frac{y}{x}\right) + 1\right)dx$

$$+\left(\frac{1}{x}\cos\left(\frac{y}{x}\right) - \frac{x}{y^2}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\right)dy = 0.$$

18. Solve: $\frac{dy}{dx} = \frac{(1 + y^2)}{xy(1 + x^2)}$.

19. Solve: $\frac{y}{x} + \frac{dy}{dx} = \frac{1}{\sin^4(xy) + \cos^2(xy)}$.

20. Solve: $(x^2 + y^2 + 1)\,dy + 2xy\,dx = 0$.

21. Solve: $y\frac{dy}{dx} + x = \frac{1}{2}\left(\frac{x^2 + y^2}{x}\right)^2$.

22. Solve: $\frac{y + \sin x\cos^2(xy)}{\cos^2(xy)}dx + \frac{x\,dy}{\cos^2(xy)} = 0$.

23. Find the general solution of the linear equation of the first order $\frac{dy}{dx} + p(x)y = q(x)$, if one particular solution $y_1(x)$ is known.

24. Find the curve such that the square of the intercept cut by any tangent off the y-axis is equal to the product of the co-ordinates of the point of tangency.

25. If $f(x)$ is a function such that

$$x\int_0^x (1 - t)f(t)\,dt = \int_0^x tf(t)\,dt,\ f(1) = 1,$$

find $f(x)$.

26. Find the equation of the curve passing through (3, 4) and satisfying the differential equation

$$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0.$$

27. Solve: $y' = \frac{y}{x} + \frac{\varphi(y/x)}{\varphi'(y/x)}$.

28. A differentiable function f satisfies the relation $f\left(\frac{x + y}{1 + xy}\right) = f(x)\cdot f(y)$ for all x, y in $R-\{-1\}$, where $f(0) \neq 0$ and $f'(0) = 1$. Find $f(x)$.

29. A differentiable function f satisfies the relation $f(xy) = xf(y) + yf(x)$ for all x, y in R^+.
If $f(1) = 1$, find $f(x)$.

30. Find the curve for which the portion of the tangent included between the coordinate axes is bisected at the point of contact.

31. Find the nature of the curve for which the length of the normal at *a* point P is equal to the radius vector of the point P.

32. Find the equation of the curve passing through (2, 2) such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point.

33. A curve is such that the length of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P.
Prove that the differential equation of the curve is

$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0$$

and hence find the equation of the curve.

34. Find the orthogonal trajectories to the curve $y^2 = a(x + a)$.

35. Find the equation of the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact.

Integer Type Questions

1. Find the degree of the differential equation

$$\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\left(\frac{d^2y}{dx^2}\right) + 5\left(\frac{dy}{dx}\right) + 4y = 0.$$

2. If the order of the differential equation of the curve $y = (c_1 + c_2)\sin(x + c_3) - c_4e^{x+c_5}$ is M, find the value of $(M - 1)$.

3. If $y = e^{4x} + 2e^{-x}$ satisfies the differential equation $\frac{d^3y}{dx^3} + A\frac{dy}{dx} + By = 0$, the value of $A - B + 4$.

4. The solution of $\frac{dy}{dx} = \frac{ax + b}{by + k}$, where $a < 0$ and $b \geq 0$, represents a parabola, find the value of $a + b + 4$.

5. The solution of the differential equation $\ln\left(\frac{dy}{dx}\right) = 3x + 4y$, $y(0) = 0$ is

$$Ae^{3x} + Be^{-4y} + C = 0,$$

where $A + B + C + 5$ is...

6. The polynomial $f(x)$ satisfies the equation $f(x + 1) = x^2 + 4x$. The area enclosed by $y = f(x - 1)$ and the curve $x^2 + y = 0$ is $\frac{16\sqrt{2}}{k}$, where k is ...

7. If the number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 = y$ is m, the value of $m + 2$ is...

8. If $f(x) + f'(x) = x$, where $f(0) = 2$, then $f(x)$ is $(x - 1) + ke^{-x}$, where $k + 2$ is ...

9. The solution of the differential equation $\frac{dz}{dx} + \frac{z}{x}\ln z = \frac{z}{x^2}(\ln z)^2$ is $\frac{1}{x\ln z} = \frac{1}{mz^2} + C$, where $m + 4$ is

10. The solution of the differential equation $\left(xy^2 - \frac{e}{x^3}\right)dx - x^2y\,dy = 0$, given $y = 0$ when $x = 1$, is $y^2 = \frac{e}{k}\left(\frac{1}{x^4} - x^2\right)$, where $k + 5$ is ...

Comprehensive Link Passages

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

Passage I

The differential equation corresponding to

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x},$$

where c_1, c_2 and c_3 are arbitrary constants and m_1, m_2 and m_3 are the roots of $m^3 - 7m + 6 = 0$ is

$$\alpha\left(\frac{d^3y}{dx^3}\right) + f\left(\frac{d^2y}{dx^2}\right) + \gamma\left(\frac{dy}{dx}\right) + \delta = 0,$$

where α, β, γ and δ are arbitrary constants.

On the basis of the above information, answer the following questions.

1. The order of the differential equation is
   (a) 1 (b) 2
   (c) 3 (d) 4
2. The degree of the differential equation is
   (a) 1 (b) 2
   (c) 3 (d) 4
3. The value of α and β, respectively, are
   (a) 0, 1 (b) 1, 0
   (c) –1, 0 (d) 0, –1
4. The value of γ is
   (a) 6 (b) –7
   (c) 2 (d) –1
5. The value of δ is
   (a) 6 (b) –7
   (c) 2 (d) –1

Passage II

The order of the differential equation is the highest order derivative appearing in the equation and the degree of the differential equation is the index power of the highest order derivative and is free from any type of radical sign.

On the basis of the above information, answer the following questions.

1. The degree of the differential equation
   $$x = 1 + \left(xy\frac{dy}{dx}\right) + \frac{1}{2}!\left(xy\frac{dy}{dx}\right)^2 + \frac{1}{3}!\left(xy\frac{dy}{dx}\right)^3 + \frac{1}{4}!\left(xy\frac{dy}{dx}\right)^3 + \ldots \text{ to } \infty \text{ is}$$
   (a) 0 (b) 1
   (c) 2 (d) not defined
2. The dgree of the differential equation
   $$\sqrt[3]{1 + \left(\frac{d^2y}{dx^2}\right)} = \sqrt[4]{1 + \left(\frac{d^3y}{dx^3}\right)} \text{ is}$$
   (a) 3 (b) 1
   (c) 2 (d) not defined
3. The order and the degree of the differential equation of the curve $y = mx + m^2 + m^3 + m^4$ is
   (a) 1, 3 (b) 1, 4
   (c) 2, 3 (d) 2, 4
4. The order and the degree of the differential equations in $\left(\frac{dy}{dx}\right) = x + 1$ is
   (a) 1, 4 (b) 2, 4
   (c) 1, not defined (d) 2, not defined

Passage III

Differential equations are solved by reducing them to get the exact differential of an expression x and y, i.e. they are reduced to the form $d[f(x, y)] = 0$

For example,

$$\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \frac{x\,dy - y\,dx}{x^2}$$

$$\Rightarrow \quad \frac{1}{2} \times \frac{2x\,dx + 2y\,dy}{\sqrt{x^2 + y^2}} = \frac{x\,dy - y\,dx}{x^2}$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{2\sqrt{x+y^2}} = -d\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad d(\sqrt{x^2+y^2}) = -d\left(\frac{y}{x}\right)$$

Thus the required solution is

$$\sqrt{x^2+y^2} + \frac{y}{x} = c.$$

On the basis of the above information, answer the following questions.

1. The general solution of $(2x^3 - xy^2)dx + (2y^3 - x^2y)dy = 0$, is
   (a) $x^2 + x^2y^2 - y^4 = c$ (b) $x^2 - x^2y^2 + y^4 = c$
   (c) $x^2 - x^2y^2 - y^4 = c$ (d) $x^2 + x^2y^2 + y^4 = c$
2. The general solution of the differential equation
   $$\frac{x\,dy}{x^2+y^2} + \left(1 - \frac{y}{x^2+y^2}\right)dx = 0,$$
   is
   (a) $x + \tan^{-1}\left(\frac{y}{x}\right) = c$ (b) $x + \tan^{-1}\left(\frac{x}{y}\right) = c$
   (c) $x - \tan^{-1}\left(\frac{y}{x}\right) = c$ (d) none.
3. The general solution of the differential equation $e^y dy + (xe^y - 2y)dy = 0$, is
   (a) $xe^y - y^2 = c$ (b) $xe^y - x^2 = c$
   (c) $ye^y + x = c$ (d) $xe^y - 1 = cy^2$.

Passage IV

The differential equation $\frac{dy}{dx} = f(x)\cdot g(y)$ can be solved by seperating the variable $\frac{dy}{g(y)} = f(x)dx$ and then integrating the function and get required result.

On the basis of the above information, answer the following questions.

1. The equation of the curve to the point (1, 0) which satisfies the differential equation $(1 + y^2)dx - xy\,dy = 0$, is
   (a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$
   (c) $x^2 + y^2 = 2$ (d) $x^2 - y^2 = 2$
2. The solution of the differential equation
   $$\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1+x^2}} = 0, \text{ is}$$
   (a) $\tan^{-1}y + \sin^{-1}x = c$ (b) $\tan^{-1}x + \sin^{-1}y = c$
   (c) $\tan^{-1}y\cdot\sin^{-1}x = c$ (d) $\tan^{-1}y\cdot -\sin^{-1}x = c$.
3. If $\frac{dy}{dx} = 1 + x + y + xy$ and $y(-1) = 0$, then y is
   (a) $e^{\frac{(1-x)^2}{2}}$ (b) $e^{\frac{(1-x)^2}{2}} - 1$
   (c) $\ln(1 + x) - 1$ (d) $1 + x$.

Passage V

A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. PQ is of constant length k.

On the basis of the above information, answer the following questions.

1. The co-odinates of Q are
   (a) $\left(x + y\frac{dy}{dx}, 0\right)$ (b) $\left(x + y\frac{dx}{dy}, 0\right)$
   (c) $\left(x - y\frac{dy}{dx}, 0\right)$ (d) $\left(x - y\frac{dx}{dy}, 0\right)$.
2. The differential equation of the curve is
   (a) $\left(y\frac{dy}{dx}\right)^2 = y^2 - k^2$ (b) $\left(y\frac{dy}{dx}\right)^2 = k^2 - y^2$
   (c) $\left(\frac{dy}{dx}\right)^2 = y^2 - k^2$ (d) $\left(\frac{dy}{dx}\right)^2 = k^2 - y^2$
3. The cartesian equation of the curve, if it passes through the point $(0, k)$ is
   (a) $x^2 + y^2 = k^2$ (b) $x^2 + y^2 = 2k^2$
   (c) $\log\left(y + \sqrt{y^2 - k^2}\right) = 0$ (d) $\sin^{-1}\left(\frac{y}{k}\right) = x + c$.

Matrix Match
(For JEE-Advanced Examination only)

Given below are matching type questions, with two columns (each having some items) each. Each item of Column I has to be matched with the items of Column II, by encircling the correct match(es).

Note: An item of Column I can be matched with more than one items of Column II. All the items of Column II have to be matched.

1. **Observe the following Columns**

	Column I		Column II
(A)	The degree of the differential equation of the curve $y^2 = 2c(x + \sqrt{c})$ is	(P)	3
(B)	The degree of the differential equation $\log\left(1 + \frac{dy}{dx}\right) = x$ is	(Q)	Not Defined
(C)	The order (L) and the degree (M) of the differential equation of the curve $y = a\sin(bx + c)$ where a, b and c are arbitrary constants, the value of $L + M$ is	(R)	4

(D)	The order (L) and the degree (M) of the family of parabolas having vertex at origin and axis along positive y-axis, then $L + M + 3$ is	(S)	5

2. **Observe the following Columns:**

	Column I	Column II	
(A)	The integrating factor of the differential equation $\frac{dy}{dx} = x^3y^2 - xy$ is	(P)	e^{-x^2}
(B)	The integrating factor of the differential equation $\frac{dy}{dx} + x(x + y) = x^3(x + y)^3 - 1$ is	(Q)	e^{-x^3}
(C)	The integrating factor of the differential equation $x^2y\,dx - (x^3 + y^3)\,dy = 0$ is	(R)	$\frac{1}{x^4}$
(D)	The integrating factor of the differential equation $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$ is	(S)	$-\frac{1}{y^4}$

3. **Observe the following Columns:**

	Column I	Column II	
(A)	$(x + y)\,dy + (x - y)\,dy = 0$, $y = 1$, when $x = 1$	(P)	$y + 2x = 3x^2y$
(B)	$x^2dy + (xy + y^2)\,dx = 0$, $y = 1$ when $x = 1$	(Q)	$\cos(y/x) = \log\lvert cx\rvert$
(C)	$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$, $y = 0$, when $x = 1$	(R)	$y = 2\frac{x}{1 - \log\lvert x\rvert}$ $(x \neq 0, x \neq e)$
(D)	$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, $y = 2$ when $x = 1$	(S)	$\log(x^2 + y^2) + 2\tan^{-1}(y/x) = \left(\frac{\pi}{2} + \log 2\right)$.

4. **Observe the following Columns:**

	Column I	Column II	
(A)	If m and n are the order and the degree of the equation $\frac{d^4y}{dx^4} = y + \left(\frac{dy}{dx}\right)^4$, then $(m + n)$ is	(P)	$\sin(y/x) = cx$
(B)	If $\frac{2 + \sin x}{i + y}\frac{dy}{dx} = -\cos x$, $y = 1$ at $x = 0$, then $y\left(\frac{\pi}{2}\right)$ is	(Q)	5
(C)	The solution of $x\frac{dy}{dx} = y + x\tan\frac{y}{x}$ is	(R)	$\sin^{-1}(y/x) = x^2 + c$
(D)	The solution of $\frac{y\frac{dy}{dx} - y}{\sqrt{x^2 + y^2}} = 2x^2$ is	(S)	1/3

Assertion and Reason

Codes

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is not the correct explanation of A.

(c) R is true and A is false.

(d) R is false and A is true.

1. **Assertion (A):** The order of all the non-vertical lines in a plane is 3.

   Reason (R): The order of all the non-vertical lines in a plane passing through the origin is 1.

2. **Assertion (A):** The order of the differential equation $\sin\left(\frac{dy}{dx}\right) = x$ is 1.

   Reason (R): The degree of the differential equation $e^{\frac{dy}{dx}} = (x + 1)$ is 2.

3. **Assertion (A):** The integrating factor of the differential equation $\frac{dy}{dx} + xy = 10$ is e^x.

   Reason (R): The integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$ is $e^{\int P dx}$.

4. **Assertion (A):** The orthogonal trajectories of the curve $xy = c$ is $x^2 - y^2 = c$.

   Reason (R): The solution of he differential equation $x\,dy + y\,dx = 0$ is $xy = c$

5. **Assertion (A):** The solution of the differential equation $x^2y\,dx = (x^3 + y^3)\,dy$ is $y = ce^{\frac{x^3}{3y^3}}$.

   Reason (R): The integrating factor of the differential equation $M\,dx + N\,dy = 0$ is $\frac{1}{Mx + Ny}$.

6. **Assertion (A):** The order of the differential equation of the curve $y = c_1e^x + c_2e^{2x} + c_3ex+5$ is 2.
Reason (R): The order of a differential equation is the highest order derivative of a differential equation.

Questions asked in Previous Years' JEE-Advanced Examinations

1. A normal is drawn at a point $P(x, y)$ of a curve, it meets the x-axis at Q. If PQ is of constant length k, prove that the differential equation describing such curve is $y\dfrac{dy}{dx} = \sqrt{k^2 - y^2}$.
Find the equation of such a curve passing through $(0, k)$ **[IIT, 1994]**
2. Let $y = f(x)$ be a curve passing through $(1, 1)$ such that the triangle formed by the co-ordinate axes and the tangent at any point of the curve lies in the first quadrant and has area 2. Form the differential equation and determine all such possible curves. **[IIT, 1995]**
3. Determine the equation of the curve passing through the origin in the form $y = f(x)$ which satisfies the differential equation $\dfrac{dy}{dx} = \sin(10x + 6y)$ **[IIT, 1996]**
4. A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at P is $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also, obtain the area bounded by the y-axis, the curve and the normal to the curve at P. **[IIT, 1996]**
5. A spherical rain drop evaporates at a rate proportional to its surface area at any instant t. The differential equation giving the rate of change of the radius of the rain drop is ... **[IIT, 1997]**
6. Let $u(x)$ and $v(x)$ satisfy the differential equations $\dfrac{du}{dx} + p(x)u = f(x)$ and $\dfrac{dv}{dx} + p(x)v = g(x)$, where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y) where $x > x_1$ does not satisfy the equation $y = u(x)$ and $y = v(x)$. **[IIT, 1997]**
7. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point $(1, 1)$. Determine the equation of the curve. **[IIT, 1998]**
8. The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4e^{x+c_5}$, where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants is ...
(a) 5 (b) 4
(c) 3 (d) 2
[IIT, 1998]
9. A curve passing through the point $(1, 1)$ has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve. **[IIT, 1999]**
10. A solution of the differential equation
$$\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$$
is
(a) $y = 2$ (b) $y = 2x$
(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$
[IIT, 1999]
11. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter is of
(a) order = 1 (b) order = 2
(c) degree = 3 (d) degree = 4.
[IIT, 1999]
12. If $x^2 + y^2 = 1$, then
(a) $yy'' - 2(y')^2 + 1 = 0$
(b) $yy'' + (y')^2 + 1 = 0$
(c) $yy'' + (y')^2 - 1 = 0$
(d) $yy'' + 2(y')^2 + 1 = 0$.
[IIT, 2000]
13. If $y(t)$ is a solution of $(1 + t)\dfrac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is
(a) $-1/2$ (b) $e + 1/2$
(c) $e - 1/2$ (d) $1/2$
[IIT, 2003]
14. A right circular cone with radius R and height H contains a liquid which evaporates at a rate propeortional to its surface in contact with air (proportonality constant $(k > 0)$. Find the after what time which the cone is empty. **[IIT, 2003]**
15. If $y = y(x)$ and $\dfrac{2 + \sin x}{y + 1}\left(\dfrac{dy}{dx}\right) = -\cos x$ and $y(0) = 1$, then $y\left(\dfrac{\pi}{2}\right)$ equals
(a) 1/3 (b) 2/3
(c) $-1/3$ (d) 1.
[IIT, 2004]

16. A curve C passes through (2, 0) and the slope at (x, y) as $\frac{(x+1)^2 + y - 3}{(x+1)}$. Find the equation of the curve and the area enclosed by the curve and the x-axis in the fourth quadrant.

[IIT, 2004]

17. If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$, the value of $y''(0)$ is

(a) 1 (b) –1
(c) π (d) $-\pi$

[IIT, 2005]

18. If the length of the tangent at any point on the curve
$y = f(x)$ intercepted between the point and the x-axis is of length 1, find the equation of the curve.

[IIT, 2005]

19. The solution of the primitive integral equation $(x^2 + y^2)dy = xy\,dx$ is $y = y(x)$.
If $y(1) = 1$ and $y(x_0) = e$, then the value of x_0 is

(a) $\sqrt{2(e^2 - 1)}$ (b) $\sqrt{2(e^2 + 1)}$
(c) $\sqrt{3}\,e$ (d) $\sqrt{\frac{e^2+1}{2}}$

[IIT, 2005]

20. For the primitive integral equation $y\,dx + y^2dy = x\,dy$, $x \in R$, $y > 0$, $y = y(x)$, $y(1) = 1$, then $y(-3)$ is

(a) 3 (b) 2
(c) 1 (d) 5

[IIT, 2005]

21. A curve $y = f(x)$ passes through (1, 1) and the tangent at $P(x, y)$ cuts the x-axis and y-axis at A and B, respectively such that $BP : AP = 3 : 1$, then

(a) the equation of the curve is $xy' - 3y = 0$.
(b) the normal at (1, 1) is $x + 3y = 4$.
(c) the curve passes through (2, 1/8).
(d) the equation of the curve is $xy' + 3y = 0$.

[IIT, 2006]

22. If y satisfies the differential equation $\frac{dy}{dx} = \frac{2}{x} + y$, $y(1) = 1$, then $(x + y + 2)^2 e^{-y}$ equals ...

[IIT, 2006]

23. The differential equation $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{y}}$ determines a family of circles with

(a) the variable radii and fixed centre at (0, 1)
(b) the variable radii and fixed centre at (0, –1)
(c) the fixed radius 1 and variable centre along the x-axis.
(d) the fixed radius 1 and variable centres along the y-axis.

[IIT, 2007]

24. Let $f(x)$ be a differentiable on $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t\to x}\left(\frac{t^2f(x) - x^2f(t)}{t - x}\right) = 1$ for each $x > 0$, then $f(x)$ is

(a) $\frac{1}{3x} + \frac{2x^2}{3}$ (b) $-\frac{1}{3x} + \frac{4x^2}{3}$
(c) $-\frac{1}{x} + \frac{2}{x^2}$ (d) $\frac{1}{x}$

[IIT, 2007]

25. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2 - 1}\,dy - y\sqrt{y^2 - 1}\,dx = 0$ satisfies $y(2) = \frac{2}{\sqrt{3}}$.

Assertion (A): $y = (x) = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right)$

Reason (R): $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explannation of A.
(c) A is true and R is false.
(d) A is false and R is true.

[IIT, 2008]

26. If $y' = y + 1$ and $y(0) = 1$, the values of $y(\ln 2)$ is ...

[IIT, 2009]

27. Find the domain of the definition of non-zero solution of the differential equation $(x - 3)^2\, y' + y = 0$

[IIT, 2009]

28. Let f be a real-valued differentiable function on R (the set of real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, the value of $f(-3)$ is ...

[IIT, 2010]

29. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. The value of $y(2)$ is ...

[IIT, 2011]

30. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$.

If $6\int_0^x f(t)\,dt = 3xf(x) - x^3 - 5$ for all $x \geq 1$, the value of $f(2)$ is ...

[IIT, 2011]

31. If $y(x)$ satisfies the differential equation

$y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$

(b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$

(d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

[IIT, 2012]

32. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$. The eaquation of the curve is

(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$

(b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$

(d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

[IIT-JEE, 2013]

33. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in $(-1, 1)$ satisfying

$f(0) = 0$, then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x) = dx$ is

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

[IIT-JEE, 2014]

34. Let $y(x)$ be a solution of the differential equation $(1 + ex)\, y' + ye^x = 1$. If $y(0) = 2$, which of the following statements (is/are) true?

(a) $y(-4) = 0$

(b) $y(2) = 0$

(c) $y(x)$ has a critical point in $(-1, 0)$

(d) $y(x)$ has no critical point in $(-1, 0)$.

[IIT-JEE, 2015]

35. Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation

$Py'' + Qy' + 1 = 0,$

where P, Q are functions of x, y and y $\left(\text{Here } y' = \frac{dy}{dx},\ y'' = \frac{dy}{dx}\right)$, which of the following statements (is/are) true?

(a) $P = y + x$

(b) $P = y - x$

(c) $P + Q = 1 - x + y + y + (y')^2$

(d) $P - Q = x + y - y' - (y')^2$

[IIT-JEE, 2015]

36. *A* solution curve of the differential equation

$(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0,\ x > 0$

passes through the point (1, 3). The solution curve

(a) intersects $y = x + 2$ exactly at one point

(b) intersects $y = x + 2$ exactly at two points

(c) intersects $y = (x + 2)^2$

(d) does not intersect $y = (x + 3)^2$

[IIT-JEE, 2016]

ANSWERS

LEVEL I

190. $(y - ce^x)(y + x - ce^{-x}) = 0$
191. $(x - y + c)(x^2 + y^2 + c) = 0$
192. $(xy - c)(x^2 - y^2 - c) = 0$
193. $(y - cx)(y^2 - x^2 - c) = 0$
196. $\ln y = cx + c^2$.
197. $y^2 = 2cx + c^2$
198. $c(4x - c)^2 = 64y$
199. $y^2 = 2cx + c^3$
200. $y = cx - 1 + 1/c$
203. $y = 2\sqrt{cx} + \tan^{-1}(c)$
204. $y = 2\sqrt{cx} + c^2$
205. $y = c^2 - \frac{c}{x}$
206. $x = \sin p + c$

LEVEL II

1. (d)	2. (b)	3. (c)	4. (c)	5. (b)
6. (a)	7. (d)	8. (b)	9. (c)	10. (b)
11. (c)	12. (a)	13. (b)	14. (a)	15. (d)
16. (a)	17. (a)	18. (b)	19. (b)	20. (b)
21. (d)	22. (b)	23. (a)	24. (b)	25. (d)
26. (b)	27. (c)	28. (a)	29. (d)	30. (a)
31. (a)	32. (b)	33. (c)	34. (d)	35. (b)
36. (c)	37. (c)	38. (a)	39. (b)	40. (a, c)
41. (a, b, c, d)		42. (a, c)	43. (a, b)	
44. (a, b, c, d)		45. (b)	46. (a, b, c, d)	
47. (a, b, c)	48. (a, b, c)	49. (a, b, c, d)		50. (b, c, d)

LEVEL III

1. $\sqrt{1-y^2}\ \ln\left|\dfrac{1+\sqrt{1-y^2}}{1-\sqrt{1-y^2}}\right| = \pm x + c$

3. $\dfrac{1}{3}\tan^{-1}\left[\dfrac{4}{5}\tan\left(4x+\tan^{-1}\dfrac{3}{4}\right)-\dfrac{3}{5}\right]-\dfrac{5x}{3}$

4. $\dfrac{4}{3}$ sq.u.

5. $x^2 + y^2 = 2x$

7. $y + \sqrt{x^2+y^2} = cx^2$

8. $\log x = c + \dfrac{1}{2}e^{-y/x}\left\{\sin\dfrac{y}{x}+\cos\dfrac{y}{x}\right\}$

9. $x + c = \dfrac{1}{3}\left[x + 2y + \dfrac{4}{3}\ln(3x+6y-1)\right]$

10. $xe^{\tan^{-1}y} + c = (\tan^{-1}y - 1)\,e^{\tan^{-1}y}$

11. $\dfrac{x^2}{y} + 2\ln y - \dfrac{1}{y} = c.$

12. $-\dfrac{x^2}{y^2} = \dfrac{2x^3}{3}\left(\dfrac{2}{3}+\ln x\right) + c$

13. $\tan x . \tan y = c$

14. $x + c = 2\sqrt{x+y+1} + \dfrac{2}{3}\ln(\sqrt{x+y+1} - 1)$
$-\dfrac{8}{3}\ln\sqrt{x+y} + 1 + 2$

15. $x + y\cdot e^{x/y} = c$

16. $(x+y+z)^5 = c(y-2x-1)^2$

17. $xe^{\tan^{-1}y} = c(y-2x-1)^2$

18. $\ln y = \dfrac{1}{xy} + c$

20. $x = ce^{\pm 2\sqrt{y/x}}$

21. $x^2 \pm y^2 = a^2$

22. $(y - e^x + c)(y + e^{-x} + c) = 0$

23. $(2y - x^2 - c)(2y + 3x^2 - c) = 0$

24. $(xy - c)(x^2 - y^2 - c) = 0$

25. $x^2 - y^2 + 4 = 0$

26. $(y+3) = (x+4)^2$

27. $y = Ce^{x^2/2} - 1$

28. $y^2 + \sqrt{y^4-1} = e^{\pm 2x}$

29. $x^2 - y^2 = kx$

30. $Q\left(x + y\dfrac{dy}{dx}, 0\right), x^2 + y^2 = 2k^2$

31. $y = x^2$

32. $y^2 = 2x^2 + 16$

33. $x^2 + y^2 - 8(x+y) + 16 = 0$

34. $(x-1)^2 + (y-1)^2 = 25$

35. $y = cx^3,\ x = cy^3$

LEVEL IV

1. 5

2. $2a\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$

3. $y = x^2$

4. $xy = 8.$

5. $cx = e - e^{-\left(\frac{x^2+y^2}{x}\right)}$

6. $((\ln y + 1) + ey)x = 1$

7. $\log|(x^2 - y^2)| + \left(\tan^{-1}\left(\dfrac{y}{x}\right)\right) = C$

8. $y\cdot x^2 = -x^2\cos x + 2x\sin x + 2\cos x$
$+\dfrac{x^3}{3}\log x - \dfrac{x^3}{9} + C$

9. $(xy)\sin(xy) = C$

10. $\dfrac{2}{\sqrt{15}}\tan^{-1}\left(\dfrac{4y+\sin x}{\sqrt{15}\sin x}\right) = \log|\sin x| + C$

11. $y\sin\left(\dfrac{y}{x}\right) = \dfrac{\pi}{2x}$

12. $y - \log|y+1| = \dfrac{2}{3}$ R1 $+ x^3 + C$

14. $y - \dfrac{3}{2}\log|(x+y)^2 + 1| = x + C$

15. $y = x - 1$

16. $y = x\log|x|$

17. $-\cos\left(\dfrac{x}{y}\right) + \sin\left(\dfrac{y}{x}\right) + x - \dfrac{1}{y} = c$

18. $(1+y^2)(1+y^2) = c^2 x^2$

19. $\frac{1}{4}(xy - \sin(2xy)) + \frac{1}{8}\left(xy + \frac{\sin(4xy)}{4}\right)$
$$+ \frac{1}{2}\left(xy + \frac{\sin 2(xy)}{2}\right) = \log|x| + c$$

20. $\frac{y^3}{3} + y + x^2 y = c$

21. $\frac{1}{x} - \frac{1}{(x^2 + y^2)} = c$

22. $y \cdot e^{p(x)x} = \int (e^{p(x)x} \cdot q(x))\,dx + c$

23. $y \cdot e^{p(x)x} = \int (e^{p(x)x} \cdot q(x))\,dx + c$

24. $2\sqrt{\frac{y}{x}} + \log|x| = c$

25. $f(x) = x^3 e^{\left(\frac{1-x}{x}\right)}$

27. $\varphi\left(\frac{y}{x}\right) = cx$

30. $xy = c$

31. $x^2 \pm y^2 = a^2$

32. $y^2 = 2x$

33. $x^2 + y^2 = cx$

34. $\left(\sqrt{x^2 + y^2}\right) = 2x + c$

35. $x^2 + y^2 = cx$

36. $x^2 + y^2 = 25,\ y = x + 1$

INTEGER TYPE

1. (2)	2. (2)	3. (3)	4. (4)	5. (5)
6. (3)	7. (4)	8. (5)	9. (6)	10. (8)

COMPREHENSIVE LINK PASSAGES

P-I :	1. (c)	2. (a)	3. (b)	4. (b)	5. (a)
P-II :	1. (b)	2. (a)	3. (b)	4. (c)	
P-III :	1. (b)	2. (a)	3. (a)		
P-IV :	1. (b)	2. (a)	3. (b)		
P-V :	1. (a)	2. (b)	3. (a)		

MATRIX MATCH

1. (A)→(P); (B)→(Q); (C)→(R); (D)→(S)
2. (A)→(P); (B)→(P); (C)→(S); (D)→(R)
3. (A)→(S), (B)→(P), (C)→(Q), (D)→(R)
4. (A)→(Q), (B)→(S), (C)→(P), (D)→(R)

ASSERTION AND REASON

1. (d)	2. (c)	3. (d)	4. (b)	5. (a)
6. (a)				

HINTS AND SOLUTIONS

Level I

1. Clearly, the order is 2 and the degree is 1.
2. Clearly, the order is 1 and the degree is 3.
3. Clearly, the order is 2 and the degree is 1.
4. We have,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt[3]{c \cdot \frac{d^2y}{dx^2}}$$

$$\Rightarrow \quad \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} = \left(c\,\frac{d^2y}{dx^2}\right)^{1/3}$$

$$\Rightarrow \quad \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = \left(c\,\frac{d^2y}{dx}\right)^2$$

Thus, order is 2 and degree is also 2.

5. We have,

$$x + \frac{dy}{dx} = \text{R}1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \left(x + \frac{dy}{dx}\right)^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \quad x^2 + 2x\frac{dy}{dx} = 1$$

So, the order is 1 and the degree is also 1.

6. We have,

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \left(y - x\frac{dy}{dx}\right)^2 = a^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

So, the order is 1 and the also degree is 1.

7. We have,

$$\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

$$\frac{d^4y}{dx^4} + \left(\frac{d^3y}{dx^3}\right) - \left(\frac{d^3y}{dx^3}\right)^3 \frac{1}{3}! + \left(\frac{d^3}{dx^3}\right)^5 \frac{1}{5}! + \ldots$$

So, the order is 4 and the degree is not defined.

8. We have,

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$$

So, the order is 2 but the degree is not defined.

9. We have,

$$e^{\frac{dy}{dx}} = (x + 1)$$

$$1 + \left(\frac{dy}{dx}\right) + \frac{1}{2}! \left(\frac{dy}{dx}\right)^2 + \frac{1}{3}! \left(\frac{dy}{dx}\right)^3 + ...$$

So, the order is 1 and the degree is not defined.

10. We have,

$$\sin\left(\frac{dy}{dx}\right) + \cos\left(\frac{dy}{dx}\right)$$

$$= x\left(\left(\frac{dy}{dx}\right) - \frac{1}{3}! \left(\frac{dy}{dx}\right)^3 + \frac{1}{5}! \left(\frac{dy}{dx}\right)^5 - ...\right)$$

$$+ \left(1 - \frac{1}{2}! \left(\frac{dy}{dx}\right)^2 + \frac{1}{4}! \left(\frac{dy}{dx}\right)^4 - \frac{1}{6}! \left(\frac{dy}{dx}\right)^6 + ...\right) = x$$

So, the order is 1 and degree is not defined.

11. Let the equation of the family of non-vertical lines in a plane is

$$y = mx + c,$$

where m and c are arbitrary constants.

Now $\quad \frac{dy}{dx} = m$ and $\frac{d^2y}{dx^2} = 0.$

From the above three equations, we should eliminate m.

Thus, the required differential equation is

$$\frac{d^2y}{dx^2} = 0.$$

12. Let the equation of the parabola be

$$y^2 = 4ax.$$

where a is an arbitrary constant.

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{y}{2}\frac{dy}{dx} = a$$

Thus, the required differential equation is

$$y^2 = 2xy\frac{dy}{dx}.$$

13. The equation of the circle be

$(x - a)^2 + y^2 = a^2,$

where a is an arbitrary constant.

$$\Rightarrow \quad x^2 + y^2 - 2ax = 0$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2a = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = a.$$

Thus, the required differential equation is

$$\left(y\frac{dy}{dx}\right)^2 + y^2 = \left(x + y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 2xy\frac{dy}{dx} = y^2 - x^2.$$

14. The equation of all parabolas whose axes parallel to x-axis is

$(y - k)^2 = 4a(x - h)$

Here, h and k are two arbitrary constants.

Differentiating both sides, we get

$$(y - k)\frac{dy}{dx} = 2a \quad ...(i)$$

Differentiating again w.r.t. x, we get

$$(y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad ...(ii)$$

Eliminating k from Eq. (i) and (ii), we get

$$2a\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3 = 0$$

which is the required differential equation.

15. The given curve is

$$y = c(x - c)^2 \quad ...(i)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2c(x - c) \quad ...(ii)$$

From Eq. (i) and (ii), we get, $c = \frac{1}{4y}\left(\frac{dy}{dx}\right)^2$

Put the value of c in Eq. (ii), we get

$$\frac{dy}{dx} = 2\cdot\frac{1}{4y}\left(\frac{dy}{dx}\right)^2\left(x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \quad 2y = \frac{dy}{dx}\left(x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \quad 8y^2 = 4xy\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$$

which is the required differential equation.

16. The given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2yy''}{b^2} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} + \frac{yy'}{b^2} = 0$$

Again differentiating w.r.t. x, we get

$$\frac{1}{a^2} + \frac{yy''}{b^2} + \frac{(y')^2}{b^2} = 0$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{xyy''}{b^2} + \frac{x(y')^2}{b^2} = 0$$

$$\Rightarrow \quad -\frac{yy'}{b^2} + \frac{xyy''}{b^2} + \frac{x(y')^2}{b^2} = 0$$

$$\Rightarrow \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$$

which is the required differential equation.

17. Given curve is

$$y = mx$$

$$\Rightarrow \quad \frac{dy}{dx} = m$$

$$= \frac{y}{x}$$

which is the required differential equation.

18. Given curve is

$$y = A\cos x + B\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = -A\sin x + B\cos x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -A\cos x - B\sin x = -y$$

Hence, the required differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

19. We have,

$$y = ae^x + be^{-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = ae^x - be^{-x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = ae^x + be^{-x} = y$$

Hence, the required differential equation is

$$\frac{d^2y}{dx^2} - y = 0$$

20. We have,

$$y = \sin(bx + c)$$

$$\Rightarrow \quad y' = b\cos(bx + c)$$

$$\Rightarrow \quad y'' = -b^2\sin(bx + c)$$

$$= -b^2 y$$

$$= -\left(\frac{y'}{\sqrt{1 - y^2}}\right)^2 y$$

$$\Rightarrow \quad y''(1 - y^2) + y(y')^2 = 0$$

$$\Rightarrow \quad (1 - y^2)\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation.

21. We have,

$$y = ae^{2x} + be^{3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2ae^{2x} + 3be^{3x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 4ae^{2x} + 9be^{3x}$$

$$= 2(2ae^{2x} + 3be^{3x}) + 3be^{3x}$$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 3be^{3x}$$

$$= 2\frac{dy}{dx} + 3\left(\frac{dy}{dx} - 2y\right)$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 + (y - a)\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 5\frac{dy}{dx} - 6y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

22. Let the equation of the circle be

$$x^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y - \frac{dy}{dx} = a\frac{dy}{dx}$$

$$\Rightarrow \quad a = \frac{x + y\dfrac{dy}{dx}}{\dfrac{dy}{dx}} = \frac{x + yy_1}{y_1}$$

Hence, the required differential equation is

$$\Rightarrow \quad x^2 + y^2 = 2\left(\frac{x + yy_1}{y_1}\right)y$$

$$\Rightarrow \quad \frac{dy}{dx}(x^2 + y^2 - 2y^2) = 2xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{(x^2 - y^2)}$$

(b) Do yourself.

23. Let the equation of the circle be

$(x - h)^2 + (y - k)^2 = r^2$

where h and k are parameters but not r.

$$\Rightarrow \quad 2(x - h) + 2(y - a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x - h) + (y - k)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 + (y - k)\frac{d^2y}{dx^2} = 0$$

24. Let the equation of the circle be

$$(x - a)^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2a(x + y) + a^2 = 0$$

$$\Rightarrow \quad 2x + 2y - 2a(1 + y_1) = 0$$

$$\Rightarrow \quad x + y - a(1 + y_1) = 0$$

$$\Rightarrow \quad a = \frac{x + y}{1 + y_1}$$

Hence, the required differential equation is

$$x^2 + y^2 - 2\left(\frac{x + y}{1 + y_1}\right)(x + y) + \left(\frac{x + y}{1 + y_1}\right)^2 = 0$$

$$\Rightarrow (1 + y_1)^2(x^2 + y^2) - 2(x + y)^2(1 + y_1) + (x + y)^2 = 0$$

25. Do yourself

26. Let the line be

$$y = mx + c$$

$$\Rightarrow \quad \frac{dy}{dx} = m$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0$$

which is the required differential equation.

27. Let the equation of the parabola be

$(y - k)^2 = 4a(x - h)$,

where h and k are parameters.

$$\Rightarrow \quad 2(y - k)\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad (y - k)\frac{dy}{dx} = 2a$$

$$\Rightarrow \quad (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

which is the required differential equation

28. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y}{b^2} = 1$$

$$\Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad -\frac{y}{b^2x}\frac{dy}{dx} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad -\frac{y}{x}\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$$

which is the required differential equation.

29. Let the equation of the parabola be

$$(x - h)^2 = 4a(y - k)$$

where h and k are parameters

$$\Rightarrow \quad 2(x - h) = 4a\frac{dy}{dx}$$

$$\Rightarrow \quad (x - h) = 2a\frac{dy}{dx}$$

$$\Rightarrow \quad 1 = 2a\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{2a}$$

which is the required differential equation.

30. Given curve is

$$y^2 = 2c(x + \sqrt{c})$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2c$$

$$\Rightarrow \quad y\frac{dy}{dx} = c$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation.

31. Do yourself.

32. Given differential equation is

$$dy = \frac{x^2 - 1}{x^2 + 1}dx.$$

Integrating, we get

$$\int dy = \int\frac{x^2 - 1}{x^2 + 1}dx$$

$$\Rightarrow \quad y = \int\left(\frac{1 - 2}{x^2 + 1}\right)dx$$

$$= x - 2\tan^{-1}x + c$$

which is the required solution.

33. Given differential equation is

$$dy = \frac{1}{e^x + 1}\, dx$$

Integrating, we get

$$\int dy = \frac{1}{e^x + 1}\, dx$$

$$\Rightarrow \quad y = \int ((e^x + 1) - \frac{e^x}{e^x} + 1)\, dx$$

$$\Rightarrow \quad y = \int \left(\frac{1 - e^x}{1 + e^x}\right) dx$$

$$= x - \log|e^x + 1| + c,$$

which is the required solution.

34. The given differential equation is

$$\frac{dy}{dx} = \frac{\cos 2x - \cos x}{1 - \cos x}$$

$$\Rightarrow \quad dy = \left(\frac{\cos 2x - \cos x}{1 - \cos x}\right) dx$$

Integrating, we get

$$\int dy = \int \frac{\cos 2x - \cos x}{1 - \cos x}\, dx$$

$$\Rightarrow \quad y = -\int \frac{(2\cos x + 1)(1 - \cos x)}{(1 - \cos x)}\, dx$$

$$= -\int (2\cos x + 1)\, dx$$

$$= -(2\sin x + x) + c$$

which is the required solution of the given differential equation.

35. The given differential equation is

$$\frac{dy}{dx} = \frac{1}{x}(x^4 + 1)$$

$$\Rightarrow \quad dy = \frac{dx}{x(x^4 + 1)}$$

Integrating, we get

$$\int dy = \int \frac{dx}{x(x^4 + 1)}$$

$$\Rightarrow \quad y = \int \frac{x^3 dx}{x^4(x^4 + 1)}$$

$$= \frac{1}{4}\int \frac{dt}{t(t + 1)}$$

$$= \frac{1}{4}\int \left(\frac{1}{t} - \frac{1}{t} + 1\right) dt$$

$$= \frac{1}{4}\log\left|\frac{t}{t + 1}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{x^4}{x^4 + 1}\right| + c$$

which is the required solution.

36. The given differential equation is

$$\frac{dy}{dx} = \frac{\sqrt{\sin x}}{\sin x \cos x}\, dx$$

$$\Rightarrow \quad dy = \frac{\sqrt{\sin x}}{\sin x \cos x}\, dx$$

Integrating, we get

$$\int dy = \int \frac{\sqrt{\sin x}}{\sin x \cos x}\, dx$$

$$\Rightarrow \quad y = \int \frac{\sqrt{\sin x}}{\sin x \cos x}\, dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx$$

(Dividing the numerator and the denuminator by $\cos^2 x$)

$$\Rightarrow \quad y = 2\sqrt{\tan x} + C$$

which is the general solution.

37. The given differential equation is

$$\frac{dy}{dx} = x^2(3x + 2x)^2$$

$$\Rightarrow \quad dy = \frac{x^2}{(3 + 2x)^2}\, dx$$

Integrating, we get

$$\int dy = \int \frac{x^2}{(3 + 2x)^2}\, dx$$

$$\Rightarrow \quad y = \int \frac{x^2 dx}{(3 + 2x)^2}$$

Let $\quad 3 + 2x = t$

$$= \frac{1}{2}\int \left(\frac{t - 3}{2}\right)^2 \cdot \frac{1}{t^2}\, dt$$

$$= \frac{1}{8}\int \left(\frac{t - 3}{t}\right)^2 dt$$

$$= \frac{1}{8}\int \left(1 - \frac{3}{t}\right)^2 dt$$

$$= \frac{1}{8}\int \left(1 - \frac{6}{t} + \frac{9}{t^2}\right) dt$$

$$= \frac{1}{8}\left(1 - 6\log t - \frac{1}{t}\right) + c$$

$$= \frac{1}{8}\left((3 + 2x) - 6\log(3 + 2x) - \frac{1}{(3 + 2x)}\right) + c$$

which is the required differential equation.

38. The given differential equation is

$$\frac{dy}{dx} = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

$$\Rightarrow \quad dy = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$$

Integrating, we get

$$\int dy = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$$

$$\Rightarrow \quad y = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$$

Let $x = \cos^2 2\theta$

$$\Rightarrow \quad dx = -2\sin 4\theta\, d\theta$$

Integrating, we get

$$\Rightarrow \quad y = \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times -2\sin 4\theta d\theta$$

$$= \int \tan\theta \times -2\sin 4\theta d\theta$$

$$= -2\int \cos 2\theta(1-\cos 2\theta)\,d\theta$$

$$= -2\int (2\cos 2\theta - 1 - \cos 4\theta)\,d\theta$$

$$= -2\left(\sin 2\theta - \theta - \frac{\sin 4\theta}{4}\right) + c$$

$$= (-2\sin 2\theta + 2\theta + \sin 2\theta\cos 2\theta) + c$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}(1-x) + c$$

which is the required solution of the given differential equation.

39. The given differential equation is

$$\frac{dy}{dx} = (\sqrt{\tan x} + \sqrt{\cot x})$$

$$\Rightarrow \quad dy = (\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

Integrating, we get

$$\int dy = \int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

$$\Rightarrow \quad y = \int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

$$= \int \left(\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\right)dx$$

$$= \sqrt{2}\int \left(\frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}}\right)dx$$

$$= \sqrt{2}\int \left(\frac{\sin x + \cos x}{\sqrt{\sin 2x}}\right)dx$$

Let $\sin x - \cos x = t$

$$\Rightarrow \quad (\cos x + \sin x)\,dx = dt$$

Also, $1 - \sin 2x = t^2$

$$\Rightarrow \quad \sin 2x = 1 - t^2$$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1}(t) + c$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + c$$

which is the required differential equation.

40. The given differential equation is

$$\frac{dy}{dx} = \frac{1}{\sin^3 x + \cos^3 x}$$

$$\Rightarrow \quad dy = \frac{dx}{\sin^3 x + \cos^3 x}$$

Integrating, we get

$$\int dy = \int \frac{dx}{\sin^3 x + \cos^3 x}$$

$$= \int \frac{dx}{\sin^3 x + \cos^3 x}$$

$$= \int \frac{dx}{(\sin x + \cos x)(1 - \sin x \cos x)}$$

$$= \frac{1}{3}\int \left(\frac{2}{(\sin x + \cos x)} + \frac{\sin x + \cos x}{(1 - \sin x \cos x)}\right)dx$$

$$= \frac{2}{3}\int \frac{dx}{(\sin x + \cos x)}$$

$$+ \frac{1}{3}\int \frac{\sin x + \cos x}{(1 - \sin x \cos x)}\,dx$$

$$= \frac{2}{3\sqrt{2}}\int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)}$$

$$+ \frac{2}{3}\int \frac{\sin x + \cos x}{(2 - \sin 2x)}dx$$

$$= \frac{2}{3\sqrt{2}}\log\left|\frac{x}{2} + \frac{\pi}{8}\right|$$

$$+ \frac{2}{3}\tan^{-1}(\sin x - \cos x) + c$$

which is the required solution of the given differential equation.

41. The given differential equation is

$$\frac{dy}{dx} = \frac{1}{(\sqrt{\sin x} + \sqrt{\sin x})^4}$$

$$\Rightarrow \quad dy = \frac{dx}{(\sqrt{\sin x} + \sqrt{\cos x})^4}$$

Integrating, we get

$$\int dy = \int \frac{dx}{(\sqrt{\sin x} + \sqrt{\cos x})^4}$$

$$\Rightarrow \quad y = \int \frac{dx}{(\sqrt{\sin x} + \sqrt{\cos x})^4}$$

$$= \int \frac{\sec^2 x\, dx}{(\sqrt{\tan x} + 1)^4}$$

$$= \frac{2t\, dt}{(t+1)^4}, \text{ (Let } \tan x = t^2)$$

$$= 2\int \frac{(t+1) - 1}{(t+1)^4}\, dt$$

$$= 2\int \left(\frac{1}{(t+1)^3} - \frac{1}{(t+1)^4}\right) dt$$

$$= 2\left(\frac{1}{-2(t+1)^2} + \frac{1}{3(t+1)^3}\right) + c$$

$$= 2\left(\frac{1}{-2(\sqrt{\tan x} + 1)^2} + \frac{1}{3(\sqrt{\tan x} + 1)^3}\right) + c$$

which is the required solution of the given differential equation.

42. Given differential equation is

$$\int \frac{dy}{(1+y)} = \int (1+x)\, dx$$

Integrating, we get

$$\frac{dy}{(1+y)} = (1+x)\, dx.$$

$$\Rightarrow \quad \log|1+y| = x + \frac{x^2}{2} + c,$$

which is the required solution.

43. Given differential equation is

$$\frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow \quad \frac{dy}{e^{-y}} = (e^x + x^2)\, dx$$

Integrating, we get

$$\int \frac{dy}{e^{-y}} = \int (e^x + x^2)\, dx$$

$$\Rightarrow \quad \int e^y dy = \int (e^x + x^2)\, dx$$

$$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + c,$$

which is the required solution.

44. The given differential equation is

$$3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$$

$$\Rightarrow \quad \frac{3e^x}{(1-e^x)}\, dx = \frac{\sec^2 y}{\tan y}\, dy$$

On Integrating, we get

$$\int \frac{3e^x}{(1-e^x)}\, dx = \int \frac{\sec^2 y}{\tan y}\, dy$$

$$\Rightarrow \quad -3\log|1 - e^x| = \log|\tan y| + c$$

which is the required solution of the given differential equation.

45. The given differential equation is

$$\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{(1+x^2)(1+x^2)} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{(1+x^2)}\,\sqrt{(1+y^2)} = -xy\frac{dy}{dx}$$

$$\frac{\sqrt{(1+x^2)}}{x}\, dx = -\frac{y}{\sqrt{(1+y^2)}}\, dx$$

Integrating, we get

$$\int \frac{\sqrt{(1+x^2)}}{x}\, dx = -\int \frac{y}{\sqrt{(1+y^2)}}\, dy$$

$$\Rightarrow \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2} - 1}{\sqrt{1+x^2} + 1}\right| + \sqrt{1+y^2} = c$$

which is the required solution of the given differential equation.

46. The given differential equation is

$$\frac{dy}{dx} = \frac{\sin y + \cos y}{x(2\log x + 1)}$$

$$\Rightarrow \quad \frac{dy}{\sin y + \cos y} = \frac{dx}{x(2\log x + 1)}$$

Integrating, we get

$$\int \frac{dy}{\sin y + \cos y} = \int \frac{dx}{x(2\log x + 1)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\int \frac{dy}{\sin\left(y + \frac{\pi}{4}\right)} = \int \frac{dx}{x(2\log x + 1)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{y}{2} + \frac{\pi}{8}\right)\right| = \frac{1}{2}\log|2\log x + 1|$$

which is the required solution of the given differential equation.

47. The given differential equation is

$$x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad (x-a)\frac{dy}{dx} = ay^2$$

$$\Rightarrow \quad y^{-2}dy = \frac{a}{(x-a)}dx$$

Integrating, we get

$$\Rightarrow \quad -\frac{1}{y} = a\log|x-a| + C$$

which is the required solution of the given differential equation.

48. The given differential equation is

$$xy^2\frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$$

$$\Rightarrow \quad xy^2\frac{dy}{dx} = (1-x^2)(1+y^2)$$

$$\Rightarrow \quad \frac{y^2}{(1+y^2)}dy = \left(\frac{1-x^2}{x}\right)dx$$

$$\Rightarrow \quad \left(1 - \frac{1}{1+y^2}\right)dy = \left(\frac{1}{x} - x\right)dx$$

Integrating, we get

$$y - \tan^{-1}y = \log|x| - \frac{x^2}{2} + c$$

which is the required solution of the given differential equation.

49. Given differential equation is

$$(x+1)\frac{dy}{dx} = 2xy$$

$$\Rightarrow \quad \frac{dy}{y} = \left(\frac{2x}{x+1}\right)dx$$

$$\Rightarrow \quad \frac{dy}{y} = 2\left(1 - \frac{1}{x+1}\right)dx$$

Integrating, we get

$$\log|y| = 2x - 2\log|x+1| + c$$

which is the required solution.

50. Given differential equation is

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$$

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$\log|\tan x| + \log|\tan y| = \log c$$

$$\Rightarrow \quad \tan x \tan y = c$$

which is the required solution.

51. Given differential equation is

$$(1+e^{2x})dy + (1+y^2)e^x\, dx = 0$$

$$\Rightarrow \quad \frac{dy}{1+y^2} + \frac{e^x}{1+ex^{2x}}dx = 0$$

Integrating, we get

$$\tan^{-1}(y) + \tan^{-1}(e^x) = \tan^{-1}c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y+e^x}{1-e^x y}\right) = \tan^{-1}c$$

$$\Rightarrow \quad \left(\frac{y+e^x}{1-e^x y}\right) = c$$

which is the required solution.

52. Given differential equation is

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$

$$\Rightarrow \quad \frac{dy}{dx} = (ex + x^2)e^y$$

$$\Rightarrow \quad e^{-y}dy = (e^x + x^2)dx$$

Integrating, we get

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

which is the required differential equation.

53. Given differential equation is

$$y\sqrt{1+x^2} + x\sqrt{1+x^2}\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, we get

$$\log|x| + \log|y| = \log c$$

$$\Rightarrow \quad xy = c$$

which is the required solution.

54. Given differential equation is

$$\sqrt{1+x^2}\,dy + \sqrt{1+y^2}\,dx = 0$$

$$\Rightarrow \quad \frac{dy}{\sqrt{1+y^2}} + \frac{dx}{\sqrt{1+x^2}} = 0$$

Integrating, we get

$$\log\left|y + \sqrt{y^2+1}\right| + \log\left|x + \sqrt{x^2+1}\right| = \log c$$

$$\Rightarrow \quad \left(y + \sqrt{y^2+1}\right)\left(x + \sqrt{x^2+1}\right) = c$$

which is the required solution.

55. Given differential equation is

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{(1+x^2)(1+y^2)} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad xy\frac{dy}{dx} = -\sqrt{(1+x^2)(1+y^2)}$$

$$\Rightarrow \quad \frac{ydy}{\sqrt{1+y^2}} = -\frac{\sqrt{1+x^2}}{x}dx$$

Intergrating, wet get

$$\sqrt{1+y^2} = \sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + c$$

which is the required solution.

56. Given differential equation is

$$xy\frac{dy}{dx} + 1 + x + y + xy = 0$$

$$\Rightarrow \quad xy\frac{dy}{dx} + (1+x)(1+y) = 0$$

$$\Rightarrow \quad \left(\frac{y}{1+y}\right)dy + \left(\frac{1+x}{x}\right)dx = 0$$

$$\Rightarrow \quad \left(1 - \frac{1}{1+y}\right)dy + \left(1 + \frac{1}{x}\right)dx = 0$$

Integrating, we get

$$y - \log|1+y| + x + \log|x| = c$$

$$\Rightarrow \quad (x+y) - \log\left|\frac{x}{1+y}\right| = c$$

57. Given differential equation is

$$(1+x)(1+y^2)dx + (1+y)(1+x^2)dy = 0$$

$$\Rightarrow \quad \left(\frac{1+x}{1+x^2}\right)dx + \left(\frac{1+y}{1+y^2}\right)dy = 0$$

Integrating, we get

$$\tan^{-1}x + \tan^{-1}y + \frac{1}{2}\log|(x^2+1)(y^2+1)| = c$$

58. Given differential equation is

$$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$$

$$\Rightarrow \quad \frac{x}{\sqrt{1-x^2}}dx + \frac{y}{\sqrt{1-y^2}}dy = 0$$

Integrating, we get

$$\sqrt{1-x^2} + \sqrt{1-y^2} = c$$

which is the required solution.

59. Given differential equation is

$$\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad y(1-ay) = (x+a)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

$$\Rightarrow \quad \frac{dx}{x+a} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)$$

Integrating, we get

$$\log|x+a| + \log c = \log|y| - \log|1-ay|$$

$$\Rightarrow \quad c(x+a) = \frac{y}{1-ay}$$

which is the required solution.

60. We have,

$$xy\frac{dx}{dy} = \frac{1+y^2}{1+x^2}(1+x+x^2)$$

$$\Rightarrow \quad \frac{ydy}{1+y^2} = \frac{(1+x+x^2)}{(1+x^2)x}dx$$

$$= \frac{(1+x+x^2)}{(1+x^2)x}dx$$

$$= \frac{dx}{x} + \frac{dx}{1+x^2}$$

$$\Rightarrow \quad \frac{1}{2}\log|1+y^2| = \log|x| + \tan^{-1}x + c$$

which is the required solution.

61. We have,

$$(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$$

$$\Rightarrow \quad x^2(1-y)dy + y^2(1+x)dx = 0$$

$$\Rightarrow \quad \frac{(1-y)}{y^2}dy + \frac{(1+x)}{x^2}dx = 0$$

Integrating, we get

$$-\frac{1}{y} - \log|y| - \frac{1}{x} + \log|x| = c$$

$$\Rightarrow \quad \log\left|\frac{x}{y}\right| = c + \left(\frac{1}{x} + \frac{1}{y}\right)$$

which is the required solution.

62. The given differential equation is

$$\frac{dy}{dx} = (x+y+1)^2 \qquad \text{...(i)}$$

Let $\quad x + y + 1 = v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = v^2$$

$$\Rightarrow \quad \frac{dv}{v^2+1} = dx$$

Integrating, we get

$$\int\frac{dv}{v^2+1} = \int dx$$

$$\Rightarrow \quad \tan^{-1}(v) = x + c$$

$$\Rightarrow \quad \tan^{-1}(x+y+1) = x + c$$

is the required solution.

63. The given differential equation is

$$\frac{dy}{dx} = \sin(x + y) + \cos(x + y) \quad ...(i)$$

Put $x + y = v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = \sin v + \cos v$$

$$\Rightarrow \frac{dv}{dx} = 1 + \sin v + \cos v$$

$$\Rightarrow \frac{dv}{1 + \sin v + \cos v} = dx$$

Integrating, we get

$$\int \frac{dv}{1 + \sin v + \cos v} = x + c$$

$$\Rightarrow \int \frac{dv}{1 + \left(\frac{1 - \tan^2(v/2)}{1 + \tan^2(v/2)}\right) + \left(\frac{2\tan(v/2)}{1 + \tan^2(v/2)}\right)} = x + c$$

$$\Rightarrow \int \frac{\sec^2(v/2)dv}{2(1 + \tan(v/2))} = x + c$$

$$\Rightarrow \log|1 + \tan(v/2)| = x + c$$

$$\Rightarrow \log|1 + \tan(v/2)| = x + c$$

$$\Rightarrow \log\left|1 + \tan\left(\frac{x + y}{2}\right)\right| = x + c$$

which is the required solution of the given differential equation.

64. The given differential equation is

$$(x + y)(dx - dy) = (dx + dy)$$

$$\Rightarrow (x + y - 1)dx = (x + y + 1)dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x + y - 1)}{(x + y + 1)} \quad ...(i)$$

Let $x + y = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$$\Rightarrow \frac{dv}{dx} - 1 = \frac{v - 1}{v + 1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v - 1}{v + 1} + 1 = \frac{2v}{v + 1}$$

$$\Rightarrow \left(\frac{v + 1}{v}\right)dv = 2dx$$

Integrating, we get

$$\int\left(1 + \frac{1}{v}\right)dv = 2x + c$$

$$\Rightarrow (v + \log|v|) = 2x + c$$

$$\Rightarrow (x + y + \log|x + y|) = 2x + c$$

$$\Rightarrow (y + \log|x + y|) = x + c$$

which is the required solution of the given differential equation.

65. The given differential equation is

$$\tan y\frac{dy}{dx} = \sin(x + y) + \sin(x - y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(x + y) + \sin(x - y)}{\tan y}$$

$$= \frac{2\sin x\cos y}{\tan y}$$

$$\Rightarrow \frac{\tan y}{\cos y}dy = 2\sin x\,dx$$

$$\Rightarrow \frac{\sin y}{\cos^2 y\,dy} = 2\sin x\,dx$$

Integrating, we get

$$\int \frac{\sin y}{\cos^2 y\,dy} = \int 2\sin x\,dx$$

$$\Rightarrow \frac{1}{\cos y} = c - 2\cos x$$

which is the required solution of the given differential equation.

66. The given differential equation is

$$\frac{dy}{dx} - x\tan(y - x) = 1$$

$$\Rightarrow \frac{dy}{dx} = 1 + x\tan(y - x)$$

Let $y - x = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} + 1$

$$\Rightarrow \frac{dv}{dx} + 1 = x\tan v + 1$$

$$\Rightarrow \frac{dv}{dx} = x\tan v + 1$$

$$\Rightarrow \cot v\,dv = x\,dx$$

Intergrating, we get

$$\int \cot v\,dv = x + c$$

$$\Rightarrow \log|\sin v| = \frac{x^2}{2} + c$$

$$\Rightarrow \log|\sin(x - y)| = \frac{x^2}{2} + c$$

which is the required solution of the given differential equation.

67. We have,

$$\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5} \quad \text{...(i)}$$

Let $x - y = v$

$\Rightarrow \quad 1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = 1 - \frac{dv}{dx}$

$\Rightarrow \quad \left(1 - \frac{dv}{dx}\right) = \frac{v+3}{2v+5}$

$\Rightarrow \quad \frac{dv}{dx} = 1 - \frac{v+3}{2v+5}$

$= \frac{2v+5-v-3}{2v+5} = \frac{v+2}{2v+5}$

$\Rightarrow \quad \left(\frac{2v+5}{v+2}\right)dv = dx$

$\Rightarrow \quad \left(2 + \frac{1}{v+2}\right)dv = dx$

Integrating, we get

$(2v + \log|v+2|) = x + c$

$\Rightarrow \quad (2(x-y) + \log|x-y+2|) = x + c$

$\Rightarrow \quad (x - 2y + \log|x-y+2|) = c$

which is the required solution.

68. We have,

$(x+y)(dx - dy) = dx + dy$

$\Rightarrow \quad \frac{d(x+y)}{(x+y)} = d(x-y)$

Integrating, we get

$\log|x+y| = (x-y) + c$

69. We have $\frac{dy}{dx} = \sec(x+y)$

$\Rightarrow \quad \frac{dv}{dx} - 1 = \sec v \quad$ (Let $v = x + y$)

$= 1 + \sec v$

$\Rightarrow \quad \frac{dv}{1+\sec v} = dx$

$\Rightarrow \quad \frac{\cos v\, dv}{1+\cos v} = dx$

$\Rightarrow \quad \frac{\cos v(1-\cos v)\, dv}{\sin^2 v} = dx$

$\Rightarrow \quad [\text{cosec}\, v \cot v - (\text{cosec}\, v - 1)]\, dv = dx$

Integrating, we get

$-\text{cosec}\, v + \cot v - v = x + c$

$\Rightarrow \quad -\text{cosec}(x+y) + \cot(x+y) = 2x + y + c$

which is the required solution.

70. We have,

$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

$\Rightarrow \quad \frac{dy}{dx} = \sin(x+y)$

$\Rightarrow \quad \frac{dv}{dx} - 1 = \sin v, \quad$ (Let $v = x + y$)

$\Rightarrow \quad \frac{dv}{dx} = 1 + \sin v$

$\Rightarrow \quad \frac{dv}{(1+\sin v)} = dx$

$\Rightarrow \quad \frac{(1-\sin v)dv}{\cos^2 v} = dx$

$\Rightarrow \quad (\sec^2 v - \sec v \tan v)\, dv = dx$

Integrating, we get

$\tan v - \sec v = x + c$

$\Rightarrow \quad \tan(x+y) - \sec(x+y) = x + c$

which is the required solution.

71. We have

$\frac{dy}{dx} = \cos(x+y+1)$

$\Rightarrow \quad \frac{dv}{dx} - 1 = \cos v$, (Let $x + y + 1 = v$)

$\Rightarrow \quad \frac{dv}{1+\cos v} = dx$

$\Rightarrow \quad \frac{1-\cos v}{\sin^2 v\, dv} = dx$

$\Rightarrow \quad (\text{cosec}^2 v - \text{cosec}\, v \,.\, \cot v)\, dv = dx$

Integrating, we get

$\text{cosec}\, v - \cot v = x + c$

$\Rightarrow \quad \text{cosec}(x+y+1) - \cot(x+y+1)$

72. We have,

$\frac{dy}{dx} = \frac{2(x+y)}{1+(x+y)^2}$

$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{2v}{1+v^2}, \; v = x + y$

$\Rightarrow \quad \frac{dv}{dx} = 1 + \frac{2v}{1+v^2}$

$= 1 + \frac{2v}{1+v^2} = \frac{(1+v)^2}{1+v^2}$

$$\Rightarrow \quad \frac{1+v^2}{(1+v)^2}\,dv = dx$$

$$\Rightarrow \quad \left(1 - \frac{2}{(1+v)^2}\right)dv = dx$$

$$\Rightarrow \quad \left(1 - \frac{2}{1+v} + \frac{2}{(1+v)^2}\right)dv = dx$$

Integrating, we get

$$v - 2\log|1+v| - \frac{2}{(1+v)} = x + c$$

$$\Rightarrow \quad y - 2\log|1+x+y| - \frac{2}{(1+x+y)} = c$$

which is the required solution.

73. We have,

$$\frac{dy}{dx} = \sin(10x+6y)$$

$$\Rightarrow \quad \frac{1}{6}\left(\frac{dv}{dx} - 10\right) = \sin v$$

$$\Rightarrow \quad \frac{dv}{dx} = 6\sin v + 10$$

$$\Rightarrow \quad \frac{dv}{6\sin v + 10} = dx$$

$$\Rightarrow \quad \frac{dv}{6\left(\frac{2\tan(v/2)}{1+\tan^2(v/2)}\right) + 10} = dx$$

$$\Rightarrow \quad \frac{\sec^2(v/2)dv}{10(1+\tan^2(v/2)) + 12\tan(v/2)} = dx$$

$$\Rightarrow \quad \frac{2\,dt}{10(1+t^2)+12t} = dx,\ t = \tan(v/2)$$

$$\Rightarrow \quad \frac{dt}{5(1+t^2)+6t} = dx$$

$$\Rightarrow \quad \frac{dt}{5t^2+6t+5} = dx$$

$$\Rightarrow \quad \frac{dt}{t^2+\frac{6}{5}t+1} = 5dx$$

$$\Rightarrow \quad \frac{dt}{\left(t+\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 5dx$$

Integrating, we get

$$\frac{5}{4}\tan^{-1}\left(\frac{t+(3/5)}{4/5}\right) = 5x + c$$

$$\Rightarrow \quad \frac{5}{4}\tan^{-1}\left(\frac{5t+3}{4}\right) = 5x + c$$

$$\Rightarrow \quad \frac{5}{4}\tan^{-1}\left(\frac{5\tan(v/2)+3}{4}\right) = 5x + c$$

$$\Rightarrow \quad \frac{5}{4}\tan^{-1}\left(\frac{5\tan(5x+3y)+3}{4}\right) = 5x + c$$

which is the required solution.

74. We have,

$$\frac{y}{x}\frac{dy}{dx} + \frac{2(x^2+y^2)-1}{x^2+y^2+1} = 0$$

$$\Rightarrow \quad \frac{1}{2}\frac{dv}{dx} - 1 + \frac{2v-1}{v+1} = 0$$

$$\Rightarrow \quad \frac{1}{2}\frac{dv}{dx} = 1 - \frac{2v-1}{v+1}$$

$$= \frac{2-v}{v+1}$$

$$\Rightarrow \quad \left(\frac{v+1}{2-v}\right)dv = 2dx$$

$$\Rightarrow \quad \left(\frac{v+1}{v-1}\right)dv = -2dx$$

$$\Rightarrow \quad \left(\frac{v-2+3}{v-2}\right)dv = -2dx$$

$$\Rightarrow \quad 1 + \left(\frac{3}{v-2}\right)dv = -2dx$$

Integrating, we get

$$(x^2+y^2) + \log|(x^2+y^2) - 2| = c - 2x$$

which is the required solution.

75. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}.$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{1+\left(\frac{y}{3}\right)^3},$$

Let $\frac{y}{x} = v$

$\Rightarrow \quad y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dy}{dx}.$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow \quad \left(\frac{v^3+1}{v^4}\right)dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{v} + v^{-4}\right)dv = -\frac{dx}{x}$$

Integrating, we get

$$\log|v| - \frac{1}{3v^3} = c - \log|x|$$

$$\Rightarrow \quad \log\left|\frac{y}{x}\right| - \frac{x^3}{3y^3} = c - \log|x|$$

$$\Rightarrow \quad \log|v| - \frac{1}{3v^3} = c,$$

which is the required

solution of the given differential equation.

76. The given differential equation is

$$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = \frac{\sqrt{x^2 + y^2}}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = \sqrt{1 + \left(\frac{y}{x}\right)^2} \qquad \text{...(i)}$$

Let $\frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dy}{dx}$$

$$\Rightarrow \quad v + x\frac{dy}{dx} - v = \sqrt{1 + v^2}$$

$$\Rightarrow \quad x\frac{dy}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log c$$

$$\Rightarrow \quad \left(v + \sqrt{1 + v^2}\right) = cx$$

$$\Rightarrow \quad \left(\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\right) = cx$$

$$\Rightarrow \quad \left(y + \sqrt{x^2 + y^2}\right) = cx^2$$

which is the required solution of the given differential equation.

77. The given differential equation is

$$(x^2 - y^2)\,dx = 2xy\,dy$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \sqrt{x^2 + y^2} \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y} - \frac{y}{x}\right)$$

Let $\frac{y}{x} = v \Rightarrow y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v} - v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{2v} - \frac{3v}{2} = \frac{1 - 3v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1 - 3v^2}\,dv = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{2v}{1 - 3v^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{3}\log|1 - 3v^2| = \log c + \log|x|$$

$$\Rightarrow \quad -\frac{1}{3}\log\left|1 - 3\left(\frac{y}{x}\right)^2\right| = \log c + \log|x|$$

which is the required solution of the given differential equation.

78. The given differential equation is

$$(1 + 2e^{x/y})\,dx + 2e^{x/y}(1 - x/y)\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 + 2e^{x/y})}{2e^{x/y}\left(\frac{x}{y} - 1\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2e^{x/y}\left(\frac{x}{y} - 1\right)}{(1 + 2e^{x/y})} \qquad \text{...(i)}$$

Let $\frac{x}{y} = v$

$$\Rightarrow \quad x = vy$$

$$\Rightarrow \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{2e^v(v - 1)}{(1 + 2e^v)}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{2e^v(v - 1)}{(1 + 2e^v)} - v$$

$$= \frac{2ve^v - 2e^v - v - 2ve^v}{1 + 2e^v}$$

$$= \frac{2e^v + v}{1 + 2e^v}$$

$$\Rightarrow \quad \frac{1 + 2e^v}{v + 2e^v}\,dv = -\frac{dy}{y}$$

$\Rightarrow \quad \log|v + 2e^v| + \log|y| = \log c$

$\Rightarrow \quad y(v + 2e^v) = c$

$\Rightarrow \quad y\left(\frac{x}{y} + 2e^{x/y}\right) = c$

$\Rightarrow \quad (x + 2ye^{x/y}) = c$

which is the required solution of the given differential equation.

79. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y - x}{y + x} \quad \text{...(i)}$$

$$\left(\text{Let } \frac{y}{x} = v \Rightarrow y = vx\right)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v - 1}{v + 1}$

$$= \frac{v - 1}{v + 1} - v$$

$$= \frac{v - 1 - v^2 - v}{v + 1}$$

$$= \frac{1 + v^2}{v + 1}$$

$\Rightarrow \quad \frac{v + 1}{v^2 + 1}dv = -\frac{dx}{x}$

$\Rightarrow \quad \frac{1}{2}\log|v^2 + 1| + \tan^{-1}(v) = c - \log|x|$

$\Rightarrow \quad \frac{1}{2}\log|x^2 + y^2| + \tan^{-1}\left(\frac{y}{x}\right) = c$

which is the required solution of the given differential equation.

80. We have,

$$x\frac{dy}{dx} = x + y$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y}{x} = 1 + \frac{y}{x}$

$\Rightarrow \quad v + x\frac{dv}{dx} = 1 + v, \ v = \frac{y}{x}$

$\Rightarrow \quad x\frac{dv}{dx} = 1$

$\Rightarrow \quad dv = \frac{dx}{x}$

Integrating, we get

$$\log|v^2 - 1| + \log|x| = \log c$$

$\Rightarrow \quad \frac{y}{x} = \log|x| + c$

which is the required solution.

81. We have,

$$\frac{dy}{dx} = \frac{y - x}{y + x} = \frac{(y/x) - 1}{(y/x) + 1}$$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v - 1}{v + 1}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v$

$$= \frac{v - 1}{v + 1} - v$$

$$= -\frac{1 + v^2}{1 + v}$$

$\Rightarrow \quad \frac{1 + v}{1 + v^2}dv = -\frac{dx}{x}$

Integrating, we get

$$\tan^{-1}(v) + \frac{1}{2}\log|1 + v^2| = c - \log|x|$$

$$\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\log|x^2 + y^2| = c$$

which is the required solution.

82. We have,

$$2xy\frac{dy}{dx} = x^2 + y^2$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} = \frac{1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{x}{y}\right)}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 - v^2}{2v}$

$\Rightarrow \quad \frac{2v\,dv}{v^2 - 1} = -\frac{dx}{x}$

Integrating, we get

$$\log|v^2 - 1| + \log|x| = \log c$$

$\Rightarrow \quad (v^2 - 1)x = c$

$\Rightarrow \quad (y^2 - x^2) = cx$

which is the required solution.

83. We have,

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2 - 2\left(\frac{y}{x}\right)}{1 - 2\left(\frac{y}{x}\right)}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v}$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{3v^2 - 3v}{1 - 2v}$$

$$\Rightarrow \quad \frac{1 - 2v}{3v(v - 1)}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{3}\left(\frac{1}{v(v - 1)} - \frac{2}{(v - 1)}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{3}\left(\frac{1}{v - 1} - \frac{1}{v} - \frac{2}{(v - 1)}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{3}\left(\frac{1}{v - 1} + \frac{1}{v}\right)dv = \frac{dx}{x}$$

Integrating, we get

$$-\frac{1}{3}\log|v(v - 1)| = \log|cx|$$

$$\Rightarrow \quad -\frac{1}{3}\log\left|\frac{y}{x}\left(\frac{y}{x} - 1\right)\right| = \log|cx|$$

which is the required solution.

84. Given differential equation is

$$x^2y\,dx - (x^3 + y^3)\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$$

$$= \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1 + v^3}, \qquad \left(\text{Let } v = \frac{y}{x}\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1 + v^3} - v$$

$$= \frac{v - v - v^4}{1 + v^3}$$

$$= -\frac{v^4}{1 + v^3}$$

$$\Rightarrow \quad \left(\frac{1 + v^3}{v^4}\right)dv = -\frac{dx}{x}$$

Integrating, we get

$$-\frac{1}{3v^3} + \log|v| = c - \log|x|$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log|y| = c$$

which is the required solution

85. Given differential equation is

$$(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\text{ dy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}$$

$$= \frac{1 - 3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{3}\right)^3 - 3\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}, \qquad \left(\text{Let } \quad v = \frac{y}{x}\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

$$= \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$$

$$= \frac{1 - v^4}{v^3 - 3v}$$

$$\Rightarrow \quad \left(\frac{v^3 - 3v}{1 - v^4}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{v^3}{1 - v^4}\right)dv - \left(\frac{3v}{1 - v^4}\right)dv = \frac{dx}{x}$$

Integrating, we get

$$\Rightarrow \quad \left(\frac{v^3}{1 - v^4}\right)dv - \left(\frac{3v}{1 - v^4}\right)dv = \frac{dx}{x}$$

Integrating, we get

$$-\frac{1}{4}\log|1 - v^4| + \frac{3}{4}\log\left|\frac{v^2 - 1}{v^2 + 1}\right| = \log|x| + \log c$$

$$\Rightarrow -\frac{1}{4}\log\left|1 - \left(\frac{y}{x}\right)^4\right| + \frac{3}{4}\log\left|\frac{\left(\frac{y}{x}\right)^2 - 1}{\left(\frac{y}{x}\right)^2 + 1}\right| = \log|xc|$$

which is the required solution.

86. The given differential equation is

$$\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sqrt{v^2 - 1} \qquad \left(\text{Let } \quad \frac{y}{x} = u\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sqrt{v^2 - 1}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{v^2 - 1}} = -\frac{dx}{x}$$

Integrating, we get

$$\log\left|v + \sqrt{v^2 - 1}\right| = \log c - \log c$$

$$\Rightarrow \quad \log\left|v + \sqrt{v^2 - 1}\right| = \log\left|\frac{c}{x}\right|$$

$$\Rightarrow \quad \left(v + \sqrt{v^2 - 1}\right) = \left(\frac{c}{x}\right)$$

$$\Rightarrow \quad \left(\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 - 1}\right) = \left(\frac{c}{x}\right)$$

$$\Rightarrow \quad \left(y + \sqrt{y^2 - x^2}\right) = c$$

which is the required solution.

87. The given differential equation is

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v + \sin(v) \qquad \left(\text{Let } \frac{y}{x} = u\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sin(v)$$

$$\Rightarrow \quad \frac{dv}{\sin(v)} = \frac{dx}{x}$$

Integrating, we get

$$\Rightarrow \quad \log\left|\tan\left(\frac{v}{2}\right)\right| = \log c + \log|x|$$

$$\Rightarrow \quad \tan\left(\frac{y}{2x}\right) = cx$$

which is the required solution.

88. The given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{y}{x}\right) - \tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \tan(v) \qquad \left(\text{Let } \frac{y}{x} = u\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan(v)$$

$$\Rightarrow \quad \frac{dv}{\tan(v)} = -\frac{dx}{x}$$

Integrating, we get

$$\Rightarrow \quad \log\left|\tan\left(\frac{\pi}{4} + \frac{v}{2}\right)\right| = \log c - \log|x|$$

$$\Rightarrow \quad \log\left|\tan\left(\frac{\pi}{4} + \frac{v}{2}\right)\right| = \log\left|\frac{c}{x}\right|$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4} + \frac{y}{2x}\right) = \left(\frac{c}{x}\right)$$

which is the required solution.

89. The given differential equation is

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\left(\frac{x}{y} - 1\right)e^{\frac{x}{y}}}{\left(1 + e^{\frac{x}{y}}\right)}$$

$$\Rightarrow \quad v + x\frac{dv}{dy} = \frac{(v-1)e^v}{1 + e^v}, \; v = \frac{x}{y}$$

$$\Rightarrow \quad x\frac{dv}{dy} = \frac{(v-1)e^v}{1 + e^v} - v$$

$$= \frac{ve^v - e^v - v - ve^v}{1 + e^v}$$

$$= \frac{-e^v - v}{1 + e^v}$$

$$\Rightarrow \quad \left(\frac{1 + e^v}{v + e^v}\right)dv = -\frac{dv}{y}$$

Integrating, we get

$$\log|v + e^v| = \log c - \log|y|$$

$$\Rightarrow \quad \left(\frac{x}{y} + e^{x/y}\right) = \left(\frac{c}{y}\right)$$

$$\Rightarrow \quad (x + y e^{x/y}) = c$$

which is the required solution.

90. The given differential equation is

$$\frac{dy}{dx} = \frac{x + 2y + 3}{2x + 3y + 4}.$$

Let $\quad x = X + a$ and $y = Y + \beta$

Thus, $$\frac{dY}{dX} = \frac{(X + \alpha) + 2(Y + \beta) + 3}{2(X + \alpha) + 3(Y + \beta) + 4}$$

$$\frac{dY}{dX} = \frac{(X + 2Y) + (\alpha + 2\beta + 3)}{(2X + 3Y) + (2a + 3\beta + 4)} \qquad \text{...(i)}$$

Let us choose α and β in such a way that

$$\alpha + 2\beta + 3 = 0 \text{ and } 2\alpha + 3\beta + 4 = 0$$

Solving, we get

$$\alpha = 1, \; \beta = -2.$$

Equation (i) reduces to

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + 3Y} \qquad \text{...(ii)}$$

which is a homogeneous equation.

Put $\quad \frac{Y}{X} = V$

$$\Rightarrow \quad \frac{dY}{dX} = V + X\frac{dV}{dX}.$$

Equation (ii) reduces to

$$V + X\frac{dV}{dX} = \frac{1 + 2V}{2 + 3V}$$

$$\Rightarrow \quad \frac{2 + 3V}{1 - 3V^2}\,dV = \frac{dX}{X}$$

Integrating, we get

$$\int \frac{2 + 3V}{1 - 3V^2}\,dV = \int \frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}}\log\left|\frac{1 + \sqrt{3}V}{1 - \sqrt{3}V}\right| - \frac{1}{2}\log|(1 - 3V^2)|$$

$$= \log C + \log|X|$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}}\log\left|\frac{X+\sqrt{3}Y}{X-\sqrt{3}Y}\right| - \frac{1}{2}\log|(X^2-3Y^2)| = \log c$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}}\log\left|\frac{(x-1)+\sqrt{3}(y+2)}{(x-1)-\sqrt{3}(y+2)}\right|$$

$$-\frac{1}{2}\log|(x^2-3y^2-2x-12y-11)| = \log c$$

which is the required solution of the given differential equation.

91. The given differential equation is

$$\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)} \qquad \text{...(i)}$$

Let $X = x + 2$ and $Y = y + 2$

Equation (i) reduces to

$$\frac{dY}{dX} = \frac{(X-2+Y+2)^2}{XY} = \frac{(X+Y)^2}{XY} \qquad \text{...(ii)}$$

Let $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

Equation (ii) reduces to

$$V + X\frac{dV}{dX} = \frac{(1+V)^2}{V}$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{(1+V)^2}{V} - V$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{(1+2V)}{V}$$

$$\Rightarrow \quad \frac{VdV}{(2V+1)} = \frac{dX}{X}$$

Integrating, we get

$$\int\frac{VdV}{(2V+1)} = \int\frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}\int\left(1-\frac{1}{(2V+1)}\right) = \int\frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}(V-\log(2V+1)) = \log|X| + c$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{Y}{X} - \log\left|2\frac{Y}{X}+1\right|\right) = \log|X| + c$$

$$\Rightarrow \quad \frac{1}{2}\left(\left(\frac{y-2}{x+2}\right) - \log\left|2\left(\frac{y-2}{x+2}\right)+1\right|\right)$$

$$= \log|x+2| + c$$

which is the required solution of the given differential equation.

97. The given differential equation is

$$y\,dx - x\,dy + \ln x\,dx = 0$$

$$\Rightarrow \quad y - x\frac{dy}{dx} + \ln x = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \ln x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = \frac{\ln x}{x} \qquad \text{...(i)}$$

which is a linear differential equation.

$$\text{IF} = e^{-\int\frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$y\cdot(\text{IF}) = \int Q.(\text{IF})\,dx + c$$

$$\Rightarrow \quad y\cdot\frac{1}{x} = \int\frac{\ln x}{x^2}dx + c$$

$$\Rightarrow \quad \frac{y}{x} = -\left(\frac{\log x}{x} + \frac{1}{x}\right) + c$$

$$\Rightarrow \quad y = -(\log x + 1) + c$$

which is the required solution of the given differential equation.

98. The differential equation is

$$x^2\frac{dy}{dx} - 3xy = 4x^4 + 2x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - 3\frac{y}{x} = 4x^2 + 2 \qquad \text{...(i)}$$

which is a linear differential equation.

$$\text{IF} = e^{-3\int\frac{dx}{x}} = e^{-3\log x} = \frac{1}{x^3}$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$y\cdot(\text{IF}) = \int Q.(\text{IF})\,dx + c$$

$$\Rightarrow \quad y\cdot\frac{1}{x^3} = \int\left(\frac{4x^2+2}{x^3}\right)dx + c$$

$$\Rightarrow \quad \frac{y}{x^3} = \int\left(\frac{4}{x} + \frac{2}{x^3}\right)dx + c$$

$$= \left(4\log x - \frac{1}{x^2}\right) + c$$

which is the required solution of the given differential equation.

99. The given differential equation is

$$\frac{dy}{dx} = y\tan x - \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} + (-\tan x)y = -\sin x \qquad \text{...(i)}$$

which is a linear differential equation.

$$\text{IF} = e^{-\int \tan x dx} = e^{\log(\cos x)} = \cos x$$

Multiplying both sides of Eq. (i) by I.F and integrating, we get

$$y \cdot (\text{I.F.}) = \int Q.(\text{I.F}) dx + c$$

$$\Rightarrow \quad y(\cos x) = -\int (\sin x \cos x) dx + c$$

$$= -\frac{1}{2}\int \sin 2x dx + c$$

$$= \frac{\cos 2x}{4} + c$$

which is the required solution of the given differential equation.

100. The given differential equation is

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2} \quad \text{...(i)}$$

which is a linear differential equation.

$$\text{IF} = e^{\int \frac{dx}{x\log x}} = e^{\log(\log x)} = \log x$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$y(\text{IF}) = \int Q(\text{IF}) dx + c$$

$$\Rightarrow \quad y \cdot \log x = \int \frac{2}{x^2} \log x dx + c$$

$$= -\frac{2}{x}(1 + \log x) + c \quad \text{...(i)}$$

which is the required solution of the given differential equation.

101. The given differential equation is

$$x dx = \left(\frac{x}{y^2} - y\right) dy$$

$$\Rightarrow \quad x\frac{dx}{dy} + y = xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = y^2 \quad \text{...(i)}$$

which is a linear differential equation.

102. The given differential equation is

$$(x + 3y + 2)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dx}{dy} - x = 3y + 2 \quad \text{...(i)}$$

which is a linear differential equation.

$$\text{IF} = e^{-\int dy} = e^{-y}$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$x \cdot (\text{IF}) = \int Q.(\text{IF}) dy + c$$

$$\Rightarrow \quad x \cdot e^{-y} = \int (3y + 2) e^{-y} dy + c$$

$$= 3(-ye^{-y} + 2e^{-y}) + c$$

$$\Rightarrow \quad x = -3(y + 2) + ce^{y}$$

which is the required solution of the given differential equation.

103. The given differential equation is

$$\frac{dy}{dx} + 2y = e^{3x} \quad \text{...(i)}$$

which is a linear differential equation.

$$\text{IF} = e^{\int 2 dx} = e^{2x}$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$y \cdot (\text{IF}) = \int Q.(\text{IF}) dx + C$$

$$\Rightarrow \quad y(e^{x}) = \int (e^{2x} \cdot e^{x}) dx + c$$

$$= \int (e^{3x}) dx + c$$

$$= \left(\frac{e^{3x}}{3}\right) + c$$

$$\Rightarrow \quad y = \left(\frac{e^{3x}}{3}\right) + ce^{-x}$$

which is the required solution.

104. The given differential equation is

$$x\frac{dy}{dx} = x + y$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 1 \quad \text{...(i)}$$

which is a linear differential equation.

$$\text{Thus,} \quad \text{IF} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF}) dx + c$$

$$\Rightarrow \quad \frac{y}{x} = \int \frac{dx}{x} + c$$

$$\Rightarrow \quad \frac{y}{x} = \log|x| + c$$

which is the required solution.

105. The given differential equation is

$$x\frac{dy}{dx} + y = xe^x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = e^x$$

which is a linear differential equation.

Thus, $\text{IF} = e^{\int \frac{dx}{x}} = e^{\log x} = x$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q.(\text{IF})\, dx + c$$

$$\Rightarrow \quad y \cdot x = \int xe^x\, dx + c$$

$$\Rightarrow \quad (xy) = e^x(x - 1) + c$$

which is the required solution.

106. The given differential equation is

$$x\frac{dy}{dx} + y = x\log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \log x \quad \text{...(i)}$$

which is a linear differential equation.

Thus, $\text{IF} = e^{\int \frac{dx}{x}} = e^{\log x} = x$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int x\log x\, dx + c$$

$$\Rightarrow \quad y \cdot x = \int x\log x\, dx + c$$

$$= \int x\log x\, dx - \frac{1}{2}\int x\, dx + c$$

$$= \left(\frac{x^2}{2}\right)\log x - \frac{x^2}{4} + c$$

which is the required solution.

107. The given differential equation is

$$\frac{dy}{dx} + \frac{y}{x} = x^3 \quad \text{...(i)}$$

which is a linear differential equation.

Thus, $\text{IF} = e^{\int \frac{dx}{x}} = e^{\log x} = x$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q.(\text{I.F.})\, dx + c$$

$$\Rightarrow \quad xy = \int x^4\, dx + c$$

$$= \frac{x^5}{5} + c$$

which is the required solution.

108. The given differential equation is

$$\frac{dy}{dx} = y\tan x - 2\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} + (-\tan x)y = -2\sin x \quad \text{...(i)}$$

which is a linear differential equation

Thus, $\text{IF} = e^{-\int \tan x\, dx} = e^{-\log(\sec x)} = x$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

$$\Rightarrow \quad y \cdot \cos x = \int (-\sin x \cdot \tan \cdot x)\, dx + c$$

$$= -\int \left(\frac{1 - \cos^2 x}{\cos x}\right) dx + c$$

$$= -\int (\sec x - \cos x)\, dx + c$$

$$= -\log|\sec x + \tan x| + \sin x + c$$

which is the required solution.

109. The given differential equation is

$$\frac{dy}{dx} - \frac{y}{x} = 2x^2 \quad \text{...(i)}$$

which is a linear differential equation.

Thus, $\text{I.F.} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{I.F}) = \int Q \cdot (\text{IF})\, dx + c$$

$$\Rightarrow \quad y \cdot \frac{1}{x} = 2\int x\, dx + c$$

$$\Rightarrow \quad \frac{y}{x} = x^2 + c$$

which is the required solution.

110. The given differential equation is

$$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2} \quad \text{...(i)}$$

which is a linear differential equation.

Thus, $\text{IF} = e^{-\int \frac{dx}{x\log x}} = e^{\log(\log x)} = x$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

$$\Rightarrow \quad y \cdot \log x = 2\int \frac{\log x}{x^2}dx + c$$

$$= 2\int (te^{-t})\,dt + c,\ t = \log x$$

$$= -2(te^{-t} - e^{-t}) + c$$

$$= -2\left(\frac{\log x - 1}{x}\right) + c$$

which is the required solution.

111. The given differential equation is

$$y\,dx - (x + 2y^2)\,dy = 0$$

$$\Rightarrow \quad y\frac{dx}{dy} + (x - y^2) = 0$$

$$\Rightarrow \quad y\frac{dx}{dy} + x = y^2$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y} = y \qquad \text{...(i)}$$

which is a linear differential equation.

Thus, $\quad \text{IF} = e^{\int \frac{dy}{y}} = e^{\log y} = y$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$x \cdot (\text{I.F}) = \int y^2 dy + c$$

$$\Rightarrow \quad x \cdot y = \int y^2 dy + c$$

$$= \frac{y^3}{3} + c$$

which is the required solution.

112. The given differential equation is

$$y\,dx + (x - y^3)\,dy = 0$$

$$\Rightarrow \quad y\frac{dx}{dy} + (x - y^3) = 0$$

$$\Rightarrow \quad y\frac{dx}{dy} + x = y^3$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y} = y^2 \qquad \text{...(i)}$$

which is a linear differential equation

Thus, $\quad \text{IF} = e^{\int \frac{dy}{y}} = e^{\log y} = y$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF})\,dy + c$$

$$\Rightarrow \quad x \cdot y = \int y^3\,dy + c$$

$$= \frac{y^4}{4} + c$$

which is the required solution.

113. The given differential equation is

$$x\frac{dx}{dy} - ay = x + 1$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(-\frac{a}{x}\right)y = x + \frac{1}{x} \qquad \text{...(i)}$$

which is a linear differential equation

Thus, $\quad \text{IF} = e^{-a\int \frac{dx}{x}} = e^{-a\log x} = e^{\log(x^{-a})} = \frac{1}{x^a}$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q.(\text{IF})\,dx + c$$

$$\Rightarrow \quad \frac{y}{x^a} = \int (x^{-a} + x^{-a-1})\,dx + c$$

$$= \left(\frac{x^{-a+1}}{1} - a + \frac{x^{-a}}{-a} + c\right)$$

which is the required solution.

114. The given differential equation is

$$(x + 1)\frac{dy}{dx} - ny = e^x(x + 1)^{n+1}$$

$$\Rightarrow \quad \frac{dy}{dx} - \left(\frac{n}{x + 1}\right)y = e^x(x + 1)^n \qquad \text{...(i)}$$

which is a linear differential equation

Thus, $\quad \text{I.F.} = e^{-n\int \frac{dx}{x+1}} = e^{-n\log(x+1)}$

$$= e^{\log\left(\frac{1}{(x+1)^n}\right)} = \frac{1}{(x + 1)^n}$$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\,dx + c$$

$$\Rightarrow \quad \frac{y}{(x + 1)^n} = \int e^x dx + c$$

$$= e^x + c$$

which is the required solution.

115. The given differential equation is

$$(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{1 + y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2} \qquad \text{...(i)}$$

which is a linear differential equation

Thus, $\text{IF} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dy + c$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y}\, dy$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = \int t e^t\, dt + c, \; t = \tan^{-1} y$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = e^t (t - 1) + c$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + c$$

which is the required solution.

116. The given differential equation is

$$(x + 3y + 2)\frac{dx}{dy} = 1$$

$$\Rightarrow \quad \frac{dx}{dy} = (x + 3y + 2)$$

$$\Rightarrow \quad \frac{dx}{dy} - x = (3y + 2) \qquad \text{...(i)}$$

which is a linear differential equation

Thus, $\text{IF} = e^{-\int dy} = e^{-y}$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$x \cdot (\text{IF}) = \int Q.(\text{IF})\, dy + c$$

$$\Rightarrow \quad x \cdot e^{-y} = \int \{3(ye^{-y}) + 2(e^{-y})\}\, dy + c$$

$$= 3(-ye^{-y} - e^{-y}) - 2e^{-y} + c$$

$$= -3(y + 1)e^{-y} - 2e^{-y} + c$$

$$\Rightarrow \quad x = -3(y + 1) - 2 + ce^{y}$$

which is the required solution.

117. The given differential equation is

$$(1 + y^2)\, dx = (xy + y^3 + y)\, dy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{xy}{y^2 + 1} + y$$

$$\Rightarrow \quad \frac{dx}{dy} + \left(-\frac{y}{y^2 + 1}\right)x = y \qquad \text{...(i)}$$

which is a linear differential equation

Thus, $\text{IF} = e^{-\int \frac{y\,dy}{y^2+1}} = e^{-\frac{1}{2}\log|y^2 + 1|} = \dfrac{1}{\sqrt{y^2 + 1}}$

Multiplying both sides of Eq. (i) by IF and integrating we get

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dy + c$$

$$\Rightarrow \quad \left(\frac{x}{\sqrt{y^2 + 1}}\right) = \int \frac{y\,dy}{\sqrt{y^2 + 1}} + c$$

$$\Rightarrow \quad \left(\frac{x}{\sqrt{y^2 + 1}}\right) = \sqrt{y^2 + 1} + c$$

which is the required solution.

118. The given differential equation is

$$\frac{dy}{dx} + \frac{y}{x}\log x = \frac{y}{x^2}(\log y)^2$$

DIviding both the sides by $y(\log y)^2$, we get

$$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} + \frac{1}{\log y} \cdot \frac{1}{x} = \frac{1}{x^2}$$

Let $\dfrac{1}{\log y} = v$

$$\Rightarrow \quad \frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} = -\frac{dv}{dx}$$

$$\Rightarrow \quad -\frac{dv}{dx} + v \cdot \frac{1}{x} = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$$

which is a linear differential equation.

Thus, $\text{IF} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = \dfrac{1}{x}.$

Hence the solution is

$$v \cdot (\text{IF}) = \int Q.(\text{IF})\, dx + c$$

$$\Rightarrow \quad v \cdot \left(\frac{1}{x}\right) = \int -\frac{1}{x^2} \cdot \left(\frac{1}{x}\right) dx + c$$

$$\Rightarrow \quad \frac{1}{x\log y} - \frac{1}{2x^2} = c$$

119. The given differential equation is

$$\frac{dy}{dx} + xy = x^3 y^6$$

$$\Rightarrow \quad \frac{1}{y^6}\frac{dy}{dx} + \frac{x}{y^5} = x^3 \qquad \text{...(i)}$$

Let $\dfrac{1}{y^5} = v \Rightarrow \dfrac{1}{y^6}\dfrac{dy}{dx} = -\dfrac{1}{5}\dfrac{dv}{dx}$

$$\Rightarrow \quad -\frac{1}{5}\frac{dv}{dx} + vx = x^3$$

$$\Rightarrow \quad \frac{dv}{dx} - 5vx = 5x^3 \qquad \text{...(ii)}$$

which is a linear differential equation.

$$\text{IF} = e^{-5\int x\,dx} = e^{-\frac{5x^2}{2}}$$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + C$$

$$\Rightarrow \quad v . e^{-\frac{5x^2}{2}} = -5\int x^3 \cdot e^{-\frac{5x^2}{2}} dx + C$$

$$\Rightarrow \quad \frac{e^{-\frac{5x^2}{2}}}{y^5} = \frac{2}{5}\left(\frac{5x^2}{2} + 1\right)e^{-\frac{5x^2}{2}} + C$$

which is the required solution of the given differential equation.

120. The given differential equation is

$$\frac{dy}{dx} + \frac{\sin 2y}{x} = x^3 \cos^2 y$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} + \frac{2\tan y}{x} = x^3 \quad \text{...(i)}$$

Let $\tan y = v$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dv}{dx} + \frac{2v}{x} = x^3 \quad \text{...(ii)}$$

which is a linear differential equation

$$\text{IF} = e^{2\int \frac{dx}{x}} = e^{2\log x} = x^2$$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

$$\Rightarrow \quad v \cdot x^2 = \int x^5 dx + c$$

$$\Rightarrow \quad x^2 \tan y = \frac{x^6}{6} + c$$

which is the required solution of the given differential equation.

121. The given differential equation is

$$\frac{dy}{dx} - x^3 y^3 + xy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + xy = x^3 y^3$$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + xy^{-2} = x^3 \quad \text{...(i)}$$

Let $y^{-2} = v \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

$$\Rightarrow \quad -\frac{1}{2}\frac{dv}{dx} + vx = x^3$$

$$\Rightarrow \quad \frac{dv}{dx} - 2vx = -2x^3 \quad \text{...(ii)}$$

which is a linear differential equation.

$$\text{IF} = e^{-2\int x dx} = e^{-x^2}$$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

$$\Rightarrow \quad v \cdot e^{-x^2} = \int x^3 e^{-x^2} dx + c$$

$$\Rightarrow \quad v \cdot e^{-x^2} = -\frac{1}{2}(x^2 + 1) + c$$

$$\Rightarrow \quad \frac{e^{-x^2}}{y^2} = -\frac{1}{2}(x^2 + 1)e^{-x^2} + c$$

which is the required solution of the given differential equation.

122. The given differential equation is

$$(1 - x^2)\frac{dy}{dx} + xy = xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{xy}{1 - x^2} = \frac{xy^2}{1 - x^2}$$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{y}(1 - x^2) = \frac{x}{1 - x^2} \quad \text{...(i)}$$

Let $\frac{1}{y} = v \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{vx}{1 - x^2} = \frac{x}{1 - x^2} \quad \text{...(ii)}$$

which is a linear differential equation.

$$\text{IF} = e^{-\int \frac{x}{1-x^2 2} dx} = e^{\frac{1}{2}\log|1-x^2|} = \sqrt{1 - x^2}$$

Multiplying both sides of Eq. (ii) by IF and integrating, we get,

$$v \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + C$$

$$\Rightarrow \quad v\sqrt{1 - x^2} = -\int \frac{x\sqrt{1 - x^2}}{1 - x^2}\, dx$$

$$= -\int \frac{x}{\sqrt{1 - x^2}}\, dx$$

$$= \sqrt{1 - x^2} + C$$

$$= \sqrt{1 - x^2} + C$$

which is the required solution of the given differential equation.

123. The given differential equation is

$$x\frac{dy}{dx} + y = x^3 y^6.$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = x^2y^6$$

$$\Rightarrow \quad \frac{1}{y^6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \quad \text{...(i)}$$

Put $y^{-5} = v$

$$\Rightarrow \quad \frac{-5}{y^6}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx}$$

$$\Rightarrow \quad -\frac{1}{5}\frac{dv}{dx} + \frac{v}{x} = x^2$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{5v}{x} = -5x^2 \quad \text{...(ii)}$$

which is a linear differential equation

Thus, $\text{IF} = e^{-5\int \frac{dx}{x}} = e^{-5\log x} = \frac{1}{x^5}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

$$\Rightarrow \quad \frac{1}{(xy)^5} = -5\int \frac{dx}{x^3} + c$$

$$\Rightarrow \quad \frac{1}{(xy)^5} = \frac{5}{2x^2} + c$$

which is the required solution.

124. The given differential equation is

$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{(x^2+1)} = \frac{e^{\tan^{-1}x}}{(x^2+1)} \quad \text{...(i)}$$

which is a linear differential equation

Thus, $\text{I.F} = e^{\int \frac{dx}{x^2+1}} = e^{\tan^{-1}x}$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$y \cdot e^{\tan^{-1}x} = \int \frac{\left(e^{\tan^{-1}x}\right)^2}{(x^2+1)} dx + c$$

$$= \int e^{2t}\, dt + c,\ t = \tan^{-1}x$$

$$= \int \frac{e^{2t}}{2} + c$$

$$\Rightarrow \quad \log|x^2 + c_1| + 2a\tan^{-1}\left(\frac{x}{C_1}\right) + y = c_2$$

$$\Rightarrow \quad y \cdot e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c$$

which is the required solution of the given differential equation.

125. The given differential equation is

$$\frac{dy}{dx}(x^2y^3 + xy) = 1$$

$$\Rightarrow \quad \frac{dx}{dy} = x^2y^3 + xy$$

$$\Rightarrow \quad \frac{dx}{dy} - xy = x^2y^3$$

$$\Rightarrow \quad \frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3 \quad \text{...(i)}$$

Let $v = -\frac{1}{x}$

$$\Rightarrow \quad \frac{dv}{dy} = \frac{1}{x^2}\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dv}{dx} + vy = y^3 \quad \text{...(ii)}$$

which is a linear differential equation

Thus, $\text{IF} = e^{\int y\, dy} = e^{\frac{y^2}{2}}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot e^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + c$$

$$\Rightarrow \quad -\frac{e^{\frac{y^2}{2}}}{x} = 2\int y\left(\frac{y^2}{2}\right)e^{\frac{y^2}{2}} dy + c$$

$$\Rightarrow \quad -\frac{e^{\frac{y^2}{2}}}{x} = (y^2 - 2)e^{\frac{y^2}{2}} dy + c$$

which is the required solution.

126. The given differential equation is

$$(y\log - 1)y\,dx = x\,dy$$

$$y^2\log x\,dx - y\,dx = x\,dy$$

$$x\frac{dy}{dx} + y = y^2\log x$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2\log x}{x}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x}$$

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{\log x}{x}, \qquad \left(\text{Let } v = \frac{1}{y}\right)$$

$$\frac{dv}{dx} - \frac{v}{x} = -\frac{\log x}{x} \quad \text{...(i)}$$

which is a linear differential equation

Thus, $\quad$ I.F. $= e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

Multing both sides of Eq. (i) IF and integrating, we get

$$v \cdot \frac{1}{x} = -\int \frac{\log x}{x^2} dx$$

$$\Rightarrow \quad \frac{v}{x} = -\int te^{-t}\, dt, \qquad (\text{Let } \quad t = \log x)$$

$$= (t + 1)e^{-t} + c$$

$$\Rightarrow \quad \frac{1}{xy} = \left(\frac{\log x + 1}{x}\right) + c$$

which is the required solution of the given differential equation.

127. The given differential equation is

$$\frac{dy}{dx} = x^3y^3 = xy$$

$$\Rightarrow \quad \frac{dy}{dx} + xy = x^3y^3$$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3y^3 \qquad \text{...(i)}$$

Let $\frac{1}{y^2} = v$

$$\Rightarrow \quad -\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad -\frac{1}{2}\frac{dv}{dx} + vx = x^2$$

$$\Rightarrow \quad \frac{dv}{dx} - 2vx = -2x^3 \qquad \text{...(ii)}$$

which is a linear differential equation

Thus, $\quad$ IF $= e^{-2\int x\, dx} = e^{-x^2}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v(\text{IF}) = \int Q(\text{IF})\, dx + c$$

$$\Rightarrow \quad v \cdot e^{-x^2} = \int 2x^3\, e^{-x^2} dx + c$$

$$= (1 + x^2)e^{-x^2} + c$$

$$\Rightarrow \quad v = (1 + x^2) + ce^{x^2}$$

$$\Rightarrow \quad \frac{1}{y^2} = (1 + x^2) + ce^{x^2}$$

which is the required solution of the given differential equation.

128. The given differential equation is

$$\frac{dy}{dx} + \frac{xy}{1 - x^2} = x\sqrt{y}.$$

$$\Rightarrow \quad \frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x\sqrt{y}}{1 - x^2} = x \qquad \text{...(i)}$$

Let $\sqrt{y} = v$

$$\Rightarrow \quad \frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1}{\sqrt{y}}\frac{dy}{dx} = 2\frac{dv}{dx}$$

$$\Rightarrow \quad 2\frac{dv}{dx} + \frac{vx}{1 - x^2} = x$$

$$\Rightarrow \quad \frac{dv}{dx} + \left(\frac{x}{2(1 - x^2)}\right)v = \frac{x}{2} \qquad \text{...(ii)}$$

which is a linear differential equation.

Thus, $\quad$ IF $= e^{\int \frac{x dx}{2(1 - x^2)}} = e^{-\frac{1}{4}\log|1 - x^2|} = \frac{1}{\sqrt[4]{1 - x^2}}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot \frac{1}{\sqrt[4]{1 - x^2}} = \frac{1}{2}\int \frac{x\, dx}{\sqrt[4]{1 - x^2}} + c$$

$$\Rightarrow \quad \frac{\sqrt{y}}{\sqrt[4]{1 - x^2}} = \frac{(1 - x^2)^{3/4}}{3} + c$$

which is the required solution.

129. The given differential equation is

$$\frac{dy}{dx} + \frac{y}{x} \cdot \log y = \frac{y}{x^2}(\log y)^2$$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x(\log y)} = \frac{1}{x^2} \qquad \text{...(i)}$$

Let $\quad \frac{1}{(\log y)} = v$

$$\Rightarrow \quad -\frac{1}{y(\log y)^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} = -\frac{dv}{dx}$$

$$\Rightarrow \quad -\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2} \qquad \text{...(ii)}$$

which is a linear differential equation.

Thus, $\quad$ IF $= e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$\frac{v}{x} = -\int \frac{dx}{x^3} + c$$

$$= \frac{1}{2x^2} + c$$

$$\Rightarrow \quad \frac{1}{x(\log y)} = \frac{1}{2x^2} + c$$

which is the required solution.

130. The given differential equation is

$$\frac{dy}{dx} + \frac{1}{x}\cdot \sin^2 y = x^3 \cdot \cos^2 y$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} + \frac{2\tan y}{x} = x^3 \quad \text{...(i)}$$

Let $\tan y = v$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dv}{dx} + \frac{2v}{x} = x^3 \quad \text{...(ii)}$$

which is a linear differential equation

Thus, $\quad \text{IF} = e^{2\int \frac{dx}{x}} = e^{2\log x} = x^2$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$\Rightarrow \quad v \cdot x^2 = \int x^5\, dx + c$$

$$= \frac{x^6}{6} + c$$

$$\Rightarrow \quad (\tan y)x^2 = \frac{x^6}{6} + c$$

which is the required solution of the given differential equation.

131. The given differential equation is

$$(xy^2 - e^{1/x^3})\, dx - x^2 y dy = 0$$

$$\Rightarrow \quad x^2 y \frac{dy}{dx} = xy^2 - e^{1/x^3}$$

$$\Rightarrow \quad y\frac{dy}{dx} = \frac{y^2}{x} - \frac{e^{1/x^3}}{x^2}$$

$$-\frac{y^2}{x} = \frac{e^{1/x^3}}{x^2} \quad \text{...(i)}$$

Let $y^2 = v$

$$\Rightarrow \quad y\frac{dy}{dx} = \frac{1}{2}\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1}{2}\frac{dv}{dx} - \frac{v}{x} = -\frac{e^{1/x^3}}{x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{2v}{x} = -\frac{2e^{1/x^3}}{x^2} \quad \text{...(ii)}$$

which is a linear differential equation

Thus, $\quad \text{IF} = e^{-2\int \frac{dx}{x}} = e^{-2\log x} = \frac{1}{x^2}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$\frac{v}{x^2} = -2\int \frac{e^{1/x^3}}{x^4} dx + c$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)^2 = \frac{2}{3} e^{1/x3} + c$$

which is the required solution.

132. The given differential equation is

$$\frac{dy}{dx} + x(x + y) = x^3 (x + y)^3 - 1$$

$$\frac{1}{(x + y)^3}\left(\frac{dy + dx}{dx}\right) + \frac{x}{(x + y)^2} = x^3$$

$$\frac{1}{(x + y)^3}\left(\frac{d(x + y)}{dx}\right) + \frac{x}{(x + y)^2} = x^3 \quad \text{...(i)}$$

put $\frac{1}{(x + y)^2} = v$

$$\Rightarrow \quad -\frac{2}{(x + y)^3}\frac{d(x + y)}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1}{(x + y)^3}\frac{d(x + y)}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

$$\Rightarrow \quad -\frac{1}{2}\frac{dv}{dx} + vx = x^3$$

$$\Rightarrow \quad \frac{dv}{dx} - 2xv = -2x^3$$

which is a linear differential equation

Thus, $\quad \text{IF} = e^{-2\int x\, dx} = e^{-x^2}$

Multiplying both sides of Eq. (ii) by IF and integrating, we get

$$v \cdot e^{-x^2} = -2\int x^3 e^{-x^2} dx + c$$

$$= (x^2 + 1)e^{-x^2} + c$$

$$\Rightarrow \quad v = (x^2 + 1) + \text{c}e^{x^2}$$

$$\Rightarrow \quad \frac{1}{(x^2+1)} = (x^2+1) + ce^{x^2}$$

which is the required solution of the given differential equation.

133. The given differential equation is

$$x\,dx + y\,dy = x\,dy - y\,dx$$

$$\Rightarrow \quad \frac{1}{2}d(x^2+y^2) = x\,dy - y\,dx$$

$$\Rightarrow \quad d(x^2+y^2) = 2(x\,dy - y\,dx)$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{(x^2+y^2)} = \frac{2(x\,dy - y\,dx)}{(x^2+y^2)}$$

$$= 2\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$$

Integrating, we get

$$\log|x^2+y^2| = 2\tan^{-1}\left(\frac{y}{x}\right) + c$$

which is the required solution of the given differential equation.

134. The given differential equation is

$$x\,dy - y\,dx = x^4\,dx$$

$$\Rightarrow \quad d\left(\frac{y}{x}\right) = x^2\,dx$$

Integrating, we get

$$\Rightarrow \quad \frac{y}{x} = \frac{x^3}{3} + c$$

which is the required solution of the given differential equation.

135. The given differential equation is

$$x\,dy + y\,dx = \sin y\,dy$$

$$\Rightarrow \quad d(xy) = \sin y\,dy$$

Integrating, we get

$$\Rightarrow \quad xy = c - \cos y$$

which is the required solution of the given differential equation.

136. The given differential equation is

$$x\,dy + y\,dx + y^2(x\,dy - y\,dx) = 0.$$

$$\Rightarrow \quad d(xy) = y^2(y\,dx - x\,dy)$$

$$\Rightarrow \quad \frac{d(xy)}{x^2y^2} = \frac{y^2(y\,dx - x\,dy)}{x^2y^2}$$

$$\Rightarrow \quad d\left(-\frac{1}{xy}\right) = -d\left(\frac{x}{y}\right)$$

Integrating, we get

$$\Rightarrow \quad -\frac{1}{xy} = -\frac{y}{x} - c$$

$$\Rightarrow \quad \frac{1}{xy} = \frac{y}{x} + c$$

which is the required solution of the given differential equation.

137. The given differential equation is

$$x\,dy + y\,dx + xy^2\,dx - x^2y\,dy = 0.$$

$$\Rightarrow \quad d(xy) = xy(x\,dx - y\,dx)$$

$$\Rightarrow \quad \frac{d(xy)}{x^2y^2} = \frac{(x\,dy - y\,dx)}{xy}$$

$$\Rightarrow \quad d\left(-\frac{1}{xy}\right) = \left(\frac{dy}{y} - \frac{dx}{x}\right)$$

Integrating, we get

$$\Rightarrow \quad -\frac{1}{xy} = \log\left(\frac{y}{x}\right) + c$$

$$\Rightarrow \quad \frac{1}{xy} + \log\left(\frac{y}{x}\right) + c = 0$$

which is the required solution of the given differential equation.

138. The given differential equation is

$$(4x - 3y)\,dx + (2y - 3x)\,dy = 0$$

$$\Rightarrow \quad 4x\,dx + 2y\,dy - 3(x\,dy + y\,dx) = 0$$

$$\Rightarrow \quad 4x\,dx + 2y\,dy - 3d(xy) = 0$$

Integrating, we get

$$\Rightarrow \quad 2x^2 + y^2 - 3xy = c$$

which is the required solution of the given differential equation.

139. The given differential equation is

$$\left(\sin y + y\sin x + \frac{1}{x}\right)dx$$

$$+\left(x\cos y - \cos x + \frac{1}{y}\right)dy = 0$$

$$\Rightarrow (\sin y\,dx + x\cos y\,dy) - (\cos x\,dy - y\sin x\,dx)$$

$$+\left(\frac{dx}{x} + \frac{dy}{y}\right) = 0$$

$$\Rightarrow d(x\sin y) - d(y\cos x) + \left(\frac{dx}{x} + \frac{dy}{y}\right) = 0$$

Integrating, we get

$\Rightarrow x\sin y - y\cos x + \log(xy) = c$

which is the required solution of the given differential equation.

140. The given differential equation is

$$\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \frac{y\,dx - x\,dy}{x^2}$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{\sqrt{x^2+y^2}} = 2d\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad d\left(\sqrt{x^2+y^2}\right) = 2d\left(\frac{y}{x}\right)$$

Integrating, we get

$$\Rightarrow \quad \left(\sqrt{x^2+y^2}\right) = 2\left(\frac{y}{x}\right) + c$$

which is the required solution of the given differential equation.

141. The given differential equation is

$$\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0$$

$$\Rightarrow \quad (x\,dy + y\,dy) + \left(\frac{\sin 2x}{y}dx - \frac{\sin^2 x}{y^2}dy\right) = 0$$

$$\Rightarrow \quad d(xy) + d\left(\frac{\sin^2 x}{y}\right) = 0$$

Integrating, we get

$$\Rightarrow \quad xy + \frac{\sin^2 x}{y} = c$$

which is the required solution of the given differential equation.

142. The given differential equation is

$$\frac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}.$$

$$\Rightarrow \quad \frac{x\,dx + y\,dy}{y\,dx - x\,dy} = \frac{x^4 + 2x^2y^2 + y^4}{x^2}$$

$$= \frac{(x^2+y^2)^2}{x^2}$$

$$\Rightarrow \quad \frac{x\,dy + y\,dx}{(x^2+y^2)^2} = \frac{y\,dx - x\,dy}{x^2}.$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{(x^2+y^2)^2} = -2d\left(\frac{y}{x}\right),$$

Integrating, we get

$$-\frac{1}{(x^2+y^2)} = c - \frac{y}{x}$$

$$\Rightarrow \quad \frac{y}{x} - \frac{1}{(x^2+y^2)} = c$$

which is the required solution of the given differential equation.

143. The given differential equation is

$$x + y\frac{dy}{dx} = \frac{a^2\left(x\dfrac{dy}{dx} - y\right)}{x^2+y^2} \qquad \text{...(i)}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$,

$$\Rightarrow \quad x^2 + y^2 = r^2, \tan\theta = \frac{y}{x}$$

and $x\,dx + y\,dy = r\,dr$, $x\,dy - y\,dx = r^2\,d\theta$

$$\Rightarrow \quad r\,dr = \frac{a^2 \cdot r^2\,d\theta}{r^2}.$$

Integrating, we get

$$\int r\,dr = \int a^2 \cdot d\theta$$

$$\Rightarrow \quad \frac{r^2}{2} = a^2\theta + c$$

$$\Rightarrow \quad \frac{(x^2+y^2)}{2} = a^2\tan^{-1}\left(\frac{y}{x}\right) + c$$

$$\Rightarrow \quad (x^2+y^2) = 2a^2\tan^{-1}\left(\frac{y}{x}\right) + k$$

which is the required solution.

144. The given differential equation is

$$x\,dx + y\,dy = (x^2+y^2)\,y\,dy$$

$$\Rightarrow \quad \frac{x\,dx + y\,dy}{x^2+y^2} = y\,dy$$

$$\Rightarrow \quad \frac{1}{2}d(\log(x^2+y^2)) = y\,dy$$

Integrating, we get

$$\frac{1}{2}(\log(x^2+y^2)) = \frac{y^2}{2} + \frac{c}{2}$$

$$\Rightarrow \quad \log(x^2+y^2) = y^2 + c$$

which is the required solution.

145. The given differential equation is

$$(x + y)dx + (x - y)dy = 0$$

$$\Rightarrow \quad (xdx - ydy) + (xdy - ydx) = 0$$

$$(xdx - ydy) + d(xy) = 0$$

Integrating, we get

$$\frac{x^2}{2} - \frac{y^2}{2} + xy = c$$

which is the required solution.

146. The given differential equation is

$$ydx + x(x - 1)dy = 0$$

$$\Rightarrow \quad ydx - xdy = -x^2 dy$$

$$\Rightarrow \quad xdy - ydx = x^2 dy$$

$$\Rightarrow \quad \frac{xdy - ydx}{x^2} = dy$$

$$\Rightarrow \quad d\left(\frac{y}{x}\right) = dy$$

Integrating, we get

$$\left(\frac{y}{x}\right) = y + c$$

which is the required solution.

147. The given differential equation is

$$ydx - x(1 - xy)dy = 0$$

$$\Rightarrow \quad \frac{ydx - xdy}{x^2} = ydy$$

$$\Rightarrow \quad d\left(\frac{y}{x}\right) = ydy$$

Integrating, we get

$$\left(\frac{y}{x}\right) = \frac{y^2}{2} + c$$

which is the required solution.

148. The given differential equation is

$$(x + y)(dx - dy) = (dx + dy)$$

$$\Rightarrow \quad (x + y)\ d(x - y) = d(x + y)$$

$$\Rightarrow \quad d(x - y) = \frac{d(x + y)}{(x + y)}$$

Integrating, we get

$$(x - y) = \log|(x + y)|$$

which is the required solution.

149. The given differential equation is

$$dx + dy = xdy + ydx$$

$$\Rightarrow \quad d(x + y) = d(xy)$$

Integrating, we get

$$(x + y) = xy + c$$

which is the required solution.

150. The given differential equation is

$$xdy - ydx = (x^2 + y^2)dx$$

$$\Rightarrow \quad \frac{xdy - ydx}{(x^2 + y^2)} = dx$$

$$\Rightarrow \quad \frac{ydx - xdy}{(x^2 + y^2)} = -dx$$

$$\Rightarrow \quad \frac{\frac{ydx - xdy}{x^2}}{\left(1 + \left(\frac{y}{x}\right)^2\right)} = -dx$$

$$\Rightarrow \quad \frac{d\left(\frac{x}{y}\right)}{\left(1 + \left(\frac{y}{x}\right)^2\right)} = -dx$$

Integrating, we get

$$\tan^{-1}\left(\frac{y}{x}\right) = c - x$$

which is the required solution.

151. The given differential equation is

$$\left(\frac{1}{x} + \frac{1}{y}\right)(xdy + ydx) = dx + dy$$

$$\Rightarrow \quad \left(\frac{x + y}{xy}\right)(xdy + ydx) = d(x + y)$$

$$\Rightarrow \quad \left(\frac{x + y}{xy}\right)d(xy) = d(x + y)$$

$$\Rightarrow \quad \frac{d(xy)}{xy} = \frac{d(x + y)}{(x + y)}$$

Integrating, we get

$$\log|xy| = \log|x + y| + \log c$$

$$\Rightarrow \quad (xy) = c(x + y)$$

which is the required solution.

152. The given differential equation is

$$(xdy + ydx)\sqrt{x^2 + y^2} = x^2ydx + xy^2dy$$

$$\Rightarrow \quad \frac{xdy + ydx}{xy} = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \quad \frac{d(xy)}{xy} = \frac{1}{2}\frac{d(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$= \frac{d\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

Integrating, we get

$$\log|xy| = \sqrt{x^2+y^2} + c$$

which is the required solution.

153. The given differential equation is

$$(x - y^3)dy = y(dx + x^2dy)$$

$$\Rightarrow \quad xdy - y^3dy = ydx + x^2ydy$$

$$\Rightarrow \quad \frac{xdy - ydx}{y} = y^2dy + x^2dy$$

$$\Rightarrow \quad \frac{xdy - ydx}{(x^2+y^2)} = y\,dy$$

$$\Rightarrow \quad \frac{d\left(\frac{x}{y}\right)}{\left(1 + \left(\frac{y}{x}\right)^2\right)} = y\,dy$$

Integrating, we get

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{y^2}{2} + c$$

which is the required solution.

154. The given differential equation is

$$(x + 2y)dy + ydx = 0$$

$$\Rightarrow \quad (xdy + ydx) + 2y\,dy = 0$$

$$\Rightarrow \quad d(xy) + 2y\,dy = 0$$

Integrating, we get

$$(xy) + y^2 = c$$

which is the required solution.

155. The given differential equation is

$$(x + \sin y)dy + ydx = 0$$

$$\Rightarrow \quad (xdy + ydx) + \sin y\,dy = 0$$

$$\Rightarrow \quad d(xy) + \sin y\,dy = 0$$

Integrating, we get

$$(xy) - \cos y = c$$

which is the required solution.

156. The given differential equation is

$$e^y dx + (xe^y - 2y)dy = 0$$

$$\Rightarrow \quad (e^y dx + xe^y dy) = 2y\,dy$$

$$\Rightarrow \quad d(xe^y) = 2y\,dy$$

Integrating, we get

$$(xe^y) = \frac{y^2}{2} + c$$

which is the required solution.

157. The given differential equation is

$$2ydy + (\cos x \cdot \cot y - y^2)dx = (2xy + \sin x \cdot \text{cosec}^2 y)dy$$

$$\Rightarrow \quad 2ydy - d(xy^2) - (\sin x\,\text{cosec}^2 y\,dy - \cos x \cot y\,dx) = 0$$

$$\Rightarrow \quad 2ydy - d(xy^2) + d(\sin x \cot dy) = 0$$

Integrating, we get

$$y^2 - xy^2 + (\sin x \cot dy) = c$$

which is the required solution.

158. The given differential equation is

$$\frac{dy}{dx} = \frac{2x - 3y + 1}{3x - 2y - 2}$$

$$\Rightarrow \quad 3xdy - 2(y + 1)dy = 2xdx - 3ydx + dx$$

$$\Rightarrow \quad 3(xdy + ydx) - 2(y + 1)dy = (2x + 1)dx$$

$$\Rightarrow \quad 3d(xy) - 2(y + 1)dy = (2x + 1)dx$$

Integrating, we get

$$3(xy) - 2\left(\frac{y^2}{2} + y\right) = \left(\frac{x^2}{2} + x\right) + c$$

which is the required solution.

159. The given differential equation is

$$x(dy + dx) = y(dx - dy)$$

$$\Rightarrow \quad \frac{xdy - ydx}{(x^2+y^2)} = \frac{xdx + ydy}{(x^2+y^2)}$$

$$\Rightarrow \quad \frac{\frac{xdy - ydx}{y^2}}{\left(1 + \left(\frac{x}{y}\right)^2\right)} = \frac{1}{2}\frac{d(x^2+y^2)}{(x^2+y^2)}$$

$$\Rightarrow \quad \frac{d\left(\frac{x}{y}\right)}{1 + \left(\frac{x}{y}\right)^2} = \frac{1}{2}\frac{d(x^2+y^2)}{(x^2+y^2)}$$

Integrating, we get

$$\tan^{-1}\left(\frac{x}{y}\right) = \frac{1}{2}\log|(x^2 + y^2)| + c$$

which is the required solution.

160. The given differential equation is

$$x(\log x - \tan^{-1} x)\,dy + ydx = \left(\frac{xy}{1 + x^2}\right)dx$$

$$\Rightarrow \quad x(\log x - \tan^{-1} x)\frac{dy}{dx} + y = \frac{xy}{1 + x^2}$$

$$\Rightarrow \quad x(\log x - \tan^{-1} x)\frac{dy}{dx} = \frac{xy}{1 + x^2} - y$$

$$\Rightarrow \quad x(\log x - \tan^{-1} x)\frac{dy}{dx} = y\left(\frac{x}{1 + x^2} - 1\right)$$

$$\Rightarrow \quad (\log x - \tan^{-1} x)\frac{dy}{dx} = y\left(\frac{x}{1 + x^2} - \frac{1}{x}\right)$$

$$\Rightarrow \quad \frac{d}{dx}\{y(\log x - \tan^{-1} x)\} = 0$$

Integrating, we get

$$y(\tan^{-1} x - \log x) = C$$

which is the required solution.

161. The given differential equation is

$$x(dy - x\,dx) + ydx = 0$$

$$\Rightarrow \quad xdy + ydx = x^2dx$$

$$\Rightarrow \quad d(xy) = x^2\,dx$$

Integrating, we get

$$xy = \frac{x^2}{3} + c$$

which is the required solution.

162. The given differential equation is

$$\Rightarrow \quad \frac{y - x\dfrac{dy}{dx}}{y + x\dfrac{dy}{dx}} = \frac{1}{x^2} + \frac{1}{y^2}$$

$$\Rightarrow \quad \frac{ydx - xdy}{ydx + xdy} = \frac{x^2 + y^2}{(xy)^2}$$

$$\Rightarrow \quad \frac{ydx - xdy}{(x^2 + y^2)} = \frac{ydx + xdy}{(xy)^2}$$

$$\Rightarrow \quad \frac{\dfrac{ydx - xdy}{y^2}}{\left(1 + \left(\frac{x}{y}\right)^2\right)} = \frac{d(xy)}{(xy)^2}$$

$$\Rightarrow \quad \frac{d\left(\frac{x}{y}\right)}{1 + \left(\frac{x}{y}\right)^2} = \frac{d(xy)}{(xy)^2}$$

Integrating, we get

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{xy} = c$$

which is the required solution.

163. The given differential equation is

$$x^3 xdy - x^2y^2dx = x^4dy + x^3ydx$$

$$\Rightarrow \quad \frac{x^2y(xdy - ydx)}{x^3} = (xdy + ydx)$$

$$\Rightarrow \quad xy\left(\frac{xdy - ydx}{x^2}\right) = d(xy)$$

$$\Rightarrow \quad d\left(-\frac{y}{x}\right) = \frac{d(xy)}{(xy)}$$

Integrating, we get

$$\log|xy| - \frac{y}{x} = c$$

which is the required solution.

164. The given differential equation is

$$dx + x(ydx + xdy) = e^{-xy}dx$$

$$\Rightarrow \quad dx + x\,d(xy) = e^{-xy}dx$$

$$\Rightarrow \quad (1 - e^{-xy})dx + x\,d(xy) = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{d(xy)}{(1 - e^{-xy})} = 0$$

Integrating, we get

$$\log|x| + \log|e^{xy} - 1| = \log c$$

$$\Rightarrow \quad x(e^{xy} - 1) = c$$

which is the required solution.

165. The given differential equation is

$$y(y^2dx + (x^2dy - xydx)) = e^{-\frac{x}{y}}\,dy$$

$$\Rightarrow \quad y^3dx + xy(xdy - ydx) = e^{-\frac{x}{y}}\,dy$$

$$\Rightarrow \quad y^3dx + xy^3\left(\frac{xdy - ydx}{y^2}\right) = e^{-\frac{x}{y}}\,dy$$

$$\Rightarrow \quad y^3 dx + xy^3 d(-xy) = e^{-\frac{x}{y}} dy$$

166. The given differential equation is

$$ydx - xdy + xy^2 dx = 0$$

$$\Rightarrow \quad xdy - ydx = xy^2 dx$$

$$\Rightarrow \quad \frac{xdy - ydx}{x^2} = xdx$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) = xdx$$

Integrating, we get

$$\left(\frac{x}{y}\right) = \frac{x^2}{2} + c$$

which is the required solution.

167. The given differential equation is

$$2\left(x - y\frac{dy}{dx}\right)(x^2 + y^2) = (x^2 - y^2)\left(y - x\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{2\left(x - y\frac{dy}{dx}\right)}{(x^2 - y^2)} = \frac{\left(y - x\frac{dy}{dx}\right)}{(x^2 + y^2)}$$

$$\Rightarrow \quad \frac{2(xdx - ydy)}{(x^2 - y^2)} = \frac{(ydx - xdy)}{(x^2 + y^2)}$$

$$\Rightarrow \quad \frac{d(x^2 - y^2)}{(x^2 - y^2)} = \frac{\left(\frac{ydx - xdy}{y^2}\right)}{\left(1 + \left(\frac{x}{y}\right)^2\right)}$$

$$= \frac{d\left(\frac{x}{y}\right)}{\left(1 + \left(\frac{x}{y}\right)^2\right)}$$

Integrating, we get

$$\log|x^2 - y^2| = \tan^{-1}\left(\frac{x}{y}\right) + c$$

which is the required solution.

168. The given differential equation is

$$(x^2 + y^2 + a^2)y\ dy = (x^2 - y^2 - a^2)x\ dx$$

$$\Rightarrow x^2 ydy + y^2 xdx + (y^3 + a^2 y)dy = (x^3 - a^2 x)dx$$

$$\Rightarrow xy(xdy + ydx) + (y^3 + a^2 y)dy = (x^3 - a^2 x)dx$$

$$\Rightarrow xyd(xy) + (y^3 + a^2 y)dy = (x^3 - a^2 x)dx$$

Integrating, we get

$$\frac{(xy)^2}{2} + \frac{y^4}{2} + \frac{a^2 y^2}{2} = \frac{x^4}{4} - \frac{a^2 x^2}{2} + c$$

which is the required solution.

169. The given differential equation is

$$(1 + xy)ydx + x(1 - xy)dy = 0$$

$$\Rightarrow \quad ydx + xdy + xy(ydx - xdy) = 0$$

$$\Rightarrow \quad d(xy) + xy(ydx - xdy) = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)} + (ydx - xdy) = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)} = (xdy - ydx)$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} = \left(\frac{xdy - ydx}{xy}\right) = \frac{dy}{y} - \frac{dx}{x}$$

Integrating, we get

$$-\frac{1}{xy} = \log\left|\frac{y}{x}\right| + c$$

which is the required solution.

170. The given differential equation is

$$\frac{xdy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$$

$$\Rightarrow \quad xdy = ydx - (x^2 + y^2)dx$$

$$\Rightarrow \quad xdy - ydx = -(x^2 + y^2)dx$$

$$\Rightarrow \quad \frac{xdy - ydx}{(x^2 + y^2)} = -dx$$

$$\Rightarrow \quad \frac{\frac{xdy - ydx}{x^2}}{\left(1 + \left(\frac{y}{x}\right)^2\right)} = -dx$$

$$\Rightarrow \quad \frac{d\left(\frac{x}{y}\right)}{\left(1 + \left(\frac{y}{x}\right)^2\right)} = -dx$$

Integrating, we get

$$\tan^{-1}\left(\frac{y}{x}\right) = c - x$$

which is the required solution.

171. The given differential equation is

$$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$$

$$\Rightarrow \quad \frac{xdx + ydy}{ydx - xdy} = \frac{(x^2 + y^2)^2}{x^2}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d(x^2 + y^2)}{(x^2 + y^2)^2} = \frac{ydx - xdy}{x^2}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d(x^2 + y^2)}{(x^2 + y^2)^2} + d\left(\frac{y}{x}\right) = 0$$

Integrating, we get

$$\frac{y}{x} - \frac{1}{2(x^2 + y^2)} = c$$

which is the required solution.

172. The given differential equation is

$$xdx + ydy = m(xdy - ydx)$$

$$\Rightarrow \quad \frac{xdx + ydy}{(x^2 + y^2)} = \frac{m(xdy - ydx)}{(x^2 + y^2)}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d(x^2 + y^2)}{(x^2 + y^2)} = \frac{md\left(\frac{y}{x}\right)}{\left(1 + \left(\frac{y}{x}\right)^2\right)}$$

Integrating, we get

$$\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2(x^2 + y^2)} = c$$

which is the required solution.

173. The given differential equation is

$$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\sin^2(x^2 + y^2)}{y^3}$$

$$\Rightarrow \quad \frac{xdx + ydy}{ydx - xdy} = \frac{x\sin^2(x^2 + y^2)}{y^3}$$

$$\Rightarrow \quad \frac{xdx + ydy}{\sin^2(x^2 + y^2)} = \frac{x(ydx + xdy)}{y^3}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d(x^2 + y^2)}{\sin^2(x^2 + y^2)} + \frac{x}{y}\, d\left(\frac{y}{x}\right)$$

Integrating, we get

$$\frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{1}{2}\cot(x^2 + y^2) = c$$

which is the required solution.

174. The given differential equation is

$$x + y\frac{dy}{dx} = \frac{a^2\left(x\frac{dy}{dx} - y\right)}{x^2 + y^2}$$

$$\Rightarrow \quad xdx + ydy = \frac{a^2(xdy - ydx)}{(x^2 + y^2)}$$

$$= \frac{a^2\left(\frac{xdy - ydx}{y^2}\right)}{\left(1 + \left(\frac{x}{y}\right)^2\right)}$$

$$= \frac{a^2 d\left(\frac{x}{y}\right)}{\left(1 + \left(\frac{x}{y}\right)^2\right)}$$

$$\Rightarrow \quad \frac{1}{2}d(x^2 + y^2) = \frac{a^2 d\left(\frac{x}{y}\right)}{\left(1 + \left(\frac{x}{y}\right)^2\right)}$$

Integrating, we get

$$\Rightarrow \quad \frac{1}{2}(x^2 + y^2) = \tan^{-1}\left(\frac{x}{y}\right) + c$$

which is the required solution.

175. The given differential equation is

$$\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}} \quad \text{...(i)}$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \quad x^2 + y^2 = r^2, \ \frac{y}{x} = \tan\theta$$

$$xdx + ydy = rdr, \ \frac{xdy - ydx}{x^2} = \sec^2\theta d\theta$$

$$\Rightarrow \quad \frac{rdr}{r^2 d\theta} = \sqrt{\frac{1 - r^2}{r^2}}$$

$$\Rightarrow \quad \frac{dr}{rd\theta} = \sqrt{\frac{1 - r^2}{r^2}}$$

$$\Rightarrow \quad \frac{dr}{d\theta} = \sqrt{1 - r^2}$$

$$\Rightarrow \quad \frac{dr}{\sqrt{1 - r^2}} = d\theta$$

Integrating, we get

$$\sin^{-1}(r) = \theta + c$$

$$\Rightarrow \quad \sin^{-1}\left(\sqrt{x^2 + y^2}\right) = \tan^{-1}\left(\frac{y}{x}\right) + c$$

which is the required solution.

176. The given differential equation is

$$\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 + x^2 - y^2}{x^2 + y^2}} \quad \text{...(i)}$$

Let $x = r\sec\theta,\ y = r\tan\theta$

$\Rightarrow\ x^2 - y^2 = r^2$

$\Rightarrow\ x\,dx - y\,dy = r\,dr,\ x\,dy - y\,dx = r^2\sec\theta\,d\theta$

$$\Rightarrow\ \frac{r\,dr}{r^2\sec\theta\,d\theta} = \sqrt{\frac{1 + r^2}{r}}$$

$$\Rightarrow\ \frac{dr}{\sqrt{1 + r^2}} = \sec\theta\,d\theta$$

Integrating, we get

$$\log|r + \sqrt{r^2 + 1}| = \log|\sec\theta + \tan\theta| + \log c$$

$$\Rightarrow\ \left(r + \sqrt{r^2 + 1}\right) = c(\sec\theta + \tan\theta)$$

$$\Rightarrow\ \left(\sqrt{x^2 - y^2} + \sqrt{(x^2 - y^2) + 1}\right) = c\left(\frac{x + y}{\sqrt{x^2 - y^2}}\right)$$

which is the required solution.

177. The given differential equation is

$$\sin\left(\frac{x}{y}\right)(y\,dx - x\,dy) = xy^3(x\,dy + y\,dx)$$

$$\Rightarrow\ \sin\left(\frac{x}{y}\right)\left(\frac{y\,dx - x\,dy}{y^2}\right) = xy\,d(xy)$$

$$\Rightarrow\ \sin\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) = xy\,d(xy)$$

Integrating, we get

$$-\cos\left(\frac{x}{y}\right) = \frac{(xy)^2}{2} + c$$

which is the required solution.

178. The given differential equation is

$$x\,dy - y\,dx + y^2 = y^2(x\,dy + y\,dx) = 0$$

$$\Rightarrow\ y^2\,d(xy) = y\,dx - x\,dy$$

$$\Rightarrow\ d(xy) = \frac{y\,dx - x\,dy}{y^2}$$

$$= d\left(\frac{x}{y}\right)$$

Integrating, we get

$$\Rightarrow\ \left(xy - \frac{x}{y}\right) = c$$

which is the required solution.

179. The given differential equation is

$$(4x^3 + e^x\sin y)\,dx + e^x\cos y\,dy = 0$$

$$\Rightarrow\ (4x^3)\,dx + (e^x\sin y\,dx + e^x\cos y\,dy) = 0$$

$$\Rightarrow\ (4x^3)\,dx + d(e^x\sin y) = 0$$

Integrating, we get

$$x^4 + e^x\sin y = c$$

which is the required solution.

180. Let the equation be

$$y = mx,$$

where m is an arbitrary constant ...(i)

Differentiating w.r.t. x, we get,

$$\frac{dy}{dx} = m \quad \text{...(ii)}$$

Eliminating m, between Eqs (i) and (ii), we get

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{...(iii)}$$

whcih is the differential equation of a family of lines.

Now replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (iii), we get,

$$-\frac{dx}{dy} = \frac{x}{y}$$

$$\Rightarrow\ \frac{dy}{y} = -\frac{dx}{x}$$

Integrating, we get

$$\int\frac{dy}{y} = -\int\frac{dx}{x}$$

$$\Rightarrow\ \log|y| = \log|c| - \log|x| = \log\left|\frac{c}{x}\right|$$

$$\Rightarrow\ xy = c$$

which is required orthogonal trajectories.

181. The given family of corves are

$$x^2 + y^2 = a^2$$

Differentiaint w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ x + y\frac{dy}{dx} = 0 \quad \text{...(i)}$$

Replacing $\frac{dy}{dx}$ by $\dfrac{\frac{dy}{dx} - \tan 45°}{1 + \frac{dy}{dx}\tan 45°} =$

$\dfrac{\frac{dy}{dx} - 1}{1 + \frac{dy}{dx}}$ in Eq. (i), we get

$$x + y\left(\frac{\frac{dy}{dx} - 1}{1 + \frac{dy}{dx}}\right) = 0$$

$$x\left(1+\frac{dy}{dx}\right)+y\left(\frac{dy}{dx}-1\right)=0$$

$$\Rightarrow \quad (x+y)\frac{dy}{dx}=(y-x)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y-x}{y+x} \qquad \text{...(ii)}$$

which is a homeogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{v-1}{v+1}-v=\frac{v-1-v^2-v}{v+1}$$

$$\Rightarrow \quad \left(\frac{v+1}{v^2+1}\right)dv=-\frac{dx}{x}$$

Integrating, we get

$$\frac{1}{2}\log|v^2+1|+\tan^{-1}v=c-\log x$$

$$\Rightarrow \quad \log|x^2+y^2|+\tan^{-1}\left(\frac{y}{x}\right)=c$$

which is the required trajectories of the given family of curves.

182. The given curve is

$$x^2+y^2=1 \qquad \text{...(i)}$$

$$\Rightarrow \quad 2ax+2y\frac{dy}{dx}=0$$

$$\Rightarrow \quad ax+y\frac{dy}{dx}=0 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii), we get

$$xy\frac{dy}{dx}+y^2=1$$

$$\Rightarrow \quad xy\frac{dy}{dx}=1-y^2$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1-y^2}{xy}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$\Rightarrow \quad -\frac{dx}{dy}=\frac{1-y^2}{xy}$$

$$\Rightarrow \quad x\,dx=\left(y-\frac{1}{y}\right)dy$$

$$\Rightarrow \quad x\,dx-y\,dy+\frac{dy}{y}=0$$

Integrating, we get

$$\Rightarrow \quad \frac{x^2}{2}-\frac{y^2}{2}+\log|y|=c$$

$$\Rightarrow \quad (x^2-y^2)+2\log|y|=c$$

which is the required orthogonal trajectories.

183. Given curve is

$$x^2+y^2-ay=0$$

$$\Rightarrow \quad 2x+2y\frac{dy}{dx}-a\frac{dy}{dx}=0$$

$$\Rightarrow \quad a=\frac{2x+2y\frac{dy}{dx}}{\frac{dy}{dx}}$$

Eliminating a between (i) and (ii), we get

$$(x^2+y^2)-\left(\frac{2x+2y\frac{dy}{dx}}{\frac{dy}{dx}}\right)y=0$$

$$\Rightarrow \quad (x^2+y^2)\frac{dy}{dx}-\left(2x+2y\frac{dy}{dx}\right)y=0$$

$$\Rightarrow \quad (x^2+y^2-2y^2)\frac{dy}{dx}=2xy$$

$$\Rightarrow \quad (x^2-y^2)\frac{dy}{dx}=2xy$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2xy}{(x^2-y^2)}$$

Replace dy/dx by $-\,dx/dy$, we get

$$-\frac{dx}{dy}=\frac{2xy}{(x^2-y^2)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

$$=\frac{\left(\frac{y}{x}\right)^2-1}{2\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{v^2-1}{2v},\ v=\frac{y}{x}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{v^2-1}{2v}-v$$

$$=-\frac{v^2-1}{2v}$$

$$\Rightarrow \quad \frac{2v\,dv}{(v^2+1)}=-\frac{dx}{x}$$

Integrating, we get

$$\log|v^2+1|+\log|x|=\log c$$

$\Rightarrow \quad x(v^2 + 1) = c$

$\Rightarrow \quad x^2 + y^2 - cx = 0$

which is the required orthogonal trajectories.

184. The given curve is

$$y^2 = 4ax \quad \text{...(i)}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx} \quad \text{...(ii)}$$

Eliminating a from Eqs (i) and (ii), we get

$$y^2 = 4x\left(\frac{y}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \quad y = 2x\frac{dy}{dx}$$

Replacing dy/dx by $-dx/dy$, we get

$$y = 2x \times -\frac{dy}{dx}$$

$$\Rightarrow \quad y\,dy + 2x\,dx = 0$$

Integrating, we get

$$\frac{y^2}{2} + x^2 = c$$

which is the required orthogonal trajectory.

185. Given curve is

$$xy = c^2$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

Replacing dy/dx by $-dx/dy$, we get

$$-\frac{dx}{dy} = -\frac{y}{x}$$

$$\Rightarrow \quad x\,dx - y\,dy = 0$$

$$\Rightarrow \quad 2x\,dx - 2y\,dy = 0$$

Integrating, we get

$$x^2 - y^2 = a^2$$

which is the required orthogonal trajectories.

186. The given curve is

$$x^2 - y^2 = c^2.$$

$$\Rightarrow \quad 2x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

Replacing dy/dx by $-dx/dy$, we get

$$-\frac{dx}{dy} = \frac{x}{y}$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, we get

$$\log|x| + \log|y| = \log(c^2)$$

$$\Rightarrow \quad xy = c^2$$

which is the required orthogonal trajectories.

187. The given equation is

$$(x + y + p)(2x + p) = 0$$

$$\Rightarrow \quad (x + y + p) = 0 \quad \text{...(i)}$$

$$\text{and} \quad (2x + p) = 0 \quad \text{...(ii)}$$

From Eq. (i), we get,

$$\frac{dy}{dx} + y + x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + y = -x$$

which is a linear differential equation.

$$\Rightarrow \quad y\,.\,e^x = -\int xe^x\,dx + c_1$$

$$= -e^x(x - 1) + c_1$$

$$\Rightarrow \quad y = -(x - 1) + c_1e^{-x} \quad \text{...(iii)}$$

Also, from Eq. (ii), we get,

$$\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad dy = -2x\,dx$$

$$\Rightarrow \quad y = c_2 - x^2 \quad \text{...(iv)}$$

Hence, the required solution is

$$(y + (x - 1) - ce^{-x})(y + x^2 - c) = 0.$$

188. The given equation is

$$p^2 - p(e^x + e^{-x}) + 1 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad p^2 - p\left(e^x + \frac{1}{e^{-x}}\right) + 1 = 0$$

$$\Rightarrow \quad p^2e^x - p(e^{2x} + 1) + e^x = 0$$

$$\Rightarrow \quad p^2e^x - pe^{2x} - p + e^x = 0$$

$$\Rightarrow \quad pe^x(p - e^x) - 1(p - e^x) = 0$$

$$\Rightarrow \quad (p - e^x)(pe^x - 1) = 0$$

$$\Rightarrow \quad (p - e^x) = 0 \quad \text{...(ii)}$$

$$\text{and} \quad (pe^x - 1) = 0 \quad \text{...(iii)}$$

From Eq. (i), we get,

$$\frac{dy}{dx} = e^x$$

$$\Rightarrow \quad dy = e^x\,dx$$

Integrating, we get

$$\Rightarrow \quad y = e^x + c_1 \quad \text{...(iv)}$$

From Eq. (iii), we get,

$$\frac{dy}{dx}\cdot e^x - 1 = 0$$

$\Rightarrow \quad e^x dy = dx$

$\Rightarrow \quad dy = e^{-x}dx$

Integrating, we get

$\Rightarrow \quad y = c_2 - e^{-x}$

Hence, the general solution of Eq. (i) is

$$(x - e^{-x} - c)(y - e^{-x} - c) = 0.$$

189. The given equation is

$$p^2 + 2py\cot x = y^2 \qquad \text{...(i)}$$

$\Rightarrow \quad (p + y\cot x)^2 = y^2(1 + \cot^2 x)$

$\Rightarrow \quad (p + y\cot x)^2 = y^2\text{cosec}^2 x$

$\Rightarrow \quad (p + y\cot x) = \pm\, y\,\text{cosec}\, x$

$\Rightarrow \quad p = y(\text{cosec}\, x - \cot x) \qquad \text{...(ii)}$

and $\quad p = -y(\text{cosec}\, x - \cot x) \qquad \text{...(iii)}$

From Eq. (ii), we get,

$$\frac{dy}{dx} = y(\text{cosec}\, x - \cot x)$$

$\Rightarrow \quad \frac{dy}{y} = (\text{cosec}\, x - \cot x)\,dx$

$\Rightarrow \quad \frac{dy}{y} = \left(\frac{1 - \cos x}{\sin x}\right)dx$

$\Rightarrow \quad \frac{dy}{y} = \tan\left(\frac{x}{2}\right)dx$

Integrating, we get

$$\log|y| = 2\log\left|\sec\left(\frac{x}{2}\right)\right| + 2\log(c_1)$$

$\Rightarrow \quad y = c_1^2\left(\sec^2\left(\frac{x}{2}\right)\right)$

From Eq. (iii), we get

$$\frac{dy}{dx} = -y(\text{cosec}\, x + \cot x)$$

$\Rightarrow \quad \frac{dy}{y} = -(\text{cosec}\, x + \cot x)\,dx$

Integrating, we get

$\Rightarrow \quad \frac{dy}{y} = -\cot\left(\frac{x}{2}\right)dx$

$\Rightarrow \quad \log y = 2\log c_2 - 2\log\left(\sin\left(\frac{x}{2}\right)\right)$

$\Rightarrow \quad y\left(\sin^2\left(\frac{x}{2}\right)\right) = c_2$

Hence, the required solution is

$$\left(y - c\left(\sec^2\left(\frac{x}{2}\right)\right)\right)\left(y\sin^2\left(\frac{x}{2}\right) - c\right) = 0.$$

190. The given differential equation is

$$p^2 + px - xy - y^2 = 0$$

$\Rightarrow \quad (p^2 - y^2) + x(p - y) = 0$

$\Rightarrow \quad (p - y)(p + x + y) = 0$

$\Rightarrow \quad (p - y) = 0,\ (p + x + y) = 0$

When $\quad (p - y) = 0$

$\Rightarrow \quad \frac{dy}{dx} = y$

$\Rightarrow \quad \frac{dy}{y} = dx$

Integrating, we get

$$\log|y| = x + c_1$$

When $\quad (p + x + y) = 0,$

$\Rightarrow \quad \frac{dy}{dx} + (x + y) = 0$

$\Rightarrow \quad \frac{dy}{dx} + y = -x$

which is a linear differential equation

Thus, $\quad \text{IF} = e^{\int dx} = e^x$

Thus, the solution is

$$y\,.\,e^x = -\int x e^x\, dx + c$$

$\Rightarrow \quad y\,.\,e^x = -(x - 1)e^x + c_2$

Hence, the required solution is

$$(\log|y| - x\ c_1)(y\,.\,e^x + (x - 1)e^x - c_2) = 0$$

191. The given differential equation is

$$p^2y + (x - y)p - x = 0$$

$\Rightarrow \quad p^2y + px - py - x = 0$

$\Rightarrow \quad py(p - 1) + x(p - 1) = 0$

$\Rightarrow \quad (py + x)(p - 1) = 0$

$\Rightarrow \quad (py + x) = 0,\ (p - 1) = 0$

When $\quad (py + x) = 0$

$\Rightarrow \quad \frac{y\,dy}{dx} + x = 0$

$\Rightarrow \quad y\,dy + x\,dx = 0$

Integrating, we get

$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} = \frac{c^2}{2}$

$\Rightarrow \quad x^2 + y^2 = c^2$

When $\quad (p - 1) = 0$

$$p = 1$$

$\Rightarrow \quad \frac{dy}{dx} = 1$

$\Rightarrow \quad dy = dx$

Integrating, we get

$$y = x + c_1$$

Hence, the solution is

$$(x^2 + y^2 - c^2)(y - x - c_1) = 0$$

192. The given differential equation is

$$(p^2 - 1)xy = (x^2 - y^2)p$$

$\Rightarrow\quad p^2xy - xy - px^2 + py^2 = 0$

$\Rightarrow\quad p^2xy - px^2 + py^2 - xy = 0$

$\Rightarrow\quad px(py - x) + y(py - x) = 0$

$\Rightarrow\quad (px + y)(py - x) = 0$

$\Rightarrow\quad (px + y) = 0, (py - x) = 0$

When $(px + y) = 0$

$\Rightarrow\quad \left(\frac{dy}{dx}x + y\right) = 0$

$\Rightarrow\quad \frac{dx}{x} + \frac{dy}{y} = 0$

Integrating, we get

$$\log x + \log y = \log c$$

$\Rightarrow\quad xy = c$

When $(py - x) = 0$

$\Rightarrow\quad \left(y\frac{dy}{dx} - x\right) = 0$

$\Rightarrow\quad xdx - ydy = 0$

Integrating, we get

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c_1^2}{2}$$

$\Rightarrow\quad x^2 - y^2 = c_1^2$

Hence, the required solution is

$$(xy - c)(x^2 - y^2 - c_1^2) = 0$$

193. The given differential equation is

$$xyp^2 - (x^2 + y^2)p + xy = 0$$

$\Rightarrow\quad xyp^2 - x^2p - y^2p + xy = 0$

$\Rightarrow\quad px(py - x) - y(py - x) = 0$

$\Rightarrow\quad (px - y)(py - x) = 0$

$\Rightarrow\quad (px - y) = 0, (py - x) = 0$

When $(px - y) = 0$

$\Rightarrow\quad \left(\frac{dy}{dx}x - y\right) = 0$

$\Rightarrow\quad \frac{dx}{x} - \frac{dy}{y} = 0$

Integrating, we get

$$\log x - \log y = \log c$$

$\Rightarrow\quad x = cy$

When $(py - x) = 0$

$\Rightarrow\quad \left(y\frac{dy}{dx} - x\right) = 0$

$\Rightarrow\quad xdx - ydy = 0$

Integrating, we get

$$x^2 - y^2 = a^2$$

Hence, the required solution is

$$(xy - c)(x^2 - y^2 - a^2) = 0$$

194. The given equation can be written as

$$x = yp + a \quad \text{...(i)}$$

Differentiaing w.r.t. y, we get

$$\frac{dx}{dy} = p + y\frac{dp}{dy}$$

$\Rightarrow\quad \frac{1}{p} = p + y\frac{dp}{dy}$

$\Rightarrow\quad \left(\frac{1}{p} - p\right) = y\frac{dp}{dy}$

$\Rightarrow\quad \left(\frac{p}{1 - p^2}\right)dp = \frac{dy}{y}$

$\Rightarrow\quad \left(\frac{-2p}{1 - p^2}\right)dp = \frac{-2dy}{y}$

Integrating, we get

$$\log|1 - p^2| = \log c - 2\log|y|$$

$\Rightarrow\quad \log|1 - p^2| = \log\left|\frac{c}{y^2}\right|$

$\Rightarrow\quad (1 - p^2) = \frac{c}{y^2}$

$\Rightarrow\quad p^2 = \left(1 - \frac{c}{y^2}\right) \quad \text{...(ii)}$

Eliminating p between Eqs (i) and (ii), we get

$$x = a \pm y\sqrt{1 - \frac{c}{y^2}}$$

which is the required solution of Eq. (i)

195. The given equation can be written as

$$x = \frac{1}{2}\left(\frac{y}{p} - y^2p^2\right) \quad \text{...(i)}$$

Differentiaing w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2py^2\frac{dp}{dy}\right)$$

$\Rightarrow\quad \frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2py^2\frac{dp}{dy}\right)$

$\Rightarrow\quad 2p = \left(p - y\frac{dp}{dy} - 2yp^4 - 2p^3y^2\frac{dp}{dy}\right)$

$\Rightarrow\quad p + 2yp^4 = -\left(y\frac{dp}{dy} - 2p^3y^2\frac{dp}{dy}\right)$

$\Rightarrow\quad p(1 + 2yp^3) = -y\frac{dp}{dy}(1 + 2p^3y)$

$\Rightarrow\quad (1 + 2yp^3)\left(p + y\frac{dp}{dy}\right) = 0$

$$\Rightarrow \quad \left(p + y\frac{dp}{dy}\right) = 0$$

$$\Rightarrow \quad d(py) = 0$$

Integrating, we get

$$py = c$$

$$\Rightarrow \quad p = \frac{y}{c} \qquad \text{...(ii)}$$

Eliminating p between Eq. (i) and (ii), we get

$$y = 2\frac{c}{y}.x + \frac{c^3}{y^3}.y^2$$

$$\Rightarrow \quad y^2 = 2cx + c^3$$

which is the required solution.

196. The given differential equation is

$$y^2 \log y = xyp + p^2 \qquad \text{...(i)}$$

$$\Rightarrow \quad x = \frac{y^2 \log y - p^2}{yp}$$

Differentiaing w.r.t. y, we get

$$\frac{dy}{dx} = \frac{yp\left(2y\log y + y - 2p\frac{dp}{dy}\right) - (y^2\log y - p^2)\left(p + y\frac{dp}{dy}\right)}{(yp)^2}$$

$$\Rightarrow \quad y^2p = yp\left(2y\log y + y - 2p\frac{dp}{dy}\right) - (y^2\log y - p^2)\left(p + y\frac{dp}{dy}\right)$$

$$\Rightarrow \quad y\frac{dp}{dy}(y^2\log y - p^2) = p(y^2\log y - p^2)$$

$$\Rightarrow \quad y\frac{dp}{dy} = p$$

$$\Rightarrow \quad \frac{dy}{p} = \frac{dp}{p}$$

Integrating, we get

$$\Rightarrow \quad \log p = \log y + \log c$$

$$\Rightarrow \quad p = cy \qquad \text{...(ii)}$$

Eliminating p between Eqs (i) and (ii), we get

$$\log y = cx + c^2$$

which is the required solution of (i)

197. The given differential equation is

$$p^2y + 2px = y \qquad \text{...(i)}$$

$$\Rightarrow \quad 2px = y - p^2y$$

$$\Rightarrow \quad x = \frac{y - p^2y}{2p}$$

Differentiating, w.r.t. y, we get

$$\frac{dx}{dy} = \frac{2p\left(1 - p^2 - 2py\frac{dp}{dy}\right) - 2(y - p^2y)\frac{dp}{dy}}{4p^2}$$

$$\Rightarrow \quad \frac{1}{p} = \frac{p\left(1 - p^2 - 2py\frac{dp}{dy}\right) - (y - p^2y)\frac{dp}{dy}}{2p^2}$$

$$\Rightarrow \quad 2p = p - p^3 - 2p^2y\frac{dp}{dy} - y\frac{dp}{dy} + p^2y\frac{dp}{dy}$$

$$\Rightarrow \quad p = -p^3 - p^2y\frac{dp}{dy} - y\frac{dp}{dy}$$

$$\Rightarrow \quad p(p^2 + 1) = -(p^2 + 1)y\frac{dp}{dy}$$

$$\Rightarrow \quad p = -y\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{dp}{p} + \frac{dy}{p} = 0$$

$$\Rightarrow \quad py = c \qquad \text{...(ii)}$$

Eliminating p between Eqs (i) and (ii), we get

$$y^2 + 2cx = c^2$$

which is the required solution.

198. $c(4x - c)^2 = 64y$

199. The given differential equation is

$$yp = 2p^2x + y^2p^4 \qquad \text{...(i)}$$

$$\Rightarrow \quad x = \frac{y - y^2p^3}{2p}$$

Differentiating, w.r.t. y, we get

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2}\frac{p\left(1 - 2yp^3 - 3y^2p^2\frac{dp}{dy}\right) - (y - y^2p^3)\frac{dp}{dy}}{p^2}$$

$$\Rightarrow p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0$$

$$\Rightarrow p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0$$

$$\Rightarrow \left(p + y\frac{dp}{dy}\right)(1 + 2yp^3) = 0$$

$$\Rightarrow \left(p + y\frac{dp}{dy}\right) = 0$$

$$\Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$$

$\Rightarrow \quad py = c$ ...(ii)

Eliminating p between Eqs (i) and (ii), we get

$$y^2 = 2cx + c^3$$

which is the required solution.

200. The given differential equation is

$$xp^2 - yp - p + 1 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad x = \frac{yp + p - 1}{p^2}$$

Differentiating, w.r.t. y, we get

$$\frac{dx}{dy} = \frac{p^2\left(p + y\frac{dp}{dy} - \frac{dp}{dy}\right) - 2(yp + p - 1)p\frac{dp}{dy}}{p^4}$$

$$\Rightarrow \quad p^3 = p^3 + p^2y\frac{dp}{dy} - p^2\frac{dp}{dy} - 2p^2y\frac{dp}{dy} - 2p^2\frac{dp}{dy} + 2p\frac{dp}{dy}$$

$$\Rightarrow \quad p^2y\frac{dp}{dy} - p^2\frac{dp}{dy} - 2p^2y\frac{dp}{dy} - 2p^2\frac{dp}{dy} + 2p\frac{dp}{dy} = 0$$

$$\Rightarrow \quad \frac{dp}{dy} = 0$$

$$\Rightarrow \quad dp = 0$$

$\Rightarrow \quad p = c$ ...(ii)

Eliminating p between Eqs (i) and (ii) we get

$$yc = c^2x - c + 1$$

which is the required solution.

201. The given equation is $y = (1 + p)\,x + ap^2$ ...(i)

Differentiaing w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = (1 + p)\cdot 1 + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow \quad p = (1 + p)\cdot 1 + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow \quad (x + 2ap)\frac{dp}{dx} = -1$$

$$\Rightarrow \quad \frac{dx}{dp} = -(x + 2ap)$$

$\Rightarrow \quad \dfrac{dx}{dp} + x = -2ap$ ...(ii)

which is a linear differential equation.

$$\text{IF} = e^{\int dp} = e^p$$

Hence, the required solution of Eq. (ii) is

$$x.e^p = -2a\int pe^p\,dp = -2ae^p(p - 1) + c$$

$\Rightarrow \quad x = -2a(p - 1) + ce^{-p}$ ...(iii)

Eliminating p between Eqs (i) and (iii), we get the required solution of Eq. (i).

202. The given equation is

$$y = yp^2 + 2px$$

$$\Rightarrow \quad y = \frac{2px}{1 - p^2}$$

Differentiaing w.r.t. x, we get

$$\frac{2\,dp}{p(p^2 - 1)} = \frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{p - 1} + \frac{1}{p + 1} - \frac{2}{p}\right)dp = \frac{dx}{x}$$

Integrating, we get

$$\log|p - 1| + \log|p + 1| - \log|p^2| = \log|x| + \log c$$

$$\Rightarrow \quad \log\left|\frac{p^2 - 1}{p^2}\right| = \log|cx|$$

$$\Rightarrow \quad \frac{p^2 - 1}{p^2} = cx$$

$$\Rightarrow \quad p^2 - 1 = p^2cx$$

$$\Rightarrow \quad p^2(1 - cx) = 1$$

$$\Rightarrow \quad p^2 = \left(\frac{1}{(1 - cx)}\right)$$

$\Rightarrow \quad p = \sqrt{\dfrac{1}{(1 - cx)}}$ ...(ii)

Eliminating p between Eqs (i) and (ii), we get

$$2x\sqrt{1 - cx} + cxy = 0.$$

which is the required solution.

203. The given differential equation is

$y = 2px + \tan^{-1}(xp^2)$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + \frac{p^2 + 2px\frac{dp}{dx}}{1 + (xp^2)^2}$$

$$\Rightarrow \quad \left(-p - 2x\frac{dp}{dx}\right)(1 + p^4x^2) = p^2 + 2px\frac{dp}{dx}$$

$$\Rightarrow \quad -p - 2x\frac{dp}{dx} - p^5x^2 - 2x^3p^4\frac{dp}{dx} = p^2 + 2px\frac{dp}{dx}$$

$$\Rightarrow -p - p^5x^2 - p^2 = + 2x\frac{dp}{dx} + 2x\frac{dp}{dx} + 2x^3p^4\frac{dp}{dx}$$

$$\Rightarrow -p\ (1 + p^4x^2 - p) = + 2x(p + 1 + x^2p^4)\frac{dp}{dx}$$

$$\Rightarrow -p = 2x\frac{dp}{dx}$$

$$\Rightarrow \frac{2dp}{p} + \frac{dx}{x} = 0$$

$$\Rightarrow 2\log p + \log x = \log c$$

$$\Rightarrow p^2x = c \quad \text{...(ii)}$$

Eliminating p between Eqs (i) and (iii), we get the required solution of Eq. (i).

$$y = 2\left(\sqrt{\frac{c}{x}}\right)x + \tan^{-1}(c)$$

$$\Rightarrow \quad y = 2\sqrt{cx} + \tan^{-1}(c)$$

which is the required solution.

204. The given differential equation is

$$x^2p^4 + 2xp = y$$

$$\Rightarrow \quad y = p^4x^2 + 2px \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2xp^4 + 4x^2p^3\frac{dp}{dx} + 2p + 2x\frac{dp}{dx}$$

$$\Rightarrow \quad p = 2xp^4 + 4x^2p^3\frac{dp}{dx} + 2p + 2x\frac{dp}{dx}$$

$$\Rightarrow \quad -2xp^4 - p = (4x^2p^3 + 2x)\frac{dp}{dx}$$

$$\Rightarrow \quad 2x(2xp^3 + 1)\frac{dp}{dx} = -p(2xp^3 + 1)$$

$$\Rightarrow \quad 2x\frac{dp}{dx} = -p$$

$$\Rightarrow \quad 2\frac{dp}{p} = -\frac{dx}{x}$$

Integrating, we get

$$2\log|p| = \log c - \log|x|$$

$$\Rightarrow \quad 2\log|p| = \log\left|\frac{c}{x}\right|$$

$$p^2 = \frac{x}{c} \quad \text{...(ii)}$$

Eliminating p between Eqs (i) and (ii), we get

$$y = 2\sqrt{cx} + c^2$$

which is the required solution.

205. The given differential equation is

$$y + px = x^4p^2 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} + p + x\frac{dp}{dx} = 2px^4\frac{dp}{dx} + 4x^3p^2$$

$$\Rightarrow \quad p + p + x\frac{dp}{dx} = 2px^4\frac{dp}{dx} + 4x^3p^2$$

$$\Rightarrow \quad (x - 2px^4)\frac{dp}{dx} = 4x^3p^2 - 2p$$

$$\Rightarrow \quad x(1 - 2px^3)\frac{dp}{dx} = 2p(2x^3p - 1)$$

$$\Rightarrow \quad x(1 - 2px^3)\frac{dp}{dx} = -2p(1 - 2x^3p)$$

$$\Rightarrow \quad x\frac{dp}{dx} = -2p$$

$$\Rightarrow \quad \frac{dp}{p} + \frac{2dx}{x} = 0$$

Integrating, we get

$$\log p + 2\log x = \log c$$

$$\Rightarrow \quad px^2 = c \quad \text{...(ii)}$$

Eliminating p between Eqs (i) and (iii), we get the required solution of Eq. (i)

$$y + \frac{c}{x} + c^2$$

which is the required solution.

206. The given differential equation is

$$y = p\sin p + \cos p \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sin p\frac{dp}{dx} + p\cos p\frac{dp}{dx} - \sin p\frac{dp}{dx}$$

$$\Rightarrow \quad p = p\cos p\frac{dp}{dx}$$

$$\Rightarrow \quad \cos p\,\frac{dp}{dx} = 1$$

$$\Rightarrow \quad \cos p\ dp = dx$$

Integrating, we get

$$\sin p = x + c \quad \text{...(ii)}$$

Eliminating p from Eqs (i) and (ii), we get

$$y = (x + c)\sin^{-1}(x + c) + \sqrt{1 - (x + c)^2}$$

which is the required solution.

207. The given differential equation can be written as

$$y = Px - \frac{2}{P} - 1 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$\frac{dP}{dx} = P + x\frac{dP}{dx} + \frac{2}{P^2}\frac{dP}{dx}$$

$\Rightarrow \quad P = P + x\frac{dP}{dx} + \frac{2}{p^2}\frac{dP}{dx}$

$\Rightarrow \quad \left(x + \frac{2}{P^2}\right)\frac{dP}{dx} = 0$

$\Rightarrow \quad \frac{dP}{dx} = 0$ or $\left(x + \frac{2}{P^2}\right) = 0$

$\Rightarrow \quad P = c$ ...(ii)

or $\quad x = -\frac{2}{p^2}$ ...(iii)

Eliminating p between Eqs (i) and (ii), we get

$$y = cx - \frac{2}{c} - 1$$

which is the genral solution of Eq. (i).

Eliminating p between Eqs (i) and (iii), we get

$$(y + 1)^2 + 8x = 0$$

which is the singular solution of Eq. (i).

208. The given differential equation is

$$P^3x - P^2y - 1 = 0$$

$\Rightarrow \quad P^2y = P^3x - 1$

$\Rightarrow \quad y = Px - \frac{1}{P^2}$

which is a Clairaut differential equation.
Hence, the solution is

$$y = cx - \frac{1}{c^2}$$

209. The given differential equation is

$$(y + 1)P - xP^2 + 2 = 0$$

$\Rightarrow \quad (y + 1)P = p^2x - 2$

$\Rightarrow \quad (y + 1) = Px - \frac{2}{p}$

$\Rightarrow \quad y = Px - \left(\frac{2}{P} + 1\right)$

which is a Clairaut differential equation.
Hence, the solution is

$$y = cx - \left(\frac{2}{c} + 1\right)$$

210. The given differential equation is

$$\sin y \cos Px - \cos y \sin Px - P = 0$$

$\Rightarrow \quad \sin(y - Px) = P$

$\Rightarrow \quad (y - Px) = \sin^{-1}(P)$

$\Rightarrow \quad y = Px + \sin^{-1}(P)$

which is a Clairaut differential equation.
Hence, the solution is

$$y = cx + \sin^{-1}(c)$$

211. The given differential equation is

$$(x - a)P^2 + (x - y)P - y = 0$$

$\Rightarrow \quad P^2x + Px - aP^2 = (P + 1)y$

$\Rightarrow \quad (P + 1)y = P(P + 1)x - aP^2$

$\Rightarrow \quad y = Px - \frac{aP^2}{(P + 1)}$

which is a Clairaut differential equation.
Hence, the solution is

$$y = cx - \frac{ac^2}{(c + 1)}$$

212. The given differential equation is

$$y = Px + a\sqrt{1 + P^2}$$

which is a Clairaut differential equation.
Hence, the solution is

$$y = cx + a\sqrt{1 + c^2}$$

213. The given differential equation is

$$y = p(x - b) + \frac{a}{p}$$

$\Rightarrow \quad y = px + \left(\frac{a}{p} - bp\right)$

which is a Clairaut differential equation.
Hence, the solution is

$$y = cx + \left(\frac{a}{c} - bc\right)$$

214. The given differential equation is

$$\frac{d^2y}{dx^2} = x + \sin x$$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = x + \sin x$

Integrating, we get

$$\frac{dy}{dx} = \frac{x^2}{2} - \cos x + c_1$$

Again integrating, we get

$\Rightarrow \quad y = \frac{x^3}{6} - \sin x + c_1x + c_2$

which is the required solution.

215. The given differential equation is

$$\frac{d^2y}{dx^2} = e^{2x} + e^x + 2014$$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = e^{2x} + e^x + 2014$

Integrating, we get

$$\frac{dy}{dx} = \frac{e^{2x}}{2} + e^x + 2014 + c_1$$

Again integrating, we get

$$y = \frac{e^{2x}}{4} + e^x + 1007x^2 + c_1x + c_2$$

which is the required solution.

216. The given differential equation is

$$\frac{d^2y}{dx^2} = \sin^2 x$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \sin^2 x$$

Integrating, we get

$$\left(\frac{dy}{dx}\right) = \frac{1}{2}\int 2\sin 2x\, dx + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \frac{1}{2}\left(x - \frac{2\sin 2x}{2}\right) + c_1$$

Again Integrating, we get

$$y = \frac{1}{2}\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right) + c_1x + c_2$$

which is the required solution.

217. The given differential equation is

$$\frac{d^2y}{dx^2} = \cos^3 x$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \cos^3 x$$

Integrating, we get

$$\left(\frac{dy}{dx}\right) = \int \cos^3 x\, dx$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \int \cos^3 x\, dx$$

$$= \left(x - \frac{\sin^3 x}{3}\right) + c_1$$

Again Integrating, we get

$$y = \int\left(x - \frac{\sin^3 x}{3}\right)dx + c_1x + c_2$$

$$\Rightarrow \quad y = \frac{x^2}{2} + \frac{1}{3}\left(x + \frac{\cos^3 x}{3}\right) + c_1x + c_2$$

which is the required solution.

218. The given differential equation is

$$\frac{d^2y}{dx^2} = \frac{1}{\sin^2 x\cos^2 x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \sec^2 x + \operatorname{cosec}^2 x$$

Integrating, we get

$$\left(\frac{dy}{dx}\right) = \tan x - \cot x + c_1$$

Again Integrating, we get

$$\Rightarrow \quad y = \log(\cos x) - \log(\sin x) + c_1x + c_2$$

$$\Rightarrow \quad y = \log(\cot x) + c_1x + c_2$$

which is the required solution.

219. The given differential equation is

$$\frac{d^2y}{dx^2} = \sin^4 x + \cos^4 x$$

$$= 1 - 2\sin^2 x\cos^2 x$$

$$= 1 - \frac{1}{2}\sin^2 x$$

$$= 1 - \frac{1}{4}(1 - \cos 4x)$$

$$= \frac{3}{4} + \frac{1}{4}\cos 4x$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{3}{4} + \frac{1}{4}\cos 4x$$

Integrating, we get

$$\left(\frac{dy}{dx}\right) = \frac{3x}{4} + \frac{1}{16}\sin 4x + c_1$$

Again Integrating, we get

$$y = \frac{3x^2}{8} + \frac{\cos(4x)}{64} + c_1x + c_2$$

which is the required solution.

220. The given differential equation is

$$\frac{d^2y}{dx^2} = xe^x$$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{dy}{dx}\right) = xe^x$$

Integrating, we get

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = (x - 1)e^x + c_1$$

Again integrating, we get

$$\Rightarrow \quad y = (x - 2)e^x + c_1x + c_2$$

which is the required solution.

221. The given differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

$$\Rightarrow \quad p\frac{dp}{dy} + y = 0$$

$$\Rightarrow \quad p\frac{dp}{dy} = -y$$

$$\Rightarrow \quad 2p\,dp = -2y\,dy$$

On integrating, we get

$$\Rightarrow \quad p^2 = c_1^2 - y^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = c_1^2 - y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{(c_1^2 - y^2)}$$

$$\Rightarrow \quad \frac{dy}{\sqrt{(c_1^2 - y^2)}} = \pm\, dx$$

Integrating, we get

$$\sin^{-1}\left(\frac{c_1}{y}\right) = c^2 \pm x$$

which is the required solution.

222. The given differential equation is

$$\frac{d^2y}{dx^2} = \frac{1}{y^3}$$

$$\Rightarrow \quad p\frac{dp}{dy} = \frac{1}{y}, \qquad \left(\text{Let } p = \frac{dy}{dx}\right)$$

$$\Rightarrow \quad p\,dp = -\frac{dy}{y^3}$$

Integrating, we get

$$\frac{p^2}{2} = \frac{1}{2y^2} + \frac{c_1}{2}$$

$$\Rightarrow \quad p^2 = \frac{c_1y^2 + 1}{y^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{c_1y^2 + 1}{y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\frac{c_1y^2 + 1}{y^2}}$$

$$\Rightarrow \quad \frac{y}{\sqrt{c_1y^2}} + 1dy = \pm\, dx$$

Again integrating, we get

$$\frac{\sqrt{c_1y^2 + 1}}{c_1} = c_2 \pm x$$

which is the required solution.

223. The given differential equation is

$$\frac{d^2y}{dx^2} = \frac{1}{4\sqrt{y}}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{4\sqrt{y}}$$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{dy}{dx} = \frac{1}{4\sqrt{y}}$$

$$\Rightarrow \quad p\frac{dp}{dy} = \frac{1}{4\sqrt{y}}$$

$$\Rightarrow \quad 4p\,dp = \frac{dy}{\sqrt{y}}$$

Integrating, we get

$$2p^2 = 2\sqrt{y} + 2c_1$$

$$\Rightarrow \quad p^2 = \sqrt{y} + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \sqrt{y} + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = +\sqrt{\sqrt{y} + c_1}$$

$$\Rightarrow \quad \frac{dy}{\sqrt{\sqrt{y} + c_1}} = \pm\, dx$$

Again integrating, we get

$$c_2 \pm x = \int \frac{dy}{\sqrt{\sqrt{y} + c_1}}$$

$$= \int \frac{2t\,dt}{\sqrt{t + c_1}}, \text{ where } y = t^2$$

$$= \int \frac{2\{(t + c_1) - c_1\}dt}{\sqrt{t + c_1}}$$

$$= 2\int (\sqrt{t + c_1} - c_1(t + c_1)^{-1/2})\,dt$$

$$= 4\left[\frac{1}{3}(t + c_1)^{3/2} - c_1\sqrt{t + c_1}\right] + c_2$$

$$= 4\left[\frac{1}{3}(\sqrt{y} + c_1)^{3/2} - c_1\sqrt{\sqrt{y} + c_1}\right] + c_2$$

which is the required solution.

224. The given differential equation is

$$a^2\frac{d^2y}{dx^2} - y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{a^2}$$

$$\Rightarrow \quad p\frac{dp}{dy} = \frac{y}{a^2}$$

$$\Rightarrow \quad p\,dp = \frac{y\,dy}{a^2}$$

Integrating, we get

$$p^2a^2 = y^2 + c_1^2$$

$$\Rightarrow \quad p^2 = \frac{y^2 + c_1^2}{a^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{y^2 + c_1^2}{a^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \pm\sqrt{\frac{y^2 + c_1^2}{a^2}}$$

$$\Rightarrow \quad \frac{dy}{\sqrt{y^2 + c_1^2}} = \pm\frac{dx}{a^2}$$

Again integrating, we get

$$\log\left|y + \sqrt{y^2 + c_1^2}\right| = c_2 \pm \frac{x}{a^2}$$

which is the required solution.

225. The given differential equation is

$$\frac{d^2y}{dx^2} = e^{2y}$$

$$\Rightarrow \quad p\frac{dp}{dy} = e^{2y}$$

$$\Rightarrow \quad p\,dp = e^{2y}\,dy$$

Integrating, we get

$$\frac{p^2}{2} = \frac{e^{2y}}{2} + \frac{c_1}{2}$$

$$\Rightarrow \quad p^2 = e^{2y} c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = e^{2y} + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \pm\sqrt{e^{2y} + c_1}$$

$$\Rightarrow \quad \frac{dy}{\sqrt{e^{2y} + c_1}} = \pm dx$$

Again integrating, we get

$$c_2 \pm x = \int\frac{dy}{\sqrt{e^{2y} + c_1}}$$

$$= \int\frac{e^{-y}dy}{\sqrt{1 + c_1 e^{-2y}}}$$

$$= -\int\frac{dt}{\sqrt{1 + c_1 t^2}}, \; t = e^{-y}$$

$$= -\frac{1}{\sqrt{c_1}}\int\frac{dt}{\sqrt{t^2 + \left(\frac{1}{\sqrt{c_1}}\right)^2}}$$

$$= -\frac{1}{\sqrt{c_1}}\log\left|t + \sqrt{t^2 + \frac{1}{\sqrt{c_1}}}\right|$$

$$= -\frac{1}{\sqrt{c_1}}\log\left|e^{-y} + \sqrt{e^{-2y} + \frac{1}{\sqrt{c_1}}}\right|$$

226. The given differential equation is

$$2\frac{d^2y}{dx^2} = 3y^2$$

$$\Rightarrow \quad 2p\frac{dp}{dy} = 3y^2$$

$$\Rightarrow \quad 2p\,dp = 3y^2\,dy$$

Integrating, we get

$$p^2 = y^3 + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = y^3 + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \pm\sqrt{y^3 + c_1}$$

When $x = -2$, $y = -1$ and $\left(\frac{dy}{dx}\right)_{x=-2} = -1$

$\therefore \quad c_1 = 2$

Thus, $\left(\frac{dy}{dx}\right) = \pm\sqrt{y^3 + 2}$

$$\Rightarrow \quad \frac{dy}{\sqrt{y^3 + 2}} = \pm dx$$

227. The given differential equation is

$$y^3\left(\frac{d^2y}{dx^2}\right) = -1$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{y^3}$$

$$\Rightarrow \quad 2p\frac{dp}{dy} = -\frac{1}{y^3}$$

$$\Rightarrow \quad 2p\,dp = -\frac{dy}{y^3}$$

Integrating, we get

$$p^2 = -\frac{1}{2y^2} + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = -\frac{1}{2y^2} + c_1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{2c_1 y^2 - 1}{2y^2}$$

When $x = -1$, $y = -1$, $\left(\frac{dy}{dx}\right)_{x=-1} = 0$

$\therefore \quad c_1 = \frac{1}{2}$

$$\therefore \quad \left(\frac{dy}{dx}\right) = \pm\sqrt{\frac{y^2 - 1}{2y^2}}$$

$$\Rightarrow \quad \frac{y\,dy}{\sqrt{y^2 - 1}} = \pm\frac{dx}{2}$$

$$\Rightarrow \quad \sqrt{y^2 - 1} = c_2 \pm \frac{x}{2}$$

which is the required solution.

228. The given differential equation is

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0 \quad \text{...(i)}$$

Let $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$$\Rightarrow \quad x\frac{dp}{dx} + p + x = 0$$

$$\Rightarrow \quad x\,dp + p\,dx + x\,dx = 0$$

$$\Rightarrow \quad d(xp) + xdx = 0$$

Integrating, we get

$$xp = -\frac{x^2}{2} + c_1$$

$$\Rightarrow \quad x\frac{dy}{dx} = -\frac{x^2}{2} + c_1$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{2} + \frac{c_1}{x}$$

$$\Rightarrow \quad dy = \left(-\frac{x}{2} + \frac{c_1}{x}\right)dx$$

Again Integrating, we get

$$y = -\frac{x^2}{4} + c_1 \log|x| + c_2$$

which is the required solution.

229. The given differential equation is

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{dy}{dx} + x$$

$$\Rightarrow \quad \frac{dp}{dx} = \frac{p}{x} + x$$

$$\Rightarrow \quad x\,dp = p\,dx + x^2dx$$

$$\Rightarrow \quad \frac{x\,dp - p\,dx}{x^2} = dx$$

$$\Rightarrow \quad d\left(\frac{p}{x}\right) = dx$$

Integrating, we get

$$\frac{p}{x} = \frac{x^2}{2} + c_1$$

$$\Rightarrow \quad p = \frac{x^3}{2} + c_1x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3}{2} + c_1x$$

$$\Rightarrow \quad dy = \left(\frac{x^3}{2} + c_1x\right)dx$$

Again integrating, we get

$$y = \left(\frac{x^4}{6} + \frac{c_1x^2}{2}\right) + c_2$$

which is the required solution.

230. The given differential equation is

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$$

$$\Rightarrow \quad yp\frac{dp}{dy} + p^2 = 1$$

$$\Rightarrow \quad yp\frac{dp}{dy} = 1 - p^2$$

$$\Rightarrow \quad \frac{p}{1-p^2}dp = \frac{dy}{y}$$

Integrating, we get

$$-\frac{1}{2}\log|1 - p^2| = \log y + \log c_1$$

$$\Rightarrow \quad \frac{1}{1-p^2} = (c_1y)^2$$

$$\Rightarrow \quad 1 - p^2 = \frac{1}{c_1^2y^2}$$

$$\Rightarrow \quad p^2 - 1 = \frac{1}{c_1^2y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{1 - \frac{1}{c_1^2y^2}}$$

$$\Rightarrow \quad \frac{c_1y}{\sqrt{c_1^2y^2 - 1}}dy = dx$$

Again integrating, we get

$$\frac{\sqrt{c_1^2y^2 - 1}}{c_1^2} = x + c_2$$

which is the required solution.

231. The given differential equation is

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad py\frac{dp}{dy} - p^2 = y^2p$$

$$\Rightarrow \quad y\frac{dp}{dy} - p = y^2$$

$$\Rightarrow \quad y\,dp - p\,dy = y^2dy$$

$$\Rightarrow \quad \frac{ydp - pdy}{y^2} = dy$$

$$\Rightarrow \quad d\left(\frac{p}{y}\right) = dy$$

Integrating, we get

$$\frac{p}{y} = y + c_1$$

$$\Rightarrow \quad p = y^2 + c_1 y$$

$$\Rightarrow \quad \frac{dy}{dx} = y^2 + c_1 y$$

$$\Rightarrow \quad \frac{dy}{y^2 + c_1 y} = dx$$

$$\Rightarrow \quad \frac{dy}{y(y + c_1)} = dx$$

$$\Rightarrow \quad \left(\frac{1}{y} - \frac{1}{y + c_1}\right)dy = dx$$

Again integrating, we get

$$\log\left|\frac{y}{y + c_1}\right| = x + c_2$$

which is the required solution.

232. Given differential equation is

$$(x + a)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (x + a)\frac{dp}{dx} + xp^2 = p$$

$$\Rightarrow \quad \frac{dp}{dx} + \frac{xp^2}{(x + a)} = \frac{p}{x + a}$$

$$\Rightarrow \quad \frac{dp}{dx} - \frac{p}{x + a} = -\frac{xp^2}{(x + a)}$$

$$\Rightarrow \quad \frac{1}{p^2}\frac{dp}{dx} - \frac{1}{p/(x + a)} = -\frac{x}{(x + a)}$$

Put $-\frac{1}{p} = v$

$$\Rightarrow \quad \frac{dv}{dx} + \frac{v}{(x + a)} = \frac{x}{(x + a)}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{dx}{x + a}} = e^{\log|x + a|} = (x + a)$$

$$\Rightarrow \quad v(x + a) = \int x\,dx$$

$$\Rightarrow \quad v(x + a) = \frac{x^2}{2 + c}$$

$$\Rightarrow \quad -\frac{1}{p}(x + a) = \frac{x^2}{2} + c$$

$$\Rightarrow \quad \frac{2(x + a)}{x^2 + C_1}\,dx + dy = 0$$

$$\Rightarrow \quad \frac{2x}{x^2 + C_1}\,dx + \frac{2a}{x^2 + C_1}\,dx + dy = 0$$

Integrating, we get

$$\log|x^2 + c_1| + 2a\tan^{-1}\left(\frac{x}{c_1}\right) + y = c_2$$

which is the reqired solution.

233. Given differential equation is

$$x\frac{d^2y}{dx^2} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x\frac{dp}{dx} - \frac{1}{4}p^2 = p$$

$$\Rightarrow \quad x\frac{dp}{dx} = \frac{p^2}{4} + p = p\left(\frac{p}{4} + 1\right)$$

$$\Rightarrow \quad \frac{4\,dp}{p(p + 4)} = \frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{p} - \frac{1}{p + 4}\right)dp = \frac{dx}{x}$$

Integrating, we get

$$\log\left|\frac{p}{p + 4}\right| = \log|x| + \log c_1$$

$$\Rightarrow \quad \frac{p}{p + 4} = c_1 x$$

$$\Rightarrow \quad \frac{p + 4}{p} = \frac{1}{c_1 x}$$

$$\Rightarrow \quad 1 + \frac{4}{p} = \frac{1}{c_1 x}$$

$$\Rightarrow \quad \frac{4}{p} = \frac{1}{c_1 x - 1}$$

$$\Rightarrow \quad \frac{4}{p} = \frac{1 - c_1 x}{c_1 x}$$

$$\Rightarrow \quad \frac{p}{4} = \frac{c_1 x}{1 - c_1 x}$$

$$\Rightarrow \quad dy = \frac{4c_1 x}{1 - c_1 x}\,dx$$

Again integrating, we get

$$y = -4x + \frac{1}{c_1}\log|c_1 x - 1| + c_2$$

which is the required solution.

234. Given differential equation is

$$y\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad py\frac{dp}{dx} = 1 - p^2$$

$$\Rightarrow \quad \frac{2p\,dp}{(p^2 - 1)} = -2\frac{dy}{y}$$

Integrating, we get

$$\log|p^2 - 1| = \log C_1 - \log|y^2|$$

$$\Rightarrow \quad \log|p^2 - 1| = \log\left|\frac{C_1}{y^2}\right|$$

$$\Rightarrow \quad (p^2 - 1) = \frac{C_1}{y^2}$$

$$\Rightarrow \quad p^2 = \frac{C_1}{y^2} + 1$$

$$= \frac{C_1 + y^2}{y^2}$$

$$\Rightarrow \quad p = \pm\sqrt{\frac{C_1 + y^2}{y^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\frac{C_1 + y^2}{y^2}}$$

$$\Rightarrow \quad \frac{y\,dy}{\sqrt{y^2 + C_1}} = \pm\,dx$$

$$\Rightarrow \quad \sqrt{y^2 + C_1} = C_2 \pm x$$

which is the required solution.

235. Given differential equation is

$$\frac{d^2y}{dx^2} = 2y\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dp}{dx} = 2yp$$

$$\Rightarrow \quad p\frac{dp}{dy} = 2yp$$

$$\Rightarrow \quad \frac{dp}{dy} = 2y$$

$$\Rightarrow \quad dp = 2y\,dy$$

Integrating, we get

$$p = y^2 + C_1$$

$$\Rightarrow \quad \frac{dy}{dx} = y^2 + C_1$$

$$\Rightarrow \quad \frac{dy}{(y^2 + C_1)} = dx$$

$$\Rightarrow \quad \frac{1}{\sqrt{C_1}}\tan^{-1}\left(\frac{y}{\sqrt{C_1}}\right) = x + C_2$$

which is the required solution.

236. Given differential equation is

$$\Rightarrow \quad y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad yp\frac{dp}{dy} - p^2 = y^2p$$

$$\Rightarrow \quad y\frac{dp}{dy} - p = y^2$$

$$\Rightarrow \quad \frac{dp}{dy} - \frac{p}{y} = y \qquad \text{...(i)}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int\frac{dy}{y}} = e^{-\log y} = \frac{1}{y}$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$\frac{p}{y} = \int dy + c_1$$

$$\frac{p}{y} = y + c_1$$

$$p = y^2 + c_1 y$$

$$\frac{dy}{dx} = y^2 + c_1 y$$

$$\frac{dy}{y(y + c_1)} = dx$$

Integrating, we get

$$\frac{1}{c_1}\log\left|\frac{y}{y + c_1}\right| = x + c_2$$

which is the required solution.

237. Given differential equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\left(\frac{dy}{dx}\right) - \frac{a}{x^2} = 0$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) = a$$

$$\Rightarrow \quad x^2\frac{dp}{dx} + xp = a$$

$$\Rightarrow \quad x\,dp + p\,dx = \frac{a\,dx}{x}$$

$$\Rightarrow \quad d(xp) = \frac{a\,dx}{x}$$

Integrating, we get

$$xp = a\log|x| + c_1$$

$$\Rightarrow \quad x\frac{dy}{dx} = a\log|x| + c_1$$

$$\Rightarrow \quad dy = \left(\frac{a\log|x|}{x} + \frac{c_1}{x}\right)dx$$

Again integrating, we get

$$y = a\frac{(\log|x|)^2}{2} + C_1\log|x| + C_2$$

which is the required solution.

238. Given differential equation is

$$y\frac{d^2y}{dx^2} - y\left(\frac{dy}{dx}\right)\ln y = \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y\frac{dp}{dx} - yp\ln y = p^2$$

$$\Rightarrow \quad \frac{1}{p^2}\frac{dp}{dx} + \frac{\ln y}{-p} = \frac{1}{y}$$

$$\Rightarrow \quad yp\frac{dp}{dy} - yp\ln y = p^2$$

$$\Rightarrow \quad y\frac{dp}{dy} - y\ln y = p$$

$$\Rightarrow \quad \frac{dp}{dy} - \ln y = \frac{p}{y}$$

$$\Rightarrow \quad \frac{dp}{dy} - \frac{p}{y} = \ln y \qquad \text{...(i)}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int \frac{dy}{y}} = e^{-\log y} = \frac{1}{y}$$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$\frac{p}{y} = \int \frac{\log y}{y} dy + c_1$$

$$\Rightarrow \quad \frac{p}{y} = \frac{(\log y)^2}{2} + c_1$$

$$\Rightarrow \quad \frac{dy}{y\left(\frac{(\log y)^2 + 2c_1}{2}\right)} = dx$$

$$\Rightarrow \quad \frac{2\,dt}{(t^2 + 2c_1)} = dx, \qquad (\text{Let } \log y = t)$$

Integrating, we get

$$\Rightarrow \quad \frac{2}{\sqrt{2c_1}} \tan^{-1}\left(\frac{t}{\sqrt{2c_1}}\right) = x + c_2$$

$$\Rightarrow \quad \sqrt{\frac{2}{c_1}} \tan^{-1}\left(\frac{\log y}{\sqrt{2c_1}}\right) = x + c_2$$

which is the required solution.

239. The slope of the curve

= The slope of the tangent

$$= \tan\psi = \frac{dy}{dx} = \frac{1}{2y}$$

$$\Rightarrow \quad 2y\,dy = dx$$

$$\Rightarrow \quad y^2 = x + c$$

which is passing through (4, 3), so $c = 5$

Hence the equation of the curve is

$$y^2 = x + 5.$$

240. Given

$$\frac{dy}{dx} = y + 2x$$

$$\Rightarrow \quad \frac{dy}{dx} + (-y) = 2x$$

which is a linear differential equation.

$$\text{IF} = e^{-\int dx} = e^{-x}$$

Hence, the solution is

$$y \cdot e^{-x} = 2\int xe^{-x}\,dx + c$$

$$\Rightarrow \quad y \cdot e^{-x} = -2e^{-x}(x + 1) + c$$

which is passing through (0, 0) then $c = 0$.

Hence, the required equation of the curve is

$$y \cdot e^{-x} = -2e^{-x}(x + 1)$$

241. Given

$$\frac{dy}{dx} = \frac{x^4 + 2xy - 1}{x^2 + 1}$$

$$= x^2 - 1 + \frac{2xy}{x^2 + 1}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2x}{x^2 + 1} y = x^2 - 1$$

which is a linear differential equation.

$$\text{IF} = e^{-\int \frac{2x}{x^2+1} dx} = \text{e}^{-\log|x^2 + 1|} = \frac{1}{x^2 + 1}$$

Hence, the solution is

$$y \cdot \frac{1}{x^2 + 1} = x - 2\tan^{-1}x + c$$

which passes through (0, 0), then $c = 0$.

Hence, the equation of the curve is

$$y = (x^2 + 1)(x - 2\tan^{-1}x)$$

242. Given

$$\frac{dy}{dx} = \frac{(x + 1)^2 + y - 3}{(x + 1)} \qquad \text{...(i)}$$

Let $x + 1 = X$, $y - 3 = Y$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dY}{dX}$$

$$\Rightarrow \quad \frac{dY}{dX} = X + \frac{Y}{X}$$

$$\Rightarrow \quad \frac{dY}{dX} + \left(-\frac{1}{X}\right)Y = X$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int \frac{1}{X} dX} = e^{-\log X} = \frac{1}{X}$$

Hence, the solution is

$$Y\cdot\frac{1}{X} = \int\left(X\cdot\frac{1}{X}\right)dX + c = X + c$$

$\Rightarrow \quad \frac{Y}{X} = X + c$

$\Rightarrow \quad \frac{y-3}{x+1} = (x+1) + c$

which is passing through (2, 0) then $c = -4$

Hence, the required equation of the curve is

$$\frac{y-3}{x+1} = (x+1) - 4$$

$\Rightarrow \quad y - 3 = (x+1)^2 - 4(x+1)$

$\Rightarrow \quad y = x^2 - 2x$

243. The equation of the tangent at any point (x, y) on the curve is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets x-axis at $A\left(x - y\frac{dy}{dx}, 0\right)$

and y-axis at $B\left(0, y - x\frac{dy}{dx}\right)$

$\therefore$ Mid-point of AB

$$= \left(\frac{1}{2}\left(x - y\frac{dy}{dx}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right)$$

It is given that,

and $\quad \frac{1}{2}\left(x - y\frac{dy}{dx}\right) = x,$

$$\frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$$

Thus, $\quad x\frac{dy}{dx} = -y$

$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x}$

$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0$

$\Rightarrow \quad \log y + \log x = \log c$

$\Rightarrow \quad \log(yx) \log c$

$\Rightarrow \quad xy = c$

which is passing through (1, 1), then $c = 1$.

Hence, the equation of the curve is

$$xy = 1.$$

244. The equation of the normal at the point (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x) \qquad ...(i)$$

The distance of perpendicular from the origin to the normal (i) is

$$= \frac{\left|y + x\frac{dy}{dx}\right|}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

Also the distance between P and x-axis is $|y|$.

Thus, $\quad \frac{\left|y + x\frac{dy}{dx}\right|}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = |y|$

$\Rightarrow \quad y^2 + \left(\frac{dx}{dy}\right)^2 x^2 + 2xy\frac{dy}{dx} = y^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$

$\Rightarrow \quad (x^2 - y^2)\left(\frac{dx}{dy}\right)^2 + 2xy\frac{dx}{dy} = 0$

$\Rightarrow \quad \frac{dx}{dy}\left((x^2 - y^2)\left(\frac{dx}{dy}\right) + 2xy\right) = 0$

$\Rightarrow \quad \frac{dx}{dy} = 0, \left(\frac{dx}{dy}\right) = \frac{y^2 - x^2}{2xy}$

Now, $\quad \frac{dx}{dy} = 0$ gives $x = k$

which is passing through (1, 1), so $k = 1$.

Thus, the equation of the curve is $x = 1$

Also, $\quad \frac{dx}{dy} = 0 = \frac{y^2 - x^2}{2xy}$

which is a homogeneous differential equation.

Let $\quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = -\frac{v^2 + 1}{2v}$

$\Rightarrow \quad \frac{2v}{v^2} + 1dv = -\frac{dx}{x}$

$\Rightarrow \quad \log|v^2 + 1| = \log c - \log x$

$\Rightarrow \quad |v^2 + 1| = \frac{c}{x}$

$\Rightarrow \quad x^2 + y^2 = cx$

which also passes through (1, 1), so $c = 2$.

Hence, the equation of the curve is $x^2 + y^2 = 2x$.

245. The Equation of the tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets the x-axis at $\left(x - y\frac{dy}{dx}, 0\right)$

Given $\quad AP = 1$

$\Rightarrow \quad AP^2 = 1$

$$\Rightarrow \quad \left(-y\frac{dx}{dy}\right)^2 + y^2 = 1$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right)^2 = \frac{1-y^2}{y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$$

$$\Rightarrow \quad \frac{\sqrt{1-y^2}}{y}dy = \pm dx$$

Integrating, we get

$$\log\left|\frac{y}{1+\sqrt{1-y^2}}\right| + \sqrt{1-y^2} = c \pm x$$

which is the required equation of the curve.

246. The slope of the line segment joining the points (x, y) and $(-4, -3)$ is

$$\frac{y+3}{x+4}$$

Thus, $\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$

$$\Rightarrow \quad \frac{dy}{y+3} = \frac{2\cdot dx}{x+4}$$

Integrating, we get

$$\log|y+3| = \log c + 2\log(x+4)$$

$$\Rightarrow \quad (y+3) = c(x+4)^2$$

which is passing through $(-2, 1)$, so $c = 1$

Hence, the required equation of the curve is

$$(y+3) = (x+4)^2$$

247. The equation of the normal at P is

$$Y - y = \frac{dx}{dy}(X - x)$$

Thus, the points A and B are $(x, 0)$ and

$$\left(x + y\frac{dy}{dx}, 0\right)$$

Given condition is

$$x + y\frac{dy}{dx} - x = \frac{x+y}{2}$$

$$\frac{dy}{dx} = \frac{x+y}{2y}$$

which is a homogeneous equation.

$$\text{Put} \quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1+v}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v}{2v} - v = \frac{1+v-2v^2}{2v}$$

$$\Rightarrow \frac{2v}{1+v-2v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{2v}{2v^2-v-1}dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{(4v-1)+1}{2v^2-v-1}dv = -\frac{2dx}{x}$$

$$\Rightarrow \frac{(4v-1)dv}{(2v^2-v-1)} + \frac{dv}{(2v^2-v-1)} = -\frac{2dx}{x}$$

$$\Rightarrow \log|2v^2-v-1| + \frac{1}{3}\log\left|\frac{2v-2}{2v-1}\right| = c - \log x$$

$$\Rightarrow \log\left|\frac{2y^2}{x^2} - \frac{y}{x} - 1\right| + \frac{1}{3}\log\left|\frac{2y-2x}{2y-x}\right| = c - \log x$$

$$\Rightarrow \log|2y^2 - xy - x^2| + \frac{1}{3}\log\left|\frac{2y-2x}{2y-x}\right| - \log x = c$$

which is the required equation of the curve.

248. The equation of the normal at P is

$$Y - y = -\frac{dx}{dy}(X - x)$$

It meets the x-axis at $A\left(x + y\frac{dy}{dx}, 0\right)$

Given condition is

$$OP^2 = OA^2$$

$$\Rightarrow \quad x^2 + y^2 = y^2\left(\frac{dy}{dx}\right)^2 + y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\frac{x}{y}$$

Taking positive sign,

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow \quad y\,dy = x\,dx$$

$$\Rightarrow \quad \frac{x^2}{2} - \frac{y^2}{2} = \frac{c^2}{2}$$

$$\Rightarrow \quad x^2 - y^2 = c^2,$$

which represents a rectangular hyperbola.

Taking negative sign,

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} = \frac{a^2}{2}$$

$$\Rightarrow \quad x^2 + y^2 = a^2.$$

which represents a circle.

Level III

(Problems for JEE-Advanced)

1. It is given that

$$\frac{dy}{dx} = \frac{x^4 + 2xy - 1}{1 + x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2x}{1 + x^2} y = \frac{x^4 - 1}{1 + x^2}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int \frac{2x}{1 + x^2} dx} = e^{-\log(x^2 + 1)} = \frac{1}{(x^2 + 1)}$$

Hence, the sloution is

$$y \cdot \frac{1}{(x^2 + 1)} = \int \frac{x^4 - 1}{(x^2 + 1)^2} dx + c$$

$$\Rightarrow \quad \frac{y}{(x^2 + 1)} = \int \frac{x^2 - 1}{(x^2 + 1)} dx + c$$

$$= \int \left(1 - \frac{2}{x^2 + 1}\right) dx + c$$

$$= (x - 2\tan^{-1} x) + c$$

which is the required solution.

2. The given differential equation is

$$y \cos\left(\frac{y}{x}\right)(x\,dy - y\,dx) + x \sin\left(\frac{y}{x}\right)(x\,dy + y\,dx) = 0$$

$$\Rightarrow \frac{y}{x} \cos\left(\frac{y}{x}\right)\left(\frac{dy}{dx} - \frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)\left(\frac{dy}{dx} + \frac{y}{x}\right) = 0$$

$$\Rightarrow v \cos v \left(v + x\frac{dv}{dx} - v\right) + \sin v \left(v + x\frac{dv}{dx} + v\right) = 0$$

$$\left(\text{Let} \quad \frac{y}{x} = v\right)$$

$$\Rightarrow xv \cos v \frac{dv}{dx} + \sin v \left(2v + x\frac{dv}{dx}\right) = 0$$

$$\Rightarrow x(v \cos v + \sin v)\frac{dv}{dx} = -2v \sin v$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{2v \sin v}{(v \cos v + \sin v)}$$

$$\Rightarrow \frac{(v \cos v + \sin v)}{v \sin v} dv = -\frac{2dx}{x}$$

$$\Rightarrow \left(\cot v + \frac{1}{v}\right) dv = -\frac{2dx}{x}$$

Integrating, we get

$$\log|\sin v| + \log|\sin|v|| = \log c - 2\log x$$

$$\Rightarrow \quad v \sin v = \frac{c}{x^2}$$

$$\Rightarrow \quad \left(\frac{y}{x}\right) \sin\left(\frac{y}{x}\right) = \frac{c}{x^2}$$

$$\Rightarrow \quad (xy) \sin\left(\frac{y}{x}\right) = c$$

when $x = 1,\ y = \frac{\pi}{2}$ then $c = \frac{\pi}{2}$

Hence, the solution is

$$(xy) \sin\left(\frac{y}{x}\right) = \frac{\pi}{2}$$

3. The given differential equation is

$$\cos^2 x \frac{dy}{dx} - (\tan 2x) y = \cos^4 x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{(\tan 2x)}{\cos^2 x} y = \cos^2 x$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int \frac{\tan(2x)}{\cos^2 x} dx} = \frac{\cos 2x}{1 + \cos 2x}$$

Hence, the solution is

$$y \cdot \left(\frac{\cos 2x}{1 + \cos 2x}\right) = \int \left(\frac{\cos 2x}{1 + \cos 2x}\right) \cos^2 x\, dx + c$$

$$\Rightarrow \quad y \cdot \left(\frac{\cos 2x}{1 + \cos 2x}\right) = \frac{1}{2} \int \cos 2x\, dx + c$$

$$= \frac{\sin 2x}{4 + c}$$

when $x = \frac{\pi}{6},\ y = \frac{3\sqrt{3}}{8}$ then $c = 0$

Hence, the required curve is

$$y = \frac{1}{2} \cdot \tan(2x) \cdot \cos 2x$$

4. The given differential equation is

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x)$$

$$\Rightarrow \quad x\frac{dy}{dx} + 2y = x \sin x + x \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2y}{x} = \sin x + \log x$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{2dx}{x}} = x^2$$

Hence, the solution is

$$y \cdot x^2 = \int (x^2 \sin x + x^2 \log x)\, dx + c$$

$$\Rightarrow \quad y = -\cos x + \frac{2\sin x}{x} + 2\frac{\cos x}{x^2} + \frac{x}{3}\log x - \frac{x}{9} + \frac{c}{x^2}.$$

5. The given differential equation is

$$x(1 - x^2)\, dy + (2x^2 y - y - 5x^3)\, dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{(2x^2 - 1)y}{x(1 - x^2)} = \frac{5x^3}{x(1 - x^2)}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{(2x^2 - 1)y}{x(1 - x^2)} = \frac{5x^2}{(1 - x^2)}$$

which is a linear differential equation

$$\therefore \quad \text{IF} = e^{\int \frac{(2x^2 - 1)}{x(1 - x^2)} dx} = \frac{1}{x\sqrt{1 - x^2}}$$

Hence, the solution is

$$y \cdot \frac{1}{x\sqrt{1 - x^2}} = \int \frac{5x}{(1 - x^2)^{3/2}} dx + c$$

$$\Rightarrow \quad \frac{y}{x\sqrt{1 - x^2}} = \frac{5}{\sqrt{1 - x^2}} + c$$

6. The equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It cuts the line $y = x$, so the point of intersection

$$= \left(\frac{y - x\frac{dy}{dx}}{1 - \frac{dy}{dx}}, \frac{y - x\frac{dy}{dx}}{1 - \frac{dy}{dx}} \right)$$

It is given that

$$\frac{y - x\frac{dy}{dx}}{1 - \frac{dy}{dx}} = 1$$

$$\Rightarrow \quad y - x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad (1 - x)\frac{dy}{dx} = (1 - y)$$

$$\Rightarrow \quad \frac{dy}{(y - 1)} = \frac{dx}{(x - 1)}$$

Integrating, we get

$$\log|x - 1| = \log|y - 1| + \log c$$

$$\Rightarrow \quad \log\left|\frac{x - 1}{y - 1}\right| = \log c$$

$$\Rightarrow \quad \left(\frac{x - 1}{y - 1}\right) = c$$

which is the required equation of the curve.

7. The given differential equation is

$$\Rightarrow \quad x\frac{dy}{dx} = -y = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} - v = \sqrt{1 + v^2}, \qquad \left(\text{Let} \quad v = \frac{y}{x}\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v + \sqrt{1 + v^2}\right| = \log c + \log|x|$$

$$\Rightarrow \quad \left(v + \sqrt{1 + v^2}\right) = cx$$

$$\Rightarrow \quad \left(y + \sqrt{y^2 + x^2}\right) = cx^2$$

which is the required solution.

8. The given differential equation is

$$\left(xe^{y/x} - y\sin\left(\frac{y}{x}\right)\right)dx + x\sin\left(\frac{y}{x}\right)dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{y}{x}\sin\left(\frac{y}{x}\right) - e^{\frac{y}{x}}}{\sin\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v\sin - e^v}{\sin v}, \qquad \left(\text{Let} \quad v = \left(\frac{y}{x}\right)\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin - e^v}{\sin v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin - e^v - v\sin v}{\sin v}$$

$$\Rightarrow \quad e^{-v}\sin v\, dv = -dv$$

Integrating, we get

$$\frac{e^{-v}\sin v}{(1 + 1)}(-\sin v - \cos v) = -x - c$$

$$\Rightarrow \quad \frac{e^{-v}\sin v}{(1 + 1)}(\sin v + \cos v) = x + c$$

$$\Rightarrow \quad \frac{e^{-\frac{y}{x}}\sin\left(\frac{y}{x}\right)}{2}\left(\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right) = x + c$$

which is the required solution.

9. The given differential equation is

$$(x + 2y)(dx - dy) = dx + dy$$

$$\Rightarrow \quad (x + 2y - 1)dx = (x + 2y + 1)dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x + 2y - 1)}{(x + 2y + 1)} \qquad \text{...(i)}$$

Let $\quad x + 2y = v$

$$\Rightarrow \quad 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1/2\left(\frac{dv}{dx} - 1\right)$$

$$\frac{1}{2}\left(\frac{dv}{dx} - 1\right) = \frac{v+1}{v-1}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2v-2}{v+1} + 1$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{3v-1}{v+1}$$

$$\Rightarrow \quad \left(\frac{v+1}{3v-1}\right)dv = dx$$

$$\Rightarrow \quad \left(\frac{3v-1+4}{3v-1}\right)dv = 3\,dx$$

$$\Rightarrow \quad \left(1 + \frac{3}{3v-1}\right)dv = 3\,dx$$

$$\Rightarrow \quad v + 4\log|3v-1| = 3x + c$$

$$\Rightarrow \quad (x+2y) + 4\log|3x+6y-1| = 3x + c$$

$$\Rightarrow \quad 2(y-x) + 4\log|3x+6y-1| = c$$

which is the required solution.

10. The given differential equation is

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

Hence, the solution is

$$x\cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y\, e^{\tan^{-1}y}}{1+y^2}\,dy + c$$

$$= \int t e^t\,dt + c,\ t = \tan^{-1}y$$

$$= (t-1)e^t + c$$

$$= (\tan^{-1}y - 1)e^{\tan^{-1}y} + c$$

which is the required solution.

11. The given differential equation is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - 2y - 1}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x^2 - y^2 - 1}{2xy}$$

$$\Rightarrow \quad 2xy\frac{dx}{dy} = x^2 - y^2 - 1$$

$$\Rightarrow \quad 2xy\frac{dx}{dy} - x^2 = -(y^2 + 1)$$

$$\Rightarrow \quad 2x\frac{dx}{dy} - \frac{x^2}{y} = -\left(y + \frac{1}{y}\right)$$

Let $x^2 = v$

$$\Rightarrow \quad 2x\frac{dx}{dy} = \frac{dv}{dy}$$

$$\Rightarrow \quad \frac{dv}{dy} - \frac{v}{y} = -\left(y + \frac{1}{y}\right)$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int \frac{dy}{y}} = e^{-\log y} = \frac{1}{y}$$

Hence, the solution is

$$v\cdot\frac{1}{y} = -\int\left(1 + \frac{1}{y^2}\right)dy + c$$

$$\Rightarrow \quad \frac{v}{y} = \left(y - \frac{1}{y}\right) + c$$

$$\Rightarrow \quad \frac{x^2}{y} = -\left(y - \frac{1}{y}\right) + c$$

$$\Rightarrow \quad \frac{x^2+1}{y} + y = c$$

which is the required solution.

12. The given differential equation is

$$x\,dy - y\,dx = xy^3(1 + \log x)\,dx$$

$$\Rightarrow \quad \frac{x\,dy - y\,dx}{y^2} = xy(1 + \log x)\,dx$$

$$\Rightarrow \quad -\frac{x}{y}\,d\left(\frac{x}{y}\right) = x^2\,dx + x^2\log x\,dx$$

Integrating, we get

$$-\frac{1}{2}\left(\frac{x}{y}\right)^2 = \frac{x^3}{3} + \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

$$\Rightarrow \quad -\frac{1}{2}\left(\frac{x}{y}\right)^2 = \frac{2x^3}{3} + \frac{x^3}{3}\log x + C$$

which is the required solution.

13. The given differential equation is

$$\sec^2 x\cdot\tan y\,dx + \sec^2 y\cdot\tan x\,dy = 0$$

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$\log\tan x + \log\tan y = \log c$$

$$\Rightarrow \quad (\tan x \tan y) = c$$

which is the required solution.

14. The given differential equation is

$$\frac{dy}{dx} = \frac{x+y-1}{\sqrt{x+y+1}} \qquad \text{...(i)}$$

Let $x + y + 1 = v^2$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = 2v\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2v\frac{dv}{dx} - 1$$

$$\Rightarrow 2v\frac{dv}{dx} - 1 = \frac{v^2 - 2}{v}$$

$$\Rightarrow 2v\frac{dv}{dx} = \frac{v^2 - 2}{v} + 1 = \frac{v^2 - 2 + v}{v}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v^2 - 2 + v}{2v^2}$$

$$\Rightarrow \frac{2v^2}{v^2 + v - 2}dv = dx$$

$$\Rightarrow \left(1 - \frac{1}{2}\frac{(2v + 1) - 5}{v^2 + v - 2}\right)dv = \frac{dx}{2}$$

$$\Rightarrow \left(1 - \frac{1}{2}\frac{(2v + 1)}{v^2 + v - 2} + \frac{5}{2}\cdot\frac{1}{(v + 2)(v - 1)}\right)dv = \frac{dx}{2}$$

$$\Rightarrow v - \frac{1}{2}\log|v^2 + v - 2| + \frac{15}{2}\log\left|\frac{v - 1}{v + 2}\right| = \frac{x}{2} + C$$

$$\Rightarrow \sqrt{x + y + 1} - \frac{1}{2}\log|(x + y + 1) + \sqrt{(x + y + 1)} - 2|$$

$$\Rightarrow + \frac{15}{2}\log\left|\frac{\sqrt{(x + y + 1)} - 1}{\sqrt{(x + y + 1)} + 2}\right| = \frac{x}{2} + C$$

which is the required solution.

15. The given differential equation is

$$(1 + e^{x/y})dx + \left(1 - \frac{x}{y}\right)e^{x/y}dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\left(\frac{x}{y} - 1\right)e^{x/y}}{(1 + e^{x/y})}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{(v - 1)e^v}{1 + e^v}, \; v = \frac{x}{y}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{(v - 1)e^v}{1 + e^v} - v$$

$$= \frac{ve^v - e^v - v - ve^v}{1 + e^v}$$

$$= -\frac{e^v + v}{1 + e^v}$$

$$\Rightarrow \quad \frac{1 + e^v}{v + e^v}dy = -\frac{dy}{y}$$

Integrating, we get

$$\log|v + e^v| = \log c - \log y$$

$$\Rightarrow \quad \log|v + e^v| = \log\left(\frac{c}{y}\right)$$

$$\Rightarrow \quad (v + e^v) = \frac{c}{y}$$

$$\Rightarrow \quad \left(\frac{x}{y} + e^{x/y}\right) = \frac{c}{y}$$

which is the required solution.

16. Do yourself.

17. The given differential equation is

$$(1 + y^2) + (x - e^{\tan^{-1}}y)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1 + y^2)\frac{dx}{dy} + (x - e^{\tan^{-1}y}) = 0$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{(1 + y^2)} = \frac{e^{\tan^{-1}}y}{(1 + y^2)}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int\frac{dy}{(1 + y^2)}} = e^{\tan^{-1}y}$$

Hence, the solution is

$$x\cdot e^{\tan^{-1}y} = \int\frac{\left(e^{\tan^{-1}y}\right)^2}{1 + y^2}dy + c$$

$$\Rightarrow \quad x\cdot e^{\tan^{-1}y} = \frac{\left(e^{\tan^{-1}y}\right)^2}{2} + C$$

which is the required solution.

18. The given differential equation is

$$y\,dx - x(1 + xy)dy = 0$$

$$\Rightarrow \quad y\,dx - x\,dy = x^2y\,dy$$

$$\Rightarrow \quad \frac{y\,dx - x\,dy}{x^2} = y\,dy$$

$$\Rightarrow \quad -d\left(\frac{y}{x}\right) = y\,dy$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) + y\,dy = 0$$

Integrating, we get

$$\left(\frac{y}{x}\right) + \frac{y^2}{2} = c$$

which is the required solution.

19. It is given that

$$\frac{dy}{dx} = -\left(\frac{x + y}{x}\right) = -1 - \frac{y}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = -1 - v, \qquad \text{(Let } v = y/x\text{)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -1 - 2v$$

$$\Rightarrow \quad \frac{dv}{2v + 1} = -\frac{dx}{x}$$

Integrating, we get

$$\log|2v+1| + \log(x^2) = \log c$$

$\Rightarrow$ $(2v+1)x^2 = c$

$\Rightarrow$ $\left(\frac{2y}{x}+1\right)x^2 = c$

$\Rightarrow$ $x^2 + 2xy = c$

which is passing thorugh the point (2, 1), so, $c = 8$

Hence, the equation of the curve is

$$x^2 + 2xy = 8$$

20. The equation of tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Putting $X = 0$, we get

$$Y = y - x\frac{dy}{dx}$$

It is given that,

$$\left(y - x\frac{dy}{dx}\right)^2 = xy$$

$\Rightarrow$ $\left(y - x\frac{dy}{dx}\right) = \sqrt{xy}$

$\Rightarrow$ $x\frac{dy}{dx} = y - \sqrt{xy}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y}{x}}$

$\Rightarrow$ $v + x\frac{dv}{dx} = v - \sqrt{v}$, $\left(\text{Let } v = \frac{y}{x}\right)$

$\Rightarrow$ $x\frac{dv}{dx} = -\sqrt{v}$

$\Rightarrow$ $\frac{dv}{\sqrt{v}} = -\frac{dx}{x}$

Integrating, we get

$\Rightarrow$ $2\sqrt{v} = c - \log|x|$

$\Rightarrow$ $2\sqrt{\frac{y}{x}} = c - \log|x|$

which is the required equation of the curve.

21. It is given that

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{x^2 + y^2}$$

$\Rightarrow$ $y^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = (x^2 + y^2)$

$\Rightarrow$ $\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \frac{(x^2 + y^2)}{y^2}$

$$= \frac{x^2}{y^2} + 1$$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2}$

$\Rightarrow$ $\left(\frac{dy}{dx}\right) = \pm\frac{x}{y}$

$\Rightarrow$ $y\,dy \mp x\,dx = 0$

$\Rightarrow$ $x\,dx \pm y\,dy = 0$

Integrating, we get

$$x^2 \pm y^2 = a^2$$

which is the required solution.

22. The given differential equation is

$$\left(\frac{dy}{dx}\right)^2 - (ex + e^{-x})\frac{dy}{dx} + 1 = 0$$

$\Rightarrow$ $p^2 - (e^x + e^{-x})p + 1 = 0$

$\Rightarrow$ $p = \frac{(e^x + e^{-x}) \pm \sqrt{(e^x + e^{-x})^2 - 4}}{2}$

$$= \frac{(e^x + e^{-x}) \pm \sqrt{(e^x - e^{-x})^2}}{2}$$

$$= \frac{(e^x + e^{-x}) \pm (e^x - e^{-x})}{2}$$

$$= e^x, e^{-x}$$

$\Rightarrow$ $\frac{dy}{dx} = e^x, \frac{dy}{dx} = e^{-x}$

Integrating, we get

$$y = e^x + c_1, \; y + e^x = c_2$$

$\Rightarrow$ $y - e^x - c_1 = 0, \; y + e^x - c_2 = 0$

Hence, the solution is

$$(y - ex - c_1)(y + e^x - c_2) = 0$$

23. The given differential equation is

$$\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = 3x^2$$

$\Rightarrow$ $p^2 + 2px - 3x^2 = 0$

$\Rightarrow$ $(p + 3x)(p - x) = 0$

$\Rightarrow$ $(p + 3x) = 0, (p - x) = 0$

$\Rightarrow$ $\left(\frac{dy}{dx} + 3x\right) = 0, \left(\frac{dy}{dx} - x\right) = 0$

$\Rightarrow$ $dy = -3x\,dx, \; dy = x\,dx$

Integrating, we get

$$y + \frac{3x^2}{2} - c_1 = 0, \; y - \frac{x^2}{2} - c_2 = 0$$

Hence, the solution is

$$\left(y + \frac{3x^2}{2} - c_1\right)\left(y - \frac{x^2}{2} - c_2\right) = 0$$

24. The given differential equation is

$$xy\left\{\left(\frac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\frac{dy}{dx}$$

$\Rightarrow \quad xy(p^2 - 1) = (x^2 - y^2)p$

$\Rightarrow \quad xyp^2 - xy = x^2p - y^2p$

$\Rightarrow \quad xyp^2 - x^2p - xy + y^2p = 0$

$\Rightarrow \quad xp(yp - p) + y(yp - x) = 0$

$\Rightarrow \quad (xp + y)(yp - x) = 0$

$\Rightarrow \quad (xp + y) = 0, (yp - x) = 0$

$\Rightarrow \quad x\frac{dy}{dx} + y = 0, y\frac{dy}{dx} = x$

$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0, ydy - xdx = 0$

$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0, xdx - ydy = 0$

Integrating, we get

$$\log|y| + \log|x| = \log c, x^2 - y^2 = a^2$$

$\Rightarrow \quad xy = c, x^2 - y^2 = a^2$

Hence, the solution is

$$(xy - c)(x^2 - y^2 - a^2) = 0.$$

25. It is given that

$$y\frac{dy}{dx} = x$$

$\Rightarrow \quad xdx - ydy = 0$

Integrating, we get

$$x^2 - y^2 = c$$

which is passing through (0, –2), so $c = -4$

Hence, the equation of the curve is

$$x^2 - y^2 = -4$$

26. Clearly, the slope of the tangent is $\frac{dy}{dx}$

Also, the slope of the line segment joining the points

(x, y) and (–4, –3) is

$$\frac{y + 3}{x + 4}.$$

It is given that

$$\frac{dy}{dx} = 2\left(\frac{y + 3}{x + 4}\right)$$

$\Rightarrow \quad \frac{dy}{(y + 3)} = \frac{2dx}{(x + 4)}$

Integrating, we get

$$\log|y + 3| = 2\log|x + 4| + \log c$$

$\Rightarrow \quad (y + 3) = (x + 4)^2$

which is passing through (–2, 1), so, $c = 1$

Hence, the equation of the curve is

$$(y + 3) = (x + 4)^2.$$

27. It is given that

$$\frac{dy}{dx} = x + xy$$

$\Rightarrow \quad \frac{dy}{dx} - xy = x$

which is a linear differential equation.

$\therefore \quad \text{IF} = e^{-\int x\,dx} = e^{-\frac{x^2}{2}}$

Hence, the solution is

$$y \cdot e^{-\frac{x^2}{2}} = \int x \cdot e^{-\frac{x^2}{2}} dx + c$$

$\Rightarrow \quad y \cdot e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c$

which is passing through (0, 1), so $c = 2$

Hence, the equation of the curve is

$$y \cdot e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + 2$$

$\Rightarrow \quad y = 2e^{\frac{x^2}{2}} - 1$

28. The given differential equation is

$$y^3\left(\frac{d^2y}{dx^2} - y\right) = 1$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = y + \frac{1}{y^3}$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = y + \frac{1}{y^3}$

Integrating, we get

$$\left(\frac{dy}{dx}\right) = \frac{y^2}{2} - \frac{1}{2y^2} = \frac{1}{2}\left(y^2 - \frac{1}{y^2}\right) + c$$

As $\quad y = 1, dy/dx = 0$, so $c = 0$

$\Rightarrow \quad \left(\frac{dx}{dy}\right) = \frac{1}{2}\left(y^2 - \frac{1}{y^2}\right)$

$\Rightarrow \quad \frac{y^2dy}{y^4 - 1} = \frac{dx}{2}$

$\Rightarrow \quad \frac{2y^2dy}{y^4 - 1} = dx$

$\Rightarrow \quad \frac{2y^2dy}{(y^2 - 1)(y^2 + 1)} = dx$

$\Rightarrow \quad \left(\frac{1}{y^2-1}+\frac{1}{y^2+1}\right)dy = dx$

Integrating, we get

$$\frac{1}{2}\log\left|\frac{y-1}{y+1}\right| + \tan^{-1}y = x + c_1$$

When $x = 0, y = 1, c_1 = \frac{\pi}{4}$

Hence, the curve is

$$\frac{1}{2}\log\left|\frac{y-1}{y+1}\right| + \tan^{-1}y = x + \frac{\pi}{4}$$

29. The equation of the normal at $P(x, y)$ is

$$Y - y = -\frac{dx}{xy}(X - x)$$

Putting $Y = 0$, we get

$$-y = -\frac{dx}{xy}(X - x)$$

$\Rightarrow \quad X = y\frac{dy}{dx} + x$

Thus, $B = \left(y\frac{dy}{dx} + x, 0\right)$

Here, $A = (x, 0)$

It is given that

$$x + y\frac{dy}{dx} - x = \frac{x+y}{2}$$

$\Rightarrow \quad y\frac{dy}{dx} = \frac{x+y}{2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{2y}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1+v}{v}, \qquad \left(\text{Let } v = \frac{y}{x}\right)$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v-2v^2}{2v}$

$\Rightarrow \quad \frac{v\,dv}{1+v-2v^2} = \frac{dx}{x}$

$\Rightarrow \quad \frac{v\,dv}{(2v+1)(v-1)} = -\frac{dx}{x}$

$\Rightarrow \quad \frac{(v-1+1)\,dv}{(2v+1)(v-1)} = -\frac{dx}{x}$

$\Rightarrow \quad \frac{dv}{(2v+1)} + \frac{1}{3}\left(\frac{dv}{v-1} - \frac{2dv}{2v+1}\right) = -\frac{dx}{x}$

$\Rightarrow \quad \frac{dv}{(2v+1)} + \frac{dv}{v-1} = -\frac{3\,dx}{x}$

Integrating, we get

$$\frac{1}{2}\log|2v+1| + \log|v-1| = \log c - 3\log x$$

$\Rightarrow \quad \frac{1}{2}\log\left|2\left(\frac{y}{x}\right)+1\right| + \log\left|\left(\frac{y}{x}\right)-1\right| = \log\left(\frac{c}{x^3}\right)$

$\Rightarrow \quad \left(\left(\frac{y}{x}\right)-1\right)\left(\sqrt{2\left(\frac{y}{x}\right)+1}\right) = \left(\frac{c}{x^3}\right)$

which is the required equation of the curve.

30. The equation of the normal at $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x)$$

Putting $Y = 0$, we get

$$-y = -\frac{dx}{dy}(X - x)$$

$\Rightarrow \quad X = y\frac{dy}{dx} + x$

Thus, $Q = \left(y\frac{dy}{dx} + x, 0\right)$

Also, it is given that,

$$PQ = k$$

$\Rightarrow \quad \left(y\frac{dy}{dx}\right)^2 + y^2 = k^2$

$\Rightarrow \quad \left(y\frac{dy}{dx}\right)^2 = k^2 - y^2$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{k^2-y^2}{y^2}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \pm\sqrt{\frac{k^2-y^2}{y^2}}$

$\Rightarrow \quad \frac{y\,dy}{\sqrt{k^2-y^2}} = \pm\, dx$

Integrating, we get

$$\sqrt{k^2-y^2} = c \mp x$$

31. It is given that

$$\frac{dy}{dx} = \frac{2y}{x}$$

$\Rightarrow \quad \frac{dy}{y} = \frac{2dx}{x}$

Integrating, we get

$$\log|y| = \log|x^2| + \log c$$

$\Rightarrow \quad y = cx^2$

which is passing through (1, 1), so, $c = 1$

Hence, the equation of the curve is

$$y = x^2$$

32. The equation of the normal at $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x)$$

$$A = \left(x + y\frac{dy}{dx}, 0\right), B = \left(0, y + x\frac{dx}{dy}\right)$$

Here, $P = (x, y)$

It is given that,

$$\frac{PB}{PA} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\sqrt{x^2 + \left(x\frac{dx}{dy}\right)^2}}{\sqrt{\left(y\frac{dv}{dx}\right)^2 + y^2}} = \frac{1}{2}$$

$$\Rightarrow \quad x^2 + \left(x\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(\left(y\frac{dv}{dx}\right)^2 + y^2\right)$$

$$\Rightarrow \quad x^2 + \frac{x^2}{p^2} = \frac{1}{4}(p^2y^2 + y^2)$$

$$\Rightarrow \quad \frac{x^2}{p^2}(p^2 + 1) = \frac{y^2}{4}(p^2 + 1)$$

$$\Rightarrow \quad \frac{x^2}{p^2} = \frac{y^2}{4}$$

$$\Rightarrow \quad y^2p^2 = 4x^2$$

$$\Rightarrow \quad yp = \pm 2x$$

$$\Rightarrow \quad y\,dv = \pm 2x\,dx$$

Integrating, we get

$$\Rightarrow \quad \frac{y^2}{2} = c_1 \pm x^2$$

$$\Rightarrow \quad y^2 = 2c_1 \pm 2x^2$$

which is passing through (0, 4) so $c_1 = 8$

Hence, the eqution of the curve is

$$y^2 = 16 \pm 2x^2$$

$$\Rightarrow \quad y^2 = 16 + 2x^2$$

Clearly, it is passing through $(\sqrt{10}, -6)$.

33. The equation of the normal at $P(x, y)$ is

$$(Y - y) = \frac{dy}{dx}(X - x)$$

Thus, $OM = x + y\frac{dy}{dx}$ and $ON = y + x\frac{dx}{xy}$

Also, it is given that,

$$\frac{1}{OM} + \frac{1}{ON} = 1$$

$$\Rightarrow \quad OM + ON = OM \cdot ON$$

$$\Rightarrow \quad \left(x + y\frac{dy}{dx}\right) + \left(y + x\frac{dx}{dy}\right)$$

$$= \left(x + y\frac{dy}{dx}\right) \cdot \left(y + x\frac{dx}{dy}\right)$$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = x + y\frac{dy}{dx}$$

$$\Rightarrow \quad (y - 1)\frac{dy}{dx} + (x - 1) = 0$$

Integrating, we get

$$(x - 1)^2 + (y - 1)^2 = c$$

which is passing through (5, 4), so $c = 25$

Hence, the equation of the curve is

$$(x - 1)^2 + (y - 1)^2 = 25.$$

Level III

(Tougher Problems for JEE-Advanced)

1. We have

$$y = C_1 \cos(2x + C_2) - C_8 3^{C_6 3^x} + C_6 \sin(x - C_7)$$

$$\Rightarrow \quad y = C_1 \cos(2x + C_2) - C_9 3^x + C_6 \sin(x - C_7)$$

Since the above equation has 5 arbitrary constants, so the order of the differential equation is 5.

2. The Equation of the family of parabolas is

$(y - k)^2 = 4a(x - h)$,

where h and k are arbitrary constants ...(i)

Differentiating w.r.t. x, we get

$$2(y - k)\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad (y - k)\frac{dy}{dx} = 2a \quad \text{...(ii)}$$

Again differentiaing w.r.t. x, we get

$$(y - k)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \quad \text{...(iii)}$$

From Eqs (ii) and (iii), we get

$$2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation.

3. Given condition is

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \quad \frac{dy}{y} = 2 \cdot \frac{dx}{x}$$

Integrating, we get

$$\log|y| = 2\log|x| + \log|c|$$

$$\Rightarrow \quad y = cx^2$$

which is passing through the curve (1, 1), so $c = 1$

Hence, the equation of the curve is

$$y = x^2$$

4. The equation of the tangent is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{...(i)}$$

Equation (i) meets the x-axis at $\left(x - y\frac{dx}{dy}, 0\right)$

and the y-axis at $\left(0, y - x\frac{dy}{dx}\right)$ respectively.

Given condition is

$$x - y\frac{dx}{dy} = 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow \quad \log y + \log x = \log c$$

$$\Rightarrow \quad xy = c$$

which is passing through (2, 4), so $c = 8$

Hence, the equation of the curve is

$$xy = 8.$$

5. The given differential equation is

$$2y\frac{dy}{dx} = e^{\left(\frac{x^2+y^2}{x}\right)} + \left(\frac{x^2 + y^2}{x} - 2x\right) \quad \text{...(i)}$$

$$\text{Let} \quad \frac{x^2 + y^2}{x} = v$$

$$\Rightarrow \quad x^2 + y^2 = vx$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = e^v + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = e^v$$

$$\Rightarrow \quad e^{-v}dv = \frac{dx}{x}$$

Integrating, we get

$$-e^{-v} = \log|x| + \log c$$

$$\Rightarrow \quad -e^{-v} = \log|cx|$$

$$\Rightarrow \quad cx = e^{-e^{-v}}$$

$$\Rightarrow \quad cx = e^{-e^{-\left(\frac{x^2+y^2}{x}\right)}}$$

which is the required solution.

6. The given differential equation is

$$x(1 - x\ln y)\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad y\frac{dx}{xy} + x = x^2\ln y \quad \text{...(i)}$$

$$\text{Let} \quad \ln y = z$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \quad y\frac{dx}{dy} = \frac{dx}{dz}$$

$$\Rightarrow \quad \frac{dx}{dz} + x = x^2 z$$

$$\Rightarrow \quad \frac{1}{x^2}\frac{dx}{dz} + \frac{1}{x} = z$$

which is a linear differential equation.

$$\text{Let} \quad \frac{1}{x} = v$$

$$\Rightarrow \quad -\frac{1}{x^2}\frac{dx}{dz} = \frac{dv}{dz}$$

$$\Rightarrow \quad -\frac{dv}{dz} + v = z$$

$$\Rightarrow \quad \frac{dv}{dz} - v = -z$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int dz} = e^{-z}$$

Hence, the solution is

$$v \cdot e^{-z} = \int -ze^{-z}dz + c$$

$$\Rightarrow \quad v \cdot e^{-z} = -(z - 1)e^{-z} + c$$

$$\Rightarrow \quad v = (z + 1) + ce^{z}$$

$$\Rightarrow \quad \frac{1}{x} = (\ln y + 1) + ce^{\ln y}$$

$$\Rightarrow \quad \frac{1}{x} = (\ln y + 1) + cy$$

which is passing through $\left(e, \frac{1}{e}\right)$, so, $c = e$

Hence, the required integral curve is

$$\frac{1}{x} = (\ln y + 1) + ey$$

$$\Rightarrow \quad ((\ln y + 1) + ey)x = 1$$

7. The given differential equation can be written as

$$\frac{2(xdx - ydv)}{(x^2 - y^2)} = -\left(\frac{ydx - xdy}{(x^2 + y^2)}\right)$$

$$\Rightarrow \quad \frac{d(x^2 - y^2)}{(x2 - y^2)} = -d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

Integrating, we get

$$\log|(x^2 - y^2)| + \left(\tan^{-1}\left(\frac{y}{x}\right)\right) = c$$

which is the required solution.

8. The given differential equation is

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x)$$

$$\Rightarrow \quad y + y + x\frac{dy}{dx} = x(\sin x + \log x)$$

$$\Rightarrow \quad x\frac{dy}{dx} + 2y = x(\sin x + \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2y}{x} = (\sin x + \log x)$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{2\int\frac{dx}{x}} = e^{2\log x} = e^{\log x2} = x^2$$

Hence, the solution is

$$y \cdot x^2 = \int x^2(\sin x + \log x)\,dx + c$$

$$\Rightarrow \quad y \cdot x^2 = -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{x^3}{9} + c$$

9. The given differential equation is

$$(xdy + ydx)\sin(xy) + (x^2ydx + xy^2dx)\cos(xy) = 0$$

$$\Rightarrow d(xy)\sin(xy) + xy(xdy + ydx)\cos(xy) = 0$$

$$\Rightarrow d(xy)\sin(xy) + xy\,d(xy)\cos(xy) = 0$$

$$\Rightarrow \sin(xy)\,d(xy) + xy\cos(xy)\,d(xy) = 0$$

On Integrating, we get

$$-\cos(xy) + (xy)\sin(xy) + \cos(xy) = c$$

$$\Rightarrow \quad (xy)\sin(xy) = c$$

which is the required solution.

10. The given differential equation is

$$\frac{dy}{dx} = \frac{2y^2\cos x + y\sin 2x + 2\cos x \cdot \sin^2 x}{\sin^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cos x\left(\frac{y^2}{\sin^2 x} + \frac{y}{\sin x} + 1\right)$$

$$\text{Let} \quad \frac{y}{\sin x} = t$$

$$\frac{dy}{dx} = t\cos x + \sin x\frac{dt}{dx}$$

$$\Rightarrow \quad t\cos x + \sin x\frac{dt}{dx} = 2\cos x(t^2 + t + 1)$$

$$\Rightarrow \quad \tan x\frac{dt}{dx} = (2t^2 + t + 2)$$

$$\Rightarrow \quad \frac{dt}{(2t^2 + t + 2)} = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\frac{dt}{\left\{\left(t + \frac{1}{4}\right)^2 + \frac{15}{16}\right\}} = \frac{dx}{x}$$

Integrating, we get

$$\frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{t + \frac{1}{4}}{\sqrt{\frac{15}{16}}}\right) = \log|\sin x| + C$$

$$\Rightarrow \quad \frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{4y + \sin x}{\sqrt{15}\sin x}\right) = \log|\sin x| + C$$

which is the required solution.

11. The given differential equation is

$$y\cos\left(\frac{y}{x}\right)(xdy - ydx) + x\sin\left(\frac{y}{x}\right)(xdy + ydx) = 0$$

$$\Rightarrow \frac{y}{x}\cos\left(\frac{y}{x}\right)\left(\frac{dy}{dx} - \frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)\left(\frac{dy}{dx} + \frac{y}{x}\right) = 0$$

$$\left(\text{Let} \quad \frac{y}{x} = u\right)$$

$$\Rightarrow v\cos v\left(v + x\frac{dv}{dx} - v\right) + \sin v\left(v + x\frac{dv}{dx} + v\right) = 0$$

$$\Rightarrow (v\cos v + \sin v)\left(x\frac{dv}{dx}\right) + 2v\sin v = 0$$

$$\Rightarrow \left(\frac{v\cos v + \sin v}{v\sin v}\right)dv + \frac{2dx}{x} = 0$$

Integrating, we get

$$\log|v\sin v| = -2\log|x| + \log c$$

$$\Rightarrow \quad v\sin v = \frac{c}{x^2}$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right) = \frac{c}{x^2}$$

$$\Rightarrow \quad y\sin\left(\frac{y}{x}\right) = \frac{c}{x}$$

when $x = 1$, $y = \frac{\pi}{2}$, then $c = \frac{\pi}{2}$

Hence, the required solution is

$$y\sin\left(\frac{y}{x}\right) = \frac{\pi}{2x}$$

12. The given differential equation is

$$\frac{dy}{dx} = \sqrt{\frac{x^4y^2 - x^6 + 2x^4y - x^6y^2 - 2x^6y + x^4}{y^2 - x^2y^2 + x^3y^2 - x^5y^2}}$$

$$= \sqrt{\frac{x^4(y^2 - x^2 + 2y - x^2y^2 - 2x^2y + 1)}{y^2(1 - x^2 + x^3 - x^5)}}$$

$$= \sqrt{\frac{x^4\{(y^2+2y+1)-x^2(y^2+2y+1)\}}{y^2(1-x^2+x^3-x^5)}}$$

$$= \frac{x^2}{y}\sqrt{\frac{\{(y+1)^2-x^2(y+1)^2\}}{(1+x^2)(1+x^3)}}$$

$$= \frac{x^2}{y}\sqrt{\frac{\{(y+1)^2(1-x^2)\}}{(1-x^2)(1+x^3)}}$$

$$= \frac{x^2(y-1)}{y\sqrt{1+x^3}}$$

$$\Rightarrow \quad \frac{y\,dv}{y+1} = \frac{x^2\,dx}{\sqrt{1+x^3}}$$

$$\Rightarrow \quad \left(1-\frac{1}{y+1}\right)dy = \frac{x^2\,dx}{\sqrt{1+x^3}}$$

Integrating, we get

$$y - \log|y+1| = \frac{2}{3}\sqrt{1+x^3} + c$$

which is the required solution.

13. Given

$$f(x) = \int_1^x \frac{\log t}{1+t+t^2}\,dt,\ x \ge 1$$

Now, $f\left(\frac{1}{x}\right) = \int_1^{1/x}\left(\frac{\log t}{1+t+t^2}\right)dt$

Let $t = \frac{1}{y} \Rightarrow dt = -\frac{1}{y^2}\,dy$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^y \left(\frac{\log\left(\frac{1}{y}\right)}{1+\frac{1}{y}+\frac{1}{y^2}}\right)\left(-\frac{1}{y^2}\right)dy$$

$$= \int_1^y \left(\frac{-\log(y)}{y^2+y+1}\right)(-1)\,dy$$

$$= \int_1^y \left(\frac{\log(y)}{y^2+y+1}\right)dy$$

$$= \int_1^x \left(\frac{\log\ (t)}{t^2+t+1}\right)dt$$

$$= f(x)$$

Hence, the result.

14. The given differential equation is

$$\frac{dy}{dx} = \frac{x^2+y^2+3x+3y+2xy+1}{x^2+y^2-3x-3y+2xy+2}$$

$$= \frac{(x+y)^2+3(x+y)+1}{(x+y)^2-3(x+y)+2}$$

$$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{v^2+3v+1}{v^2-3v+1}, \quad (\text{Let } v = x+y)$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v^2+3v+1}{v^2-3v+1} + 1$$

$$= \frac{2v^2+2}{v^2-3v+1}$$

$$\Rightarrow \quad \frac{v^2-3v+1}{v^2+1}\,dv = 2\,dx$$

$$\Rightarrow \quad \left(1-\frac{3}{2}\left(\frac{2v}{v^2+1}\right)\right)dv = 2\,dx$$

Integrating, we get

$$v - \frac{3}{2}\log|v^2+1| = 2x + C$$

$$\Rightarrow \quad y - \frac{3}{2}\log|(x+y)^2+1| = x + C$$

which is the required solution.

15. Given

$$f(0) = -1$$

Put $x = 0,\ y = 1$, then $f(1) = 1$

Now, $f(x+h) + f(x)\cdot f(h) = f(xh+1)$

$$\Rightarrow \quad (f(x+h)-f(x)) + (f(x)\cdot f(h) - f(x)) = f(xh+1) - f(1)$$

$$\Rightarrow \quad \frac{f(x+h)-f(x)}{h} + \frac{f(x)f(h-1)}{h} = \frac{f(xh+1)-f(1)}{xh}\cdot x$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\left(\frac{f(x+h)-f(x)}{h}\right) + f(x)\left(\frac{f(h)-1}{h)}\right)\right) = \lim_{h\to 0}\left(\frac{f(xh+1)-f(1)}{xh}\right)x$$

$$\Rightarrow \quad f'(x) + f(x)\cdot f'(0) = xf'(1)$$

$$\Rightarrow \quad \frac{dy}{dx} + py = qx$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int p\,dx} = e^{px}$$

Hence, the solution is

$$y\cdot e^{px} = q\int x e^{px}dx + c$$

$$\Rightarrow \quad y\cdot e^{px} = q\int x e^{px}dx + c$$

$$\Rightarrow \quad y\cdot e^{x} = \int x e^{x}dx + c$$

$\Rightarrow \quad y \cdot e^x = (x+1)e^x + c$

$\Rightarrow \quad y = (x-1) + ce^{-x}$

when $x = 1, y = 1$, then $c = 0$

Thus, the equation of the given curve is

$$y = x - 1.$$

16. Put $x = 1, y = 1$, then $f(1) = 0$.

Now, $f(x)(1+h)) = xf(1+h) + (1+h)f(x)$

$\Rightarrow \quad f(x + xh) - f(x) = xf(1+h) + hf(x)$

$\Rightarrow \quad f(x+xh) - f(x)$

$$= x(f(1+h) - f(1)) + hf(x)$$

$$\Rightarrow \left(\frac{f(xh+x) - f(x)}{xh}\right)\cdot x = \frac{x(f(1+h)-f(1))}{h} + \frac{hf(x)}{h}$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{f(xh+x)-f(x)}{xh}\right)x = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)x + f(x)$$

$\Rightarrow \quad xf'(x) = xf'(1) + f(x)$

$\Rightarrow \quad xf'(x) = x + f(x)$

$\Rightarrow \quad x\dfrac{dy}{dx} = x + y$

$\Rightarrow \quad \dfrac{dy}{dx} = 1 + \dfrac{y}{x}$

$\Rightarrow \quad \dfrac{dy}{dx} + \left(-\dfrac{1}{x}\right)y = 1$

which is a linear differential equation.

$\therefore \quad \text{IF} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \dfrac{1}{x}$

Hence, the solution is

$$\frac{y}{x} = \int \frac{dx}{x} + c$$

$\Rightarrow \quad \dfrac{y}{x} = \log|x| + c$

when $x = 1, y = 0$, then $c = 0$

Thus, the equation of the given curve is

$$y = x\log|x|$$

17. The given differential equation can be written as

$$\left(\frac{dx}{y} - \frac{xdy}{y^2}\right)\sin\left(\frac{x}{y}\right) + \left(\frac{dy}{x} - \frac{ydx}{x^2}\right)\cos\left(\frac{y}{x}\right) + dx + \frac{dy}{y^2} = 0$$

$$\Rightarrow \sin\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) + \cos\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) + dx + \frac{dy}{y^2} = 0$$

Integrating, we get

$$-\cos\left(\frac{x}{y}\right) + \sin\left(\frac{y}{x}\right) + x - \frac{1}{y} = c$$

which is the required differential equation.

18. The given differential equation is

$$\frac{dy}{dx} = \frac{(1+y^2)}{xy(1+x^2)}$$

$\Rightarrow \quad \dfrac{ydy}{1+y^2} = \dfrac{dx}{x(1+x^2)}$

$\Rightarrow \quad \dfrac{ydy}{(1+y^2)} = \left(\dfrac{1}{x} - \dfrac{x}{1+x^2}\right)dx$

Integrating, we get

$$\log|1+y^2| = 2\log|x| - \log|1+x^2| + 2\log c$$

$\Rightarrow \quad (1+y^2)(1+y^2) = c^2x^2$

which is the required solution.

19. The given differential equation is

$$\frac{y}{x} = \frac{dy}{dx} = \frac{1}{\sin^4(xy) + \cos^2(xy)}$$

Let $xy = v$

$\Rightarrow \quad \dfrac{y}{x} + \dfrac{dy}{dx} = \dfrac{1}{x}\dfrac{dv}{dx}$

$\Rightarrow \quad \dfrac{1}{x}\dfrac{dv}{dx} = \dfrac{1}{\sin^4 v + \cos^2 v}$

$\Rightarrow \quad (\sin^4 v + \cos^2 v)dv = \dfrac{dx}{x}$

$\Rightarrow \quad \dfrac{1}{4}(2\sin^2 v)^2 dv + \dfrac{1}{2}(2\cos^2 v)dv = \dfrac{dx}{x}$

$\Rightarrow \quad \dfrac{1}{4}(1 - \cos^2 v)^2 dv + \dfrac{1}{2}(1 + \cos^2 v)dv = \dfrac{dx}{x}$

$$\Rightarrow \frac{1}{4}(1-\cos^2 v)dv + \frac{1}{8}(1+\cos^4 v)dv + \frac{1}{2}(1+\cos^2 v)dv = \frac{dx}{x}$$

Integrating, we get

$$\Rightarrow \frac{1}{4}(v - \sin^2 v) + \frac{1}{8}\left(v + \frac{\sin^4 v}{4}\right) + \frac{1}{2}\left(v + \frac{\sin 2v}{2}\right) = \log|x| + c$$

$$\Rightarrow \frac{1}{4}(xy - \sin(2xy)) + \frac{1}{8}\left(xy + \frac{\sin(4xy)}{4}\right)$$

$$+\frac{1}{2}\left(xy+\frac{\sin 2(xy)}{2}\right)=\log|x|+c$$

which is the required solution.

20. The given differential equation can be written as

$$(y^2+1)\,dy+x^2dy+2xy\,dx=0$$

$$\Rightarrow\quad (y^2+1)\,dy+d(x^2y)=0$$

Integrating, we get

$$\frac{y^3}{3}+y+x^2y=c$$

which is the required solution.

21. The given differential equation is

$$y\frac{dy}{dx}+x=\frac{1}{2}\left(\frac{x^2+y^2}{x}\right)^2$$

$$\Rightarrow\quad y\frac{dy}{dx}+x=\frac{1}{2x^2}(x^2+y^2)^2$$

$$\Rightarrow\quad \frac{ydy+xdx}{(x^2y^2)^2}=\frac{dx}{2x^2}$$

$$\Rightarrow\quad \frac{d(x^2+y^2)}{(x^2+y^2)^2}=\frac{dx}{x^2}$$

Integrating, we get

$$-\frac{1}{(x^2+y^2)}=c-\frac{1}{x}$$

$$\Rightarrow\quad \frac{1}{x}-\frac{1}{(x^2+y^2)}=c$$

which is the required solution.

22. The given differential equation is

$$(y+\sin x\cos^2(xy))\,dx+x\,dy=0$$

$$\Rightarrow\quad \left(y+x\frac{dy}{dx}\right)+\sin x\cos^2(xy)=0$$

$$\left(\text{Let}\quad xy=v\Rightarrow y+x\frac{dy}{dx}=\frac{dv}{dx}\right)$$

$$\Rightarrow\quad \frac{dv}{dx}+\sin x\cos^2 v=0$$

$$\Rightarrow\quad \frac{dv}{dx}=-\sin x\cos^2 v$$

$$\Rightarrow\quad \sec^2 v\,dv=-\sin x\,dx$$

Integrating, we get

$$\Rightarrow\quad \tan(xy)-\cos x=c$$

which is the required solution.

23. The given differential equation is

$$\frac{dy}{dx}+p(x)y=q(x)$$

which is a linear differential equation.

$$\therefore\quad \text{IF}=e^{\int p(x)\,dx}=e^{p(x)x}$$

Hence, the solution is

$$y\cdot e^{p(x)x}=\int\left(e^{p(x)x}\cdot q(x)\right)dx+c$$

24. The equation of tangent to the curve at (x, y) is

$$Y-y=\frac{dy}{dx}(X-x)$$

Putting $X=0$, we get

$$Y=y-x\frac{dy}{dx}$$

It is given that,

$$\left(y-x\frac{dy}{dx}\right)^2=xy$$

$$\Rightarrow\quad \left(y-x\frac{dy}{dx}\right)=\sqrt{xy}$$

$$\Rightarrow\quad x\frac{dy}{dx}+\sqrt{xy}=y$$

$$\Rightarrow\quad \frac{dy}{dx}+\sqrt{\frac{y}{x}}=\frac{y}{x}$$

$$\Rightarrow\quad v+x\frac{dv}{dx}+\sqrt{v}=v,\ v=\frac{v}{x}$$

$$\Rightarrow\quad x\frac{dv}{dx}=\sqrt{v}=0$$

$$=-\sqrt{v}$$

$$\Rightarrow\quad \frac{dv}{\sqrt{v}}+\frac{dx}{x}=0$$

Integrating, we get

$$2\sqrt{v}+\log|x|=c$$

$$\Rightarrow\quad 2\sqrt{\frac{y}{x}}+\log|x|=c$$

which is the required solution.

25. We have

$$x\int_0^x(1-t)f(t)\,dt=\int_0^x tf(t)\,dt,\ f(1)=1$$

Applying Newton and Leibnitz formula, we get

$$x(x-1)f(x)+\int_0^x(1-t)f(t)\,dt=xf(x)$$

$$\Rightarrow\quad (x-x^2-x)\,f(x)+\int_0^x(1-t)f(t)\,dt=0$$

$$\Rightarrow\quad (-x^2)\,f(x)+\int_0^x(1-t)f(t)\,dt=0$$

$$\Rightarrow\quad (-x^2)f'(x)-2xf(x)+(1-x)f(x)=0$$

$$\Rightarrow\quad (x^2)f'(x)+(1-3x)f(x)=0$$

$$\Rightarrow \quad (x^2)\frac{dy}{dx} + (1 - 3x)y = 0$$

$$\Rightarrow \quad \frac{dy}{y} + \left(\frac{1 - 3y}{x^2}\right)dx = 0$$

$$\Rightarrow \quad \frac{dy}{y} + \left(\frac{1}{x^2} - \frac{3}{x}\right)dx = 0$$

Integrating, we get

$$\log|y| - 3\log|x| - \frac{1}{x} = c$$

$$\Rightarrow \quad \log\left|\frac{y}{x^3}\right| - \frac{1}{x} = c$$

when $x = 1$, $y = 1$, then $c = -1$

Hence, the solution is

$$\log\left|\frac{y}{x^3}\right| - \frac{1}{x} + 1 = 0$$

$$\Rightarrow \quad \log\left|\frac{y}{x^3}\right| = \frac{1}{x} - 1 = \frac{1 - x}{x}$$

$$\Rightarrow \quad \frac{y}{x^3} = \frac{1 - x}{x}$$

$$= e^{\left(\frac{1-x}{x}\right)}$$

$$\Rightarrow \quad y = x^3 e^{\left(\frac{1-x}{x}\right)}$$

$$\Rightarrow \quad f(x) = x^3 e^{\left(\frac{1-x}{x}\right)}$$

26. The given differential equation is

$$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$$

$$\Rightarrow \quad yp^2 + (x - y)p - x = 0, \text{ where } \frac{dy}{dx} = p$$

$$\Rightarrow \quad yp^2 - yp + xp - x = 0$$

$$\Rightarrow \quad yp(p - 1) + x(p - 1) = 0$$

$$\Rightarrow \quad (yp - 1)(p - 1) = 0$$

$$\Rightarrow \quad (yp + x) = 0, (p - 1) = 0$$

$$\Rightarrow \quad x\,dx + y\,dy = 0, dy = dx$$

Integrating, we get

$$x^2 + y^2 = a^2, y = x + b$$

which is passing through (3, 4), so $a^2 = 25$ and $b = 1$

Hence, the equations of the curves are

$$x^2 + y^2 = 25, y = x + 1$$

27. The given differential equation is

$$y' = \frac{y}{x} + \frac{\varphi(y/x)}{\varphi'(y/x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \frac{\varphi(y/x)}{\varphi'(y/x)}$$

Let $y = vx$

Thus,

$$v + x\frac{dv}{dx} = v + \frac{\varphi(v)}{\varphi'(v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{\varphi(v)}{\varphi'(v)}$$

$$\Rightarrow \quad \frac{\varphi'(v)}{\varphi(v)}dv = \frac{dx}{x}$$

Integrating, we get

$$\log|\varphi(v)| = \log|x| + \log c$$

$$\Rightarrow \quad \log|\varphi(v)| = \log|cx|$$

$$\Rightarrow \quad |\varphi(v) = cx$$

$$\Rightarrow \quad \varphi\left(\frac{y}{x}\right) = cx$$

which is the required solution.

28. Clearly, $f(0) = 1$

We have,

$$f\left(\frac{x + h}{1 + xh}\right) - f(x)\cdot f(h) = 0$$

$$\Rightarrow \quad \frac{\left[f\left(1 + \frac{h - x^2h}{1 + xh}\right) - f(x)\cdot\right]}{\left(\frac{h - x^2h}{1 + xh}\right)\cdot h}\left(\frac{h - x^2h}{1 + xh}\right)$$

$$= \frac{f(x)\cdot f(h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x)\cdot(1 - x^2) = f(x)\cdot f'(0)$$

$$\Rightarrow \quad f'(x)\cdot(1 - x^2) = f(x)$$

$$\Rightarrow \quad \frac{dy}{dx}(1 - x^2) = y$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{dx}{(1 - x^2)}$$

Integrating, we get

$$\log|y| = \log\left|\frac{1 - x}{1 + x}\right| + \log c$$

$$\Rightarrow \quad y = c\left(\frac{1 - x}{1 + x}\right)$$

when $f(0) = 1$, then we get, $c = 1$

Thus, $y = \left(\frac{1 - x}{1 + x}\right)$

which is the required solution.

29. Let $x = 1 = y$, then $f(1) = 0$

Now, $f(x(1+h))$

$$= xf(1+h) + (1+h)f(x)$$

$$\Rightarrow x \cdot \frac{f(x+xh) - f(x)}{xh} = \frac{x\{f(1+h) - 0\}}{h} + \frac{hf(x)}{h}$$

$$\Rightarrow \lim_{x \to 0}\left(x \cdot \frac{f(x+xh) - f(x)}{xh}\right)$$

$$= \lim_{x \to 0}\left(\frac{x\{f(1+h) - 0\}}{h} + \frac{hf(x)}{h}\right)$$

$$\Rightarrow xf'(x) = xf'(1) + f(x)$$

$$\Rightarrow xf'(x) = x + f(x) (\because f'(1) = 1)$$

$$\Rightarrow x\frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{y}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v,\ v = \frac{y}{x}$$

$$\Rightarrow x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating, we get

$$\Rightarrow v = \log|x| + c$$

$$\Rightarrow \frac{y}{x} = \log|x| + c$$

when $x = 1$, $y = 0$, then $c = 0$

Thus, $\frac{y}{x} = \log|x|$

$$\Rightarrow y = x\log|x|$$

which is the required solution.

30. The equation of tangent to the curve at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Put $Y = 0$, then $A = \left(x - y\frac{dx}{dy}, 0\right)$

and $X = 0$, then $B = \left(0, y - x\frac{dy}{dx}\right)$

It is given that,

$$\frac{x - y\frac{dx}{dy}}{2} = x \text{ and } \frac{y - x\frac{dy}{dx}}{2} = y$$

$$\Rightarrow \quad x - y\frac{dx}{dy} = 2x \text{ and } y - x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad y\frac{dx}{dy} = -x \text{ and } x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, we get

$$\log|x| + \log|y| = \log|c|$$

$$\Rightarrow \quad \log|xy| \log c$$

$$\Rightarrow \quad xy = c$$

which is the required equation of the curve.

31. It is given that

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad y^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = (x^2 + y^2)$$

$$\Rightarrow \quad \left(1 + \left(\frac{dy}{dx}\right)^2\right) = \left(\left(\frac{x}{y}\right)^2 + 1\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \left(\frac{x}{y}\right)^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \pm\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad xdx \pm ydy = 0$$

Integrating, we get

$$x^2 \pm y^2 = a^2$$

32. It is given that

$$\frac{dy}{dx} = \frac{1}{y}$$

$$\Rightarrow \quad ydy = dx$$

Integrating, we get

$$\frac{y^2}{2} = x + c$$

which is passing through (2, 2), so, $c = 0$

Hence, the eqution of the curve is

$$y^2 = 2x$$

33. The tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

If p be the length of perpendicular from the origin, then

$$p = \left|\frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

It is given that

$$\left|\frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right| = x$$

$\Rightarrow \quad \left(x\frac{dy}{dx} - y\right)^2 = x^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$

$\Rightarrow \quad y^2 - 2xy\frac{dy}{dx} - x^2 = 0$

Hence, the result.

Also, $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{x}{y}\right)}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}, \qquad$ (Let $y = vx$)

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$

$= -\frac{v^2 + 1}{2v}$

$\Rightarrow \quad \frac{2v}{v^2 + 1}\,dv = -\frac{dx}{x}$

$\Rightarrow \quad \frac{2v}{v^2 + 1}\,dv + \frac{dx}{x} = 0$

Integrating, we get

$\log|v^2 + 1| + \log|x| = \log c$

$\Rightarrow \quad (v^2 + 1)x = c$

$\Rightarrow \quad \left(\left(\frac{y}{x}\right)^2 + 1\right)x = c$

$\Rightarrow \quad x^2 + y^2 = cx$

which is the required equation of the curve.

34. The given curve is

$y^2 = 4a(x + a) \qquad$ ...(i)

$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$

$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx} \qquad$ ...(ii)

Eliminating a between Eqs (i) and (ii) we get

$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$

$\Rightarrow \quad y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$

Replacing dy/dx by $-dx/dy$, we get

$y^2 = 2xy\left(-\frac{dx}{xy}\right) + y^2\left(\frac{dx}{xy}\right)^2$

$\Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 + 2xy\left(\frac{dy}{dx}\right) - y^2 = 0$

$\Rightarrow \quad y^2p^2 + 2xyp - y^2 = 0,\ p = \frac{dy}{dx}$

$\Rightarrow \quad (yp + x)^2 = x^2 + y^2$

$\Rightarrow \quad (yp + x) = \sqrt{x^2 + y^2}$

$\Rightarrow \quad x + y\frac{dy}{dx} = \sqrt{x^2 + y^2}$

$\Rightarrow \quad \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = dx$

$\Rightarrow \quad \frac{2xdx + 2ydy}{\sqrt{x^2 + y^2}} = 2dx$

$\Rightarrow \quad d\left(\sqrt{x^2 + y^2}\right) = 2dx$

Integrating, we get

$\left(\sqrt{x^2 + y^2}\right) = 2x + c$

which is the required orthogonal trajectory.

35. The equation of the tangent to the curve at (x, y) is

$Y - y = \frac{dy}{dx}(X - x)$

It is given that

$\left|\frac{0 - y + x\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right| = x$

$\Rightarrow \quad 2xy\frac{dy}{dx} = y^2 - x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{(y/x)^2 - 1}{2(y/x)}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \qquad \left(\text{Let } \frac{y}{\eta} = v\right)$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$

$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^2 + 1}{2v}$

$\Rightarrow \quad \frac{2v}{v^2 + 1}\,dv = -\frac{dx}{x}$

Integrating, we get

$\log|v^2 + 1| = \log c - \log|x|$

$\Rightarrow \quad (v^2 + 1) = \frac{c}{x}$

$\Rightarrow \quad x^2 + y^2 = cx$

which is the required solution.

Questions Asked in Past IIT-JEE Examination

3. The given differential equation is

$$\frac{dy}{dx} = \sin(10x + 6y) \quad ...(i)$$

Let $10x + 6y = v$

$$\Rightarrow 10 + 6\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{6}\left(\frac{dv}{dx} - 10\right)$$

$$\Rightarrow \frac{1}{6}\left(\frac{dv}{dx} - 10\right) = \sin v$$

$$\Rightarrow \frac{dy}{dx} = 6\sin v + 10$$

$$\Rightarrow \int \frac{dv}{(6\sin v + 10)} = \int dx$$

$$\Rightarrow \frac{1}{2}\int \frac{dv}{(3\sin v + 5)} = x + c$$

$$\Rightarrow \frac{1}{2}\int \frac{dv}{\left(\frac{3 \cdot 2\tan(v/2)}{1 + \tan^2(v/2) + 5}\right)} = x + c$$

$$\Rightarrow \frac{1}{2}\int \frac{\sec^2(v/2)\,dv}{(5 + 5\tan^2(v/2) + 6\tan(v/2))} = x + c$$

$$\Rightarrow \int \frac{dt}{(5 + 5t^2 + 6t)} = x + c,$$

(Let $\tan(v/2) = t$)

$$\Rightarrow \frac{1}{5}\int \frac{dt}{\left(t^2 + \frac{6}{5}t + 1\right)} = x + c$$

$$\Rightarrow \frac{1}{5}\int \frac{dt}{\left(\left(t + \frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right)} = x + c$$

$$\Rightarrow \frac{1}{5} \times \frac{5}{4}\tan^{-1}\left(\frac{5t + 3}{4}\right) = x + c$$

$$\Rightarrow \frac{1}{4}\tan^{-1}\left(\frac{5\tan(5x + 3y) + 3}{4}\right) = x + c$$

which is passing through origin so $c = \frac{1}{4}\tan^{-1}\left(\frac{3}{4}\right)$.

Thus, the equation of the curve is

$$\frac{1}{4}\tan^{-1}\left(\frac{5\tan(5x + 3y) + 3}{4}\right) = x + \frac{1}{4}\tan^{-1}\left(\frac{3}{4}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{5\tan(5x + 3y) + 3}{4}\right) = 4x + \tan^{-1}\left(\frac{3}{4}\right)$$

$$\Rightarrow \left(\frac{5\tan(5x + 3y) + 3}{4}\right) = \tan\left(4x + \tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$\Rightarrow (5x + 3y) = \tan^{-1}\left(\frac{4}{5}\tan\left(4x + \tan^{-1}\left(\frac{3}{4}\right)\right) - \frac{3}{5}\right)$$

4. A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at P is $a(y - 1) + (x - 1)$. If the slope of the tangent at any point on the curve is propor tional to the ordinate of the point, determine the equation of the curve. Also, otain the area bounded by the y-axis, the curve and the normal to the curve at P.

[IIT-1996]

5. A spherical rain drop evaporates at a rate proportional to its surface area at any instant t. The differential equation giving the rate of change of the radius of the rain drop is.......

[IIT-1997]

6. Let $u(x)$ and $v(x)$ satisfy the differential

equations $\frac{du}{dx} = p(x)u = f(x)$ and

$\frac{dv}{dx} + p(x)v = g(x)$, where $p(x)$, $f(x)$ and $g(x)$

are continuous functions.

If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$. Prove that any point (x, y) where $x > x_1$ does not satisfy the equation $y = u(x)$ and $y = v(x)$.

[IIT-1997].

7. The equation of the tangent at any point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets the axes at $A\left(x - y\frac{dy}{dx}, 0\right)$ and

$B\left(0, y - x\frac{dy}{dx}\right)$ respectively.

$\therefore$ Mid-point AB is $\left(\frac{1}{2}\left(x - y\frac{dx}{dy}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right)$.

It is given that

$$\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x \text{ and } \frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$$

$$\Rightarrow x - y\frac{dx}{dy} = 2x$$

$$\Rightarrow -y\frac{dx}{dy} = x$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|y| = \log c - \log|x|$$

$$\Rightarrow \quad xy = c$$

which is passing through (1, 1), so $c = 1$

Hence, the equation of the curve is

$$xy = 1$$

8. The given curve is

$$y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x + c_5}$$

$$\Rightarrow \quad y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{c_5} \cdot e^x$$

$$\Rightarrow \quad y = A\cos(x + c_3) - B \cdot e^x$$

Since the given curve has 3 arbitrary constants, so, the order of the differential equation represented by the given curve is 3.

9. The equation of the normal at any point $P(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x) \qquad \text{...(i)}$$

It is given that

$$\left| \frac{y + x\frac{dx}{dy}}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} \right| = |y|$$

$$\Rightarrow \quad \left(y + x\frac{dx}{dy}\right)^2 = y^2\left(1 + \left(\frac{dx}{dy}\right)^2\right)$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right)^2 (x^2 - y^2) + 2xy\frac{dx}{dy} = 0$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right) = 0, \ (x^2 - y^2)\frac{dx}{dy} + 2xy = 0$$

$$= 0, \ \frac{dx}{dy} = \frac{2xy}{(x^2 - y^2)}$$

Now, $\left(\frac{dx}{dy}\right) = 0 \Rightarrow x = c$

which is passing through (1, 1), so $c = 1$

Hence, the equation of the curve is $x = 1$

Also, $\quad \frac{dx}{dy} = \frac{2xy}{(x^2 - y^2)}$

$$= \frac{(x^2 - y^2)}{2xy}$$

$$= \frac{\left(1 - \left(\frac{y}{x}\right)^2\right)}{2\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = -\frac{1 - v^2}{2v}, \qquad (\text{Let } \ y = vx)$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1 - v^2}{2v} - v = \frac{-1 + v^2 - 2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1 + v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1 + v^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{2v}{1 + v^2}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|1 + v^2| = \log c - \log|x|$$

$$\Rightarrow \quad (1 + v^2) = \frac{c}{x}$$

$$\Rightarrow \quad x^2 + y^2 = cx$$

which is passes through (1, 1), so $c = 2$

Hence, the equation of the curve is

$$x^2 + y^2 = 2x$$

10. The given differential equation is

$$\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$$

$$\Rightarrow \quad y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y = px + p^2, \text{ where } p = \frac{dy}{dx}$$

which is a Clairauts differential equation

Thus, the solution is

$$y = cx + c^2$$

$$\Rightarrow \quad y = 2x + 4$$

11. The given curve is

$$y^2 = 2c(x + \sqrt{c}) \qquad \text{...(i)}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2c$$

$$\Rightarrow \quad c = y\frac{dy}{dx} \qquad \text{...(ii)}$$

From Eq. (i) and (ii), we get

$$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{y\frac{dy}{dx}}\right)$$

$$\Rightarrow \quad \left(\frac{y}{2}\frac{dx}{dy} - x\right)^2 = y\frac{dy}{dx}$$

$$\Rightarrow \quad \left(\frac{y}{2} - x\frac{dy}{dx}\right)^2 = y\left(\frac{dy}{dx}\right)^3$$

Thus, the order of the differential equation is 3.

12. The given expression is $x^2 + y^2 = 1$

Differentiating w.r.t. x, we get

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

Again differentiating, w.r.t. x, we get

$$1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad 1 + y'^2 + y \cdot y'' = 0$$

$$\Rightarrow \quad y \cdot y'' + (y')^2 + 1 = 0$$

13. The given differential equation is

$$(1 + t)\frac{dy}{dt} - ty = 1$$

$$\Rightarrow \quad \frac{dy}{dt} + \left(\frac{-t}{t+1}\right)y = \frac{1}{t+1}$$

which is a linear differential equation.

Thus, IF $= e^{-\int \frac{t}{t+1}dt} = e^{-\int \frac{(t+1)-1}{t+1}dt}$

$$= e^{\log(t+1)} - t = (t + 1)e^{-1}$$

Therefore, the solution is

$$y \cdot (t + 1)e^{-1} = \int e^{-1}dt + c$$

$$\Rightarrow \quad y \cdot (t + 1)e^{-1} = -e^{-t} + c$$

Put $t = 0$, $y = 1$, then $c = 2$

Hence, the equation of the curve is

$$y \cdot (t + 1)e^{-t} = -e^{-t} + 2$$

Put $t = 1$, then

$$2y \cdot e^{-1} = -e^{-1} + 2$$

$$\Rightarrow \quad 2y = -1 + 2e$$

$$\Rightarrow \quad y = e - \frac{1}{2}$$

14. A right circular cone with radius R and height H contains a liquid which evaporates at a rate propeortional to its surface in contact with air (proportonality constant $k > 0$). Find the after which the cone is empty

15. The given differential equation is

$$\frac{2 + \sin x}{y + 1}\left(\frac{dy}{dx}\right) = -\cos x$$

$$\Rightarrow \quad \frac{dy}{y+1} = \frac{-\cos x}{2 + \sin x}dx$$

$$\Rightarrow \quad \int \frac{dy}{y+1} = \int \frac{-\cos x}{2 + \sin x}dx$$

$$\Rightarrow \quad \log|y + 1| = \log c - \log|2 + \sin x|$$

$$\Rightarrow \quad (y + 1)(2 + \sin x) = c$$

Put $x = 0$, $y = 1$, then $c = 4$

Hence, the equation of the curve is

$$(y + 1)(2 + \sin x) = 4$$

When $x = \frac{\pi}{2}$, then

$$y + 1 = \frac{4}{3}$$

$$\Rightarrow \quad y = \frac{4}{3} - 1 = \frac{1}{3}$$

16. Given,

$$\frac{dy}{dx} = \frac{(x+1)^2 + (y-3)}{(x+1)}$$

$$= (x + 1) + \frac{y}{(x+1)} - \frac{3}{(x+1)}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{(x+1)} = (x + 1) - \frac{3}{(x+1)}$$

which is a linear differential equation.

Thus, IF $= e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = \frac{1}{(x+1)}$

Therefore, the solution is

$$\frac{y}{x+1} = \int\left(1 - \frac{3}{(x+1)^2}\right)dx$$

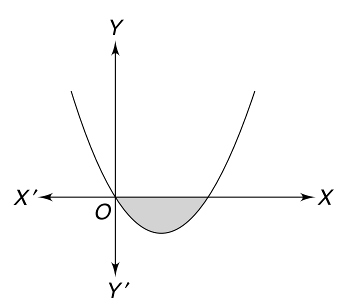
$$\Rightarrow \quad \frac{y}{y+1} = x + \frac{3}{x+1} + c$$

Put $x = 2$ and $y = 0$, then $c = -3$

Hence, the equation of the curve is

$$\frac{y}{x+1} = x + \frac{3}{x+1} - 3$$

$$\Rightarrow \quad y = x^2 + x + 3 - 3x - 3 = x^2 - 2x.$$



Hence, the required area

$$= \int_0^2 [0 - (x^2 - 2x)]\,dx$$

$$= \left(x^2 - \frac{x^3}{3}\right)_0^2$$

$$= \left(4 - \frac{8}{3}\right) = \frac{4}{3} \text{ sq.u.}$$

17. The given relation is

$$x\cos y + y\cos x = \pi$$

Put $x = 0$, then $y = \pi$

$$\cos y - x\sin y\frac{dy}{dx} - y\sin x + \cos x\frac{dy}{dx} = 0$$

Put $x = 0,\ y = \pi$, then $\frac{dy}{dx} = -\cos(\pi) = 1$

$$\Rightarrow \quad -\sin y\frac{dy}{dx} - \sin y\frac{dy}{dx} - x\cos y\left(\frac{dy}{dx}\right)^2$$

$$-x\sin y\left(\frac{d^2y}{dx^2}\right) - y\cos x - \sin\left(\frac{dy}{dx}\right)$$

$$-\sin x\left(\frac{dy}{dx}\right) + \cos x\left(\frac{d^2y}{dx^2}\right) = 0$$

Put $x = 0,\ y = \pi$ and $\frac{dy}{dx} = 1$, then

$$(-\sin\pi - \sin\pi - 1) + \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 1$$

18. The equation of tangent at any point (x, y) to the curve $y = f(x)$ is

$$Y - y = f'(x)(X - x)$$

It meets the x-axis at $\left(x - y\frac{dx}{dy}, 0\right)$.

It is given that

$$AP = 1$$

$$\Rightarrow \quad AP^2 = 1$$

$$\Rightarrow \quad \left(-y\frac{dx}{xy}\right)^2 + y^2 = 1$$

$$\Rightarrow \quad y^2\left(\frac{dx}{dy}\right)^2 = 1 - y^2$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right)^2 = \frac{1 - y^2}{y^2}$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right) = \sqrt{\frac{1 - y^2}{y^2}}$$

$$= \sqrt{\frac{y^2}{1 - y^2}}$$

$$\Rightarrow \quad \frac{\sqrt{1 - y^2}}{y}dy = dx$$

$$\Rightarrow \quad \int \frac{\sqrt{1 - y^2}}{y}\,dy = \int dx$$

$$\Rightarrow \quad \int \frac{y\sqrt{1 - y^2}}{y^2}dy = x + c$$

$$\Rightarrow \quad \int \frac{t^2}{t^2 - 1}\,dt = x + c, \qquad (\text{Let } t^2 = (1 - y^2))$$

$$\Rightarrow \quad \int \left(1 + \frac{1}{t^2 - 1}\right)dt = x + c$$

$$\Rightarrow \quad t + \frac{1}{2}\log\left|\frac{t - 1}{t + 1}\right| = x + c$$

$$\Rightarrow \quad \sqrt{1 - y^2} + \frac{1}{2}\log\left|\frac{\sqrt{1 - y^2} - 1}{\sqrt{1 - y^2} + 1}\right| = x + c$$

19. The given differential equation is

$$(x^2 + y^2)\,dy = xy\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} \qquad \text{...(i)}$$

which is a homogeneous differential equation.

$$\left(\text{Let } \ y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1 + v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1 + v^2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - v - v^3}{1 + v^2} = -\frac{v^3}{1 + v^2}$$

$$\Rightarrow \quad \left(\frac{1 + v^2}{v^3}\right)dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \int \left(\frac{1 + v^2}{v^3}\right)dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{2v^2} + \log|v| = c - \log|x|$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log\left|\frac{y}{x}\right| = c - \log|x|$$

$\Rightarrow \quad -\frac{x^2}{2y^2} + \log|y| = c$

When $x = 1, y = 1$, then $c = -\frac{1}{2}$

Thus, the equation of the curve is

$$-\frac{x^2}{2y^2} + \log|y| = -\frac{1}{2}$$

When $x = x_0, y = e$, then

$$-\frac{x_0^2}{2e^2} + 1 = -\frac{1}{2}$$

$\Rightarrow \quad -\frac{x_0^2}{2e^2} = -1 - \frac{1}{2} = -\frac{3}{2}$

$\Rightarrow \quad x_0^2 = 3e^2$

$\Rightarrow \quad x_0 = \sqrt{3}\,e$

20. The given differential equation is

$$y\,dx + y^2\,dy = x\,dy$$

$\Rightarrow \quad ydx - xdy = -y^2dy$

$\Rightarrow \quad \frac{ydx - xdy}{y^2} = -dy$

$\Rightarrow \quad d\left(\frac{x}{y}\right) = -dy$

$\Rightarrow \quad \int d\left(\frac{x}{y}\right) = -\int dy$

$\Rightarrow \quad \frac{x}{y} = c - y$

When $x = 1, y = 1$, then $c = 2$

Thus, the equation of the curve is

$$\frac{x}{y} = 2 - y$$

When $x = -3$, then

$$2 - y = -\frac{3}{y}$$

$\Rightarrow \quad y - 2 = \frac{3}{y}$

$\Rightarrow \quad y^2 - 2y - 3 = 0$

$\Rightarrow \quad (y - 3)(y + 1) = 0$

$\Rightarrow \quad y = 3, -1$

Since $y > 0$, so the value of $y = 3$.

21. The equation of any tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets the x-axis at $A\left(x - y\frac{dx}{dy}, 0\right)$

and $\quad B\left(0, y - x\frac{dy}{dx}\right)$

Since $P(x, y)$ divides the line AB in the ratio $3:1$, we get

$\Rightarrow \quad \frac{1}{4}\left(y - x\frac{dy}{dx}\right) = y, \frac{3}{4}\left(x - y, \frac{dx}{dy}\right) = x$

$\Rightarrow \quad x\frac{dy}{dx} = -3y$

$\Rightarrow \quad \frac{dy}{y} = -3\frac{dx}{x}$

$\Rightarrow \quad \int \frac{dy}{y} = -3\int \frac{dx}{x}$

$\Rightarrow \quad \log|y| = \log c - 3\log|x|$

$\Rightarrow \quad x^3y = c$

As $f(1) = 1$, we get $c = 1$

Thus, the equation of the curve is

$$x^3y = 1$$

$\Rightarrow \quad y = \frac{1}{x^3}$

which passes through (2, 1/8).

Also, $\quad \frac{dy}{dx} = -\frac{3}{x^4}$

$$m = \left(\frac{dx}{dy}\right)_{(1,1)} = -3$$

$\therefore$ The equation of normal at (1,1) is

$$y - 1 = \frac{1}{3}(x - 1)$$

$\Rightarrow \quad 3y - 3 = x - 1$

$\Rightarrow \quad x - 3y + 2 = 0$

22. The given differential equation is

$$\frac{dy}{dx} = \frac{2}{x + y}$$

Let $\quad x + y = v$

$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{2}{v}$

$\Rightarrow \quad \frac{dv}{dx} = \frac{v + 2}{v}$

$\Rightarrow \quad \left(\frac{v}{v + 2}\right)dv = dx$

$\Rightarrow \quad \int\left(1 - \frac{2}{v + 2}\right)dv = \int dx$

$\Rightarrow \quad v - 2\log|v + 2| = x + C$

$\Rightarrow \quad x + y - 2\log|x + y + 2| = x + C$

$\Rightarrow \quad y - 2\log|x + y + 2| = C$

When $x = 1$, $y = 1$, then

$\Rightarrow \quad c = 1 - 2\log 4 = \log\left(\frac{e}{16}\right)$

Thus, the equation of the curve is

$$y - 2\log|x + y + 2| = \log\left(\frac{e}{16}\right)$$

$\Rightarrow \quad y = \log|x + y + 2|^2 + \log\left(\frac{e}{16}\right)$

$\Rightarrow \quad \left((x + y + 2)^2\left(\frac{e}{16}\right)\right) = e^y$

$\Rightarrow \quad |(x + y + 2)^2 e^{-y})| = \left(\frac{16}{e}\right)$

23. The given differential equation is

$$\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}$$

$\Rightarrow \quad \frac{y\,dv}{\sqrt{1 - y^2}} = dx$

$\Rightarrow \quad \int\frac{y\,dv}{\sqrt{1 - y^2}} = \int dx$

Integrating, we get

$$-\sqrt{1 - y^2} = x + c$$

$\Rightarrow \quad (x + c)^2 + y^2 = 1$

which represents a circle with the centre $(c, 0)$ and the radius 1.

24. Given

$$\lim_{t\to x}\left(\frac{t^2 f(x) - x^2 f(t)}{t - x}\right) = 1$$

$\Rightarrow \quad \lim_{t\to x}\left(\frac{2tf(x) - x^2 f'(t)}{1}\right) = 1$

$\Rightarrow \quad 2xf(x) = x^2 f'(x) = 1$

$\Rightarrow \quad 2xy - x^2\frac{dy}{dx} = 1$

$\Rightarrow \quad x^2\frac{dy}{dx} - 2xy = -1$

$\Rightarrow \quad \frac{dy}{dx} - \frac{2}{x}y = -\frac{1}{x^2}$

which is a linear differential equation.

$\therefore \quad \text{IF} = e^{-2\int\frac{dx}{x}} = e^{-2\log x} = \frac{1}{x^2}$

Hence, the solution is

$$\frac{y}{x^2} = -\int\frac{dx}{x^4} + c$$

$\Rightarrow \quad \frac{y}{x^2} = \frac{1}{3x^3} + c$

As $\quad f(1) = 1$, then $c = 1 - 1/3 = 2/3$

Therefore, the equation of the curve is

$$\frac{y}{x^2} = \frac{1}{3x^3} + \frac{3}{2}$$

$\Rightarrow \quad y = \frac{1}{3x} + \frac{3x^2}{2}$

25. The given differential equation is

$$x\sqrt{x^2 - 1}\,dy - y\sqrt{y^2 - 1}\,dx = 0$$

$\Rightarrow \quad x\sqrt{x^2 - 1}\,dy = y\sqrt{y^2 - 1}\,dx$

$\Rightarrow \quad \frac{dy}{y\sqrt{y^2 - 1}} = \frac{dx}{x\sqrt{x^2 - 1}}$

$\Rightarrow \quad \int\frac{dy}{y\sqrt{y^2 - 1}} = \int\frac{dx}{x\sqrt{x^2 - 1}}$

$\Rightarrow \quad \sec^{-1}(y) = c + \sec^{-1}(x)$

When $\quad x = 2$, $y = \frac{2}{\sqrt{3}}$, then

$\Rightarrow \quad c = -\sec^{-1}(2) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$\Rightarrow \quad c = -\frac{\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{6}$

Thus, the equation of the curve is

$$\sec^{-1}(y) = \sec^{-1}(x) - \frac{\pi}{6} \qquad \text{...(i)}$$

$\Rightarrow \quad y = \sec\left(\sec^{-1}(x) - \frac{\pi}{6}\right)$

Thus, the statement (i) is trure

Also, we can write Eq. (i) as

$$\cos^{-1}\left(\frac{1}{y}\right) = \cos^{-1}\left(\frac{1}{x}\right) - \frac{\pi}{6}$$

$\Rightarrow \quad \frac{1}{y} = \cos\left(\cos^{-1}\left(\frac{1}{x}\right) - \frac{\pi}{6}\right)$

$\Rightarrow \quad \frac{1}{y} = \cos\left(\cos^{-1}\left(\frac{1}{x}\right)\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\cos^{-1}\left(\frac{1}{x}\right)\right)\sin\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \frac{1}{y} = \frac{1}{x}\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{x^2}}$

Thus, the statement (ii) is false.

26. Given differential equation is

$$y' = y + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = y + 1$$

$$\Rightarrow \quad \frac{dy}{y} + 1 = dx$$

$$\Rightarrow \quad \int \frac{dy}{y+1} = \int dx$$

$$\Rightarrow \quad \log|y + 1| = x + c$$

when $x = 0, y = 1$, then $c = \log 2$

Hence, the equation of the curve is

$$\log|y + 1| = x + \log 2$$

When $x = \ln 2$, then

$$y + 1 = 4 \Rightarrow y = 3$$

27. The given differential equation is

$$(x - 3)^2 y' + y = 0$$

$$\Rightarrow \quad (x-3)^2 \frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{(x-3)^2}$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{(x-3)^2}$$

$$\Rightarrow \quad \int \frac{dy}{y} = -\int \frac{dx}{(x-3)^2}$$

$$\Rightarrow \quad \log|y| = C + \frac{1}{x-3}$$

$$\Rightarrow \quad y = e^{C + \frac{1}{x-3}} = e^{\frac{1}{x-3}} \cdot e^c = A \cdot e^{\frac{1}{x-3}}$$

Thus, the domain of the above function is

$$R - \{3\}$$

28. The equation of tangent at any point (x, y) to the curve $y = f(x)$ is

$$Y - y = f'(x)(X - x)$$

Intercept on y-axis is

$$y - xf'(x)$$

It is given that,

$$y - xf'(x) = x^3$$

$$\Rightarrow \quad y - x\frac{dy}{dx} = x^3$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -x^2$$

which is a linear differential equation.

Thus $\quad \text{IF} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$

Hence, the solution is

$$\frac{y}{x} = -\int x\,dx + c$$

$$\Rightarrow \quad \frac{y}{x} = c - \frac{x^2}{2}$$

When $x = 1, y = 1$, then $c = 3/2$

Hence, the equation of the curve is

$$\frac{y}{x} = \frac{3}{2} - \frac{x^2}{2}$$

When $x = -3$, then

$$y = \frac{3x}{2} - \frac{x^3}{2} = -\frac{9}{2} + \frac{27}{2} = \frac{18}{2} = 9$$

29. The given differential equation is

$$y'(x) + y(x)g'(x) = g(x)g'(x)$$

$$\Rightarrow \quad \frac{dy}{dx} + g'(x)y = g(x)g'(x)$$

which is a linear differential equation.

Thus. $\quad \text{IF} = e^{\int g'(x)\,dx} = e^{g(x)}$

Therefore, the solution is

$$y \cdot e^{g(x)} = \int (g(x) \int g'(x)) e^{g(x)} dx$$

$$\Rightarrow \quad y \cdot e^{g(x)} = e^{g(x)}(g(x) - 1) + c$$

When $x = 0, y = 0$, then $c = 1$

Thus, the curve is

$$y \cdot e^{g(x)} = e^{g(x)}(g(x) - 1) + 1$$

When $x = 2$, then $y = (0 - 1) + 1 = 0$.

30. Given equation is

$$6\int_0^x f(t)\,dt = 3xf(x) - x^3 - 5$$

$$\Rightarrow \quad 6f(x) = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow \quad 3f(x) = 3xf'(x) - 3x^2$$

$$\Rightarrow \quad f(x) = xf'(x) - x^2$$

$$\Rightarrow \quad y = x\frac{dy}{dx} - x^2$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = x$$

which is a linear differential equation

$$\text{IF} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

Hence, the solution is

$$\frac{y}{x} = \int dx + c$$

$$\Rightarrow \quad \frac{y}{x} = x + c$$

When $x = 1$, $y = 2$, then $c = 1$

Hence, the equation of the curve is

$$\frac{y}{x} = x + 1$$

when $x = 2$, then $y = 6$.

Thus, the value of $f(2)$ is 6.

31. The given differential equation is

$$y' - y\tan x = 2x\sec x$$

$$\Rightarrow \quad \frac{dy}{dx} - \tan x \cdot y = 2x\sec x$$

which is a linear differential equation.

Thus, $\text{IF} = e^{-\int \tan x\, dx} = e^{\log\cos x} = \cos x$

Hence, the solution is

$$y\cdot\cos x = \int 2x\cdot\sec x\cdot\cos x\, dx + c$$

$$\Rightarrow \quad y\cdot\cos x = \int 2x\cdot dx + c$$

$$\Rightarrow \quad y\cdot\cos x = x^2 + c$$

when $x = 0$, $y = 0$, then $c = 0$

Thus, the equation of the curve is

$$y\cdot\cos x = x^2$$

When $x = \frac{\pi}{4}$, then $y\cdot\frac{1}{\sqrt{2}} = \frac{\pi^2}{16}$

$$\Rightarrow \quad y = \frac{\pi^2}{8\sqrt{2}}$$

When $x = \frac{\pi}{3}$, then $y = \frac{2\pi^2}{9}$

Also, $\frac{dy}{dx} - \tan\left(\frac{\pi}{3}\right)\cdot\left(\frac{2\pi^2}{9}\right) = 2\cdot\frac{\pi}{3}\cdot 2$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{3}\cdot\left(\frac{2\pi^2}{9}\right) + \frac{4\pi}{3}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\pi^2}{3\sqrt{3}} + \frac{4\pi}{3}$$

Again, $\frac{dy}{dx} - \tan\left(\frac{\pi}{4}\right)\cdot\left(\frac{\pi^2}{8\sqrt{2}}\right) = 2\cdot\frac{\pi}{4}\cdot\sqrt{2}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\pi^2}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}$$

32. Given $\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$ ...(i)

which is a homogeneous differential equation.

Let $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v + \sec v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sec v$$

$$\Rightarrow \quad \frac{dv}{\sec v} = \frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{dv}{\sec v} = \int\frac{dx}{x}$$

$$\Rightarrow \quad \int\cos v\, dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \sin(v) = \log|x| + C$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \log|x| + C$$

which is passing through $\left(1, \frac{\pi}{6}\right)$, so, $c = 1/2$

Hence, the equation of the curve is

$$\sin\left(\frac{y}{x}\right) = \log|x| + \frac{1}{2}$$

33. The given differential equation is

$$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{xy}{1 - x^2} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{-\int\frac{x}{1 - x^2}dx} = e^{\frac{1}{2}\log(1 - x^2)} = \sqrt{1 - x^2}$$

Hence, the solution is

$$y\cdot\sqrt{1 - x^2} = \int(x^4 + 2x)\, dx + c$$

$$\Rightarrow \quad y\cdot\sqrt{1 - x^2} = \frac{x^5}{5} + x^2 + c$$

which is passing through (0, 0), so $c = 0$

Hence, the equation of the curve is

$$y\cdot\sqrt{1 - x^2} = \frac{x^5}{5} + x^2$$

$$\Rightarrow \quad y = \frac{x^5}{5\sqrt{1 - x^2}} + \frac{x^2}{\sqrt{1 - x^2}}$$

Now, $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\left(\frac{x^5}{5\sqrt{1 - x^2}} + \frac{x^2}{\sqrt{1 - x^2}}\right)dx$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{x^2}{\sqrt{1-x^2}}\right) dx$$

$$= 2\cdot\int_{0}^{\frac{\sqrt{3}}{2}} \left(\frac{x^2}{\sqrt{1-x^2}}\right) dx$$

$$= 2\cdot\int_{0}^{\frac{\pi}{3}} \sin^2\theta d\theta, \quad (\text{Let } x = \sin\theta)$$

$$= \int_{0}^{\frac{\pi}{3}} (1-\cos 2\theta)\, d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi}{3} - \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

34. We have $(1+e^x)y' + ye^x = 1$

$$\Rightarrow \quad (1+e^x)\frac{dy}{dx} + y\cdot e^x = 1$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{e^x}{(1+e^x)}\right)\cdot y = \frac{1}{1+e^x}$$

which is a linear differential equation.

Thus, $\quad \text{IF} = e^{\int \frac{e^x}{(1+e^x)}dx} = e^{\log|1+ex|} = (1+e^x)$

Multiplying both sides of Eq. (i) by IF and integrating, we get

$$\Rightarrow \quad y\cdot(1+e^x) = \int dx + c$$

$$\Rightarrow \quad y\cdot(1+e^x) = x + c$$

When $\quad x = 0,\ y = 2$, then $c = 4$

Hence, the solution is

$$y\cdot(1+e^x) = x+4$$

$$\Rightarrow \quad y = \frac{x+4}{(1+e^x)}$$

Thus, $\quad y(-4) = 0$

$$\Rightarrow \quad y' = \frac{(1+e^x)-(x+4)e^x}{(1+e^x)^2}$$

Let $\quad g(x) = \dfrac{(1+e^x)-(x+4)e^x}{(1+e^x)^2}$

Clearly, $\quad g(0) = \dfrac{2-4}{2^2} < 0$

and $\quad g(-1) = \dfrac{\left(1+\frac{1}{e}\right)-\frac{3}{e}}{\left(1+\frac{1}{e}\right)^2} = \dfrac{\left(1-\frac{2}{e}\right)}{\left(1+\frac{1}{e}\right)^2} > 0$

$$g(0)\cdot g(-1) < 0$$

Hence, $g(x)$ has a root in between $(-1, 0)$.

35. Let the family of circles be

$$x^2+y^2-ax-ay+c=0$$

Differentiating w.r.t. x, we get

$$2x+2y\frac{dy}{dx} - a - a\frac{dy}{dx} = 0 \quad \text{...(i)}$$

Again differentiating w.r.t. x, we get

$$2+2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - a\frac{d^2y}{dx^2} = 0 \quad \text{...(ii)}$$

Eliminating a between Eqs (i) and (ii), we get

$$2+2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - \left(\frac{2x+2y\frac{dy}{dx}}{1+\frac{dy}{dx}}\right)\frac{d^2y}{dx^2} = 0$$

$$\therefore \quad \left(\frac{(x+2)^2}{3}\right) > \frac{4}{3} > 1$$

$$\Rightarrow 2+2yy''+2(y')^2 - \left(\frac{2x+2yy'}{1+y'}\right)y'' = 0$$

$$\Rightarrow 1+yy''+(y')^2 - \left(\frac{x+yy'}{1+y'}\right)y'' = 0$$

$$\Rightarrow (y-x)y'' + (1+y'+y'^2)y' + 1 = 0$$

Thus, $\quad P = (y-x)$ and $Q = (1+y'+y'^2)$

and $\quad P+Q = (-x+1+y+y'+y'^2)$

36. Given curve is

$$(x^2+xy+4x+2y+4)\frac{dy}{dx} - y^2 = 0,\ x > 0$$

$$\Rightarrow \quad \{(x+2)^2 + y(x+2)\}\frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow \quad \{(x+2)^2 + y(x+2)\}\frac{dy}{dx} = y^2$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{y\,d(x+2)-(x+2)\,dy}{(x+2)^2}$$

$$\Rightarrow \quad \frac{dy}{y} = -d\left(\frac{y}{(x+2)}\right)$$

Integrating, we get

$$\log|y| = c - \frac{y}{(x+2)}$$

When $x = 1,\ y = 3$, then $c = 1 + \log 3$

Thus, $\log\left(\frac{y}{3}\right) + \frac{y}{(x+2)} = 1$

For $y = (x + 2)$, then

$$\log\left(\frac{x+2}{3}\right) = 0$$

$\Rightarrow$ $\left(\frac{x+2}{3}\right) = 1$

$\Rightarrow$ $x = 1$

For $y = (x + 2)^2$, then

$$\log\left(\frac{(x+2)^2}{3}\right) + (x+2) = 1$$

Now for $x > 0,\ \left(\frac{(x+2)^2}{3}\right) > \frac{4}{3} > 1$

$$\log\left(\frac{(x+2)^2}{3}\right) + (x+2) > 2$$

So, it has no solution.

CHAPTER

5 Vectors

CONCEPT BOOSTER

1.1 Introduction

A study of motion involves the introduction of a variety of quantities that are used to describe the physical world. For examples, distance, displacement, speed, velocity, acceleration, force, mass, momentum, energy, work, power, etc. All these quantities can by divided into two categories—vectors and scalars. A *vector quantity* is a quantity that is fully described by both magnitude and direction. On the other hand, a *scalar quantity* is a quantity that is described by its magnitude only. The emphasis of this unit is to understand some fundamentals about vectors and to apply the fundamentals in order to understand motion and forces that occur in two dimensions.

1.2 Physical Quantities

The physical quantity is divided into two categories:

(a) scalar quantities
(b) vector quantities

(a) Scalar Quantities

A quantity which has only magnitude and not related to any fixed direction in the space is called the scalar quantity.

For example, mass, length, volume, density, time, temperature, etc., are all scalar quantities.

If the unit of measurement is defined, a real number is sufficient to represent a scalar quantity. Thus, in this chapter, we shall represent scalars by real numbers.

(b) Vector Quantities

A quantity which has magnitude as well as direction is called a vector.

For exmaple, force, velocity, acceleration, displacement, momentum, etc. are all vector quantities.

Notes: Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

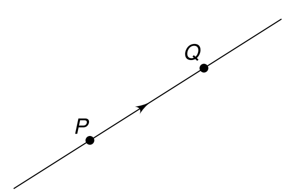
For Example, the rotation of a rigid body through finite angles have both magnitude and direction but do not satisfy the vector law of addition. Therefore, it is not a vector.

1.3 Representation of Vectors

Directed Line Segment

Any given protion of a given straight line where the two end-points are distinguished as the initial and the terminal points is called a directed line segment.

The directed line segment with the initial point P and the terminal point Q is denoted by the symbol **PQ** or $\overrightarrow{PQ}$.



The two end-points P and Q are not interchangeable.

A directed line segment is called a vector if it has three following characteristics.

(i) **Length:** The length of $\overrightarrow{PQ}$ is denoted by the symbol $|\overrightarrow{PQ}|$.

(ii) **Support:** The line of unlimited length of which a directed line segment is a part is called the *support*.

(iii) **Sense:** The sense of the directed line segment is from its initial point to its terminal point.

We genrally denote the vector by bold letter or by a single letter with an arrow or by a letter with a bar over its head,

i.e. **a**, $\vec{a}$, $\bar{a}$ denote the vector $\overrightarrow{PQ}$.

1.4 Types of Vectors

(i) Zero or Null Vector

A vector whose initial or terminal points are identical or coincident, is called a zero vector. It is also known as *null vector*. It is denoted by 0 and its direction is indeterminate.

Thus $\overrightarrow{AA}$, $\overrightarrow{BB}$ and $\overrightarrow{CC}$ are zero vectors. Geometrically, it represents a point.

(ii) Proper Vector

Any non-zero vector is called a proper vector.

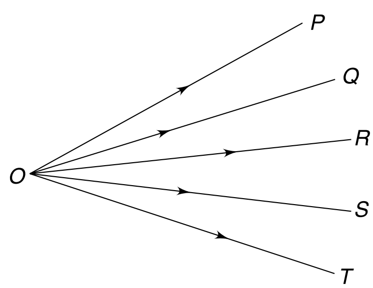
(iii) Unit Vector

If the magnitude of a vector is unity, it is called a unit vector.

If $|\vec{a}| = 1$, then **a** is called a unit vector. It is generally denoted as $\hat{a}$ and is defined as $\hat{a} = \frac{a}{|\vec{a}|}$.

(iv) Co-initial Vectors

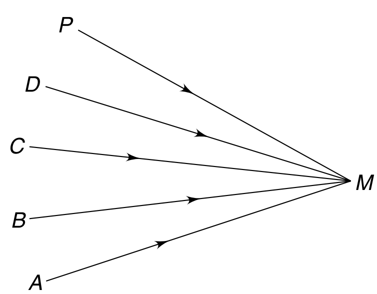
Two or more vectors are said to be co-initial vectors, if they have the same initial point.



Here, $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{OS}$, $\overrightarrow{OT}$ are co-initial vectors.

(v) Co-terminal Vectors

Two or more vectors are said to be co-terminal vectors if they have the same terminal point.



Here, $\overrightarrow{PM}$, $\overrightarrow{DM}$, $\overrightarrow{CM}$, $\overrightarrow{BM}$, $\overrightarrow{AM}$ are co-terminal vectors.

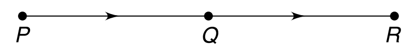
(vi) Collinear or Parallel Vectors

Two vectors are said to be collinear vectors if their supports are parallel or the same irrespective of their direction.

Collinear vectors are also called *parallel vectors*.

Like Vectors

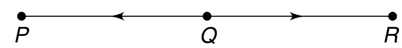
Collinear vectors are called like vectors if their direction are the same.



Here, $\overrightarrow{PQ}$, and $\overrightarrow{QR}$ vectors are like vectors.

Unlike Vectors

Collinear vectors are called unlike vectors if their directions are opposite.



Here, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are unlike vectors.

Difference between parallel and collinear vectors

Every collinear vectors are parallel, whereas every parallel vectors need not be a collinear.

Notes

1. Two non-zero vectors **a** and **b** are collinear (or parallel) if $\mathbf{a} = \lambda \mathbf{b}$, $\forall\ \lambda \in R$.
2. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two collinear (or parallel) vectors, then
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

(vii) Coplanar Vectors

Three or more vectors are said to be coplanar if they lie in the same plane, or they are parallel to the same plane.Three vectors **a**, **b** and **c** are coplanar if any one of their is a linear combination of the other two vectors.

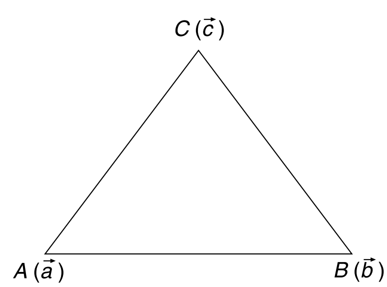
i.e $\quad \mathbf{a} = x\mathbf{b} + y\mathbf{c}$

or $\quad \mathbf{b} = x\mathbf{a} + y\mathbf{c}$

or $\quad \mathbf{c} = x\mathbf{a} + y\mathbf{b}$

(viii) Free Vector

If the origin of a vector is not specified, it is called a free vector.



Here, $\quad \overrightarrow{OA}, = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

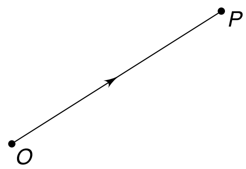
Thus $\vec{a}$, $\vec{b}$, and $\vec{c}$ are free vectors.

(ix) Localized Vector

For a vector of given magnitude and direction, if its initial point is fixed in space, the vector is called a localized vector.

(x) Position Vector

Let O be the origin and P be a point in the space. The position vector of P is **OP**.



If **a** and **b** are the position vectors of the points A and B, then

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}.$$

(xi) Negative of a Vector

The vector, which has the same magnitude as the vector **a** but has the direction opposite to that of **a**, is called the negative of **a** and is written as – **a**.

(xii) Equality of Two Vectors

Two vectors are said to be equal if they have

(a) the same length
(b) the same or parallel supports
(c) the same sense.

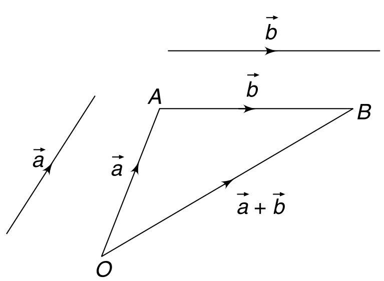
1.5 Algebra of Vectors

It is possible to develop an algebra of vectors which is useful in the study of geometry, mechanics and other branches of applied mathematics.

1.5.1 Addition of Vectors

Let **a** and **b** be two given vectors. Take a point O in the space.

Let $\overrightarrow{OA}$, = **a**, $\overrightarrow{AB}$ = **b**, so that the terminal point of $\vec{a}$ is the initial point of $\vec{b}$.



The vector $\overrightarrow{OB}$ is defined as the vector sum (or resultant) of $\vec{a}$ and $\vec{b}$.

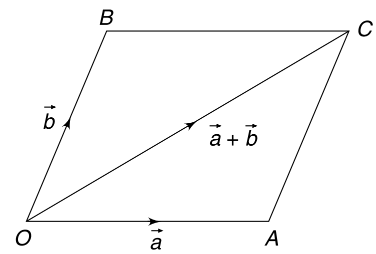
Thus, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$ which is known as the triangle law of vector addition.

It is also stated as, If two vectors are represented by the two sides of a triangle taken in order, their sum is represented by the third side of the triangle, taken in reverse order.

Parallelogram Law of Addition of Vectors

Let the parallelogram $OACB$, where

$$\overrightarrow{OA} = \vec{a} \text{ and } \overrightarrow{OB} = \vec{b}$$



Thus, $$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB}$$
$$= \overrightarrow{OA} + \overrightarrow{OB}$$
$$= \vec{a} + \vec{b}.$$

This method of addition of two vectors is called the parallelogram law of addition vectors.

Properties of Addition of Vectors

(i) Addition of vectors is commutative, i.e.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(ii) Addition of vectors is associative, i.e.

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

(iii) Additive identity exists, i.e.

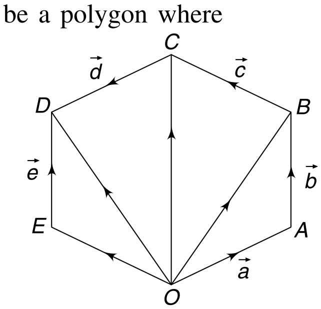
$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$$

(iv) Additive inverse also exists, i.e.

$$\vec{a} + (\overrightarrow{-a}) = \vec{0}\ (\overrightarrow{-a}) + \vec{a}$$

Polygon Law of Vector Addition

Let OABCDE be a polygon where



$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{c}$, $\overrightarrow{CD} = \vec{d}$ and $\overrightarrow{DE} = \vec{e}$.

Applying the triangle law, we have.

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$$

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = (\vec{a} + \vec{b}) + \vec{c}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = (\vec{a} + \vec{b} + \vec{c}) + \vec{d}$$

$$\overrightarrow{OE} = \overrightarrow{OD} + \overrightarrow{DE} = (\vec{a} + \vec{b} + \vec{c} + \vec{d}) + \vec{e}$$

Thus, to find the sum of more than two vectors, a polygon is formed.

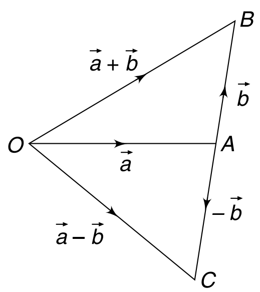
Therefore, this method is known as the polygon law of vector addition.

Note: If the initial point of the first vector and the final point of the last vector are the same, the sum of the vectors is a zero vector.

1.5.2 Subtraction of Vectors

If **a** and **b** be any two vectors, the subtraction of **b** from **a** is defined as the addition of **–b** to **a** and is written as **a** + (–**b**) = **a** – **b**, i.e.

$$\vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$$



1.5.3 Multiplication of a Vector by a Scaler

Let m be any scaler and **a** be any vector. Then its product $m\vec{a}$ is called the multiplication of a vector by a scaler.

If $\vec{a}$ and $\vec{b}$ be two vectors and m, n are scalers, then

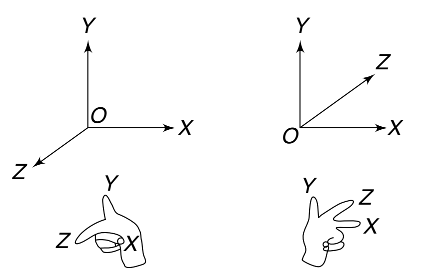
(i) $m(\vec{a}) = m\vec{a} = (\vec{a})m$

(ii) $m(n\vec{a}) = (mn)\vec{a}$

(iii) $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

(iv) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

1.6 Left and Right Handed Orientation



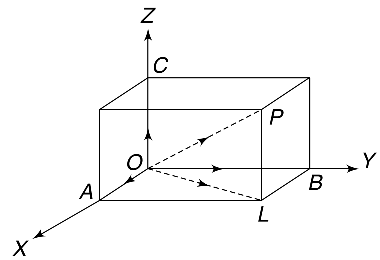
Right Handed System   Left Handed System

(i) If the rotation from OX to OY is in the anti-clockwise direction and OZ is directed upwards, the system is called the right handed system.

(ii) If the rotation from OX to OY is in the clockwise direction and OZ is directed upwards, the system is called the left handed system.

1.7 Position Vector of a Point in a Space

Let O be a fixed point, known as the origin, and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis, respectively, in such a way that they form a right handed system.



The plane XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane, respectively.

Let P be a point in a space such that its distances from yz-, zx- and xy-plane be a, b, c respectively and $\vec{i}$, $\vec{j}$, $\vec{k}$ are the vectors along x, y and z axes, respectively.

Let $OA = a$, $OB = b$ and $OC = c$

Now, $\mathbf{OP} = \mathbf{OL} + \mathbf{LP}$

$= \mathbf{OA} + \mathbf{AL} + \mathbf{LP}$

$= \mathbf{OA} + \mathbf{OB} + \mathbf{OC}$.

$= a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

and $|\mathbf{OP}| = \sqrt{a^2 + b^2 + c^2}$.

1.8 Linear Combination

A vector **r** is said to be a linear combination of the vectors **a, b, c,** ..., if there exist scalars x, y, z, ... such that

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$$

1.9 Linearly Dependent Vectors

A system of vectors $\mathbf{a_1}$, $\mathbf{a_2}$, ..., $\mathbf{a_n}$ is said to be linearly dependent if there exist scalars x_1, x_2, ..., x_n (not all zero) such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}.$$

1.10 Linearly Independent Vectors

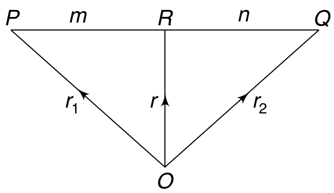
A system of vectors $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$ is said to be linearly independent if there exist scalars x_1, x_2,...., x_n (all zero) such that $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \mathbf{0}$

1.11 Section Formulae

1.11.1 Internal Section

If a point $\mathbf{R}(\mathbf{r})$ divides the line segment joining the points $\mathbf{P}(\mathbf{r_1})$ and $\mathbf{Q}(\mathbf{r_2})$ internally in the ratio $m:n$, then

$$\mathbf{r} = \frac{m\mathbf{r}_2 + n\mathbf{r}_1}{m + n}.$$



From the figure, we have by using triangle rule,

$$\frac{\mathbf{PR}}{\mathbf{RQ}} = \frac{m}{n}$$

$$\Rightarrow \quad \frac{\mathbf{OR} - \mathbf{OP}}{\mathbf{OQ} - \mathbf{OR}} = \frac{m}{n}$$

$$\Rightarrow \quad \frac{\mathbf{r} - \mathbf{r}_1}{\mathbf{r}_2 - \mathbf{r}} = \frac{m}{n}$$

$$\Rightarrow \quad n(\mathbf{r} - \mathbf{r}_1) = m(\mathbf{r}_2 - \mathbf{r})$$

$$\Rightarrow \quad \mathbf{r}(m + n) = m\mathbf{r}_2 + n\mathbf{r}_1$$

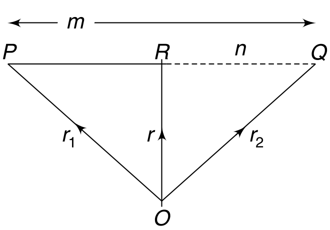
$$\Rightarrow \quad \mathbf{r} = \frac{m\mathbf{r}_2 + n\mathbf{r}_1}{m + n}$$

1.11.2 External Section

If a point **R(r)** divides the line segment joining the points $\mathbf{P}(\mathbf{r}_1)$ and $\mathbf{Q}(\mathbf{r}_2)$ externally in the ratio $m:n$, then

$$\mathbf{r} = \frac{m\mathbf{r}_2 - n\mathbf{r}_1}{m - n}.$$

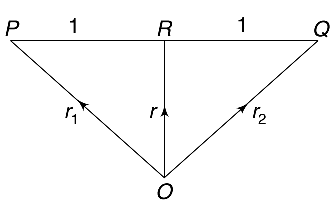
It can be proved by using triangle rule.



1.11.3 Mid-Point Formula

If a point **R(r)** divides the line segment joining the points $\mathbf{P}(\mathbf{r}_1)$ and $\mathbf{Q}(\mathbf{r}_2)$ internally in the ratio 1 : 1, then

$$\mathbf{r} = \frac{\mathbf{r}_2 + \mathbf{r}_1}{\mathbf{2}}.$$

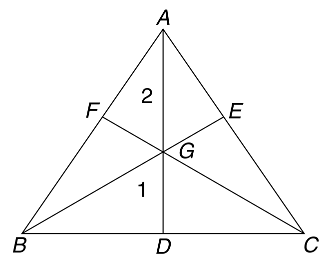


1.11.4 Centroid

The point of intersection of the medians of a triangle is called its centroid.

Let $A(\mathbf{r}_1)$, $B(\mathbf{r}_2)$ and $C(\mathbf{r}_3)$ be the vertices of a triangle ABC and $G(r)$ be its cen troid. Then the position vector **r** of the centroid is given by

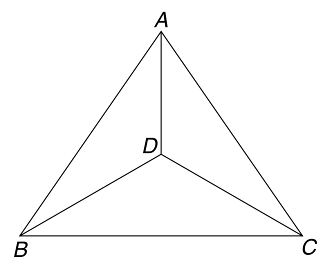
$$\mathbf{r} = \frac{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3}{3}.$$



1.11.5 Centroid of a Tetrahedron

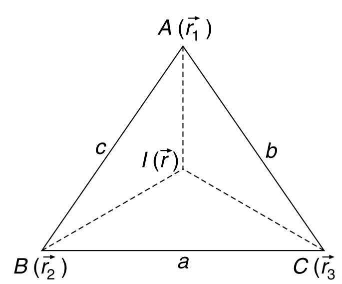
Let the position vectors of the points A, B, C and D are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. The position vector **r** of the centroid is

$$\mathbf{r} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$$



1.11.6 Incentre

The point of intersection of the angle bisectors of a triangle is called its incentre.



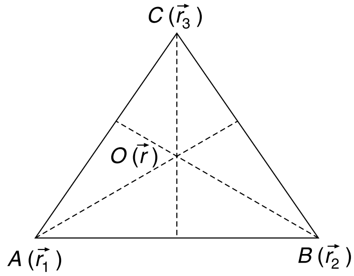
Let the position vectors of the points A, B, C are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively, and **r** be the position vector of the incentre I. The position vector **r** of the incentre is given by

$$\mathbf{r} = \frac{a\mathbf{r}_1 + b\mathbf{r}_2 + c\mathbf{r}_3}{a + b + c}$$

1.11.7 Circumcentre

The point of intersection of the perpendicular bisector of a triangle is called its circumcentre.

Let the position vectors of the points A, B, C are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively, and $\mathbf{r}$ be the position vector of the circumcentre O. The position vector $\mathbf{r}$ of the incentre is given by

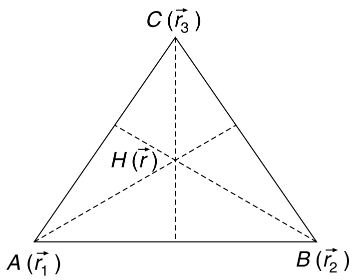


$$\mathbf{r} = \frac{(\sin 2A)\mathbf{r}_1 + (\sin 2B)\mathbf{r}_2 + (\sin 2C)\mathbf{r}_3}{\sin 2A + \sin 2B + \sin 2C}$$

1.11.8 Orthocentre

The point of intersection of the perpendiculars of a triangle is called its orthocentre.

Let the position vectors of the points A, B, C are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively and $\mathbf{r}$ be the position vector of the orthocentre H. The position vector of the orthocentre is

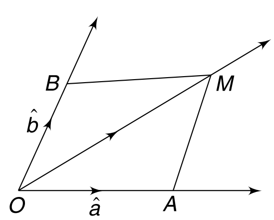


$$\mathbf{r} = \frac{(\tan A)\mathbf{r}_1 + (\tan B)\mathbf{r}_2 + (\tan C)\mathbf{r}_3}{\tan A + \tan B + \tan C}$$

1.12 Bisector of Angle between Vectors A and B

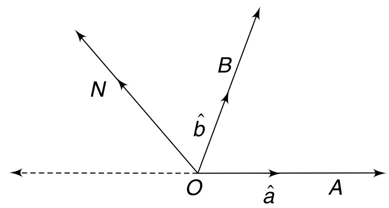
(i) If **a** is not parallel to **b**, the vectors along the bisectors (internal) of angle between **a** and **b** is given by

$$\overrightarrow{OM} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$$



(ii) If **a** is not parallel to **b**, the vectors along the bisectors (external) of the angle the between **a** and **b** is given by

$$\overrightarrow{ON} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} - \frac{\vec{b}}{|\vec{b}|}\right)$$

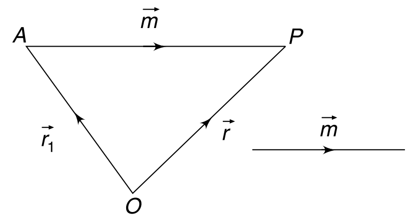


1.13 Straight Line

1.13.1 Equation of a Line Passing Thorugh a Point and Parallel to a Vector

The equation of a line passing through a point A with position vector $\mathbf{r}_1$ and parallel to a vector **m** is given by

$$\mathbf{r} = \mathbf{r}_1 + \lambda\mathbf{m}$$



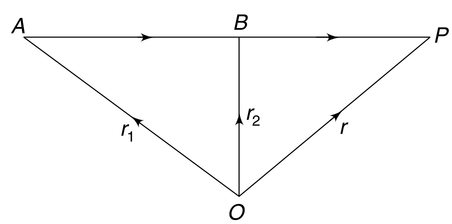
Now, $\quad \mathbf{OP} = \mathbf{OA} + \mathbf{AP}$

$\Rightarrow \quad \mathbf{OP} = \mathbf{OA} + \lambda\mathbf{M}$

$\Rightarrow \quad \mathbf{r} = \mathbf{r}_1 + \lambda\mathbf{m}$

1.13.2 Equation of a Line Passing Through Two Points A($\mathbf{r}_1$) and B($\mathbf{r}_2$) is

$$\mathbf{r} = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1)$$



Now, $\quad \mathbf{OP} = \mathbf{OA} + \mathbf{AP}$

$\Rightarrow \quad \mathbf{OP} = \mathbf{OA} + \lambda\,\mathbf{AB}$

$\Rightarrow \quad \mathbf{OA} + \lambda(\mathbf{OB} - \mathbf{OA})$

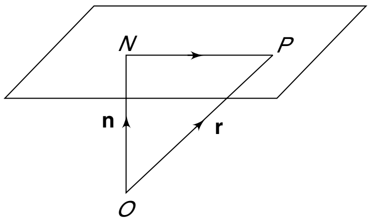
$\Rightarrow \quad \mathbf{r} = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1)$.

1.14 Plane

1.14.1 Vector Equation

The vector equation of a plane passing through a point having position vector **a** and the normal to a vector **n** is given by

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{d}$$



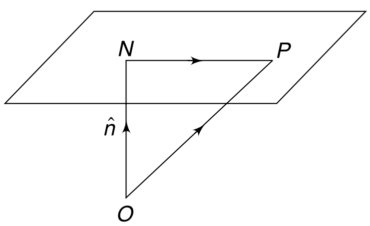
Let $\quad$ **OP** = **r** and **ON** = **n**

It is given that

$$\mathbf{ON} \perp \mathbf{NP}$$
$$\Rightarrow \quad \mathbf{NP}\cdot\mathbf{ON} = 0$$
$$\Rightarrow \quad (\mathbf{OP} - \mathbf{ON})\cdot\mathbf{ON} = 0$$
$$\Rightarrow \quad (\mathbf{r} - \mathbf{n})\cdot\mathbf{n} = 0$$
$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{n} - \mathbf{n}\cdot\mathbf{n} = 0$$
$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{n}$$
$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{n} = d.$$

1.14.2 Equation of a Plane in Normal Form

The vector equation of a plane normal to a unit vector and at a distance from the origin is $\mathbf{r}\cdot\hat{n} = d$.



Let O be the origin and ON be the perpendicular from O to the given plane such that $\mathbf{ON} = d\hat{n}$.

Let P be a point on the plane with position vector **r** so that

$$\mathbf{OP} = \mathbf{r}$$

Now, $\quad \mathbf{ON} \perp \mathbf{NP}$

$$\Rightarrow \quad \mathbf{NP}\cdot\mathbf{ON} = 0$$
$$\Rightarrow \quad (\mathbf{OP} - \mathbf{ON})\cdot\mathbf{ON} = 0$$
$$\Rightarrow \quad (\mathbf{r} - d\hat{n})\cdot d\hat{n} = 0$$
$$\Rightarrow \quad (\mathbf{r}\cdot d\hat{n} - d\hat{n}\cdot d\hat{n}) = 0$$
$$\Rightarrow \quad (\mathbf{r}\cdot\hat{n} - d) = 0$$
$$\Rightarrow \quad \mathbf{r}\cdot\hat{n} = d$$

which is the required equation of the plane.

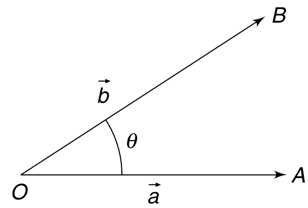
Scalar Product of Two Vectors

2.1 Definition

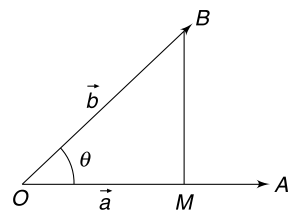
If the product of two vectors is a scalar, it is known as scaler product or dot product of two vectors.

Let **a** and **b** be two non-zero vectors and θ the angle between them. Its scalar product is denoted as $\mathbf{a}\cdot\mathbf{b}$ and is defined as

$$\mathbf{a}\cdot\mathbf{b} = ab\cos(\theta),\ 0 \le \theta \le \pi.$$



2.2 Geometrical Interpretation of a·b



We have
$$\mathbf{a}\cdot\mathbf{b} = ab\cos(\theta)$$
$$= a[b\cos(\theta)]$$
$$= a\,OM$$

Thus, $\quad OM = \dfrac{\mathbf{a}\cdot\mathbf{b}}{a}$

Projection of **b** on **a** is $\dfrac{\mathbf{a}\cdot\mathbf{b}}{a}$.

Thus, geometrically product of two vectors **a** and **b** is defined as the product of the magnitude of the first vector and the projection of the 2nd vector on the 1st vector.

2.3 Properties of Dot Product of Two Vectors

(i) Dot product is commutative, i.e.

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

Now
$$\mathbf{b}\cdot\mathbf{a} = ba\cos(-\theta)$$
$$= ba\cos\theta$$
$$= ab\cos\theta$$
$$= \mathbf{a}\cdot\mathbf{b}$$

(ii) If θ acute, then $\mathbf{a}\cdot\mathbf{b} > 0$

(iii) If θ obtuse, then $\mathbf{a}\cdot\mathbf{b} < 0$

(iv) If a and b are perpendiculars, then $\mathbf{a}\cdot\mathbf{b} = 0$

(v) If a and b are like vectors, then $\mathbf{a}\cdot\mathbf{b} = ab$

(vi) If a and b are unlike vectors, then $\mathbf{a}\cdot\mathbf{b} = -ab$

(vii) Maximum value of **a, b** is ab

(viii) Minimum value of $\mathbf{a}\cdot\mathbf{b}$ is $-ab$

(ix) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = \mathbf{a}^2$

(x) Projection of **a** on **b** = $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$.

(xi) Projection of **b** on **a** = $\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}$.

(xii) *Orthonormal triad vectors*

If three vectors which are mutually perpendicular to each other, they are called orthonormal triad vectors.

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be three unit vectors which are mutually perpendicular to each other.

Thus $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k}$

and $\hat{i}.\hat{j} = 0 = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k}$

Therefore,

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

(xiii) If $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $b = b_1\hat{i} + b_2\hat{j} + + b_3\hat{k}$,

then $\mathbf{a}\cdot\mathbf{b} = a_1b_2 + a_2b_2 + a_3b_3$

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Thus, $$\cos(\theta) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

(xiv) Any vector **a** can be written as

$$\mathbf{a} = (a\cdot\hat{i})\hat{i} + (a\cdot\hat{j})\hat{j} + (a\cdot\hat{k})\hat{k}$$

(xv) $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2\sum(\mathbf{a}\cdot\mathbf{b})$

(xvi) If **a, b, c** are coplanar vectors, then

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix} = 0$$

As we know that three coplanar vectors are always linearly dependent.

Thus, $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$ ...(i)

Taking dot product **a** and **b** with Eq. (i), we get

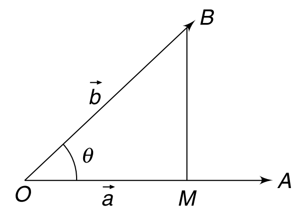
$x(\mathbf{a}\cdot\mathbf{a}) + y(\mathbf{a}\cdot\mathbf{b}) + z(\mathbf{a}\cdot\mathbf{c}) = 0$ ...(ii)

$x(\mathbf{b}\cdot\mathbf{a}) + y(\mathbf{b}\cdot\mathbf{b}) + z(\mathbf{b}\cdot\mathbf{c}) = 0$ ...(iii)

Eliminating x, y and z from Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix} = 0$$

2.4 Component of a Vector $\vec{B}$ along and Perpendicular to Vector $\vec{A}$



Here, $\mathbf{OB} = \mathbf{OM} + \mathbf{MB}$

$\mathbf{b} = \mathbf{OM} + \mathbf{MB}$

Component of a vector **b** along **a**

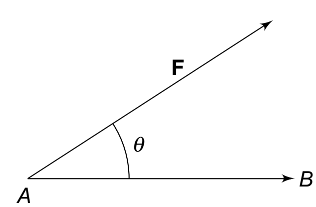
$$= \mathbf{OM}$$
$$= (OM)\hat{a}$$
$$= (OB\cos\theta)\hat{a}$$
$$= (b\cos\theta)\hat{a}$$
$$= \left(\frac{b\,\mathbf{a}\cdot\mathbf{b}}{ab}\right)\hat{a}$$
$$= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{a}\right)\hat{a}$$
$$= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{a}\right)\frac{\mathbf{a}}{a}$$
$$= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}.$$

Component of a vector **b** perpendicular to a vector **a**

$$= \mathbf{MB}$$
$$= \mathbf{OB} - \mathbf{OM}$$
$$= \mathbf{b} - \mathbf{OM}$$
$$= \mathbf{b} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

2.5 Physical Significance of the Dot Prodcut of Two Vectors

A force acting on a particle is said to be work done if the particle is displaced in a direction which is not perpendicular to the force applied.



Let a force **F** acts on a particle P and the particle P gets displaced from the position A to B, where AB is not perpendicular to the applied force **F**.

Let $AB = d$.

$$\therefore \text{ Work done} = \mathbf{F}\,\mathbf{d}(\cos\theta)$$
$$= \mathbf{F}\cdot\mathbf{d}$$
$$= (\text{Force})\cdot(\text{Displacement}).$$

***Note*:** The work done by the resultant of the number of forces $\mathbf{F_1}, \mathbf{F_2}, ..., \mathbf{F_n}$ in a displacement **d** of a particle is equal to the sum of the work done by the forces separately, i.e.

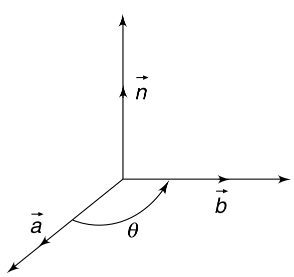
$$\text{Work done} = \mathbf{F_1}\cdot\mathbf{d} + \mathbf{F_2}\cdot\mathbf{d} + ... + \mathbf{F_n}\cdot\mathbf{d}.$$
$$= (\mathbf{F_1} + \mathbf{F_2} + ... + \mathbf{F_n})\cdot\mathbf{d}$$
$$= \mathbf{R}\cdot\mathbf{d}$$

where $\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} + ... + \mathbf{F_n}$

CROSS PRODUCT OF TWO VECTORS

3.1 Introduction

If the product of two vectors is a vector, it is known as the vector product or cross product or outer product of two vectors.

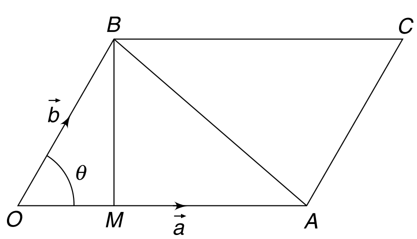


Let **a** and **b** be two vectors. The cross product of two vectors is denoted as $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ and is defined as $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (ab\sin\theta)\hat{n}$,

where θ the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and $\hat{n}$ is unit vector perpendicular to the plane of $\vec{\mathbf{a}}$ and **b**.

Here **a, b,** $\hat{n}$ form a right handed system.

3.2 Geometrical Interpretation of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$



Let $OACB$ be a parallelogram with O origin.

Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$ and $\angle AOB = \theta$

Area of a parallelogram $OACB$

$$= (\text{base})(\text{height})$$
$$= (OA)(BM)$$
$$= (a)(b\sin\theta)$$
$$= (ab\sin\theta)$$
$$= |\mathbf{a} \times \mathbf{b}|$$

Thus, $|\mathbf{a} \times \mathbf{b}|$ reprtesents the area of a parallelogram with adjacent sides represented by the vectors **a** and **b**.

3.3 Properties of Vector Product of Two Vectors

(i) Vector product is not commutative, i.e.
$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

(ii) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(iii) If **a** and **b** are parallel vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

(iv) If **a** and **b** are collinear vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

(v) $(m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) = m(\mathbf{a} \times \mathbf{b})$, where m is any scalar.

(vi) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} + \mathbf{c})$.

(vii) Unit vector perpendicular to the plane of **a** and **b** is

$$\hat{n} = \pm\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}.$$

(viii) A vector of the magnitude λ and perpendicular to the plane of **a** and **b** is

$$\pm\lambda\frac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}.$$

(ix) If θ be the angle between two vectors **a** and **b**, then

$$\sin(\theta) = \frac{|\mathbf{a} \times \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

(x) The area of ΔOAB is given by

$$\frac{1}{2}|\mathbf{a} \times \mathbf{b}|,$$

where

$\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$

(xi) The area of $\Delta ABC = \frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$, where **a**, **b** and **c** are the position vectors of the points A, B and C, respectively.

(xii) The area of a parallelogram $= \frac{1}{2}|\mathbf{d_1} \times \mathbf{d_2}|$,

where d_1 and d_2 are the diagonals of the parallelogram.

(xiii) If A, B and C are three collinear, points then
$(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} \times \mathbf{c} \times \mathbf{a}) = \mathbf{0},$
where **a**, **b** and **c** are the position vectors of the points A, B and C, respectively.

(xiv) A unit vector perpendicular to the plane of

$\angle ABC$ is given by

$$\hat{n} = \pm \frac{(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})}{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} + \mathbf{a}|}.$$

(xv) If **a, b** and $\mathbf{a} \times \mathbf{b}$ form a right handed system, any vector **r** is **a** linear combination of **a**, **b** and $\mathbf{a} \times \mathbf{b}$, i.e.

$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$,

where x, y and z are scalars.

(xvi) *Orthonormal Triad Vectors*

If three vectors which are mutually perpendicular to each other, they are called orthonormal triad vectors.

Let $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors which are mutually perpendicular to each other.

Thus $\hat{i} \times \hat{i} = \mathbf{0} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k}$

and $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

Therefore,

$\times$	$\hat{i}$,	$\hat{j}$,	$\hat{k}$
$\hat{i}$,	$\mathbf{0}$	$\hat{k}$	$-\hat{j}$
$\hat{j}$,	$-\hat{k}$	$\mathbf{0}$	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	$\mathbf{0}$

(xvii) If $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(xviii) *Lagranges identity*

For any two vectors **a** and **b**,

$$(\mathbf{a} \times \mathbf{b})^2 = (ab)^2 - (\mathbf{a} \cdot \mathbf{b})^2 = \begin{vmatrix} \mathbf{a.a} & \mathbf{a.b} \\ \mathbf{a.b} & \mathbf{b.b} \end{vmatrix}.$$

Scalar Triple Product of Vectors

4.1 Introduction

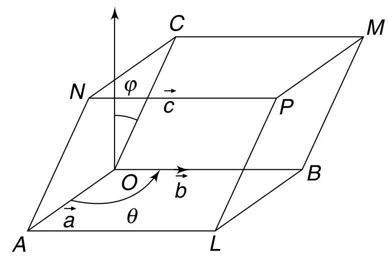
If the product of three vectors is a scalar, then it is known as scalar triple or box product or mixed product of three vectors.

Let a, b and c are three vectors. Its scalar triple product is denoted as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = abc \sin\theta \cos\varphi$$

where θ the angle between $\vec{a}$ and $\vec{b}$ and φ the angle between **c** and $\mathbf{a} \times \mathbf{b}$.

Geometricel interpretation of [**a**, **b**, **c**]



Here OA, OB and OC are the coterminous edges of a parallelopiped represented by the vectors **a, b** and **c** respectively.

We know that

$$\begin{aligned}[\mathbf{a}, \mathbf{b}, \mathbf{c}] &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= abc \sin\theta \cos\varphi \\ &= (ab\sin\theta)(c\cos\varphi) \\ &= |\mathbf{a} \times \mathbf{b}|(c\cos\varphi) \\ &= (\text{area of the II}^{\text{gm}}\ OALB) \times (\text{height}) \\ &= \text{Volume of a parallelopiped.}\end{aligned}$$

4.2 Properties of Scalar Triple Product of Vectors

(i) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$,

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

(iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$

i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$

(iv) $[k\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = k[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, where $k \in R$

(v) $[l\mathbf{a}\ m\mathbf{b}\ n\mathbf{c}] = lmn[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(vi) $[\mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{d}] = [\mathbf{a}, \mathbf{c}, \mathbf{d}] + [\mathbf{b}, \mathbf{c}, \mathbf{d}]$

(vii) If any two vectors are same, then

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

(viii) If any two vectors are parallel or collinear, then

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

(ix) If any one of them be a zero vector, then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

(x) If **a, b, c** are coplanar vectors, then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

(xi) If **a, b, c** form a right handed system, then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0$

(xii) If **a, b, c** form a left handed system, then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] < 0$

(xiii) For any three vectors **a, b** and **c,**
$[\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

(xiv) If **a, b, c** are coplanar vectors, then
a + b, + b + c, c + a are also coplanar vectors

(xv) If **a** and **b** are non-zero and non-collinear vectors, then
$[\mathbf{a}, \mathbf{b}, \mathbf{i}]\mathbf{i} + [\mathbf{a}, \mathbf{b}, \mathbf{j}]\mathbf{j} + [\mathbf{a}, \mathbf{b}, \mathbf{k}]\mathbf{k} = \mathbf{a} \times \mathbf{b}$

(xvi) For any three vectors **a, b, c,**

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

(xvii) If **a, b, c, m, n** are non-zero vectors, then

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\cdot[\mathbf{m} \times \mathbf{n}] = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \\ \mathbf{a}\cdot\mathbf{n} & \mathbf{b}\cdot\mathbf{n} & \mathbf{c}\cdot\mathbf{n} \end{vmatrix}$$

(xix) Any vector **r** can be expressed as a linear combination of three non-coplaner vectors **a, b, c,** then

$$\mathbf{r} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}$$

i.e. $$\mathbf{r} = \frac{[\mathbf{rbc}]}{[\mathbf{abc}]}\mathbf{a} + \frac{[\mathbf{rca}]}{[\mathbf{abc}]}\mathbf{b} + \frac{[\mathbf{rab}]}{[\mathbf{abc}]}\mathbf{c}$$

(xx) If four points **a**, **b**, **c** and **d** are coplanar vectors, then
$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{d}] + [\mathbf{b}\,\mathbf{c}\,\mathbf{d}] + [\mathbf{c}\,\mathbf{a}\,\mathbf{d}]$

VECTOR TRIPLE PRODUCT

5.1 Introduction

If the product of three vectors is also a vector, then it is known as vector triple product.

Let **a**, **b** and **c** be any three vectors, then its vector triple product is denoted as $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and is defined as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

5.2 Geometrical significance of $a \times (b \times c)$

Let $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

$\Rightarrow$ $\mathbf{r} \perp \mathbf{a}$ and $\mathbf{r} \perp (\mathbf{b} \times \mathbf{c})$

$\Rightarrow$ $\mathbf{r} \perp \mathbf{a}$ and **r** lies in the plane of **b** and **c**

$\Rightarrow$ **r** is orthogonal to **a** and coplanar with **b** and **c**

5.3 Explanation of Vector Triple Product

Since **r** is coplanar with **b** and **c**, then we can write

$$\mathbf{r} = l\mathbf{b} + m\mathbf{c}$$

Thus, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = l\mathbf{b} + m\mathbf{c}$

$\mathbf{a}\cdot(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = l(\mathbf{a}\cdot\mathbf{b}) + m(\mathbf{a}\cdot\mathbf{c})$

$0 = l(\mathbf{a}\cdot\mathbf{b}) + m(\mathbf{a}\cdot\mathbf{c})$

$l(\mathbf{a}\cdot\mathbf{b}) + m(\mathbf{a}\cdot\mathbf{c}) = 0$

$l(\mathbf{a}\cdot\mathbf{b}) = -m(\mathbf{a}\cdot\mathbf{c})$

$$\frac{(\mathbf{a}\cdot\mathbf{b})}{m} = -\frac{(\mathbf{a}\cdot\mathbf{c})}{l} = \lambda(\text{say})$$

Therefore, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda[(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}]$

Let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Now,

$$(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{i} \times (\mathbf{i} - \mathbf{j}) = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$

Also, $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$= \mathbf{i} + \mathbf{j} - \mathbf{i} - \mathbf{j} - \mathbf{k} = -\mathbf{k}$

Thus, $-\mathbf{k} = \lambda \times -\mathbf{k}$

$\Rightarrow$ $\lambda = 1$

Hence, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

5.4 Properties of Vector Triple Product

(i) A unit vector perpendicular to **a** and coplanar with **b** and **c** is given by

$$\pm\frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|}.$$

(ii) If **a, b** and **c** be three non-zero vectors, then
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

(iii) If **a, b** and **c** be three non-zero vectors, then
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$
are coplaner.

(iv) The vector product of three vectors is not associative, i.e.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

(v) If **a, b** and **c** be three non-zero vectors, then

$$[(\mathbf{a} \times \mathbf{b}), (\mathbf{b} \times \mathbf{c}), (\mathbf{c} \times \mathbf{a})] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$$

Proof: We have

$(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})$

$= \mathbf{d} \times (\mathbf{c} \times \mathbf{a})$ (Let $\mathbf{d} = (\mathbf{b} \times \mathbf{c})$]

$= (\mathbf{d}\cdot\mathbf{a})\mathbf{c} - (\mathbf{d}\cdot\mathbf{c})\mathbf{a}$

$= (\mathbf{b} \times \mathbf{c}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b} \times \mathbf{c}\cdot\mathbf{c})\mathbf{a}$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{c} - 0$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c},]\,\mathbf{c}$

Thus, $[(\mathbf{a} \times \mathbf{b}), (\mathbf{b} \times \mathbf{c}), (\mathbf{c} \times \mathbf{a})]$

$= (\mathbf{a} \times \mathbf{b})\cdot((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}))$

$= (\mathbf{a} \times \mathbf{b}) \cdot [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{c}$

$= ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$

(vi) If **a, b** and **c** be coplanar vectors, then $(\mathbf{a} \times \mathbf{b}), (\mathbf{b} \times \mathbf{c}), (\mathbf{c} \times \mathbf{a})$ are also coplanar.

Proof: As we know that, if **a, b, c** are coplanar vectors, then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

We have $[(\mathbf{a} \times \mathbf{b}), (\mathbf{b} \times \mathbf{c}), (\mathbf{c} \times \mathbf{a})]$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = 0$

Thus, $(\mathbf{a} \times \mathbf{b}), (\mathbf{b} \times \mathbf{c}), (\mathbf{c} \times \mathbf{a})$ are coplanar.

(vii) For any vector **a**,

$\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = 2\mathbf{a}$

Proof: We have

$\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) = (\mathbf{i} \cdot \mathbf{i})\mathbf{a} - (\mathbf{i} \cdot \mathbf{a})\mathbf{i}$

$= \mathbf{a} - (\mathbf{i} \cdot \mathbf{a})\mathbf{i}$

$= \mathbf{a} - a_1\mathbf{i}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Similarly, $\mathbf{j} \times (\mathbf{a} \times \mathbf{j}) = \mathbf{a} - a_2\mathbf{j}$

and $\mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = \mathbf{a} - a_3\mathbf{k}$

Thus, $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})$

$= \mathbf{a} - \mathbf{a}_1\mathbf{i} + \mathbf{a} - a_2\mathbf{j} + \mathbf{a} - a_3\mathbf{k}$

$= 3\mathbf{a} - (\mathbf{a}_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$

$= 3\mathbf{a} - \mathbf{a}$

$= 2\mathbf{a}$

(viii) If **a** and **b** be two non-zero vectors, then

$\mathbf{i} \times ((\mathbf{a} \times \mathbf{b}) \times \mathbf{i}) + \mathbf{j} \times ((\mathbf{a} \times \mathbf{b}) \times \mathbf{j})$

$+ \mathbf{k} \times [(\mathbf{a} \times \mathbf{b}) \times \mathbf{k}] = 2(\mathbf{a} \times \mathbf{b}).$

Proof: Replace **a** by $\mathbf{a} \times \mathbf{b}$ in property (vii), we get the required result.

(ix) If $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, then $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0$

Proof: We have

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$

$\Rightarrow \quad -(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = -(\mathbf{c} \cdot \mathbf{d})\mathbf{a}$

$\Rightarrow \quad (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = 0$

$\Rightarrow \quad (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0$

(x) If **a, b, c** and **d** are four coplanar vectors then,

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}, \mathbf{d}] + [\mathbf{b}, \mathbf{c}, \mathbf{d}] + [\mathbf{c}, \mathbf{a}, \mathbf{d}]$

Proof: Let the position vectors of the points **A, B, C** and **D** are **a, b, c** and **d**, respectively.

Thus, $\mathbf{AB} = \mathbf{b} - \mathbf{a}$, $\mathbf{AC} = \mathbf{c} - \mathbf{a}$, $\mathbf{AD} = \mathbf{d} - \mathbf{a}$

Given **A, B, C** and **D** are coplanar, so, $\mathbf{AB} \cdot \mathbf{AC}$, **AD** are coplanar

Therefore, $\mathbf{AB} \cdot (\mathbf{AC} \times \mathbf{AD}) = 0$

$\Rightarrow (\mathbf{b} - \mathbf{a}) \cdot ((\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})) = 0$

$\Rightarrow (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} \times \mathbf{d} - \mathbf{c} \times \mathbf{a} - \mathbf{a} \times \mathbf{d}) = 0$

$\Rightarrow (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{a}) - \mathbf{b} \cdot (\mathbf{a} \times \mathbf{d}) - (\mathbf{a} \cdot \mathbf{c} \times \mathbf{d}) = 0$

$\Rightarrow [\mathbf{b}, \mathbf{c}, \mathbf{d}] - [\mathbf{b}, \mathbf{c}, \mathbf{a}] - [\mathbf{b}, \mathbf{a}, \mathbf{d}] = [\mathbf{a}, \mathbf{c}, \mathbf{d}]$

$\Rightarrow [\mathbf{b}, \mathbf{c}, \mathbf{d}] - [\mathbf{b}, \mathbf{a}, \mathbf{d}] - [\mathbf{a}, \mathbf{c}, \mathbf{d}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}]$

$\Rightarrow [\mathbf{b}, \mathbf{c}, \mathbf{a}] + [\mathbf{a}, \mathbf{b}, \mathbf{d}] + [\mathbf{c}, \mathbf{a}, \mathbf{d}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}]$

$\Rightarrow [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{b}, \mathbf{c}, \mathbf{d}] + [\mathbf{a}, \mathbf{b}, \mathbf{d}] + [\mathbf{c}, \mathbf{a}, \mathbf{d}]$

$\Rightarrow [\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}, \mathbf{d}] + [\mathbf{b}, \mathbf{c}, \mathbf{d}] + [\mathbf{c}, \mathbf{a}, \mathbf{d}]$

Hence, the result.

5.5 Scaler Product of Four Vectors

If **a, b, c** and **d** are four non-zero vectors, then the scalar product of $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{c} \times \mathbf{d})$ is called scalar product of four vectors and is denoted as $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ and is defined as

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} (\mathbf{a} \cdot \mathbf{c}) & (\mathbf{a} \cdot \mathbf{d}) \\ (\mathbf{b} \cdot \mathbf{c}) & (\mathbf{b} \cdot \mathbf{c}) \end{vmatrix}$$

We have $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{p}, \quad$ [Let $\mathbf{p} = (\mathbf{c} \times \mathbf{d})$]

$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{p})$

$= \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{d}))$

$= \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d})$

$= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

$$= \begin{vmatrix} (\mathbf{a} \cdot \mathbf{c}) & (\mathbf{a} \cdot \mathbf{d}) \\ (\mathbf{b} \cdot \mathbf{c}) & (\mathbf{b} \cdot \mathbf{c}) \end{vmatrix}$$

5.5.1 If **a** and **b** lie in a plane normal to the plane containing **c** and **d**, then

$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

Proof: It is given that $(\mathbf{c} \times \mathbf{d})$ is perpendicular to the plane containing **c** and **d**.

But **a** and **b** are normal to the plane containing **c** and **d**

Thus, **c** and **d** lie in the plane containing **a** and **b**.

Also $(\mathbf{a} \times \mathbf{b})$ is perpendicular to the plane containing **a** and **b**. Therefore, $(\mathbf{a} \times \mathbf{b})$ is perpendicular to $(\mathbf{c} \times \mathbf{d})$ also.

Hence, $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 0$

5.5.2 If **a, b, c** and **d** are four non-zero vectors, then

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})=0$$

Proof: We have,

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})$$

$$=\begin{vmatrix}(\mathbf{a}\cdot\mathbf{c}) & (\mathbf{a}\cdot\mathbf{d})\\ (\mathbf{b}\cdot\mathbf{c}) & (\mathbf{b}\cdot\mathbf{d})\end{vmatrix}+\begin{vmatrix}(\mathbf{b}\cdot\mathbf{a}) & (\mathbf{b}\cdot\mathbf{d})\\ (\mathbf{c}\cdot\mathbf{a}) & (\mathbf{c}\cdot\mathbf{d})\end{vmatrix}+\begin{vmatrix}(\mathbf{c}\cdot\mathbf{b}) & (\mathbf{c}\cdot\mathbf{d})\\ (\mathbf{a}\cdot\mathbf{b}) & (\mathbf{a}\cdot\mathbf{d})\end{vmatrix}$$

$$=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$+(\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d})-(\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{d})$$
$$+(\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d})$$
$$=0$$

5.6 Vector Product of Four Vectors

If **a, b, c** and **d** are four non-zero vectors, the vector product of $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$ is called the vector product of four vectors and is denoted as $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ and is defined as

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$$
$$=[\mathbf{a},\mathbf{b},\mathbf{d}]\mathbf{c}-[\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d}$$
$$=[\mathbf{a},\mathbf{c},\mathbf{d}]\mathbf{b}-[\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a}$$

Proof: We have,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$$
$$=\mathbf{p}\times(\mathbf{c}\times\mathbf{d}),\qquad[\text{Let } \mathbf{a}\times\mathbf{b}=\mathbf{p}]$$
$$=(\mathbf{p}\cdot\mathbf{d})\mathbf{c}-(\mathbf{p}\cdot\mathbf{c})\mathbf{d}$$
$$=((\mathbf{a}\times\mathbf{b})\cdot\mathbf{d})\mathbf{c}-((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\mathbf{d}$$
$$=[\mathbf{a},\mathbf{b},\mathbf{d}]\mathbf{c}-[\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d}$$

Also, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$

$$=(\mathbf{a}\times\mathbf{b})\times\mathbf{q},\qquad[\text{Let } \mathbf{q}=(\mathbf{c}\times\mathbf{d})]$$
$$=(\mathbf{a}\cdot\mathbf{q})\mathbf{b}-(\mathbf{b}\cdot\mathbf{q})\mathbf{a}$$
$$=[\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\mathbf{b}-[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\mathbf{a}$$
$$=[\mathbf{a},\mathbf{c},\mathbf{d}]\mathbf{b}-[\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a}$$

Hence, the result.

5.7 Geometrical Interpretation of $(a\times b)\times(c\times d)$

$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is a vector perpendicular to $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$.

But $(\mathbf{a}\times\mathbf{b})$ is perpendicular to the plane containing **a** and **b**.

Thus, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ represents a vector coplanar with **a** and **b**.

Similarly $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ represents a vector coplanar with **c** and **d**.

Hence, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ represents a vector parallel to the line of intersections of two planes, one being parallel to the plane containing **a** and **b** and the other one parallel to the plane containing **c** and **d**.

5.7.1 If **a**, **b**, **c** and **d** are four non-zero vectors, then $a\times[b\times(c\times d)]$

$$=(\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\times\mathbf{d})$$

Proof: We have,

$$\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]$$
$$=\mathbf{a}\times[(\mathbf{b}\cdot\mathbf{d})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}]$$
$$=(\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\times\mathbf{d})$$

Hence, the result.

5.7.2 If **a, b, c** and **d** are coplanar vectors then $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{0}$

Proof: As we know that $(\mathbf{a}\times\mathbf{b})$ is perpendicular to **a** and **b** vectors and $(\mathbf{c}\times\mathbf{d})$ is also perpendicular to **c** and **d** vectors.

Since **a, b, c** and **d** are coplanar vectors, so $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$ are perpendicular to the same plane.

Thus, $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$ are parallel.

Therefore, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{0}$

5.7.3 If **b, c** and **d** are three non-coplanar vectors, then $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})+(\mathbf{c}\times\mathbf{d})+(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})$ is parallel to **a.**

Proof: We have,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=[\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{c}-[\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d}\qquad\text{...(i)}$$

Again,

$$(\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b})$$
$$=-(\mathbf{d}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})$$
$$=-[\mathbf{d},\mathbf{b},\mathbf{c}]\mathbf{a}+[\mathbf{d},\mathbf{b},\mathbf{a}]\mathbf{c}$$
$$=-[\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a}-[\mathbf{a},\mathbf{b},\mathbf{d}]\mathbf{c}\qquad\text{...(ii)}$$

Also,

$$(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})$$
$$=-(\mathbf{b}\times\mathbf{c})\times(\mathbf{a}\times\mathbf{d})$$
$$=-[\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a}+[\mathbf{b},\mathbf{c},\mathbf{a}]\mathbf{d}$$
$$=-[\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a}+[\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d}\qquad\text{...(iii)}$$

Adding Eqs. (i), (ii) and (iii), we get

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})+(\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b})$$
$$+(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})=-2[\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a}$$

Therefore, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})+(\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b})$ $+(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})$ is parallel to **a.**

5.8 Reciprocal System of Vectors

Let **a, b** and **c** be three non-zero vectors such that $[\mathbf{a},\mathbf{b},\mathbf{c}]\neq 0$. The reciprocal system of vectors are denoted as $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ and are defined as

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a},\mathbf{b},\mathbf{c}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}\text{ and }\mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}.$$

Properties of Reciprocal System of Vectors

If **a**, **b**, **c** and **a′**, **b′**, **c′** be reciprocal system of vectors, then

(i) $\mathbf{a}\cdot\mathbf{a}' = 1 = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}'$

Proof: We have,

$$\mathbf{a}\cdot\mathbf{a}' = \mathbf{a}\cdot\left(\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right)$$

$$= \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 1$$

Similarly, $\mathbf{b}\cdot\mathbf{b}' = 1$ and $\mathbf{c}\cdot\mathbf{c}' = 1$

(ii) $\mathbf{a}\cdot\mathbf{a}' + \mathbf{b}\cdot\mathbf{b}' + \mathbf{c}\cdot\mathbf{c}' = 0$

(iii) $\mathbf{a}\cdot\mathbf{b}' = 0 = \mathbf{a}\cdot\mathbf{c}'$, $\mathbf{b}\cdot\mathbf{c}' = 0 = \mathbf{b}\cdot\mathbf{a}'$
and $\mathbf{c}\cdot\mathbf{a}' = 0 = \mathbf{c}\cdot\mathbf{b}'$

Proof: We have

$$\mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\left(\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right)$$

$$= \left(\frac{\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right) = 0$$

Also, $$\mathbf{a}\cdot\mathbf{c}' = \mathbf{a}\cdot\left(\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right)$$

$$= \left(\frac{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right) = 0$$

(iv) $[\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{a}', \mathbf{b}', \mathbf{c}'] = 1$

Proof: We have,

$$[\mathbf{a}', \mathbf{b}', \mathbf{c}'] = \mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}')$$

$$= \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\cdot\left(\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c},]}\times\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right)$$

$$= \frac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^3}[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}]$$

$$= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c},]^2}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^3}$$

$$= \frac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Thus, $[\mathbf{a}', \mathbf{b}', \mathbf{c}'][\mathbf{a}, \mathbf{b}, \mathbf{c}] = 1$

(v) $\mathbf{a}\times\mathbf{a}' + \mathbf{b}\times\mathbf{b}' + \mathbf{c}\times\mathbf{c}' = 0$

(vi) $\mathbf{a}'\times\mathbf{b}' + \mathbf{b}'\times\mathbf{c}' + \mathbf{c}' + \mathbf{a}' = \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Proof: We have,

$$\mathbf{a}'\times\mathbf{b}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\times\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c},]}$$

$$= \frac{(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$

$$= \frac{\mathbf{d}\times(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$

$$= \frac{(\mathbf{d}\cdot\mathbf{a})\mathbf{c} - (\mathbf{d}\cdot\mathbf{c})\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$

$$= \frac{(\mathbf{b}\times\mathbf{c}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\times\mathbf{c}\cdot\mathbf{c})\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$

$$= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{c} - \mathbf{0}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$

$$= \frac{\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Similarly, $\mathbf{b}'\times\mathbf{c}' = \dfrac{\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

and $\mathbf{c}'\times\mathbf{a}' = \dfrac{\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Hence, $\mathbf{a}', \times\mathbf{b}' + \mathbf{b}'\times\mathbf{c}' + \mathbf{c}'\times\mathbf{a}' = \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

(vii) $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}'+\mathbf{b}'+\mathbf{c}') = 3.$

(viii) The system of unit vectors **i, j, k** be its own reciprocal.

(ix) If **a′**, **b′** and **c′** be the reciprocal system of **a, b** and **c,** then **a, b** and **c** is the reciprocal system of **a′**, **b′** and **c′**,

(x) If **a, b** and **c** be three non-coplanar vectors and **a′**, **b′** and **c′** be its reciprocal system of vectors, then any vector **r** can be expressed as

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a}')\mathbf{a} + (\mathbf{r}\cdot\mathbf{b}')\mathbf{b} + (\mathbf{r}\cdot\mathbf{c}')\mathbf{c}$$

or $$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a})\mathbf{a}' + (\mathbf{r}\cdot\mathbf{b})\mathbf{b}' + (\mathbf{r}\cdot\mathbf{c})\mathbf{c}'$$

Proof: Since **a**, **b**, **c** be three non-coplanar vectors, any vector **r** can be expressed as a linear combination of **a**, **b** and **c**.

Thus, $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, where $x, y, z \in R$

Taking dot product of $(\mathbf{b}\times\mathbf{c})$, we have

$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) = x\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + y\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) + z\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})$

$\Rightarrow \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) = x\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + 0 + 0 = x\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$\Rightarrow \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) = x\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$\Rightarrow x = \dfrac{\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})} = \dfrac{\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

$\Rightarrow x = \mathbf{r}\cdot\mathbf{a}'$

Similarly, $y = \mathbf{r}\cdot\mathbf{b}', = z = \mathbf{r}\cdot\mathbf{c}'$

(xi) If **a, b** and **c** be three non-coplanar vectors, any vector **r** can be expressed as a linear combination of **a, b** and **a × b,** i.e.

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b}).$$

EXERCISES

Level I

(Problems based on Fundamentals)

DOT PRODUCT OF VECTORS

1. If $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.
2. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, prove that $(\mathbf{a} + \mathbf{b})$ is perpendicular to $(\mathbf{a} - \mathbf{b})$.
3. If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$ such that $(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$, find the value of $(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$.
4. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular unit vectors, find $|\mathbf{a} + \mathbf{b} - \mathbf{c}|$.
5. Find the projection of the vector $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ on the vector $\mathbf{b} = \mathbf{j} - 2\mathbf{j} = \mathbf{i} - \mathbf{j}$.
6. Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ such that $2\mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{a} + 2\mathbf{b} = \mathbf{i} - \mathbf{j}$.
7. Find the unit vector perpendicular to each of the vectors $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.
8. Find a unit vector in xy-plane that makes an angle of 45° with the vector $\mathbf{i} + \mathbf{j}$ and an angle of 60° with the vector $3\mathbf{i} - 4\mathbf{j}$.
9. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be mutually perpendicular vectors of equal magnitude, show that $(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is equally inclined to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
10. Find a vector of magnitude $\sqrt{51}$ which makes equal angles with the vectors
$\mathbf{a} = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, $\mathbf{b} = \frac{1}{5}(-4\mathbf{i} - 3\mathbf{k})$ and $\mathbf{c} = \mathbf{j}$.
11. Find a vector of magnitude 4, which is equally inclined to vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{k} + \mathbf{i}$.
12. Find a unit vector in the plane of $(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and perpendicular to $(2\mathbf{i} + \mathbf{j} + \mathbf{k})$.
13. If $\hat{a}$ and $\hat{b}$ are unit vectors and θ the angle between them, show that $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|$.
14. If $\hat{a}$, and $\hat{b}$ are unit vectors perpendicular to each other, find the value of $(\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}})\cdot(\hat{\boldsymbol{b}} - \hat{\boldsymbol{c}})$.
15. Let $\hat{a}$ and $\hat{b}$ be two unit vectors and θ the angle between them. If $(\hat{a} + \hat{b})$ is a unit vector, find $\sin(\theta)$.
16. If $\hat{a}$ and $\hat{b}$ be two unit vectors inclined to x-axis at angles 30° and 120° respectively, find the value of $|\hat{a} + \hat{b}|$.
17. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors and their lengths are a and b, show that $\left(\frac{\mathbf{a}}{a^2} - \frac{\mathbf{b}}{b^2}\right)^2 = \left(\frac{\mathbf{a} - \mathbf{b}}{ab}\right)^2$
18. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$, and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. If $\hat{n}$ is a unit vector such that $\mathbf{u}\cdot\hat{n} = 0$ and $\mathbf{v}\cdot\hat{n} = 0$, then find $|\mathbf{w}\cdot\hat{n}|$.
19. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be unit vectors such that $\mathbf{a}$ is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{3}$, find $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.
20. If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$ and $\mathbf{a} \perp (\mathbf{b} + \mathbf{c})$ $\mathbf{b} \perp (\mathbf{c} + \mathbf{a})$ and $\mathbf{c} \perp (\mathbf{a} + \mathbf{b})$, find $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.
21. Let $\hat{a}$ and $\hat{b}$ are two unit vectors and the angle between them is 60°, find the value of $|\hat{a} + \hat{b}/\hat{a} - \hat{b}|$.
22. If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = \frac{\sqrt{17}}{3}$ such that $(\mathbf{a} + \mathbf{b} + 3\mathbf{c}) = 0$, find the value of $(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$.
23. Constant forces $\mathbf{F_1} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{F_2} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{F_3} = \mathbf{j} - \mathbf{k}$ act on a particle at a point A. Find the work done when the particle is displaced from the point A to B where $\mathbf{A} = 4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.
24. Prove that, by vector method,
$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$.
25. Prove that, by vector method,
$a = b\cos(C) + c\cos(B)$.
26. Prove that, by vector method,
$\cos(A - B) = \cos A\cdot\cos B + \sin A\cdot\sin B$
27. Prove that, by vector method,
$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(b)$

CROSS PRODUCT OF TWO VECTORS

28. If $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$.
29. Find the area of a triangle whose adjacent sides are determined by the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.
30. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
31. Find the area of a parallelogram whose diagonals are given by the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.
32. Find the area of a triangle whose vertices are (3, –1, 2), (1, –1, –1) and (4, –3, 1).
33. Find a unit vector perpendicular to each of the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.
34. Prove that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times (\mathbf{c} + \mathbf{a}) + \mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{0}$.
35. If $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$, show that $(\mathbf{a} - \mathbf{d})$ is parallel to $(\mathbf{b} - \mathbf{c})$.
36. If $(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$, prove that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

37. Prove that $|\mathbf{a} \times \hat{i}|^2 + |\mathbf{a} \times \hat{j}|^2 + |\mathbf{a} \times \hat{k}|^2 = 2a^2$.
38. Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be three unit vectors such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{0}$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{6}$, prove that $\mathbf{a} = \pm 2(\mathbf{b} \times \mathbf{c})$.
39. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{c}$, prove that $\mathbf{b} = \mathbf{c} + \lambda \mathbf{a}$, $\forall \lambda \in R$.
40. Let $\mathbf{A} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{C} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Find a vector $\mathbf{R}$ satisfying $\mathbf{R} \times \mathbf{B} = \mathbf{R} \times \mathbf{C}$ and $\mathbf{R} \cdot \mathbf{A} = 1$.
41. Find the number of vectors of unit length perpendicular to the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.
42. Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$, find a vector $\mathbf{c}$ for which **a, b** and **c** form a right handed system.
43. Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$. Find the point of intersection of the lines $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$ and $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$.
44. If **a** and **b** are two non-zero vectors such that $\mathbf{r} \cdot \mathbf{a} = 0$ and $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, find $\mathbf{r}$.
45. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{j} - \mathbf{k}$ such that $\mathbf{a} \cdot \mathbf{b} = 3$ and $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, find **b.**
46. Find the angle between the vectors $[\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}]$ and $(\mathbf{a} \times \mathbf{b})$.
47. Find the area of a square whose one diagonal is $(3\mathbf{i} + 4\mathbf{j})$.
48. A constant force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4$ is applied at the point $(1, -2, 2)$. Find the vector moment of $\mathbf{F}$ about the point $(2, -1, 3)$.
49. Forces $(5\mathbf{i} + \mathbf{k})$ and $(-5\mathbf{i} - \mathbf{k})$ act at the points $P(9, -1, 2)$ and $Q(3, -2, 1)$, respectively. Find the moment of the couple.
50. Find the torque about the point $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ of a force $4\mathbf{i} + \mathbf{k}$ acting through the point $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.
51. In a ΔABC, $a = BC$, $b = CA$ and $c = AB$, by using vector method, prove that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
52. Prove, by the vector method,
$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$
53. Prove, by vector method,
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

SCALER TRIPLE PRODUCT OF VECTORS

54. For any three vectors **a, b** and **c,** prove that $[\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.
55. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ be unit coplanar vectors, find $[2\mathbf{a} - \mathbf{b}, 2\mathbf{b} - \mathbf{c}, 2\mathbf{c} - \mathbf{a}]$.
56. Find $[\mathbf{a}, \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{b} + \mathbf{c}]$.
57. If **a**, **b**, and **c** be three non-coplanar vectors, find $[\mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{c}]$.
58. Find the value of $[\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{c}, \mathbf{c} - \mathbf{a}]$.
59. If **a, b** and **c** are three non-zero and non-coplanar vectors such that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$, find the angle between **a** and **b**.
60. For non-zero vectors **a**, **b** and **c**, if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 4$, find the value of $\frac{1}{2}(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$.
61. If the vectors $2\mathbf{i} - \mathbf{j} + \lambda\mathbf{k}$, $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ are coplanar vectors, find the value of λ.
62. Find the volume of a parallelopiped whose coterminous edges are $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
63. Find the volume of a parallelopiped whose coterminous edges are $\mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} + \mathbf{c} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} + \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$.
64. The volume of a parallelopiped determined by the vectors **a, b** and **c** is 5. Find the volume of the parallelopiped determined by the vectors $\mathbf{3}(\mathbf{a} + \mathbf{b})$, $(\mathbf{b} + \mathbf{c})$ and $\mathbf{2}(\mathbf{c} + \mathbf{a})$.
65. If the volume of the parallelopiped formed by the vectors $\mathbf{i} + a\mathbf{j} + \mathbf{k}$, $\mathbf{j} + a\mathbf{k}$ and $a\mathbf{i} + \mathbf{k}$, $a > 0$, is minimum, find the value of **a**.
66. Let V_1 be the volume of the parallelopiped formed by three non-zero vectors **a**, **b** and **c** and V_2 be the volume of the parallelopiped formed by the vectors $\mathbf{p} = \mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$, $\mathbf{q} = 3\mathbf{a} - \mathbf{b} + \mathbf{c}$ and $\mathbf{r} = \mathbf{a} - \mathbf{b} + 2\mathbf{c}$. Find $\left(\frac{V_1}{V_2}\right)$.
67. If the vectors $a\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + b\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + c\mathbf{k}$, $(a, b, c \neq 1)$ are coplanar, then find the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$.
68. For any three vectors **a, b** and **c**, prove that
$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$
69. The edges of a parallelopiped are of unit length and parallel to non-coplaner unit vectors **a, b** and **c** such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \frac{1}{2}$. Find the volume of the parallelopiped.
70. If **a** and **b** be non-zero and non-collinear vectors, prove that $[\mathbf{a}, \mathbf{b}, \mathbf{i}]\mathbf{i} + [\mathbf{a}, \mathbf{b}, \mathbf{j}]\mathbf{j} + [\mathbf{a}, \mathbf{b}, \mathbf{k}]\mathbf{k} = \mathbf{a} \times \mathbf{b}$.
71. If **a, b, c, m** and **n** are non-zero vectors, prove that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}](\mathbf{m} \times \mathbf{n}) = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \\ \mathbf{a}\cdot\mathbf{n} & \mathbf{b}\cdot\mathbf{n} & \mathbf{c}\cdot\mathbf{n} \end{vmatrix}$$

72. If a vector **r** can be expressed as a linear combination of three non-coplanar vectors **a**, **b** and **c**, prove that

$$\mathbf{r} = \frac{[\mathbf{r}\,\mathbf{b}\,\mathbf{c}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{a} + \frac{[\mathbf{r}\,\mathbf{c}\,\mathbf{a}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{b} + \frac{[\mathbf{r}\,\mathbf{a}\,\mathbf{b}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{c}$$

73. If **a** and **b** be unit vectors such that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \frac{1}{4}$, find the angle between **a** and **b**.
74. If $\alpha = x(\mathbf{a} + \mathbf{b}) + y(\mathbf{b} \times \mathbf{c}) + z(\mathbf{c} \times \mathbf{a})$ and $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \frac{1}{8}$, find $(x + y + z)$.
75. Find the volume of the tetrahedron with edges $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.
76. If **a**, **b** and **c** are unit vectors satisfying $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = 9$, find $|2\mathbf{a} + 5\mathbf{b} + 5\mathbf{c}|$.
77. The three vectors $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{k} + \mathbf{i}$ taken two at a time form three planes. The three unit vectors drawn perpendicular to three planes form a parallelopiped. Prove that its volume is $\frac{4}{\sqrt{3}}$ c.u.
78. If **a**, **b** and **c** are three non-coplanar vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$ and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$, find $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + 4$.

VECTOR TRIPLE PRODUCT

79. Find a unit vector which is orthogonal to $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and is coplanar with the vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$.
80. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{a}\cdot\mathbf{b} = 1$, $\mathbf{a} \times \mathbf{b} = \mathbf{j} - \mathbf{k}$, find **b**.
81. If **a, b** and **c** are three non-parallel unit vectors such that $(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \frac{1}{2}\mathbf{b}$, find the angles which **a** makes with **b** and **c**.
82. If **a, b** and **c** are three non-coplanar unit vectors such that

$$(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \frac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c}),$$

where **b** and **c** are non parallel, find the angle between **a** and **b**.

83. If $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + \mathbf{b} = 0$ such that $|\mathbf{a}| = 1$ $|\mathbf{b}|$ and $|\mathbf{c}| = 2$, find the angle between **a** and **c.**
84. If $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, prove that $(\mathbf{a}\cdot\mathbf{c}) = 0$.
85. Let **a, b** and **c** be three non-zero vectors such that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}.$$

If θ be the angle between **b** and **c**, find the value of $\sin\theta$.

86. If **a** and **b** be mutually perpendicular unit vectors satisfying $\mathbf{r}\cdot\mathbf{a} = 0$, $\mathbf{r}\cdot\mathbf{b} = 1$ and $[\mathbf{r}, \mathbf{a}, \mathbf{b}] = 1$, find **r**.
87. If $|\mathbf{a}| = 7$, $|\mathbf{b}| = 1 = |\mathbf{c}|$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \frac{1}{2}\mathbf{a}$, where **a** and **b** are non-collinear vectors, find $|\mathbf{a} \times \mathbf{c}|$.
88. Simplify: $(\mathbf{d} + \mathbf{a})\cdot\mathbf{a} \times \mathbf{b} \times (\mathbf{c} \times \mathbf{d})]$.
89. Solve: $p\mathbf{x} + (\mathbf{x} \times \mathbf{a}) = \mathbf{b}$.
90. If $[\mathbf{x}\,\mathbf{a}\,\mathbf{b}] = 0$, $\mathbf{x}\cdot\mathbf{a} = 0$, $\mathbf{x}\cdot\mathbf{b} = 1$, $\mathbf{a}\cdot\mathbf{b} = 0$, find **x**.
91. Find the value of
$[\mathbf{a} \times (\mathbf{b} + \mathbf{c}), \mathbf{b} \times (\mathbf{c} - 2\mathbf{a}), \mathbf{c} \times (\mathbf{a} + 3\mathbf{b})]$
if $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 2$.
92. Find the value of
$[(\mathbf{a} - \mathbf{b}), (\mathbf{a} - \mathbf{b} - \mathbf{c}), (\mathbf{a} + 2\mathbf{b} - \mathbf{c})]$ if $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.
93. Find the value of
$[(2\mathbf{a} \times 3\mathbf{b}), (3\mathbf{b} \times 4\mathbf{c}), (4\mathbf{c} \times 2\mathbf{a})]$ if $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Vector Equations

94. Solve: $\mathbf{x} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, where **a** and **b** are non-zero and non-collinear vectors.
95. Solve: $\mathbf{x} \times \mathbf{a} = \mathbf{b}$ and $\mathbf{x}\cdot\mathbf{a} = 0$.
96. Solve: $\mathbf{x} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{x}\cdot\mathbf{c} = 0$, where **c** is not perpendicular to **b.**
97. Solve: $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$.
98. Solve: $\mathbf{x} + \mathbf{y} = \mathbf{a}$, $\mathbf{x} \times \mathbf{y} = \mathbf{b}$, $\mathbf{x}\cdot\mathbf{a} = 1$

Level II

Mixed Problems (More than one options are correct)

1. If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}, \vec{b} < 0$ and $|\vec{a}\cdot\vec{b}| < |\vec{a} \times \vec{b}|$, the angle between $\vec{a}$ and $\vec{b}$ is
   (a) π (b) $\frac{\pi}{3}$
   (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$
2. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors of equal magnitudes. The angle between each pair of vectors is $\frac{\pi}{3}$ such that $|\vec{a} + \vec{b} + c| = \sqrt{6}$, the value of $(|\vec{a}| + 2)$ is
   (a) 2 (b) 3
   (c) 1 (d) 0
3. If a, b and c are three unit vectors inclined to each other at angle θ, the maximum value of θ is
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
   (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$
4. The number of vectors of unit length perpendicular to vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (0, 1, 1)$ is

(a) 1 (b) 2
(c) 3 (d) infinite

5. Let α, β and γ be distinct real numbers. The points with position vectors $\alpha\hat{i}$, $\beta\hat{j}$, $\gamma\hat{k}$, $\beta\hat{i}$, $\gamma\hat{j}$, $\alpha\hat{k}$ and $\gamma\hat{i}$, $\alpha\hat{j}$, $\beta\hat{k}$
(a) are collinear
(b) form an equilateral triangle
(c) form a scalene triangle
(d) form a right-angle triangle.

6. If P and Q have position vectors $\vec{a}$ and $\vec{b}$ relative to the origin O such that X and Y divide $\overrightarrow{PQ}$ internally and externally respectively in the ratio 2:1, the vector $\overrightarrow{XY}$ is
(a) $\frac{3}{2}(\vec{b}-\vec{a})$ (b) $\frac{4}{3}(\vec{a}-\vec{b})$
(c) $\frac{5}{6}(\vec{b}-\vec{a})$ (d) $\frac{4}{3}(\vec{b}-\vec{a})$

7. If the three points with position vectors $(1, \vec{a}, \vec{b})$, $(\vec{a}, 2, \vec{b})$ and $(\vec{a}, \vec{b}, 3)$ are collinear, the value of $\vec{a}+\vec{b}$ is
(a) 1 (b) 4
(c) 5 (d) none.

8. The acite angle between the medians drawn from the acute angle of an isosceles right-angled triangle is
(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$
(c) $\cos^{-1}\left(\frac{4}{5}\right)$ (d) $\cos^{-1}\left(\frac{2}{3}\right)$

9. If $\vec{e}_1$ and $\vec{e}_2$ are two unit vectors and θ the angle between them, then $\cos\left(\frac{\theta}{2}\right)$ is
(a) $\frac{1}{2}|\vec{e}_1+\vec{e}_2|$ (b) $\frac{1}{2}|\vec{e}_1-\vec{e}_2|$
(c) $\frac{1}{2}|\vec{e}_1\times\vec{e}_2|$ (d) $\frac{1}{2}|\vec{e}_1\cdot\vec{e}_2|$

10. If the vectors $(3\vec{p}+\vec{q})$, $(5\vec{p}-3\vec{q})$ and $(2\vec{p}+\vec{q})$, $(4\vec{p}-2\vec{q})$ are pairwise mutually perpendicular vectors such that θ the angle between $\vec{p}$ and $\vec{q}$, the value of $\sin(\theta)$ is
(a) $\frac{\sqrt{55}}{4}$ (b) $\frac{\sqrt{55}}{8}$
(c) $\frac{3}{16}$ (d) $\frac{\sqrt{247}}{16}$

11. The set of values of c for which the angle between the vectors $cx\hat{i}-6\hat{j}+3\hat{k}$, $x\hat{i}-2\hat{j}+2cx\hat{k}$ is acute for every x in R is
(a) $\left(0, \frac{4}{3}\right)$ (b) $\left[0, \frac{4}{3}\right]$
(c) $\left(\frac{11}{9}, \frac{4}{3}\right)$ (d) $\left[0, \frac{4}{3}\right)$

12. Let $\vec{u}=\hat{i}+\hat{j}$, $\vec{v}=\hat{i}-\hat{j}$ and $\vec{w}=\hat{i}-2\hat{j}+3\hat{k}$
If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n}$, the value of $|\vec{w}\cdot\hat{n}|$ is
(a) 1 (b) 2
(c) 3 (d) 0

13. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$, the angle between $\vec{a}$ and $\vec{b}$ is
(a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$
(c) $\frac{5\pi}{3}$ (d) $\frac{\pi}{3}$

14. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors of the lengths 3, 4 and 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ to $\vec{a}+\vec{c}$ and $\vec{c}$ to $\vec{a}+\vec{b}$, the value of $|\vec{a}+\vec{b}+\vec{c}|$ is
(a) $2\sqrt{5}$ (b) $2\sqrt{2}$
(c) $10\sqrt{5}$ (d) $5\sqrt{2}$

15. If a vector $\vec{a}$ of length 50 is collinear with the vector $\vec{b}=6\hat{i}-8\hat{j}-\hat{k}$ and makes an acute angle with the positive z-axis, then
(a) $\vec{a}=4\vec{b}$ (b) $a=-4\vec{b}$
(c) $\vec{b}=4\vec{a}$ (d) none.

16. Let $\vec{a}$ and $\vec{b}$ be two non-parallel unit vectors in a plane. If $(\alpha\vec{a}+\vec{b})$ bisects the internal angle between $\vec{a}$ and $\vec{b}$, then α is
(a) 1/2 (b) 1
(c) 2 (d) −1

17. Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ are three unit vectors such that $(\hat{a}+\hat{b}+\hat{c})$ is also a unit vector. If pairwise angle between $\hat{a}, \hat{b}, \hat{c}$ are θ_1, θ_2 and θ_3, spectively, the value of $\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)$ is
(a) 3 (b) −3
(c) 1 (d) −1

18. Given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ each two of which are non-collinear. Further $(\vec{a}+\vec{b})$ is collinear with $\vec{c}$ and $(\vec{c}+\vec{b})$ is collinear with $\vec{a}$ and each length of $\vec{a}$, $\vec{b}$ is $\sqrt{2}$. The value of $(\vec{a}\cdot\text{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$ is
(a) 3 (b) −3
(c) 0 (d) not defined.

19. If $\vec{p}=3\vec{a}-5\vec{b}$, $\vec{q}=2\vec{a}+\vec{b}$, $\vec{r}=\vec{a}+4\vec{b}$ and $\vec{s}=-\vec{a}+\vec{b}$ are four vectors such that sine of ($\vec{p}$ and $\vec{q}$) is 1 and sine of ($\vec{r}$ and $\vec{s}$) is also 1, the value of cosine of ($\vec{a}$ and $\vec{b}$) is
(a) $-\frac{19}{5\sqrt{43}}$ (b) 0

(c) 1 (d) $\frac{19}{5\sqrt{43}}$

20. If $\vec{a}$ and $\vec{b}$ are non-zero and non-collinear and the linear combination $(2x - y)\vec{a} + 4\vec{b} = 5\vec{a} + (x - 2y)\vec{b}$ holds good for all real x and y, then the value of $x + y$ is
(a) -3 (b) 1
(c) 17 (d) 3.

21. If $\vec{p}$ the position vector of the orthocentre and $\vec{q}$ the position vector of the centroid of the triangle ABC and origin O is the circumcentre. If $\vec{p} = k\vec{q}$, the value of k is
(a) 3 (b) 2
(c) 1/3 (d) 2/3

22. If the vectors $\hat{i} - 2x\hat{j} - 3y\hat{k}$ and $\hat{i} + 3x\hat{j} + 2y\hat{k}$ are orthogonal to each other, the locus of the point is
(a) a/an circle (b) ellipse
(c) a/an parabola (d) straight line.

23. A tangent is drawn to the curve $x^2y = 8$ at a point $A(x_1, y_1)$ where $x_1 = 2$. The tangent cuts the x-axis at a point B. The value of $(\overrightarrow{AB} \cdot \overrightarrow{OB})$ is
(a) 3 (b) -3
(c) 6 (d) -6.

24. If $\vec{e}_1$ and $\vec{e}_2$ be two unit vectors and θ the angle between them, then $\sin\left(\frac{\theta}{2}\right)$ is
(a) $\frac{1}{2}|\vec{e}_1 + \vec{e}_2|$ (b) $\frac{1}{2}|\vec{e}_1 - \vec{e}_2|$
(c) $\frac{1}{2}|\vec{e}_1 \times \vec{e}_2|$ (d) $\frac{1}{2}|\vec{e}_1 \cdot \vec{e}_2|$

25. If $\vec{u} = \vec{a} - \vec{b}$, $\vec{v} = \vec{a} + \vec{b}$ and $|\vec{a}| = 2 = |\vec{b}| = (\vec{a} \cdot \vec{b})$, the value of $|\vec{u} \times \vec{v}|$ is
(a) 0 (b) 2
(c) 4 (d) 16.

26. If $\vec{a} = (1, 1, 1)$ are given vectors, then a vector $\vec{b}$ satisfying the conditions $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$ is
(a) (5, 2, 2) (b) $\left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$
(c) $\left(\frac{2}{3}, \frac{5}{3}, \frac{2}{3}\right)$ (d) $\left(\frac{2}{3}, \frac{2}{3}, \frac{5}{3}\right)$.

27. If $|\vec{a}| = 1$, $|\vec{b}| = 2$ and and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$, the value of $\{(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})\}^2$ is
(a) 225 (b) 250
(c) 275 (d) 300

28. The set of values of m for which the vectors $\hat{i} + \hat{j} + m\hat{k}$, $\hat{i} + \hat{j} + (m + 1)\hat{k}$ and $\hat{i} - \hat{j} + m\hat{k}$ are non-coplanar, is
(a) R (b) $R - \{1\}$
(b) $R - \{-2\}$ (d) φ

29. Given $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$,
$\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$, $(\vec{a} \cdot \vec{c}) = 4$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$, then
(a) $[\vec{a}, \vec{b}, \vec{c}]^2 = |\vec{a}|$ (b) $[\vec{a}, \vec{b}, \vec{c}]^2 = |\vec{a}|$
(c) $[\vec{a}, \vec{b}, \vec{c}]^2 = 0$ (d) $[\vec{a}, \vec{b}, \vec{c}] = |\vec{a}|^2$

30. Let a, b, c be three distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + a\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is
(a) AM of a and b (b) GM of a and b
(c) HM of a and b (d) equal to zero.

31. For three vectors $\vec{u}, \vec{v}, \vec{w}$, which of the following expressions is not equal to any of the following three?
(a) $\vec{u} \cdot (\vec{v} \times \vec{w})$ (b) $(\vec{v} \times \vec{w}) \cdot \vec{u}$
(c) $\vec{v} \cdot (\vec{u} \times \vec{w})$ (d) $(\vec{u} \times \vec{v}) \cdot \vec{w}$

32. The vector perpendicular to the vectors $\vec{a} = (2, -3, 1)$, $\vec{b} = (1, -2, 3)$ and satisfies the condition $\vec{c} \cdot (\hat{i} + 2\hat{j} + 7\hat{k})$, the vector $\vec{c}$ is
(a) (7, 5, 1) (b) $(-7, -5, 1)$
(c) $(1, 1, -1)$ (d) (1, 2, 3)

33. If $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$, the vector $\vec{c}$ such that $\vec{a} \cdot \vec{c} = 2$ and $\vec{a} \times \vec{c} = \vec{b}$ is
(a) $\frac{1}{3}(3\hat{i} + 2\hat{j} + 5\hat{k})$ (b) $\frac{1}{3}(-\hat{i} + 2\hat{j} + 5\hat{k})$
(c) $\frac{1}{3}(\hat{i} + 2\hat{j} + 5\hat{k})$ (d) $\frac{1}{3}(3\hat{i} + 2\hat{j} + \hat{k})$

34. The altitude of a parallelopiped whose coterminous edges are the vectors $\vec{A} = (\hat{i} + \hat{j} + \hat{k})$ $\vec{B} = (2\hat{i} + 4\hat{j} - \hat{k})$ and $\vec{C} = (\hat{i} + \hat{j} + 3\hat{k})$ with $\vec{A}$ and $\vec{B}$ as the sides of the base of the parallelopiped, is
(a) $\frac{2}{\sqrt{19}}$ (b) $\frac{4}{\sqrt{19}}$
(c) $\frac{3}{\sqrt{19}}$ (d) $\frac{2\sqrt{2}}{\sqrt{19}}$.

35. If the vectors $a\hat{i} + b\hat{j} + c\hat{k}$, $b\hat{i} + c\hat{j} + a\hat{k}$ and $c\hat{i} + a\hat{j} + b\hat{k}$ are coplanar and a, b and c are distinct, then

(a) $a^3 + b^3 + c^3 = 1$ (b) $a + b + c = 1$

(c) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ (d) $a + b + c = 0$.

36. If $\vec{u}, \vec{v}, \vec{w}$ are vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$, then $\lfloor \vec{u} + \vec{v}, \vec{v} + \vec{w}, \vec{w} + \vec{u} \rfloor$ is

(a) 1 (b) 0

(c) –1 (d) None.

37. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors, then $(\vec{r}\cdot\vec{a})\vec{a} + (\vec{r}\cdot\vec{a})\vec{b} + (\vec{r}\cdot\vec{a})\vec{c}$ is

(a) $\vec{r}$ (b) $2\vec{r}$

(c) $-\vec{r}$ (d) $-2\vec{r}$

38. Let $\vec{a} = \hat{i} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is

(a) (3, –1, 1) (b) (3, 1, –1)

(c) (–3, 1, 1) (d) (–3, –1, –1)

39. If $|\vec{a}| = 1 = |\vec{b}|$ and $|\vec{c}| = 2$ and $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = \vec{0}$, the acute angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

40. If $\vec{b}$ and $\vec{c}$ are two mutually perpendicular unit vectors and $\vec{a}$ is any vector, then

$$(\vec{a}\cdot\vec{b})\vec{b} + (\vec{a}\cdot\vec{c})\vec{c} + \frac{a\cdot(\vec{b}\times\vec{c})}{(\vec{b}\times\vec{c}^2)}(\vec{b}\times\vec{c}) \text{ is}$$

(a) 0 (b) $\vec{a}$

(c) $2\vec{a}$ (d) $\frac{\vec{a}}{2}$

Level III

(Problems for JEE-Advanced)

1. Three force vectors **P, Q** and **R** of 15 KN each acts along AB, BC and CA, respectively. The position vectors **OA**, **OB** and **OC** are given in metres as **OA** = 2**i** – 4**j** + 3**k**, **OB** = 5**i** + 3**j** – 2**k** and **OC** = –2**i** + 2**j** + 3**k**.
Find the resultant force vector **S** of the vectors **P, Q** and **R**. **[Roorkee-JEE, 1983]**

2. Let **a** and **b** are two vectors satisfying the conditions **a** × **b** = **0**. Does it imply that one of the vectors **a** and **b** must be a null vector. Give one example in support of your answer.
[Roorkee-JEE, 1984]

3. Constant forces **P** = 2**i** – 5**j** + 6**k** and **Q** = –**i** + 2**j** – **k** act on a particle. Determine the work done when the particle is displaced from a point A with position vector 4**i** – 3**j** – 2**k** to a point B with a position vector 6**i** + **j** – 3**k**.
[Roorkee-JEE, 1984]

4. Prove that for any two vectors **a** and **b**
$(\mathbf{a} \times \mathbf{b})^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$
[Roorkee-JEE, 1985]

5. If $(\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{c}) \neq \mathbf{0}$, show that $\mathbf{a} + \mathbf{c} = k\mathbf{b}$, where k is a scalar.
[Roorkee-JEE, 1985]

6. Find the value of the constant S such that the scalar product of the vector (**i** + **j** + **k**) with the unit vector parallel to the sum of the vectors (2**i** + 4**j** – 5**k**) and (S**i** + 2**j** + 3**k**) is equal to 1. **[Roorkee-JEE, 1985]**

7. Find **a** such that the vectors (2**i** – **j** + **k**), (**i** + 2**j** – 3**k**) and (3**i** + a**j** + 5**k**) are coplanar.
[Roorkee-JEE, 1986]

8. If **a** and **b** are non-null vectors and |**a** + **b**| = |**a** – **b**|, show that **a** and **b** are perpendicular to each other.
[Roorkee-JEE, 1986]

9. Find **a** vector of magnitude $\sqrt{51}$ which makes equal angles with the three vectors $\mathbf{a} = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, $\mathbf{b} = \frac{1}{5}(-4\mathbf{i} - 3\mathbf{k})$ and **c** = **j**.
[Roorkee-JEE, 1987]

10. Three vectors **a** = (12, 4, 3), **b** = (8, –12, –9) and **c** = (33, –4, –24) define a parallelopiped. Evaluate the length of its edges, area of the faces and its volume.
[Roorkee-JEE, 1988]

11. It is given that $\mathbf{x} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$, $\mathbf{y} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$ and $\mathbf{z} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$, where **a**, **b** and **c** are non-coplanar vectors.
Show that **x, y** and **z** also form a non-coplanar system. Find the value of $\mathbf{x}\cdot(\mathbf{a} + \mathbf{b}) + \mathbf{y}\cdot(\mathbf{b} + \mathbf{c}) + \mathbf{z}\cdot(\mathbf{a} + \mathbf{b})$.
[Roorkee-JEE, 1989]

12. It is given that $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$, $\mathbf{r}\cdot\mathbf{a} = 0$, $\mathbf{a}\cdot\mathbf{b} \neq 0$ What is the geometrical meaning of these equations

separately? If the above three statements hold good simultaneously, determine the vector **r** in terms of **a, b** and **c.**

[Roorkee-JEE, 1989]

13. Let **a**, **b**, **c** are three unit vectors, **a** + **b** + **c** = **0.** Then **a**·**b** + **b**·**c** + **c**·**a** is
(a) –3/2 (b) 0
(c) –1 (d) 1

[Roorkee-JEE, 1990]

14. Let **a** = x**i** + y**j** + z**k**, **b** = **j**. For what **c, a, b, c** form a right handed system?
(a) 2**i** – x**k** (b) **0**
(c) –z**i** × x**k** (d) y**j**

[Roorkee-JEE, 1990]

15. Let **a, b, c** are in the same plane. Which of the following is correct?
(a) **a**·**b** × **c** = 0 (b) **a**·**b** × **c** = 1
(c) **a**·**b** × **c** = 2 (d) **a**·**b** × **c** = 3

[Roorkee-JEE, 1990]

16. Given that the vectors **A, B, C** form a triangle such that **A = B + C**. Find a, b, c and d such that the area of the triangle is $5\sqrt{6}$, where **A** = a**i** + b**j** + c**k**, **B** = d**i** + 3**j** + 4**k**, **C** = 3**i** + **j** – 2**k**.

[Roorkee-JEE, 1990]

17. If **a** and **b** be two unit vectors, the vector (**a** + **b**) × (**a** × **b**) is
(a) perpendicular to (**a** – **b**)
(b) parallel to (**a** – **b**)
(c) equal to 2(**a** – **b**)
(d) equal to 2(**a** – **b**)

[Roorkee-JEE, 1991]

18. If the position vectors of points A and B with respect to origin O are **a** = 2**i** – **j** + 2**k** and **b** = **i** + 2**j** – 2**k**, respectively, the projection of the vector **a** + **b** + **a** × **b** on a line perpendicular to the plane OAB is
(a) 15 (b) 10
(c) $\sqrt{75}$ (d) $\sqrt{65}$

[Roorkee-JEE, 1991]

19. If two vectors **a** and **b** are such that **a**·**b** = **0**, the solution of **x** + **a** = **b** for all x
(a) is unique
(b) is possible only, when **a** = **b**
(c) does not exist
(d) will be infinitely many.

[Roorkee-JEE, 1991]

20. Let $\hat{a}$ be a unit vector and **b** be a non-zero vector not parallel to $\hat{a}$. Find the angles of the triangle, two sides of each are represented by the vectors

$$\sqrt{3}\,(\hat{\mathbf{a}} \times \mathbf{b}) \text{ and } \mathbf{b} - (\hat{\mathbf{a}}\cdot\mathbf{b})\hat{\mathbf{a}}$$

[Roorkee-JEE, 1991]

21. Unit vector in the xy-plane that makes an angle of 45° with the vector **i** + **j** and an angle of 60° with the vector 3**i** – 4**j** is
(a) **i** (b) $\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$
(c) $\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$ (d) none

[Roorkee-JEE, 1992]

22. Find the value of λ that the vectors 2**i** – **j** + **k**, **i** + 2**j** – 3**k** and 3**i** + λ**j** + 5**k** are coplaner
(a) –4 (b) 0
(c) 2 (d) 4

[Roorkee-JEE, 1992]

23. In parallelogram $ABCD$, the interrior bisectors of the consecutive angles B and C intersect at P. Find $\angle BPC$.

[Roorkee-JEE, 1992]

24. If the sides of an angle are given by the vectors **a** = **i** – 2**j** + 2**k** and **b** = 2**i** + **j** + 2**k**, the internal bisector is
(a) 3**i** + **j** + 4**k** (b) $\frac{1}{3}$(3**i** – **j** + 4**k**)
(c) $\frac{1}{3}$(–**i** – 3**j**) (d) 3**i** – **j** – 4**k**

[Roorkee-JEE, 1993]

25. Let **a, b** and **c** are three vectors such that **a** × **b** = **c**, **b** × **c** = **a,** then
(a) $a = 1$, $b = c$ (b) $c = 1$, $a = b$
(c) $b = 2$, $c = 2a$ (d) $b = 1$, $c = a$

[Roorkee-JEE, 1993]

26. Let **a, b** and **c** are in the same plane. Which of the following is correct?
(a) **a**·**b** × **c** = 0 (b) **a**·**b** × **c** = 1
(c) **a**·**b** × **c** = 2 (d) **a**·**b** × **c** = 1

[Roorkee-JEE, 1994]

27. If two vectors **a** and **b** are such that **a**·**b** = **0**, the solution of **x** × **a** = **b** for all x.
(a) is unique
(b) is possible only, when **a** = **b**
(c) does not exist
(d) will be infinitely many.

[Roorkee-JEE, 1994]

28. Solve the following system of simultaneous equations for vectors **x** and **y**

$$\mathbf{x} + \mathbf{y} = \mathbf{a},\ \mathbf{x} \times \mathbf{y} = \mathbf{b},\ \mathbf{x} \cdot \mathbf{a} = 1$$

[Roorkee-JEE, 1994]

29. The vector $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ bisects the angle between vector **c** and $3\mathbf{i} + 4\mathbf{j}$. Determine the unit vector along **a**.

(a) $-\frac{1}{15}(3\mathbf{i} - 8\mathbf{jz} + 12\mathbf{k})$

(b) $-\frac{1}{15}(11\mathbf{i} + 10\mathbf{j} + 2\mathbf{k})$

(c) $-\frac{1}{15}(3\mathbf{i} + 8\mathbf{j} + 12\mathbf{k})$

(d) none. **[Roorkee-JEE, 1995]**

30. Find the scalers α and β if $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + (\mathbf{a} \cdot \mathbf{b})\mathbf{b} = (4 - 2\beta - \sin\alpha)\mathbf{b} + (\beta^2 - 1)\mathbf{c}$ and $(\mathbf{c} \cdot \mathbf{c})\mathbf{a} = \mathbf{c}$, where **b** and **c** are non-collinear.

[Roorkee-JEE, 1995]

31. $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$, where **a, b, c** are non-zero vectors holds, if

(a) $\mathbf{a} \cdot \mathbf{b} = 0$ (b) $\mathbf{b} \cdot \mathbf{c} = 0$

(c) $\mathbf{c} \cdot \mathbf{a} = 0$ (d) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$

[Roorkee-JEE, 1996]

32. Let **x, y** and **z** be unit vectors such that $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{a}$, $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{b}$ $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = \mathbf{c}$, $\mathbf{a} \cdot \mathbf{x} = \frac{3}{2}, \mathbf{a} \cdot \mathbf{y} = \frac{7}{4}$ and $|\mathbf{a}| = 2$. Find **x**, **y** and **z** in terms of **a**, **b**, and **c**.

[Roorkee-JEE, 1996]

33. Vectors **x, y** and **z** each of magnitude $\sqrt{2}$ makes an angle of 60° with each other. If $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{a}$, $\mathbf{y} \times (\mathbf{z} \times \mathbf{x}) = \mathbf{b}$ and $(\mathbf{x} \times \mathbf{y}) = \mathbf{c}$, find **x, y, z** in terms of **a, b** and **c**.

[Roorkee-JEE, 1997]

34. Which of the following is a true statement?

(a) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is coplaner with **c**

(b) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to **a**

(c) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to **b**

(d) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to **c**.

[Roorkee-JEE, 1998]

35. If A, B, C and D are any four points, then $\mathbf{AB} \cdot \mathbf{CD} + \mathbf{BC} \cdot \mathbf{AD} + \mathbf{CA} \cdot \mathbf{BD}$ is equal to

(a) 1 (b) 0

(c) –1 (d) None.

[Roorkee-JEE, 1998]

36. If $\mathbf{p} = 2\mathbf{a} - 3\mathbf{b}$, $\mathbf{q} = \mathbf{a} - 2\mathbf{b} + \mathbf{c}$, $\mathbf{r} = -3\mathbf{a} + \mathbf{b} + 2\mathbf{c}$, where **a, b, c** are non-zero are non-coplanar vectors, then the vector $-2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ is equal to

(a) $\mathbf{p} - 4\mathbf{q}$ (b) $\frac{7}{5}\mathbf{p} + \frac{1}{5}\mathbf{r}$

(c) $2\mathbf{p} - 3\mathbf{q} - \mathbf{r}$ (d) $-4\mathbf{p} - 2\mathbf{r}$

[Roorkee-JEE, 1998]

37. The vector **c** directed along the bisectors of the angle between the vectors $\mathbf{a} = 7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, where $|\mathbf{c}| = 3\sqrt{6}$ is given by

(a) $\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$

(c) $-\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ (d) $\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$

[Roorkee-JEE, 1998]

38. **a, b, c** three non-coplanar vectors such that $\mathbf{r_1} = \mathbf{a} - \mathbf{b} + \mathbf{c}, \mathbf{r_2} = \mathbf{b} + \mathbf{c} - \mathbf{a}, \mathbf{r_3} = \mathbf{c} + \mathbf{a} - \mathbf{b}$, and $\mathbf{r} = 2\mathbf{a} - 2\mathbf{b} + 4\mathbf{c}$, if $\mathbf{r} = \mathbf{x_1 r_1} + \mathbf{x_2 r_2} + \mathbf{x_3 r_3}$, then

(a) $x_1 = \frac{7}{3}$ (b) $x_1 + x_3 = 3$

(c) $x_1 + x_2 + x_3 = 4$ (d) $x_2 + x_3 = 2$

[Roorkee-JEE, 1998]

39. If $\mathbf{x} \times \mathbf{y} = \mathbf{a}$, $\mathbf{y} \times \mathbf{z} = \mathbf{b}$, $\mathbf{x} \cdot \mathbf{b} = \mathbf{c}$, $\mathbf{x} \cdot \mathbf{y} = \mathbf{1}$ and $\mathbf{y} \cdot \mathbf{z} = \mathbf{1}$, find **x, y** and **z** in terms of **a, b** and **c.**

[Roorkee-JEE, 1998]

40. If **a, b, c** are non-coplanar vectors and **d** is a unit vector, find the value of $|(\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \times \mathbf{b})|$ independent of **d.**

[Roorkee-JEE, 1999]

41. If $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find a unit vector normal to the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} - \mathbf{c}$.

[Roorkee-JEE, 2000]

42. Given that the vectors **a** and **b** are perpendicular to each other, find vector **v** in terms of **a** and **b** satisfying the equations

$\mathbf{v} \cdot \mathbf{a} = 0$, $\mathbf{v} \cdot \mathbf{b} = 1$ and $[\mathbf{v}, \mathbf{a}, \mathbf{b}] = 1$

[Roorkee-JEE, 2000]

43. **a, b, c** are three unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{b} + \mathbf{c})$. Find the angle between vectors **a** and **b** given that vectors **b** and **c** are nonparallel.

[Roorkee-JEE, 2000]

44. The diagonals of a parallelogram are given by the vectors $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - \mathrm{k}$. Determine its sides and the area also.

[Roorkee-JEE, 2001]

45. Find the value of λ that **a, b, c** are all non-zero and $(-4\mathbf{i} + 5\mathbf{j})a + (3\mathbf{i} - 3\mathbf{j} + \mathbf{k})b$ $(\mathbf{i} + \mathbf{j} + \mathbf{k})\mathbf{c} = \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

[Roorkee-JEE, 2001]

46. Find the vector r which is perpendicular to $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + 8 = \mathbf{0}$.

[Roorkee-JEE, 2001]

47. Two vertices of a triangle are $-\mathbf{i} + 3\mathbf{j}$ and $2\mathbf{i} + 5\mathbf{j}$ and its orthocentre is at $\mathbf{i} + 2\mathbf{j}$. Find the position vector of the third vertex.

[Roorkee-JEE, 2001]

Level III

(Tougher Problems for JEE-Advanced)

1. If $\vec{a}$ and $\vec{b}$ be two unit vectors, find the range of $\frac{3}{2}|\vec{a} + \vec{b}| + 2|\vec{a} - \vec{b}|$.
2. Find the unit vector $\hat{a}$ which makes an angle $\left(\frac{\pi}{4}\right)$ with axis of z and is such that $(\hat{a} + \hat{i} + \hat{j})$ is a unit vector.
3. If **r** and **s** are non-zero vectors and the sacalar b is chosen in such a way that $|\vec{r} + b\vec{s}|$ is minimum, find the value of $|r + b\vec{s}|^2 + |b\vec{s}|^2$.

$$\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k},$$

4. Given that $\vec{v} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$, and

$(\vec{u} \cdot \vec{R} - 10)\hat{i} + (\vec{v} \cdot \vec{R}\ 20)\hat{j} + (\vec{w} \cdot \vec{R} - 20)\hat{k} = 0$

Find the vector $\vec{R}$.

5. Suppose the vectors **a**, **b**, **c** on a plane satisfy the conditions $|\vec{a}| = |\vec{b}| = |\vec{c}| = |a + \vec{\mathbf{b}}| = 1$; $c \cdot a = 0$ and $\vec{b} \cdot \vec{c} > 0$. Find the angle between $(2\vec{a} + \vec{b})$ and $\vec{b}$.
6. Find the minimum area of the triangle whose vertices are $A(-1, 1, 2)$, $B(1, 2, 3)$ and $C(t, 1, 1)$, where t is a real number.
7. Suppose the vectors a, b, c on a plane satisfy the conditions $|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{a} + \vec{b}| = 1$; $c \cdot a = 0$ and $\vec{b} \cdot \vec{c} > 0$. If the vector $\vec{c}$ is a linear combination $\lambda\vec{a} + \mu + \vec{b}$, find the ordered pair $(\lambda \cdot \mu)$.
8. Given that $\vec{a}$ and $\vec{b}$ are two unit vectors such that the angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. If $\vec{c}$ be **a** vector in the plane of $\vec{a}$ and $\vec{b}$ such that $|\vec{c}| = 4\ \vec{c} \times \vec{b} = 2\vec{a} \times \vec{b}$ and $\vec{c} = \lambda\vec{a} \times \mu\vec{b}$, find the
   (i) values of λ
   (ii) sum of the values of μ
   (iii) product of all values of μ.
9. Let $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 4$, $\vec{a} \cdot \vec{b} = 2$. If $\vec{c} = (2\vec{a} \times \vec{b}) - 3\vec{b}$, find the angle between $\vec{b}$ and $\vec{c}$.
10. Given that $\vec{a}$, $\vec{b}$, $\vec{p}$ and $\vec{q}$ are four vectors such that $\vec{a} + \vec{b} = \mu\vec{p}$, $\vec{b} \cdot \vec{q} = 0$ and $(\vec{b})^2 = 1$, where μ is a scaler, prove that

$|(\vec{a} \cdot \vec{q})\vec{p} - (\vec{p} \cdot \vec{q})\vec{a}| = |\vec{p} \cdot \vec{q}|$

11. Find a vector $\vec{v}$, which is coplanar with with the vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} + \hat{j} - 2\hat{k}$ and is orthogonal to the vector $2\hat{i} + \hat{j} + \hat{k}$.
It is given that the projection of $\vec{v}$ along the vector $\hat{i} - \hat{j} + \hat{k}$ is equal to $6\sqrt{3}$.
12. Let $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.
Find the numbers α, β and γ such that

$$\alpha\vec{a} = \beta\vec{b} + \gamma\vec{c} = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

13. Let a 3-dimentional vector $\vec{V}$ satisfies the condition $2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j}) = 2\hat{i} + 2\hat{k}$.
If $3|\vec{V}| = \sqrt{m}$, $m \in N$, find m.
14. Let $\vec{A} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{B} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{C} = \hat{j} + \hat{k}$.
If the vector $\vec{B} \times \vec{C} = x\vec{A} + y\vec{B} + z\vec{C}$, where x, y and z are scalars, find the value of $(100x + 10y + 8z)$.
15. Find the angle between any edge and face, which is not containing the edge of a regular tetrahedron and also find the angle between the two faces of a regular tetrahedron.

INTEGER TYPE QUESTIONS

1. If $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 1$, find the value of

$$\frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}} + \frac{\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})}{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}} + \frac{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}}$$

2. If **a, b** and **c** are non-coplanar vectors and **p, q, r** are defined as

$$\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]},\ \mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{c}\,\mathbf{a}\,\mathbf{b}]},\ \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]},$$

find the value of

$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$

3. Let $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$
and $\vec{c} = \lambda\hat{i} + \hat{j} + (2\lambda - 1)\hat{k}$. If $\vec{c}$

is parallel to the plane of the vectors $\vec{a}$ and $\vec{b}$, find the value of $(\lambda^2 + 1)$.

4. Let $\vec{r}$ be a vector perpendicular to $(\vec{a} + \vec{b} + \vec{c})$ where $[\vec{a}, \vec{b}, \vec{c}] = 2$. If $\vec{r} = P(\vec{b} \times \vec{c}) + Q(\vec{c} \times \vec{a}) + R(\vec{a} \times \vec{b})$, find the value of $(P + Q + R + S)$.
5. Let $|\vec{a}| = 1$, $|\vec{b}| = 1$, $|\vec{c}| = 2$ and $\vec{a} \perp (\vec{b} + \vec{c})$, $\vec{b} \perp (\vec{c} + \vec{a})$, $\vec{c} \perp (\vec{a} + \times \vec{b})$ such that $m = |\vec{a} + \vec{b} + \vec{c}|$, find the value of $(m^2 + 1)$.
6. If a, b and c are the pth, qth and rth terms of an HP and $\vec{u} = (q - r)\hat{i} + (r - b)\hat{j} + (p - q)\hat{k}$ and $\vec{v} = \frac{\hat{i}}{a} + \frac{\hat{j}}{b} + \frac{\hat{k}}{c}$ such that $m = |(\vec{u} \cdot \vec{v}) + 2|$, find the value of $(m + 4)$.
7. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + (2\lambda - 1)\hat{k}$.

   If $\vec{c}$ is parallel to the plane of $\vec{a}$ and $\vec{b}$, find the value of $(\lambda^2 + \lambda + 2)$.
8. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors of magnitudes 3, 4 and 5 respectively and $\vec{a} \perp \vec{b} + \vec{c}$, $\vec{b} \perp \vec{c} + \vec{a}$ and $\vec{c} \perp \vec{a} + \vec{b}$ such that $m = |\vec{a} + \vec{b} + \vec{c}|$, find the value of $\left(\frac{m}{\sqrt{2} + 3}\right)$.
9. If the three points with position vectors $(1, a, b)$, $(a, 2, b)$ and $(a, b, 3)$ are collinear in the space, find the value of $(a + b)$.
10. Let $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i} + 2\hat{j} - 3\hat{k}$. If $\hat{n}$ is a unit vector such that $\hat{u} \cdot \hat{n} = 0$ and $v \cdot \hat{n} = 0$, find the value of $|\hat{w} \cdot \hat{n}|$.
11. If $\vec{a}$ and $\vec{b}$ are non-zero and non-collinear and the linear combination $(2x - y)\vec{a} + 4\vec{b} = 5\vec{a} + (x - 2y)\vec{b}$ holds for real x and y, find the value of $(x + y + 2)$.
12. Given the vectors $\vec{u} = 2\hat{i} - \hat{j} - \hat{k}$, $\vec{v} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{w} = \hat{i} - 3\hat{k}$. If the volume of the parallelopiped having $-c\vec{u}$, $\vec{v}$ and $c\vec{w}$ are concurrent edges is 8, find the positive integral value of c.
13. Consider three vectors $\vec{p} = \hat{i} + \hat{j} + \hat{k}$, $\vec{q} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{r} = \hat{i} + \hat{j} + 3\hat{k}$ and s be a unit vector. If $(\vec{p} \times \vec{q}) \times \vec{r} = u\vec{p} + v\vec{q} + w\vec{r}$, find the value of $(u + v + w + 4)$.
14. Consider three vectors $\vec{p} = \hat{i} + \hat{j} + \hat{k}$, $\vec{q} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{r} = \hat{i} + \hat{j} + 3\hat{k}$ and $\vec{s}$ be a unit vector. Find the value of

   $|(\vec{p} \cdot \vec{s})(\vec{q} \times \vec{r}) + (\vec{q} \cdot \vec{s})(\vec{r} \times \vec{p}) + (\vec{r} \cdot \vec{s})(\vec{p} \times \vec{q})|$

Linked Comprehension Type (For JEE-Advanced Examination Only)

Passage I

Let the vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ are coplanar such that $\vec{x} = a\hat{i} + \hat{j} - \hat{k}$, $\vec{y} = \hat{i} + b\hat{j} - \hat{k}$ and $\vec{z} = \hat{i} + \hat{j} - c\hat{k}$.

(i) The value of $\left(\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}\right)$ is

(a) 1 (b) 2
(c) 0 (d) –1

(ii) The value of $\left(\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}\right)$ is

(a) –1 (b) –2
(c) –3 (d) 0

(iii) The value of $\lfloor \vec{x} + \vec{y}, \vec{y} + \vec{z}, \vec{z} + \vec{z} \rfloor$ is

(a) 1 (b) 2
(c) 0 (d) –1

Passage II

Consider the three vectors $\vec{p} = \hat{i} + \hat{j} + \hat{k}$, $\vec{q} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{r} = \hat{i} + \hat{j} + 3\hat{k}$ also $\vec{s}$ be a unit vector.

(i) $\vec{p}$, $\vec{q}$, $\vec{r}$ are

(a) linearly dependent vectors
(b) can form the sides of a possible triangle
(c) $(\vec{q} - \vec{r})$ is orthogonal to $\vec{p}$
(d) any one can be expressed as a linear combination of the other two.

(ii) If $(\vec{p} \times \vec{q}) \times \vec{r} = u\vec{p} + v\vec{q} + w\vec{r}$, the value of $(u + v + w + 2)$ is

(a) 6 (b) 4
(c) 0 (d) 2

(iii) The value of

$|(\vec{p} \cdot \vec{s})(\vec{q} \times \vec{r}) + (\vec{q} \cdot \vec{s})(\vec{r} \times \vec{p}) + (\vec{r} \cdot \vec{s})(\vec{p} \times \vec{q})|$ is

(a) 4 (b) 8
(c) 18 (d) 2.

Passage III

Consider the three vectors $\vec{p}$, $\vec{q}$ and $\vec{r}$ such that $\vec{p} = \hat{i} + \hat{j} + \hat{k}$, $\vec{q} = \hat{i} - \hat{j} + \hat{k}$ and $(\vec{p} \times \vec{r}) = \vec{q} + c\vec{p}$ and $(\vec{p} \cdot \vec{q}) = 2$, then

(i) the value of $\lfloor \vec{p}, \vec{q}, \vec{r} \rfloor$ is

(a) $-\frac{4}{3}$ (b) $-\frac{8}{3}$
(c) 2 (d) 1

(ii) the value of $\lfloor \vec{p} + \vec{q}, \vec{q} + \vec{r}, \vec{r} + \vec{p} \rfloor$ is

(a) $-\frac{4}{3}$ (b) $-\frac{8}{3}$

(c) $-\frac{16}{3}$ (d) 2

(iii) the value of $\lfloor p \times q, q \times r, r \times p \rfloor$ is

(a) $\frac{4}{3}$ (b) $\frac{16}{9}$

(c) $\frac{64}{9}$ (d) 1

Passage IV

The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that

$\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = \frac{1}{2}$.

(i) The value of $\lfloor \hat{a} \times \hat{b}, \times \hat{b} \times \hat{c}, \hat{c} \times \hat{a} \rfloor$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{3}$ (d) $\frac{1}{9}$

(ii) The volume of the parallelopiped with coterminous edges are given by $\hat{a}, \hat{b}, \hat{c}$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$.

(iii) The volume of the tetrahedron formed by the vectors $\hat{a}, \hat{b}, \hat{c}$ is

(a) $\frac{1}{6\sqrt{2}}$ (b) $\frac{1}{3\sqrt{2}}$

(c) $\frac{1}{4\sqrt{2}}$ (d) $\frac{1}{12\sqrt{2}}$.

Passage V

If $\alpha + \beta + \gamma = 2$ and $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ and also $\hat{k} \times (\hat{k} \times \vec{a}) = \vec{a}$.

(i) The value of $(\alpha^2 + 1)$ is

(a) 2 (b) 1

(c) –1 (d) 0.

(ii) The value of $(\beta^2 + \beta + 1)$ is

(a) 1 (b) 2

(c) –1 (d) 0

(iii) The value of $(\gamma^2 - \gamma)$ is

(a) –2 (b) 0

(c) 4 (d) 2

Passage VI

Let $\vec{u}, \vec{v}, \vec{w}$ be three unit vectors such that $\vec{u} + \vec{v} + \vec{w} = \vec{a}$, $\vec{a}\cdot\vec{u} = \frac{3}{2}$, $\vec{a}\cdot\vec{v} = \frac{7}{4}$ and $|\vec{a}| = 2$.

(i) The value of $\vec{a}\cdot\vec{w}$ is

(a) 3/4 (b) 1/4

(c) 1/2 (d) –1/2

(ii) The value of $\vec{u}\cdot\vec{w}$ is

(a) –3/4 (b) 1/2

(c) –1/4 (d) 1/6

(iii) The value of $\vec{u}\cdot\vec{v}$ is

(a) 1/2 (b) 3/4

(c) –1/2 (d) –3/4.

Passage VII

Let $\vec{a}, \vec{b}$, and $\vec{c}$ be three vectors with magnitude 4 such that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = \frac{1}{2}$.

(i) The height of the parallelopiped whose adjacent edges by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is

(a) $\sqrt{\frac{2}{3}}$ (b) $2 \times \sqrt{\frac{2}{3}}$

(c) $3 \times \sqrt{\frac{2}{3}}$ (d) $4 \times \sqrt{\frac{2}{3}}$

(ii) The volume of the prism whose adjacent edges by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is

(a) $2\sqrt{2}$ (b) $3\sqrt{2}$

(c) $4\sqrt{2}$ (d) $16\sqrt{2}$

(iii) The volume of the tetrahedron whose adjacent edges by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is

(a) $\frac{4\sqrt{2}}{3}$ (b) $\frac{8\sqrt{2}}{3}$

(c) $\frac{16\sqrt{2}}{3}$ (d) $\frac{32\sqrt{2}}{3}$

Passage VIII

Let $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-parallel unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}(\vec{b} + \vec{c})$.

(i) The angle between $\vec{a}$ and $\vec{c}$ is

(a) 90° (b) 60°

(c) 30° (d) none.

(ii) The angle between $\vec{a}$ and $\vec{b}$ is

(a) 90° (b) 60°

(c) 120° (d) 0°

(iii) The value of $|\vec{a} \times \vec{b}|^2 + |\vec{a} \times \vec{c}|^2$ is

(a) 1 (b) 2
(c) 0 (d) 4.

Passage IX

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 1$, $|\vec{c}|$, $|\vec{b}| = 4$ and $|\vec{b} \times \vec{c}| = \sqrt{15}$ and also $\vec{b} - 2\vec{c} = \lambda\vec{a}$.

(i) The angle between the vectors $\vec{b}$ and $\vec{c}$ is

(a) $\sin^{-1}\left(\frac{1}{4}\right)$ (b) $\cos^{-1}\left(\frac{1}{4}\right)$

(c) $\tan^{-1}\left(\frac{1}{4}\right)$ (d) $\cos^{-1}\left(\frac{1}{4}\right)$

(ii) The value of λ is

(a) ± 4 (b) ± 3
(c) ± 2 (d) ± 6

(iii) The angle between the vectors **a** and **b** is

(a) $\pi - \cos^{-1}\left(\frac{7}{8}\right)$

(b) $\pi + \cos^{-1}\left(\frac{7}{8}\right)$

(c) $-\pi + \cos^{-1}\left(\frac{7}{8}\right)$

(d) $\pi - \cos^{-1}\left(\frac{1}{8}\right)$.

Passage X

Vectors **x, y** and **z** each of magnitude $\sqrt{2}$ makes an angle of 60° with each other.

If $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{a}$, $\mathbf{y} \times (\mathbf{z} \times \mathbf{x}) = \mathbf{b}$ and $(\mathbf{x} \times \mathbf{y}) = \mathbf{c}$.

(i) The vector $\vec{x}$ is

(a) $\mathbf{x} = \frac{(\mathbf{a} + \mathbf{b}) \times \mathbf{c} - (\mathbf{a} + \mathbf{b})}{2}$

(b) $\mathbf{x} = \frac{(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})}{2}$

(c) $\mathbf{x} = \frac{(\mathbf{a} - \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})}{3}$

(d) $\mathbf{x} = \frac{(\mathbf{a} + \mathbf{b}) \times \mathbf{c} - (\mathbf{a} - \mathbf{b})}{3}$

(ii) The vector $\vec{y}$ is

(a) $\mathbf{y} = \frac{(\mathbf{a} - \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})}{\mathbf{2}}$

(b) $\mathbf{y} = \frac{(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})}{\mathbf{2}}$

(c) $\mathbf{y} = \frac{(\mathbf{a} - \mathbf{b}) \times \mathbf{c} + (\mathbf{a} - \mathbf{b})}{\mathbf{2}}$

(d) $\mathbf{y} = \frac{(\mathbf{a} - \mathbf{b}) \times \mathbf{c} - (\mathbf{a} + \mathbf{b})}{\mathbf{2}}$

(iii) The vector $\vec{z}$ is

(a) $\mathbf{z} = \frac{\mathbf{b} - \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{c}}{\mathbf{2}}$

(b) $\mathbf{z} = \frac{\mathbf{b} + \mathbf{a} - (\mathbf{a} + \mathbf{b}) \times \mathbf{c}}{\mathbf{2}}$

(c) $\mathbf{z} = \frac{\mathbf{b} - \mathbf{a} + (\mathbf{a} + \mathbf{b}) \times \mathbf{c}}{\mathbf{2}}$

(d) $\mathbf{z} = \frac{\mathbf{b} + \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{c}}{\mathbf{2}}$.

Passage XI

Let **x, y** and **z** be three unit vectors such that

$\vec{x} + \vec{y} + \vec{z} = \vec{a}$, $\vec{x} \times (\vec{y} \times \vec{z}) = \vec{b}$

$(\vec{x} \times \vec{y}) \times \vec{z}$, $\vec{c}$, $\vec{a} \cdot \vec{x} = \frac{3}{2}$, $\vec{a} \cdot \vec{y} = \frac{7}{4}$ and $|\vec{a}| = 2$.

(i) The vector $\vec{x}$ is

(a) $\mathbf{x} = \mathbf{a} + \frac{4}{3}\mathbf{b} + \frac{8}{3}\mathbf{c}$

(b) $\mathbf{x} = \mathbf{a} - \frac{4}{3}\mathbf{b} + \frac{8}{3}\mathbf{c}$

(c) $\mathbf{x} = \mathbf{a} + \frac{4}{3}\mathbf{b} - \frac{8}{3}\mathbf{c}$

(d) $\mathbf{x} = -\mathbf{a} + \frac{4}{3}\mathbf{b} - \frac{8}{3}\mathbf{c}$

(ii) The vector $\vec{y}$ is

(a) $\mathbf{y} = \mathbf{a} - 4\mathbf{c}$
(b) $\mathbf{y} = -4\mathbf{c}$
(c) $\mathbf{y} = \mathbf{a} + 4\mathbf{c} + 3\mathbf{b}$
(d) $\mathbf{y} = 2\mathbf{b} - 4\mathbf{c}$

(iii) The vector $\vec{z}$ is

(a) $\mathbf{z} = \frac{2}{3}(\mathbf{c} - \mathbf{b})$

(b) $\mathbf{z} = \frac{1}{3}(\mathbf{c} - \mathbf{b})$

(c) $\mathbf{z} = \frac{4}{3}(\mathbf{c} - \mathbf{b})$

(d) $\mathbf{z} = \frac{5}{3}(\mathbf{c} - \mathbf{b})$.

Passage XII

If $\vec{x} + \vec{y} = \vec{a}$, $\vec{x} \times \vec{y} = \vec{b}$, $\vec{x} \cdot \vec{a} = 1$.

(i) The vector $\vec{x}$ is

(a) $\vec{x} = \frac{1}{a^2}\left(\vec{a} + \vec{a} \times \vec{b}\right)$

(b) $\vec{x} = \frac{1}{a^2}\left(\vec{a} - \vec{a} \times \vec{b}\right)$

(c) $\vec{x} = -\frac{1}{a^2}(\vec{a} + \vec{a} \times \vec{b})$

(d) $\vec{x} = \frac{1}{a^2}(\vec{a} + \vec{b} \times \vec{a})$

(ii) The vector $\vec{y}$ is

(a) $\vec{y} = \vec{a} + \frac{1}{a^2}(\vec{a} + \vec{a} \times \vec{b})$

(b) $\vec{y} = \vec{a} - \frac{1}{a^2}(\vec{a} + \vec{a} \times \vec{b})$

(c) $\vec{y} = -\vec{a} + \frac{1}{a^2}(\vec{a} + \vec{a} \times \vec{b})$

(d) $\vec{y} = \vec{a} + \frac{1}{a^2}(\vec{a} - \vec{a} \times \vec{b})$.

(iii) The vector $(\vec{x} + \vec{y})$ is

(a) $\vec{a}$ (b) $-\vec{a}$

(c) $2\vec{a}$ (d) $3\vec{a}$.

Matrix Match
(For JEE-Advanced Examination Only)

1. **Match the following Columns:**

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

	Column I		Column II
(A)	The projection of $\vec{a}$ on $\vec{b}$ is	(P)	$\sqrt{\frac{26}{2}}$
(B)	The projection of $\vec{b}$ on $\vec{a}$ is	(Q)	$\frac{\sqrt{26}}{4}$
(C)	The area of a triangle formed by the vectors $\vec{a}$ and $\vec{b}$ is	(R)	$\frac{1}{3}$
(D)	The area of a parallelogram having diagonals $\vec{a}$ and $\vec{b}$ is	(S)	$\frac{1}{\sqrt{3}}$

2. **Match the following Columns:**

	Column I		Column II
(A)	If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} - \vec{b}\rvert$, the angle between $\vec{a}$ and $\vec{b}$ is	(P)	60°
(B)	If $\lvert\vec{a} + 2\vec{b}\rvert = \lvert\vec{a} - 2\vec{b}\rvert$, the angle between $\vec{a}$ and $\vec{b}$ is	(Q)	45°
(C)	If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} - 2\vec{b}\rvert$ and $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert$, the angle between $\vec{a}$ and $\vec{b}$ is	(R)	0°
(D)	If $\lvert\vec{a} + 2\vec{b}\rvert = \lvert\vec{a} - 3\vec{b}\rvert$, and $\lvert\vec{a}\rvert = 1$, the angle between $\vec{a}$ and $\vec{b}$ is	(S)	90°

3. **Match the following Columns:**

	Column I		Column II
(A)	If **a, b** and **c** be three unit vectors such that $\vec{a} + \vec{b} - \vec{c} = 0$, the angle between $\vec{a}$ and $\vec{b}$ is	(P)	60°
(B)	If **a, b** and **c** be unit vectors such that **a** is perpendicular to **b** and **c** and $\lvert\vec{a} + \vec{b} + \vec{c}\rvert = 1$, the angle between **b** and **c** is	(Q)	30°
(C)	If $\vec{a} + \vec{b} + \vec{c} = 0$ and $\lvert\vec{a}\rvert = 1$, $\lvert\vec{b}\rvert = 5$ $\lvert\vec{c}\rvert = 7$, the angle between $\vec{a}$ and $\vec{b}$ is	(R)	180°
(D)	If $\lvert\vec{a} \cdot \vec{b}\rvert = \sqrt{3}\lvert\vec{a} \times \vec{b}\rvert$, the angle between **a** and **b** is	(S)	120°

4. **Match the following Columns:**

	Column I		Column II
(A)	If **a** be any vector, the value of $\lvert(\lvert\vec{a} \cdot \hat{i}\rvert^2 + \lvert\vec{a} \cdot \hat{j}\rvert^2 + \lvert\vec{a} \cdot \hat{k}\rvert^2)^{1/2}\rvert$ is	(P)	$2\lvert\vec{a}\rvert$
(B)	If **a** be any vector, the value of $\lvert \hat{i} \times (\vec{a} \times \hat{i})\rvert^2\ \hat{j} - (\vec{a} \times \hat{j})\rvert^2 - (\vec{a} \times \hat{k})\rvert$	(Q)	$\sqrt{2}\lvert\vec{a}\rvert$
(C)	If **a** be any vector, the value of $(\lvert(\lvert\vec{a} \times \hat{i}\rvert^2 + \lvert\vec{a} \times \hat{j}\rvert^2 + \lvert\vec{a} \times \hat{k}\rvert^2)^{1/2}\rvert$	(R)	$\sqrt{3}\lvert\vec{a}\rvert$
(D)	If a be any vector, the value of $\lvert(\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k}\rvert$ is	(S)	$\lvert\vec{a}\rvert$

5. **Match the following Columns:**

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplaner vectors such that $[\vec{a}, \vec{b}, \vec{c}] = 2$.

	Column I		Column II
(A)	The value of $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}]$ is	(P)	12
(B)	the value of $[\vec{a} + 2\vec{b}, 2\vec{b} + 3\vec{c}, 3\vec{c} + \vec{a}]$ is	(Q)	16

(C)	The value of $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right]$ is	(R)	9
(D)	The value of $\left[\vec{a}\times 2\vec{b}, 2\vec{b}\times\vec{c}, \vec{c}\times\vec{a}\right]$ is	(S)	4

6. **Match the following Columns:**
If **a, b, c** and **a′**, **b′**, **c′** be the reciprocal systems of vectors.

Column I		Column II	
(A)	The value of $\mathbf{a}\cdot\mathbf{a}' + \mathbf{b}\cdot\mathbf{b}' + \mathbf{c}\cdot\mathbf{c}'$ is	(P)	2
(B)	The value of $\mathbf{a}'(\mathbf{a}+\mathbf{b}) + \mathbf{b}'\cdot(\mathbf{b}+\mathbf{c}) + \mathbf{c}'\cdot(\mathbf{c}+\mathbf{a})$ is	(Q)	0
(C)	The value of $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}'+\mathbf{b}'+\mathbf{c}')$ is	(R)	1
(D)	The value of $(\mathbf{a}\times\mathbf{a}' + \mathbf{b}\times\mathbf{b}' + \mathbf{c}\times\mathbf{c}')$ is	(S)	3

7. **Match the following Columns:**
If $\alpha+\beta+\gamma=2$ and $\vec{a}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, $\hat{i}+\hat{k}(\hat{k}\times a) = \vec{0}$.

Column I		Column II	
(A)	The value of $(\alpha+1)$ is	(P)	1
(B)	The value of (β^2+1) is	(Q)	2
(C)	The value of $(\lambda+1)$ is	(R)	0
(D)	The value of $(\beta+\gamma)$ is	(S)	3

8. **Match the following Columns:**
Let $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=\hat{j}+\hat{k}$, and $\vec{c}=\alpha a+\beta b$. If the vectors $\hat{i}-2\hat{j}+\hat{k}$, $3\hat{i}+2\hat{j}-\hat{k}$ and the vector $\vec{c}$ are coplanar.

Column I		Column II	
(A)	The value of $(\alpha+3\beta+2)$ is	(P)	1
(B)	The value of $\left(\frac{\alpha}{\beta}+4\right)$ is	(Q)	4
(C)	The value of $3\left(\frac{\beta}{\alpha}+\frac{2}{3}\right)$ is	(R)	6
(D)	The value of $3\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)+11$ is	(S)	2

9. **Match the following Columns:**
Given that $\vec{A}$, $\vec{B}$, $\vec{C}$ form a triangle such that $\vec{A}=\vec{B}+\vec{C}$. If $\vec{A}=a\hat{i}+\hat{j}+b\hat{j}+c\hat{k}$ and $\vec{B}=d\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{C}=3\hat{i}+\hat{j}-2\hat{k}$ and the area of the triangle is $5\sqrt{6}$ sq.u.

Column I		Column II	
(A)	The value of $(a+b+c)$ is	(P)	15
(B)	The value of $(a+c+d)$ is	(Q)	11
(C)	The value of $(b+c+d)$ is	(R)	19
(D)	The value of $(a+b+c+d)$ is	(S)	14

10. **Match the following Columns:**
If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be unit vectors satisfying
$|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$.

Column I		Column II	
(A)	The value of $\|2\vec{a}+5\vec{b}+4\vec{c}\|$ is	(P)	1
(B)	The value of $\|2\vec{a}+3\vec{b}+3\vec{c}\|$ is	(Q)	5
(C)	The value of $\|2\vec{a}+\vec{b}+\vec{c}\|$ is	(R)	4
(D)	The value of $\|2\vec{a}+7\vec{b}+7\vec{c}\|$ is	(S)	3

11. **Match the following Columns:**

Column I		Column II	
(A)	The volume of the parallelopiped determined by the vectors $\vec{a}, \vec{b}, \vec{c}$ is 2. The volume of the parallelopiped determined by the vectors $2(\vec{a}\times\vec{b})$, $3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	(P)	100
(B)	The volume of the parallelopiped determined by the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is 5. The volume of the parallelopiped determined by the vectors $3(\vec{a}+\vec{b})$, $(\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	(Q)	30
(C)	The area of a triangle with adjacent sides determined by the vectors $\vec{a}$ and $\vec{b}$ is 20. The area of the triangle with adjacent sides determined by the vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	(R)	24
(D)	The area of a parallelogram with adjacent sides determined by the vectors $\vec{a}$ and $\vec{b}$ is 30. The area of the parallelogram with adjacent sides determined by the vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	(S)	60

Questions asked in Previous Years' JEE-Advanced Examinations

1. The value of $\mathbf{A}\cdot\{(\mathbf{B}+\mathbf{C})\times(\mathbf{A}+\mathbf{B}+\mathbf{C})\}$ is
(a) 0 (b) $[\mathbf{ABC}]+[\mathbf{BCA}]$
(c) $[\mathbf{ABC}]$ (d) none
[IIT-JEE, 1981]

2. Let **A, B, C** be there unit vectors. Suppose that $\mathbf{A}\cdot\mathbf{B} = \mathbf{0} = \mathbf{A}\cdot\mathbf{C}$ and the angle between **B** and **C** is $\pi/6$. Then $\mathbf{A} = \pm 2(\mathbf{B}\times\mathbf{C})$. Is it true or false?
[IIT-JEE, 1981]

3. For non-zero-vectors a, b, c $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$ holds if and only if
(a) $\mathbf{a}\cdot\mathbf{b}=0$, $\mathbf{b}\cdot\mathbf{c}=0$
(b) $\mathbf{c}\cdot\mathbf{a}=0$, $\mathbf{b}\cdot\mathbf{c}=0$
(c) $\mathbf{c}\cdot\mathbf{a}=0$, $\mathbf{a}\cdot\mathbf{b}=0$
(d) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0$
[IIT-JEE, 1982]

4. $A_1, A_2, ..., A_n$ are the vertices of a regular polygon with n sides and O its centre. Show that
$$\sum_{\mathbf{i=1}}^{\mathbf{n-1}}(\mathbf{OA_i}\times\mathbf{OA_i+1}) = (1-n)(\mathbf{OA_2}\times\mathbf{OA_1}+\mathbf{1}).$$
[IIT-JEE, 1982]

5. Find all values of λ such that $(x, y, z) \neq (0, 0, 0)$ and
$\mathbf{x}(\mathbf{i}, \mathbf{j}, 3\mathbf{k}) + \mathbf{y}(3\mathbf{i}-3\mathbf{j}+\mathbf{k}) + \mathbf{z}(-4\mathbf{i}+5\mathbf{j}) = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$,
where **i**, **j**, **k** are unit vectors along the co-ordinate axes.
[IIT-JEE, 1982]

6. The points with position vectors $60\mathbf{i}+3\mathbf{j}$, $40\mathbf{i}-8\mathbf{j}$, $a\mathbf{i}-52\mathbf{j}$ are collinear if
(a) $a=-40$ (b) $a=40$
(c) $a=20$ (d) None.
[IIT-JEE, 1983]

7. If $\mathbf{X}\cdot\mathbf{A} = \mathbf{0} = \mathbf{X}\cdot\mathbf{B} = \mathbf{X}\cdot\mathbf{C}$ for some non-zero vector **X**, then $[\mathbf{A\,B\,C}] = 0$. Is it true or false?
[IIT-JEE, 1983]

8. The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i}-3\mathbf{j}$, $\mathbf{OB} = \mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{OC} = 3\mathbf{i}-\mathbf{k}$ is
(a) 4/13 (b) 4
(c) 2/7 (d) None.
[IIT-JEE, 1983]

9. If c be given non-zero scalar and **A** and **B** be given non zero vectors such that $\mathbf{A}\perp\mathbf{B}$. Find the vector **X** which satisfies the equations $\mathbf{A}\cdot\mathbf{X} = \mathbf{c}$, $\mathbf{A}\times\mathbf{X} = \mathbf{B}$
[IIT-JEE, 1983]

10. A vector **A** has components A_1, A_2, A_3 in a right handed rectangular cartesian co-ordinate system $OXYZ$. The co-ordinate system is rotated about the x-axis through an angle $\frac{\pi}{2}$. Find the components of **A** in the new co-ordinate system in terms of A_1, A_2, A_3
[IIT-JEE, 1983]

11. The points with position vectors $a+b$, $a-b$ and $a+k\,b$ are collinear for all real values of k. Is it true or false?
[IIT-JEE, 1984]

12. Let $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$ be three non-zero vector such that **c** is a unit vector perpendicular to both the vectors **a** and **b.** If the angle between **a** and **b** be $\pi/6$, then
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ is equal to}$$
(a) 0
(b) 1
(c) $\frac{1}{4}\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)$
(d) $\frac{3}{4}\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)\left(c_1^2+c_2^2+c_3^2\right)$
[IIT-JEE, 1986]

13. A vector **a** has components $2p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter-clockwise sense. If, with respect to the new system, **a** has component $p+1$ and 1, then
(a) $p \neq 0$
(b) $p = 1$ or $p = -1/3$
(c) $p = -1$ or $p = 1/3$
(d) $p = 1$ or $p = -1$
[IIT-JEE, 1986]

14. The position vectors of the points A, B, C and D are $3\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $-\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $4\mathbf{i}+5\mathbf{j}+\lambda\mathbf{k}$ respectively. If the points A, B, C and D lie on a plane, find the value λ.
[IIT-JEE, 1986]

15. The number of vectors of unit length perpendicular to vectors $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (0, 1, 1)$ is
(a) 1 (b) 2
(c) 3 (d) infinite
[IIT-JEE, 1987]

16. If the vectors $a\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{i}+b\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+c\mathbf{k}$, $(a, b, c \neq 1)$ are coplanar, the value of
$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} \text{ is ...}$$
[IIT-JEE, 1987]

17. Let $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\mathbf{b}$ and $\mathbf{c}$ respectively, are given by ...

[IIT-JEE, 1987]

18. If A, B, C, D are four points in the space, prove that $|\mathbf{AB} \times \mathbf{CD} \times \mathbf{BC} \times \mathbf{AD} \times \mathbf{CA} \times \mathbf{BD}|$
$$= 4(\text{area of } \Delta ABC)$$

[IIT-JEE, 1987]

19. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors and $\mathbf{p}, \mathbf{q}$ $\mathbf{r}$ are vectors defined by the relations
$$\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \text{ and } \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]},$$
the value of the expression
$(\mathbf{a} + \mathbf{b})\cdot\mathbf{p} + (\mathbf{b} + \mathbf{c})\mathbf{q} + (\mathbf{c} + \mathbf{a})\cdot\mathbf{r}$ is equal to
(a) 0 (b) 1
(c) 2 (d) 3

[IIT-JEE, 1988]

20. The components of a vector $\mathbf{a}$ along and perpendicular to a non-zero vector $\mathbf{b}$ are ... and ... respectively.

[IIT-JEE, 1988]

21. Let $OACB$ be a parallelogram with O at the origin and OC a diagonal. Let D be the mid-point of OA. Using vectors method, prove that BD and CO intersect in the same ratio. Determine the same ratio.

[IIT-JEE, 1988]

22. For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$,
$(\mathbf{a} - \mathbf{b})\cdot\{(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})\} = 2a\cdot(\mathbf{b} \times \mathbf{c})$.
Is it true or false?

[IIT-JEE, 1989]

23. If vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar, show that
$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix} = \mathbf{0}$$

[IIT-JEE, 1989]

24. In a triangle OAB, E is the mid-point of BO and D is a point on AB such that $AD:DB = 2:1$. If OD and AE intersect at P, determine the ratio $OP:OD$, using vector method.

[IIT-JEE, 1989]

25. Let $\mathbf{A} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{C} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Determine a vector $\mathbf{R}$ satifying $\mathbf{R} \times \mathbf{B} = \mathbf{C} \times \mathbf{B}$ and $\mathbf{R}\cdot\mathbf{A} = 0$.

[IIT-JEE, 1990]

26. Determine the value of c so that for all real x, the vectors $cx\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $x\mathbf{i} + 2\mathbf{j} + 2cx\mathbf{k}$ make an obtuse angle with each other.

[IIT-JEE, 1991]

27. Given that $\mathbf{a} = (1, 1, 1)$, $\mathbf{c} = (0, 1 -1,)$ $\mathbf{a}\cdot\mathbf{b} = 3$ and $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then $\mathbf{b} = ...$

[IIT-JEE, 1991]

28. A unit vector coplanar with $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$ is ...

[IIT-JEE, 1992]

29. Let a, b, c be distinct non-negative numbers. If the vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $\mathbf{i} + \mathbf{k}$ and $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ lie on a plane, then c is
(a) the AM of a and b
(b) the GM of a and b
(c) The HM of a and b
(d) equal to zero.

[IIT-JEE, 1993]

30. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ be three vectors. A vector in the plane of $\mathbf{b}$ and $\mathbf{c}$, whose projection on $\mathbf{a}$ is $\sqrt{\frac{2}{3}}$, is
(a) $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$
(c) $-2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ (d) $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

[IIT-JEE, 1993]

31. In a triangle ABC, D and E are points on BC and AC respectively, such that $BD = 2DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. Find the ratio $\frac{BP}{PE}$, using vector method.

[IIT-JEE, 1993]

32. Let $\mathbf{p}$ and $\mathbf{q}$ be the position vectors of P and Q, respectively, with respect to O and $|\mathbf{p}| = p$, $|\mathbf{q}| = q$. The points R and S divide PQ internally and externally in the ratio $2:3$ respectively. If OR and OS are perpendicular, then
(a) $9p^2 = 4q^2$ (b) $4p^2 = 9q^2$
(c) $9p = 4q$ (d) $4p = 9q$

[IIT-JEE, 1994]

33. Let α, β and γ be distinct real numbers. The points with position vectors $\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, $\beta\mathbf{i} + \gamma\mathbf{j} + \alpha\mathbf{k}$ and $\gamma\mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$
(a) are collinear
(b) form an equilateral triangle
(c) form a scalene triangle
(d) form a right angled triangle.

[IIT-JEE, 1994]

34. The vector $\mathbf{d} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ is
(a) a unit vector.

(b) makes an angle $\frac{\pi}{3}$ with the vector.

$(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$

(c) parallel to the vector $\left(-\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}\right)$

(d) perpendicular to the vector $(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

[IIT-JEE, 1994]

35. A unit vector perpendicular to the plane determined by the points

$P(1, -1, 2)$, $Q(2, 0, -1)$ and $R(0, 2, 1)$ is ...

[IIT-JEE, 1994]

36. If the vectors **a**, **b**, **c**, **d** are not coplanar, prove that the vector $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$ is parallel to **a**.

[IIT-JEE, 1994]

37. Let $\mathbf{a} = \mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{k} - \mathbf{i}$. If d be a unit vector such that $\mathbf{a}\cdot\mathbf{d} = \mathbf{0} = [\mathbf{b}, \mathbf{c}, \mathbf{d}]$, then d equals

(a) $\pm\left(\frac{\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{6}}\right)$ (b) $\pm\left(\frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{3}}\right)$

(c) $\pm\left(\frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}\right)$ (d) $\pm\,\mathbf{k}$

[IIT-JEE, 1995]

38. If **a**, **b** and **c** be non-coplanar unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c})$, the angle between **a** and **b** is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

[IIT-JEE, 1995]

39. Let **u, v** and **w** be the vectors such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$. If $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$, $|\mathbf{w}| = 5$, the value of $\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}$ is

(a) 47 (b) –25

(c) 0 (d) 25

[IIT-JEE, 1995]

40. If **a, b** and **c** are three non-coplaner vectors,

$(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\{(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})\}$ is

(a) 0 (b) $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

(c) $2[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ (d) $-[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

[IIT-JEE, 1995]

41. A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors **i**, **i** + **j** and the line determined by the vectors **i** – **j**, **i** + **k**. The angle between **a** and the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is ...

[IIT-JEE, 1996]

42. If **b** and **c** any two non-collinear unit vectors and **a** be any vector, then

$$(\mathbf{a}\cdot\mathbf{b})\mathbf{b} + (\mathbf{a}\cdot\mathbf{c})\mathbf{c} + \frac{\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})}{|(\mathbf{b} \times \mathbf{c})|}(\mathbf{b} \times \mathbf{c}) = \ldots$$

[IIT-JEE, 1996]

43. The position vectors of the vertices A, B, C of a tetrahedron $ABCD$ are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, **i** and 3**i,** respectively. The altitude from vertex D to the opposite face ABC, meets the median line through A of the $\Delta\, ABC$ at E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$. Find the position vector of E for all its possible positions.

[IIT-JEE, 1996]

44. Let **p, q** and **r** be three mutually perpendicular vectors of the same magnitude. If a vector **x** satisfies the equation $\mathbf{p} \times [(\mathbf{x} - \mathbf{q}) \times \mathbf{p}] + \mathbf{q} \times [(\mathbf{x} - \mathbf{r}) \times \mathbf{q}] + \mathbf{r} \times [(\mathbf{x} - \mathbf{p}) \times \mathbf{r}] = \mathbf{0}$, then **x** is given by

(a) $\frac{1}{2}(\mathbf{p} + \mathbf{q} - 2\mathbf{r})$ (b) $\frac{1}{2}(\mathbf{p} + \mathbf{q} + \mathbf{r})$

(c) $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r})$ (d) $\frac{1}{3}(2\mathbf{p} + \mathbf{q} - \mathbf{r})$

[IIT-JEE, 1997]

45. Let $OA = \mathbf{a}$, $\mathbf{OB} = 10\mathbf{a} + 2\mathbf{b}$ and $\mathbf{OC} = \mathbf{b}$ where O, A and C are non-collinear points. Let p denotes the area of the quadrilateral $OABC$ and q denotes the area of the parallelogram with OA and OC as adjacent sides. If $p = kq$ then $k = \ldots$

[IIT-JEE, 1997]

46. Let **a**, **b**, and **c** be three vectors having magnitudes 1, 1 and 2, respectively. If $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + \mathbf{b} = \mathbf{0}$, the acute angle between **a** and **c** is ...

[IIT-JEE, 1997]

47. If **A**, **B** and **C** are vectors such that $|\mathbf{B}| = |\mathbf{C}|$. Prove that $|(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} + \mathbf{C})| \times (\mathbf{B} \times C)\cdot(\mathbf{B} + \mathbf{C}) = 0$

[IIT-JEE, 1997]

48. Let **a**, **b** and **c** be non-coplaner unit vectors, equally inclined to one another at an angle θ.

If $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$,

find scalars p, q and r in terms of θ.

[IIT-JEE, 1997]

49. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$ are linearly independent vectors and $|c| = \sqrt{3}$, then

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 1, \beta = \pm 1$

(c) $\alpha = -1, \beta = \pm 1$ (d) $\alpha = \pm 1, \beta = 1$

[IIT-JEE, 1998]

50. For the vectors **u, v** and **w**, which of the following expressions is not equal to any of the remaining three?

(a) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ (b) $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}$
(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})$ (d) $(\mathbf{u}\times\mathbf{w})\cdot\mathbf{w}$

[IIT-JEE, 1998]

51. Which of the following expressions are meaningful?
(a) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ (b) $(\mathbf{u}\cdot\mathbf{w})\cdot\mathbf{w}$
(c) $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$ (d) $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$

[IIT-JEE, 1998]

52. For any two vectors u and v, prove that
(i) $(\mathbf{u}\cdot\mathbf{v})^2 + |\mathbf{u}\times\mathbf{v}|^2 = (u^2v^2)$
(ii) $(1+|\mathbf{u}|^2)(1+|\mathbf{v}|^2)$
$= |\mathbf{u}+\mathbf{v}+(\mathbf{u}\times\mathbf{v})|^2 + (1-(\mathbf{u}\cdot\mathbf{v}))^2$

[IIT-JEE, 1998]

53. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-collinear unit vectors. If $\mathbf{u} = \mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}$ and $\mathbf{v} = \mathbf{a}\times\mathbf{b}$, then $|\mathbf{v}|$ is
(a) $|\mathbf{u}|$ (b) $|\mathbf{u}| + |\mathbf{u}\cdot\mathbf{a}|$
(c) $|\mathbf{u}| + |\mathbf{u}\cdot\mathbf{b}|$ (d) $|\mathbf{u}| + \mathbf{u}\cdot(\mathbf{a}+\mathbf{b})$

[IIT-JEE, 1999]

54. Let $\mathbf{a} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = \mathbf{i}+\mathbf{j}$. If $\mathbf{c}$ be a vector such that $\mathbf{a}\cdot\mathbf{c} = |\mathbf{c}|$, $|\mathbf{c}-\mathbf{a}| = 2\sqrt{2}$ and the angle between $(\mathbf{a}\times\mathbf{b})$ and $\mathbf{c}$ is 30°, the value of $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|$ is
(a) 2/3 (b) 3/2
(c) 2 (d) 3.

[IIT-JEE, 1999]

55. Let $\mathbf{a} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{b} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$ and a unit vector $\mathbf{c}$ be coplanar. If $\mathbf{c}$ is perpendicular to $\mathbf{a}$, then $\mathbf{c}$ is
(a) $\frac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k})$ (b) $\frac{1}{\sqrt{2}}(-\mathbf{i}-\mathbf{j}-\mathbf{k})$
(c) $\frac{1}{\sqrt{5}}(-\mathbf{i}-2\mathbf{j})$ (d) $\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k})$

[IIT-JEE, 1999]

56. Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors. If w be a vector such that $\mathbf{w}\times(\mathbf{w}\times\mathbf{u}) = \mathbf{v}$, prove that $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| \le \frac{1}{2}$ and that the equality holds if and only if $\mathbf{u}$ is perpendicular to $\mathbf{v}$.

[IIT-JEE, 1999]

57. If the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ from the sides BC, CA and AB, respectively, of a triangle ABC, then
(a) $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a} = \mathbf{0}$
(b) $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$
(c) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a}$
(d) $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = \mathbf{0}$

[IIT-JEE, 2000]

58. Let the vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be such that $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{0}$. Let P_1 and P_2 be the two planes determined by the pair of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{d}$ respectively, then the angle between P_1 and P_2 is
(a) 0 (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

[IIT-JEE, 2000]

59. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit coplanar vectors, then the value of $[2\mathbf{a}-\mathbf{b}\ 2\mathbf{b}-\mathbf{c}\ 2\mathbf{c}-\mathbf{a}]$ is
(a) 0 (b) 1
(c) $-\sqrt{3}$ (d) $\sqrt{3}$

[IIT-JEE, 2000]

60. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors, then $|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2$ does not exceed
(a) 4 (b) 9
(c) 8 (d) 6

[IIT-JEE, 2000]

61. Let $\mathbf{a} = \mathbf{i}-\mathbf{k}$, $\mathbf{b} = x\mathbf{i}+\mathbf{j}+(1-x)\mathbf{k}$ and $\mathbf{c} = y\mathbf{i}+x\mathbf{j}+(1+x-y)\mathbf{k}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on
(a) only x (b) only y
(c) neither x nor y (d) both x and y.

[IIT-JEE, 2001]

62. Find 3-dimensional vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ satisfying $\mathbf{v_1}\cdot\mathbf{v_1} = 4$, $\mathbf{v_1}\cdot\mathbf{v_2} = -2$ and $\mathbf{v_1}\cdot\mathbf{v_3} = 6$, $\mathbf{v_2}\cdot\mathbf{v_2} = 2$, $\mathbf{v_2}\cdot\mathbf{v_3} = 5$ and $\mathbf{v_3}\cdot\mathbf{v_3} = 29$.

[IIT-JEE, 2001]

63. Let $\mathbf{A}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j}$ and $\mathbf{B}(t) = g_1(t)\mathbf{i} + g_2(t)\mathbf{j}$, $t \in [0, 1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are non-zero vectors for all t and $\mathbf{A}(0) = 2\mathbf{i}+3\mathbf{j}$, $\mathbf{A}(1) = 6\mathbf{i}+2\mathbf{j}$, $\mathbf{B}(0) = 3\mathbf{i}+2\mathbf{j}$, $\mathbf{B}(1) = 2\mathbf{i}+6\mathbf{j}$, show that $\mathbf{A}(t)$, $\mathbf{B}(t)$ are parallel for some t.

[IIT-JEE, 2001]

64. Let V be the volume of the parallelopiped formed by the vectors $a = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $c = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.
If a_r, b_r, c_r where $r = 1, 2, 3$, are non-negative real numbers and
$\sum_{r=1}^{3}(a_r+b_r+c_r) = 3L$,
Show that $V \le L^3$.

[IIT-JEE, 2002]

65. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors such that $\mathbf{a}+2\mathbf{b}$ and $5\mathbf{a}-4\mathbf{b}$ are perpendicular to each other, the angle between $\mathbf{a}$ and $\mathbf{b}$ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

[IIT-JEE, 2002]

66. Let $\mathbf{v} = 2\,\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{w} = \mathbf{i} + 3\,\mathbf{k}$ and $\mathbf{u}$ is a unit vector, the maximum value of $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is

(a) -1 (b) $\sqrt{10} + \sqrt{6}$

(c) $\sqrt{59}$ (d) $\sqrt{60}$

[IIT-JEE, 2002]

67. The value of **a** be such so that the volume of a parallelopiped formed by the vectors $\mathbf{i} + a\mathbf{j} + \mathbf{k}$, $\mathbf{j} + a\mathbf{k}$, $a\mathbf{i} + \mathbf{k}$ becomes minimum, is

(a) -3 (b) 3

(c) $1/\sqrt{3}$ (d) $\sqrt{3}$

[IIT-JEE, 2003]

68. If **u, v, w** are three non-coplaner unit vectors and α, β, γ are the angle between **u** and **v**, **v** and **w** and **w** and **u** respectively, and **x, y, z** are unit vectors along the bisectors of the angle α, β, γ respectively. Prove that $[\mathbf{x} \times \mathbf{y}, \mathbf{y} \times \mathbf{z}, \mathbf{z} \times \mathbf{x}]$

$$= \frac{1}{16}\,[u\ v\ w]^2 \sec^2\left(\frac{\alpha}{2}\right)\sec^2\left(\frac{\beta}{2}\right)\sec^2\left(\frac{\gamma}{2}\right)$$

[IIT-JEE, 2003]

69. If **a, b, c** and **d** are distinct vectors satisfying relation $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$. Prove that $\mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$

[IIT-JEE, 2004]

70. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{a}.\ \mathbf{b} = 1$ and $\mathbf{a} \times \mathbf{b} = \mathbf{j} - \mathbf{k}$, then **b** is

(a) $\mathbf{i} - \mathbf{j} + \mathbf{k}$ (b) $2\mathbf{j} - \mathbf{k}$

(c) $\mathbf{i}$ (d) $2\mathbf{i} - \mathbf{k}$

[IIT-JEE, 2004]

71. The unit vector which is orthogonal to the vector $5i + 2j + 6k$ and is coplanar with the vectors $2i + j + k$ and $i - j + k$ is

(a) $\frac{1}{\sqrt{41}}(2\mathbf{i} - 6\mathbf{j} + \mathbf{k})$

(b) $\frac{1}{\sqrt{29}}(2\mathbf{i} - 5\mathbf{j})$

(c) $\frac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k})$

(d) $\frac{1}{\sqrt{69}}(2\mathbf{i} - 8\mathbf{j} + \mathbf{k})$

[IIT-JEE, 2004]

72. Let **v** be a unit vector along the incident ray, **w** be a vector along the reflected ray and **a** be a unit vector along the outward normal to the plane mirror at the point of incidence *P*. Express **w** in terms of **a** and **v**.

[IIT-JEE, 2005]

73. Let **a, b, c** be three non-coplanar vectors and

$$\mathbf{b}_1 = \mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a},\ \mathbf{b}_2 = \mathbf{b} + \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a},$$

$$\mathbf{c}_1 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} + \frac{\mathbf{c}\cdot\mathbf{b}}{|\mathbf{c}|^2}\mathbf{b}_1,$$

$$\mathbf{c}_2 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \frac{\mathbf{b}_1\cdot\mathbf{c}}{|\mathbf{b}_1|^2}\mathbf{b}_1,$$

$$\mathbf{c}_3 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}|^2}\mathbf{a} + \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{b}_1,$$

and $\mathbf{c}_4 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}|^2}\mathbf{a} - \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}_1|^2}\mathbf{b}_1,$

the triplet of pairwise orthogonal vectors is

(a) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_1)$ (b) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_2)$

(c) $(\mathbf{a}, \mathbf{b}_2, \mathbf{c}_2)$ (d) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_3)$

[IIT-JEE, 2005]

74. Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. A vector in the plane of **a** and **b**, whose projection on **c** is $1/\sqrt{3}$, is

(a) $4\mathbf{i} - \mathbf{j} + \mathbf{k}$ (b) $3\mathbf{i} + \mathbf{j} + \mathbf{k}$

(c) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (d) $4\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

[IIT-JEE,2006]

75. The number of distinct real values of γ which the vectors $-\gamma^2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \gamma^2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \gamma^2\mathbf{k}$ are coplanar is

(a) 0 (b) 1

(c) 2 (d) 3

[IIT-JEE, 2007]

76. Let **a, b** and **c** be unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, Which one of the following is correct?

(a) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = 0$

(b) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq 0$

(c) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{c} \neq 0$

(d) $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ are mutually perpendicular.

[IIT-JEE, 2007]

77. Let the vectors **PQ, QR, RS, ST, TU** and **UP** represent the sides of a regular hexagon.

Statement I: $\mathbf{PQ} \times (\mathbf{RS} + \mathbf{ST}) \neq \mathbf{0}$

Statement II: $\mathbf{PQ} \times \mathbf{RS} = \mathbf{0}$ and $\mathbf{PQ} \times \mathbf{ST} \neq \mathbf{0}$

[IIT-JEE, 2007]

78. Let two non-collinear unit vectors **a** and **b** form an acute angle. *A* point *P* moves so that at any time *t*, the position vector **OP** (where *O* is the origin) is given by $\mathbf{a}\cos t + \mathbf{b}\sin t$, where *P* is the farthest from origin *O*, let *M* be the length of *OP* and *u* be the unit vector along *OP*. Then

(a) $\mathbf{u} = \dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$ and $M = (1 + \mathbf{a}\cdot\mathbf{b})^{1/2}$

(b) $\mathbf{u} = \dfrac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}-\mathbf{b}|}$ and $M = (1 + \mathbf{a}\cdot\mathbf{b})^{1/2}$

(c) $\mathbf{u} = \dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$ and $M = (1 + 2\mathbf{a}\cdot\mathbf{b})^{1/2}$

(d) $\mathbf{u} = \dfrac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}-\mathbf{b}|}$ and $M = (1 + 2\mathbf{a}\cdot\mathbf{b})^{1/2}$

[IIT-JEE, 2008]

79. The edges of a parallelopiped are of length and parallel to non-coplanar unit vectors **a, b, c** such that $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = \frac{1}{2}$. The volume of the parallelopiped is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{3}}$

[IIT-JEE, 2008]

80. If **a, b, c** and **d** are unit vectors such that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 1$ and $(\mathbf{a}\times\mathbf{c}) = \frac{1}{2}$, then

(a) **a, b, c** are non-coplanar

(b) **b, c, d** are non-coplanar

(c) **b, d** are non-parallel

(d) **a, d** are parallel and **b, c** are parallel

[IIT-JEE, 2009]

81. The volume of the parallelopiped with its edges represented by the vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+2\mathbf{j}$ and $\mathbf{i}+\mathbf{j}+\pi\mathbf{k}$ is...

[IIT-JEE, 2009]

82. Angle between vectors **a** and **b** where **a**, **b** and **c** are unit vectors satisfying **a** + **b** + R3**c** = **0** is ...

[IIT-JEE, 2009]

83. Let P, Q, R, S be the points on the plane with position vectors $-2\mathbf{i}-\mathbf{j}$, $4\mathbf{i}$, $3\mathbf{i}+3\mathbf{j}$ and $-3\mathbf{i}+2\mathbf{j}$ respectively. The quadrilateral $PQRS$ must be a

(a) parallelogram, which is neither a rhombus nor a rectangle

(b) square

(c) rectangle but not a square

(d) rhombus but not a square.

[IIT-JEE, 2010]

84. Two adjacent sides of a parallelogram $ABCD$ are given by $AB = 2\mathbf{i} + 10\mathbf{j} + 11\mathbf{k}$ and $\mathbf{AD} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, the cosine of the angle α is given by

(a) 8/9 (b) $\sqrt{17}/9$

(c) 1/9 (d) $4\sqrt{5}/9$

[IIT-JEE, 2010]

85. Let non-zero vectors **a, b** and **c** such that $\mathbf{a}\cdot\mathbf{b} = 0$, $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}+\mathbf{c}) = 0$ and $2|\mathbf{b}+\mathbf{c}| = |\mathbf{b}-\mathbf{a}|$.
If $\mathbf{a} = \mu\mathbf{b} + 4\mathbf{c}$, the possible values of μ are ...

[IIT-JEE, 2010]

86. If **a** and **b** be vectors in the space given by $\mathbf{a} = \dfrac{\mathbf{i}-2\mathbf{j}}{\sqrt{5}}$ and $\dfrac{2\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{14}}$, then the value of $(2\mathbf{a}+\mathbf{b})\cdot\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-2\mathbf{b})\}$ is

[IIT-JEE, 2010]

87. Let $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b} = \mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{c} = \mathbf{i}-\mathbf{j}-\mathbf{k}$ be three vectors. A vector **v** in the plane of **a** and **b**, whose projection on **c** is $\frac{1}{\sqrt{3}}$, is given by

(a) $\mathbf{i}-3\mathbf{j}-3\mathbf{k}$ (b) $-3\mathbf{i}-3\mathbf{j}+\mathbf{k}$

(c) $3\mathbf{i}-\mathbf{j}+3\mathbf{k}$ (d) $\mathbf{i}+3\mathbf{j}-3\mathbf{k}$

[IIT-JEE, 2011]

88. The unit vector(s) which is/are coplanar with vectors $\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and perpendicular to the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is given by

(a) $\mathbf{j}-\mathbf{k}$ (b) $-\mathbf{i}+\mathbf{j}$

(c) $\mathbf{i}-\mathbf{j}$ (d) $-\mathbf{j}+\mathbf{k}$

[IIT-JEE, 2011]

89. Let $\mathbf{a} = \mathbf{i}-\mathbf{k}$, $\mathbf{b} = -\mathbf{i}+\mathbf{j}$ and $\mathbf{c} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ be three given vectors. If **r** be a vector such that $\mathbf{r}\times\mathbf{b} = \mathbf{c}\times\mathbf{b}$ and $\mathbf{r}\cdot\mathbf{a} = 0$, the value of $\mathbf{r}\cdot\mathbf{b}$ is ...

[IIT-JEE, 2011]

90. If $\mathbf{a} = \mathbf{j} + \sqrt{3}\mathbf{k}$, **b** = –**j** + R3**k** and $\mathbf{c} = 2\sqrt{3}\mathbf{k}$ form a triangle, the internal angle of the triangle between **a** and **b** is.

[IIT-JEE, 2011]

91. If **a**, **b** and **c** are unit vectors satisfying $|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2 = 9$, then $|2\mathbf{a}+5\mathbf{b}+5\mathbf{c}|$ is.

[IIT-JEE, 2012]

92. If **a** and **b** are vectors such that $|\mathbf{a}+\mathbf{b}| = \sqrt{29}$ and $\mathbf{a}\times(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = (2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\times\mathbf{b}$, a possible value of $(\mathbf{a}+\mathbf{b})\cdot(-7\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ is

(a) 0 (b) 3

(c) 4 (d) 8.

[IIT-JEE, 2012]

93. Let $\mathbf{PR} = 3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{SQ} = \mathbf{i}-3\mathbf{j}-4\mathbf{k}$ determine diagonals of a parallelogram $PQRS$ and $\mathbf{PT} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ be another vector. The volume of the parallelopiped determined by the vectors **PT, PQ** and **PS** is

(a) 5 (b) 20
(c) 10 (d) 30

[IIT-JEE, 2013]

94. The volume of the parallelopiped determined by the vectors **a, b** and **c** is 2. The volume of the parallelopiped determined the vectors (**a** × **b**), **3**(**b** × **c**) and (**c** × **a**) is
(a) 100 (b) 30
(c) 24 (d) 60

[IIT-JEE, 2013]

95. The volume of the parallelopiped determined by the vectors **a, b** and **c** is 5. The volume of the parallelopiped determined the vectors **3**(**a** + **b**), (**b** + **c**) and **2**(**c** + **a**) is
(a) 100 (b) 30
(c) 24 (d) 60

[IIT-JEE, 2013]

96. The area of a triangle with adjacent sides determined by the vectors **a** and **b** is 20. The area of the triangle with adjacent sides determined by the vectors (**2a** + **3b**) and (**a** – **b**) is
(a) 100 (b) 30
(c) 24 (d) 60

[IIT-JEE, 2013]

97. The area of a parallelogram with adjacent sides determined by the vectors **a** and **b** is 30. The area of the parallelogram with adjacent sides determined by the vectors (**a** + **b**) and **a** is
(a) 100 (b) 30
(c) 24 (d) 60

[IIT-JEE, 2013]

98. Let **x, y** and **z** be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If **a** be a non-zero vector perpendicular to **x** and (**y** × **z**) and **b** is another non-zero vector perpendicular to **y** and (**z** × **x**), then
(a) $\mathbf{b} = (\mathbf{b}\cdot\mathbf{z})(\mathrm{z} - \mathbf{x})$
(b) $\mathbf{a} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{y} - \mathbf{z})$
(c) $(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{y})(\mathbf{b}\cdot\mathbf{z})$
(d) $\mathbf{a} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{z} - \mathbf{y})$

[IIT-JEE, 2014]

99. If **a**, **b** and **c** are three non-coplaner unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$, where p, q and r are scalears, find the value of $\left(\frac{p^2 + 2q^2 + r^2}{q^2}\right)$ is... **[IIT-JEE, 2014]**

100. Let PQR be a triangle. $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$, $\vec{c} = \overrightarrow{PQ}$
If $|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}$, and $\vec{b}\cdot\vec{c} = 24$, which of the following is (are) true?
(a) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$
(b) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$
(c) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$
(d) $\vec{a}\cdot\vec{b} = -72$

[IIT-JEE, 2015]

101. In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and $\alpha = 2 + \sqrt{3}\beta$, find the possible values of $|\alpha|$.

[IIT-JEE, 2015]

102. Suppose that $\vec{p}$, $\vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3.
Let the components of a vector $\vec{s}$, along $\vec{p}$, $\vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of a vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r})$, $(\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x, y and z, respectively, find the value of $2x + y + z$.

[IIT-JEE, 2015]

103. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in R^3 and $\vec{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + \hat{k})$. Given that there exists a vector $\vec{v}$ in R^3 such that $|\vec{u} \times \vec{v}| = 1$ and $\hat{w}.(\vec{u} \times \vec{v}) = 1$.
Which of the following statement (s) is (are) correct?
(a) There is exactly one choice for such $\vec{v}$.
(b) There are infinitely many choices for such $\vec{v}$.
(c) If $\vec{u}$ lies in the xy-plane, then $|u1| = |u2|$.
(d) If $\vec{u}$ lies in the xz-plane, then $2|u1| = |u3|$.

[IIT-JEE, 2016]

ANSWERS

LEVEL II

1. (d)	2. (b)	3. (c)	4. (b)	5. (b)
6. (d)	7. (b)	8. (c)	9. (a)	10. (b)
11. (d)	12. (c)	13. (d)	14. (d)	15. (b)
16. (b)	17. (d)	18. (b)	19. (d)	20. (b)
21. (a)	22. (a)	23. (**)	24. (b)	25. (a)
26. (b)	27. (c)	28. (a)	29. (d)	30. (c)
31. (c)	32. (a)	33. (b)	34. (d)	35. (d)
36. (b)	37. (a)	38. (b)	39. (a)	40. (b)

LEVEL-IV

1. $[-5, 5]$
2. $-\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}$
3. $|\vec{r}|^2$
4. $-\hat{i} + 2\hat{j} + 5\hat{k}$
5. $\frac{\pi}{2}$
6. $\frac{\sqrt{3}}{2}$.
7. $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$
8. (i) 2 (ii) –1 (iii) 12
9. $\frac{5\pi}{6}$
11. $9(-\hat{j} + \hat{k})$
12. $\alpha = -1, \beta = -2, \gamma = 3$.
13. $m = 6$
14. 101
15. $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{3}\right)$

INTEGER TYPE QUESTIONS

1. 3	2. 3	3. 3	4. 4	5. 7
6. 6	7. 2	8. 8	9. 4	10. 3
11. 3	12. 2	13. 2	14. 4	

COMPREHENSIVE LINK PASSAGES

P-I :	(i) (a)	(ii) (b)	(iii) (c)
P-II :	(i) (c)	(ii) (b)	(iii) (a)
P-III :	(i) (b)	(ii) (c)	(iii) (c)
P-IV :	(i) (a)	(ii) (c)	(iii) (a)
P-V :	(i) (b)	(ii) (a)	(iii) (d)
P-VI :	(i) (a)	(ii) (c)	(iii) (b)
P-VII :	(i) (d)	(ii) (d)	(iii) (c)
P-VIII :	(i) (b)	(ii) (c)	(iii) (a)
P-IX :	(i) (b)	(ii) (a)	(iii) (a)
P-X :	(i) (a)	(ii) (b)	(iii) (c)
P-XI :	(i) (a)	(ii) (b)	(iii) (c)
P-XII :	(i) (a)	(ii) (b)	(iii) (a)

MATRIX MATCH

1.	(A)→(R),	(B)→(S),	(C)→(P),	(D)→(P)
2.	(A)→(S),	(B)→(S),	(C)→(P),	(D)→(P)
3.	(A)→(S),	(B)→(R),	(C)→(P),	(D)→(Q)
4.	(A)→(S),	(B)→(P),	(C)→(Q),	(D)→(S)
5.	(A)→(S),	(B)→(P),	(C)→(S),	(D)→(Q)
6.	(A)→(S),	(B)→(S),	(C)→(S),	(D)→(Q)
7.	(A)→(P),	(B)→(P),	(C)→(S),	(D)→(Q)
8.	(A)→(S),	(B)→(P),	(C)→(P),	(D)→(P)
9.	(A)→(S),	(B)→(P),	(C)→(Q),	(D)→(R)
10.	(A)→(S),	(B)→(P),	(C)→(P),	(D)→(Q)
11.	(A)→(R),	(B)→(S),	(C)→(P),	(D)→(Q)

HINTS AND SOLUTIONS

Level I

1. Let θ the angle between **a** and **b**.

Now, $\mathbf{a}\cdot\mathbf{b} = ab\cos(\theta)$

$\Rightarrow \cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}$

$\Rightarrow \cos(\theta) = \frac{6 - 6 + 0}{\sqrt{14}\sqrt{13}} = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

Hence, the angle between **a** and **b** is $\frac{\pi}{2}$.

2. We have,

$$\mathbf{a} + \mathbf{b} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

Also,

$$\mathbf{a} - \mathbf{b} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = (-2\mathbf{i} + \mathbf{j} - 5\mathbf{k})$$

Now,

$$(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = -8 + 3 + 5 = 0$$

Thus, $(\mathbf{a} + \mathbf{b})$ is perpendicular to $(\mathbf{a} - \mathbf{b})$.

3. Given,

$(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$\Rightarrow |(\mathbf{a} + \mathbf{b} + \mathbf{c})|^2 = 0$

$\Rightarrow |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$\Rightarrow 9 + 16 + 25 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$\Rightarrow 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = -50$

$\Rightarrow (\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = -25$

4. We have,

$$|(\mathbf{a}+\mathbf{b}-\mathbf{c})|^2$$
$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c})$$
$$= 1 + 1 + 1 + 0$$
$$= 3$$
$$\Rightarrow \quad |\mathbf{a}+\mathbf{b}-\mathbf{c}| = \sqrt{3}$$

5. As we know that the projection of **a** on **b**

$$= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.$$
$$= \frac{3-2+2}{\sqrt{1+4+1}} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

6. Given $2\mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j}$ and $\mathbf{a}+2\mathbf{b} = \mathbf{i}-\mathbf{j}$

Solving, we get

$$\mathbf{a} = \frac{1}{3}(\mathbf{i}+3\mathbf{j}) \text{ and } \mathbf{b} = \frac{1}{3}(\mathbf{i}-3\mathbf{j})$$

Since, $\mathbf{a}\cdot\mathbf{b} = ab\cos(\theta)$

$$\Rightarrow \quad \cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}$$
$$\Rightarrow \quad \cos(\theta) = \frac{\frac{1}{9}(1-9)}{\frac{10}{9}} = -\frac{8}{10} = -\frac{4}{5}$$
$$\Rightarrow \quad (\theta) = \cos^{-1}\left(-\frac{4}{5}\right)$$

7. Let $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

We have to find $\hat{r} = \frac{\mathbf{r}}{|\mathbf{r}|}$.

Here, **r** is perpendicular to the given vectors, so

$$a - 2b + c = 0$$

and $\quad 2a + b - 3c = 0$

Solving, we get

$$\frac{a}{6-1} = \frac{b}{2+3} = \frac{c}{1+4}$$
$$\Rightarrow \quad \frac{a}{5} = \frac{b}{5} = \frac{c}{5}$$
$$\Rightarrow \quad a = b = c = \lambda \text{(say)}$$

Thus, $\hat{r} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\lambda(i+j+k)}{\lambda\sqrt{3}} = \frac{(i+j+k)}{\sqrt{3}}$.

8. Let the vector be $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

We have to find $\hat{r} = \frac{\mathbf{r}}{|\mathbf{r}|}$.

Let $\quad \mathbf{a} = \mathbf{i}+\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$

Now,

$$\mathbf{r}\cdot\mathbf{a} = ra\cos(45°)$$
$$\Rightarrow \quad x + y = \sqrt{x^2+y^2}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad x + y = \sqrt{x^2+y^2} \quad \text{...(i)}$$

and

$$\mathbf{r}\cdot\mathbf{b} = rb\cos(60°)$$
$$\Rightarrow \quad 3x - 4y = \sqrt{x^2+y^2}\cdot 5\cdot\frac{1}{2}$$
$$\Rightarrow \quad \frac{2}{5}(3x-4y) = \sqrt{x^2+y^2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$x = 13y$$

Thus, $\hat{r} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$

$$= \frac{13y\mathbf{i}+y\mathbf{j}}{\sqrt{169y^2+y^2}}$$
$$= \frac{13\mathbf{i}+\mathbf{j}}{\sqrt{170}}.$$

9. Given $\quad \mathbf{a}\cdot\mathbf{b} = 0 = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$

Let $\quad |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = \lambda$

Now, $\quad (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{a} = |\mathbf{a}+\mathbf{b}+\mathbf{c}||\mathbf{a}|\cos(\theta_1)$

$$\Rightarrow \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}||\mathbf{a}|\cos(\theta_1) = (\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c})$$
$$\Rightarrow \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}||\mathbf{a}|\cos(\theta_1) = |\mathbf{a}|^2$$
$$\Rightarrow \quad \cos(\theta_1) = \frac{|\mathbf{a}|}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|} = \frac{\lambda}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$$

Similarly, $\cos(\theta_2) = \frac{|\mathbf{b}|}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|} = \frac{\lambda}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$

and $\quad \cos(\theta_3) = \frac{|\mathbf{c}|}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|} = \frac{\lambda}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$

Thus, $\quad \cos(\theta_1) = \cos(\theta_2) = \cos(\theta_3)$

$$\Rightarrow \quad (\theta_1) = (\theta_2) = (\theta_3).$$

10. Let $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $|\mathbf{r}| = \sqrt{51}$.

It is given that

$$\cos(\theta_1) = \cos(\theta_2) = \cos(\theta_3)$$
$$\Rightarrow \quad \frac{\mathbf{a}\cdot\mathbf{r}}{|\mathbf{a}||\mathbf{r}|} = \frac{\mathbf{b}\cdot\mathbf{r}}{|\mathbf{b}||\mathbf{r}|} = \frac{\mathbf{c}\cdot\mathbf{r}}{|\mathbf{c}||\mathbf{r}|}$$
$$\Rightarrow \quad \frac{\mathbf{a}\cdot\mathbf{r}}{|\mathbf{a}|} = \frac{\mathbf{b}\cdot\mathbf{r}}{|\mathbf{b}|} = \frac{\mathbf{c}\cdot\mathbf{r}}{|\mathbf{c}|}$$
$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{r} = \mathbf{b}\cdot\mathbf{r} = \mathbf{c}\cdot\mathbf{r}$$

$(\because \ |\mathbf{a}| = 1 = |\mathbf{b}| = |\mathbf{c}|)$

$$\Rightarrow \quad \frac{x-2y+2z}{3} = \frac{-4x-3z}{5} = \frac{y}{1}$$

$\Rightarrow \quad \frac{x - 2y + 2z}{3} = \frac{-4x - 3z}{5} = \frac{y}{1} = \lambda(\text{say})$

Solving, we get

$x = -5\lambda,\ y = \lambda,\ z = 5\lambda$

Also,

$|\mathbf{r}| = \sqrt{51}$

$\Rightarrow \quad x^2 + y^2 + z^2 = 51$

$\Rightarrow \quad 25\lambda^2 + \lambda^2 + 25\lambda^2 = 51$

$\Rightarrow \quad 51\lambda^2 = 51$

$\Rightarrow \quad \lambda^2 = 1$

$\Rightarrow \quad \lambda = \pm 1$

$\Rightarrow \quad \mathbf{r} = \pm(-5\mathbf{i} + \mathbf{j} + 5\mathbf{k})$

11. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $|\mathbf{r}| = 4$.
It is given that

$\cos(\theta_1) = \cos(\theta_2) = \cos(\theta_3)$

$\Rightarrow \quad \frac{\mathbf{a}\cdot\mathbf{r}}{|\mathbf{a}||\mathbf{r}|} = \frac{\mathbf{b}\cdot\mathbf{r}}{|\mathbf{b}||\mathbf{r}|} = \frac{\mathbf{c}\cdot\mathbf{r}}{|\mathbf{c}||\mathbf{r}|}$

$\Rightarrow \quad \frac{\mathbf{a}\cdot\mathbf{r}}{|\mathbf{a}|} = \frac{\mathbf{b}\cdot\mathbf{r}}{|\mathbf{b}|} = \frac{\mathbf{c}\cdot\mathbf{r}}{|\mathbf{c}|}$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{r} = \mathbf{b}\cdot\mathbf{r} = \mathbf{c}\cdot\mathbf{r}$

$\left(\because \ |\mathbf{a}| = \frac{1}{\sqrt{2}} = |\mathbf{b}| = |\mathbf{c}|\right)$

$\Rightarrow \quad x + y = y + z = x + z$

$\Rightarrow \quad x + y = y + z = x + z = \lambda(\text{say})$

Solving, we get

$x = -\frac{1}{2} = y = z$

Also,

$|\mathbf{r}| = 4$

$\Rightarrow \quad x^2 + y^2 + z^2 = 4$

$\Rightarrow \quad \frac{\lambda^2}{4} + \frac{\lambda^2}{4} + \frac{\lambda^2}{4} = 4$

$\Rightarrow \quad \frac{3\lambda^2}{4} = 4$

$\Rightarrow \quad \lambda^2 = \frac{16}{3}$

$\Rightarrow \quad \lambda = \pm\frac{4}{\sqrt{3}}$

Therefore, $\mathbf{r} = \pm\frac{4}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

12. Let $\quad \mathbf{a} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k})$,

$\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

and $\quad \mathbf{c} = (2\mathbf{i} + \mathbf{j} + \mathbf{k})$

and $\mathbf{r}$ be a vector

such that $\mathbf{r} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

It is given that $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$ lie in the same plane, so

$\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}) = 0$

$\Rightarrow \quad \begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0$

$\Rightarrow \quad 3x - y - z = 0$ ...(i)

Also,

$\mathbf{r}\cdot\mathbf{c} = 0$

$\Rightarrow \quad 2x + y + z = 0$ ...(ii)

Solving Eqs. (i) and (ii), we get

$\frac{x}{0} = \frac{y}{-5} = \frac{z}{5}$

$\Rightarrow \quad \frac{x}{0} = \frac{y}{-1} = \frac{z}{1} = \lambda(\text{say})$

Therefore, $\hat{r} = \frac{\mathbf{r}}{|\mathbf{r}|}$

$\Rightarrow \quad \hat{r} = \pm(\mathbf{j} + \mathbf{k})$

13. We have,

$|(\hat{a} - \hat{b})|^2$

$= |\hat{a}| + |\hat{b}|^2 - 2(\hat{a}\cdot\hat{b})$

$= [1 + 1 - 2(\hat{a}\cdot\hat{b})]$

$= 2 - 2(\hat{a}\cdot\hat{b})$

$= 2 - 2\cos\theta$

$= 2(1 - \cos\theta)$

$= 2 \times 2\sin^2\left(\frac{\theta}{2}\right)$

$\Rightarrow \quad |\hat{a} - \hat{b}| = 2\sin\left(\frac{\theta}{2}\right)$

$\Rightarrow \quad \sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|$

14. Given $\quad \hat{a}\cdot\hat{b} = 0 = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a}$

and $\quad |\hat{a}| = 1 = |\hat{b}| = |\hat{c}|$

We have,

$(\hat{\mathbf{a}} - \hat{\mathbf{b}})\cdot(\hat{\mathbf{b}} - \hat{\mathbf{c}})$

$= (\hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{c} - \hat{b}\cdot\hat{b} + \hat{b}\cdot\hat{c})$

$= (0 - 0 - \hat{b}\cdot\hat{b} + 0)$

$= -|\hat{b}|^2$

$= -1.$

15. Given $|\hat{a}| = 1 = |\hat{b}| = |\hat{a} + \hat{b}|$

We have,

$|\hat{a} + \hat{b}| = 1$

$\Rightarrow |\hat{a} + \hat{b}|^2 = 1$

$\Rightarrow |\hat{a}|^2 + \hat{b}|^2 + 2(\hat{a}\cdot\hat{b}) = 1$

$\Rightarrow 1 + 1 + 2(\hat{a}\cdot\hat{b}) = 1$

$\Rightarrow 2\cos(\theta) = 1 - 2 = -1$

$\Rightarrow \cos(\theta) = -\frac{1}{2}$

Thus, $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$

$\Rightarrow \sin(\theta) = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

16. It is given that, $\mathbf{a}\cdot\hat{\mathbf{i}} = |\mathbf{a}||\hat{\mathbf{i}}|\cos(30°) = \frac{\sqrt{3}}{2}$

and $\mathbf{b}\cdot\hat{\mathbf{i}} = |\mathbf{b}||\hat{\mathbf{i}}|\cos(120°) = -\frac{1}{2}$

Clearly, the angle between **a** and **b** is 90°.

Thus, $(\hat{a}\cdot\hat{b}) = 0$

We have,

$|\hat{a} + \hat{b}|^2 = |\hat{a}|^2 + |\hat{b}|^2 + 2(\hat{a}\cdot\hat{b})$

$= 1 + 1 + 0$

$= 2$

$\Rightarrow |\hat{a} + \hat{b}| = \sqrt{2}$

17. Given,

$$\left(\frac{\mathbf{a}}{a^2} - \frac{\mathbf{b}}{b^2}\right)^2 = \frac{a^2}{a^4} + \frac{b^2}{b^4} - \frac{2(\mathbf{a}\cdot\mathbf{b})}{a^2b^2}$$

$$= \frac{1}{a^2} + \frac{1}{a^2} - \frac{2(\mathbf{a}\cdot\mathbf{b})}{a^2b^2}$$

$$= \frac{a^2 + b^2 - 2(\mathbf{a}\cdot\mathbf{b})}{a^2b^2}$$

$$= \left(\frac{\mathbf{a} - \mathbf{b}}{ab}\right)^2$$

18. Let $\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $|\hat{\mathbf{n}}| = 1$.

Now,

$\mathbf{u}\cdot\hat{n} = 0$

$\Rightarrow x + y = 0$

$\Rightarrow x = -y$ ...(i)

Also, $\mathbf{v}\cdot\hat{n} = 0$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$ ...(ii)

From Eqs. (i) and (ii), we get

$x = 0 = y$

Also,

$|\hat{\mathbf{n}}| = 1$

$\Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$

$\Rightarrow x^2 + y^2 + z^2 = 1$

$\Rightarrow 0 + 0 + z^2 = 1$

$\Rightarrow z = \pm 1$

Thus, $\mathbf{r} = \pm\mathbf{k}$

Now, $|(\mathbf{w}\cdot\hat{\mathbf{n}})| = |-3| = 3.$

19. Given $\mathbf{a}\cdot\mathbf{b} = 0 = \mathbf{a}\cdot\mathbf{c}$

Also, $\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Now,

$|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

$= 1 + 1 + 1 + 2\left(0 + \frac{1}{2} + 1\right)$

$= 4$

Thus, $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = 2.$

20. Given

$2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

Now,

$|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

$= 9 + 16 + 25 + 0$

$= 50$

Thus, $2|\mathbf{a} + \mathbf{b} + \mathbf{c}| = 5.$

21. We have,

$$\left|\frac{\hat{a} + \hat{b}}{\hat{a} - \hat{b}}\right|^2 = \frac{|\hat{a} + \hat{b}|^2}{|\hat{a} - \hat{b}|^2}$$

$$= \frac{|\hat{a}|^2 + |\hat{b}|^2 + 2(\hat{a}\cdot\hat{b})}{|\hat{a}|^2 + |\hat{b}|^2 - 2(\hat{a}\cdot\hat{b})}$$

$$= \frac{1 + 1 + 2\cos(60°)}{1 + 1 - 2\cos(60°)}$$

$$= \frac{1 + 1 + 1}{1 + 1 - 1}$$

$= 3.$

Thus, $\left|\dfrac{\hat{a}+\hat{b}}{\hat{a}-\hat{b}}\right| = \sqrt{3}$

22. Given,

$$(\mathbf{a}+\mathbf{b}+3\mathbf{c}) = \mathbf{0}$$

$$\Rightarrow \quad (\mathbf{a}+\mathbf{b}+\mathbf{c}) = (-2\mathbf{c})$$

We have,

$$|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = |(-2\mathbf{c})|^2$$

$$\Rightarrow |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = 4|\mathbf{c}|^2$$

$$\Rightarrow 1 + 16 + \frac{17}{9} + 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = \frac{68}{9}$$

$$\Rightarrow 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = \frac{68-17}{9} - 17$$

$$\Rightarrow 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = \frac{51}{9} - 17$$

$$\Rightarrow 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = \frac{17}{3} - 17$$

$$\Rightarrow 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = -\frac{34}{3}$$

$$\Rightarrow 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = -\frac{17}{3}$$

23. Let the resultant force be **F** such that

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}$$

i.e. $\mathbf{F} = 2\mathbf{j} - \mathbf{k}$

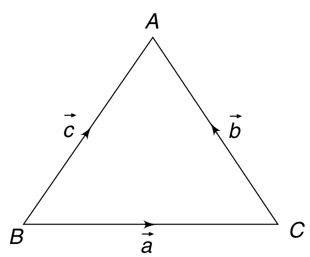
Now, $\mathbf{AB} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

$\therefore$ Work done = (Force)·(Displacement)

$$= (2j-k)\cdot(2i+4j-k)$$

$$= 8 + 1 = 9.$$

24. Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$



We have $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = 0$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$$

$$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$$

$$\Rightarrow (\vec{b} + \vec{c})^2 = (-\vec{a})^2$$

$$\Rightarrow b^2 + c^2 + 2b\cdot c = a^2$$

$$\Rightarrow b^2 + c^2 + 2bc\cos(\pi - A) = a^2$$

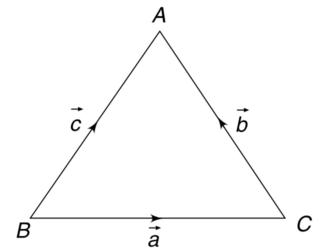
$$\Rightarrow b^2 + c^2 + 2bc\cos(A) = a^2$$

$$\Rightarrow 2bc\cos(A) = b^2 + c^2 - \text{a2}$$

$$\Rightarrow \cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence, the result.

25. Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$



We have,

$$\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$$

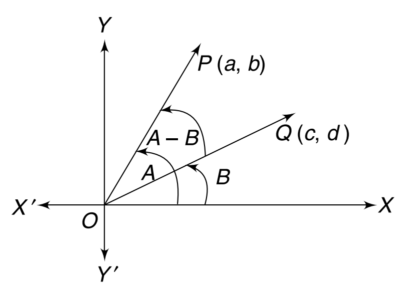
$$\Rightarrow \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = -\vec{a}\cdot\vec{a}$$

$$\Rightarrow ab\cos(\pi - C) + ac\cos(\pi - B) = -a^2$$

$$\Rightarrow -ab\cos(C) - ac\cos(B) = -a^2$$

$$\Rightarrow a = b\cos(C) + c\cos(B)$$

26.



Here, $\overrightarrow{OP} = a\hat{i} + b\hat{j}$

and $\overrightarrow{OQ} = c\hat{i} + d\hat{j}$

Now, $\overrightarrow{OP}\cdot\overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}|\cos(\angle \text{POQ})$

Now, $(a\hat{i} + b\hat{j})\cdot(c\hat{i} + d\hat{j})$

$$= \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}\cos(A-B)$$

$$\Rightarrow \cos(A-B)$$

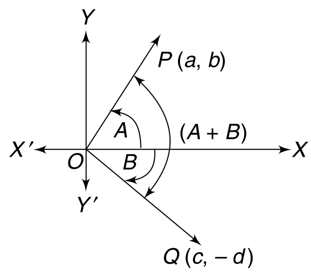
$$= \frac{(a\hat{i} + b\hat{j})\cdot(c\hat{i} + d\hat{j})}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$$

$$= \frac{ac + bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$$

$$= \frac{a}{\sqrt{a^2+b^2}}\cdot\frac{c}{\sqrt{c^2+d^2}} + \frac{b}{\sqrt{a^2+b^2}}\cdot\frac{d}{\sqrt{c^2+d^2}}$$

$$= \cos A\cdot\cos B + \sin A\cdot\sin B$$

27.



Here, $\overrightarrow{OP} = a\hat{i} + b\hat{j}$

and $\overrightarrow{OQ} = c\hat{i} + d\hat{j}$

Now,

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}|\cos(\angle POQ)$$

$$\Rightarrow \quad ac - bd = (OP)(OQ)\cos(A + B)$$

$$\Rightarrow \quad \cos(A + B) = \frac{ac - bd}{(OP)(OQ)}$$

$$= \frac{a}{OP}\cdot\frac{c}{OQ} - \frac{b}{OP}\cdot\frac{d}{OQ}$$

$$= \cos(A)\cos(B) - \sin(A)\sin(B)$$

28. We have,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{vmatrix}$$

$$= 8\mathbf{i} + 11\mathbf{j} - 5\mathbf{k}$$

29. Area of a triangle $= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

Now,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 5 & 7 & 0 \end{vmatrix}$$

$$= (21 - 20)\mathbf{k}$$

$$= \mathbf{k}$$

Thus, the area of the triangle $= \frac{1}{2}$ sq.u

30. Now,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= 8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$$

$$= 8(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

Area of the parallelogram

$$= |\mathbf{a}\times\mathbf{b}|$$

$$= 8\sqrt{3} \text{ sq.u.}$$

31. Now,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & -4 & 1 \end{vmatrix}$$

$$= 5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$$

Area of the parallelogram

$$= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

$$= \frac{1}{2}\times\sqrt{25 + 4 + 49}$$

$$= \frac{\sqrt{78}}{2} \text{ sq.u.}$$

32. Let $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,

$\mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$

and $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Now, $\mathbf{ab} = -2\mathbf{i} - 3\mathbf{k}$

and $\mathbf{ac} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\therefore \quad \mathbf{ab}\times\mathbf{ac} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -3 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= -6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$$

Area of the triangle $= \frac{1}{2}|\mathbf{ab}\times\mathbf{ac}|$

$$= \frac{1}{2}\sqrt{36 + 25 + 16} = \frac{\sqrt{77}}{2} \text{ sq.u}$$

33. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.

A unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$

$$= \pm\frac{(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$$

Now,

$$(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 3 & -4 & -1 \end{vmatrix}$$

$$= 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

$$= 5(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

Thus, the unit vector $= \pm \dfrac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$

$= \pm \dfrac{1}{\sqrt{3}} (\mathbf{i} + \mathbf{j} - \mathbf{k})$.

34. We have,

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times (\mathbf{c} + \mathbf{a}) + \mathbf{c} \times (\mathbf{a} + \mathbf{b})$

$= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}$

$= \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c}$

$= \mathbf{0}$

35. Given $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ ...(i)

and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{c} \times \mathbf{d} - \mathbf{b} \times \mathbf{d}$

$\Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = (\mathbf{c} - \mathbf{b}) \times \mathbf{d}$

$\Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) - (\mathbf{c} - \mathbf{b}) \times \mathbf{d} = 0$

$\Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) + (\mathbf{c} - \mathbf{b}) \times \mathbf{d} = \mathbf{0}$

$\Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) - \mathbf{d} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$

$\Rightarrow (\mathbf{a} - \mathbf{d}) \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$

Thus, $(\mathbf{a} - \mathbf{d})$ is parallel to $(\mathbf{b} - \mathbf{c})$.

36. Given,

$(\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\Rightarrow \mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{0} = \mathbf{0}$

$\Rightarrow \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$

$\Rightarrow \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$

$\Rightarrow \mathbf{a} \times \mathbf{b} = -\mathbf{a} \times \mathbf{c}$

$\Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$ ...(i)

Also,

$(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$\Rightarrow \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{b} \times \mathbf{0} = \mathbf{0}$

$\Rightarrow \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$

$\Rightarrow \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = 0$

$\Rightarrow \mathbf{b} \times \mathbf{c} = -\mathbf{b} \times \mathbf{a}$

$\Rightarrow \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$ ...(ii)

From Eqs. (i) and (ii), we get

$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$

37. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Now, $(\mathbf{a} \times \mathbf{i})$

$= (a_1\mathbf{i} \times \mathbf{i} + a_2\mathbf{j} \times \mathbf{i} + a_3\mathbf{k} \times \mathbf{i})$

$= (-a_2\mathbf{k} - a_3\mathbf{j})$

Thus, $|(\mathbf{a} \times \mathbf{i})|^2 = a_2^2 + a_3^2$

Similarly, $|(\mathbf{a} \times \mathbf{j})|^2 = a_1^2 + a_3^2$

and $|(\mathbf{a} \times \mathbf{k})|^2 = a_1^2 + a_2^2$

Therefore,

$|\mathbf{a} \times \hat{i}|^2 + |\mathbf{a} \times \hat{j}|^2 + |\mathbf{a} \times \hat{k}|^2$

$= (a_2^2 + a_3^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2)$

$= 2(a_1^2 + a_2^2 + a_3^2)$

$= 2a^2$

38. Given $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{a} \cdot \mathbf{c}$

Thus, $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{c}$.

A unit vector perpnedicular to $\mathbf{b}$ and $\mathbf{c}$

$= \pm \dfrac{(\mathbf{b} \times \mathbf{c})}{|\mathbf{b} \times \mathbf{c}|}$

$= \pm \dfrac{(\mathbf{b} \times \mathbf{c})}{|\mathbf{b}||\mathbf{c}| \sin\left(\frac{\pi}{6}\right)}$

$= \pm \dfrac{(\mathbf{b} \times \mathbf{c})}{1/2}$

$= \pm 2(\mathbf{b} \times \mathbf{c})$.

39. Given,

$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$

$\Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0$

$\therefore (\mathbf{b} - \mathbf{c})$ is parallel to $\mathbf{a}$

$\Rightarrow (\mathbf{b} - \mathbf{c}) = \lambda\mathbf{a}$

$\Rightarrow \mathbf{b} - \mathbf{c} + \lambda\mathbf{a}, \lambda \in R$

40. Given,

$\mathbf{R} \times \mathbf{B} = \mathbf{R} \times \mathbf{C}$

$\Rightarrow \mathbf{R} \times (\mathbf{B} - \mathbf{C}) = \mathbf{0}$

$\therefore R$ is parallel to $(\mathbf{B} - \mathbf{C})$

$\Rightarrow \mathbf{R} = \lambda(\mathbf{B} - \mathbf{C})$

$\Rightarrow \mathbf{R} = \lambda(-3\mathbf{i} + 4\mathbf{j} - 6k)$ ...(i)

Also,

$\mathbf{R} \cdot \mathbf{A} = 1$

$\Rightarrow -6\lambda - 6\lambda = 1$

$\Rightarrow \lambda = -\dfrac{1}{12}$

Thus, $\mathbf{R} = -\dfrac{1}{2}(-3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})$.

41. Now,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$= \mathbf{i} - \mathbf{j} + \mathbf{k}$

Unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$

$$= \pm \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

$$= \pm (i - \mathbf{j} + \mathbf{k})$$

Thus, the number of vectors of unit length = 2.

42. As we know that, if **a**, **b** and **c** form a right handed system, then

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 0 & 1 & 0 \end{vmatrix} = -z\mathbf{i} + xk$$

43. Given,

$$\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a} \text{ and } \mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} \text{ and } \mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} \times \mathbf{a} = -\mathbf{r} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} \times \mathbf{a} + \mathbf{r} \times \mathbf{b} = \mathbf{0}$$

$$\Rightarrow \quad \mathbf{r} \times (\mathbf{a} + \mathbf{b}) = \mathbf{0}$$

$\therefore$ r is parallel to $(\mathbf{a} + \mathbf{b})$

$$\Rightarrow \quad \mathbf{r} = \lambda(\mathbf{a} + \mathbf{b})$$

$$\Rightarrow \quad \mathbf{r} = \lambda(\mathbf{a} + \mathbf{b})$$

$$\Rightarrow \quad \mathbf{r} \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{b})$$

$$\Rightarrow \quad \mathbf{r} \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow \quad \mathbf{a} \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow \quad \lambda = 1$$

Therefore, $\mathbf{r} = (\mathbf{a} \times \mathbf{b})$

$$= 3\mathbf{i} + \mathbf{j} - \mathbf{k}$$

44. Given that

$$\mathbf{r} \times \mathbf{a} = \mathbf{b} \text{ and } \mathbf{r} \cdot \mathbf{a} = 0$$

$$\Rightarrow \quad \mathbf{r} \times \mathbf{a} = \mathbf{b}$$

$$\Rightarrow \quad \mathbf{a} \times (\mathbf{r} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{a})\mathbf{r} - (\mathbf{a} \cdot \mathbf{r})\mathbf{a} = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad |\mathbf{a}|^2\mathbf{r} - 0 = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}$$

45. Given $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b} = 3$

Let $\mathbf{b} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Thus, $x + y + z = 3$

and
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$\Rightarrow \quad (z - y)\mathbf{i} + (x - z)\mathbf{j} + (y - x)\mathbf{k} = \mathbf{j} - \mathbf{k}$$

Comparing, we get

$$(z - y) = 0, \ (x - z) = 1, \ (y - x) = -1$$

Solving, we get

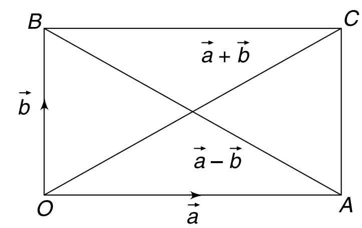
$$\Rightarrow \quad x = \frac{5}{3}, \ y = \frac{2}{3} = z$$

Thus, $\mathbf{b} = \frac{1}{3}(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

46. The vector $[\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}]$ lies in the plane of **a** and **b** and the vector $(\mathbf{a} \times \mathbf{b})$ is perpendicular to each of **a** and **b**

Thus, the angle between $(\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b})$ and $(\mathbf{a} \times \mathbf{b}) = 90°$.

47.



Here, $|\mathbf{a}| = |\mathbf{b}|$ and $(\mathbf{a} \cdot \mathbf{b}) = 0$

Given $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$

Now, $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a} \cdot \mathbf{b})$

$$\Rightarrow \quad |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 = 2|\mathbf{a}|^2$$

$$\Rightarrow \quad 2|\mathbf{a}|^2 = |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} + \mathbf{b}|^2 = (3^2 + 4^2) = 25$$

$$\Rightarrow \quad |\mathbf{a}|^2 = \frac{25}{2}$$

$\therefore$ Area of a square $= |\mathbf{a}|^2 = \frac{25}{2}$.

48. Let $\mathbf{A} = (1, -2, 2)$ and $\mathbf{B} = (2, -1, 3)$

Thus, $\mathbf{r} = \mathbf{BA} = \mathbf{OA} - \mathbf{OB} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

Therefore, Momentum about the point B

$$= \mathbf{r} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= 6\mathbf{i} - 7\mathbf{j} + \mathbf{k}$$

49. Here, $\mathbf{PQ} = -6\mathbf{i} - \mathbf{j} - \mathbf{k}$

Thus, moment of the couple

$$= \mathbf{M}$$

$$= \mathbf{PQ} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -1 & -1 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= -\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

50. Here, $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\Rightarrow \quad \mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

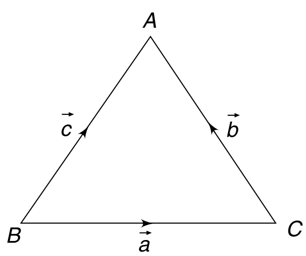
Thus, torque about the point

$$= \mathbf{r} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 4 & 0 & 1 \end{vmatrix}$$

$$= -2\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}.$$

51. Given ΔABC, $a = BC$, $b = CA$, $c = AB$



Clearly, $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

$\Rightarrow \quad |ab\sin(\pi - C)| = |bc\sin(\pi - A)|$

$$= |ca\sin(\pi - B)|$$

$\Rightarrow \quad ab\sin C = bc\sin A = ca\sin B$

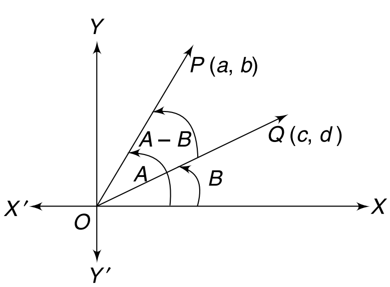
$\Rightarrow \quad \dfrac{ab\sin C}{abc} = \dfrac{bc\sin A}{abc} = \dfrac{ca\sin B}{abc}$

$\Rightarrow \quad \dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\Rightarrow \quad \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Hence, the result.

52.



Here, $\overrightarrow{OP} = a\hat{i} + b\hat{j}$

and $\overrightarrow{OQ} = c\hat{i} + d\hat{j}$

Now, $OP \times OQ = (ad - bc)\hat{k}$

$\Rightarrow \quad [(OP)(OQ)\sin(\angle POQ)]\hat{k} = (ad - bc)\hat{k}$

$\Rightarrow \quad -\sin(A - B) = \dfrac{(ad - bc)}{(OP)(OQ)}$

$$= \frac{a}{OP}\cdot\frac{d}{OQ}\cdot\frac{b}{OP}\cdot\frac{c}{OQ}$$

$$= \cos A \cdot \sin B - \sin A \cdot \cos B$$

$\Rightarrow \quad \sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

Hence, the result.

54. We have,

$[\mathbf{a} + \mathbf{b}, + \mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}]$

$= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})$

$= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$

$= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a})$

$= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{a} \cdot \mathbf{b} \times \mathbf{a} + \mathbf{a} \cdot \mathbf{c} \times \mathbf{a})$

$+ (\mathbf{b} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{b} \cdot \mathbf{b} \times \mathbf{a} + \mathbf{b} \cdot \mathbf{c} \times \mathbf{a})$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

55. We have,

$[2\mathbf{a} - \mathbf{b}, 2\mathbf{b} - \mathbf{c}, 2\mathbf{c} - \mathbf{a}]$

$= (2\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{b} - \mathbf{c}) \times (2\mathbf{c} - \mathbf{a})$

$= (2\mathbf{a} - \mathbf{b}) \cdot (4\mathbf{b} \times \mathbf{c} - 2\mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a})$

$= (2\mathbf{a} - \mathbf{b}) \cdot (4\mathbf{b} \times \mathbf{c} + 2\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a})$

$= (8\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} - \mathbf{b} \cdot \mathbf{c} \times \mathbf{a})$

$= 7[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= 0, \quad \because$ **a, b** and **c** are coplaner vectors

56. We have,

$[\mathbf{a}, \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{b} + \mathbf{c}]$

$= \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b})$

$= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{a} \cdot \mathbf{c} \times \mathbf{b})$

$= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} - \mathbf{a} \cdot \mathbf{c} \times \mathbf{b})$

$= 0.$

57. We have,

$[\mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{c}]$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{b} + \mathbf{c})$

$= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{b} \cdot \mathbf{a} \times \mathbf{c} + \mathbf{c} \cdot \mathbf{b} \times \mathbf{a})$

$= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} - \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} + \mathbf{c} \cdot \mathbf{b} \times \mathbf{a})$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}] - [\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

58. We have,

$[\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{c}, \mathbf{c} - \mathbf{a}]$

$= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})$

$= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a})$

$= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a})$

$= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} - \mathbf{b} \cdot \mathbf{c} \times \mathbf{a})$

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}] - [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= 0.$

59. We have,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$$

$\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow \quad |\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin(\theta) \cos(\theta) = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow \quad \sin(\theta) \cos(\varphi) = 1$

$\Rightarrow \quad \sin(\theta) = 1, \cos(\varphi) = 1$

$\Rightarrow \quad \theta = \frac{\pi}{2}, \varphi = 0$

Hence, the angle between **a** and **b** is $\theta = \frac{\pi}{2}$

60. We have,

$\frac{1}{2}(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= \frac{1}{2}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} + (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{b} + (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c})$

$= \frac{1}{2}((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} + 0 + 0)$

$= \frac{1}{2}(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$

$= \frac{1}{2}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= \frac{1}{2} \times 4 = 2$

61. Since the given vectors are coplanar, so

$$\begin{vmatrix} 2 & -1 & \lambda \\ 1 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 6 - 5 + \lambda(-2 + 3) = 0$

$\Rightarrow \quad 1 + \lambda = 0$

$\Rightarrow \quad \lambda = -1$

62. The volume of the parallelopiped

$$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

$$= \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 3(-3) + 2(-3) + 0$$

$$= |-3| = 3 \text{ c.u.}$$

63. We have,

$$2(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 6\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}$$

$\Rightarrow \quad (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$

Thus, $\quad \mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{k}, \mathbf{c} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

Hence, the volume of the parallelopiped

$$= \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 3 & -5 \end{vmatrix} = 1 \text{ c.u.}$$

64. Given $\quad [\mathbf{a}, \mathbf{b}, \mathbf{c}] = 5$

Now, $\quad [3(\mathbf{a} + \mathbf{b}), (\mathbf{b} + \mathbf{c}), 2(\mathbf{c} + \mathbf{a})]$

$= 3.2[(\mathbf{a} + \mathbf{b}), (\mathbf{b} + \mathbf{c}), 2(\mathbf{c} + \mathbf{a})]$

$= 6[(\mathbf{a} + \mathbf{b}), (\mathbf{b} + \mathbf{c}), 2(\mathbf{c} + \mathbf{a})]$

$= 6 \times 2\,[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= 6 \times 2 \times 5$

$= 60.$

65. The volume of the parallelopiped,

$$V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$$= 1 + a(a^2 - 1)$$

$$= a^3 - a + 1$$

$\Rightarrow \quad \frac{dV}{da} = 3a^2 - 1$

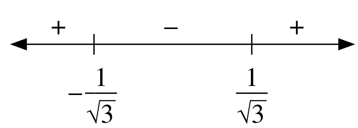
For maximum or minimum,

$$\frac{dV}{da} = 0$$

gives

$\Rightarrow \quad 3a^2 - 1 = 0$

$\Rightarrow \quad a = \pm\frac{1}{\sqrt{3}}$



By sign scheme, the point of minima is $a = \frac{1}{\sqrt{3}}$.

67. Since the given vectors are coplanar, so

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$\Rightarrow \quad \begin{vmatrix} a & 1 - a & 1 - a \\ 1 & b - 1 & 0 \\ 1 & 0 & c - 1 \end{vmatrix} = 0, \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$

$\Rightarrow \quad a(b-1)(c-1)(a-1)(c-1) + (1-b)(1-b) = 0.$

$\Rightarrow \quad (1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$

Dividing both the sides by $(1-a)(1-b)(1-c)$, we get

$$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

$$\Rightarrow \quad \left(1 + \frac{a}{1-a}\right) + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

$$\Rightarrow \quad \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

68. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$,

$\mathbf{b} = \mathbf{b}_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $c = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

We have, $[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$

$$= [\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

69. As we know that,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix}$$

$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

Thus, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \dfrac{\sqrt{3}}{2\sqrt{2}}$

Therefore, the volume of the parallelopiped

$$= \frac{\sqrt{3}}{2\sqrt{2}}$$

70. Let $\quad \mathbf{a} \times \mathbf{b} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

We have,

$[\mathbf{a}, \mathbf{b}, \mathbf{i}]\mathbf{i} = ((\mathbf{a} \times \mathbf{b})\cdot\mathbf{i})\mathbf{i}$

$= a_1\mathbf{i}$

Similarly, $[\mathbf{a}, \mathbf{b}, \mathbf{j}]\mathbf{j} = a_2\mathbf{j}$

and $\quad [\mathbf{a}, \mathbf{b}, \mathbf{k}]\mathbf{k} = a_3\mathbf{k}$

Thus,

$[\mathbf{a}, \mathbf{b}, \mathbf{i}]\mathbf{i} + [\mathbf{a}, \mathbf{b}, \mathbf{j}]\mathbf{j} + [\mathbf{a}, \mathbf{b}, \mathbf{k}]\mathbf{k}$

$= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$= (\mathbf{a} \times \mathbf{b})$

Hence, the result.

71. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$,

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $c = \mathbf{a} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

Also, let $\mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}$

and $\quad \mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$

Now,

$[\mathbf{a}, \mathbf{b}, \mathbf{c}](\mathbf{m} \times \mathbf{n})$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \\ \mathbf{a}\cdot\mathbf{n} & \mathbf{b}\cdot\mathbf{n} & \mathbf{c}\cdot\mathbf{n} \end{vmatrix}$$

72. Given $\quad \mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

Taking scalar product of $(\mathbf{b} \times \mathbf{c})$

$\mathbf{r}\cdot(\mathbf{b} \times \mathbf{c}) = x\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) + y\mathbf{b}\cdot(\mathbf{b} \times \mathbf{c}) + z\mathbf{c}\cdot(\mathbf{b} \times \mathbf{c})$

$\Rightarrow \quad \mathbf{r}\cdot(\mathbf{b} \times \mathbf{c}) = x\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$

$\Rightarrow \quad [\mathbf{r}, \mathbf{b}, \mathbf{c}] = x[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\Rightarrow \quad x = \dfrac{[\mathbf{r}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Similarly, $y = \dfrac{[\mathbf{r}, \mathbf{c}, \mathbf{a}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$ & $z = \dfrac{[\mathbf{r}, \mathbf{a}, \mathbf{b}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Hence, $\quad \mathbf{r} = \dfrac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\mathbf{a} + \dfrac{[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\mathbf{b} + \dfrac{[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\mathbf{c}$

73. Given,

$$[\mathbf{a}\ \mathbf{b}, \mathbf{a} \times \mathbf{b}] = \frac{1}{4}$$

$$\Rightarrow \quad (\mathbf{a} \times \mathbf{b})\cdot(\mathbf{a} \times \mathbf{b}) = \frac{1}{4}$$

$\Rightarrow \quad |(\mathbf{a} \times \mathbf{b})|^2 = \frac{1}{4}$

$\Rightarrow \quad |(\mathbf{a} \times \mathbf{b})| = \frac{1}{2}$

$\Rightarrow \quad ab\sin(\theta) = \frac{1}{2}$

$\Rightarrow \quad \sin(\theta) = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}.$

74. Given $\boldsymbol{\alpha} = \mathbf{x}(\mathbf{a} \times \mathbf{b}) + \mathbf{y}(\mathbf{b} \times \mathbf{c}) + \mathbf{z}(\mathbf{c} \times \mathbf{a})$

$\Rightarrow \quad \boldsymbol{\alpha} \cdot \mathbf{c} = \mathbf{x}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + \mathbf{y}(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c} + \mathbf{z}(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{c}$

$= x(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = x[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\Rightarrow \quad x = \frac{\boldsymbol{\alpha} \cdot \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Similarly, $y = \frac{\boldsymbol{\alpha} \cdot \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$, $z = \frac{\boldsymbol{\alpha} \cdot \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Thus,

$x + y + z$

$= \frac{\boldsymbol{\alpha} \cdot \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{\boldsymbol{\alpha} \cdot \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{\boldsymbol{\alpha} \cdot \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

$= \frac{\boldsymbol{\alpha} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

$= \frac{\boldsymbol{\alpha} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})}{1/8}$

$= 8\boldsymbol{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}).$

75. Volume of the tetrahedron

$= \frac{1}{6}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= \frac{1}{6} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$= \frac{1}{6}(-1 + 2 + 3)$

$= \frac{2}{3}$ c.u.

76. Given $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = 9$

$\Rightarrow \quad 2(a^2 + b^2 + c^2) - 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = 9$

$\Rightarrow \quad 6 - 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = 9$

$\Rightarrow \quad 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = -3$

$\Rightarrow \quad (a^2 + b^2 + c^2) + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = 3 - 3$

$\Rightarrow \quad |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = 0$

$\Rightarrow \quad (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

Now,

$|2\mathbf{a} + 5\mathbf{b} + 5\mathbf{c}| = |2\mathbf{a} + 5(\mathbf{b} + \mathbf{c})|$

$= |2\mathbf{a} + 5(-\mathbf{a})$

$= |-3(\mathbf{a})|$

$= |-3||\mathbf{a}|$

$= 3.$

77. Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{k} + \mathbf{i}$

Now,

$(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (\mathbf{i} - \mathbf{j} + \mathbf{k})$

$\hat{m} = \frac{(\mathbf{a} \times \mathbf{b})}{|(\mathbf{a} \times \mathbf{b})|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

similarly, $\hat{n} = \frac{(\mathbf{b} \times \mathbf{c})}{|(\mathbf{b} \times \mathbf{c})|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$

and $\hat{p} = \frac{(\mathbf{c} \times \mathbf{a})}{|(\mathbf{c} \times \mathbf{a})|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$

Volume of a parallelopiped

$= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$

$= \frac{1}{3\sqrt{3}} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix} \quad (C_2 \to C_2 + C_1)$

$= \frac{2}{3\sqrt{3}}(1 + 1)$

$= \frac{4}{3\sqrt{3}}$ c.u

78. Given $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$

and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$

$\Rightarrow \quad d = -\mathbf{b} - \mathbf{c} - \beta\mathbf{a}$

$\Rightarrow \quad \alpha\mathbf{d} = -\alpha\mathbf{b} - \alpha\mathbf{c} - \alpha\beta\mathbf{a}$

$\Rightarrow \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{b} - \alpha\mathbf{c} - \alpha\beta\mathbf{a}$

Comparing, the co-efficients of the vectors, we get

$\alpha = -1, \ \alpha\beta = -1$

$\Rightarrow \quad \alpha = -1, \ \beta = 1$

Now.

$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + 4 = \alpha\mathbf{d} + \mathbf{d} + 4$

$= -\mathbf{d} + \mathbf{d} + 4$

$= 4.$

79. Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$,

$\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

and $\mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

We have to find $\pm \dfrac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|}$.

Now,

$$(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

Also,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 6 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= 21\mathbf{i} - 7\mathbf{k}$$

Thus,

$$\pm \frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|}$$

$$= \pm \frac{(21\mathbf{i} - 7\mathbf{k})}{\sqrt{441 + 49}}$$

$$= \pm \frac{(21\mathbf{i} - 7\mathbf{k})}{7\sqrt{10}}$$

$$= \pm \frac{(3\mathbf{i} - 7\mathbf{k})}{\sqrt{10}}$$

80. We have,

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b}$$
$$= \mathbf{a} - |\mathbf{a}|^2\mathbf{b}$$
$$= \mathbf{a} - 3\mathbf{b}$$

$$\Rightarrow \quad 3\mathbf{b} = \mathbf{a} - \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) \quad \text{...(i)}$$

Now,

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

Thus, from Eq. (i), we get

$$3\mathbf{b} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (-2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\Rightarrow \quad 3\mathbf{b} = 3\mathbf{i}$$

$$\Rightarrow \quad \mathbf{b} = \mathbf{i}$$

81. We have,

$$(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \frac{1}{2}\mathbf{b}$$

$$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \frac{1}{2}\mathbf{b}$$

$$= \frac{1}{2}, (\mathbf{a} \cdot \mathbf{b}) = 0$$

$$\Rightarrow \quad ac\cos(\theta_1) = \frac{1}{2} \text{ and } ab\cos(\theta_2) = 0$$

$$\Rightarrow \quad \cos(\theta_1) = \frac{1}{2} \text{ and } \cos(\theta_2) = 0$$

$$\Rightarrow \quad (\theta_1) = \frac{\pi}{3}, (\theta_2) = \frac{\pi}{2}$$

Hence, the result.

82. We have,

$$(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \frac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c})$$

$$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \frac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c})$$

$$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{c}) = \frac{1}{\sqrt{2}}, (\mathbf{a} \cdot \mathbf{b}) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad ac\cos(\theta_1) = \frac{1}{\sqrt{2}} \text{ and } ab\cos(\theta_2) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos(\theta_1) = \frac{1}{\sqrt{2}} \text{ and } \cos(\theta_2) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta_1 = \frac{\pi}{4} \text{ and } \theta_2 = \frac{\pi}{4}$$

$$\Rightarrow \quad \theta_2 = \frac{\pi}{4}$$

Hence, the angle between **a** and **b** is $\dfrac{\pi}{4}$

83. Given,

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + \mathbf{b} = \mathbf{0}$$

$$\Rightarrow \quad \mathbf{a} \times (\mathbf{a} \times \mathbf{c}) = -\mathbf{b}$$

$$\Rightarrow \quad |\mathbf{a} \times (\mathbf{a} \times \mathbf{c})| = |-\mathbf{b}|$$

$$\Rightarrow \quad |\mathbf{a}||(\mathbf{a} \times \mathbf{c})|\sin\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow \quad |\mathbf{a}||(\mathbf{a} \times \mathbf{c})| = 1$$

$$\Rightarrow \quad |(\mathbf{a} \times \mathbf{c})| = 1$$

$$\Rightarrow \quad ac\sin\theta = 1$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Hence, the angle between **a** and **c** is $\dfrac{\pi}{6}$.

84. We have,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = 0$$

$$\Rightarrow \quad \{(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\} \cdot \{(\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}\} = 0$$

$$\Rightarrow \quad ((\mathbf{a} \cdot \mathbf{c})\mathbf{b})^2 - (\mathbf{a} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{a})$$
$$- (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a})(\mathbf{c} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a}) = \mathbf{0}$$

$$\Rightarrow \quad ((\mathbf{a} \cdot \mathbf{c})\mathbf{b})^2 - (\mathbf{a} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{a}) = 0$$

$$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{c})\{(\mathbf{a} \cdot \mathbf{c})\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c}) = 0$$

$$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{c}) = 0, \{(\mathbf{a} \cdot \mathbf{c})\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c})\} = 0$$

Hence, $(\mathbf{a} \cdot \mathbf{c}) = 0$

85. Given,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b} = -\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad (\mathbf{c}\cdot\mathbf{b}) = -\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad |\mathbf{c}||\mathbf{b}|\cos(\theta) = -\frac{1}{3}|\mathbf{b}||\mathbf{c}|$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{3}$$

$$\Rightarrow \quad \sin\theta = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

Hence, the result.

86. Given,

$$\mathbf{a}\cdot\mathbf{b} = 0$$

Let $\mathbf{r} = m\mathbf{a} + n\mathbf{b} + p(\mathbf{a} \times \mathbf{b})$ ...(i)

Now, $\mathbf{r}\cdot\mathbf{a} = m(\mathbf{a}\cdot\mathbf{a}) + n(\mathbf{a}\cdot\mathbf{b}) + p(\mathbf{a}\cdot(\mathbf{a} \times \mathbf{b}))$

$\Rightarrow \quad m = 0$

Also, $\mathbf{r}\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}) + n(\mathbf{b}\cdot\mathbf{b}) + p(\mathbf{b}\cdot(\mathbf{a} \times \mathbf{b}))$

$\Rightarrow \quad 1 = 0 + n(\mathbf{b}\cdot\mathbf{b}) + 0$

$\Rightarrow \quad n = 1$

Again, $\mathbf{r}\cdot\mathbf{a} \times \mathbf{b} = m(\mathbf{a}\cdot\mathbf{a} \times \mathbf{b}) + n(\mathbf{b}\cdot\mathbf{a} \times \mathbf{b}) + p[(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{a} \times \mathbf{b})]$

$\Rightarrow \quad 1 = 0 + 0 + p|(\mathbf{a} \times \mathbf{b})|^2$

$\Rightarrow \quad p = 1$

Hence, from Eq. (i),

$$\mathbf{r} = \mathbf{b} + (\mathbf{a} \times \mathbf{b})$$

87. We have,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \frac{1}{2}\mathbf{a}$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} = \frac{1}{2}\mathbf{a}$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c}) = 0,\ (\mathbf{a}\cdot\mathbf{b}) = 0,\ (\mathbf{b}\cdot\mathbf{a}) = 0,\ (\mathbf{b}\cdot\mathbf{c}) = -\frac{1}{2}$$

Now, $(\mathbf{a}\cdot\mathbf{c}) = 0$

$\Rightarrow \quad ac\cos(\theta_1) = 0$

$\Rightarrow \quad \cos(\theta_1) = 0 = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \theta_1 = \left(\frac{\pi}{2}\right)$

Thus,

$$|(\mathbf{a} \times \mathbf{c})| = ac\sin\left(\frac{\pi}{2}\right) = 7.\ 1.\ 1 = 7.$$

88. We have,

$(\mathbf{d} + \mathbf{a})\cdot[\mathbf{a} \times \mathbf{b} \times (\mathbf{c} \times \mathbf{d})]$

$= (\mathbf{d} + \mathbf{a})\cdot(\mathbf{a} \times [(\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b} \times \mathbf{c})\mathbf{d}]$

$= (\mathbf{d} + \mathbf{a})\cdot[(\mathbf{a} \times \mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a} \times \mathbf{d})(\mathbf{b}\cdot\mathbf{c})]$

$= [\mathbf{d}\cdot(\mathbf{a} \times \mathbf{c})(\mathbf{b}\cdot\mathbf{d}) + \mathbf{a}\cdot(\mathbf{a} \times \mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}\cdot(\mathbf{a} \times \mathbf{d})(\mathbf{b}\cdot\mathbf{c}) - \mathbf{a}\cdot(\mathbf{a} \times \mathbf{d})(\mathbf{b}\cdot\mathbf{c})]$

$= [\mathbf{a},\ \mathbf{c},\ \mathbf{d}](\mathbf{b}\cdot\mathbf{d})$

89. Given,

$$p\mathbf{x} + (\mathbf{x} \times \mathbf{a}) = \mathbf{b}$$

$$\Rightarrow \quad \mathbf{p}(\mathbf{a}\cdot\mathbf{x}) + \mathbf{a}\cdot(\mathbf{x} \times \mathbf{a}) = \mathbf{a}\cdot\mathbf{b}$$

$$\Rightarrow \quad \mathbf{p}(\mathbf{a}\cdot\mathbf{x}) + 0 = \mathbf{a}\cdot\mathbf{b}$$

$$\Rightarrow \quad \mathbf{p}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}\cdot\mathbf{b}$$

$$\Rightarrow \quad \mathbf{p} = \frac{(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{a}\cdot\mathbf{x})} \qquad \text{...(i)}$$

Also, $p\mathbf{x} + (\mathbf{x} \times \mathbf{a}) = \mathbf{b}$

$$\Rightarrow \quad \mathbf{p}(\mathbf{a} \times \mathbf{x}) + \mathbf{a} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{p}(\mathbf{a} \times \mathbf{x}) + (\mathbf{a}\cdot\mathbf{a})\mathbf{x} - (\mathbf{a}\cdot\mathbf{x})\mathbf{a} = \mathbf{a} \times \mathbf{b}$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{a})\mathbf{x} = (\mathbf{a}\cdot\mathbf{x})\mathbf{a} - \mathbf{a} \times \mathbf{b} - \mathbf{p}(\mathbf{a} \times \mathbf{x})$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{a})\mathbf{x} = \frac{(\mathbf{a}\cdot\mathbf{b})}{\mathbf{p}}\mathbf{a} - \mathbf{a} \times \mathbf{b} - \mathbf{p}(\mathbf{a} \times \mathbf{x})$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{a})\mathbf{x} = \frac{(\mathbf{a}\cdot\mathbf{b})}{\mathbf{p}}\mathbf{a} - \mathbf{a} \times \mathbf{b} - \mathbf{p}(p\mathbf{x} - \mathbf{b})$$

$$\Rightarrow \quad (a^2 + p^2)\mathbf{x} = \frac{(\mathbf{a}\cdot\mathbf{b})}{p}\mathbf{a} - \mathbf{a} \times \mathbf{b} - p\mathbf{b}$$

$$\Rightarrow \quad \mathbf{x} = \frac{\frac{(\mathbf{a}\cdot\mathbf{b})}{p}\mathbf{a} - \mathbf{a} \times \mathbf{b} - p\mathbf{b}}{(a^2 + p^2)}$$

90. Given $[\mathbf{x}\ \mathbf{a}\ \mathbf{b}] = 0$

$\Rightarrow$ $\mathbf{x}$, $\mathbf{a}$, $\mathbf{b}$ are coplanar vectors

$\Rightarrow \quad \mathbf{x} = m\mathbf{a} + n\mathbf{b}$

$\Rightarrow \quad \mathbf{x}\cdot\mathbf{a} = m(\mathbf{a}\cdot\mathbf{a}) + n(\mathbf{b}\cdot\mathbf{a})$

$\Rightarrow \quad \mathbf{x}\cdot\mathbf{a} = m(\mathbf{a}\cdot\mathbf{a}) + n(\mathbf{b}\cdot\mathbf{a})$

$\Rightarrow \quad 0 = m|\mathbf{a}|^2 + 0$

$\Rightarrow \quad m = 0$

Also, $x = m\mathbf{a} + n\mathbf{b}$

$\Rightarrow \quad \mathbf{x}\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}) + n(\mathbf{b}\cdot\mathbf{b})$

$\Rightarrow \quad 1 = 0 + |\mathbf{b}|^2$

$\Rightarrow \quad n = \dfrac{1}{|\mathbf{b}|^2}$

Thus, $\quad \mathbf{x} = m\mathbf{a} + n\mathbf{b} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2} = \dfrac{\hat{\mathbf{b}}}{|\mathbf{b}|}$

91. We have,

$[\mathbf{a} \times (\mathbf{b} + \mathbf{c}), \mathbf{b} \times (\mathbf{c} - 2\mathbf{a}), \mathbf{c} \times (\mathbf{a} + 3\mathbf{b})]$

$= [\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}, \mathbf{b} \times \mathbf{c} - 2\mathbf{b} \times \mathbf{a}, \mathbf{c} \times \mathbf{a} + 3\mathbf{c} \times \mathbf{b}]$

$= [\mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a}, \mathbf{b} \times \mathbf{c} + 2\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{a} - 3\mathbf{b} \times \mathbf{c}]$

$= [\mathbf{x} - \mathbf{z}, \mathbf{y} + 2\mathbf{x}, \mathbf{z} - 3\mathbf{y}]$

where $\mathbf{x} = \mathbf{a} \times \mathbf{b}$, $\mathbf{y} = \mathbf{b} \times \mathbf{c}$, $\mathbf{z} = \mathbf{c} \times \mathbf{a}$

$= (\mathbf{x} - \mathbf{z}) \cdot (\mathbf{y} + 2\mathbf{x}) \times (\mathbf{z} - 3\mathbf{y})$

$= (\mathbf{x} - \mathbf{z}) \cdot (\mathbf{y} \times \mathbf{z} - 6\mathbf{x} \times \mathbf{y} + 2\mathbf{x} \times \mathbf{z})$

$= \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) + 6\mathbf{z} \cdot (\mathbf{x} + \mathbf{y})$

$= 7\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$

$= 7[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$

$= 7[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

$= 28$

92. We have,

$[(\mathbf{a} - \mathbf{b}), (\mathbf{a} - \mathbf{b} - \mathbf{c}), (\mathbf{a} + 2\mathbf{b} - \mathbf{c})]$

$= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b} - \mathbf{c}) \times (\mathbf{a} + 2\mathbf{b} - \mathbf{c})$

$= (\mathbf{a} - \mathbf{b}) \cdot [2\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} - \mathbf{c} \times \mathbf{a} - 2\mathbf{c} \times \mathbf{b}]$

$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - 2\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + 2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

$= 3[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$= 15$

93. We have,

$[(2\mathbf{a} \times 3\mathbf{b}), (3\mathbf{b} \times 4\mathbf{c}), (4\mathbf{c} \times 2\mathbf{a})]$

$= [\mathbf{x} \times \mathbf{y}, \mathbf{y} \times \mathbf{z}, \mathbf{z} \times \mathbf{x}]$

where $2\mathbf{a} = \mathbf{x}$, $3\mathbf{b} = \mathbf{y}$, $4\mathbf{c} = \mathbf{z}$

$= [\mathbf{x}, \mathbf{y}, \mathbf{z}]^2$

$= [2\mathbf{a}, 3\mathbf{b}, 4\mathbf{c}]^2$

$= 4.\ 9.\ 16[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$

$= 576 \times 4 = 2304$

94. Given,

$\mathbf{x} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad (\mathbf{x} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

$\Rightarrow \quad (\mathbf{x} - \mathbf{a}) = \lambda\mathbf{b}$, where $\lambda \in R$

$\Rightarrow \quad \mathbf{x} = \mathbf{a} + \lambda\mathbf{b}$.

95. We have,

$\mathbf{x} \times \mathbf{a} = \mathbf{b}$

$\Rightarrow \quad (\mathbf{x} \times \mathbf{a}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{x} \times \mathbf{a}) = -\mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{a})\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\mathbf{a} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad |\mathbf{a}|^2\mathbf{x} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{x} = \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}$.

96. Given,

$\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

$\Rightarrow \quad (\mathbf{r} - \mathbf{a}) = \lambda\mathbf{b}, \lambda \in R$

$\Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

$\Rightarrow \quad \mathbf{r} \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) + \lambda(\mathbf{b} \cdot \mathbf{c})$

$\Rightarrow \quad \lambda = -\dfrac{(\mathbf{a} \cdot \mathbf{c})}{(\mathbf{b} \cdot \mathbf{c})}$

Thus, $\quad \mathbf{r} = \mathbf{a} - \left(\dfrac{(\mathbf{a} \cdot \mathbf{c})}{(\mathbf{b} \cdot \mathbf{c})}\right)\mathbf{b}$.

97. Given,

$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$

Taking scalar product of $(\mathbf{b} \times \mathbf{c})$, we get

$x\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + y\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + z\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$

$\Rightarrow \quad x\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + 0 + 0 = \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$

$\Rightarrow \quad x\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$

$\Rightarrow \quad x = \dfrac{\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} = \dfrac{[\mathbf{d}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

Similarly, we can easily find that

$y = \dfrac{[\mathbf{d}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ and $z = \dfrac{[\mathbf{d}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

98. We have,

$\mathbf{x} + \mathbf{y} = \mathbf{a}$

$\Rightarrow \quad (\mathbf{x} + \mathbf{y}) \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{a}$

$\Rightarrow \quad \mathbf{x} \cdot \mathbf{a} + \mathbf{y} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{a}$

$\Rightarrow \quad 1 + \mathbf{y} \cdot \mathbf{a} = |\mathbf{a}|^2$ ...(i)

Also, $\quad \mathbf{x} \times \mathbf{y} = \mathbf{b}$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{y})\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\mathbf{y} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow \quad (|\mathbf{a}|^2 - 1)\mathbf{x} - \mathbf{y} = \mathbf{a} \times \mathbf{b}$ [from Eq. (i)] ...(ii)

Again $\quad \mathbf{x} + \mathbf{y} = \mathbf{a}$ ...(iii)

Adding Eqs. (ii) and (iii), we get

$(|\mathbf{a}|^2)\mathbf{x} = (\mathbf{a} \times \mathbf{b} + \mathbf{a})$

$$\Rightarrow \quad x = \frac{(\mathbf{a} \times \mathbf{b} + \mathbf{a})}{|\mathbf{a}|^2}$$

From Eq. (iii), we get

$$y = \mathbf{a} - \mathbf{x} = \mathbf{a} - \frac{(\mathbf{a} \times \mathbf{b} + \mathbf{a})}{|\mathbf{a}|^2}$$

Hence, $\mathbf{x} = \frac{(\mathbf{a} \times \mathbf{b} + \mathbf{a})}{|\mathbf{a}|^2}, \mathbf{y} = \mathbf{a} - \frac{(\mathbf{a} \times \mathbf{b} + \mathbf{a})}{|\mathbf{a}|^2}$

Level III

1. We have,

$$\mathbf{AB} = 3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$$

$$\mathbf{BC} = -7\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

and $\mathbf{CA} = 4\mathbf{i} - 6\mathbf{j}$

Unit vectors along **AB, BC, CA** are

$$\frac{1}{\sqrt{83}}(3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k})$$

$$\frac{1}{\sqrt{75}}(-7\mathbf{i} + \mathbf{j} + 5\mathbf{k})$$

and $\frac{1}{\sqrt{52}}(4\mathbf{i} - 6\mathbf{j})$.

Since the forces **P, Q, R** are of 15 KN each, so

$$\mathbf{P} = \frac{15}{\sqrt{83}}(3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k})$$

$$\mathbf{Q} = \frac{15}{\sqrt{75}}(-7\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$

and $\mathbf{R} = \frac{15}{\sqrt{52}}(4\mathbf{i} - 6\mathbf{k})$

Thus, the resultant S is given by

$= \mathbf{P} + \mathbf{Q} + \mathbf{R}$

$$= \frac{15}{\sqrt{83}}(3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}) + \frac{15}{\sqrt{75}}(-7\mathbf{i} - \mathbf{j} + 5\mathbf{k}) + \frac{15}{\sqrt{52}}(4\mathbf{i} - 6\mathbf{j})$$

$$= 15\left[\left(\frac{3}{\sqrt{83}} - \frac{7}{\sqrt{75}} + \frac{4}{\sqrt{52}}\right)\mathbf{i} + \left(\frac{7}{\sqrt{83}} - \frac{1}{\sqrt{75}} + \frac{6}{\sqrt{52}}\right)\mathbf{j} + \left(-\frac{5}{\sqrt{83}} + \frac{5}{\sqrt{75}}\right)\mathbf{k}\right]$$

2. As $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ does not imply that the vectors **a** and **b** are null vectors, let

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} \text{ and } b = 3\mathbf{j} + 4\mathbf{k}$$

Thus,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{vmatrix}$$

$$= 12\mathbf{i} - 8\mathbf{j} + 6\mathbf{k} \neq 0$$

3. Resultant of **P** and **Q**

$$= (2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$= (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}).$$

Displacement,

$$= \mathbf{AB} = (6\mathbf{i} + \mathbf{j} - 3\mathbf{k}) - (4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$$

$$= (2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$$

Workdone = Force · Displacement

$$= (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})\cdot(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$$

$$= |2 - 12 - 5|$$

$$= |-15| = 15\text{units}.$$

4. We have,

$$(\mathbf{a} \times \mathbf{b})^2 = (ab\sin\theta)^2$$

$$= a^2b^2\sin^2\theta$$

$$= a^2b^2(1 - \cos^2\theta)$$

$$= a^2b^2 - a^2b^2\cos^2\theta$$

$$= a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$$

5. Given,

$$(\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{c}) \neq \mathbf{0}$$

$$= (\mathbf{b} \times \mathbf{c})$$

$$= -(\mathbf{c} \times \mathbf{b})$$

$$\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) + (\mathbf{c} \times \mathbf{b}) = \mathbf{0}$$

$$\Rightarrow \quad (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} = \mathbf{0}$$

$\Rightarrow \quad \mathbf{a} \times \mathbf{c} = k\mathbf{b}$, where k is a scalar.

6. Sum of the vectors

$$= (2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + (S\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$= (2 + S)\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

Unit vector parallel to the sum of the vectors

$$= \frac{(2 + S)\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{(2 + S)^2 + 36 + 4}}$$

$$= \frac{(2 + S)\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{S^2 + 4S + 44}}$$

Also, it is given that

$$(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot\frac{(2 + S)\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{S^2 + 4S + 44}} = 1$$

$\Rightarrow \quad (2 + S) + 6 - 2 = \sqrt{S^2 + 36 + 44}$

$\Rightarrow \quad (S + 6)^2 = (S^2 + 4S + 44)$

$\Rightarrow \quad S^2 + 12S + 36 = (S^2 + 4S + 44)$

$\Rightarrow \quad 8S = 8$

$\Rightarrow \quad S = 1$

7. Since the vectors $(2\mathbf{i} - \mathbf{j} + \mathbf{k})$,

$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and $(3\mathbf{i} + a\mathbf{j} + 5\mathbf{k})$

are coplanar, so,

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

$\Rightarrow \quad 3(3 - 2) - a(-6 - 1) + 5(4 + 1) = 0$

$\Rightarrow \quad 3 + 7a + 25 = 0$

$\Rightarrow \quad 7a = -28$

$\Rightarrow \quad a = -4.$

8. Given,

$|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

$\Rightarrow \quad |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$

$\Rightarrow \quad a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow \quad 4\mathbf{a}\cdot\mathbf{b} = 0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = 0$

Thus, **a** and **b** are perpendicular to each other.

9. Let $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\Rightarrow \quad x^2 + y^2 + z^2 = 51 \qquad ...(i)$

Let the vector **r** makes an angle θ with the vectors **a, b** and **c**.

Thus, $\quad \cos\theta = \dfrac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}||a|} = \dfrac{\mathbf{r}\cdot\mathbf{b}}{|\mathbf{r}||\mathbf{b}|} = \dfrac{\mathbf{r}\cdot\mathbf{c}}{|\mathbf{r}||\mathbf{c}|}$

$\Rightarrow \quad \dfrac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}||a|} = \dfrac{\mathbf{r}\cdot\mathbf{c}}{|\mathbf{r}||\mathbf{c}|}$ and $\dfrac{\mathbf{r}\cdot\mathbf{b}}{|\mathbf{r}||\mathbf{b}|} = \dfrac{\mathbf{r}\cdot\mathbf{c}}{|\mathbf{r}||\mathbf{c}|}$

$\Rightarrow \quad \dfrac{\frac{1}{3}(x - 2y + 2z)}{\sqrt{51}} = \dfrac{y}{\sqrt{51}}; \dfrac{\frac{1}{5}(-4x - 3z)}{\sqrt{51}} = \dfrac{y}{\sqrt{51}}$

$\Rightarrow \quad x - 5y + 2z = 0;\ 4x + 5y + 3z = 0$

Solving, we get

$\dfrac{x}{-5} = \dfrac{y}{1} = \dfrac{z}{5} = \lambda(\text{say}) \qquad ...(ii)$

From Eqs. (i) and (ii), we get

$25\lambda^2 + \lambda^2 + 25\lambda^2 = 51$

$\Rightarrow \quad 51\lambda^2 = 51$

$\Rightarrow \quad \lambda^2 = 1$

$\Rightarrow \quad \lambda = \pm 1$

Thus, the vector **r** is

$= \pm(-5\mathbf{i} + \mathbf{j} + 5\mathbf{k}).$

10. We have $|\mathbf{a}| = \sqrt{144 + 16 + 9} = 13$

$|\mathbf{b}| = \sqrt{64 + 144 + 81} = \sqrt{289} = 17$

and $\quad |\mathbf{c}| = \sqrt{1089 + 16 + 576} = \sqrt{1681} = 41$

Thus, the length of the edges are 13, 17 and 41 respectively.

$\therefore$ Area of the faces

$= |\mathbf{a} \times \mathbf{b}|, |\mathbf{b} \times \mathbf{c}|, |\mathbf{c} \times \mathbf{a}|$

$= 220, 435, 455$

Volume of the parallelopiped

$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$$= \begin{vmatrix} 12 & 4 & 3 \\ 8 & -12 & -9 \\ 33 & -4 & -24 \end{vmatrix}$$

$= 3696$

11. Since **a**, **b** and **c** are non-coplanar vectors, so

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0.$

Now,

$[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \mathbf{x}\cdot(\mathbf{y} \times \mathbf{z})$

$= \dfrac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \cdot \left(\dfrac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \times \dfrac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\right)$

$= \dfrac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^3}[\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}]$

$= \dfrac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^3}[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$

$= \dfrac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \neq 0.$

Thus, **x, y** and **z** are non-coplanar vectors.

Now, $\mathbf{x}\cdot(\mathbf{a} + \mathbf{b}) + \mathbf{y}\cdot(\mathbf{b} + \mathbf{c}) + \mathbf{z}\cdot(\mathbf{a} + \mathbf{b})$

$$= \frac{(\mathbf{b} \times \mathbf{c})\cdot(\mathbf{a} + \mathbf{b}) + (\mathbf{c} \times \mathbf{a})\cdot(\mathbf{b} + \mathbf{c}) + (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} + \mathbf{a})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$= \dfrac{(\mathbf{b} \times \mathbf{c})\cdot\mathbf{a} + (\mathbf{c} \times \mathbf{a})\cdot\mathbf{b} + (\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

$= \dfrac{[\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

$= \dfrac{3[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \boldsymbol{c}]}$

$= 3$

12. Given,

$$\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$$

$\Rightarrow$ $(\mathbf{r} - \mathbf{c}) \times \mathbf{b} = \mathbf{0}$

$\Rightarrow$ $(\mathbf{r} - \mathbf{c})$ is parallel to $\mathbf{b}$.

or $\mathbf{r}$ is coplanar with the vectors $\mathbf{b}$ and $\mathbf{c}$.

Also, $\mathbf{r} \cdot \mathbf{a} = 0$

$\Rightarrow$ $\mathbf{r}$ is perpendicular to $\mathbf{a}$.

Again $\mathbf{a} \cdot \mathbf{b} \neq 0$

$\Rightarrow$ neither $\mathbf{a}$ nor $\mathbf{b}$ is a null vector or $\mathbf{a}$ is not perpendicular to $\mathbf{b}$.

13. Given,

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

$\Rightarrow$ $(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = 0$

$\Rightarrow$ $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{2}(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = 0$

$\Rightarrow$ $1 + 1 + 1 + \mathbf{2}(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = 0$

$\Rightarrow$ $\mathbf{2}(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = -3$

$\Rightarrow$ $(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = \frac{-3}{2}$

14. As we know that, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form $\mathbf{a}$ right handed system, so

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}$$

$\Rightarrow$ $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow$ $\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 0 & 1 & 0 \end{vmatrix}$

$$= x\mathbf{k} - z\mathbf{i} = -z\mathbf{i} + x\mathbf{k}$$

15. Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are lie in the same plane, so they are coplanar.

Thus, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

$\Rightarrow$ $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$

16. Given,

$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$

$\Rightarrow$ $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = d\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$\Rightarrow$ $a = d + 3,\ b = 4,\ c = 2.$

We have,

$$(\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 4 & 2 \\ d & 3 & 4 \end{vmatrix}$$

$$= 10\mathbf{i} - (4a - 2d)\mathbf{j} + (3a - 4d)\mathbf{k}$$

$$= 10\mathbf{i} - (4d + 12 - 2d)\mathbf{j} + (3d + 9 - 4d)\mathbf{k}$$

$$= 10\mathbf{i} - (2d - 12)\mathbf{j} + (9 - d)\mathbf{k}$$

Given the area of the triangle is $5\sqrt{6}$.

$\Rightarrow$ $\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = 5\sqrt{6}$

$\Rightarrow$ $|\mathbf{A} \times \mathbf{B}| = 10\sqrt{6}$

$\Rightarrow$ $\sqrt{100 + (2d + 12)^2 + (9 - d)^2} = 10\sqrt{6}$

$\Rightarrow$ $100 + (2d + 12)^2 + (9 - d)^2 = 600$

$\Rightarrow$ $(2d + 12)^2 + (9 - d)^2 = 500$

$\Rightarrow$ $4d^2 + 48d + 144 + 81 - 18d + d^2 = 500$

$\Rightarrow$ $5d^2 + 30d - 275 = 0$

$\Rightarrow$ $d^2 + 6d - 55 = 0$

$\Rightarrow$ $(d + 11)(d - 5) = 0$

$\Rightarrow$ $d = -11, 5$

Hence, $a = 8,\ b = 4,\ c = 2,\ d = 5$

or $a = -8,\ b = 4,\ c = 2,\ d = -11$

17. We have,

$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

$$= \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{a} \times \mathbf{b})$$

$$= (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \mathbf{b})\mathbf{a} - (\mathbf{b} \cdot \mathbf{a})\mathbf{b}$$

$$= \mathbf{a} - \mathbf{b} + \mathbf{a} - \mathbf{b}$$

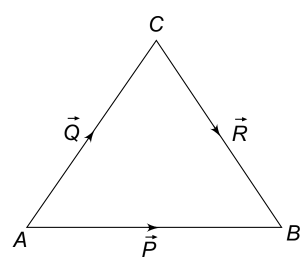
$$= 2(\mathbf{a} - \mathbf{b})$$

18. Ans. (c)

19. Ans. (d)

20. Also, Let $\hat{j}$, $\hat{k}$ be unit vectors on $\hat{a}$ so that $\hat{a}$, $\hat{j}$, $\hat{k}$ form a right handed system

Let $\vec{b} = b_1\hat{a} + b_2\hat{j} + b_3\hat{k}$



Then $\vec{P} = \sqrt{3}(\hat{a} \times \vec{b})$

$$= \sqrt{3}(b_2\hat{k} - b_3\hat{j})$$

and $\vec{Q} = \vec{b} - (\hat{a} \cdot \vec{b})\hat{a}$

$$= (b_2\hat{j} + b_3\hat{k})$$

Then $\cos A = \frac{\vec{P}\vec{Q}}{|\vec{P}||\vec{Q}|} = 0$

$$A = 90°$$

Also, $\vec{R} = \vec{P} - \vec{Q}$

$$= (\sqrt{3}b_2 - b_3)\hat{k} - (\sqrt{3}b_2 - b_2)\hat{j}$$

Now, $\cos B = \dfrac{\vec{P}\vec{R}}{|\vec{P}||\vec{R}|}$

$$= \frac{3(b_2^2 + b_3^2) - 2\sqrt{3}\,b_2 b_3}{2\sqrt{3}(b_2^2 + b_3^2)}$$

Thus, $B = \cos^{-1}\left(\dfrac{3(b_2^2 + b_3^2) - 2\sqrt{3}\,b_2 b_3}{2\sqrt{3}(b_2^2 + b_3^2)}\right)$

Also, $C = 180° - 90° - B = 90° - B$

Hence, the result,

Question asked in Past IIT-JEE Examinations

21. Let $\mathbf{x}$ be a unit vector such that $\mathbf{x} = a\mathbf{i} + b\mathbf{j}$

Thus, $|\mathbf{x}| = 1$

$\Rightarrow$ $|a\mathbf{i} + b\mathbf{j}| = 1$

$\Rightarrow$ $\sqrt{a^2 + b^2} = 1$

$\Rightarrow$ $a^2 + b^2 = 1$ ...(i)

Also

$(a\mathbf{i} + b\mathbf{j})\cdot(\mathbf{i} + \mathbf{j}) = |(a\mathbf{i} + b\mathbf{j})|\cdot|(\mathbf{i} + \mathbf{j})|\cos(45°)$

$\Rightarrow$ $a + b = 1\cdot\sqrt{2}\cdot\dfrac{1}{\sqrt{2}}$

$\Rightarrow$ $a + b = 1$ ...(ii)

Again,

$(a\mathbf{i} + b\mathbf{j})\cdot(3\mathbf{i} - 4\mathbf{j}) = |(a\mathbf{i} + b\mathbf{j})|\cdot|(3\mathbf{i} - 4\mathbf{j})|\cos(60°)$

$\Rightarrow$ $3a - 4b = 1.5\cdot\dfrac{1}{2}$

$\Rightarrow$ $3a - 4b = \dfrac{5}{2}$ ...(ii)

On solving (ii) and (iii), we get

$$a = \frac{13}{14} \text{ and } b = \frac{1}{14}$$

Thus, $\mathbf{x} = \dfrac{1}{14}(13\mathbf{i} - \mathbf{j})$

22. Since the given vectors are coplanar, so

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$\Rightarrow$ $3(3 - 2) - \lambda(-6 - 1) + 5(4 + 1) = 0$

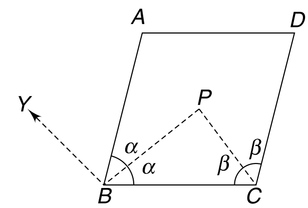
$\Rightarrow$ $3(3 - 2) - \lambda(-6 - 1) + 5(4 + 1) = 0$

$\Rightarrow$ $3 + 7\lambda + 25 = 0$

$\Rightarrow$ $7\lambda + 28 = 0$

$\Rightarrow$ $\lambda = -4$

23. Let BP as x-axis and BY as y-axis.



Let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors along x-axis and y-axis, respectively.

Here, $2\alpha + 2\beta = \pi$

$\Rightarrow$ $\alpha + \beta = \dfrac{\pi}{2}$

Let θ the angle between **BP** and **CP**

Now, $\mathbf{CP} = \mathbf{i}\cos(\alpha + \beta) + \mathbf{j}\sin(\alpha + \beta)$

$$= \mathbf{i}\cos\left(\frac{\pi}{2}\right) + \mathbf{j}\sin\left(\frac{\pi}{2}\right) = \mathbf{j}$$

Thus, $\mathbf{BP}\cdot\mathbf{CP} = \mathbf{i}\cdot\mathbf{j} = \dfrac{\pi}{2}$.

Therefore, the angle between the angle bisectors of angles B and C is 90°.

24. As we know that, a vector in the direction of the bisector of the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$= \frac{1}{3}(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

25. We $\mathbf{a} = \mathbf{b} \times \mathbf{c}$

$= \mathbf{b} \times (\mathbf{a} \times \mathbf{b})$

$= (\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}$

$= |\mathbf{b}|^2\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}$

Comparing, we get

$|\mathbf{b}|^2 = 1, (\mathbf{b}\cdot\mathbf{a}) = 0$

$\Rightarrow$ $|\mathbf{b}| = 1, \mathbf{a}\perp\mathbf{b}$

Also, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

$= (\mathbf{b} \times \mathbf{c}) \times \mathbf{b}$

$= (\mathbf{b}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b}$

$= |\mathbf{b}|^2\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b}$

Comparing, we get

$|\mathbf{b}|^2 = 1, (\mathbf{b}\cdot\mathbf{c}) = 0$

$\Rightarrow$ $|\mathbf{b}| = 1, \mathbf{b} \perp \mathbf{c}$

Thus, $|\mathbf{b}| = \mathbf{1}$ and $\mathbf{a} = \mathbf{c}$

26. Ans. (a)

27. Ans. (d)

28. Given,

$$\mathbf{x} \times \mathbf{y} = \mathbf{b}$$

$\Rightarrow$ $(\mathbf{x} \times \mathbf{y}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$

$\Rightarrow$ $\mathbf{a} \times (\mathbf{x} \times \mathbf{y}) = -\mathbf{b} \times \mathbf{a}$

$= \mathbf{a} \times \mathbf{b}$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{y})\mathbf{x} - (\mathbf{a}\cdot\mathbf{x})\mathbf{y} = \mathbf{a} \times \mathbf{b}$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{y})\mathbf{x} - \mathbf{y} = \mathbf{a} \times \mathbf{b}$ ...(i)

Also, $\mathbf{x} + \mathbf{y} = \mathbf{a}$ ...(ii)

$\Rightarrow$ $\mathbf{x}\cdot\mathbf{a} + \mathbf{y}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}$

$\Rightarrow$ $1 + \mathbf{y}\cdot\mathbf{a} = a^2$

$\Rightarrow$ $\mathbf{y}\cdot\mathbf{a} = a^2 - 1$ ...(iii)

From Eqs. (i) and (iii), we get

$(a^2 - 1)\mathbf{x} - \mathbf{y} = \mathbf{a} \times \mathbf{b}$ ...(iv)

Adding and subtracting Eq. (ii) from Eq. (iv), we get

$$\mathbf{x} = \frac{1}{a^2}(\mathbf{a} + \mathbf{a} \times \mathbf{b})$$

and $\mathbf{y} = \mathbf{a} - \frac{1}{a^2}(\mathbf{a} + \mathbf{a} \times \mathbf{b})$.

29. Ans. (b)

30. We have,

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + (\mathbf{a}\cdot\mathbf{b})\mathbf{b} = (4 - 2\beta - \sin\alpha)\mathbf{b} + (\beta^2 - 1)\mathbf{c}$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\cdot\mathbf{b})\mathbf{b}$

$= (4 - 2\beta - \sin\alpha)\mathbf{b} + (\beta^2 - 1)\mathbf{c}$

Comparing, we get

$(\mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{b}) = 4 - 2\beta - \sin\alpha$ ...(i)

and $(\beta^2 - 1) = -(\mathbf{a}\cdot\mathbf{b})$

$\Rightarrow$ $\beta^2 = 1(\mathbf{a}\cdot\mathbf{b})$ ...(ii)

Also $(\mathbf{c}\cdot\mathbf{c})\mathbf{a} = \mathbf{c}$

$\Rightarrow$ $(\mathbf{c}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{c}) = (\mathbf{c}\cdot\mathbf{c})$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c}) = 1$ ...(iii)

From Eqs. (i) and (iii), we get

$1 + (\mathbf{a}\cdot\mathbf{b}) = 4 - 2\beta - \sin\alpha$ ...(iv)

Adding Eqs. (ii) and (iv), we get

$$4 - 2\beta - \sin\alpha + \beta^2 = 2$$

$\Rightarrow$ $\beta^2 - 2\beta + 2 = \sin\alpha$

$\Rightarrow$ $(\beta - 1)^2 + 1 = \sin\alpha$

It is possible only when

$$(\beta - 1) = 0 \text{ and } \sin\alpha = 1$$

$\Rightarrow$ $\beta = $ and $\alpha = (4n + 1)\frac{\pi}{2}, n \in I$

31. Given $|(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow$ $|a||b||c|\sin\theta\cos\varphi = |a||b||c|$

$\Rightarrow$ $\sin\theta\cos\varphi = 1$

$\Rightarrow$ $\sin\theta = 1$ and $\cos\varphi = 1$

$\Rightarrow$ $\theta = \frac{\pi}{2}$ and $\varphi = 0$

Similarly, $|(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow$ $|(\mathbf{b} \times \mathbf{c})\cdot\mathbf{a}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow$ $\mathbf{b} \perp \mathbf{c}$

Again $|(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow$ $|(\mathbf{c} \times \mathbf{a})\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow$ $\mathbf{a} \perp \mathbf{c}$

Thus, $\mathbf{a} \perp \mathbf{b}$, $\mathbf{a} \perp \mathbf{c}$ and $\mathbf{b} \perp \mathbf{c}$

32. Given,

$$\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{a}$$

$\Rightarrow$ $(\mathbf{x} + \mathbf{y} + \mathbf{z})\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}$

$\Rightarrow$ $(\mathbf{x}\cdot\mathbf{a} + \mathbf{y}\cdot\mathbf{a} + \mathbf{z}\cdot\mathbf{a}) = |\mathbf{a}|^2$

$\Rightarrow$ $\left(\frac{3}{2} + \frac{7}{4} + \mathbf{z}\cdot\mathbf{a}\right) = 4$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{z}) = 4 - \frac{13}{4} = \frac{3}{4}$ ...(i)

Also,

$$\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{a}$$

$\Rightarrow$ $\mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{x} + \mathbf{z}\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{x}$

$\Rightarrow$ $1 + \mathbf{y}\cdot\mathbf{x} + \mathbf{z}\cdot\mathbf{x} = \frac{3}{2}$

$\Rightarrow$ $\mathbf{y}\cdot\mathbf{x} + \mathbf{z}\cdot\mathbf{x} = \frac{1}{2}$ ...(ii)

Similarly, $\mathbf{x}\cdot\mathbf{y} + \mathbf{z}\cdot\mathbf{y} = \frac{3}{4}$ ...(iii)

and $\mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z} = -\frac{1}{4}$ ...(iv)

Adding Eqs. (ii), (iii) and (iv), we get

$$\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{z} + \mathbf{z}\cdot\mathbf{y} = \frac{1}{2}$$

Hence, $\mathbf{x}\cdot\mathbf{y} = \frac{3}{4}$, $\mathbf{y}\cdot\mathbf{z} = 0$, $\mathbf{z}\cdot\mathbf{x} = -\frac{1}{4}$

Now, $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{b}$

$\Rightarrow$ $(\mathbf{x}\cdot\mathbf{z})\mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\mathbf{z} = \mathbf{b}$

$\Rightarrow$ $-\frac{1}{4}\mathbf{y} - \frac{3}{4}\mathbf{z} = \mathbf{b}$ ...(v)

Also, $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = \mathbf{c}$

$\Rightarrow \quad \mathbf{z} \times (\mathbf{x} \times \mathbf{y}) = -\mathbf{c}$

$\Rightarrow \quad (\mathbf{z} \cdot \mathbf{y})\mathbf{x} - (\mathbf{z} \cdot \mathbf{x})\mathbf{y} = -\mathbf{c}$

$\Rightarrow \quad 0\mathbf{x} + \frac{1}{4}\mathbf{y} = -\mathbf{c}$

$\Rightarrow \quad \mathbf{y} = -4\mathbf{c}$ ...(vi)

From Eqs. (v) and (vi), we get

$$\mathbf{c} - \frac{3}{4}\mathbf{z} = \mathbf{b}$$

$\Rightarrow \quad \mathbf{z} = \frac{4}{3}(\mathbf{c} - \mathbf{b})$

Finally, $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{a}$

$\Rightarrow \quad \mathbf{x} = \mathbf{a} - \mathbf{y} - \mathbf{z}$

$\Rightarrow \quad \mathbf{x} = \mathbf{a} + \frac{4}{3}\mathbf{b} + \frac{8}{3}\mathbf{c}$

33. We have,

$$\mathbf{x} \cdot \mathbf{y} = \sqrt{2} \cdot \sqrt{2} \cos(60°) = 2 \cdot \frac{1}{2}$$

Thus, $\mathbf{x} \cdot \mathbf{y} = 1 = \mathbf{y} \cdot \mathbf{z} = \mathbf{z} \cdot \mathbf{x}$

$\Rightarrow \quad \mathbf{x} \cdot \mathbf{x} = 2 = \mathbf{y} \cdot \mathbf{y} = \mathbf{z} \cdot \mathbf{z}$

Now, $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{a}$

$\Rightarrow \quad (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z} = \mathbf{a}$

$\Rightarrow \quad \mathbf{y} - \mathbf{z} = \mathbf{a}$ ...(i)

Also, $\mathbf{y} \times (\mathbf{z} \times \mathbf{x}) = \mathbf{b}$

$\Rightarrow \quad (\mathbf{y} \cdot \mathbf{z})\mathbf{z} - (\mathbf{y} \cdot \mathbf{z})\mathbf{x} = \mathbf{b}$

$\Rightarrow \quad \mathbf{z} - \mathbf{x} = \mathbf{b}$ ...(ii)

Adding Eq. (i) and (ii), we get

$\mathbf{y} - \mathbf{x} = \mathbf{a} + \mathbf{b}$ ...(iii)

Again, $(\mathbf{x} \times \mathbf{y}) = \mathbf{c}$

$\Rightarrow \quad \mathbf{x} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times \mathbf{c}$

$\Rightarrow \quad (\mathbf{x} \cdot \mathbf{y})\mathbf{x} - (\mathbf{x} \cdot \mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{c}$

$\Rightarrow \quad \mathbf{x} - 2\mathbf{y} = \mathbf{x} \times \mathbf{c}$ ...(iv)

Finally, $\mathbf{x} \times \mathbf{y} = \mathbf{c}$

$\Rightarrow \quad \mathbf{y} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \times \mathbf{c}$

$\Rightarrow \quad (\mathbf{y} \cdot \mathbf{y})\mathbf{x} - (\mathbf{y} \cdot \mathbf{x})\mathbf{y} = \mathbf{y} \times \mathbf{c}$

$\Rightarrow \quad 2\mathbf{x} - \mathbf{y} = \mathbf{y} \times \mathbf{c}$ ...(v)

Subtracting Eq. (iv) from Eq. (v), we get

$\mathbf{x} + \mathbf{y} = (\mathbf{y} - \mathbf{x}) \times \mathbf{c}$

$\Rightarrow \quad \mathbf{x} + \mathbf{y} = (\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ ...(vi)

Solving Eqs. (iii) and (vi), we get

$$\mathbf{x} = \frac{(\mathbf{a} + \mathbf{b}) \times \mathbf{c} - (\mathbf{a} + \mathbf{b})}{2}$$

$$\mathbf{y} = \frac{(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})}{2}$$

and $\mathbf{z} = \dfrac{\mathbf{b} - \mathbf{a} + (\mathbf{a} + \mathbf{b}) \times \mathbf{c}}{2}$

34. Let $\mathbf{r} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

$\Rightarrow \quad \mathbf{r} \perp (\mathbf{a} \times \mathbf{b})$ and $\mathbf{r} \perp \mathbf{c}$

$\Rightarrow \quad [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \perp \mathbf{c}$

35. Let D be the origin.

and $\mathbf{A} = \mathbf{i}, \mathbf{B} = \mathbf{i} + \mathbf{j}, \mathbf{C} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Now, $\mathbf{AB} \cdot \mathbf{CD} + \mathbf{BC} \cdot \mathbf{AD} + \mathbf{CA} \cdot \mathbf{BD}$

$= -1 + 0 + 1$

$= 0.$

36. Since **a**, **b** and **c** are non-zero and non-coplanar vectors, so any vector can be expressed as a linear combination of these three vectors, i.e.

$$-2\mathbf{a} + 3\mathbf{b} - \mathbf{c} = l\mathbf{p} + m\mathbf{q} + n\mathbf{r}$$

where $l, m, n \in R$

$\Rightarrow \quad -2\mathbf{a} + 3\mathbf{b} - \mathbf{c} = l(2\mathbf{a} - 3\mathbf{b}) + m(\mathbf{a} - 2\mathbf{b} + \mathbf{c}) + n(-3\mathbf{a} + \mathbf{b} + 2\mathbf{c})$

$= (2l + m - 3n)\mathbf{a} + (-3l - 2m + n)\mathbf{b} + (m + 2n)\mathbf{c}$

Comparing, we get

$(2l + m - 3n) = -2,$

$(-3l - 2m + n) = 3$

and $(m + 2n) = -1$

Solving, we get

$$l = 0, m = -\frac{7}{5}, n = \frac{1}{5}$$

Therefore, $-2\mathbf{a} + 3\mathbf{b} - \mathbf{c} = -\frac{7}{5}\mathbf{q} + \frac{1}{5}\mathbf{r}.$

37. According to the question,

$$\mathbf{c} = \lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right), \quad \text{where } \lambda \in R$$

$\Rightarrow \quad \mathbf{c} = \lambda\left(\frac{7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}}{9} + \frac{-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}}{3}\right)$

$\Rightarrow \quad \mathbf{c} = \lambda\left(\frac{\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}}{9}\right)$

It is given that,

$$|\mathbf{c}| = 3\sqrt{6}$$

$\Rightarrow \quad |\mathbf{c}|^2 = 54$

$\Rightarrow \quad 54 = \lambda^2\left(\frac{1 + 49 + 4}{81}\right)$

$\Rightarrow \quad \lambda^2 = 81$

$\lambda = \pm 9$

Therefore, $\mathbf{c} = \pm 9\left(\frac{\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}}{9}\right)$

$= \pm(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

38. Given $\mathbf{r} = \mathbf{x_1 r_1} + \mathbf{x_2 r_2} + \mathbf{x_3 r_3}$

$\Rightarrow \quad (2\mathbf{a} - 3\mathbf{b} + 4\mathbf{c})$

$= (\mathbf{a} - \mathbf{b} + \mathbf{c})x_1 + (\mathbf{b} + \mathbf{c} - \mathbf{a})x_2 + (\mathbf{c} + \mathbf{a} - \mathbf{b})x_3$

$= (x_1 - x_2 + x_3)\mathbf{a} + (-x_1 + x_2 - x_3)\mathbf{b}$

$+(x_1 + x_2 + x_3)\mathbf{c}$

Comparing, we get

$x_1 - x_2 + x_3 = 2$

$\Rightarrow \quad -x_1 + x_2 - x_3 = -2$

$\Rightarrow \quad x_1 + x_2 + x_3 = 4$

On solving, we get,

$$x_1 = \frac{7}{2}, x_2 = 1, x_3 = -\frac{1}{2}$$

39. Given,

$\mathbf{x} \times \mathbf{y} = \mathbf{a}$

$\Rightarrow \quad \mathbf{b} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{b} \times \mathbf{a}$

$\Rightarrow \quad (\mathbf{b}\cdot\mathbf{y})\mathbf{x} - (\mathbf{b}\cdot\mathbf{x})\mathbf{y} = \mathbf{b} \times \mathbf{a}$ ...(i)

Also, $\quad \mathbf{y} \times \mathbf{z} = \mathbf{b}$

$\Rightarrow \quad \mathbf{y}\cdot(\mathbf{y} \times \mathbf{z}) = \mathbf{y}\cdot\mathbf{b}$

$\Rightarrow \quad 0 = \mathbf{y}\cdot\mathbf{b}$

$\Rightarrow \quad \mathbf{y}\cdot\mathbf{b} = 0$ ...(ii)

From Eqs. (i) and (ii), we get

$-(\mathbf{b}\cdot\mathbf{x})\mathbf{y} = \mathbf{b} \times \mathbf{a}$

$\Rightarrow \quad \mathbf{y} = -\frac{\mathbf{b} \times \mathbf{a}}{(\mathbf{b}\cdot\mathbf{x})} = -\frac{\mathbf{b} \times \mathbf{a}}{\mathbf{c}}$ ...(iii)

Again $\quad \mathbf{x} \times \mathbf{y} = \mathbf{a}$

$\Rightarrow \quad \mathbf{y} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \times \mathbf{a}$

$\Rightarrow \quad (\mathbf{y}\cdot\mathbf{y})\mathbf{x} - (\mathbf{y}\cdot\mathbf{x})\mathbf{y} = \mathbf{y} \times \mathbf{a}$

$\Rightarrow \quad |\mathbf{y}|^2\mathbf{x} - (\mathbf{y}\cdot\mathbf{x})\mathbf{y} = \mathbf{y} \times \mathbf{a}$

$\Rightarrow \quad |\mathbf{y}|^2\mathbf{x} - \mathbf{y} = \mathbf{y} \times \mathbf{a}$ $\quad [\because (\mathbf{y}\cdot\mathbf{x}) = 1]$

$\Rightarrow \quad |\mathbf{y}|^2\mathbf{x} = \mathbf{y} + \mathbf{y} \times \mathbf{a}$

$\Rightarrow \quad \mathbf{x} = \frac{\mathbf{y} + \mathbf{y} \times \mathbf{a}}{|\mathbf{y}|^2}$

$$= \frac{-\left(\frac{\mathbf{b} \times \mathbf{a}}{\mathbf{c}}\right) - \left(\frac{\mathbf{b} \times \mathbf{a}}{\mathbf{c}}\right) \times \mathbf{a}}{\left(\frac{\mathbf{b} \times \mathbf{a}}{\mathbf{c}}\right)^2}$$

$$= \frac{\left(\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{c}}\right) - \left(\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{c}}\right) \times \mathbf{a}}{\left(\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{c}}\right)^2}$$

$$= \frac{\mathbf{c}((\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) \times \mathbf{a})}{(\mathbf{a} \times \mathbf{b})^2}$$

Finally, $\quad \mathbf{y} \times \mathbf{z} = \mathbf{b}$

$\Rightarrow \quad \mathbf{y} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \times \mathbf{b}$

$\Rightarrow \quad (\mathbf{y}\cdot\mathbf{z})\mathbf{y} - (\mathbf{y}\cdot\mathbf{y})\mathbf{z} = \mathbf{y} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{y} - |\mathbf{y}|^2\mathbf{z} = \mathbf{y} \times \mathbf{b}$ $\quad [\therefore (\mathbf{y}\cdot\mathbf{z}) = 1]$

$\Rightarrow \quad |\mathbf{y}|^2\mathbf{z} = \mathbf{y} - \mathbf{y} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{z} = \frac{\mathbf{y} - \mathbf{y} \times \mathbf{b}}{|\mathbf{y}|^2}$

$$= \frac{\left(\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{c}}\right) - \left(\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{c}}\right) \times \mathbf{b}}{\left(\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{c}}\right)^2}$$

$$= \frac{\mathbf{c}((\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{b}) \times \mathbf{b})}{(\mathbf{a} \times \mathbf{b})^2}.$$

40. Let $\mathbf{r} = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b} \times \mathbf{c}) + (\mathbf{b}\cdot\mathbf{d})(\mathbf{c} \times \mathbf{a}) + (\mathbf{c}\cdot\mathbf{d})(\mathbf{a} \times \mathbf{b})$

$\Rightarrow \quad \mathbf{r} = \alpha(\mathbf{b} \times \mathbf{c}) + \beta(\mathbf{c} \times \mathbf{a}) + \gamma(\mathbf{a} \times \mathbf{b})$

where $\alpha = (\mathbf{a}\cdot\mathbf{d})$, $\beta = (\mathbf{b}\cdot\mathbf{d})$, $\gamma = (\mathbf{c}\cdot\mathbf{d})$

Now, $\quad \mathbf{r}\cdot\mathbf{a} = \alpha[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\mathbf{r}\cdot\mathbf{b} = \beta[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\mathbf{r}\cdot\mathbf{c} = \gamma[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

Thus, $\mathbf{r}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) = (\alpha + \beta + \gamma)[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\Rightarrow \quad \mathbf{r}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) = (\mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d})[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\Rightarrow \quad \mathbf{r}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\Rightarrow \quad (\mathbf{r} - \mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}])(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$

$\Rightarrow \quad (\mathbf{r} - \mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}]) = \mathbf{0}$ [$\because$ **a**, **b**, **c** are non-coplanar]

$\Rightarrow \quad \mathbf{r} - \mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

$\Rightarrow \quad \mathbf{r} = \mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$\Rightarrow \quad |\mathbf{r}| = |\mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$

$\Rightarrow \quad |\mathbf{r}| = |[\mathbf{a}, \mathbf{b}, \mathbf{c}]| \qquad (\because \mathbf{d}$ is a unit vector)

which is independent of d.

Thus,

$$|(\mathbf{a}\cdot\mathbf{d})(\mathbf{b} \times \mathbf{c}) + (\mathbf{b}\cdot\mathbf{d})(\mathbf{c} \times \mathbf{a}) + (\mathbf{c}\cdot\mathbf{d})(\mathbf{a} \times \mathbf{b})|$$
$$= [\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

41. Let $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Now,

$$\mathbf{a} + \mathbf{b} = (\mathbf{i} + \mathbf{j} - \mathbf{k}) + (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$
$$= (3\mathbf{j} + \mathbf{k})$$

and

$$\mathbf{b} - \mathbf{c} = (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$
$$= 3\mathbf{k}$$

It is given that,

$$\mathbf{r}\cdot(\mathbf{a} + \mathbf{b}) = 0 \text{ and } \mathbf{r}\cdot(\mathbf{b} - \mathbf{c}) = 0$$

$\Rightarrow \quad 3y + z = 0$ and $3z = 0$

$\Rightarrow \quad y = 0$ and $z = 0$

Since $\mathbf{r}$ is a unit vector, so

$$x^2 + y^2 + z^2 = 1$$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

Thus, $\quad \mathbf{r} = \pm\mathbf{i}$

$\Rightarrow \quad \hat{\mathbf{r}} = \pm\mathbf{i}$

Hence, the result.

42. Here, $\mathbf{v}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are coplanar.

Thus,

$$\mathbf{v} = \alpha\mathbf{b} + \beta(\mathbf{a} \times \mathbf{b})$$

$\Rightarrow \quad \mathbf{v}\cdot\mathbf{b} = \alpha\mathbf{b}\cdot\mathbf{b} + \beta(\mathbf{a} \times \mathbf{b})\cdot\mathbf{b}$

$\Rightarrow \quad \mathbf{v}\cdot\mathbf{b} = \alpha|\mathbf{b}|^2 + 0$

$\Rightarrow \quad 1 = \alpha|\mathbf{b}|^2 + 0$

$\Rightarrow \quad \alpha = \dfrac{1}{|\mathbf{b}|^2}$

Now, $\quad [\mathbf{v}, \mathbf{a}, \mathbf{b}] = \mathbf{1}$

$\Rightarrow \quad \mathbf{v}\cdot(\mathbf{a} \times \mathbf{b}) = \mathbf{1}$

$\Rightarrow \quad (\alpha\mathbf{b} + \beta(\mathbf{a} \times \mathbf{b}))\cdot(\mathbf{a} \times \mathbf{b}) = \mathbf{1}$

$\Rightarrow \quad (\alpha\mathbf{b}\cdot(\mathbf{a} \times \mathbf{b}) + \beta(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{a} \times \mathbf{b})) = \mathbf{1}$

$\Rightarrow \quad (\alpha 0 + \beta(\mathbf{a} \times \mathbf{b})^2) = 1$

$\Rightarrow \quad \beta = \dfrac{1}{|(\mathbf{a} \times \mathbf{b})|^2}$

Therefore, $\quad \mathbf{v} = \dfrac{\mathbf{b}}{|\mathbf{b}^2|} + \dfrac{(\mathbf{a} \times \mathbf{b})}{|(\mathbf{a} \times \mathbf{b})|^2}$

43. Given,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{b} + \mathbf{c})$$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \dfrac{\mathbf{1}}{\mathbf{2}}(\mathbf{b} + \mathbf{c})$

Comparing, we get

$$\mathbf{a}\cdot\mathbf{c} = \frac{1}{2}, \ \mathbf{a}\cdot\mathbf{b} = -\frac{1}{2}$$

$\Rightarrow \quad \cos\theta_1 = \dfrac{1}{2}$ and $\cos\theta_2 = -\dfrac{1}{2}$

$\Rightarrow \quad \theta_1 = \dfrac{\pi}{3}$ and $\theta_2 = \dfrac{2\pi}{3}$

Thus, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{2\pi}{3}$.

44. Area of a given parallelogram

$$= \frac{1}{2}|\mathbf{d}_1 \times \mathbf{d}_2|$$
$$= \left|\frac{1}{2}\begin{vmatrix} i & j & k \\ 2 & 3 & -6 \\ 3 & -4 & -1 \end{vmatrix}\right|$$
$$= \left|\frac{1}{2}(-27\mathbf{i} - 16\mathbf{i} - 17\mathbf{k})\right|$$
$$= \frac{1}{2}\sqrt{729 + 256 + 289}$$
$$= \frac{1}{2} \times \sqrt{1274}$$
$$= \frac{39.5}{2} = 19.75$$

45. We have,

$$(-4\mathbf{i} + 5\mathbf{j})a + (3\mathbf{i} - 3\mathbf{j} + \mathbf{k})b\ (\mathbf{i} + \mathbf{j} + 3\mathbf{k})c$$
$$= \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

$$(-4a + 3b + c)\mathbf{i} + (5a - 3b + c)\mathbf{j} + (b + c)\mathbf{k}$$
$$= \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

Comparing, we get

$(-4a + 3b + c) = \lambda a$,

$(5a - 3b + c) = \lambda b$

and $(b + 3c) = \lambda c$.

Thus, $\quad \begin{vmatrix} -4-\lambda & 3 & 1 \\ 5 & -3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$

$\Rightarrow \quad \lambda^3 + 4\lambda^2 - 25\lambda = 0$

$\Rightarrow \quad \lambda(\lambda^2 + 4\lambda - 25) = 0$

$\Rightarrow \quad \lambda = 0$ and $(\lambda^2 + 4\lambda - 25) = 0$

$\Rightarrow \quad \lambda = 0$ and $\lambda = \dfrac{-4 \pm \sqrt{116}}{2}$

$\Rightarrow \quad \lambda = 0$ and $\lambda = -2 \pm \sqrt{29}$

46. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Given,

$$\mathbf{r}\cdot\mathbf{a} = 0,\ \mathbf{r}\cdot\mathbf{b} = 0$$

$$x - 2y + 5z = 0$$

$\Rightarrow$ $2x + 3y - z = 0$

Thus, $\dfrac{x}{2-15} = \dfrac{y}{10+1} = \dfrac{z}{3+4}$

$\Rightarrow$ $\dfrac{x}{-13} = \dfrac{y}{11} = \dfrac{z}{7} = \lambda(\text{say})$ ...(i)

Also, $\mathbf{r}\cdot(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + 8 = 0$

$\Rightarrow$ $2x + y + z + 8 = 0$ ...(ii)

From Eqs (i) and (ii), we get

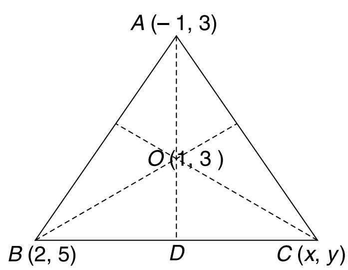
$$-26\lambda + 11\lambda + 7\lambda + 8 = 0$$

$\Rightarrow$ $-8\lambda + 8 = 0$

$\Rightarrow$ $\lambda = 1$

Hence, $\mathbf{r} = -13\mathbf{i} + 11\mathbf{j} + 7\mathbf{k}$

47. Let the position vector of the third vertex be $x\mathbf{i} + y\mathbf{j}$.



Here, $\mathbf{AC} = (x+1)\mathbf{i} + (y-3)\mathbf{j}$

$\Rightarrow$ $\mathbf{BO} = \mathbf{i} + 3\mathbf{j}$

Since $\mathbf{AC} \perp \mathbf{BO}$

$\Rightarrow$ $\mathbf{AC}\cdot\mathbf{BO} = 0$

$\Rightarrow$ $(x+1) + 3(y-3) = 0$

$\Rightarrow$ $x + 3y = 8$ ...(i)

Also, $\mathbf{BC} = (x-2)\mathbf{i} + (y-5)\mathbf{j}$

and $\mathbf{OA} = 2\mathbf{i} - \mathbf{j}$

Again, $\mathbf{BC}\cdot\mathbf{OA}$

$\Rightarrow$ $2(x-2) - (y-5) = 0$

$\Rightarrow$ $2x - y = 1$ ...(ii)

On solving Eqs (i) and (ii), we get

$$x = \frac{5}{7},\ y = \frac{17}{7}$$

Hence, the position vector of the thrid vertex is

$$= \frac{5}{7}\mathbf{i} + \frac{17}{7}\mathbf{j}.$$

Level II

(Tougher Problems for JEE Advanced)

1. We have,

$$|\vec{a} + \vec{b}|^2 = a^2 + b^2 + 2(\vec{a}\cdot\vec{b})$$
$$= 1 + 1 + 2\cos\theta$$
$$= 2(1 + \cos\theta)$$
$$= 2.2\cos^2\left(\frac{\theta}{2}\right)$$
$$= 4\cos^2\left(\frac{\theta}{2}\right)$$

$\Rightarrow$ $|\vec{a} + \vec{b}| = 2\cos\left(\dfrac{\theta}{2}\right)$

$\Rightarrow$ $\dfrac{3}{2}\left|\vec{a} + \vec{b}\right| = 3\cos\left(\dfrac{\theta}{2}\right)$

Also,

$$|\vec{a} - \vec{b}|^2 = a^2 + b^2 - 2(\vec{a}\cdot\vec{b})$$
$$= 1 + 1 - 2\cos\theta$$
$$= 2(1 - \cos\theta)$$
$$= 4\sin^2\left(\frac{\theta}{2}\right)$$

$\Rightarrow$ $|\vec{a} - \vec{b}| = 2\sin\left(\dfrac{\theta}{2}\right)$

$\Rightarrow$ $2|\vec{a} - \vec{b}| = 4\sin\left(\dfrac{\theta}{2}\right)$

Thus,

$$\frac{3}{2}\left|\vec{a} + \vec{b}\right| + 2|\vec{a} - \vec{b}| = 3\cos\left(\frac{\theta}{2}\right) + 4\sin\left(\frac{\theta}{2}\right)$$

Hence, the range of $\dfrac{3}{2}\left|\vec{a} - \vec{b}\right| + 2|\vec{a} - \vec{b}|$

$$= [-5, 5].$$

2. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$,

where

$$x^2 + y^2 + z^2 = 1$$

It is given that,

$$\vec{a}\cdot\vec{k} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $z = \dfrac{1}{\sqrt{2}}$

Also, $|\vec{a} + \hat{i} + \hat{j}| = 1$

$\Rightarrow$ $|(x+1)\hat{i} + (y+1)\hat{j} + z\hat{k}| = 1$

$\Rightarrow$ $(x+1)^2 + (y+1)^2 + z^2 = 1$

$\Rightarrow$ $(x+1)^2 + (y+1)^2 + \dfrac{1}{2} = 1$

$\Rightarrow$ $(x+1)^2 + (y+1)^2 = \dfrac{1}{2}$

$\Rightarrow \quad x^2 + y^2 + 2(x + y) + 2 = \frac{1}{2}$

$\Rightarrow \quad \frac{1}{2} + 2(x + y) + 2 = \frac{1}{2}$

$\Rightarrow \quad x + y = -1 \quad$ ...(i)

Also, $\quad x^2 + y^2 = \frac{1}{2} \quad$ ...(ii)

From Eq. (i) and (ii), we get

$x = -\frac{1}{2} = y,\ z = \frac{1}{\sqrt{2}}$

Thus, $\quad \vec{a} = -\frac{1}{2}(\hat{i} + \hat{j}) + \frac{\hat{k}}{\sqrt{2}}$

3. It is given that $|\vec{r} + \vec{bs}|$ is minimum when $b = 0$

Hence, the value of $|\vec{r} + \vec{bs}|^2 + |\vec{bs}|^2$

$= |\vec{r}|^2$

4. Let $\quad \vec{R} = a\hat{i} + b\hat{j} + c\hat{k}$.

It is given that

$(\vec{u}\cdot\vec{R} - 10)\hat{i} + (\vec{v}\cdot\vec{R} - 20)\hat{j} + (\vec{w}\cdot\vec{R})\hat{k} = 0$

$\Rightarrow \quad (a - 2b + 3c)\hat{i} + (2a + b + 4c)\hat{j}$

$+(a + 3b + 3c)\hat{k} = 10\hat{i} - 20\hat{j} - 20\hat{k}$

Comparing the co-efficients, we get

$a - 2b + 3c = 10,$

$2a + b + 4c = -20$

and $a + 3b + 3c = -20$

Solving, we get

$a = -1,\ b = 2,\ c = 5$

Hence, $\quad \vec{R} = -\hat{i} + 2\hat{j} + 5\hat{k}$.

5. It is given that,

$|\vec{a} + \vec{b}| = 1$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = 1$

$\Rightarrow \quad a^2 + b^2 + 2(\vec{a}\cdot\vec{b}) = 1$

$\Rightarrow \quad 1 + 1 + 2(\vec{a}\cdot\vec{b}) = 1$

$\Rightarrow \quad (\vec{a}\cdot\vec{b}) = -\frac{1}{2}$

Now,

$(2\vec{a} + \vec{b})\cdot\vec{b} = (2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b})$

$= 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$

$= 2\left(-\frac{1}{2}\right) + 1$

$= -1 + 1$

$= 0.$

Thus, the angle between $(2\vec{a} + \vec{b})$ and $\vec{b}$ is $\frac{\pi}{2}$.

6. Now, $\quad \overrightarrow{AB} = 2\hat{i} + \hat{j} + \hat{k}$

and $\quad \overrightarrow{AC} = (t + 1)\hat{i} - \hat{k}$

Thus,

$ar(\Delta ABC) = \frac{1}{2}\left|(\overrightarrow{AB} \times \overrightarrow{AC})\right|$

$= \left|\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 1\\ (t+1) & 0 & -1\end{vmatrix}\right|$

$= \left|\frac{1}{2}\left(-\hat{i} + (t + 3)\hat{j} - (t + 1)\hat{k}\right)\right|$

$= \frac{1}{2}\sqrt{1 + (t + 3)^2 + (t + 1)^2}$

$= \frac{1}{2}\sqrt{2t^2 + 8t + 11}$

Let $\quad z = 2t^2 + 8t + 11$

$\Rightarrow \quad \frac{dz}{dt} = 4t + 8$

$\Rightarrow \quad \frac{d^2z}{dt^2} = 4 > 0$

$\Rightarrow \quad z$ is minimum

$\Rightarrow \quad$ Area is minimum.

For maximum or minimum,

$\frac{dz}{dt} = 0$ gives $4t + 8 = 0$

$t = -2$

Thus, the minimum area is

$= \frac{1}{2}\sqrt{8 - 16 + 11} = \frac{3}{\sqrt{2}}$

7. Given,

$|\vec{a} + \vec{b}| = 1$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = 1$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = 1$

$\Rightarrow \quad 1 + 1 + 2(\vec{a}\cdot\vec{b}) = 1$

$\Rightarrow \quad (\vec{a}\cdot\vec{b}) = -\frac{1}{2}$.

It is also given that,

$\vec{c} = \lambda\vec{a} + \mu\vec{b}$

$\Rightarrow \quad |\vec{c}|^2 = \lambda^2 + \mu^2 + 2\lambda\mu(\vec{a}\cdot\vec{b})$

$\Rightarrow \quad 1 = \lambda^2 + \mu^2 + 2\lambda\mu\left(-\frac{1}{2}\right)$

$= \lambda^2 + (2\lambda)^2 - \lambda(2\lambda)$

$\Rightarrow \quad 3\lambda^2 = 1$

$\Rightarrow \quad \lambda^2 = \frac{1}{3}$

$\Rightarrow \quad \lambda = \frac{1}{\sqrt{3}}$

Again $\quad \mu = 2\lambda = \frac{2}{\sqrt{3}}$

Thus, $\quad (\lambda, \mu) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

8. It is given that,

$$\vec{a} \cdot \vec{b} = ab\cos(\theta)$$

$\Rightarrow \quad \cos\theta = \cos(\cos^{-1})\left(\frac{1}{2}\right) = \frac{1}{4}$

Also, $\quad \vec{c} \times \vec{b} = \overrightarrow{2a} \times \vec{b}$

$\Rightarrow \quad (\vec{c} - \overrightarrow{2a}) \times \vec{b} = \vec{0}$

$\therefore \quad (\vec{c} - \overrightarrow{2a})$ is parallel to $\vec{b}$

$\Rightarrow \quad (\vec{c} - \overrightarrow{2a}) = m\vec{b}$

$\Rightarrow \quad \vec{c} = \overrightarrow{2a} + m\vec{b}$

$\quad = \lambda\vec{a} + \mu\vec{b}$

Thus, $\quad \lambda = 2, m = \mu$

$\therefore \quad \vec{c} = \overrightarrow{2a} + \overrightarrow{mb}$

$\Rightarrow \quad |\vec{c}|^2 = 4|\vec{a}|^2 + m^2|\vec{b}|^2 + 4m(\vec{a} \cdot \vec{b})$

$\Rightarrow \quad 16 = 4 + m^2 + 4m\left(\frac{1}{4}\right)$

$\Rightarrow \quad m^2 + m - 12 = 0$

Sum of the roots = –1

and the product of the roots = 12

Hence, sum of the values of $\mu = -1$

and the product of all values of $\mu = 12$.

9. Given, $\quad \vec{c} = (\overrightarrow{2a} \times \vec{b}) - 3\vec{b}$

Now, $\quad \vec{b} \cdot \vec{c} = -3|\vec{b}|^2 = -48$

$\quad \vec{a} \cdot \vec{c} = -3(\vec{a} \cdot \vec{b}) = -6$

Also, $\quad |\vec{c}|^2 = |(2\vec{a} \times \vec{b})|^2 + 9|\vec{b}|^2 - 0$

$\quad = 4a^2b^2\sin^2\theta + 144$

$\quad = 64\sin^2\theta + 144$

$\quad = 64 \times \frac{3}{4} + 144 = 48 + 144 = 192$

$\Rightarrow \quad |\vec{c}| = \sqrt{192} = 8\sqrt{3}$

Again, $\quad \vec{b} \cdot \vec{c} = -48$

$\Rightarrow \quad |\vec{b}||\vec{c}|\cos\varphi = -48$

$\Rightarrow \quad \cos\varphi = \frac{-48}{4 \times 8\sqrt{3}} = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \varphi = \frac{5\pi}{6}$

10. Given,

$$\vec{a} + \vec{b} = \mu\vec{p}$$

$\Rightarrow \quad \vec{a} \cdot \vec{q} + \vec{b} \cdot \vec{q} = \mu(\vec{p} \cdot \vec{q})$

$\Rightarrow \quad \vec{a} \cdot \vec{q} = \mu(\vec{p} \cdot \vec{q})$

Now.,

$$|(\vec{a} \cdot \vec{q})\vec{p} - (\vec{p} \cdot \vec{q})\vec{a}| = |(\vec{p} \cdot \vec{q})|\ |\mu\vec{p} - \vec{a}|$$

$$= |(\vec{p} \cdot \vec{q})|\ |\vec{b}|$$

$$= |(\vec{p} \cdot \vec{q})|$$

11. Let $\quad \vec{a} = \hat{i} + \hat{j} - 2\hat{k}$

$\quad \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

and $\quad \vec{c} = -2\hat{i} + \hat{j} + \hat{k}$

Now,

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= -3\hat{i} - 3\hat{j} - 3\hat{k}$$

Also,

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ -3 & -3 & -3 \end{vmatrix}$$

$$= 3\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= 3\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= 3\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= -9(\hat{i} - \hat{j})$$

12. It is given that,

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

$$\Rightarrow \quad \alpha\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 6 \end{pmatrix}$$

$\Rightarrow \quad \alpha + 2\beta + \gamma = -2$

$\Rightarrow \quad \beta + \gamma = -5$

$\Rightarrow \quad -3\alpha - \gamma = 6$

Solving, we get

$\alpha = 1, \beta = -2, \gamma = 3.$

13. Let $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$

$\vec{X} = \hat{i} + 2\hat{j}$

and $\vec{Y} = 2\hat{i} + \hat{k}$

Now,

$$\vec{V} \times \vec{X} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ 1 & 2 & 0 \end{vmatrix}$$

$$= -2c\hat{i} + c\hat{j} + (2a - b)\hat{k}$$

$\Rightarrow \quad 2\vec{V} + \vec{V} \times \vec{X}$

$$= 2(a - c)\hat{i} + (2b + c)\hat{j} + (2a - b + 2c)\hat{k}$$

$\Rightarrow \quad 2(a - c)\hat{i} + (2b + c)\hat{j} + (2a - b + 2c)\hat{k}$

$$= (2\hat{i} + \hat{k})$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$, we get

$a - c = 1, 2b + c = 0, 2a - b + 2c = 1.$

Solving, we get

$$a = \frac{7}{9}, b = \frac{1}{9}, c = -\frac{2}{9}$$

Now,

$3|\vec{V}| = \sqrt{m}$

$\Rightarrow \quad 9|\vec{V}|^2 = m$

$\Rightarrow \quad m = 9 \times \left(\frac{49 + 1 + 4}{81}\right)$

$$= 9 \times \left(\frac{54}{81}\right) = 6$$

14. It is given that,

$$x\vec{A} + y\vec{B} + z\vec{C} = \vec{B} \times \vec{C}$$

$$= 2(\hat{i} - \hat{j} + \hat{k})x(\hat{i} - 2\hat{j} + 3\hat{k})$$

$$+ y(2\hat{i} + \hat{j} - \hat{k}) + z(\hat{j} + \hat{k})$$

$$= 2(\hat{i} - \hat{j} + \hat{k})$$

Comparing the co-efficients of the unit vectors, we get

$x + 2y = 2$

$-2x + y + z = -2$

and $3x - y + z = 2$

Solving, we get

$$x = 1, y = \frac{1}{2}, z = -\frac{1}{2}$$

Hence, the value of

$(100x + 10y + 8z)$

$= 100 + 5 - 4$

$= 101$

15. Do yourself

Integer Type Questions

1. We have,

$$\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}} + \frac{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}} + \frac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}}$$

$$= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$= 1 + 1 + 1$

$= 3$

2. We have,

$(\mathbf{a} + \mathbf{b})\cdot\mathbf{p} + (\mathbf{b} + \mathbf{c})\cdot\mathbf{q} + (\mathbf{c} + \mathbf{a})\cdot\mathbf{r}$

$$= \frac{(\mathbf{a} + \mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{(\mathbf{b} + \mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{(\mathbf{c} + \mathbf{a})\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{0}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) + \mathbf{0}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) + \mathbf{0}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$= 3$

3. It is given that, $\vec{c}$ is parallel to the plane of the vectors $\vec{a}$ and $\vec{b}$, i.e.

$\mathbf{c} \perp (\mathbf{a}\times\mathbf{b})$

$\Rightarrow \quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 0$

$$\Rightarrow \quad \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ \lambda & 1 & (2\lambda - 1) \end{vmatrix} = 0$$

$\Rightarrow \quad (6\lambda - 3 + 1) + 2(4\lambda - 2 + \lambda) + 3(2 - 3\lambda) = 0$

$\Rightarrow \quad (6\lambda - 2) + 2(5\lambda - 2) + 3(2 - 3\lambda) = 0$

$\Rightarrow \quad (6\lambda - 2) + (10\lambda - 4) + 3(6 - 9\lambda) = 0$

$\Rightarrow \quad 7\lambda = 0$

$\Rightarrow \quad \lambda = 0$

Hence, the value of $(\lambda^2 + 2)$ is 2.

4. We have,

$$\mathbf{r}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$$

$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{a} + \mathbf{r}\cdot\mathbf{b} + \mathbf{r}\cdot\mathbf{c} = 0$$

Now, $\mathbf{r}\cdot\mathbf{a} = P[\mathbf{b}, \mathbf{c}, \mathbf{a}]$

$\mathbf{r}\cdot\mathbf{b} = Q[\mathbf{c}, \mathbf{a}, \mathbf{b}]$

and $\mathbf{r}\cdot\mathbf{c} = R[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

Adding, we get

$$P[\mathbf{b}, \mathbf{c}, \mathbf{a}] + Q[\mathbf{c}, \mathbf{a}, \mathbf{b}] + R[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$$

$$\Rightarrow \quad (P + Q + R) = 0$$

Hence, the value of $(P + Q + R + 5) = 5$.

5. It is given that

$$\vec{a}\perp(\vec{b} + \vec{c}),\ \vec{b}\perp(\vec{c} + \vec{a}), \vec{c}\perp(\vec{a} + \times \vec{b})$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{b} + \vec{c}) = 0,$$

$$\vec{b}\cdot(\vec{c} + \vec{a}) = 0$$

and $\vec{c}\cdot(\vec{a} + \vec{b}) = 0$

Thus, $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = 0$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2$

$$= a^2 + b^2 + c^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$= a^2 + b^2 + c^2$$

$$= 1 + 1 + 4$$

$$= 6$$

Hence, the value of $(m^2 + 1) = 7$.

6. In an AP,

$$t_p = A + (p - 1)d = \frac{1}{a}$$

$$t_q = A + (q - 1)d = \frac{1}{b}$$

$$t_r = A + (r - 1)d = \frac{1}{c}$$

Thus, $p - q = \frac{1}{d}\left(\frac{b - a}{ab}\right)$

$$q - r = \frac{1}{d}\left(\frac{c - b}{cb}\right)$$

$$r - p = \frac{1}{d}\left(\frac{a - c}{ac}\right)$$

Now, $\mathbf{u}\cdot\mathbf{v}$

$$= \{(q - r)\hat{i} + (r - p)\hat{j} + (p - q)\hat{k}\}\cdot\left(\frac{\hat{i}}{a} + \frac{\hat{j}}{b} + \frac{\hat{k}}{c}\right)$$

$$= \frac{1}{abcd}\left(\frac{1}{b} - \frac{1}{c} + \frac{1}{c} - \frac{1}{a} + \frac{1}{a} - \frac{1}{b}\right)$$

$$= 0$$

Thus, $m = 0$

Hence, the value of $(m + 4) = 4$.

7. Clearly, $\mathbf{p} = \dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$, $\mathbf{q} = \dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$, $\mathbf{r} = \dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$

Hence, the value of $(\vec{a} + \vec{b} + c)\cdot(\vec{p} + \vec{q} + \vec{r})$

$$= \frac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{\mathbf{b}\cdot\mathbf{c}\times\mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= 3$$

8. We have,

$$|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$$

$$\Rightarrow \quad 2(a^2 + b^2 + c^2) - 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 9$$

$$\Rightarrow \quad 6 - 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 9$$

$$\Rightarrow \quad -2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 3$$

$$\Rightarrow \quad 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = -3$$

Now, $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$

$$= a^2 + b^2 + c^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{a})$$

$$= 3 - 3 = 0$$

Thus, $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = 0$

$$\Rightarrow \quad \mathbf{a} + \mathbf{c} = -\mathbf{a}$$

Hence, the value of

$$|2\vec{a} + 5\vec{b} + 5\vec{c}| = |2\mathbf{a} + 5(\mathbf{b} + \mathbf{c})|$$

$$= |2\mathbf{a} - 5\mathbf{a}|$$

$$= |3\mathbf{a}|$$

$$= 3$$

9. Let $\vec{p} = \hat{i} + a\hat{j} + b\hat{k}$,

$$\vec{q} = a\hat{i} + 2\hat{j} + b\hat{k}$$

and $\vec{r} = a\hat{i} + b\hat{j} + 3\hat{k}$

Now, $\overrightarrow{pq} = (a - 1)\hat{i} + (2 - a)\hat{j}$

and $\overrightarrow{qr} = +(b - 2)\hat{j} + (3 - b)\hat{k}$

Since the vectors are collinear, so

$$(a - 1)\hat{i} + (2 - a)\hat{j} = \lambda\{(b - 2)\hat{j} + (3 - b)\hat{k}\}$$

$$\Rightarrow \quad (a - 1) = 0,\ (3 - b) = 0$$

$$\Rightarrow \quad a = 1,\ b = 3$$

Hence, the value of $(a + b) = 4$.

10. Let $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$.

Thus, $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$

Now, $\vec{u}\cdot\hat{n} = 0 \Rightarrow a + b = 0$

$\Rightarrow \vec{v}\cdot\hat{n} = 0 \Rightarrow a - b = 0$

Solving, we get

$a = 0,\ b = 0$ and $c = 1$

Thus, $\vec{n} = \hat{k}$

Hence, the value of $|\vec{w}\cdot\hat{n}| = 3$.

11. Clearly,

$2x - y = 5$

and $x - 2y = 4$

Solving, we get

$3x = 6$

$\Rightarrow x = 2$

and so $y = -1$

Hence, the value of $(x + y + 2) = 3$.

12. It is given that,

$|-\vec{cu}, \vec{v}, \vec{cw}| = 8$

$$\Rightarrow c^2\begin{vmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & -1 \end{vmatrix} = 8$$

$$\Rightarrow c^2\begin{vmatrix} -2 & 1 & -1 \\ 1 & -1 & 3 \\ 1 & 0 & 0 \end{vmatrix} = 8 \qquad (C_3 \to C_1 + C_3)$$

$\Rightarrow c^2(3 - 1) = 8$

$\Rightarrow c^2 = 4$

$\Rightarrow c = 2$

13. Now,

$$(\mathbf{p} \times \mathbf{q}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & -1 \end{vmatrix} = -5\hat{i} + 3\hat{j} + 2\hat{k}$$

Thus,

$$(\mathbf{p} \times \mathbf{q}) \times \mathbf{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 7\hat{i} + 17\hat{j} - 8\hat{k}$$

It is given that,

$$(\vec{p} \times \vec{q}) \times \vec{r} = u\vec{p} + v\vec{q} + w\vec{r}$$

$$7\hat{i} + 17\hat{j} - 8\hat{k} = (\hat{i} + \hat{j} + \hat{k})u$$

$$+(2\hat{i} + 4\hat{j} + \hat{k})v + (\hat{i} + \hat{j} + 3\hat{k})w$$

$$= (u + 2v + w)\hat{i} + (u + 4v + w)\hat{j} + (u - v + 3w)\hat{k}$$

Comparing the co-efficients, we get

$(u + 2v + w) = 7$

$(u + 4v + w) = 17$

and $(u - v + 3w) = -8$

Solving, we get

$u = -3,\ v = 5$ and $w = 0$

Hence, the value of

$(u + v + w + 4) = -3 + 5 + 0 + 4$

$= 6.$

14. Let $\vec{s} = x(\vec{p} + \vec{q}) + y(\vec{q} + \vec{r}) + z(\vec{r} + \vec{p})$

Now, $\vec{p}\cdot\vec{s} = y(\vec{p}\cdot(\vec{q} \times \vec{r})) = y[\vec{p}, \vec{q}, \vec{r}]$

$$\Rightarrow y = \frac{\vec{p}\cdot\vec{s}}{[\vec{p}, \vec{q}, \vec{r}]}$$

Similarly, $x = \frac{\vec{r}\cdot\vec{s}}{[\vec{p}, \vec{q}, \vec{r}]}$ and $z = \frac{\vec{q}\cdot\vec{s}}{[\vec{p}, \vec{q}, \vec{r}]}$

Thus,

$$\Rightarrow \vec{s} = \frac{(\vec{p}\cdot\vec{s})(\vec{q} \times \vec{r})}{[\vec{p}, \vec{q}, \vec{r}]} + \frac{(\vec{q}\cdot\vec{s})(\vec{r} \times \vec{p})}{[\vec{p}, \vec{q}, \vec{r}]} + \frac{(\vec{r}\cdot\vec{s})(\vec{p} \times \vec{q})}{[\vec{p}, \vec{q}, \vec{r}]}$$

$$\Rightarrow \vec{s}[\vec{p}, \vec{q}, \vec{r}] = (\vec{p}\cdot\vec{s})(\vec{q} \times \vec{r}) + (\vec{q}\cdot\vec{s})(\vec{r} \times \vec{p}) + (\vec{r}\cdot\vec{s})(\vec{p} \times \vec{q})$$

$$\Rightarrow |(\vec{p}\cdot\vec{s})(\vec{q} \times \vec{r}) + (\vec{q}\cdot\vec{s})(\vec{r} \times \vec{p}) + (\vec{r}\cdot\vec{s})(\vec{p} \times \vec{q})|$$

$= |\vec{s}\,[\vec{p}, \vec{q}, \vec{r}]|$

$= |\vec{s}|\ |[\vec{p}, \vec{q}, \vec{r}]|$

$= 1 \times 4$

$= 4$

Questions asked in Past IIT-JEE Examinations

1. We have,

$\mathbf{A}\cdot\{(\mathbf{B} + \mathbf{C}) \times (\mathbf{A} + \mathbf{B} + \mathbf{C})\}$

$= \mathbf{A}\cdot\{\mathbf{B} \times \mathbf{A} + \mathbf{B} \times \mathbf{B} + \mathbf{B} \times \mathrm{C} + \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B} + \mathbf{C} \times \mathbf{C}\}$

$= \mathbf{A}\cdot\{\mathbf{B} \times \mathbf{A} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}\}$

$= \mathbf{A}\cdot(\mathbf{B} \times \mathbf{A}) + \mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) + \mathbf{A}\cdot(\mathbf{C} \times \mathbf{A}) + \mathbf{A}\cdot(\mathbf{C} \times \mathbf{B})$

$= 0 + \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) + 0 + \mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$

$= [\mathbf{A}\ \mathbf{B}\ \mathbf{C}] - [\mathbf{A}\ \mathbf{B}\ \mathbf{C}]$

$= 0$

2. We have,

$$|(\mathbf{B}\times\mathbf{C})| = BC\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Now, $\mathbf{A} = \pm\dfrac{(\mathbf{B}\times\mathbf{C})}{|(\mathbf{B}\times\mathbf{C})|} = \pm 2(\mathbf{B}\times\mathbf{C}).$

3. We have

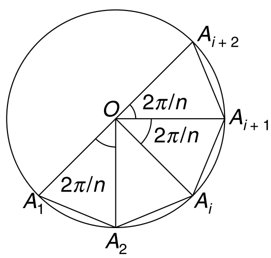
$|\mathbf{a}|\ |\mathbf{b}|\ |\mathbf{c}| = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$

= Volume of **a** parallelopiped having three adjacent sides as **a, b, c**.

= Volume of a rectangular parallelopiped having **a, b, c** as adjacenat sides.

$\Rightarrow\ \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

4. Let $|OA_i| = R$, for every i



Let **p** be a unit vector perpendicular to the plane of the regular polygon.

We have, for $1 \le i \le n-1$

$$\mathbf{OA}_i \times \mathbf{OA}_{i+1} = \left[(OA_i)(OA_{i+1})\sin\left(\frac{2\pi}{n}\right)\right]\mathbf{p}$$

$$= \left[R^2\sin\left(\frac{2\pi}{n}\right)\right]\mathbf{p}$$

Thus,

$$\sum_{i=1}^{n-1}(OA_i \times OA_{i+1}) = (n-1)\left(R^2\sin\left(\frac{2\pi}{n}\right)\right)\mathbf{p}$$

Also, $(\mathbf{OA_2}\times\mathbf{OA_1}) = -\left(R^2\sin\left(\frac{2\pi}{n}\right)\right)\mathbf{p}$

Thus,

$$\sum_{i=1}^{n-1}(\mathbf{OA_i}\times\mathbf{OA_{i+1}}) = (1-n)(\mathbf{OA_2}\times\mathbf{OA_1})$$

5. Given

$\mathbf{x}(\mathbf{i}+\mathbf{j}+3\mathbf{k}) + \mathbf{y}(3\mathbf{i}-3\mathbf{j}+\mathbf{k}) + \mathbf{z}(-4\mathbf{i}+5\mathbf{j})$

$= \lambda(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$\Rightarrow\quad x + 3y - 4z = \lambda x$

$x - 3y + 5z = \lambda y$

and $\quad 3x + y = \lambda z$

$\Rightarrow\quad (1-\lambda)x + 3y - 4z = 0$

$x - (3+\lambda)y + 5z = 0$

and $\quad 3x + y - \lambda z = 0$

Eliminating x, y, z, we get

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(\lambda+3) & 5 \\ 3 & 1 & -5 \end{vmatrix} = 0$$

$\Rightarrow\quad \lambda^3 + 2\lambda^2 + \lambda = 0$

$\Rightarrow\quad \lambda(\lambda^2 + 2\lambda + 1) = 0$

$\Rightarrow\quad \lambda(\lambda+1)^2 = 0$

$\Rightarrow\quad \lambda = -1, 0$

6. Let $\quad \mathbf{OP} = \mathbf{p} = 60\mathbf{i} + 3\mathbf{j},$

$\mathbf{OQ} = \mathbf{q} = 40\mathbf{i} - 8\mathbf{j}$

and $\quad \mathbf{OR} = \mathbf{r} = a\mathbf{i} - 52\mathbf{j}$

Since the points are collinear, so

$$\begin{vmatrix} 60 & 3 & 1 \\ 40 & -8 & 1 \\ a & -52 & 1 \end{vmatrix} = 0$$

$$\Rightarrow\quad \begin{vmatrix} 60 & 3 & 1 \\ -20 & -11 & 0 \\ a-60 & -55 & 0 \end{vmatrix} = 0 \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\Rightarrow\quad \begin{vmatrix} -20 & -11 \\ a-60 & -55 \end{vmatrix} = 0$$

$$\Rightarrow\quad \begin{vmatrix} -20 & 1 \\ a-60 & 5 \end{vmatrix} = 0$$

$\Rightarrow\quad -100 - a + 60$

$\Rightarrow\quad a = -40.$

7. Since, $\quad \mathbf{X}\cdot\mathbf{A} = 0 = \mathbf{X}\cdot\mathbf{B} = \mathbf{X}\cdot\mathbf{C}$

$\Rightarrow\quad$ **A, B, C** are coplanar

Thus, $\quad [\mathbf{A}, \mathbf{B}, \mathbf{C}] = 0$

8. $V = [\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC}]$

$$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= 2(-1 - 0) + 3(-1 + 3)$$
$$= -2 + 6$$
$$= 4.$$

9. Given,

$$\mathbf{A}\cdot\mathbf{B} = 0,\ \mathbf{A}\cdot\mathbf{X} = c,\ \mathbf{A}\times\mathbf{X} = \mathbf{B}$$

Now,

$$\mathbf{A}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\mathbf{B}$$

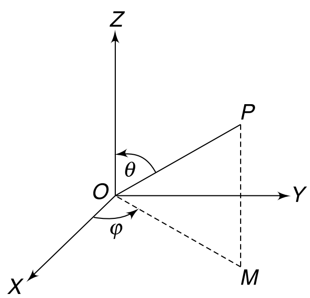
$$\Rightarrow \quad (\mathbf{A}\cdot\mathbf{X})\mathbf{A} - (\mathbf{A}\cdot\mathbf{A})\mathbf{X} = \mathbf{A}\times\mathbf{B}$$

$$\Rightarrow \quad c\mathbf{A} - A^2\mathbf{X} = \mathbf{A}\times\mathbf{B}$$

$$\Rightarrow \quad A^2\mathbf{X} = c\mathbf{A} - \mathbf{A}\times\mathbf{B}$$

$$\Rightarrow \quad \mathbf{X} = \frac{c\mathbf{A} - \mathbf{A}\times\mathbf{B}}{A^2}$$

10. Let **i, j, k** be the unit vectors along the positive directions of x, y and z axes respectively.



Here, $A_1 = r\sin\theta\cos\varphi,$

$A_2 = r\sin\theta\sin\varphi$

and $A_3 = r\cos\theta.$

When the x-axis is rotated through an angle of $\frac{\pi}{2}$, the new components of A are B_1, B_2, B_3 respectively.

Thus, $B_1 = r\sin\theta\cos\left(\varphi + \frac{\pi}{2}\right)$

$= -r\sin\theta\cos = -A_1$

$B_2 = r\sin\theta\sin\left(\varphi + \frac{\pi}{2}\right) = r\sin\theta\cos\varphi$

$= A_1$

and $B_3 = r\cos\theta = A_3.$

11. The given statement is true.
Let the position vectors of the points A, B and C are $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + k\mathbf{b}$ respectively.
Then $\mathbf{AB} = -2\mathbf{b}$
Similarly, $\mathbf{BC} = (k + 1)\mathbf{b}$
Clearly, **AB** is parallel to **BC** for all k in R
Thus, A, B and C are collinear

12. Given $(\mathbf{a}\times\mathbf{b}) = (a\ b)\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}(ab)$
Now,

$$[c\ a\ b] = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$= |\mathbf{c}|\,|\mathbf{a}\times\mathbf{b}|\cos(0)$$
$$= |\mathbf{a}\times\mathbf{b}|$$
$$= \frac{1}{2}(ab)$$

Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [a\ b\ c]^2$$
$$= \frac{1}{4}a^2b^2$$
$$= \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

13. As the length of the vectors remain the same.

$$(2p)^2 + 1 = (p + 1)^2 + 1$$

$$\Rightarrow \quad 4p^2 = (p + 1)^2$$

$$\Rightarrow \quad p + 1 = \pm 2p$$

$$\Rightarrow \quad p = 1, -\frac{1}{3}.$$

14. Given **A, B, C** and **D** are coplanar, so **AB, AC** and **AD** are also coplanar.

Now, $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

$= -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$

Also, $AC = OC - OA$

$= (-\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

$= 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}.$

Finally, $AD = OD - OA$

$= (4\mathbf{i} + 5\mathbf{j} + \lambda\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

$= \mathbf{i} + 7\mathbf{j} + (\lambda + 1)\mathbf{k}$

Given,

$$[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AB}] = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ 2 & 3 & 3 \\ 1 & 7 & (\lambda+1) \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(15 + 9) - 7(-3 + 6) + (1 + 1)(-3 - 10) = 0$$

$$\Rightarrow \quad 24 - 21 - 13(\lambda + 1) = 0$$

$$\Rightarrow \quad 13(\lambda + 1) = 3$$

$$\Rightarrow \quad (\lambda + 1) = \frac{3}{13}$$

$$\Rightarrow \quad \lambda = \frac{3}{13} - 1 = -\frac{10}{13}$$

15. Number of unit vectors $= \pm\dfrac{(\mathbf{a}\times\mathbf{b})}{|(\mathbf{a}\times\mathbf{b})|}$

16. Given $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0 \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$

$\Rightarrow a(b-1)(c-1) - (1-a)(c-1) + (1-a)(1-b) = 0$

$\Rightarrow a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$

Divide both the sides by $(1-a)(1-b)(1-c)$, we get

$$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

$$\Rightarrow 1 + \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

17. Let $\vec{c} = \alpha\hat{i} + \beta\hat{j}$

Also, $\vec{b} \cdot \vec{c} = 0$

$4\alpha + 3\beta = 0$

$\frac{\alpha}{3} = \frac{\beta}{-4} = \lambda$

Let $\vec{a} = x\hat{i} + y\hat{j}$ be the required vectors

Projection of a along $b = \frac{\mathbf{a}\cdot\mathbf{b}}{b} = 1$

$\frac{4x+3y}{\sqrt{16+9}} = 1$

$4x + 3y = 5$ ...(i)

Also, projection of a along c is 2

$\frac{\mathbf{a}\cdot\mathbf{c}}{c} = 2$

$\frac{\alpha x + \beta y}{\sqrt{\alpha^2 + \beta^2}} = 2$

$3x - 4y = 10$ ...(ii)

Solving (i) and (ii), we get,

$x = 2, y = -1$

Hence, the required vector is

$2\hat{i} - \hat{j}$

18. Let the position vectors of the points A, B, C and D be **a**, **b**, **c** and **d** respectively w.r.t the origin O

Now, $\mathbf{AB} = \mathbf{b} - \mathbf{a}$, $\mathbf{AD} = \mathbf{d} - \mathbf{a}$

$\mathbf{BC} = \mathbf{c} - \mathbf{b}$, $\mathbf{BD} = \mathbf{d} - \mathbf{b}$

$\mathbf{CD} = \mathbf{d} - \mathbf{c}$, $\mathbf{CA} = \mathbf{a} - \mathbf{c}$

Now, $|\mathbf{AB} \times \mathbf{CD} + \mathbf{BC} \times \mathbf{AD} + \mathbf{CA} \times \mathbf{BD}|$

$= 2|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$ ...(i)

Also, area of a triangle ΔABC

$= \frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$

From, (i) and (ii), we get,

$|\mathbf{AB} \times \mathbf{CD} + \mathbf{BC} \times \mathbf{AD} + \mathbf{CA} \times \mathbf{BD}|$

$= 4\,\text{ar}(\Delta ABC)$

19. $(\mathbf{a} + \mathbf{b})\cdot\mathbf{p} + (\mathbf{b} + \mathbf{c})\cdot\mathbf{q} + (\mathbf{c} + \mathbf{a})\cdot\mathbf{r}$

$$= \frac{(\mathbf{a}+\mathbf{b})\times(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} + \frac{(\mathbf{b}+\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} + \frac{(\mathbf{c}+\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$$= \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} + \frac{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} + \frac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$$= \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} + \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} + \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$= 3$

20. Component of **a** along **b**

$= (\text{Projection of } \mathbf{a} \text{ on } \mathbf{b})\hat{b}$

$= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\hat{b}$

$= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|}$

$= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

Component of **a** perpendicular to **b**

$= \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

$= \frac{|\mathbf{b}|^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$

$= \frac{\mathbf{b} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{b}|^2}.$

21. Do Yourself

By the help of internal section formula

22. $(\mathbf{a} - \mathbf{b})\cdot\{(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})\}$

$= (\mathbf{a} - \mathbf{b})\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} - \mathbf{c}\times\mathbf{c} + \mathbf{c}\times\mathbf{a})$

$= (\mathbf{a} - \mathbf{b})\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a})$

$= (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} - \mathbf{a}\cdot\mathbf{b}\times\mathbf{a} + \mathbf{a}\cdot\mathbf{c}\times\mathbf{a})$

$- (\mathbf{b}\cdot\mathbf{b}\times\mathbf{c} - \mathbf{b}\cdot\mathbf{b}\times\mathbf{a} + \mathbf{b}\cdot\mathbf{c}\times\mathbf{a})$

$= (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c}\times\mathbf{a})$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 0$

23. Given $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0.$

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $c = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

Then

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} \mathbf{a} & \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} \\ \mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \\ \mathbf{c} & \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= 0$$

($\because$ **a**, **b**, **c** are coplanar vectors.)

24. Do yourself

By the help of internal section formula and the vector equation of a line.

25. Given,

$\mathbf{R} \times \mathbf{B} = \mathbf{C} \times \mathbf{B}$ and $\mathbf{R}\cdot\mathbf{A} = 0.$

Now, $(\mathbf{R} - \mathbf{B}) \times \mathbf{C} = 0$

$\Rightarrow \quad \mathbf{R} = \mathbf{B} + \lambda\mathbf{C}$

Also,

$\mathbf{R}\cdot\mathbf{A} = 0$

$\Rightarrow \quad (\mathbf{B} + \lambda\mathbf{C})\times\mathbf{A} = 0$

$\Rightarrow \quad \mathbf{B}\cdot\mathbf{A} + \lambda(\mathbf{C}\cdot\mathbf{A}) = 0$

$\Rightarrow \quad (2 + 1) + \lambda(8 + 7) = 0$

$\Rightarrow \quad 15\lambda + 3 = 0$

$\Rightarrow \quad \lambda = -\dfrac{1}{5}$

Hence, the vector

$$\mathbf{R} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - \frac{1}{5}(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k})$$

$$= \frac{1}{5}(\mathbf{i} + 8\mathbf{j} - 2\mathbf{k})$$

26. Let $\mathbf{a} = cx\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

and $\mathbf{b} = x\mathbf{i} + 2\mathbf{j} + 2cx\mathbf{k}$

Given,

$\mathbf{a}\cdot\mathbf{b} < 0$

$\Rightarrow \quad cx^2 - 12 - 6cx < 0$

$\Rightarrow \quad cx^2 - 6cx - 12 < 0$

$\Rightarrow \quad c < 0$ and $D < 0$

$\Rightarrow \quad c < 0$ and $36c^2 + 48c < 0$

$\Rightarrow \quad c < 0$ and $3c^2 + 4c < 0$

$\Rightarrow \quad c < 0$ and $c(3c + 4) < 0$

$\Rightarrow \quad c < 0$ and $-\dfrac{4}{3} < c < 0$

Hence, the value of $c = \left(-\dfrac{4}{3}, 0\right)$

27. Given,

$\mathbf{a}\cdot\mathbf{b} = 3$ and $\mathbf{a} \times \mathbf{b} = \mathbf{c}$

Now,

$\mathbf{a}\times(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \mathbf{c}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{c}$

$\Rightarrow \quad 3\mathbf{a} - 3\mathbf{b} = \mathbf{a} \times \mathbf{c}$

$\Rightarrow \quad 3\mathbf{b} = 3\mathbf{a} - \mathbf{a} \times \mathbf{c}$

$= (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (-2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= 3\mathbf{i}$

$\Rightarrow \quad \mathbf{b} = \mathbf{i}$

28. We have,

$$(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \mathbf{i} - \mathbf{j}$$

Therefore,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

Thus, $\hat{r} = \dfrac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|} = \dfrac{1}{\sqrt{11}}(\mathbf{i} + \mathbf{j} - 3\mathbf{k}).$

29. Given $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$\Rightarrow \quad a(0 - c) - a(b - c) + c(c) = 0$

$\Rightarrow \quad c^2 = ab$

Thus, c is the GM of a and b.

30. Let **v** be a vector in the plane of **b** and **c**.

Then

$\mathbf{v} = \mathbf{b} + \lambda c$

$= (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda(i + \mathbf{j} - 2\mathbf{k})$

$= (1 + \lambda)\mathbf{i} + (2 + \lambda)\mathbf{j} - (2\lambda + 1)\mathbf{k}$

Now, the projection of **v** on $\mathbf{a} = \sqrt{\dfrac{2}{3}}.$

$\Rightarrow \quad \dfrac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{a}|} = \sqrt{\dfrac{2}{3}}$

$\Rightarrow \quad \dfrac{2(1+\lambda) - (2+\lambda) - (2\lambda+1)}{\sqrt{6}} = \dfrac{\sqrt{2}}{\sqrt{3}}$

$\Rightarrow \quad -1 - 1 = 2$

$\Rightarrow \quad 1 = -3$

Therefore, $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

31. Do yourself

By the help of internal section formula and basic Geometry.

32. Given $\mathbf{OP} = \mathbf{p}$ and $\mathbf{OQ} = \mathbf{q}$

Thus, $\mathbf{OR} = \dfrac{3\mathbf{p} + 2\mathbf{q}}{5}$ and $\mathbf{OS} = \dfrac{3\mathbf{p} - 2\mathbf{q}}{1}$

Now,

$\mathbf{OR} \perp \mathbf{OS} = 0$

$\Rightarrow \quad \mathbf{OR}\cdot\mathbf{OS}$

$\Rightarrow \quad \left(\dfrac{3\mathbf{p} + 2\mathbf{q}}{5}\right)\cdot\left(\dfrac{2\mathbf{p} - 2\mathbf{q}}{1}\right) = 0$

$\Rightarrow \quad 9\mathbf{p}\cdot\mathbf{p} = 4\mathbf{q}\cdot\mathbf{q}$

$\Rightarrow \quad 9\mathbf{p}^2 = 4\mathbf{q}^2.$

33. Let $\mathbf{OP} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$,

$\mathbf{OQ} = \beta\mathbf{i} + \gamma\mathbf{j} + \alpha\mathbf{k}$

and $\mathbf{OR} = \gamma\mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$

Now, $\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$

$= (\beta - \alpha)\mathbf{i} + (\varphi - \beta)\mathbf{j} + (\alpha - \gamma)\mathbf{k}$

Also, $\mathbf{PR} = \mathbf{OR} - \mathbf{OP}$

$= (\gamma - \alpha)\mathbf{i} + (\alpha - \beta)\mathbf{j} + (\beta - \gamma)\mathbf{k}$

Again, $\mathbf{QR} = \mathbf{OR} - \mathbf{OQ}$

$= (\gamma - \beta)\mathbf{i} + (\alpha - \gamma)\mathbf{j} + (\beta - \alpha)\mathbf{k}$

Thus, $|\mathbf{PQ}| = |\mathbf{QR}| = |\mathbf{PR}|$

$= \sqrt{(\beta - \alpha)^2 + (\gamma - \beta)^2 + (\alpha - \gamma)^2}$

Therefore, $\Delta\, PQR$ is an equilateral triangle.

34. Given,

$\mathbf{d} = \dfrac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Clearly $|\mathbf{d}| = \dfrac{1}{3}\cdot 3 = 1$

Thus, $\mathbf{d}$ is a unit vector.

Also, $\mathbf{d} = -\dfrac{2}{3}\left(-\mathbf{i} + \mathbf{j} - \dfrac{1}{2}\mathbf{k}\right)$

So, $\mathbf{d}$ is parallel to $\left(-\mathbf{i} + \mathbf{j} - \dfrac{1}{2}\mathbf{k}\right)$.

Now, $\mathbf{d}\cdot(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

$= \dfrac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})\cdot(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

$= \dfrac{1}{3}(6 - 4 - 2) = 0$

Thus, $\mathbf{d}$ is parallel to $(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

35. Unit vector $= \pm\dfrac{\mathbf{QP}\times\mathbf{QR}}{|\mathbf{QP}\times\mathbf{QR}|}$

$= \pm\dfrac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k})$

36. We have,

$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$

$= \mathbf{p}\times(\mathbf{c}\times\mathbf{d}),\ \mathbf{p} = (\mathbf{a}\times\mathbf{b})$

$= (\mathbf{p}\cdot\mathbf{d})\mathbf{c} - (\mathbf{p}\cdot\mathbf{c})\mathbf{d}$

$= ((\mathbf{a}\times\mathbf{b})\cdot\mathbf{d})\mathbf{c} - ((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\mathbf{d}$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}.$

Similarly,

$(\mathbf{a}\times\mathbf{c})\times(\mathbf{b}\times\mathbf{d}) = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

and $(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c}) = [\mathbf{a}\ \mathbf{d}\ \mathbf{c}]\mathbf{b} - [\mathbf{a}\ \mathbf{d}\ \mathbf{b}]\mathbf{c}$

Therefore, $\mathbf{x} = (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) + (\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d} + [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d} + [\mathbf{a}\ \mathbf{d}\ \mathbf{c}]\ \mathbf{b} - [\mathbf{a}\ \mathbf{d}\ \mathbf{b}]\mathbf{c}$

$= -2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

$= -2[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a}$

Thus, $\mathbf{x}$ is parallel $\mathbf{a}$.

37. Given $[\mathbf{b}\ \mathbf{c}\ \mathbf{d}] = 0$

$\Rightarrow$ $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are coplanar vectors

Thus, $\mathbf{d} = \mathbf{b} + \lambda\mathbf{c}$

$= (\mathbf{j} - \mathbf{k}) + \lambda(-\mathbf{i} + \mathbf{k})$

$= (-\lambda)\mathbf{i} + \mathbf{j} + (\lambda - 1)\mathbf{k}$

Also,

$\mathbf{a}\cdot\mathbf{d} = 0$

$\Rightarrow \quad -\lambda - 1 = 0$

$\Rightarrow \quad \lambda = -1$

Therefore, $\mathbf{d} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Hence, $\hat{\mathbf{d}} = \pm\dfrac{\mathbf{d}}{|\mathbf{d}|} = \pm\dfrac{(\mathbf{i} + \mathbf{j} - 2\mathbf{k})}{\sqrt{6}}.$

38. Given $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \dfrac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c})$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \frac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c})$

Comparing the components of **b** and **c**, we get,

$$(\mathbf{a}\cdot\mathbf{b}) = -\frac{1}{\sqrt{2}} \text{ and } (\mathbf{a}\cdot\mathbf{c}) = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad ab\cos\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{3\pi}{4}$

39. Given,

$\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$

$\Rightarrow \quad (\mathbf{u} + \mathbf{v} + \mathbf{w})^2 = 0$

$\Rightarrow \quad u^2 + v^2 + w^2 + 2(\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}) = -50$

$\Rightarrow \quad 2(\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}) = -50$

$\Rightarrow \quad (\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}) = -25$

40. Given,

$(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\{(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})\}$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\{(\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c})\}$

$= (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\{(\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c})\}$

$= \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) + \mathbf{b}\cdot(\mathbf{a} \times \mathbf{c}) + \mathbf{c}\cdot(\mathbf{b} \times \mathbf{a})$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

41. Let N_1 = normal to the plane parallel to **i** and **i** + **j**

$= \mathbf{i} \times (\mathbf{i} + \mathbf{i}) = \mathbf{k}$

and N_2 = normal to the plane parallel to **i** – **j**, **i** + **k**

$= (\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + \mathbf{k})$

$= -\mathbf{i} - \mathbf{j} + \mathbf{k}.$

Note that **a** is parallel to $\mathbf{N}_1 \times \mathbf{N}_2$

So, we can consider $\mathbf{a} = \mathbf{N}_1 \times \mathbf{N}_2$

$= \mathbf{k} \times (-\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= (\mathbf{i} - \mathbf{j})$

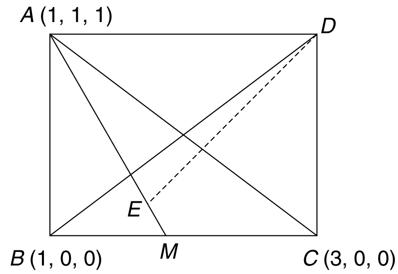
Let θ the angle between **a** and **i** – 2**j** + 2**k**.

Thus, $\quad \cos\theta = 1 + \frac{2}{\sqrt{2}.3} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

42. If **b** is perpendicular to **c**, the given expression is equal to **a**. Otherwise can not be calculated.

43. Let $DE = p$.



Volumne of the tetrahedron

$$ABCD = \frac{2\sqrt{2}}{3}$$

$\Rightarrow \quad \frac{1}{6}|\mathbf{BC} \times \mathbf{BA}|\, p = \frac{2\sqrt{2}}{3}$

$\Rightarrow \quad |\mathbf{BC} \times \mathbf{BA}|\, p = 4\sqrt{2}$

$\Rightarrow \quad 2\sqrt{2}p = 4\sqrt{2}$

$\Rightarrow \quad p = 2$

Here, $\Delta\, ADE$ is a right-angled triangle.

Thus, $\quad AE^2 = AD^2 - DE^2 = 16 - 4 = 12$

$\Rightarrow \quad AE = 2\sqrt{3}$

Now, $\quad \frac{\mathbf{OM} = \mathbf{OB} + \mathbf{OC}}{2} = 2\mathbf{i}$

Thus, $\quad \mathbf{AM} = \mathbf{OM} - \mathbf{OA}$

$= 2\mathbf{i} - \mathbf{i} - \mathbf{j} - \mathbf{k} = (\mathbf{i} - \mathbf{j} - \mathbf{k})$

$\Rightarrow \quad AM = \sqrt{3}$

If E lies on AM produced, then M is the mid-point of AE. Then

$$\frac{1}{2}(\mathbf{e} + (\mathbf{i} + \mathbf{j} + \mathbf{k})) = 2\mathbf{i}$$

$\Rightarrow \quad e = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$

If E lies on MA produced, then A divides the join of E and M in the ratio 2:1. Thus,

$$\frac{2(2\mathbf{i}) + e}{2 + 1} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$\Rightarrow \quad \mathbf{e} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

44. Let $\quad |\mathbf{p}| = |\mathbf{q}| = |\mathbf{r}| = \lambda$

Also, $\quad \mathbf{p}\cdot\mathbf{q} = 0 = \mathbf{q}\cdot\mathbf{r} = \mathbf{r}\cdot\mathbf{p}$

Let $\quad \mathbf{x} = \alpha\,\mathbf{p} + \beta\,\mathbf{q} + \gamma\,\mathbf{r},$

(where $\alpha, \beta, \gamma, \in R$)

Now,

$\mathbf{p} \times ((\mathbf{x} - \mathbf{q}) \times \mathbf{p})$

$= (\mathbf{p}\cdot\mathbf{p})(\mathbf{x} - \mathbf{q}) - [\mathbf{p}\cdot(\mathbf{x} - \mathbf{q})]\,\mathbf{p}$

$= (\mathbf{p}\cdot\mathbf{p})(\mathbf{x} - \mathbf{q}) - (\mathbf{p}\cdot\mathbf{x} - \mathbf{p}\cdot\mathbf{q})\,\mathbf{p}$

$= (\mathbf{p}\cdot\mathbf{p})(\mathbf{x}-\mathbf{q}) - (\mathbf{p}\cdot\mathbf{x})\mathbf{p}$

$= p^2(\mathbf{x}-\mathbf{q}) - (\alpha p^2)\mathbf{p}$

$= \lambda^2(\mathbf{x}-\mathbf{q}) - \lambda^2\,\alpha\mathbf{p}$

$= \lambda^2[(\mathbf{x}-\mathbf{q}) - \alpha\mathbf{p}]$ ...(i)

Similarly,

$\mathbf{q}\times[(\mathbf{x}-\mathbf{r})\times\mathbf{q}] = \lambda^2[(\mathbf{x}-\mathbf{r})-\beta\mathbf{q}]$ ...(ii)

$\mathbf{r}\times[(\mathbf{x}-\mathbf{p})\times\mathbf{r}] = \lambda^2[(\mathbf{x}-\mathbf{p})-\gamma\mathbf{r}]$ ...(iii)

Adding Eqs. (i), (ii) and (iii), we get

$\mathbf{0} = \lambda^2(3\mathbf{x} - (\mathbf{p}+\mathbf{q}+\mathbf{r}) - x)$

$\Rightarrow \quad \mathbf{x} = \frac{1}{2}(\mathbf{p}+\mathbf{q}+\mathbf{r}).$

45. Given,

$p = \text{ar}(\text{quad } OABC)$

$= \text{ar}(\Delta OAB) + \text{ar}(\Delta OCB)$

$= \frac{1}{2}|\mathbf{a}\times(10\mathbf{a}+2\mathbf{b})| + \frac{1}{2}|\mathbf{b}\times(10\mathbf{a}+2\mathbf{b})|$

$= \frac{1}{2}|2\mathbf{a}\times\mathbf{b}| + \frac{1}{2}|10\mathbf{b}\times\mathbf{a}|$

$= |\mathbf{a}\times\mathbf{b}| + 5|\mathbf{b}\times\mathbf{a}|$

$= 6|\mathbf{a}\times\mathbf{b}|$

Also, $q = \text{ar}(\text{parallelogram } OABC)$

$= |\mathbf{a}\times\mathbf{b}|$

Thus, $p = 6q$

$\Rightarrow \quad k = 6.$

46. Given,

$\mathbf{a}\times(\mathbf{a}\times\mathbf{c}) + \mathbf{b} = 0$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{a}\times\mathbf{c}) = -\mathbf{b}$

$\Rightarrow \quad |\mathbf{a}\times(\mathbf{a}\times\mathbf{c})| = |-\mathbf{b}|$

$\Rightarrow \quad |\mathbf{a}|\,|(\mathbf{a}\times\mathbf{c})|\sin\left(\frac{\pi}{2}\right) = 1$

$\Rightarrow \quad |\mathbf{a}|\,|(\mathbf{a}\times\mathbf{c})| = 1$

$\Rightarrow \quad |(\mathbf{a}\times\mathbf{c})| = 1$

$\Rightarrow \quad ac\sin\theta = 1$

$\Rightarrow \quad \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$

47. We have,

$[(\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})\ (\mathbf{B}\times\mathbf{C})\ (\mathbf{B}+\mathbf{C})]$

$= (\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})\cdot[(\mathbf{B}\times\mathbf{C})\times(\mathbf{B}+\mathbf{C})]$

$= (\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})\cdot[(\mathbf{B}\times\mathbf{C})+(\mathbf{C}\times\mathbf{B})]$

$= (\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})\cdot[(\mathbf{B}\times\mathbf{C})-(\mathbf{B}\times\mathbf{C})]$

$= (\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})\cdot 0$

$= 0.$

48. Given,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{vmatrix}, \qquad (\text{where } \lambda = \cos\theta)$$

$= (1-\lambda^2) - \lambda(\lambda-\lambda^2) + \lambda(\lambda^2-\lambda)$

$= 1 - \lambda^2 - \lambda^2 + \lambda^3 + \lambda^3 - \lambda^2$

$= 2\lambda^3 - 3\lambda^2 + 1.$

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = 2\lambda^3 - 3\lambda^2 + 1$

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \sqrt{2\lambda^3 - 3\lambda^2 + 1}$

$= m$, (say)

Here, $m = (1-\cos\theta)\sqrt{1+2\cos\theta}$

Taking dot product of

$\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} = p\mathbf{a} + p\mathbf{b} + r\mathbf{c}$ with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we get

$p + q\cos\theta + r\cos\theta = m$ ...(i)

$p\cos\theta + q + r\cos\theta = 0$ ...(ii)

$p\cos\theta + q\cos\theta + r = m$ ...(iii)

Adding, we get

$(p+q+r)(1+2\cos\theta)$

$\Rightarrow \quad (p+q+r) = \frac{2m}{(1+2\cos\theta)}$ ...(iv)

Multiplying Eq. (iv) by $\cos\theta$ and subtracting from Eq. (i), we get,

$m - \frac{2m\cos\theta}{1+2\cos\theta} = p(1-\cos\theta)$

$\Rightarrow \quad p = \frac{m}{(1+2\cos\theta)(1-\cos\theta)} = \frac{1}{\sqrt{1+2\cos\theta}}$

Similarly,

$r = \frac{1}{\sqrt{1+2\cos\theta}}$ and $q = \frac{-2\cos\theta}{\sqrt{1+2\cos\theta}}$

49. Given that

$\vec{a} = (\hat{i}+\hat{j}+\hat{k})$, $\vec{b} = (4\hat{i}+3\hat{j}+4\hat{k})$

and $\vec{c} = (\hat{i}+\alpha\hat{j}+\beta\hat{k})$ are linearly independent vectors $\vec{c} = l\vec{a} + m\vec{b}$ for some scalers l and m not all zeroes.

$\hat{i}+\alpha\hat{j}+\beta\hat{k} = (l+4m)\hat{i} + (l+3m)\hat{j} + (l+4m)\hat{k}$

$l + 4m = 1$

$l + 3m = \alpha$

Thus, $l + 4m = \beta$

On solving, we get, $\beta = 1$

Also, given that, $|c| = \sqrt{3}$

$$1 + \alpha^2 + \beta^2 = 3$$
$$\alpha = \pm 1$$

50. Ans. (c)
51. Ans. (a)
52. We have,

(i) $(u \cdot v)^2 + |(u \times v)|^2$

$$= u^2 v^2 \cos^2\theta + u^2 v^2 \sin^2\theta$$
$$= u^2 v^2 (\cos^2\theta + \sin^2\theta)$$
$$= u^2 v^2$$

(ii) $|\mathbf{u} + \mathbf{v} + (\mathbf{u} \times \mathbf{v})|^2 + (1 - (\mathbf{u}\cdot\mathbf{v}))^2$

$$= (\mathbf{u} + \mathbf{v})^2 + |(\mathbf{u} \times \mathbf{v})|^2 + 2(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} \times \mathbf{v}) + (1 - (\mathbf{u}\cdot\mathbf{v}))^2$$
$$= (\mathbf{u} + \mathbf{v})^2 + |(\mathbf{u}\cdot\mathbf{v})^2| + (\mathbf{1} - (\mathbf{u}\cdot\mathbf{v}))^2$$
$$= u^2 + v^2 + 2\mathbf{u}\cdot\mathbf{v} + u^2 v^2 \sin^2\theta + 1 - 2\mathbf{u}\cdot\mathbf{v} + u^2 v^2 \cos^2\theta$$
$$= u^2 + v^2 + u^2 v^2 + 1$$
$$= (1 + u^2)(1 + v^2)$$
$$= (1 + |\mathbf{u}|^2)(1 + |\mathbf{v}|^2).$$

53. We have $|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = ab\sin\theta = \sin\theta$

Also,

$$|\mathbf{u}|^2 = a^2 - 2(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{b})^2 b^2$$
$$= 1 - 2(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\cdot\mathbf{b})^2$$
$$= 1 - (\mathbf{a}\cdot\mathbf{b})^2 = 1 - \cos^2\theta = \sin^2\theta$$

$\Rightarrow \quad |\mathbf{u}| = \sin\theta = |\mathbf{v}|$

Again,

$$\mathbf{u}\cdot\mathbf{b} = (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{b})$$
$$= (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})b^2$$
$$= (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b}) = 0$$

Thus, $\quad |\mathbf{v}| = |\mathbf{u}| + |\mathbf{u}\cdot\mathbf{b}|$

54. Given,

$$|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$$

$\Rightarrow \quad |\mathbf{c} - \mathbf{a}|^2 = (2\sqrt{2})^2 = 8$

$\Rightarrow \quad \mathbf{c}^2 + \mathbf{a}^2 - 2\mathbf{c}\cdot\mathbf{a} = 8$

$\Rightarrow \quad c^2 + 9 - 2c = 8$

$\Rightarrow \quad c^2 - 2c + 1 = 8$

$\Rightarrow \quad (c - 1)^2 = 0$

$\Rightarrow \quad c = 1$

We have,

$$|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}|$$
$$= |\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}|\sin(30°)$$
$$= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$
$$= \frac{3}{2}, \text{ where } (\mathbf{a} \times \mathbf{b}) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

55. Given $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$.

Thus, $\quad \mathbf{c} = \mathbf{a} + \lambda\mathbf{b}$

$$= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$
$$= (2 + \lambda)\mathbf{i} + (1 + 2\lambda)\mathbf{j} + (1 - \lambda)\mathbf{k}$$

Also, $\quad \mathbf{c}$ is perpendicular to $\mathbf{a}$

$\Rightarrow \quad \mathbf{c}\cdot\mathbf{a} = 0$

$\Rightarrow \quad 2(2 + \lambda) + (1 + 2\lambda) + (1 - \lambda) = 0$

$\Rightarrow \quad 4 + 2\lambda + 1 + 2\lambda + 1 - \lambda = 0$

$\Rightarrow \quad 3\lambda + 6 = 0$

$\Rightarrow \quad \lambda + 2 = 0$

$\Rightarrow \quad \lambda = -2$

So, $\quad \mathbf{c} = -3\mathbf{j} + 3\mathbf{k}$

Thus, $\quad \hat{c} = \frac{1}{\sqrt{2}}(-\mathbf{j} + \mathbf{k})$

56. Let u and v are not parallel.

Let w be a linear combination of u, v and $\mathbf{u} \times \mathbf{v}$

Then $\quad \mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u} \times \mathbf{v})$.

$$\mathbf{w} \times \mathbf{u} = \beta(\mathbf{v} \times \mathbf{u}) + \gamma((\mathbf{u} \times \mathbf{v}) \times \mathbf{u})$$
$$= \beta(\mathbf{v} \times \mathbf{u}) + \gamma((\mathbf{u}\cdot\mathbf{u})\mathbf{v} - (\mathbf{v}\cdot\mathbf{u})\mathbf{u})$$

Now, $\quad \mathbf{w} + \mathbf{w} \times \mathbf{u} = \mathbf{v}$

$\Rightarrow \quad \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u} \times \mathbf{v}) + \gamma\mathbf{v} - \gamma(\mathbf{v}\cdot\mathbf{u})\mathbf{u} + \beta(\mathbf{v} \times \mathbf{u}) = \mathbf{v}$

$\Rightarrow (\alpha - \gamma(\mathbf{v}\cdot\mathbf{u}))\mathbf{u} + (\beta + \gamma)\mathbf{v} + (\gamma - \beta)(\mathbf{u} \times \mathbf{v}) = \mathbf{v}$

$\Rightarrow (\alpha - \gamma(\mathbf{v}\cdot\mathbf{u})) = 0,\ (\beta + \gamma) = 1,\ (\gamma - \beta) = 0$

$\Rightarrow \beta = \gamma = \frac{1}{2},\ \alpha = \frac{1}{2}(\mathbf{v}.\mathbf{u})$

Also, $\quad (\mathbf{u} \times \mathbf{v})\cdot\mathbf{w} = \alpha(0) + \beta(0) + \gamma(\upsilon + \mathbf{v})$

$$= \gamma(\upsilon \times \mathbf{v})$$

$\Rightarrow \quad |(\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}|^2 = \gamma^2|(\upsilon \times \mathbf{v})|^2$

$$= \gamma^2(u^2 v^2 - (\mathbf{u}.\mathbf{v})^2)$$
$$= \frac{1}{4}(1 - (\mathbf{u}\cdot\mathbf{v})^2)$$
$$\le \frac{1}{4}$$

$\Rightarrow \quad |(\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}| \le \frac{1}{2}$

and the equality holds iff $\mathbf{u}\cdot\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other.

57. Given $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a} + \mathbf{b} = -\mathbf{c}$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{a} + \mathbf{b}) = -\mathbf{a} \times \mathbf{c}$

$\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times \mathbf{a}$ ...(i)

Also, $\mathbf{a} + \mathbf{c} = -\mathbf{b}$

$\Rightarrow \quad \mathbf{b} \times (\mathbf{a} + \mathbf{c}) = -\mathbf{b} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{b} \times \mathbf{c} = -\mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

58. Let $\mathbf{N}_1$ = normal vector to plane $P_1 = \mathbf{a} \times \mathbf{b}$

and $\mathbf{N}_2$ = normal vector to plane $p_2 = \mathbf{c} \times \mathbf{d}$

Now,

$$\mathbf{N}_1 \times \mathbf{N}_2 = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$$

Thus, $\mathbf{N}_1$ and $\mathbf{N}_2$ are parallel.

Therefore, the angle between P_1 and P_2 is 0.

59. Given $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

Now, $[2\mathbf{a} - \mathbf{b}\ \ 2\mathbf{b} - \mathbf{c}\ \ 2\mathbf{c} - \mathbf{a}]$

$= (2\mathbf{a} - \mathbf{b}) \cdot \{(2\mathbf{b} - \mathbf{c}) \times (2\mathbf{c} - \mathbf{a})\}$

$= (2\mathbf{a} - \mathbf{b}) \cdot \{4(\mathbf{b} \times \mathbf{c}) - 2(\mathbf{b} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a})\}$

$= 8(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) - (\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}))$

$= 8[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 7[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 0.$

60. We have,

$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2$

$= 2(a^2 + b^2 + c^2) - 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$

$= 6 - 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$ ...(i)

Also,

$(a^2 + b^2 + c^2) + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c})^2 \geq 0$

$\Rightarrow \quad 3 + \mathbf{2}(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) \geq 0$

$\Rightarrow \quad \mathbf{2}(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) \geq -3$

$\Rightarrow \quad -\mathbf{2}(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) \geq 3$ ...(ii)

From Eqs (i) and (ii), we get

$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 \leq 6 + 3 = 9$

61. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & (1-x) \\ y & x & 1+x-y \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix}$ $(C_3 \to C_3 + C_1)$

$= 1 + x - x$

$= 1$

= neither depends on x nor y.

62. Thre vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ involve 9 unknown quantities. Since, we are given six equations involving 9 unknown quantities, it is not possible to have uniquely 3 vectors-satisfying these conditions.

As $\mathbf{v}_1 \cdot \mathbf{v}_2 = 4$, let us consider $\mathbf{v}_1 = 2\mathbf{k}$.

Also, let $\mathbf{v}_2 = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and $\mathbf{v}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Now, $\mathbf{v}_1 \cdot \mathbf{v}_2 = -2 \Rightarrow 2c = -2 \Rightarrow c = -1$

Now, $\mathbf{v}_1 . \mathbf{v}_3 = 6 \Rightarrow 2z = -6 \Rightarrow z = -3$

Now, $\mathbf{v}_2 \cdot \mathbf{v}_2 = 2,\ \mathbf{v}_2 \cdot \mathbf{v}_3 = 5,\ \mathbf{v}_3 \cdot \mathbf{v}_3 = 29$

$\therefore \quad a^2 + b^2 + c^2 = 2 \Rightarrow a^2 + b^2 = 2 - 1 = 1$

$x^2 + y^2 + z^2 = 29 \Rightarrow x^2 + y^2 = 20$

and $ax + by + cz = 5$

$\Rightarrow \quad ax + by = 5 - 3 = 2.$

Put $b = 0$, we get,

$a^2 = 1 \Rightarrow a = \pm 1$

and $ax = 2 \Rightarrow x = \pm 2$

Now, $x^2 + y^2 = 20 \Rightarrow y^2 = 20 - 4 = 16$

$\Rightarrow \quad y = \pm 4.$

Thus, one possible set of vectors is

$\mathbf{v}_1 = 2\mathbf{k},\ \mathbf{v}_2 = \mathbf{i} + 2\mathbf{k},\ \mathbf{v}_3 = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$

63. $\mathbf{A}(t)$ is parallel to $\mathbf{B}(t)$ for some t in [0, 1]

$\dfrac{f_1(t)}{g_1(t)} = \dfrac{f_2(t)}{g_2(t)}$ for some t in [0, 1]

Let $h(t) = f_1(t) g_2(t) - f_2(t) g_1(t)$

$h(0) = f_1(0) g_2(0) - f_2(0) g_1(0)$

$= 2.2 - 3.3 = -5 < 0$

$h(1) = f_1(1) g_2(1) - f_2(1) g_1(1)$

$= 6.6 - 2.2 = 32 > 0$

since h is a continuous function and $h(0)h(1) < 0$

Then there is some t in [0, 1] for which $h(1) = 0$

Thus, $\mathbf{A}(t)$ is parallel to $\mathbf{B}(t)$ for this t.

64. We have,

$V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$

We assume that

$(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) \geq (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$

As we know that, AM ≥ GM

$\Rightarrow \quad \frac{1}{3}(a_1 + b_2 + c_3) \geq (a_1\ b_2\ c_3)^{1/3}$

$\Rightarrow \quad (a_1\ b_2\ c_3) \geq \frac{1}{27}(a_1 + b_2 + c_3)^3$

Similarly, $\quad (a_2\ b_3\ c_1) \geq \frac{1}{27}(a_2 + b_3 + c_1)^3$

and $\quad (a_3\ b_1\ c_2) \geq \frac{1}{27}(a_3 + b_1 + c_2)^3$

Thus,

$(a_1\ b_2\ c_3) + (a_2\ b_3\ c_1) + (a_3\ b_1\ c_2)$

$\leq \frac{1}{27}[(a_1 + b_2 + c_3)^3 + (a_2 + b_3 + c_1)^3 + (a_3 + b_1 + c_2)^3]$

$\leq \frac{1}{27}[(a_1 + b_2 + c_3) + (a_2 + b_3 + c_1) + (a_3 + b_1 + c_2)]^3$

Since, $\quad x^3 + y^3 + z^3 \leq (x + y + z)^3$ for $x, y, z \geq 0$

$$= \frac{1}{27}\left(\sum_{r=1}^{3}(a_r + b_r + c_r)\right)^3$$

$$= \frac{1}{27}(3L)^3$$

$$= L^3$$

Thus, $\quad V \leq L^3$.

65. Given,

$$(\mathbf{a} + 2\mathbf{b})\cdot(5\mathbf{a} - 4\mathbf{b}) = 0$$

$\Rightarrow \quad 5a^2 - 4(\mathbf{a}\cdot\mathbf{b}) + 10(\mathbf{a}\cdot\mathbf{b}) - 8b^2 = 0$

$\Rightarrow \quad 5 + 6(\mathbf{a}\cdot\mathbf{b}) - 8 = 0$

$\Rightarrow \quad 6(\mathbf{a}\cdot\mathbf{b}) = 3$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b}) = \frac{1}{2}$

$\Rightarrow \quad ab\cos\theta = \frac{1}{2}$

$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad \theta = \frac{\pi}{2}$.

66. We have,

$$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$$

$$= \mathbf{u}\cdot\mathbf{r}$$

$$= ur\cos\theta$$

$$= r\cos\theta, \text{ where } \mathbf{r} = \mathbf{u}\times\mathbf{v}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 7\mathbf{j} - \mathbf{k}$$

$$r = |\mathbf{r}| = \sqrt{9 + 49 + 1} = \sqrt{59}$$

Thus, the maximum value of

$$[u\ v\ w] = \sqrt{59}.$$

67. Given $\quad V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$

$$= 1 + a(a^2 - 1)$$

$$= a^3 - a + 1$$

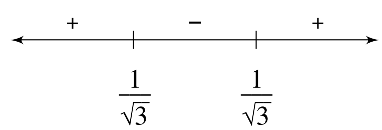
$\Rightarrow \quad \frac{dV}{da} = 3a^2 - 1$

For maximum or minimum, $\frac{dV}{da} = 0$

$\Rightarrow \quad 3a^2 - 1 = 0$

$\Rightarrow \quad a^2 = \frac{1}{3}$

$\Rightarrow \quad a = \pm\frac{1}{\sqrt{3}}$



Hence, the value of $a = \frac{1}{\sqrt{3}}$.

68. Given $\mathbf{x} = \mathbf{u} + \mathbf{v}$, $\mathbf{y} = \mathbf{v} + \mathbf{w}$, $\mathbf{z} = \mathbf{w} + \mathbf{u}$

Thus, $\hat{\mathbf{x}} = \frac{\mathbf{u} + \mathbf{v}}{|\mathbf{u} + \mathbf{v}|}\ \hat{\mathbf{y}} = \frac{\mathbf{v} + \mathbf{w}}{|\mathbf{v} + \mathbf{w}|}, \hat{\mathbf{z}} = \frac{\mathbf{w} + \mathbf{u}}{|\mathbf{w} + \mathbf{u}|}$.

We have,

$$(\mathbf{u} + \mathbf{v})^2 = u^2 + v^2 + 2\mathbf{u}\cdot\mathbf{v}$$

$$= 1 + 1 + 2\cos\alpha$$

$$= 1 + 1 + 2\cos\alpha$$

$$= 2(1 + \cos\alpha)$$

$$= 2\cdot 2\cos^2\left(\frac{\alpha}{2}\right)$$

$\Rightarrow \quad |(\mathbf{u} + \mathbf{v})| = 2\cos\left(\frac{\alpha}{2}\right)$

Similarly, $|(\mathbf{v} + \mathbf{w})| = 2\cos\frac{\alpha}{2}$

and $\quad |(\mathbf{w} + \mathbf{u})| = 2\cos \_\_$

Thus, $\mathbf{x} = 1/2(\mathbf{u} + \mathbf{v})\sec\_\_$,

$\mathbf{y} = 1/2(\mathbf{v} + \mathbf{w})\sec\_\_$

and $\mathbf{z} = \_\_(\mathbf{w} + \mathbf{u})\sec \gamma/2$

Now, $[\mathbf{x} \times \mathbf{y}\ \mathbf{y} \times \mathbf{z}\ \mathbf{z} \times \mathbf{x}] = [\mathbf{x}\ \mathbf{y}\ \mathbf{z}]^2$

$$= \left[\frac{1}{8}[\mathbf{u} + \mathbf{v}\ \mathbf{v} + \mathbf{w}\ \mathbf{w} + \mathbf{u}]\sec\left(\frac{\alpha}{2}\right)\sec\left(\frac{\beta}{2}\right)\sec\left(\frac{\gamma}{2}\right)\right]^2$$

$$= \left[\frac{1}{8}\cdot 2[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]\sec\left(\frac{\alpha}{2}\right)\sec\left(\frac{\beta}{2}\right)\sec\left(\frac{\gamma}{2}\right)\right]^2$$

$$= \left[\frac{1}{4}[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]\sec\left(\frac{\alpha}{2}\right)\sec\left(\frac{\beta}{2}\right)\sec\left(\frac{\gamma}{2}\right)\right]^2$$

$$= \frac{1}{16} \times \left[\sec^2\left(\frac{\alpha}{2}\right)\sec^2\left(\frac{\beta}{2}\right)\sec^2\left(\frac{\gamma}{2}\right)\right][\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

69. Given,

$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$

$\Rightarrow$ $\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{c} \times \mathbf{d} - \mathbf{b} \times \mathbf{d}$

$\Rightarrow$ $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = (\mathbf{c} - \mathbf{b}) \times \mathbf{d} = \mathbf{d} \times (\mathbf{b} - \mathbf{c})$

$\Rightarrow$ $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) - \mathbf{d} \times (\mathbf{b} - \mathbf{c}) = 0$

$\Rightarrow$ $(\mathbf{a} - \mathbf{d}) \times (\mathbf{b} - \mathbf{c}) = 0$

$\therefore$ $(\mathbf{a} - \mathbf{d})$ is parallel to $(\mathbf{b} - \mathbf{c})$

$\Rightarrow$ $(\mathbf{a} - \mathbf{d})\cdot(\mathbf{b} - \mathbf{c}) \neq 0$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} \neq \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$.

Hence, the result.

70. We have,

$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b}$

$\Rightarrow$ $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} - \mathbf{a}^2\mathbf{b}$

$= \mathbf{a} - 3\mathbf{b}$

$\Rightarrow$ $3\mathbf{b} = \mathbf{a} - \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$

$\Rightarrow$ $3\mathbf{b} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (-2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\Rightarrow$ $3\mathbf{b} = 3\mathbf{i}$

$\Rightarrow$ $\mathbf{b} = \mathbf{i}$

71. To find $\hat{r} = \dfrac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|}$

We have,

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

Now,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & 6 \\ 2 & -1 & -3 \end{vmatrix}$$

$= 27\mathbf{j} - 9\mathbf{k}$

$= 9(3\mathbf{j} - \mathbf{k})$

Thus, $\hat{r} = \dfrac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|} = \dfrac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k})$

72. Note that **a** is directed along the internal bisector of the triangle formed by the vectors - **v** and **w**

Thus, $k\mathbf{a} = -\mathbf{v} + \mathbf{w}$

$\Rightarrow$ $k^2a^2 = v^2 + w^2 - 2(\mathbf{v}\cdot\mathbf{w})$

$\Rightarrow$ $k^2 = 1 + 1 - 2vw\cos(\pi - 2\theta)$

$\Rightarrow$ $k^2 = 2 + 2\cos 2\theta$

$\Rightarrow$ $k^2 = 2(1 + \cos 2\theta) = 2.2\cos^2\theta$

$\Rightarrow$ $k = 2\cos\theta$

$\Rightarrow$ $k = 2(\mathbf{a}\cdot(-\mathbf{v}))$

Thus, $\mathbf{w} = \mathbf{v} + k\mathbf{a} = \mathbf{v} - 2(\mathbf{a}\cdot(-\mathbf{v}))$.

73. Ans. (b)

We have $\mathbf{a}\cdot\mathbf{b}_1 = \mathbf{a}\cdot\mathbf{b} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}(\mathbf{a}\cdot\mathbf{a}) = 0$.

$$\mathbf{a}\cdot\mathbf{c}_2 = \mathbf{a}\cdot\mathbf{c} - \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|^2}(\mathbf{a}\cdot\mathbf{a}) - \frac{\mathbf{b}_1\cdot\mathbf{c}}{|\mathbf{b}_1|^2}(\mathbf{b}_1\cdot\mathbf{a})$$

$= \mathbf{a}\cdot\mathbf{c} - \mathbf{a}\cdot\mathbf{c} = 0$

and $\mathbf{b}_1\cdot\mathbf{c}_2 = \mathbf{b}_1\cdot\mathbf{c} - \dfrac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}(\mathbf{b}_1\cdot\mathbf{a}) - \dfrac{\mathbf{b}_1\cdot\mathbf{c}}{|\mathbf{b}|^2}(\mathbf{b}_1\cdot\mathbf{b}_1)$

$= \mathbf{b}_1\cdot\mathbf{c}_2 - \mathbf{0} - \mathbf{b}_1\cdot\mathbf{c} = 0$

Thus, the required triplet is $(\mathbf{a}\ \mathbf{b}_1\ \mathbf{c}_2)$.

74. Ans.(c)

Let v be a vector.

A vector v in the plane of a and b

So, v, a, b are coplaner vectors.

Thus, $\mathbf{v} = \mathbf{a} + \lambda\mathbf{b}$

$\Rightarrow$ $\mathbf{v} = (1 + \lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (1 + \lambda)\mathbf{k}$

$\Rightarrow$ $\mathbf{v} = (1 + \lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (1 + \lambda)\mathbf{k}$

Also, the projection of v on c is $\dfrac{1}{\sqrt{3}}$

$\Rightarrow$ $\dfrac{\mathbf{v}\cdot\mathbf{c}}{|\mathbf{c}|} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow$ $\dfrac{(1 + \lambda) + (2 - \lambda) - (1 + \lambda)}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow$ $(2 - \lambda) = 1$

$\Rightarrow$ $\lambda = 1$

Therefore, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

75. Ans. (c)

Given $\begin{vmatrix} -\lambda 2 & 1 & 1 \\ 1 & -\lambda 2 & 1 \\ 1 & 1 & -\lambda 2 \end{vmatrix} = 0$

$\Rightarrow \quad -\lambda^2(\lambda^4 - 1) - 1(\lambda^2 - 1) + 1(\lambda^2 + 1) = 0$

$\Rightarrow \quad -\lambda^2(\lambda^4 - 1) + (\lambda^2 + 1) + (\lambda^2 + 1) = 0$

$\Rightarrow \quad -\lambda^6 + 3\lambda^2 + 2 = 0$

$\Rightarrow \quad \lambda^6 - 3\lambda^2 - 2 = 0$

$\Rightarrow \quad a^3 - 3a - 2 = 0,\ a = \lambda^2$

$\Rightarrow \quad a^2(a - 2) + 2a(a - 2) + 1(a - 2) = 0$

$\Rightarrow \quad (a - 2)(a^2 + 2a + 1) = 0$

$\Rightarrow \quad (a - 2)(a + 1)^2 = 0$

$\Rightarrow \quad (a - 2) = 0,\ (a + 1)^2 = 0$

$\Rightarrow \quad a = 2,\ a = -1$

$\Rightarrow \quad \lambda^2 = 2,\ \lambda^2 = -1$

$\Rightarrow \quad \lambda^2 = 2$

$\Rightarrow \quad \lambda = \pm\sqrt{2}$

Thus, the number of real values of λ is 2.

76. Ans. (b)

Given $\quad \mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a} + \mathbf{b} = -\mathbf{c}$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{a} + \mathbf{b}) = -\mathbf{a} \times \mathbf{c}$

$\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times \mathbf{a}$ ...(i)

Also, $\quad \mathbf{a} + \mathbf{c} = -\mathbf{b}$

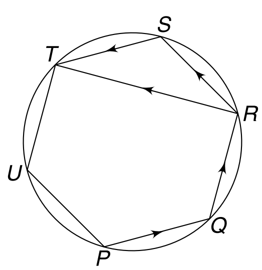
$\Rightarrow \quad \mathbf{b} \times (\mathbf{a} + \mathbf{c}) = -\mathbf{b} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{b} \times \mathbf{c} = -\mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b}$ ...(ii)

From (i) and (ii), we get,

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

77. We have $\mathbf{RS} + \mathbf{ST} = \mathbf{RT}$



As PQ and TR are not parallel to each other.

$$\mathbf{PQ} \times (\mathbf{RS} \times \mathbf{ST}) \neq 0$$

Thus, statement-I is true.

Next, since PQ and RS are not parallel,

so, $\quad \mathbf{PQ} \times \mathbf{RS} \neq 0$

Thus, statement-II is false.

78. Ans. (a)

Given $\quad \mathbf{OP} = \mathbf{a}\cos t + \mathbf{b}\sin t$

$|\mathbf{OP}|^2 = |\mathbf{a}\cos t|^2 = |\mathbf{b}\sin t|^2 + 2(a \cdot b)\sin t \cos t$

$= \cos^2 t + \sin^2 t + (\mathbf{a} \cdot \mathbf{b})\sin 2t$

$= 1 + (\mathbf{a} \cdot \mathbf{b})\sin 2t$

Thus, $\quad |OM| = |OP|_{\max} = \sqrt{1 + (\mathbf{a} \cdot \mathbf{b})},\ t = \dfrac{\pi}{4}$

$\Rightarrow \quad \mathbf{OM} = \dfrac{1}{\sqrt{2}}\mathbf{a} + \dfrac{1}{\sqrt{2}}\mathbf{b} = \dfrac{1}{\sqrt{2}}(\mathbf{a} + \mathbf{b})$

Therefore, $\hat{\mathbf{u}} = \dfrac{\mathbf{OM}}{|\mathbf{OM}|} = \dfrac{\mathbf{a} + \mathbf{b}}{|\mathbf{a} + \mathbf{b}|}$

79. We have,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix}$$

$$= 1\left(1 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \left(\frac{3}{4} - \frac{1}{8} - \frac{1}{8}\right) = \frac{2}{4} = \frac{1}{2}$$

$\Rightarrow \quad [\mathbf{a}, \mathbf{b}, \mathbf{c}] = \dfrac{1}{\sqrt{2}}$

80. Let θ_1 and θ_2 be the angles between **a** and **b**, and **c** and **d** and $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit vectors perpendicular to the plane of **a** and **b**, and **c** and **d.**

Now, $\quad \mathbf{a} \times \mathbf{b} = (ab\sin\theta_1)\mathbf{n}_1$

and $\quad \mathbf{c} \times \mathbf{d} = (cd\sin\theta_2)\mathbf{n}_2$

Let φ the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

Given $\quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$

$\Rightarrow \quad (\sin\theta_1)(\sin\theta_2)(\mathbf{n}_1 \cdot \mathbf{n}_2) = 1$

$\Rightarrow \quad (\sin\theta_1)(\sin\theta_2)(\cos\varphi) = 1$

It is possible only when $\theta_1 = \dfrac{\pi}{2} = \theta_2,\ \varphi = 0$

Since $\varphi = 0$, so $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.

Thus, **a**, **b**, **c** and **d** are coplaner vectors.

Also, $\quad \mathbf{a} \cdot \mathbf{c} = \dfrac{1}{2}$

$\Rightarrow \quad \cos\theta_3 = \dfrac{1}{2} = \cos\left(\dfrac{\pi}{3}\right)$

$\Rightarrow \quad \theta_3 = \left(\dfrac{\pi}{3}\right).$

Since the angle between **a** and **c** is $\dfrac{\pi}{3}$ and **b** and **d** is also $\dfrac{\pi}{3}$. So **b** and **d** are non-parallel.

81. We have,

$$V = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix}$$

$$= \pi(2 - 1)$$

$$= \pi.$$

82. Given,

$$\mathbf{a} + \mathbf{b} = -\sqrt{3}c$$

$$\Rightarrow \quad (\mathbf{a} + \mathbf{b})^2 = (-\sqrt{3}c)^2$$

$$\Rightarrow \quad a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = 3c^2$$

$$\Rightarrow \quad 1 + 1 + 2\mathbf{a}\cdot\mathbf{b} = 3$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \frac{1}{2}$$

$$\Rightarrow \quad ab\cos\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

83. Let M be the mid-point of PR. Then the position vector of M is

$$\frac{1}{2}(-2\mathbf{i} - \mathbf{j} + 3\mathbf{i} + 3\mathbf{j}) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

Let N be the mid-point of QS. Then the position vector of N is

$$= \frac{1}{2}(4\mathbf{i} - 3\mathbf{i} + 2\mathbf{j}) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

Thus, PQRS is a parallelogram.

Also, $PQ = |\mathbf{PQ}| = |4\mathbf{i} - (-2\mathbf{i} - \mathbf{j})| = |6\mathbf{i} + \mathbf{j}|$

$$= \sqrt{36 + 1} = \sqrt{37}$$

And $QR = |\mathbf{PR}| = |3\mathbf{i} + 3\mathbf{j} - 4\mathbf{i}| = |-\mathbf{i} + 3\mathbf{j}|$

$$= \sqrt{1 + 9} = \sqrt{10}$$

Therefore, $PQRS$ is not a rhombus.

Also, $PR = |\mathbf{PR}| = |3\mathbf{i} + 3\mathbf{j} - (-2\mathbf{i} - \mathbf{j})|$

$$= |5\mathbf{i} + 4\mathbf{j}| = \sqrt{41}$$

And $QS = |\mathbf{QS}| = |-3\mathbf{i} + 2\mathbf{j} - (-4\mathbf{i})|$

$$= |\mathbf{i} + 2\mathbf{j}| = \sqrt{1 + 4} = \sqrt{5}$$

So, $PQRS$ is not a rectangle.

Therefore, $PQRS$ is a parallelogram, which is neither a rhombus nor a rectangle.

84. Let θ be the angle between AB and AD.

Thus, $\cos\theta = \dfrac{\mathbf{AB}\cdot\mathbf{AD}}{|\mathbf{AB}||\mathbf{AD}|}$

$$= \frac{-2 + 20 + 22}{\sqrt{4 + 100 + 121}\sqrt{1 + 4 + 4}}$$

$$= \frac{40}{15.3} = \frac{8}{9}$$

Note that $\alpha = \left(\frac{\pi}{2} - \theta\right)$

Thus, $\cos\alpha = \cos\left(\frac{\pi}{2} - \theta\right)$

$$= \sin\theta$$

$$= \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{8}{9}\right)^2}$$

$$= \sqrt{\frac{81 - 64}{81}} = \sqrt{\frac{71}{81}} = \sqrt{\frac{17}{9}}.$$

85. Given,

$$\mathbf{a} = \mu\mathbf{b} + 4\mathbf{c}$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \mu\mathbf{b}\cdot\mathbf{b} + 4\mathbf{b}\cdot\mathbf{c}$$

$$\Rightarrow \quad \mu b^2 + 4\mathbf{b}\cdot\mathbf{c} = 0 \quad \text{...(i)}$$

Also,

$$\mathbf{a}\cdot\mathbf{a} = \mu(\mathbf{a}\cdot\mathbf{b}) + 4(\mathbf{a}\cdot\mathbf{c})$$

$$\Rightarrow \quad |\mathbf{a}|^2 = 4(\mathbf{a}\cdot\mathbf{c}) \quad \text{...(ii)}$$

Given,

$$(\mathbf{b} - \mathbf{a})\cdot(\mathbf{b} + \mathbf{c}) = 0$$

$$\Rightarrow \quad (\mathbf{b}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{c}) = 0$$

$$\Rightarrow \quad |\mathbf{b}|^2 + (\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c}) = 0 \quad \text{...(iii)}$$

Again,

$$2|\mathbf{b} + \mathbf{c}| = |\mathbf{b} - \mathbf{a}|$$

$$\Rightarrow \quad 4|\mathbf{b} + \mathbf{c}|^2 = |\mathbf{b} - \mathbf{a}|^2$$

$$\Rightarrow \quad 4(b^2 + c^2 + 2\mathbf{b}\cdot\mathbf{c}) = (b^2 + a^2 - 2(\mathbf{a}\cdot\mathbf{b}))$$

$$\Rightarrow \quad 4(b^2 + c^2 + 2\mathbf{b}\cdot\mathbf{c}) = (b^2 + a^2)$$

$$\Rightarrow \quad 3b^2 + c^2 + 8(\mathbf{b}\cdot\mathbf{c}) = a^2 \quad \text{...(iv)}$$

Eliminating $(\mathbf{b}\cdot\mathbf{c})$ and a^2 from Eqs (i), (ii), (iii) and (iv), we get

$$(2\mu^2 - 10\mu)|\mathbf{b}|^2 = 0$$

$$\Rightarrow \quad (2\mu^2 - 10\mu) = 0$$

$$\Rightarrow \quad \mu(\mu - 5) = 0$$

$$\Rightarrow \quad \mu = 0, 5.$$

86. Given $|\mathbf{a}| = 1 = |\mathbf{b}|$ and $a.b = 0$

Now, $(2\mathbf{a} + \mathbf{b})\cdot\{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - \mathbf{2b})\}$

$$= (2\mathbf{a} + \mathbf{b})\cdot\{(2\mathbf{b} - \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})\}$$

$$= (2\mathbf{a} + \mathbf{b})\cdot\{(2\mathbf{b} \times (\mathbf{a} \times \mathbf{b}) - \mathbf{a} \times (\mathbf{a} \times \mathbf{b}))\}$$

$$= (2\mathbf{a} + \mathbf{b})\cdot\{(2(\mathbf{b}\cdot\mathbf{b})\mathbf{a} - 2(\mathbf{b}\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a} + (\mathbf{a}\cdot\mathbf{a})\mathbf{b})\}$$

$= (2\mathbf{a} + \mathbf{b})\cdot(2b^2\mathbf{a} + a^2\mathbf{b})$

$= (2\mathbf{a} + \mathbf{b})\cdot(2\mathbf{a} + \mathbf{b})$

$= (4a^2 + b^2 + 4\mathbf{a}\cdot\mathbf{b})$

$= 4 + 1 + 0$

$= 5$

87. Given **v, a,** and **b** are coplaner, so,

$\mathbf{v} = \mathbf{a} + \lambda\mathbf{b}$

$= (\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= (1 + \lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + (1 + \lambda)\mathbf{k}$

Also, the projection of **v** on **c** is $\frac{1}{\sqrt{3}}$.

$\Rightarrow \quad \frac{\mathbf{v}\cdot\mathbf{c}}{|\mathbf{c}|} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \left|\frac{(1+\lambda) - (1-\lambda) - (1+\lambda)}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad (1 - \lambda) = 1$

$\Rightarrow \quad \lambda = 2$

Thus, $\mathbf{v} = 3\mathbf{i} - \mathbf{i} + 3\mathbf{k}$

88. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$,

$\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

and $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Now, $(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

$= -3\mathbf{i} + \mathbf{j} + \mathbf{k}$

Also, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{-3} & \mathbf{1} & \mathbf{1} \end{vmatrix}$

$= -4\mathbf{j} + 4\mathbf{k}$

$= 4(-\mathbf{j} + \mathbf{k})$.

Thus, $\frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|} = (-\mathbf{j} + \mathbf{k})$.

89. We have,

$\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{r} \times \mathbf{b}) = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})\mathbf{r} - (\mathbf{a}\cdot\mathbf{r})\mathbf{b} = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})\mathbf{r} = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$

$\Rightarrow \quad -\mathbf{r} = \mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = 3\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

$\Rightarrow \quad \mathbf{r} = -3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$

Thus, $\mathbf{r}.\mathbf{b} = 3 + 6 + 0 = 9$.

90. We have,

$|\mathbf{a}| = 2 = |\mathbf{b}|$ and $\mathbf{a}\cdot\mathbf{b} = -1 + 3 = 2$

Let θ be the angle between them.

Then $\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$

$= \frac{4 + 4 - 12}{8} = -\frac{1}{2}$

Thus, $\theta = \frac{2\pi}{3}$.

91. Given $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = \mathbf{9}$

$\Rightarrow \quad 2(a^2 + b^2 + c^2) - 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 9$

$\Rightarrow \quad 6 - 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 9$

$\Rightarrow \quad 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = -3$

$\Rightarrow \quad 2(a^2 + b^2 + c^2) - 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 3 - 3$

$\Rightarrow \quad |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = 0$

$\Rightarrow \quad (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$

Now,

$|\mathbf{2a} + \mathbf{5b} + \mathbf{5c}| = |\mathbf{2a} + \mathbf{5}(\mathbf{b} + \mathbf{c})|$

$= |\mathbf{2a} + \mathbf{5}(-\mathbf{a})|$

$= |-3(\mathbf{a})|$

$= |-3|\ |\mathbf{a}|$

$= 3$

92. Given,

$\mathbf{a} \times (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{b}$

$\Rightarrow \quad \mathbf{a} \times (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -\mathbf{b} \times (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$\Rightarrow \quad (\mathbf{a} + \mathbf{b}) \times (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = 0$

$\Rightarrow \quad (\mathbf{a} + \mathbf{b}) = \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$\Rightarrow \quad |(\mathbf{a} + \mathbf{b})| = |\lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})|$

$\Rightarrow \quad \sqrt{29}\,\lambda^2 = \sqrt{29}$

$\Rightarrow \quad |\lambda| = 1$

$\Rightarrow \quad \lambda = \pm 1$

Thus, $(\mathbf{a} + \mathbf{b}) = \pm(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

Now, $(\mathbf{a} + \mathbf{b})\cdot(-7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$= \pm(-14 + 6 + 12)$

$= \pm 4$

93. When diagonal are given, then area of a parallelogram

$PQRS = \frac{1}{2}|\mathbf{PQ} \times \mathbf{SQ}|$

Volume of the parallelopiped

$= ((\mathbf{PQ} \times \mathbf{PS})) \cdot \mathbf{PT})$

$= \left[2 \times \frac{1}{2} \times (\mathbf{PR} \times \mathbf{SQ}) \cdot \mathbf{PT}\right]$

$= [(\mathbf{PR} \times \mathbf{SQ})] \cdot \mathbf{PT})$

$= \begin{vmatrix} 3 & 1 & -2 \\ 1 & -3 & -4 \\ 1 & 2 & 3 \end{vmatrix}$

$= |3(-9 + 8) - (3 + 4) - 2(2 + 3)|$

$= |-3 - 7 - 10|$

$= 20$

94. Given $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 2$

Now,

$[2(\mathbf{a} \times \mathbf{b})3(\mathbf{b} \times \mathbf{c})(\mathbf{c} \times \mathbf{a})]$

$= 6[(\mathbf{a} \times \mathbf{b})(\mathbf{b} \times \mathbf{c})(\mathbf{c} \times \mathbf{a})]$

$= 6[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

$= 24$

95. Given $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 5$

Now, $[3(\mathbf{a} + \mathbf{b})(\mathbf{b} + \mathbf{c})2(\mathbf{c} + \mathbf{a})]$

$= 6[(\mathbf{a} + \mathbf{b})(\mathbf{b} + \mathbf{c})(\mathbf{c} + \mathbf{a})]$

$= 6 \times 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 60$

96. Given $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

$\Rightarrow$ $|\mathbf{a} \times \mathbf{b}| = 0$

Now, $\frac{1}{2}|(2\mathbf{a} + 3\mathbf{b}) \times (\mathbf{a} - \mathbf{b})|$

$= \frac{1}{2}|-2\mathbf{a} \times \mathbf{b} + 3\mathbf{b} \times \mathbf{a}|$

$= \frac{1}{2}|-2\mathbf{a} \times \mathbf{b} - 3\mathbf{a} \times \mathbf{b}|$

$= \frac{1}{2}|-5\mathbf{a} \times \mathbf{b}|$

$= \frac{5}{2}|\mathbf{a} \times \mathbf{b}|$

$= 100$

97. Given $|\mathbf{a} \times \mathbf{b}| = 30$

Now, $|\mathbf{a} \times (\mathbf{a} + \mathbf{b})|$

$= |\mathbf{a} \times \mathbf{a} + \mathbf{a} + \mathbf{b}|$

$= |\mathbf{0} + \mathbf{a} \times \mathbf{b}|$

$= |\mathbf{a} \times \mathbf{b}|$

$= 30.$

98. It is given that $\mathbf{a}$ is in the direction of $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$

$\Rightarrow$ $\mathbf{a} = \lambda_1(\mathbf{x} \times (\mathbf{y} \times \mathbf{z})) = \lambda_2((\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z})$

$\Rightarrow$ $\mathbf{a} = \lambda_1\left(\left(2 \times \frac{1}{2}\right)\mathbf{y} - \left(2 \times \frac{1}{2}\right)\mathbf{z}\right) = \lambda_1(\mathbf{y} - \mathbf{z})$

$\Rightarrow$ $\mathbf{a} \cdot \mathbf{y} = \lambda_1(\mathbf{y} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{z})$

Thus, $\mathbf{a} = \mathbf{a} \cdot \mathbf{y}(\mathbf{y} - \mathbf{z})$

Similarly, $\mathbf{b} = (\mathbf{b} \cdot \mathbf{z})(\mathbf{z} - \mathbf{x})$

Now, $\mathbf{a} \cdot \mathbf{b} = (\mathbf{a} \cdot \mathbf{y})(\mathbf{b} \cdot \mathbf{z})\{(\mathbf{y} - \mathbf{z}) \cdot (\mathbf{z} - \mathbf{x})\}$

$= (\mathbf{a} \cdot \mathbf{y})(\mathbf{b} \cdot \mathbf{z})(1 - 1 - 2 + 1)$

$= -(\mathbf{a} \cdot \mathbf{y})(\mathbf{b} \cdot \mathbf{z}).$

99. Given $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$

$\Rightarrow$ $\mathbf{a} \cdot [(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c})] = p + q(\mathbf{a} \cdot \mathbf{b}) + r(\mathbf{a} \cdot \mathbf{c})$

$\Rightarrow$ $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = p + q(\mathbf{a} \cdot \mathbf{b}) + r(\mathbf{a} \cdot \mathbf{c})$

$\Rightarrow$ $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = p + \frac{q}{2} + \frac{r}{2}$

$\Rightarrow$ $p + \frac{q}{2} + \frac{r}{2} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$ ...(i)

$\Rightarrow$ $\frac{p}{2} + q + \frac{r}{2} = 0$ ...(ii)

$\frac{p}{2} + \frac{q}{2} + r = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2$ ...(iii)

Also,

$[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}$

$= \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix}$

$= 1\left(1 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right)$

$= \left(\frac{3}{4} - \frac{1}{8} - \frac{1}{8}\right) = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow$ $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \frac{1}{\sqrt{2}}$

From Eqs (i), (ii) and (iii), we get

$q = r = -q$

Now, $\left(\frac{p^2 + 2q^2 + q^2}{q^2}\right) = \frac{q^2 + 2q^2 + q^2}{q^2} = 4$

100. We have,

$$|\vec{b} + \vec{c}| = |\vec{a}|$$

$\Rightarrow$ $|\vec{b} + \vec{c}|^2 = |\vec{a}|^2$

$\Rightarrow$ $|\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$

$\Rightarrow$ $48 + |\vec{c}|^2 + 48 = 144$

$\Rightarrow$ $|\vec{c}|^2 = 144 - 96 = 48$

$\Rightarrow$ $|\vec{c}| = 4\sqrt{3}$

Now, $\dfrac{|\vec{c}|^2}{2} - |\vec{a}| = \dfrac{48}{2} - 12 = 24 - 12 = 12$

Also,

$$|\vec{a} + \vec{b}| = |\vec{c}|$$

$\Rightarrow$ $|\vec{a} + \vec{b}|^2 = |\vec{c}|^2$

$\Rightarrow$ $|\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = |\vec{c}|^2$

$\Rightarrow$ $144 + 48 + 2(\vec{a}\cdot\vec{b}) = 48$

$\Rightarrow$ $(\vec{a}\cdot\vec{b}) = -72$

Again,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$\Rightarrow$ $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

Thus,

$$|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = |\vec{a} \times \vec{b} + \vec{a} \times \vec{b}|$$
$$= 2|\vec{a} \times \vec{b}|$$
$$= 2\sqrt{a^2b^2 - (\vec{a}\cdot\vec{b})^2}$$
$$= 2\sqrt{144 \times 48 - (-72)^2}$$
$$= 2\sqrt{144 \times 48 - 72 \times 72}$$
$$= 2\sqrt{72(2 \times 48 - 72)}$$
$$= 2\sqrt{72(96 - 72)}$$
$$= 2\sqrt{72 \times 24}$$
$$= 2 \times 24\sqrt{3}$$
$$= 48\sqrt{3}$$

Hence, the result.

101. It is given that

$$\left|\frac{\alpha\sqrt{3} + \beta}{2}\right| = \sqrt{3}$$

$$\alpha\sqrt{3} + \beta = \pm 2\sqrt{3} \quad \text{...(i)}$$

Given $\alpha = 2 + \sqrt{3}\beta$ ...(ii)

From Eqs (i) and (ii), we get

$$\alpha = -1 \text{ or } 2$$

Thus, $|\alpha| = 1$ or 2

102. We have,

$$\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$$
$$= x(-\vec{p} + \vec{q} + \vec{r}) + y(\vec{p} - \vec{q} + \vec{r}) + z(-\vec{p} - \vec{q} + \vec{r})$$
$$= (-x + y - z)\vec{p} + (x - y - z)\vec{q} + (x + y + z)\vec{r}$$

$\Rightarrow$ $(-x + y - z) = 4$

$\Rightarrow$ $(x - y - z) = 3$

$\Rightarrow$ $(x + y + z) = 5$

Solving, we get

$$x = 4,\ y = \frac{9}{2},\ z = -\frac{7}{2}$$

Hence, the value of $2x + y + z = 9$

103. We have,

$$\hat{w}\cdot(\hat{u} \times \vec{v})$$

$\Rightarrow$ $|\hat{w}|\,|\hat{u} \times \vec{v}|\cos\alpha = 1$

$\Rightarrow$ $\cos\alpha = 1$

$\Rightarrow$ $\hat{w} \perp \hat{u}$ and $\hat{w} \perp \vec{v}$

As it is given that there exists a vector $\vec{v}$.

$\Rightarrow$ $\hat{w}$ must be perpendicular to $\hat{u}$

Hence, infinitely many such vectors $\vec{v}$ exist.

If $\hat{u} = u_1\hat{i} + u_2\hat{j}$

$\Rightarrow$ $\hat{u}\cdot\vec{w} = 0$

$\Rightarrow$ $(u_1 + u_2) = 0$

$\Rightarrow$ $u_1 = -u_2$

$\Rightarrow$ $|u_1| = |-u_2| = |u_2|$

If $u = u_1\hat{i} + u_3\hat{k}$

$\Rightarrow$ $\vec{u}\cdot\vec{w} = 0$

$\Rightarrow$ $u_1 + 2u_3 = 0$

$\Rightarrow$ $|u_1| = 2|u_3|$

CHAPTER

6 3D-Co-ordinate Geometry

CONCEPT BOOSTER

1.1 Introduction

In earlier classes, we learnt about points, lines, cicles and conic sections in two-dimensional geometry. In 2D-geometry, a point represented by an ordered pair (x, y) for which both x and y are real numbers.

In the space, each body has length, breadth and height, i.e. each body exits in three-dimensional space. Thus, three independent quantities are required to represent any point in a space and so three axes are required to represent these quantities.

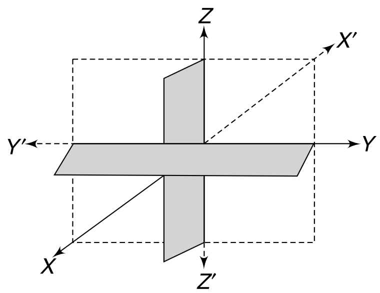
1.2 Rectangular co-ordinate system

The cartesian system of three lines which are mutually perpendicular to each other is called rectangular co-ordinate system.

When three mutually perpendicular planes intersect at a point, the mutually perpendicular lines are obtained and these lines also pass through that point. If we assume the point of intersection as the origin, these three planes are known as co-ordinate planes and the three lines are known as co-ordinate axes.

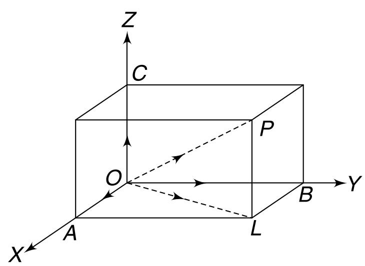
Octants

Every plane bisects the space. Hence three-co-ordinate plane divide the space into eight parts. Each part is called an octant.



1.3 Position vector of a point in a space

Let O be a fixed point, known as the origin, and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis, respectively, in such a way that they form a right handed system.



The plane XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane, respectively.

Let P be a point in a space such that its distances from yz-, zx- and xy-planes be a, b and c, respectively and i, j and k are the vectors along x, y and z axes, respectively, i.e.

$$OA = a,\ OB = b \text{ and } OC = c$$

Now, $$\mathbf{OP} = \mathbf{OL} + \mathbf{LP}$$
$$= \mathbf{OA} + \mathbf{AL} + \mathbf{LP}$$
$$= \mathbf{OA} + \mathbf{OB} + \mathbf{OC}.$$
$$= a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

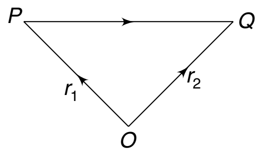
and $$|\mathbf{OP}| = \sqrt{a^2 + b^2 + c^2}.$$

Notes:

1. Equation of x-axis: $y = 0$, $z = 0$
2. Equation of y-axis: $x = 0$, $z = 0$
3. Equation of z-axis: $x = 0$, $y = 0$
4. Equation of xy-plane: $z = 0$
5. Equation of yz-plane: $x = 0$
6. Equation of zx-plane: $y = 0$

1.4 Distance between two points in space

From the figure, it is clear that



$$\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$$

$$= \mathbf{r_2} - \mathbf{r_1}$$

$$|\mathbf{PQ}| = |\mathbf{r_2} - \mathbf{r_1}|$$

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$

Then $\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$

$$= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Thus, the distance between two points,

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.4.1 Distance from Origin

Let O be the origin and $P(x, y, z)$ be any point, then

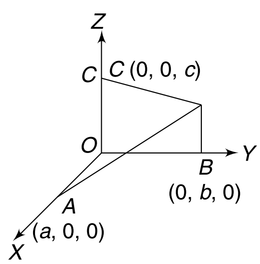
$$OP = \sqrt{x^2 + y^2 + z^2}.$$

1.4.2 Distance of a Point from Co-ordinate Axes

The distance from a point to x-axis $= \sqrt{b^2 + c^2}$

The distance from a point to y-axis $= \sqrt{a^2 + c^2}$

and the distance from a point to z-axis $= \sqrt{a^2 + b^2}$



1.5 Section formulae

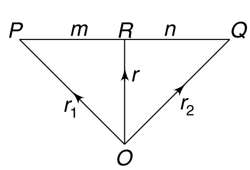
1.5.1 Internal Section

If a point $\mathbf{R(r)}$ divides the line segment joining the points $\mathbf{P(r_1)}$ and $\mathbf{Q(r_2)}$ internally in the ratio $m:n$, then

$$\mathbf{r} = \frac{m\mathbf{r_2} + n\mathbf{r}_1}{m + n}.$$

where $\frac{PR}{RQ} = \frac{m}{n}$

$$\Rightarrow \frac{\mathbf{PR}}{\mathbf{RQ}} = \frac{m}{n}$$



$$\Rightarrow \frac{\mathbf{OR} - \mathbf{OP}}{\mathbf{OQ} - \mathbf{OR}} = \frac{m}{n}$$

$$\Rightarrow \frac{\mathbf{r} - \mathbf{r_1}}{\mathbf{r_2} - \mathbf{r}} = \frac{m}{n}$$

$$\Rightarrow n(\mathbf{r} - \mathbf{r_1}) = m(\mathbf{r_2} - \mathbf{r})$$

$$\Rightarrow \mathbf{r}(m + n) = m\mathbf{r_2} + n\mathbf{r_1}$$

$$\Rightarrow \mathbf{r} = \frac{m\mathbf{r_2} + n\mathbf{r_1}}{m + n}$$

$$\Rightarrow (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

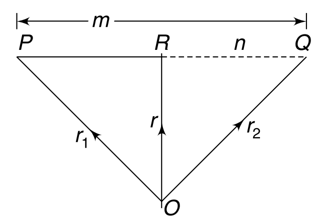
$$= \frac{m(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) + n(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})}{m + n}$$

$$\Rightarrow x = \frac{mx_2 + nx_1}{m + n}, y = \frac{my_2 + ny_1}{m + n}, z = \frac{mz_2 + nz_1}{m + n}$$

1.5.2 External Section

If a point $\mathbf{R(r)}$ divides the line segment joining the points $\mathbf{P(r_1)}$ and $\mathbf{Q(r_2)}$ externally in the ratio $m:n$, then

$$\mathbf{r} = \frac{m\mathbf{r_2} - n\mathbf{r_1}}{m - n}.$$



$$x = \frac{mx_2 - nx_1}{m - n},$$

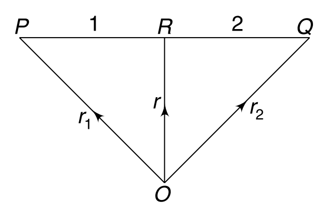
$$y = \frac{my_2 - ny_1}{m - n},$$

$$z = \frac{mz_2 - mz_1}{m - n}$$

1.5.3 Mid-Point Formula

If a point $\mathbf{R(r)}$ divides the line segment joining the points $\mathbf{P(r_1)}$ and $\mathbf{Q(r_2)}$ internally in the ratio $1:1$, then

$$\mathbf{r} = \frac{\mathbf{r_2} + \mathbf{r_1}}{\mathbf{2}}.$$

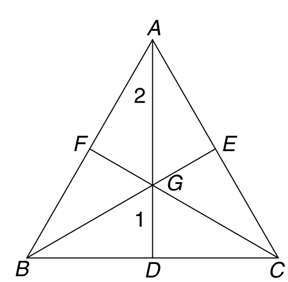


Thus, the co-ordinates of R are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

1.5.4 Centroid

The point of intersection of the medians of a triangle is called the centroid of the triangle.



Let $A(\mathbf{r_1})$, $B(\mathbf{r_2})$ and $C(\mathbf{r_3})$ be the vertices of a triangle ABC and $G(\mathbf{r})$ be its centroid. Then the position vector of the centroid is

$$\mathbf{r} = \frac{\mathbf{r_1} + \mathbf{r_2} + \mathbf{r_3}}{3}.$$

Let the position vectors of OA, OB and OC are $\mathbf{r_1}$, $\mathbf{r_2}$ and $\mathbf{r_3}$, respectively.

Let the position vectors of G be $\mathbf{r}$.

Now, the position vector of D is

$$\mathbf{OD} = \frac{\mathbf{r_2} + \mathbf{r_3}}{\mathbf{2}}$$

As we know that, the centroid divides the median in the ratio 2 : 1. Thus, the position vector of G,

$$= \mathbf{OG} = \mathbf{r}$$

$$= \frac{2 \cdot \dfrac{\mathbf{r_2} + \mathbf{r_3}}{2} + 1 \cdot \mathbf{r_1}}{1 + 2}.$$

$$= \frac{\mathbf{r_1} + \mathbf{r_2} + \mathbf{r_3}}{\mathbf{3}}$$

Thus, the co-ordinates of the centroid G are

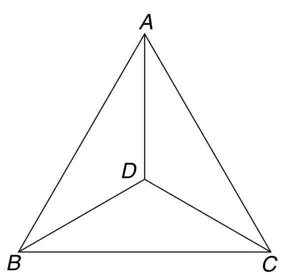
$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

Centroid of a Tetrahedron

Let the co-ordinates of A, B, C and D are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4), respectively.

Then the co-ordinates of its vertices are

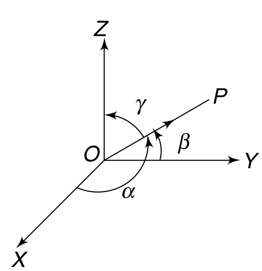
$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$



1.6 Direction cosines and direction ratios of a vector or a line

1.6.1 Direction cosines

If a line makes an angle α, β, γ with the positive direction of the co-ordinate axes. Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as the direction cosines of the given line and are, generally, denoted as l, m and n, respectively.



Thus, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

1.6.2 Direction Ratios

A set of three numbers a, b, c, which are proportional to the direction cosines l, m, n respectively of a line, are called the direction ratios.

Thus, $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$.

1.6.3 Refult-1

Let $\mathbf{OP} = \mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and i, j, k be the unit vectors along the x-axis, y-axis and z-axis, respectively

Now, $\mathbf{r} \cdot \mathbf{i} = a$

$\Rightarrow$ $|\mathbf{r}||\mathbf{i}|\cos\alpha = a$

$\Rightarrow$ $\cos\alpha = \dfrac{a}{|\mathbf{r}|}$

$\Rightarrow$ $l = \dfrac{a}{|\mathbf{r}|}$

Similarly, $m = \dfrac{b}{|\mathbf{r}|}$ and $n = \dfrac{c}{|\mathbf{r}|}$

i.e. $l = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$

and $n = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$.

1.6.4 Refult-2

If l, m, n be the direction cosines of a line, then

$$l^2 + m^2 + n^2 = 1$$

1.6.5 Refult-3

Any vector $\mathbf{r}$ can be expressed as

$$\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

Let $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

$\Rightarrow$ $\dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{a}{|\mathbf{r}|}\mathbf{i} + \dfrac{b}{|\mathbf{r}|}\mathbf{j} + \dfrac{c}{|\mathbf{r}|}\mathbf{k}$

$= l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

$\Rightarrow$ $\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$

1.6.6 Refult-4

If a vector $\mathbf{r}$ having direction cosines l, m, n, the projection of $\mathbf{r}$ on the co-ordinate axes are given by

$$l|\mathbf{r}|, m|\mathbf{r}|, n|\mathbf{r}|.$$

1.6.7 Refult-5

The projection of the segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line having direction cosines l, m, n is given by

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

1.6.8 Refult-6

If the co-ordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), the direction ratios of the line PQ are $a = x_2 - x_1$, $b = y_2 - y_1$, $c = z_2 - z_1$ and the direction cosines of the line PQ are

$$l = \frac{x_2 - x_1}{|\mathbf{PQ}|},\ m = \frac{y_2 - y_1}{|\mathbf{PQ}|},\ n = \frac{z_2 - z_1}{|\mathbf{PQ}|}$$

1.6.9 Direction Cosines of the Axes

Since the positive x-axis makes angles 0°, 90°, 90° with the axes of x, y and z, respectively, then the direction cosines of x-axis are (1, 0, 0).

Similarly, the direction cosines of y-axis and z-axis are (0, 1, 0) and (0, 0, 1) respectively.

1.6.10 Angle between Two Vectors

Let $\mathbf{m} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$

and $\mathbf{n} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

Then $\mathbf{m} \cdot \mathbf{n} = mn\cos\theta$

$$\Rightarrow \cos\theta = \frac{\mathbf{m}\cdot\mathbf{n}}{mn} = \frac{\mathbf{m}\cdot\mathbf{n}}{|m||n|}$$

$$= \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{a_1a_2 + b_1b_2 + c_1c_2}{\lambda_1\lambda_2}$$

where $\lambda_1 = \sqrt{a_1^2 + b_1^2 + c_1^2}$, $\lambda_2 = \sqrt{a_2^2 + b_2^2 + c_2^2}$

$$\Rightarrow \cos\theta = \frac{a_1}{\lambda_1}\frac{a_2}{\lambda_2} + \frac{b_1}{\lambda_1}\frac{b_2}{\lambda_2} + \frac{c_1}{\lambda_1}\frac{c_2}{\lambda_2}$$

$$= l_1l_2 + m_1m_2 + n_1n_2$$

(i) **Condition of perpendicularity**

Here, $\theta = 90°$

$\Rightarrow \cos\theta = 0$

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$

(ii) **Condition of parallelism**

Here, $\theta = 0°$

$\Rightarrow \cos\theta = 1$

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 1$

$\Rightarrow (l_1l_2 + m_1m_2 + n_1n_2)^2 = 1$

$\Rightarrow (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2)$

$\Rightarrow (l_1m_2 - l_2m_1)^2 + (m_1n_2 - n_2m_1)^2 + (n_1l_2 - l_2n_1)^2 = 0$

$\Rightarrow (l_1m_2 - l_2m_1) = 0,\ (m_1n_2 - n_2m_1) = 0,\ (n_1l_2 - l_2n_1) = 0$

$\Rightarrow l_1m_2 = l_2m_1;\ m_1n_2 = n_2m_1;\ n_1l_2 = l_2n_1$

$$\Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

Straight Line

2.1 Definition

A straight line in a space can be determined uniquely if

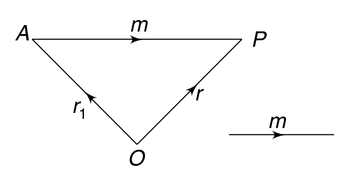
(i) it passes through a fixed point and is parallel to a fixed line.

(ii) it passes through two fixed points.

(iii) it is the intersection of two given non-parallel planes.

2.2 Equation of a Line Passing Thorugh a Point and Parallel to a Vector

The equation of a line passing through a point A with position vector $\mathbf{r_1}$ and parallel to a vector $\mathbf{m}$ is



$\mathbf{r} = \mathbf{r}_1 + \lambda\mathbf{m}$

Now, $\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$

$\Rightarrow \quad = \mathbf{OA} + \lambda\mathbf{m}$

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\mathbf{r_1} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

and $\mathbf{m} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Then $\mathbf{r} - \mathbf{r_1} = \lambda\mathbf{m}$

$\Rightarrow (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k} = \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

Thus, $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Notes:

1. In case of a line, the direction ratios and the direction cosines are the same.
2. Any point on the line can be considered as

$$(a\lambda + x_1,\ b\lambda + y_1 + c\lambda + z_1)$$

2.3 Equation of a Line Passing Through Two Points $A(\mathbf{r}_1)$ and $B(\mathbf{r}_2)$ is $\mathbf{r} = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1)$

Now, $\quad \mathbf{OP} = \mathbf{OA} + \mathbf{AP}$

$\Rightarrow \quad \mathbf{OP} + \mathbf{OA} + \lambda\mathbf{AB}$

$\Rightarrow \quad \mathbf{OA} + \lambda(\mathbf{OB} - \mathbf{OA})$

$\Rightarrow \quad \mathbf{r} = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1).$

Let $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\therefore \quad \mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

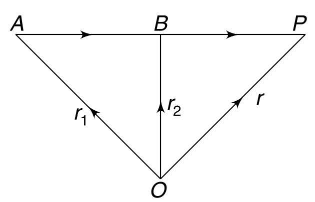
$\quad \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

Then $\quad (\mathbf{r} - \mathbf{r}_1) = \mathrm{l}(\mathbf{r}_2 - \mathbf{r}_1)$

$\Rightarrow \quad (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$

$\quad = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 + z_1)\mathbf{k}$

$$\Rightarrow \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$



2.4 Angle between Two Straight Lines

Let $\quad L_1 : \mathbf{r} = \mathbf{r}_1 + \lambda\mathbf{m}$

and $\quad L_2 : \mathbf{r} = \mathbf{r}_2 + \mu\mathbf{n}$

Then $\quad \mathbf{m}\cdot\mathbf{n} = mn\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{\mathbf{m}\cdot\mathbf{n}}{mn}$$

Let $\quad \mathbf{m} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$

and $\quad \mathbf{n} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

$$\text{Thus,} \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

1. **Condition of perpendicularity**

   When $\theta = \dfrac{\pi}{2}$

   $\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

2. **Condition of parallelism**

   When $\theta = 0$

   $$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} + \frac{c_1}{c_2}$$

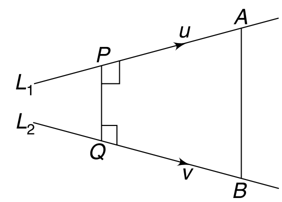
2.5 Skew Lines

Two lines are said to be skew lines if they are neither parallel nor intersecting. Clearly, the skew lines can never be coplnar.

2.5.1 Shortest Distance Between Two Skew Lines

Let $\quad L_1 : \mathbf{r} = \mathbf{r}_1 + \lambda\mathbf{u}$

and $\quad L_1 : \mathbf{r} = \mathbf{r}_2 + \mu\mathbf{v}$



Let $\quad \mathbf{OA} = \mathbf{r}_1$ and $\mathbf{OB} = \mathbf{r}_2$

Here, $\quad L_1 \,||\, \mathbf{u}$ and $L_2 \,||\, \mathbf{v}$

$\Rightarrow \quad \mathbf{PQ} \perp L_1$ and $\mathbf{PQ} \perp L_2$

$\Rightarrow \quad \mathbf{PQ} \perp \mathbf{u}$ and $\mathbf{PQ} \perp \mathbf{v}$

$\Rightarrow \quad \mathbf{PQ} \,||\, \mathbf{u} \times \mathbf{v}$

Thus, $\quad$ The shortest distance $= PQ$

$\quad = $ Projection of $\mathbf{AB}$ on $\mathbf{PQ}$

$$= \frac{\mathbf{AB}\cdot\mathbf{PQ}}{|\mathbf{PQ}|}$$

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}.$$

Notes 1: If two straight lines intersect, the shortest distance between them is zero.

The Plane

3.1 Definition

A plane is a surface such that if any two points on it are taken, the line joining them lies completely on it.

3.2 General Form

A first degree equation represents a plane. The general equation of a plane is given by

$$ax + by + cz + d = 0.$$

Notes:

1. Equation of xy-plane is $z = 0$.
2. Equation of yz-plane is $x = 0$
3. Equation of zx-plane is $y = 0$.
4. Equation of a plane passing through the origin is given by $ax + by + cz = 0$.

3.3. Equation of a plane passing through a point (x_1, y_1, z_1)

The equation of a plane passing through a point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

which is passing through (x_1, y_1, z_1).

Thus, $\quad ax_1 + vy_1 + cz_1 + d = 0 \quad$ ...(ii)

Subtracting Eqs. (ii) from Eq. we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

3.4 Equation of a plane passing through three non-collinear points

The equation of any plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

The equation of any plane passing through (x_1, y_1, z_1) is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{...(i)}$$

When it is passing through (x_2, y_2, z_2) and (x_3, y_3, z_3), then

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \quad \text{...(ii)}$$

$$a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0 \quad \text{...(iii)}$$

Eliminating a, b, c from Eqs (i), (ii) and (iii), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

which is the required equation of the plane.

3.5 Coplanarity of four points

The points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ and $S(x_4, y_4, z_4)$ are coplanar, then

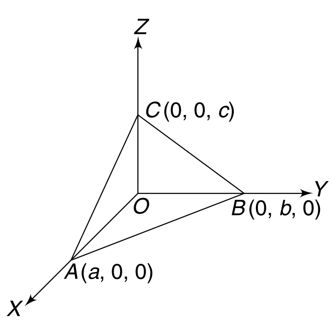
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

3.6 Intercept form of a plane

The equation of a plane in intercept form is given by

$$\frac{x}{a}, \frac{y}{b}, \frac{z}{c} = 1,$$

where a, b, c are the lengths of x, y and z axes, respectively.



Let the equation of the plane be passing through $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$.

$$Ax + By + Cz + D = 0 \quad \text{...(i)}$$

So, $\quad A \cdot a + D = 0 \Rightarrow A = -\frac{D}{a}$

and $\quad B \cdot b + D = 0 \Rightarrow B = -\frac{D}{b}$

$$C \cdot c + D = 0 \Rightarrow C = -\frac{D}{c}.$$

Putting the values of A, B and C in Eq. (i), we get

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

which is the required equation of the plane.

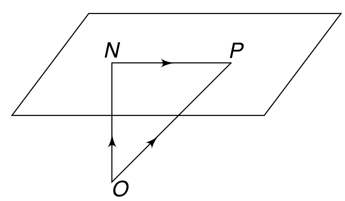
3.7 Normal to a plane

A line perpendicular to a plane is called the normal to the plane. Clearly every line in a plane is perpendicular to the normal to the plane.

3.8 Vector form

The vector equation of a plane passing through a point having position vector **a** and normal to a vector **n** is given by

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{d}$$



Let $\quad \mathbf{OP} = \mathbf{r}$ and $\mathbf{ON} = \mathbf{n}$

Now, $\quad \mathbf{ON} \perp \mathbf{NP}$

$\Rightarrow \quad \mathbf{NP} \cdot \mathbf{ON} = 0$

$\Rightarrow \quad (\mathbf{OP} - \mathbf{ON}) \cdot \mathbf{ON} = 0$

$\Rightarrow \quad (\mathbf{r} - \mathbf{n}) \cdot \mathbf{n} = 0$

$\Rightarrow \quad \mathbf{r} \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{n} = 0$

$\Rightarrow \quad \mathbf{r} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{n}$

$\Rightarrow \quad \mathbf{r} \cdot \mathbf{n} = d.$

3.9 Equation of a plane in normal form

The vector equation of a plane normal to a unit vector $\hat{n}$ and at a distance from the origin is given by

$$\mathbf{r} \cdot \hat{n} = d$$

Let O be the origin and ON be the perpendicular from O to the given plane such that

$$\mathbf{ON} = d\hat{n}.$$

Let P be a point on the plane with position vector **r** so that **OP = r.**

Now, $\mathbf{ON} \perp \mathbf{NP}$

$\Rightarrow \mathbf{NP}\cdot\mathbf{ON} = 0$

$\Rightarrow (\mathbf{OP} - \mathbf{ON})\cdot\mathbf{ON} = 0$

$\Rightarrow (\mathbf{r}\cdot d\hat{n})\cdot d\hat{n} = 0$

$\Rightarrow (\mathbf{r}\cdot d\hat{n} - d\hat{n}\cdot d\hat{n}) = 0$

$\Rightarrow (\mathbf{r}\cdot\hat{n} - d) = 0$

$\Rightarrow \mathbf{r}\cdot\hat{n} = d$

which is the required equation of the plane.

3.10 Normal form of a Plane

If l, m, n be the direction cosines of the normal to a given plane and p be the length of the perpendicular from the origin to the plane, the equation of the plane is given by

$$lx + my + nz = p.$$

As we know that the normal form of a plane is given by

$$\mathbf{r}\cdot\hat{n} = d$$

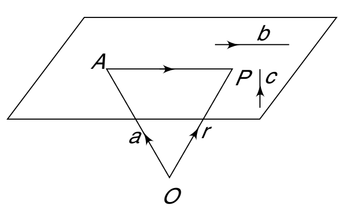
Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\hat{n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

The equation of the plane becomes

$$lx + my + nz = p.$$

3.11 Theorem

Equation of a plane passes through a point A with position vector **a** and is parallel to the given vectors **b** and **c.**



Let r be the position vector of any point P in the plane, then $\mathbf{AP} = \mathbf{OP} - \mathbf{OA}$

$= \mathbf{r} - \mathbf{a}$

Since, the vectors $\mathbf{r} - \mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar

So, $[\mathbf{r} - \mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$\Rightarrow [\mathbf{r}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$\Rightarrow [\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

which is the required equation of the plane.

3.12 Angle between two planes

The angle between two planes is defined as the angle between their normals.

Let $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$ and θ be the angle between them. Then

(i) Vector form: $\cos\theta = \dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$.

(ii) Castesian form:

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

where $\mathbf{n}_1 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

and $\mathbf{n}_2 = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

Condition of Perpendicularity

$$a_1b_1 + a_2b_2 + a_3b_3 = 0$$

Condition of Parallelism

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

3.13 Angle between a line and a plane

The angle between a line and a plane is the angle between the line and the normal to the plane.

Let the equation of the plane be

$$a_1x + b_1y + c_1z + d = 0 \quad \text{...(i)}$$

and the equation of the line be

$$\frac{x - x_1}{a_2} = \frac{y - y_1}{b_2} = \frac{z - z_1}{c_2}$$

Thus, $\cos(90° - \theta)$

$$= \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of Perpendicularity

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Condition of parallelism

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

3.14 Equation of a plane parallel to a given plane

The equation of a plane parallel to another plane $ax + by + cz + d = 0$ is

$$ax + by + cz + k = 0.$$

3.15 Equation of a plane parallel to the axes

3.15.1 Equation of a plane parallel to x-axis

Let the equation of any plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

and the equation of x-axis is

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} \quad \text{...(ii)}$$

Since the plane (i) is parallel to the line (ii), so,

$$a\cdot 1 + b.0 + c.0 = 0$$

$\Rightarrow a = 0$

Putting the value of a in Eq. (i), we get

$$by + cz + d = 0.$$

which is the required equation of the plane.

3.15.2 Equation of a plane parallel to *y*-axis

Let the equation of any plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

and the equation of y-axis is

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{0} \quad \text{...(ii)}$$

Since the plane (i) is parallel to the line (ii), so,

$$a \cdot 0 + b \cdot 1 + c \cdot 0 = 0$$

$$\Rightarrow \quad b = 0$$

Putting the value of b in Eq. (i), we get

$$ax + cz + d = 0.$$

which is the required equation of the plane.

3.15.3 Equation of a plane parallel to z-axis

Let the equation of any plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

and the equation of z-axis is

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1} \quad \text{...(ii)}$$

Since the plane (i) is parallel to the line (ii), so,

$$a \cdot 0 + b \cdot 0 + c \cdot 1 = 0$$

$$\Rightarrow \quad c = 0$$

Putting the value of a in (i), we get

$$ax + by + d = 0.$$

which is the required equation of the plane.

3.15.4 Equation of a plane parallel to *xy*-plane

The equation of xy-plane is given by

$$z = 0$$

So the equation of any plane parallel to xy-plane is

$$z + k = 0$$

3.15.5 Equation of a plane parallel to *yz*-plane

The equation of yz-plane is given by

$$x = 0$$

So the equation of any plane parallel to yz-plane is

$$x + k = 0$$

3.15.6 Equation of a plane parallel to *zx*-plane

The equation of zx-plane is given by

$$y = 0$$

So the equation of any plane parallel to zx-plane is

$$y + k = 0.$$

3.16 Equation of a Plane Passing Thorough (x_1, y_1, z_1), (x_2, y_2, z_2) and Parallel to the Line Having Direction Ratios a, b, c

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0.$$

Let the equation of the plane be

$$Ax + By + Cz + D = 0 \quad \text{...(i)}$$

which is passing through (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\therefore \quad A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

and $\quad A(x - x_2) + B(y - y_2) + C(z - z_2) = 0$

Also, $\quad A \cdot a + B \cdot b + C \cdot c = 0$

Eliminating, A, B and C from above three equations, we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0$$

Hence, the result.

3.17 Equation of the Plane Passing Through a Point (x_1, y_1, z_1) and Parallel to Two Lines Having Direction Ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

The equation of any plane passing through (x_1, y_1, z_1) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \quad \text{...(i)}$$

Since the plane (i) is parallel to two lines, so

$$Aa_1 + Bb_1 + Cc_1 = 0 \quad \text{...(ii)}$$

$$Aa_2 + Bb_2 + Cc_2 = 0 \quad \text{...(iii)}$$

Eliminating A, B, C from Eqs (i), (ii) and (iii), we get

$$\begin{vmatrix} x \quad x_1 & y \quad y_1 & z \quad z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

which is the required equation of the plane.

3.18 Equation of a Plane Passing Through the Line of Intersection of Planes

The equation of a plane passing through the line of intersection of planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

and $\quad a_2x + b_2y + c_2z + d_2 = 0$

is $\quad a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0.$

3.19 Distance of a point from a plane

The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

3.20 Distance between two parallel planes

The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$\left| \frac{d - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|.$$

3.21 Sides of a plane

We know that a plane divides the three-dimensional space in two equal parts. Two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are on the same side or opposite side of the plane $ax + by + cz + d = 0$ if

$$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0 \text{ or } < 0.$$

3.22 Intersection of a line and a plane

Let $\quad L: \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ ...(i)

and $\quad P: Ax + By + Cz + D = 0$ ....(ii)

Any point on the line (i) can be considered as

$$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda) \quad \text{...(iii)}$$

If it lies on the plane (ii), then

$$A(x_1 + a\lambda) + B(y_1 + b\lambda) + C(z_1 + c\lambda) + D = 0$$

$$\Rightarrow \quad (Ax_1 + By_1 + Cz_1 + D) + \lambda(A_a + B_b + C_c) = 0$$

$$\Rightarrow \quad \lambda = -\frac{(Ax_1 + By_1 + Cz_1 + D)}{(Aa + Bb + Cc)}$$

Substituting the values of λ in Eqs (iii), we get the required co-ordinates of the point of intersection.

3.23 Condition for a line to lie in a plane

If the line $\quad L: \dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$

lies in the plane $P: ax + by + cz + d = 0$, then

(i) $ax_1 + by_1 + cz_1 + d = 0$

(ii) $al + bm + cn = 0$

Since the line $L = 0$ lies in a plane $P = 0$, therefore the point (x_1, y_1, z_1) on $L = 0$ will also lie in the plane $P = 0$.

So, $\quad ax_1 + by_1 + cz_1 + d = 0$

Also, since the line $L = 0$ lies in the plane $P = 0$, therefore, the normal to the plane is also normal to the line.

Thus, $\quad al + bm + cn = 0.$

3.24 Condition of coplanarity of two lines

Let $\quad L_1: \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$

and $\quad L_2: \dfrac{x - x_1}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$

If two lines L_1 and L_2 are coplanar, then

$$\begin{vmatrix} x_2 \quad x_1 & y_2 \quad y_1 & z_2 \quad z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

and the equation of the plane containing them is

$$\begin{vmatrix} x \quad x_1 & y \quad y_1 & z \quad z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} x \quad x_2 & y \quad y_2 & z \quad z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

3.25 Equation of the planes bisecting the angle between the planes

The equation of the planes bisecting the angle between the planes $a_1x + b_1y + c_2z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\left| \frac{a_1x + b_1y + c_1y + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \right| = \left| \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

3.26 Bisector of the acute and obtuse angles between two planes

Let the equation of the two planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{...(i)}$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{...(ii)}$$

First we make $d_1 > 0$, $d_2 > 0$

(i) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, the origin lies in the obtuse angle between two planes and the equation of the obtuse angle bisector is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_1y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and the acute angle bisector is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, the origin lies in the acute angle between the two planes and the equation of the bisector of the acute angle is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and the obtuse angle is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

3.27 Foot of Perpendicular of a Point w.r.t. a Plane

The foot of the perpendicular M of a point $P(x_1, y_1, z_1)$ w.r.t. a plane $ax + by + cz + d = 0$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{(a^2 + b^2 + c^2)}$$

3.28 Image of a Point w.r.t. a Palne

The image Q of a point $P(x_1, y_1, z_1)$ w.r.t. a plane $ax + by + cz + d = 0$ is

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -\frac{2(ax_1 + by_1 + cz_1 + d)}{(a_2 + b_2 + c_2)}$$

where $Q = (x_2, y_2, z_2)$.

3.29 Equation of the Plane Containing a Given Line and Parallel to a Given Line

The equation of the plane containing a given line

$$L_1: \frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

and also parallel to another line

$$L_2: \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

The equation of the plane containing $L_1 = 0$ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{...(i)}$$

where $al_2 + bm_1 + cn_1 = 0$ ...(ii)

Since $L_2 = 0$ is parallel to the plane (i), therefore the normal to the plane (i) is also normal to the line $L_2 = 0$.

Thus, $al_1 + bm_2 + cn_2 = 0$ ...(iii)

Eliminating a, b, c from Eqs (i), (ii) and (iii), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

3.30 Equation of the Plane Containing Two Given Lines

The equation of the plane containing two lines

$$L_1: \frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

and

$$L_2: \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Exercises

Level 1

(Problems based on Fundamentals)

ABC of 3D-Geometry

1. Find the distance between the points $P(3, 4, 5)$ and $Q(-1, 2, -3)$.
2. Find the shortest distance of the point $P(a, b, c)$ from x-axis.
3. If the line passing through the points $(5, 1, a)$ and $(3, b, 1)$ crosses the yz-plane at $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$, find the value of $a + b + 2$.
4. Show that the points $P(0, 7, 10)$, $Q(-1, 6, 6)$ and $C(-4, 9, 6)$ are the vertices of an isosceles right-angled triangle.
5. Find the locus of a point, the sum of distances from $(1, 0, 0)$ and $(-1, 0, 0)$ is 10.
6. Find the ratio in which the plane $x - 2y + 3z = 17$ divides the line joining the points $(-2, 4, 7)$ and $(3, -4, 8)$.
7. Find the co-ordinates of points which trisects the line joining the points $P(-3, 2, 4)$ and $Q(0, 4, 7)$.
8. If the centre of a tetrahedron $OABC$ are $(1, 2, -1)$, where $A = (a, 2, 3)$, $B = (1, b, 2)$, $C = (2, 1, c)$ respectively, find the distance of $P(a, b, c)$ from the origin.
9. Find the ratio in which the line joining the points $P(-2, 3, 7)$ and $Q(3, -5, 8)$ is divided by the yz-plane.

10. The vertices of a triangle are $A(5, 4, 6)$, $B(1, -1, 3)$ and $C(4, 3, 2)$. The internal bisector of $\angle BAC$ meets BC in D. Find AD.

Direction cosines and direction ratios

11. If a line is equally inclined with the axes, find its direction cosines.
12. Find the angle at which the vector $(4\mathbf{i} + 4\mathbf{j} + \mathbf{k})$ is inclined to the axes.
13. If a line makes an angle α, β, γ with the co-ordinate axes, find the value of

    (i) $\Sigma \sin^2 \alpha$ (ii) $\Sigma \cos(2\alpha)$.
14. A line OP makes with the x-axis and y-axis at an angle of 120°, 60°, respectively. Find the angle made by the line with the z-axis.
15. A vector $\mathbf{r}$ is inclined to x-axis and to y-axis at 60°. If $|\mathbf{r}| = 8$, find the vector $\mathbf{r}$.
16. Find the projection of the line joining (1, 2, 3) and (–1, 4, 2) on the line having direction ratios (2, 3, –6).
17. What is the angle between the lines whose direction cosines are $\left(-\frac{3}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{3}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$?
18. Find the angle between any two diagonals of a cube.
19. Find the angle between one diagonal of a cube and a diagonal of one face.
20. A line makes angles α, β, γ with the four diagonal of a cube, find the value of

    (i) $\Sigma \cos^2 \alpha$

    (ii) $\Sigma \sin^2 \alpha$

    (iii) $\Sigma \cos(2\alpha)$
21. Find the angle between the lines whose direction cosines are satisfy the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$.
22. Find the direction cosines of the two lines which are connected by the relation $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$.
23. Find the angle between the lines which are connected by the relation $l + m + n = 0$ and $2lm + 2ln + mn = 0$.
24. Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are

    perpendicular if $a^2(u + v) + b^2(u + w) + c^2(u + v) = 0$

    and parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$.
25. If a variable line in two adjacent positions has direction cosines (l, m, n) and $(l + \delta l, m + \delta m, n + \delta n)$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Striaght Line

26. Find the equation of a line passing through (2, 3, 4) and parallel to the line $\frac{x-3}{3} = \frac{y+1}{4} = \frac{z-7}{5}$.
27. Find the equation of a line passing through (–3, 2, –4) and parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$.
28. Find the point where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x - 2y - z = 7$.
29. Find the equation of the line through $3x - 4y + 5z = 10$, $2x + 2y - 3z = 4$ and parallel to $x = 2y = 3z$..
30. Prove that the lines $4x = 3y = -z$ and $3x = y = -4z$ are perpendicular.
31. If the lines $x = az + b$, $y = cz + d$ and $x = a'z + b'$, $y = c'z + d'$ are perpendicular, prove that $a \cdot a' + c \cdot c' = -1$.
32. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.
33. Find the foot of the perpendicular from the point (1, 2, 3) to the line $\frac{x-6}{3} = \frac{y-1}{2} = \frac{z-7}{3}$ and also find the length of the perpendicular.
34. Find the image of a point (1, 6. 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-7}{3}$.
35. Find the equation of the line which can be drawn from the point (1, –1, 0) to intersect the lines $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-3}{4}$ and $\frac{x-4}{4} = \frac{y}{5} = \frac{z+1}{2}$. orthogonally.
36. Find the equation of the line through (2, –3, 1) parallel to the plane $2x - y + z = 6$ so as to meet the line $\frac{x-2}{2} = \frac{y}{-3} = \frac{z-2}{-1}$ at right angle.

Shortest distance between two lines

37. Find the shortest distance between the lines $\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = -3\mathbf{i} - 7\mathbf{j} + 6\mathbf{k} + \mu(-3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$.
38. Find the shortest distance between the lines

$$L_1: \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

and $$L_2: \frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+2}{3}.$$

39. Prove that the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0$, $x + z = 0$, $x + y = 0$ and $x + y + z = \sqrt{3}\,a$ is $a\sqrt{2}$.

40. If $2d$ be the shortest distance between the lines $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ and $\frac{x}{a} - \frac{z}{c} = \frac{1}{y} = 0$, prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{d^2}$.

Plane

41. Find the equation of the plane through the points $P(1, 1, 1)$, $Q(3, -1, 2)$ and $R(-3, 5, -4)$.
42. Show that the four points (1, 2, 3), (2, 1, 4) (3, –1, 3) and (2, 2, 6) are coplanar.
43. Find the intercepts of the plane $2x - 4y + 5z = 20$ on the co-ordinate axes.
44. Find the equation of the plane passes through the points (3, 0, 0), (0, 4, 0) and (0, 0, 5).
45. Find the equations of the plane through the points (0, 4, –3), (6, –4, 3) other than the plane through the origin, which cut off from the axes intercepts, whose sum is zero,
46. A plane meets the co-ordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r). Find the equation of the plane.
47. A variable plane cuts the co-ordinate axes in A, B, C and is of constant distance $3p$ from the origin. Find the locus of the centroid of the triangle ABC.
48. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the co-ordinate axes at the points A, B, C respectively. Find the area of the ΔABC.
49. A variable plane cuts the co-ordinate axes in A, B, C and is of constant distance p from the origin. Find the locus of the centroid of the tetrahedron $OABC$.
50. If a variable plane forms a tetrahedron of constant volume $64k^3$ with the co-ordinate planes, find the locus of the centroid of the tetrahedron.
51. Find the angle between the planes $x - 2y + 2z + 2014 = 0$ and $2x + y + 2z + 2015 = 0$.
52. Prove that the planes $3x - 2y + 3z + 10 = 0$ and $2x - 3y - 4z + 7 = 0$ are perpendicular.
53. Find the equation of the plane passing through (1, 2, 3) and the perpendicular to the planes $2x + 3y + 4z + 7 = 0$, $3x + 4y + 5z + 10 = 0$
54. Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane $3x + 4y + z + 5 = 0$.
55. Prove that the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ is parallel to the plane $4x + 4y - 5z + 2 = 0$.
56. Find the equation of a plane passing through (2, 3, –1) and parallel to $x - 2y + 3z + 10 = 0$.
57. Find the equation of the plane passing through the points (2, 3, –4), (1, –1, 3) and parallel to x-axis.
58. Find the equation of the plane passing through the points (1, 2, 3), (4, 3, 1) and parallel to y-axis
59. Find the equation of the plane passing through the points (3, 4, 5), (–2, 3, –1) and parallel to z-axis.
60. Find the equation of the plane passing through (1, 2, 3) and parallel to xy-plane.
61. Find the equation of a plane parallel to yz-plane and passes through the point (2, 3, 4).
62. Find the equation of the plane passing through (3, 4, 5) and parallel to zx-plane.
63. Find the equation of the plane passing through the points (2, –1, 0), (3, –4, –5) and parallel to the line $2x = 3y = 4z$.
64. Find the equation of the plane passing through (2, 3, 4) and parallel to the lines $x = 2y = 3z$ and $2x = 5y = z$.
65. Find the equation of the plane passing through the intersection of the planes $2x + 5y - 5z = 6$ and $2x + 7y - 8z = 7$ and the point (–1, 4, 3).
66. Find the equation of the plane passing through the point (2, –1, 1) and the line $4x - 3y + 5 = 0 = y - 2z - 5$.
67. Find the equation of the plane passing through the intersection of the planes $2x + 3y + 10z = 8$ $2x - 3y + 7z = 2$ and the perpendicular to $3x - 2y + 4z = 5$.
68. Find the equation of the plane passing through the line $x + y + z + 3 = 0$, $2x - y + 3z + 1 = 0$ and parallel to $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.
69. Find the length of the perpendicular from a point $P(1, 2, 3)$ to the plane $5x + 4y - 3z - 1 = 0$.
70. Find the equation of the line through (–1, 3, 2), perpendicular to the plane $x + 2y + 2z = 3$ and also find the co-ordinates of its foot.
71. Find the distance of the point (1, 2, 3) from the plane $x + y + z = 11$ measured parallel to $\frac{x+1}{1} = \frac{y+2}{-2} = \frac{z-7}{2}$.
72. Find the locus of a point, the sum of the squares of whose distances from the planes $x + y + z = 0$, $x - z = 0$, $x - 2y + z = 0$ is 9.
73. Two system of the co-ordinate axes have the same origin. If a plane cuts them at distances a, b, c and p, q, r respectively from the origin. Prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$.
74. Find the distance of the point (4, 1, 1) from the lines $x + y + z - 4 = 0 = x - 2y - 2z - 4$.
75. Find the distance between the planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$
76. Find the co-ordinates of the point of intersection of the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ with the plane $2x - 2y - z = 7$.

77. Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

78. Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar and also find the equation of the plane where they lie.

79. Find the equation of the plane passing through the line of intersection of the planes $4x - 5y - 4z = 4$ and $2x + y + 2z = 8$ at the point (1, 2, 3).

80. Find the eqution of the plane bisecting the acute angle between the planes $x + 2y + 2z - 3 = 0$ and $3x + 4y + 12z + 1 = 0$.

81. Find the image of the point (2, −3, 4) with respect to the palne $4x + 2y - 4z + 3 = 0$.

82. Find the equation of the image of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane $3x - 3y + 10z - 26$.

83. Find the equation of the plane containning the line $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+1}{2}$ and is parallel to the line $\frac{x-4}{2} = \frac{y-1}{-3} = \frac{z+3}{5}$.

84. Find the equation of the plane containing the lines $\frac{x+1}{3} = \frac{y-1}{5} = \frac{z+3}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$.

Level II

(Mixed Problems)

1. The shortest distance of the point (a, b, c) from the x-axis is
   (a) $\sqrt{a^2 + b^2}$ (b) $\sqrt{a^2 + c^2}$
   (c) $\sqrt{b^2 + c^2}$ (d) $\sqrt{a^2 + b^2 + c^2}$

2. The equation of the x-axis is
   (a) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (b) $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$
   (c) $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$ (d) $\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$

3. If the direction cosines of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$, then c is
   (a) 1 (b) ±2
   (c) ±3 (d) $\pm\sqrt{3}$

4. The number of lines, through the origin makes equal angles with the axes, is
   (a) 1 (b) 2
   (c) 4 (d) 0

5. The foot of perpendicular from (a, b, c) on the z-axis is
   (a) $(a, 0, 0)$ (b) $(0, b, 0)$
   (c) (0, 0, 0) (d) $(0, 0, c)$.

6. The angle between any two diagonals of a cube is
   (a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(\sqrt{2})$
   (c) $\cot^{-1}(\sqrt{2})$ (d) $\cos^{-1}\left(\frac{2}{3}\right)$

7. The angle between a diagonal of a unit cube and a diagonal of a face is
   (a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$
   (c) $\cos^{-1}\left(\sqrt{\frac{1}{3}}\right)$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

8. A line makes an angle α, β and γ with the axes, the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is
   (a) 1 (b) 2
   (c) −1 (d) 0.

9. A line makes an angle α, β and γ with the axes, then the value of $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma)$ is
   (a) 1 (b) 2
   (c) −1 (d) 0.

10. The shortest distance from the origin to the plane $2x + 3y + 6z - 21 = 0$ is
    (a) 2 (b) 1
    (c) 3 (d) 4

11. A plane meets the co-ordinate axes at A, B and C, respectively. If the centroid of the triangle is (2, 2, 2), the equation of the plane is
    (a) $x + y + z = 5$ (b) $x + y + z = 6$
    (c) $x + y + z = 9$ (d) $x + y + z = 4$

12. The locus of the first degree equation in x, y, z reprsents a
    (a) stline (b) plane
    (c) sphere (d) conicoid

13. If the plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(k)$ with x-axis, then k is
    (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{2}{7}$
    (c) $\frac{\sqrt{2}}{3}$ (d) 1

14. The distance between the parallel lines $2x - 3y + 6z + 5 = 0$ and $6x - 9y + 18z + 21 = 0$ is
    (a) $\frac{1}{7}$ (b) $\frac{3}{7}$
    (c) $\frac{2}{7}$ (d) $\frac{4}{7}$.

15. The equation of the plane passing through (1, 2, 3) and parallel to $2x + y + 2z + 10 = 0$ is

(a) $2x + y + 2z - 10 = 0$
(b) $2x + y + 2z - 12 = 0$
(c) $2x + y + 2z + 12 = 0$
(d) $2x + y + 2z + 10 = 0$

16. The angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

17. The line $x = 1, y = 2$ is
(a) parallel to x-axis (b) parallel to y-axis
(c) parallel to z-axis (d) none of these

18. The line $y = 2$ and $z = 3$ is
(a) parallel to x-axis (b) parallel to y-axis
(c) parallel to z-axis (d) none of these

19. The equation of a line passing through (2, 3, 4) and perpendicular to the plane $2x + 3y - z = 5$ is
(a) $\frac{x-2}{2} = \frac{y-3}{1} = \frac{z-4}{3}$
(b) $\frac{x-2}{2} = \frac{y+3}{1} = \frac{z-4}{3}$
(c) $\frac{x-2}{2} = \frac{y+3}{1} = \frac{z-4}{3}$
(d) $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-4}{5}$

20. The line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is parallel to the plane
(a) $3x + 4y + 5z = 7$ (b) $2x + 3y + 5z = 10$
(c) $2x + y - 2z = 0$ (d) $2x + 3y + 2z = 5$

21. The angle between the plane $3x + 6y - 2z + 5 = 0$ and the line $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-3}{2}$ is
(a) $\cos^{-1}\left(\frac{4}{21}\right)$ (b) $\cos^{-1}\left(\frac{8}{21}\right)$
(c) $\sin^{-1}\left(\frac{4}{21}\right)$ (d) $\sin^{-1}\left(\frac{8}{21}\right)$.

22. The co-ordinates of the point of intersection of the line $\frac{x-6}{-1} = \frac{y+1}{0} = \frac{z+3}{4}$ and the plane $x + y - z = 3$ is
(a) (2, 1, 0) (b) (1, 2, –6)
(c) (5, 1, 2) (d) (5, –1, 1)

23. The shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is
(a) $\frac{1}{6}$ (b) $\frac{1}{\sqrt{6}}$
(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{3}$

24. The plane passing through (3, 2, 0) and the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$ is
(a) $x - y + z = 1$ (b) $x + y + z = 5$
(c) $x + 2y - z = 1$ (d) $2x - y + z = 5$.

25. If the projection of $\overrightarrow{PQ}$ on OX, OY and OZ are respectively 12, 3 and 4, the magnitude of $\overrightarrow{PQ}$ is
(a) 169 (b) 13
(c) 12 (d) 144

26. The distance of the plane passing through (1, 1, 1) and perpendicular to the line $\frac{x-1}{3} = \frac{y-1}{0} = \frac{z-1}{4}$ from the origin is
(a) $\frac{3}{4}$ (b) $\frac{4}{3}$
(c) $\frac{7}{5}$ (d) 1

27. A variable plane at a distance of 1 unit from the origin cuts the co-ordinate axes at A, B and C. If the centroid $D(x, y, z)$ of ΔABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then k is
(a) 1 (b) 3
(c) 9 (d) 4

28. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{4-z}{k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if k is
(a) 1 (b) 2
(c) 0 (d) 4

29. The equation of the obtuse angle bisector between the planes $3x - 2y + 6z + 8 = 0$ and $2x - y + 2z + 3 = 0$ is
(a) $5x - y - 4z = 3$ (b) $5x + y - 4z = 5$
(c) $4x + y - 5z = 5$ (d) $7x + 3y - 9z$

30. The plane $x - y - z = 4$ is rotated through an angle 90° about its line of intersection with the plane $x + y + 2z = 4$. Then the new position of the plane is
(a) $5x + 3y + 2z = 4$ (b) $5x + y + 4z = 20$
(c) $4x + y + 5z = 20$ (d) $4x + 5y + z = 20$

31. The value of m for which the straight lines $3x - 2y + z + 3 = 0 = 4x - 3y + 4z + 1$ are parallel to the plane $2x - y + mz = 2$ is
(a) 6 (b) 8
(c) –2 (d) –11

32. A straight line is given by $r = (1 + t)\hat{i} + 3t\hat{j} + (1 - t)\hat{k}$, where t in R. If this line lies in the plane $x + y + cz = d$, the value of $(c + d - 5)$ is

(a) 8 (b) 9
(c) 4 (d) 2

33. Let L_1: $\frac{x-2}{1} = \frac{y-1}{0} = \frac{z+1}{2}$

and L_2: $\frac{x-3}{1} = \frac{y-1}{1} = \frac{z}{-1}$

and let π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane π from the origin is

(a) $\sqrt{\frac{2}{7}}$ (b) $\frac{1}{7}$
(c) $\sqrt{6}$ (d) none of these

34. If the planes $x = cy + bz$, $y = az + bx$ and $z = bx + ay$ pass through the same line, the value of $a^2 + b^2 + c^2 + 2ab + 2$ is
(a) 2 (b) 1
(c) 3 (d) 4

35. The equation of the plane passing through (0, 2, 4) and containing the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{2-z}{2}$ is
(a) $x - 2y + 4z - 12 = 0$
(b) $5x + y + 9z - 38 = 0$
(c) $10x - 12y - 9z + 60 = 0$
(d) $7x + 5y - 3z + 2 = 0$

36. The two lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ will be perpendicular if and only if
(a) $aa' + bb' + cc' + 1 = 0$
(b) $aa' + bb' + cc' = 0$
(c) $(a + a') + (b + b') + (c + c') = 0$
(d) $aa' + cc' + 1 = 0$

37. If the line $\frac{x-2}{3} = \frac{1-y}{6} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + b = 0$. Then the value of $(a + b + 1)$ is
(a) 2 (b) 1
(c) 3 (d) 4

38. If the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$; $\frac{x-1}{3}, \frac{2-y}{1} = \frac{z-3}{4}$ and $\frac{x+k}{3} = \frac{y-1}{2} = \frac{z-2}{h}$ are concurrent, then
(a) $h = -2, k = -6$ (b) $h = 6, k = 2$
(c) $h = 1/2, k = 2$ (d) $h = 2, k = 1/2$

39. If the four points $P(2 - x, 2, 2)$, $Q(2, 2 - y, 2)$ $R(2, 2, 2 - z)$ and $S(1, 1, 1)$ are coplanar, then
(a) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$
(b) $x + y + z = 1$
(c) $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = 1$
(d) $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 2$

40. The equation of the right bisector plane of the line segment joining the points (2, 3, 4) and (6, 7, 8) is
(a) $x + y + z = 15$ (b) $x + 2y + z = 15$
(c) $3x + 2y + z = 15$ (d) $x + 2y + 5z = 15$

41. The shortest distance from the plane $12x + 4y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$ is
(a) 39 (b) 26
(c) 11 (d) 13

42. The intersection of the spheres $x^2 + y^2 + z^2 + 7z - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the sphere and the plane
(a) $2x - y - z = 1$ (b) $x - 2y - z = 1$
(c) $x - y - 2z = 1$ (d) $x - y - z = 1$

43. The plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius
(a) 3 (b) 1
(c) 2 (d) $\sqrt{2}$

44. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid-point of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$, then a is
(a) -1 (b) 1
(c) -2 (d) 2

45. The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the sphere and the plane. Then
(a) $2x - y - z = 1$ (b) $x - 2y - z = 1$
(c) $x - y - 2z = 1$ (d) $x - y - z = 1$.

Level III

(Problems for JEE-Advanced)

1. Show that the straight lines whose direction cosines are given by the equation $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are perpendicular if $a^2(u + v) + b^2(u + w) + c^2(u + v) = 0$ and parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$.

2. If P be any point on the plane $lx + my + nz = p$ and Q be a point on the line OP such that $OP \cdot OQ$, find the locus of the point Q.

3. A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the co-ordinate axes at A, B and C. If the plane through A, B and C and parallel to the planes $x = 0$, $y = 0$, $z = 0$ intersect at Q, find the locus of Q.

4. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.

5. The plane $x - y - z = 4$ is rotated through an angle of 90° about the line of intersection with the plane $x + y + 2z = 4$. Find its equation in the new position.
6. If the planes $x - cy - bz = 0$, $cy - y + az = 0$ and $bx + ay - z = 0$ passing through a straight line, find the value of $a^2 + b^2 + c^2 + 2abc$.
7. Find the co-ordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which are at a distance of 3 units from the point (1, −2, 3).
8. Find the distance of the point (1, 0, −3) from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{6-z}{6}$.
9. Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{2-z}{2}$.
10. Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane $x + 2y + z = 9$.
11. Find the shortest distance between the lines $2x + y + z - 1 = 0 = 3x + y + 2z - 2$ and $x = y = z$.
12. Find the distance of the point of intersection of the lines $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ from (1, −4, 7).
13. If the lines $x = ay + b$, $z = cy + d$ and $x = a_1y + b_1$, $z = c_1y + d_1$ are perpendicular such that $aa_1 + bb_1 + k = 0$, find the value of $(k^2 + 4)$.
14. Find the distance of the point(1, 2, 3) from the plane $x + y + z = 11$ measured parallel to $\frac{x+1}{1} = \frac{y-12}{-2} = \frac{z-7}{2}$.
15. If a variable plane forms a tetrahedron of constant volume $64k^2$ with the co-ordinate planes, find the locus of the centroid of the tetrahedron.
16. If $2d$ be the shortest distance between the lines $\frac{y}{b} + \frac{z}{c} = 1$, $x = 0$ and $\frac{x}{a} - \frac{z}{c} = 1$, $y = c$, prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{d^2}$.
17. If the foot of the perpendicular from the origin to a plane is $P(a, b, c)$, find the equation of the plane.
18. Find the equation of a plane which passes through the point of intersection of lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$ and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and at the greatest distance from the point (0, 0, 0).
19. If the equation of the plane through the straight line $\frac{x-3}{2} = \frac{y+2}{-3} = \frac{z}{6}$ and perpendicular to $x - y + z + 2 = 0$ is $ax + by + cz + 4 = 0$, find the value of $a + b + c + 2$.
20. Find the equation of the plane through (2, 3, 4) and also which is at the maximum distance from the origin.
21. A sphere of radius r passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of ΔABC lies on the sphere.
22. A plane passes through a fixed point (f, g, h) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$.
23. A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC.
24. Find the equation of a sphere circumscribing the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
25. Find the equation of a sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$ and $2x + 3y + 4z = 8$ is a great circle.

Level III

(Tougher Problems for JEE-Advanced)

1. Show that the straight lines whose direction cosines are given by $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are
   (i) perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$
   (ii) parallel if $\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0$.
2. If the edges of a rectangular parallelopiped are a, b and c, show that the angles between the four diagonals are given by $\cos^{-1}\left(\frac{a^2}{a^2} + \frac{b^2}{b^2} + \frac{c^2}{c^2}\right)$.
3. A variable plane passes through the fixed point (a, b, c) and meets the axes of reference in P, Q, R. Show that the locus of the point of intersection of the plane through P, Q, R parallel to the co-ordinate planes is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.
4. A variable plane makes intercepts on the co-ordinate axes, the sum of whose squares is constant and equal to k^2. Find the locus of the foot of the perpendicular from the origin to the plane.
5. Find the incentre of the tetrahedron formed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$.
6. If the image of the point (3, 5, 7) in the plane $2x + y + z = 16$ is (a, b, c), find the value of $a^2 + b^2 + c^2 + 18$.

7. If θ be the acute angle between the lines $\frac{x-1}{a} = \frac{y+1}{b} = \frac{z}{c}$ and $\frac{x+1}{a} = \frac{y-3}{b} = \frac{z-1}{c}$,where $a > b > c$ and a, b and c are the zeroes of $x^3 + x^2 - 4x - 4 = 0$, find θ.
8. If the planes $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ meet in a line, show that the line of intersection of these planes is $\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$
9. Let the plane $x - y - z = 2$ is rotated through 90° about its line, of intersection with the plane $x + 2y + z = 2$. Find the equation of the plane in the new position.
10. A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the co-ordinate axes at A, B and C. If the planes through A, B and C parallel to the planes $x = 0$, $y = 0$ and $z = 0$ intersect at Q, find the locus of Q.

Integer Type Questions

1. If the angle between the lines whose direction cosines are given by

$$3l + m + 5n = 0$$

and

$$6mn - 2nl + 5lm = 0$$

is $\cos^{-1}\left(\frac{\alpha}{\beta}\right)$, find the value of $(\alpha + \beta)$.
2. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, find the value of

$$3(\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta).$$

3. If the image of the point $P(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is $Q(\alpha, \beta, \gamma)$, find the vaue of $(\alpha + \beta + \gamma)$.
4. If $P(a, b, c)$ be any point on the plane $3x + 2y + z = 7$, find the least value of $2(a^2 + b^2 + c^2)$.
5. Find the distance of the point $(1, 0, -3)$ from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{6-z}{6}$.
6. Find the shortest distance between the lines $x + y + 2z - 3 = 0 = 2x + 3y + 4z - 4$ and the z-axis.
7. Find the number of planes that are equidistant from four non-coplanar points.
8. If the distance between the plane $x - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, find the value of $|d|$.
9. If the angle between two faces of a regular tetrahedron be θ, find the value of $(3\cos\theta + 2)$.
10. Find the number of lines, which are equally inclined to the axes.
11. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, find the value of $(k + 5)$.
12. If the straight lines $\frac{x-2}{1} = \frac{y-3}{2} = \frac{\frac{9}{2}-z}{k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z}{1}$ intersect, find the value of k.
13. If the planes $x = cy + bz$, $y = cx + az$ and $z = bx + ay$ pass through a straight line, find the value of $a^2 + b^2 + c + 2abc + 2$.
14. If the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, find the value of k.
15. If the plane $ax - by + cz = 0$ contains the line $\frac{x-a}{a} = \frac{y-2d}{b} = \frac{z-c}{c}$, find the value of $\left(\frac{b}{d}\right)$.

Comprehensive Link Passages

Passage I

A line makes angles α, β, γ with the co-ordinate axes, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

1. If $\alpha + \beta = 90°$, then γ is
   (a) 0 (b) 60°
   (c) 90° (d) 120°
2. If $\alpha - \beta = 90°$, then γ is
   (a) 30° (b) 60°
   (c) 120° (d) 90°
3. The direction cosines of a line equally inclined to the axes are
   (a) $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ (b) $-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}$,
   (c) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (d) $-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Passage II

A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a unit cube.

1. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is
   (a) 2/3 (b) 4/3
   (c) 8/3 (d) 10/3
2. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \cos^2\delta$ is
   (a) 1/3 (b) 4/3
   (c) 8/3 (d) 15/3

3. $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) + \cos(2\delta)$ is
   (a) 1/3 (b) 4/3
   (c) –8/3 (d) –4/3

Passage III

The volume of the tetrahedron $ABCD$ with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

1. The volume of a tetrahedron with vertices (0, 0, 0), (2, 0, 0), (0, 3, 0) and (0, 0, 4) is
   (a) 1 (b) 4
   (c) 6 (d) 2
2. The volume of the terahedron with vertices (1, 2, 3), (2, 0, 0), (0, 4, 0) and (0, 0, 3) is
   (a) 2 (b) 6
   (c) 4 (d) 8
3. The volume of tetrahedron formed by the planes $x + y = 0$, $y + z = 0$, $z + x = 0$ and $x + y + z = 1$ is
   (a) 2/3 (b) 4/3
   (c) 8/3 (d) 10/3
4. The volume of the tetrahedron formed by the planes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, $\frac{x}{a} + \frac{y}{b} = 0$, $\frac{y}{b} + \frac{z}{c} = 0$ and $\frac{x}{a} + \frac{z}{c} = 0$ is
   (a) $\frac{1}{3}abc$ (b) $\frac{4}{3}abc$
   (c) $\frac{2}{3}abc$ (d) $\frac{10}{3}abc$

Passage IV

Let $\vec{r}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$
and $\vec{r}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Then $\cos(\theta) = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

1. The angle between two diagonals of a unit cube is
   (a) $\tan^{-1}(3)$ (b) $\tan^{-1}\left(\frac{1}{3}\right)$
   (c) $\tan^{-1}(2\sqrt{2})$ (d) $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$
2. The angle between one diagonal of a unit cube and a diagonal of a face is
   (a) $\cot^{-1}(\sqrt{2})$ (b) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
   (c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$
3. If θ be the angle between the vector $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and the z-axis, then $\sin\theta$ is
   (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$
   (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{6}}$

Passage V

Let L_1: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$

and L_2: $\frac{x - x_2}{a_2} = \frac{y - y_2}{y_2} = \frac{z - z_2}{c_2}$

are perpendicular or, if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

and parallel if

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

1. The number of lines that are equally inclined with the axes is
   (a) 8 (b) 6
   (c) 4 (d) 3
2. If the lines $L_1 : x = ay + b$, $z = cy + d$ and $L_2 : x = x'y + b'$, $z = c'y + d'$ are perpendicular, then
   (a) $aa' + cc' = 0$ (b) $aa' + cc' = -1$
   (c) $aa' - cc' = 0$ (d) $aa' - cc' = 1$
3. Let L_1: $\frac{1 - x}{3} = \frac{7y - 14}{25} = \frac{z - 3}{2}$
   and L_2: $\frac{7 - 7x}{3p} = \frac{y - 5}{1} = \frac{6 - z}{5}$.
   If L_1 meets L_2 at right angles, the value of $(11p - 64)$ is
   (a) 5 (b) 6
   (c) 7 (d) 8.

Passage VI

Two lines L_1: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$

and L_2: $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

1. If two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are coplanar, the point of intersection is
   (a) (1, 1, 1) (b) (1, –1, –1)
   (c) (–1, –1, –1) (d) (1, 1, –1)

2. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ are coplanar, the value of k is
(a) 3/2 (b) 9/2
(c) −2/9 (d) −3/2

3. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=z-5$ are coplanar, if
(a) $k=0$ (b) $k=1$
(c) $k=2$ (d) $k=-3$

Passage VII

Let L_1: $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$

and L_2: $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$

and $A = (2, 3, 5)$.

1. The shortest distance between the lines L_1 and L_2 is
(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{1}{\sqrt{6}}$ (d) $\frac{1}{2}$

2. The equation of the plane passing through A and containing the line L_1 is
(a) $2x - z + 1 = 0$ (b) $2x - y - z + 2 = 0$
(c) $2x - y + 3z + 2 = 0$ (d) $3x - y + 5z + 10 = 0$

3. The point $A = (2, 3, 5)$. is shifted to the line L_2 by a distance 1. Then the co-ordinates of the new position of A is
(a) $\left(\frac{7}{5}, \frac{11}{5}, \frac{4}{5}\right)$ (b) $\left(\frac{7}{5}, \frac{4}{5}, \frac{11}{5}\right)$
(c) $\left(\frac{7}{5}, \frac{11}{5}, 4\right)$ (d) $\left(\frac{7}{5}, \frac{4}{5}, 11\right)$

Passage VIII

Let $A = (2, 3, 1)$, $P: 2x + y + z = 6$

and L: $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{4}$

1. The foot of perpendicular from the point A to the line L is
(a) $\left(\frac{11}{21}, \frac{43}{21}, \frac{37}{21}\right)$ (b) $\left(\frac{11}{21}, \frac{37}{21}, \frac{43}{21}\right)$
(c) $\left(-\frac{11}{21}, \frac{37}{21}, -\frac{43}{21}\right)$ (d) $\left(\frac{11}{21}, -\frac{43}{21}, \frac{37}{21}\right)$

2. The image of the point $A(2, 3, 1)$ w.r.t. the plane $2x + y + z$ is
(a) $\left(\frac{2}{3}, \frac{7}{3}, \frac{1}{3}\right)$ (b) $\left(-\frac{2}{3}, \frac{7}{3}, -\frac{1}{3}\right)$
(c) $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{7}{3}\right)$ (d) $\left(-\frac{2}{3}, -\frac{7}{3}, \frac{1}{3}\right)$

3. The image of the line L with respect to the plane P is
(a) $\frac{3x-1}{4}=\frac{3y-5}{2}=\frac{3z-8}{3}$
(b) $\frac{3x-1}{4}=\frac{3y-4}{2}=\frac{3z-5}{3}$
(c) $\frac{3x-1}{4}=\frac{3y-5}{2}=\frac{8-3z}{1}$
(d) $\frac{3x-1}{4}=\frac{5-3y}{3}=\frac{3z-8}{5}$

Matrix Match
(For JEE-Advanced Examination Only)

1. **Match the following Columns:**
A line makes angles α, β, γ with the co-ordinate axes.

Column I		Column II	
(A)	$\Sigma\cos^2\alpha$	(P)	3
(B)	$\Sigma\sin^2\alpha$	(Q)	−1
(C)	$\Sigma\cos(2\alpha)$	(R)	2
(D)	$\Sigma(\cos^2\alpha + \sin^2\alpha)$	(S)	1

2. **Match the following Columns:**
A line makes angles α, β, γ, δ with the four diagonals of a cube.

Column I		Column II	
(A)	$\Sigma\cos^2\alpha$	(P)	4
(B)	$\Sigma\sin^2\alpha$	(Q)	4/3
(C)	$\Sigma\cos(2\alpha)$	(R)	8/3
(D)	$\Sigma(\cos^2\alpha + \sin^2\alpha)$	(S)	−4/3

3. **Match the following columns:**

Column I		Column II	
(A)	The angle between any two diagonals of a cube is	(P)	$\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$
(B)	The angle between one diagonal of a unit cube and a diagonal of a face is	(Q)	$\sin^{-1}\left(\frac{1}{3}\right)$
(C)	The angle between a diagonal of a cube and a diagonal of a face intersecting it	(R)	$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$
(D)	The angle between the diagonal of the faces of the cube through the same vertex is	(S) (T)	$\cot^{-1}(\sqrt{2})$ $\cos^{-1}\left(\frac{1}{3}\right)$

4. **Match the following Columns:**

Column I		Column II	
(A)	The direction cosines of two lines are connected with $l + m + n = 0$ and $l^2 + m^2 + n^2 = 0$, the angle between them is	(P)	60°
(B)	The angle between the planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ is	(Q)	45°
(C)	The angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$ is	(R)	90°
(D)	The angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ and the plane $x + y + 4 = 0$ is	(S)	120°

5. **Match the following Columns:**

Column I		Column II	
(A)	If the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies on the plane $2x - 4y + z = 7$, then k is	(P)	−3
(B)	The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{4-z}{k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k is	(Q)	7
(C)	If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is	(R) (S)	0 9/2
(D)	The shortest distance between the lines $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-6}{5}$ and $\frac{x-5}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ is	(T)	5/2

6. **Match the following Columns:**

Column I		Column II	
(A)	The co-ordinates of the point of intersection of the line $\frac{x-1}{1} = \frac{y+2}{3} = \frac{z-2}{2}$ and the plane $3x + 4y + 5z = 15$ is	(P)	(2, 3, 5)
(B)	The co-ordinates of the foot of the perpendicular from the point (2, 6, 3) to the line $\frac{x}{2} = \frac{y-1}{2} = \frac{z-2}{3}$ is	(Q)	(5, 10, −6)
(C)	The point of intersection of the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-6}{1}$ and $\frac{x+3}{7} = \frac{y+3}{-6} = \frac{z+3}{1}$ is	(R)	(1, 3, 5)
(D)	If the lines $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ and $\frac{x-3}{3} = \frac{y-2}{4} = \frac{z-1}{5}$ are	(S)	(−9, −14, −19)

7. **Match the following Columns:**

Let $P(0, 3, -2)$, $Q(3, 7, -1)$ and $R(1, -3, -1)$ be three given points. Let L_1 be the line passing through P and Q and L_2 be the line through R and parallel to the vector $\vec{v} = \hat{i} + \hat{k}$.

Column I		Column II	
(A)	The perpendicular distance of P from L_2 is	(P)	$7\sqrt{3}$
(B)	The shortest distance between L_1 and L_2 is	(Q)	2
(C)	Area of the ΔPQR is	(R)	6
(D)	The distance from (0, 0, 0) to the plane PQR is	(S)	$\frac{19}{\sqrt{147}}$

8. **Match the following Columns:**

Column I		Column II	
(A)	The shortest distance from the point (3, 4, 5) to the x-axis is	(P)	5
(B)	The shortest distance from the point (3, 2, 4) to y-axis is	(Q)	$\sqrt{41}$
(C)	If the plane $2x - 3y + 6z = 11$ makes an angle $\sin^{-1}(k)$ with the x-axis, then $7k$ is	(R)	9
(D)	A straight line $\vec{r} = (1+t)\hat{i} + 5t\hat{j} + (1-t)\hat{k}$ where $t \in R$. If this line lies in the plane $x + y + cz = d$, the value of $(c + d)$ is	(S) (T)	2 4

9. **Match the following Columns:**

Column I		Column II	
(A)	The volume of the tetrahedron, whose vertices are (0, 0, 0), (2, 0, 0), (0, 3, 0) and (0, 0, 4) is	(P)	6
(B)	The volume of the tetrahedron whose vertices are (0, 1, 2), (3, 0, 1), (4, 3, 6) and (2, 3, 2) is	(Q)	4
(C)	The volume of the tetrahedron formed by the planes whose equations are $x + y = 0$, $y + z = 0$, $z + x = 0$ and $x + y + z = 1$ is	(R)	20
(D)	The volume of the tetrahedron formed by the planes whose equations are $\frac{x}{2}+\frac{y}{3}=0$, $\frac{y}{3}+\frac{z}{4}=0$, $\frac{z}{4}+\frac{x}{2}=0$ and $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ is	(S) (T)	16 2/3

10. **Match the following Columns:**

Column I		Column II	
(A)	If $2x^2 + 3y^2 + 4z^2 + 2kxy + 4yz = 0$ represents two planes, the value of k^2 is	(P)	2
(B)	If $x^2 + 5y^2 + 5z^2 + 2kxy + 4xz = 0$ represents two planes, the value of $5k^2$ is	(Q)	5
(C)	If $x^2 + y^2 + 4z^2 - 2kyz = 0$ represents two planes, the value of $(k^2 + 2)$ is	(R)	10
(D)	If $9x^2 + y^2 + z^2 - 6kxy = 0$ represents two planes, the value of $(k^2 + 7)$ is	(S) (T)	8 1

Questions asked in Previous Years' JEE-Advanced Examinations

1. The value of k, for which the line $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, is
(a) 7 (b) 6
(c) no real value (d) -7
[IIT-JEE, 2003]

2. Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1).
[IIT-JEE, 2003]

3. If P be the point (2, 1, 6), find a point Q such that PQ is perpendicular to the plane in $x + y - 2z = 3$ and the mid-point of PQ lies on it.
[IIT-JEE, 2003]

4. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect at a point, the value of k is
(a) 3/2 (b) 9/2
(c) 2/9 (d) $-3/2$
[IIT-JEE, 2004]

5. A parallelopiped T has one of its face as $ABCD$. The face opposite to $ABCD$ is $A'B'C'D'$. The parallelopiped T is compressed to another parallelopiped S with the face $ABCD$ remaining the same but $A'B'C'D'$ changes to $A''B''C''D''$. If the volume of S is 90% of T, show that the locus of A'' is a plane.
[IIT-JEE, 2004]

6. π_1 and π_2 are two planes passing through the origin. L_1 and L_2 are two lines passing through the origin such that L_1 lies on π_1 not on π_2 and L_2 lies on π_2 not on π_1. Show that there exist three points A, B, C whose permutation A', B', C' can be choosen such that
(a) A is on L_1, B on π_1 but not on L_1 and C not on π_1.
(b) A' is on L_2, B' on π_2 but not on L_2 and C' not on π_2.
[IIT-JEE, 2004]

7. A plane π passes through the point (1, 1, 1) and is parallel to the vectors $\mathbf{b} = (1, 0, -1)$ and $\mathbf{c} = (-1, 1, 0)$. If π meets the axes in A, B and C, find the volume of the tetrahedron $OABC$.
[IIT-JEE, 2004]

8. Find the equation of the planes passing through the lines $2x - y + z = 3$, $3x + y + z = 5$ and which at a distance of $\frac{1}{\sqrt{6}}$ from the point (2, 1, –1).
[IIT-JEE, 2005]

9. A variable plane at a distance of 1 unit from the origin cuts the co-ordinate axes at A, B and C. If the centroid $G(x, y, z)$ of ΔABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, the value of k is
(a) 9 (b) 3
(c) 1 (d) 1/3
[IIT-JEE, 2005]

10. A plane π is perpendicular to the two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ and passes through the point (1, –2, 1). The distance of π from the point (1, 2, 2) is
(a) 0 (b) 1
(c) $\sqrt{2}$ (d) $2\sqrt{2}$
[IIT-JEE, 2006]

11. A point (α, β, γ) lies on the plane $x + y + z = 2$ Let $\mathbf{a} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$ and $\mathbf{k} \times (\mathbf{k} \times \mathbf{a}) = \mathbf{0}$, then γ equals ....

[IIT-JEE, 2006]

12. A line L is perpendicular to $x + 2y + 2z = 0$ and passes through (0, 1, 0). The perpendicular distance L from the origin equals ....

[IIT-JEE, 2006]

13. A plane passes through (1, 2, 3) and is perpendicular to two planes $x = 0$ and $y = 0$. The distance of the plane from the point (0, –1, 0) equals ....

[IIT-JEE, 2006]

14. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Statement I: The parametric equations of the line of intersection of given planes are $x = 3\ 14t$, $y = 1 + 2t$, and $z = 15t$.

Statement II: The vector $14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ is parallel to the given planes.

[IIT-JEE, 2007]

15. Consider the following linear equations:

$ax + by + cz = 0,$

$bx + cy + az = 0,$

$cx + ay + bz = 0.$

Match the conditions/expressions in Column I with statements in Column II.

Column I		**Column II**	
(A)	$a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$,	(P)	the equations represent planes meeting at a single point.
(B)	$a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$,	(Q)	the equations represent the line $x = y = z$
(C)	$a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$,	(R)	the equations represent the identical planes
(D)	$a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$,	(S)	the equations represent the whole of the three-dimensional space.

[IIT-JEE, 2007]

16. Consider the planes

$P_1 : x - y + z = 1,$

$P_2 : x + y - z = -1,$

$P_3 : x - 3y + 3z = 2.$

Let L_1, L_2 and L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1 and P_1 and P_2, respectively.

Statement I: At least two of the lines L_1, L_2 and L_3 are non-parallel.

Statement II: The three planes do not have a common point.

[IIT-JEE, 2008]

17. Consider the lines

$L_1: \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$

$L_2: \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

(i) The unit vector perpendicular to both L_1 and L_2 is

(a) $\frac{\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}}{\sqrt{99}}$ (b) $\frac{-\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}}{5\sqrt{3}}$

(c) $\frac{-\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}}{5\sqrt{3}}$ (d) $\frac{-7\mathbf{i} + 7\mathbf{j} + \mathbf{k}}{\sqrt{99}}$

(ii) The shortest distance between L_1 and L_2 is

(a) 0 (b) $\frac{17}{\sqrt{3}}$

(c) $\frac{41}{5\sqrt{3}}$ (d) $\frac{17}{5\sqrt{3}}$

(iii) The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is

(a) $\frac{2}{\sqrt{75}}$ (b) $\frac{7}{\sqrt{75}}$

(c) $\frac{13}{\sqrt{75}}$ (d) $\frac{23}{\sqrt{75}}$

[IIT-JEE, 2008]

18. A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the co-ordinate axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals

(a) 1 (b) $\sqrt{2}$

(c) $\sqrt{3}$ (d) 2

[IIT-JEE, 2009]

19. Let $P(3, 2, 6)$ be a point in space and Q be a point on the line $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(-3\mathbf{i} + \mathbf{j} + 5\mathbf{k})$. Then the value of λ for which the vector $\mathbf{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(a) 1/4 (b) –1/4

(c) 1/8 (d) –1/8

[IIT-JEE, 2009]

20. The equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

(a) $x + 2y - 2z = 0$ (b) $3x + 2y - 2z = 0$

(c) $x - 2y + z = 0$ (d) $5x + 2y - 4z = 0$

[IIT-JEE, 2010]

21. If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, the foot of the perpendicular from P to the plane is

(a) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (b) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$

(c) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (d) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

[IIT-JEE, 2010]

22. If the distance between the plane $x - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{1}$ is $\sqrt{6}$, then $|d|$ is...

[IIT-JEE, 2010]

24. Let a, b, c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix} = [0\ 0\ 0] \ \dots \ (E).$$

If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x + y + z = 1$, the value of $7a + b + c$ is

(a) 0 (b) 12

(c) 7 (d) 6.

[IIT-JEE, 2011]

25. The point P is the intersection of the straight line joining the points $Q(2, 3, 5)$ and $R(1, -1, 4)$ with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point $T(2, 1, 4)$ to QR, the length of the line segment PS is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 2 (d) $2\sqrt{2}$

[IIT-JEE, 2012]

26. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$ is

(a) $5x - 11y + z = 17$ (b) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(c) $x + y + z = \sqrt{3}$ (d) $x - \sqrt{y} = 1 - \sqrt{2}$

[IIT-JEE, 2012]

27. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, the planes containing these two lines is (are)

(a) $y + 2z = -1$ (b) $y + z = -1$

(c) $y - z = -1$ (d) $y - 2z = -1$

[IIT-JEE, 2012]

28. The perpendiculars are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of the perpendiculars lie on the line

(a) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (b) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(c) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (d) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

[IIT-JEE, 2013]

29. A line L passing through the origin is perpendicular to the lines $L_1: (3 + t)\mathbf{i} + (-1 + 2t)\mathbf{j} + (4 + 2t)\mathbf{k}$ and $L_2: (3 + 2s)\mathbf{i} + (3 + 2s)\mathbf{j} + (2 + s)\mathbf{k}$, where $-\infty < t, s < \infty$

Then, the coordinate(s) of the point(s) on L_2 at a distance of $\sqrt{17}$ from the point of intersection of L_1 and L_1 is

(a) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (b) $(-1, -1, 0)$

(c) $(1, 1, 1,)$ (d) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

[IIT-JEE, 2013]

30. Two lines $L_1: x = 5, \frac{y}{3-\alpha} = \frac{z}{2}$ and $L_2: x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar, then α can take the values

(a) 1 (b) 2

(c) 3 (d) 4

[IIT-JEE, 2013]

31. Consider the lines $L_1: \frac{x-1}{2} = \frac{y}{-1} = \frac{x+3}{1}$, $L_2: \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes $P_1: 7x + y + 2z = 3$, $P_2: 3x + 5y - 6z = 4$ Let $ax + by + cz$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2 perpendicular to planes P_1 and P_2 Match List I with List II and select the correct answer using the code given below the lists.

List I		**List II**	
P	$a =$	1	13
Q	$b =$	2	–3
R	$c =$	3	1
S	$d =$	4	–2.

Codes

	P	Q	R	S
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3.

[IIT-JEE, 2013]

32. From a point $P(\lambda, \lambda, \lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, the possible value(s) of λ is(are)
(a) $\sqrt{2}$ (b) 1
(c) -1 (d) $-\sqrt{2}$
[IIT-JEE, 2014]

33. In R^3, consider the planes $P_1: y = 0$ and $P_2: x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, which of the following relations is(are) true?
(a) $2\alpha + \beta + 2\gamma + 2 = 0$
(b) $2\alpha - \beta + 2\gamma + 4 = 0$
(c) $2\alpha + \beta - 2\gamma - 10 = 0$
(d) $2\alpha - \beta + 2\gamma - 8 = 0$
[IIT-JEE, 2015]

34. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1: x + 2y - z + 1 = 0$ and $P_2: 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?
(a) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (b) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$
(c) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (d) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$
[IIT-JEE, 2015]

35. Consider a pyramid $OPQRS$ located in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The bases $OPQR$ of the pyramid is a square with $OP = 3$. The point S is directly above the mid-point T of the diagonal OQ such that $TS = 3$. Then
(a) the acute angle between OQ and OS is $\frac{\pi}{3}$
(b) the equation of the plane containing ΔOQS is $x - y = 0$
(c) the length of the perpendicular from P to the ΔOQS is $\frac{3}{\sqrt{2}}$
(d) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$
[IIT-JEE, 2016]

36. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is
(a) $x + y - 3z = 0$ (b) $3x + z = 0$
(c) $x - 4y + 7z = 0$ (d) $2x - y = 0$
[IIT-JEE, 2016]

ANSWERS

LEVEL II

1. (c)	2. (a)	3. (d)	4. (c)	5. (d)
6. (a)	7. (b)	8. (b)	9. (c)	10. (b)
11. (b)	12. (b)	13. (b)	14. (c)	15. (a)
16. (b)	17. (c)	18. (a)	19. (d)	20. (c)
21. (c)	22. (d)	23. (b)	24. (a)	25. (b)
26. (c)	27. (c)	28. (c)	29. (a)	30. (b)
31. (c)	32. (c)	33. (a)	34. (c)	35. (c)
36. (d)	37. (a)	38. (d)	39. (a)	40. (a)
41. (b)	42. (a)	43. (b)	44. (c)	45. (a)

LEVEL III

2. $P(lx + my + nz) = (x^2 + y^2 + z^2)$
3. $\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$
5. $(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = k^2$
6. $\left(\frac{a}{3 + \sqrt{3}}, \frac{a}{3 + \sqrt{3}}, \frac{a}{3 + \sqrt{3}}\right)$
8. 101
12. 15
13. $(-3, 5, 2)$
14. $\frac{x + 1}{11} = \frac{y - 1}{9} = \frac{1 - z}{15}$
16. $\frac{3x - 1}{4} = \frac{3y - 5}{2} = \frac{8 - 3z}{1}$
17. $\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$
18. $4x + 3y + 5z = 50$
19. 2
20. $2x + 3y + 4z = 29$
21. $9(x^2 + y^2 + z^2) = 4r^2$
22. $\frac{f}{x} + \frac{g}{y} + \frac{h}{z} = 2$
23. $(x^2 + y^2 + z^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4r^2$
24. $x^2 + y^2 + z^2 - ax - by - cz = 0$
25. $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

LEVEL IV

4. $(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = k^2$

5. $\left(\frac{a}{3 + \sqrt{3}}, \frac{a}{3 + \sqrt{3}}, \frac{a}{3 + \sqrt{3}}\right)$

6. (–3, 5, 2)

14. $\frac{x + 1}{11} = \frac{y - 1}{9} = \frac{1 - z}{15}$

9. $\frac{3x - 1}{4} = \frac{3y - 5}{2} = \frac{8 - 3z}{1}$

10. $\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$

Integer Type Questions

1. (7)	2. (4)	3. (8)	4. (7)	5. (7)
6. (2)	7. (7)	8. (6)	9. (3)	10. (4)
11. (5 or 2)	12. (4)	13. (3)	14. (7)	15. (2)

Comprehensive Link Passages

Passage I

1. (c) 2. (d) 3. (c)

Passage II

1. (b) 2. (c) 3. (d)

Passage III

1. (b) 2. (c) 3. (a) 4. (c)

Passage IV

1. (c) 2. (a) 3. (c) 4. (b)

Passage V

1. (a) 2. (b) 3. (b)

Passage VI

1. (c) 2. (b) 3. (a, d)

Passage VII

1. (c) 2. (a) 3. (c)

Passage VIII

1. (b) 2. (a) 3. (c)

Matrix Match

1. (A)→S,	(B)→R,	(C)→S,	(D)→P
2. (A)→Q,	(B)→R,	(C)→S,	(D)→P
3. (A)→T,	(B)→P,	(C)→S,	(D)→R
4. (A)→P,	(B)→R,	(C)→P,	(D)→Q
5. (A)→Q,	(B)→P,	(C)→S,	(D)→R
6. (A)→Q,	(B)→P,	(C)→R,	(D)→S
7. (A)→R,	(B)→Q,	(C)→P,	(D)→S
8. (A)→Q,	(B)→P,	(C)→S,	(D)→R
9. (A)→Q,	(B)→P,	(C)→T,	(D)→S
10. (A)→Q,	(B)→P,	(C)→T,	(D)→S

Hints and Solutions

Level I

1. The required distance

$= \sqrt{(-1 - 3)^2 + (2 - 4)^2 + (-3 - 5)^2}$

$= \sqrt{16 + 4 + 64}$

$= \sqrt{84} = 2\sqrt{21}$

2. Let the point $A(a, 0, 0)$ lies on the x-axis

Thus, the shortest distance of $P(a, b, c)$ from x-axis

= shortest distance between $P(a, b, c)$ and $A(a, 0, 0)$

$= \sqrt{(a - a)^2 + b^2 + c^2}$

$= \sqrt{b^2 + c^2}$

3. Let the line AB meet the plane at C, then the co-ordinats of E is

$$\phi\left(0, \frac{17}{2}, -\frac{13}{2}\right).$$

Let the point C divides AB in the ratio $\lambda : 1$, so

$3\lambda + 5 = 0$

$\Rightarrow \quad \lambda = -\frac{5}{3}$

Also,

$\frac{\lambda b + 1}{\lambda + 1} = \frac{17}{2}$

$\Rightarrow \quad \frac{-\frac{5}{3} b + 1}{-\frac{5}{3} + 1} = \frac{17}{2}$

$\Rightarrow \quad \frac{3 - 5b}{2} = \frac{17}{2}$

$\Rightarrow \quad 5b - 3 = 17$

$\Rightarrow \quad b = 4$

Again,

$\frac{\lambda + a}{\lambda + 1} = -\frac{13}{2}$

$\Rightarrow \quad 2\lambda + 2a = -13\lambda + 13$

$\Rightarrow \quad 2a = 13 - 15\lambda$

$= 13 + 15 \cdot \frac{5}{3}$

$= 13 + 25 = 38$

$\Rightarrow \quad a = 19$

Hence,

$$a + b + 2 = 19 + 4 + 2 = 25$$

4. Now, $PQ = \sqrt{1 + 1 + 16} = 3\sqrt{2}$

$QR = \sqrt{9 + 9 + 0} = 3\sqrt{2}$

and $RP = \sqrt{16 + 4 + 16} = \sqrt{36} = 6$

Thus, $PQ^2 + QR^2 = 18 + 18 = 36 = RP^2$

Therefore, the given triangle is right-angled isosceles.

5. Let the point be $P(x, y, z)$.

Consider $A = (1, 0, 0)$ and $B = (-1, 0, 0)$

Given $PA + PB = 10$

$\Rightarrow \sqrt{(x-1)^2 + y^2 + z^2} + \sqrt{(x+1)^2 + y^2 + z^2} = 10$

$\Rightarrow \sqrt{(x-1)^2 + y^2 + z^2} = 10 - \sqrt{(x+1)^2 + y^2 + z^2}$

$\Rightarrow (x-1)^2 + y^2 + z^2 = 100 + (x+1)^2 + y^2 + z^2 - 20\sqrt{(x-1)^2 + y^2 + z^2}$

$\Rightarrow (x-1)^2 - (x+1)^2 - 100 = -20\sqrt{(x+1)^2 + y^2 + z^2}$

$\Rightarrow -4x - 2 - 100 = -20\sqrt{(x-1)^2 + y^2 + z^2}$

$\Rightarrow -2x - 51 = -10\sqrt{(x-1) + y^2 + z^2}$

$\Rightarrow (2x + 51)^2 = 100((x-1)^2 + y^2 + z^2)$

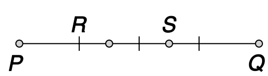
$\Rightarrow (4x^2 + 204x + 260) = 100(x^2 - 2x + 1 + y^2 + z^2)$

$\Rightarrow (96x^2 + 100y^2 + 100z^2) = (404x + 2501)$

which is the locus of the given point.

6. **

7. Let R divides P and Q internally in the ratio $1:2$.



Then the co-ordinates of R

$$= \left(\frac{0-6}{1+2}, \frac{4+4}{1+2}, \frac{7+8}{1+2}\right)$$

$$= \left(-2, \frac{8}{3}, 5\right)$$

Again, let S divides P and Q internally in the ratio $2:1$. Then the co-ordinates of S

$$= \left(\frac{0-3}{1+2}, \frac{8+2}{1+2}, \frac{14+4}{1+2}\right)$$

$$= \left(-1, \frac{10}{3}, 6\right).$$

8. We have,

$$\frac{a+1+2}{4} = 1 \Rightarrow a = 1$$

$$\frac{2+b+1}{4} = 2 \Rightarrow b = 5$$

and $\frac{2+3+c}{4} = -1 \Rightarrow c = -9$

Thus, the required distance from $P(a, b, c)$ to origin

$$= \sqrt{a^2 + b^2 + c^2}$$

$$= \sqrt{1 + 25 + 81} = \sqrt{107}.$$

9. Let the line PQ divides the xy-plane at R in the ratio $\lambda:1$.

Clearly, x-co-ordinate of the point R is zero.

Thus,

$$\frac{3\lambda - 2}{\lambda + 1} = 0$$

$\Rightarrow 3\lambda - 2 = 0$

$\Rightarrow \lambda = \frac{2}{3} = 2:3$

Therefore, the point R divides the yz-plane in the ratio $2:3$.

10. Now, $AB = \sqrt{4^2 + 5^2 + 3^2} = \sqrt{50} = 5\sqrt{2}$

and $AC = \sqrt{1 + 1 + 4^2} = \sqrt{81} = 3\sqrt{2}$

As we know that, if AD is the internal bisector of $\angle BAC$, then

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$\Rightarrow \frac{5\sqrt{2}}{3\sqrt{2}} = \frac{5}{3}$

Thus, D divides BC internally in the ratio $5:3$, so therefore the co-ordinates of D

$$= \left(\frac{20+3}{8}, \frac{15-3}{8}, \frac{10+9}{8}\right)$$

$$= \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$$

Thus, the length of AD

$$= \sqrt{\left(5 - \frac{23}{8}\right)^2 + \left(4 - \frac{7}{8}\right)^2 + \left(6 - \frac{19}{8}\right)^2}$$

$$= \sqrt{\left(\frac{17}{8}\right)^2 + \left(\frac{25}{8}\right)^2 + \left(\frac{26}{8}\right)^2}$$

$$= \sqrt{\frac{1755}{64}}$$

$$= \sqrt{\frac{1755}{8}}$$

11. Here, $\alpha = \beta = \gamma$

So, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

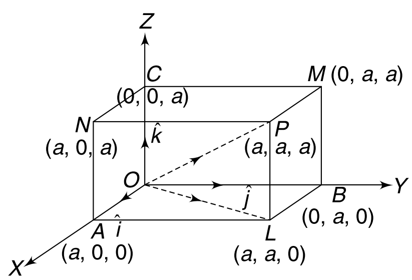
$\Rightarrow \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Thus, the direction cosines of the line are

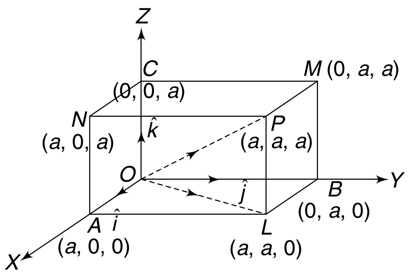
$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ or } \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

12. Let $\mathbf{r} = (4\mathbf{i} + 8\mathbf{j} + \mathbf{k})$ and then it makes angles α, β, γ with x, y and z axes respectively.
    Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ unit vectors along the x, y and z axes respectively.
    Now, $\hat{\mathbf{r}} \cdot \hat{\mathbf{i}} = 4$
    $\Rightarrow \quad |\mathbf{r}||\mathbf{i}| \cos\alpha = 4$
    $\Rightarrow \quad \cos\alpha = \dfrac{4}{|\mathbf{r}|} = \dfrac{4}{9}$
    $\Rightarrow \quad \alpha = \cos^{-1}\left(\dfrac{4}{9}\right)$
    Smilarly, we can easily find that
    $$\beta = \cos^{-1}\left(\frac{8}{9}\right), \gamma = \cos^{-1}\left(\frac{1}{9}\right)$$
13. we have
    (i) $\Sigma \sin^2\alpha$
    $= \sin^2\alpha + \sin^2\beta + \sin^2\gamma$
    $= 1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma$
    $= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$
    $= 3 - 1 = 2$
    (ii) $\Sigma \cos(2\alpha)$
    $= \cos(2\alpha) + \cos(2\beta) + \cos(2\gamma)$
    $= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1$
    $= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$
    $= 2 - 3 = -1$
14. Here, $\alpha = 120°$ and $\beta = 60°$
    As we know that,
    $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 1$$
    $\Rightarrow \quad \cos^2(120°) + \cos^2(60°) + \cos^2\gamma = 1$
    $\Rightarrow \quad \left(-\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2 + \cos^2\gamma = 1$
    $\Rightarrow \quad \cos^2\gamma = 1 - \left(\dfrac{1}{4} + \dfrac{1}{4}\right) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$
    $\Rightarrow \quad \cos\gamma = \pm\dfrac{1}{\sqrt{21}}$
    $\Rightarrow \quad \gamma = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$
15. Here, $l = \cos(60°) = m$
    As we know that,
    $$l^2 + m^2 + n^2 = 1$$
    $\Rightarrow \quad \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2 + n^2 = 1$
    $\Rightarrow \quad n^2 = 1 - \left(\dfrac{1}{4}, \dfrac{1}{4}\right) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$
    $\Rightarrow \quad n = \pm\dfrac{1}{\sqrt{2}}$
    Therefore,
    $$\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$
    $\Rightarrow \quad = 8\left(\dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j} \pm \dfrac{1}{\sqrt{2}}\mathbf{k}\right)$
    $\Rightarrow \quad = (4\mathbf{i} + 4\mathbf{j} \pm 4\sqrt{2}\,\mathbf{k}).$
16. As we know that the projection of the segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line having direction cosines l, m, n is
    $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$
    $$= \frac{1}{7}(-1 - 1) + \frac{1}{7}(4 - 2) + \frac{1}{7}(2 - 3)$$
    $$= -\frac{2}{7} + \frac{2}{7} - \frac{1}{7}$$
    $$= -\frac{1}{7}$$
17. Let θ be the angle between them. Then
    $$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$
    $$= \frac{9}{16} + \frac{1}{16} - \frac{3}{4}$$
    $$= \frac{5}{8} - \frac{6}{8} = -\frac{1}{8}$$
    $\Rightarrow \quad \theta = \cos^{-1}\left(-\dfrac{1}{8}\right)$
18. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the x, y and z axes, respectively.



Here, the four diangonals of a cube are
$$\mathbf{OP} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$$
$$\mathbf{AM} = -a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$$
$$\mathbf{BN} = a\mathbf{i} - a\mathbf{j} + a\mathbf{k}$$
$$\mathbf{CL} = a\mathbf{i} + a\mathbf{j} - a\mathbf{k}$$
Now, $\mathbf{OP} \cdot \mathbf{AM} = -a^2 + a^2 + a^2$
$\Rightarrow \quad |\mathbf{OP}||\mathbf{AM}|\cos\theta = a^2$
$\Rightarrow \quad a\sqrt{3}a\sqrt{3}\cos\theta = a^2$
$\Rightarrow \quad \cos\theta = \dfrac{a^2}{3a^2} = \dfrac{1}{3}$
Thus, $\theta = \cos^{-1}\left(\dfrac{1}{3}\right)$

19. Let **i**, **j**, **k** are the unit vectors along the x, y and z axes, respectively.



Here, the diagonal of the cube = $\mathbf{OP} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$ and a diagonal of the face = $\mathbf{OL} = a\mathbf{i} + a\mathbf{j} + 0\mathbf{k}$

Thus, $\mathbf{OP} \cdot \mathbf{OL} = a^2 + a^2 + 0$

$\Rightarrow |\mathbf{OP}||\mathbf{OL}| \cos\varphi = 2a^2$

$\Rightarrow a\sqrt{3}\, a\sqrt{2} \cos\varphi = 2a^2$

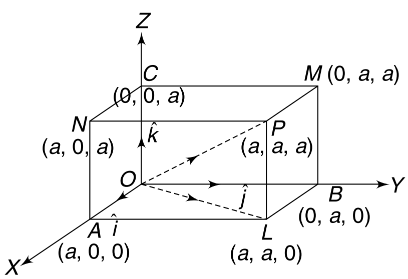
$\Rightarrow \cos\varphi = \dfrac{2a^2}{\sqrt{2}\sqrt{3a^2}}$

$= \dfrac{\sqrt{2}}{\sqrt{3}} = \sqrt{\dfrac{2}{3}}$

$\Rightarrow \varphi = \cos^{-1}\left(\sqrt{\dfrac{2}{3}}\right)$

20. Let the direction cosines of a line be l, m, n, i.e.
$\mathbf{r} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

Let **i**, **j**, **k** are the unit vectors along the x, y and z axes, respectively.



Hence, the four diangonals of a cube are

$\mathbf{OP} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$

$\mathbf{AM} = -a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$

$\mathbf{BN} = a\mathbf{i} - a\mathbf{j} + a\mathbf{k}$

$\mathbf{CL} = a\mathbf{i} + a\mathbf{j} - a\mathbf{k}$

Then $\cos\alpha = \dfrac{1}{\sqrt{3}}(l + m + n)$,

$\cos\beta = \dfrac{1}{\sqrt{3}}(-l + m + n)$,

$\cos\gamma = \dfrac{1}{\sqrt{3}}(l - m + n)$

and $\cos\delta = \dfrac{1}{\sqrt{3}}(l + m - n)$

Now,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= \dfrac{1}{3}(l + m + n)^2 + \dfrac{1}{3}(-l + m + n)^2 + \dfrac{1}{3}(l - m + n)^2 + \dfrac{1}{3}(l + m - \text{n})^2$

$= \dfrac{4}{3}(l^2 + m^2 + n^2)$

$= \dfrac{4}{3}$

21. We have

$l + m + n = 0$

$\Rightarrow n = -1(l + m)$

Now, $l^2 + m^2 - n^2 = 0$

$\Rightarrow l^2 + m^2 = n^2 = (1 + m)^2$

$\Rightarrow 2lm = 0$

$\Rightarrow l = 0, m = 0$

When $l = 0$, then $m = -n$

Thus, $\dfrac{l}{0} = \dfrac{m}{-1} = \dfrac{n}{1}$

$\Rightarrow \dfrac{l}{0} = \dfrac{m}{-1} = \dfrac{n}{1} = \dfrac{1}{\sqrt{2}}$

When $m = 0$, then $l = -n$

Thus $\dfrac{l}{-1} = \dfrac{m}{0} = \dfrac{n}{1}$

$\Rightarrow \dfrac{l}{-1} = \dfrac{m}{0} = \dfrac{n}{1} = \dfrac{1}{\sqrt{2}}$

Let θ be the angle between them.

Then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$\Rightarrow \cos\theta = 0 + 0 + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3}$

22. We have,

$l - 5m + 3n = 0$

$\Rightarrow l = 5m - 3n$

Now,

$7l^2 + 5m^2 + 3n^2 = 0$

$\Rightarrow 7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$

$\Rightarrow 7(25m^2 + 9n^2 - 30mn) + 5m^2 - 3n^2 = 0$

$\Rightarrow 180m^2 - 210mn + 60n^2 = 0$

$\Rightarrow 6m^2 - 7mn + 2n^2 = 0$

$\Rightarrow 6m^2 - 3mn - 4mn + 2n^2 = 0$

$\Rightarrow 3m(2m - n) - 2n(2m - n) = 0$

$\Rightarrow (3m - 2n)(2m - n) = 0$

$\Rightarrow (3m - 2n) = 0, (2m - n) = 0$

Now,

$(3m - 2n) = 0$

$\Rightarrow 3m = 2n$

$\Rightarrow \quad \frac{m}{2} = \frac{n}{3}$

$\Rightarrow \quad \frac{m}{2} = \frac{n}{2} = \frac{5m - 3n}{5.2 - 3.3} = \frac{l}{1}$

$\Rightarrow \quad \frac{l}{1} = \frac{m}{2} = \frac{n}{3} = \frac{1}{\sqrt{14}}$

Also, $\quad 2m - n = 0$

$\Rightarrow \quad \frac{m}{1} = \frac{n}{2}$

$\frac{5m - 3n}{5.1 - 3.2} = \frac{l}{-1}$

$\Rightarrow \quad \frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{1}{\sqrt{6}}$

Hence, the direction cosines are

$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

23. We have,

$l + m + n = 0$

$\Rightarrow \quad n = -(l + m)$

Now,

$2lm + 2ln + mn = 0$

$\Rightarrow \quad 2lm + n(2l + m) = 0$

$\Rightarrow \quad 2lm - (l + m)(2l + m) = 0$

$\Rightarrow \quad (l + m)(2l + m) - 2lm = 0$

$\Rightarrow \quad (l^2 + m^2 + 3lm - 2lm) = 0$

$\Rightarrow \quad (l^2 + m^2 + lm) = 0$

$\Rightarrow \quad \left(\frac{l}{m}\right)^2 + \left(\frac{l}{m}\right) + 1 = 0$

Let its roots are $\frac{l_1}{m_1}, \frac{l_2}{m_2}$.

Thus, product of the roots = –1

$\Rightarrow \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = -1$

$\Rightarrow \quad \frac{l_1 l_2}{1} = \frac{m_1 m_2}{-1}$

$\Rightarrow \quad \frac{l_1 l_2}{1} = \frac{m_1 m_2}{-1} = \frac{(l_1 l_2 + m_1 m_2)}{(1 - 1)}$

$\Rightarrow \quad \frac{l_1 l_2}{1} = \frac{m_1 m_2}{-1} = \frac{n_1 n_2}{0}$

$\Rightarrow \quad \frac{l_1 l_2}{1} = \frac{m_1 m_2}{-1} = \frac{n_1 n_2}{0} = \frac{1}{\sqrt{2}}$

Let θ be the angle between them.

Then $\quad \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 = \theta$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

24. We have,

$al + bm + cn = 0$

$\Rightarrow \quad l = -\frac{(bm + cn)}{a}$

Now, $\quad ul^2 + vm^2 + wn^2 = 0.$

$\Rightarrow \quad u\left(\frac{bm + cn}{a}\right)^2 + vm^2 + wn^2 = 0$

$\Rightarrow \quad u(bm + cn)^2 + va^2m^2 + wa^2n^2 = 0.$

$\Rightarrow \quad u(b^2m^2 + c^2n^2 + 2bcmn) + va^2m^2 + wa^2n^2 = 0$

$\Rightarrow \quad (ub^2 + va^2)m^2 + 2ubcmn + (uc^2 + wa^2)n^2 = 0$

$\Rightarrow \quad (ub^2 + va^2)\left(\frac{m}{n}\right)^2 + 2ubc\left(\frac{m}{n}\right) + (uc^2 + wa^2) = 0 \quad$ ...(i)

Let its roots are $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$.

(a) If two straight lines are parallel, the Eq. (i) will provide us equal roots.

Thus, $D = 0$

$\Rightarrow \quad (2ubc)^2 - 4(b^2u + a^2v)(c^2u + a^2w) = 0$

$\Rightarrow \quad (ubc)^2 - (b^2u + a^2v)(c^2u + a^2w) = 0$

$\Rightarrow \quad (b^2u + a^2v)(c^2u + a^2w) - (ubc)^2 = 0$

$\Rightarrow \quad a^2b^2uw + a^2c^2uv + a^4vw = 0$

$\Rightarrow \quad b^2uw + c^2uv + a^2vw = 0$

$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$

(b) When two straight lines are perpendicular, the, product of the roots $= \frac{uc^2 + wa^2}{ub^2 + va^2}$

$\Rightarrow \quad \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{uc^2 + wa^2}{ub_2 + va^2}$

$\Rightarrow \quad \frac{m_1 m_2}{uc^2 + wa^2} = \frac{n_1 n_2}{ub^2 + va^2}$

Similarly, eliminating n, we get

$\frac{l_1 l_2}{wb^2 + vc^2} = \frac{m_1 m_2}{uc^2 + wa^2}$

Thus, $\frac{l_1 l_2}{wb^2 + vc^2} = \frac{m_1 m_2}{uc^2 + wa^2} = \frac{n_1 n_2}{ub^2 + va^2}$

Two lines are perpendicular, if

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow \quad (wb^2 + vc^2) + (uc^2 + wa^2) + (ub^2 + va^2) = 0$

$\Rightarrow \quad a^2(u + v) + b^2(u + w) + c^2(u + v) = 0$

Hence, the result.

25. We have

$l^2 + m^2 + n^2 = 1$

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

$\Rightarrow \quad (\delta l)^2 + (\delta m)^2 + (\delta n)^2 + 2(l \cdot \delta l + m\delta m + n\delta n) = 0$

Now, $\delta\theta$ be the angle between two positions.

Then $\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$$= 1 + (l \cdot \delta + m\delta m + n\delta n)$$

$$= 1 - \left(\frac{(\delta l)^2 + (\delta m)^2 + (\delta n)^2}{2}\right)$$

$$\Rightarrow \left(\frac{(\delta l)^2 + (\delta m)^2 + (\delta n)^2}{2}\right) = 1 - \cos(\delta\theta)$$

$$\Rightarrow (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 2(1 - \cos(\delta\theta))$$

$$= 4\left(\sin^2\left(\frac{\delta\theta}{2}\right)\right)$$

$$= 4 \cdot \left(\frac{\delta\theta}{2}\right)^2$$

$$\left(\text{since } \delta\theta \text{ is very small, } \Rightarrow \sin\left(\frac{\delta\theta}{2}\right) \to \left(\frac{\delta\theta}{2}\right)\right)$$

$$= (\delta\theta)^2$$

$$= (\delta n)^2$$

26. Equation of the line passing through (2, 3, 4) and parallel to the given line is $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$

27. Equation of a line passing through (–3, 2, –4) and parallel to the given vector is $\frac{x+3}{2} = \frac{y-2}{3} = \frac{z+4}{7}$.

28. Any point on the given line can be considered as $(2\lambda + 1, 2 - 3\lambda, 4\lambda - 3)$. It is also a point on the plane. So,

$$2(2\lambda + 1) - 2(2 - 3\lambda) - (4\lambda - 3) = 7$$

$$\Rightarrow 4\lambda + 6\lambda - 4\lambda = 7 - 2 + 4 - 3$$

$$\Rightarrow 6\lambda = 6$$

$$\Rightarrow \lambda = 1$$

Hence, the co-ordinates of the plane is (3, –1, 1).

29. Equation of a line passing through (a, b, c) and parallel to $x = 2y = 3z$ is

$$\frac{x-a}{1} = \frac{y-b}{1/2} = \frac{z-c}{1/3}$$

which is passing through $3x - 4y + 5z = 10$, and $2x + 2y - 3z = 4$.

So, $3a - 4b + 5c = 0$

$$2a + 2b - 2c = 0$$

Solving, we get

$$\frac{a}{8-10} = \frac{b}{10+6} = \frac{c}{6+8}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{16} = \frac{c}{14}$$

$$\Rightarrow \frac{a}{-1} = \frac{b}{8} = \frac{c}{7}$$

Hence, the equation of the line is

$$\frac{x+1}{1} = \frac{y-8}{1/2} = \frac{z-7}{1/3}$$

30. The given lines $4x = 3y = -z$ and $3x = y = -4z$ can be written as

$$L_1: \frac{x}{3} = \frac{y}{4} = \frac{z}{-12}$$

$$L_2: \frac{x}{4} = \frac{y}{-12} = \frac{z}{-3}$$

Now, $a_1a_2 + b_1b_2 + c_1c_2 = 12 - 48 + 36 = 0$

Hence, the lines are perpendicular.

31. The given lines can be written as

$$L_1: \frac{x-b}{a} = \frac{y-d}{c} = \frac{z}{1}$$

and $L_2: \frac{x-b'}{a'} = \frac{y-d'}{c'} = \frac{z}{1}$

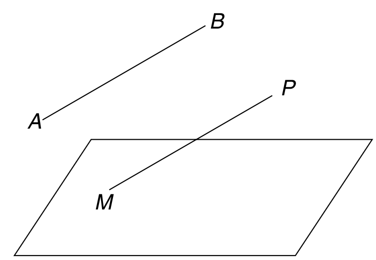
Since the given lines are perpendicular, so

$$a \cdot a' + c \cdot c' + 1 = 0$$

$$\Rightarrow a \cdot a' + c \cdot c' = -1.$$

32. Equation of any line passing through (1, –2, 3) and parallel to $\frac{x}{2} = \frac{y}{2} = \frac{z}{-6}$ is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6}$$



Any point on the given line can be considered as $M(2\lambda + 1, 3\lambda - 2, 3 - 6\lambda)$ which is also a point of the plane. So,

$$(2\lambda + 1) - (3\lambda - 2) + (3 - 6\lambda) = 5$$

$$\Rightarrow -7\lambda + 6 = 5$$

$$\Rightarrow \lambda = \frac{1}{7}$$

Hence, the point M is $\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$.

Hence, the required distance

$$= \sqrt{\left(\frac{9}{5} - 1\right)^2 + \left(-\frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2}$$

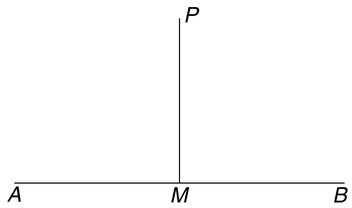
$$= \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}$$

$$= \sqrt{\frac{49}{49}}$$

$$= 1.$$

33. Given line AB is

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}.$$



Any point on the given line can be considered as $M(3\lambda + 6, 2\lambda + 7, 7 - 2\lambda)$.

Let the point P be (1, 2, 3).

Therefore, the direction ratios of PM

$$= (3\lambda + 6 - 1, 2\lambda + 7 - 2, 7 - 2\lambda - 3)$$

$$= (3\lambda + 5, 2\lambda + 5, 4 - 2\lambda)$$

Since PM is perpendicular to the given line, so

$$3(3\lambda + 5) + 2(2\lambda + 5) - 2(4 - 2\lambda) = 0$$

$\Rightarrow \quad 17\lambda + 17 = 0$

$\Rightarrow \quad \lambda = -1$

Thus, the point M is (3, 5, 9).

Therefore, the length of perpendicular

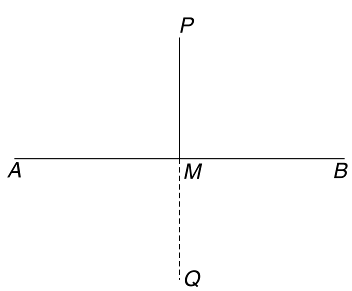
$$= \sqrt{(3-1)^2 + (5-2)^2 + (9-3)^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49} = 7$$

34. Let $AB: \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Any point on the given line AB is $(\lambda, 2\lambda + 1, 3\lambda + 2)$.



Now, the direction ratios of PM

$$= (\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3)$$

$$= (\lambda - 1, 2\lambda - 5, 3\lambda - 1)$$

Since PM is perpendicular to AB, so

$$(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$$

$\Rightarrow \quad 14\lambda = 14$

$\Rightarrow \quad \lambda = 1.$

Hence, the point M is (1, 3, 5).

Let $Q(\alpha, \beta, \gamma)$ be the image of P.

Therefore,

$$\frac{\alpha+1}{2} = 1, \ \frac{\beta+6}{2} = 3, \ \frac{\gamma+3}{2} = 5$$

$\Rightarrow \quad \alpha = 1, \beta = 0, \gamma = 7$

Hence, the required image of (1, 6, 3) is (1, 0, 7).

35. The equation of any line passing through (1, –1, 0) with the direction ratios a, b and c is

$$\frac{x-1}{a} = \frac{y+1}{b} = \frac{z-0}{c} \quad \text{...(i)}$$

which is orthogonal to $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-3}{4}$

and $\quad \frac{x-4}{4} = \frac{y}{5} = \frac{z+1}{2}.$

So, $\quad 2a + 3b + 4c = 0$

and $\quad 4a + 5b + 2c = 0$

Solving, we get

$$\frac{a}{6-20} = \frac{b}{16-4} = \frac{c}{10-12}$$

$\Rightarrow \quad \frac{a}{-4} = \frac{b}{12} = \frac{c}{-2}$

$\Rightarrow \quad \frac{a}{2} = \frac{b}{-6} = \frac{c}{1}$

Hence, the equation of the required line is

$$\frac{x-1}{2} = \frac{y+3}{b} = \frac{z-1}{c}$$

36. The equation of any line passing through (2, –3, 1) with the direction ratios

$$\frac{x-2}{a} = \frac{y+3}{b} = \frac{z-1}{c} \quad \text{...(i)}$$

The line (i) is parallel to $2x - y + z = 6$

and perpendicular to $\frac{x-2}{2} = \frac{y}{-3} = \frac{z-2}{-1}$

So, $\quad 2a - b + c = 0$

and $\quad 2a - 3b - c = 0$

Solving, we get

$$\frac{a}{1+3} = \frac{b}{2+2} = \frac{c}{-6+2}$$

$\Rightarrow \quad \frac{a}{4} = \frac{b}{4} = \frac{c}{-4}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$

$\Rightarrow \quad \frac{x-1}{1} = \frac{y+3}{1} = 1 - z$

37. Here, $\quad \mathbf{r}_1 = 3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$

$\quad \mathbf{r}_2 = -3\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$

and $\quad \mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\quad \mathbf{v} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

Now, $(\mathbf{r}_2 - \mathbf{r}_1) = -6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k}$$

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{36 + 225 + 9} = \sqrt{270} = 3\sqrt{30}$$

Also, $(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})$

$= (-6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k})\cdot(-6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k})$

$= 36 + 225 + 9 = 270$

Hence, the required shortest distance

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}$$

$$= \frac{270}{\sqrt{270}} = \sqrt{270}$$

38. Here, $\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$\mathbf{r}_2 = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

and $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

Now, $(\mathbf{r}_2 - \mathbf{r}_1) = -2\mathbf{j} - 2\mathbf{k}$

and $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 5 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 17\mathbf{i} + \mathbf{j} - 11\mathbf{k}$$

$\Rightarrow$ $|(\mathbf{u} \times \mathbf{v})| = \sqrt{289 + 1 + 121} = \sqrt{411}$

Hence, the required shortest distance

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}$$

$$= \frac{20}{\sqrt{411}}.$$

39. The equation of the given lines

$$\mathbf{y} + \mathbf{z} = 0,\ \mathbf{x} + \mathbf{z} = 0,\ \mathbf{x} + \mathbf{y} = 0$$

$x + y + z = \sqrt{3}a$ can be written as

$$L_1: \frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

$$L_2: \frac{x}{1} = \frac{y}{-1} = \frac{z}{-1}$$

Here, $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{v} = \mathbf{i} - \mathbf{j}$

Now, $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= -\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$\Rightarrow$ $|(\mathbf{u} \times \mathbf{v})| = \sqrt{1 + 1 + 4} = \sqrt{6}$

Also, $(\mathbf{r}_2 - \mathbf{r}_1) = \sqrt{3}a\mathbf{k}$

Hence, the required shortest distance

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}$$

$$= \frac{2\sqrt{3}a}{\sqrt{6}} = a\sqrt{2}$$

40. Given lines $\frac{y}{b} = \frac{z}{c} = 1,\ x = 0$ and $\frac{x}{a} - \frac{z}{c} = 1,\ y = 0$ can be written as

$$L_1: \frac{x}{0} = \frac{y - b}{b} = \frac{z}{-c}$$

and $$L_2: \frac{x - a}{a} = \frac{y}{0} = \frac{z}{c}$$

Here, $\mathbf{r}_1 = b\mathbf{j}$

$\mathbf{r}_2 = a\mathbf{i}$

and $\mathbf{u} = b\mathbf{j} - c\mathbf{k}$

$\mathbf{v} = a\mathbf{i} + c\mathbf{k}$

Now, $(\mathbf{r}_2 - \mathbf{r}_1) = ai - b\mathbf{j}$

and $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & b & -c \\ a & 0 & c \end{vmatrix}$$

$$= bc\mathbf{i} - ac\mathbf{j} - ab\mathbf{k}$$

$\Rightarrow$ $|(\mathbf{u} \times \mathbf{v})| = \sqrt{(bc)^2 + (ac)^2 + (ab)^2}$

Given shortest distance $= 2d$

$$\Rightarrow \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|} = 2d$$

$$\Rightarrow \frac{abc + abc}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}} = 2d$$

$$\Rightarrow \frac{2abc}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}} = 2d$$

$$\Rightarrow \frac{abc}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}} = d$$

$$\Rightarrow \frac{(abc)^2}{((bc)^2 + (ac)^2 + (ab)^2)} = d^2$$

$$\Rightarrow \frac{1}{d^2} = \frac{((bc)^2 + (ac)^2 + (ab)^2)}{(abc)^2}$$

$$\Rightarrow \frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

41. Hence, the equation of the plane is

$$\begin{vmatrix} x - 1 & y - 1 & z - 1 \\ 2 & -2 & 1 \\ -4 & 4 & -5 \end{vmatrix} = 0$$

$\Rightarrow \quad 6(x - 1) + 6(y - 1) = 0$

$\Rightarrow \quad x + y = 2$

42. Hence, the equation of the plane is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & -1 & 1 \\ 2 & -3 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad 3(x - 1) + 2(y - 2) - (z - 3) = 0$

$\Rightarrow \quad 3x + 2y - z = 4$

Now, $\quad 3.2 + 2.2 - 6 - 4 = 6 + 4 - 10 = 0$

Therefore, the given points are coplanar.

43. Given plane is $2x - 4y + 5z = 20$

$\Rightarrow \quad \frac{x}{10} + \frac{y}{-5} + \frac{z}{4} = 1$

which is the required intercept form of the plane.

44. The equation of any plane passing through (3, 0, 0), (0, 4, 0) and (0, 0, 5) is

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 1$$

45. Let the equation of the plane be

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where

$a + b + c = 0$ ...(i)

which is passing through $(0, 4, -3)$, $(6, -4, 3)$. So

$$\frac{4}{b} - \frac{3}{c} = 1$$

and $\quad \frac{6}{a} - \frac{4}{b} + \frac{3}{c} = 1$

Solving, we get

$a = 3, b = -2, c = -1$ or $a = 3, b = 6, c = -9$

Hence, the equation of the plane is

$$\frac{x}{3} - \frac{y}{2} - \frac{z}{1} = 1$$

or $\quad \frac{x}{3} + \frac{y}{6} - \frac{z}{9} = 1$

46. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Clealy, $\quad \frac{a}{3} = p, \frac{b}{3} = q, \frac{c}{3} = r$

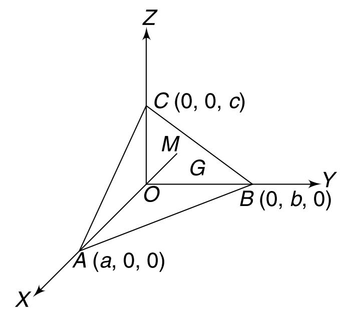
$\Rightarrow \quad a = 3p, b = 3q, c = 3r$

Hence, the equation of the plane is

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

47. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$



Let the centroid G be (α, β, γ).

Clearly, $\quad a = 3\alpha, b = 3\beta, c = 3\gamma$

It is given that, $OM = 3p$

$\Rightarrow \quad \left| \frac{0 + 0 + 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \right| = 3p$

$\Rightarrow \quad \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) = \frac{1}{9p^2}$

$\Rightarrow \quad \left(\frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2} \right) = \frac{1}{9p^2}$

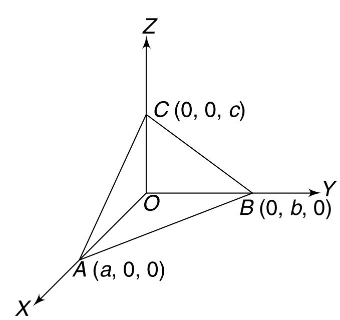
$\Rightarrow \quad \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \right) = \frac{1}{p^2}$

Hence, the locus of the centroid is

$$\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) = \frac{1}{p^2}$$

48. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



Thus, the area of ΔABC

$$= \frac{1}{2} |(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})|$$

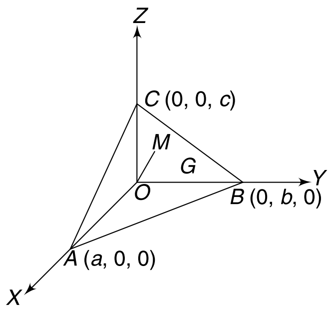
$$= \frac{1}{2} |(\vec{a}\hat{i} \times \vec{b}\hat{j} + \vec{b}\hat{j} \times \vec{c}\hat{k} + \vec{c}\hat{k} \times \vec{a}\hat{i})|$$

$$= \frac{1}{2} |(ab)\hat{k} + (bc)\hat{i} + (ca)\hat{j}|$$

$$= \frac{1}{2} \sqrt{(ab)^2 + (bc)^2 + (ca)^2}$$

49. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



Let the centroid be $G(\alpha, \beta, \gamma)$

Clearly, $\frac{a}{4} = \alpha, \frac{b}{4} = \beta, \frac{c}{4} = \gamma$

$\Rightarrow \quad a = 4\alpha, b = 4\beta, c = 4\gamma$

It is given that $OM = p$

$$\Rightarrow \quad \left|\frac{0 + 0 + 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right| = p$$

$$\Rightarrow \quad \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \frac{1}{p^2}$$

$$\Rightarrow \quad \left(\frac{1}{(4\alpha)^2} + \frac{1}{(4\beta)^2} + \frac{1}{(4\gamma)^2}\right) = \frac{1}{p^2}$$

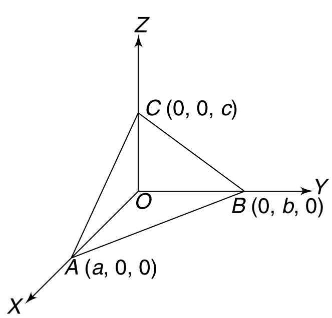
$$\Rightarrow \quad \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) = \frac{16}{p^2}$$

Hence, the locus of $G(\alpha, \beta, \gamma)$ is

$$\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = \frac{16}{p^2}.$$

50. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



It is given that, the volume of the tetrahedron

$$OABC = 64k^3$$

$$\Rightarrow \quad \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 64k^3$$

$$\Rightarrow \quad \frac{1}{6}(abc) = 64k^3$$

$$\Rightarrow \quad (abc) = 384\ k^3$$

Let the centroid be $G(\alpha, \beta, \gamma)$.

Clearly,

$$\frac{a}{4} = \alpha, \frac{b}{4} = \beta, \frac{c}{4} = \gamma$$

$$\Rightarrow \quad a = 4\alpha, b = 4\beta, c = 4\gamma$$

Therefore,

$$64(\alpha\ \beta\ \gamma) = 384k^3$$

$$\Rightarrow \quad (\alpha\ \beta\ \gamma) = 6k^3$$

Hence, the locus of $G(\alpha, \beta, \gamma)$ is

$$xyz = 6k^3$$

51. Let θ the angle between them. Thus,

$$\cos(\theta) = \frac{2 - 2 + 4}{\sqrt{1 + 4 + 4}\sqrt{4 + 1 + 4}}$$

$$= \frac{4}{9}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{4}{9}\right).$$

52. We have

$$a_1b_1 + a_2b_2 + a_3b_3$$
$$= 6 + 6 - 12$$
$$= 0$$

Thus, the given planes are perpendicular to each other.

53. Let the equation of the plane passing through (1, 2, 3,) is $a(x - 1) + b(y - 2) + c(z - 3) = 0$

which is perpendicular to the given planes

$2x + 3y + 4z + 7 = 0$ and $3x + 4y + 5z + 10 = 0$

So, $\quad 2a + 3b + 4c = 0$

and $\quad 3a + 4b + 3c = 0$

Solving, we get

$$\frac{a}{15 - 16} = \frac{b}{12 - 10} = \frac{c}{8 - 9}$$

$$\Rightarrow \quad \frac{a}{-1} = \frac{b}{2} = \frac{c}{-1}$$

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$$

Hence, the equation of the plane is

$$(x - 1) - 2(y - 2) + (z - 3) = 0$$

$$\Rightarrow \quad x - 2y + z = 0$$

54. Do yourself.
55. Do yourself.
56. Equation of any plane parallel to $x - 2y + 3z + 10 = 0$ is $x - 2y + 3z + k = 0$ which is passing through (2, 3, –1). So,
$$2 - 6 - 3 + k = 0$$
$$\Rightarrow \quad k = 7$$
Hence, the equation of the plane is $x - 2y + 3z + 7 = 0$.
57. Let the equation of the plane be $by + cz + d = 0$. which is passing through the points (2, 3, –4) and (1, –1, 3).
Thus, $3b - 4c + d = 0$
and $b - c + d = 0$
Solving, we get
$$\frac{b}{-4+1} = \frac{c}{1-3} = \frac{d}{-3+4}$$
$$\Rightarrow \quad \frac{b}{-3} = \frac{c}{-2} = \frac{d}{1}$$
$$\Rightarrow \quad \frac{b}{3} = \frac{c}{2} = \frac{d}{-1}$$
Hence, the equation of the plane is
$$3y + 2z - 1 = 0$$
58. Do yourself
59. Do yourself.
60. The equation of any plane parallel to xy-plane is $z + k = 0$
which is passing through (1, 2, 3). So,
$$3 + k = 0$$
$$\Rightarrow \quad k = -3$$
Hence, the equation of the plane is $z = 3$.
61. Do yourself.
62. Do yourself.
63. Given line is
$$2x = 3y = 4z$$
$$\Rightarrow \quad \frac{2x}{12} = \frac{3y}{12} = \frac{4z}{12}$$
$$\Rightarrow \quad \frac{x}{6} = \frac{y}{4} = \frac{z}{3}$$
Hence, the required equation of the plane is
$$\begin{vmatrix} x-2 & y+1 & z \\ 1 & -3 & -5 \\ 6 & 4 & 3 \end{vmatrix} = 0$$
$$\Rightarrow \quad 11(x-2) - 33(y+1) + 22z = 0$$
$$\Rightarrow \quad (x-2) - 3(y+1) + 2z = 0$$
$$\Rightarrow \quad x - 3y + 2z = 5.$$
64. Given lines are
$$x = 2y = 3z \text{ and } 2x = 5y = z.$$
$$\Rightarrow \quad \frac{x}{6} = \frac{y}{3} = \frac{z}{2} \text{ and } \frac{x}{5} = \frac{y}{2} = \frac{z}{10}$$
Hence, the equation of the plane passing through (2, 3, 4) and parallel to the above lines is
$$\begin{vmatrix} x-2 & y-3 & z-4 \\ 6 & 3 & 2 \\ 5 & 2 & 10 \end{vmatrix} = 0$$
$$\Rightarrow \quad 22(x-2) - 50(y-3) - 3(z-4) = 0$$
$$\Rightarrow \quad 26x - 50y - 3z = 110.$$
65. The equation of any plane passing through the line of intersection of planes
$2x + 5y - 5z = 6$ and $2x + 7y - 8z = 7$ is
$(2x + 5y - 5z - 6) + \lambda(2x + 7y - 8z - 7) = 0$
which is passing through (–1, 4, 3).
$$\Rightarrow \quad (-2 + 20 - 15 - 6) + \lambda(-2 + 28 - 24 - 7) = 0$$
$$\Rightarrow \quad -3 - 5\lambda = 0$$
$$\Rightarrow \quad \lambda = -\frac{3}{5}$$
Hence, the required equation of the plane is
$$(2x + 5y - 5z - 6) - \frac{3}{5}(2x + 7y - 8z - 7) = 0$$
$$\Rightarrow \quad 4x + 4y - z + 51 = 0.$$
66. The equation of any plane passing through the line of intersection of the planes $4x - 3y + 5 = 0$ and $y - 2z - 5. = 0$ is $(4x - 3y + 5) + \lambda(y - 2z - 5) = 0$ which is passing through (2, –1, 1) so,
$$\lambda = 1/2.$$
Hence, the equation of the required plane is
$$(4x - 3y + 5) + \frac{1}{2}(y - 2z - 5) = 0$$
$$\Rightarrow \quad 8x - 5y - 2z + 5 = 0$$
67. The equation of any plane passing through the line of intersection of the planes $2x + 3y + 10z = 8$ and $2x - 3y + 7z = 2$
is
$$(2x + 3y + 10z - 8) + \lambda(2x - 3y + 7z - 2) = 0$$
$$\Rightarrow \quad (2 + 2\lambda)x + (3 - 3\lambda)y + (10 + 7\lambda)z - (2\lambda + 8) = 0$$
which is perpendicular to $3x - 2y + 4z = 5$. So
$$3(2 + 2\lambda) - 2(3 - 3\lambda) + 4(10 + 7\lambda) = 0$$
$$\Rightarrow \quad 40 + 40\lambda = 0$$
$$\Rightarrow \quad \lambda = -1$$
Hence, the equation of the plane is
$$(2x + 3y + 10z - 8) - (2x - 3y + 7z - 2) = 0$$
$$\Rightarrow \quad 6y + 3z - 6 = 0$$
$$\Rightarrow \quad 2y + z - 2 = 0$$
68. Equation of any plane through the intersection of the planes $x + y + z + 3 = 0$, and $2x - y + 3z + 1 = 0$ is

$(x + y + z + 3) + \lambda(2x - y + 3z + 1) = 0$

$\Rightarrow \quad (1 + 2\lambda)x + (1 - \lambda)y + (3\lambda + 1)z + (\lambda + 3) = 0$

which is parallel to $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.

So, $\quad (1 + 2\lambda) + 2(1 - \lambda) + 3(3\lambda + 1) = 0$

$\Rightarrow \quad 9\lambda - 6$

$\Rightarrow \quad \lambda = -\frac{2}{3}$

Hence, the equation of the plane is

$$(x + y + z + 3) - \frac{2}{3}(2x - y + 3z + 1) = 0$$

$\Rightarrow \quad 3(x + y + z + 3) - 2(2x - y + 3z + 1) = 0$

$\Rightarrow \quad -x + 5y - 3z + 7 = 0$

$\Rightarrow \quad x - 5y + 3z - 7 = 0.$

69. Hence, the length of the perpendicular from $P(1, 2, 3)$ to the plane $5x + 4y - 3z - 1 = 0$ is

$$\left|\frac{5 + 8 - 9 - 1}{\sqrt{25 + 16 + 9}}\right| = \frac{3}{5\sqrt{2}}$$

70. Equation of any line passing through (–1, 3, 2) and normal to the given plane is

$$\frac{x + 1}{1} = \frac{y - 3}{2} = \frac{z - 2}{2}$$

Any point on the above line can be written as $(\lambda - 1, 2\lambda + 3, 2\lambda + 2)$ which lies in the given plane. So,

$$(\lambda - 1) + 2(2\lambda + 3) + 2(2\lambda + 2) = 3$$

$\Rightarrow \quad 9\lambda = -6$

$\Rightarrow \quad \lambda = -\frac{2}{3}$

Hence, the co-ordinates of the feet of the perpendicular are $\left(-\frac{5}{3}, \frac{5}{3}, \frac{2}{3}\right)$.

71. The equation of any line through (1, 2, 3) parallel to the given line is $\frac{x - 1}{1} = \frac{y - 2}{-2} = \frac{z - 3}{2}$.

Any point on the above line can be considered as $(\lambda + 1, 2 - 2\lambda, 2\lambda + 3$

which is also a common point of the plane.

Thus, $\quad \lambda + 1 + 2 - 2\lambda + 2\lambda + 3 = 11$

$\lambda = 5$

Therefore, the point is (6, –8, 13).

Hence, the required distance

$$= \sqrt{(6 - 1)^2 + (-8 - 2)^2 + (13 - 3)^2}$$

$$= \sqrt{25 + 100 + 100}$$

$$= \sqrt{225}$$

$$= 15$$

72. Let the point be $p(\alpha, \beta, \gamma)$

It is given that,

$$PM_1^2 + PM_2^2 + PM_3^2 = 9$$

$$\Rightarrow \left(\frac{\alpha + \beta + \gamma}{\sqrt{3}}\right)^2 + \left(\frac{\alpha - \gamma}{\sqrt{2}}\right)^2 + \left(\frac{\alpha - 2\beta + \gamma}{\sqrt{3}}\right)^2 = 9$$

$\Rightarrow \quad 2(\alpha + \beta + \gamma)^2 + 3(\alpha - \gamma)^2 + 2(\alpha - 2\beta + \gamma)^2 = 9$

$\Rightarrow \quad 7\alpha^2 + 10\beta^2 + 7\gamma^2 + 2\alpha\gamma - 4\alpha\beta - 4\beta\gamma = 9$

Hence, the locus of $p(\alpha, \beta, \gamma)$

$= 7x^2 + 10y^2 + 7z^2 + 2xz - 4xy - 4yz = 9.$

73. Let the equation of the planes be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$

Let OM and ON be the distances from the origin to the given planes. Thus

$$OM = ON$$

$$\Rightarrow \quad \left|\frac{0 + 0 + 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right| = \left|\frac{0 + 0 + 0 - 1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}}\right|$$

$$\Rightarrow \quad \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}\right)$$

Hence, the result.

74. Let a, b, c be the direction ratios of the line.

Since, the line is perpendicular to the normals to the plane, so

$$a + b + c = 0$$

and $\quad a - 2b - 2c = 0$

Solving, we get

$$\frac{a}{-2 + 2} = \frac{b}{1 + 2} = \frac{c}{-2 - 1}$$

$$\Rightarrow \quad \frac{a}{0} = \frac{b}{3} = \frac{c}{-3}$$

Let the point of intersection of the line with $z = 0$ plane.

Putting $z = 0$ in the given equations we get

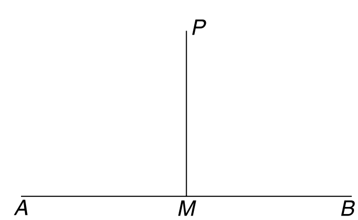
$$x + y = 4, \ x - 2y = 4$$

Solving, we get

$$x = 4, \ y = 0$$

Hence, the equation of the line is

$$\frac{x - 4}{0} = \frac{y}{3} = \frac{z}{-3}$$



Any point on the line be $M(4, 3\lambda, -3\lambda)$.

Now, the direction ratios of PM be $(0, 3\lambda - 1, -3\lambda - 1)$

Here, $PM \perp AB$

So, $0 + 3(3\lambda - 1) - 3(-3\lambda - 1) = 0$

$\Rightarrow \lambda = 0$

Hence, the point M is (4, 0, 0).

Therefore, the required distance

$$= \sqrt{0 + 1^2 + 1^2} = \sqrt{2}.$$

75. Hence the required distance between the planes

$$x + 2y - 2z + 1 = 0$$

and $2x + 4y - 4z + 5 = 0$

is $\left|\dfrac{(5/2) - 1}{\sqrt{1 + 4 + 4}}\right| = \dfrac{3/2}{3} = \dfrac{1}{2}$

76. Any point on the given line

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$$

can be considered as $(2\lambda + 1, 2 - 3\lambda, 4\lambda - 3)$ which is the point of intersection. So it is also a point of the given plane.

Thus,

$$2(2\lambda + 1) - 2(2 - 3\lambda) - (4\lambda - 3) = 7$$

$\Rightarrow 6\lambda = 6$

$\Rightarrow \lambda = 1.$

Hence, the required point is (3, –1, 1).

77. Equation of any plane passing through (1, 2, 0) is

$$a(x - 1) + b(y - 2) + c(z - 0) = 0 \quad \text{...(i)}$$

which contains the given line. So,

$$a(-3 - 1) + b(1 - 2) + c(2 - 0) = 0$$

$\Rightarrow 4a + b - 2c = 0$ ...(ii)

and $3a + 4b - 2c = 0$ ...(iii)

Solving Eqs. (ii) and (iii), we get

$$\frac{a}{6} = \frac{b}{2} = \frac{c}{13}$$

Hence, the equation of the plane is

$$6(x - 1) + 2(y - 2) + 13(z - 0) = 0$$

$\Rightarrow 6x + 2y + 13z - 10 = 0.$

78. Now, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$$= \begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$= 0$

Hence, the given lines are coplanar.

Now, the equation of the plane where they lie is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$\Rightarrow -(x - 1) + 2(y - 2) - (z - 3) = 0$

$\Rightarrow -x + 2y - z = 0$

$\Rightarrow x - 2y + z = 0$

79. The Equation of any plane passing through the line of intersection of the planes

$4x - 5y - 4z = 4$ and $2x + y + 2z = 8$

is $(4x - 5y - 4z - 4) + \lambda(2x + y + 2z - 8) = 0$

which is passing through (1, 2, 3). Thus

$$2 - 22k = 0$$

$\Rightarrow k = \dfrac{1}{11}$

Hence, the equation of the plane is

$$(4x - 5y - 4z - 4) + \frac{1}{11}(2x + y + 2z - 8) = 0.$$

$\Rightarrow 13x + 3 \cdot y + 9z = 46.$

80. The given planes can be written as

$$-x - 2y - 2z + 3 = 0$$

and $3x + 4y + 12z + 1 = 0$

Now, $a_1a_2 + b_1b_2 + c_1c_2 = -3 - 8 - 24 < 0$

Thus, the acute angle bisector is

$$\left(\frac{-x - 2y - 2z + 3}{\sqrt{9}}\right) = -\left(\frac{3x + 4y + 12z + 1}{\sqrt{169}}\right)$$

$\Rightarrow 11x + 19y + 31z = 18.$

81. The equation of a line passing through (2, –3, 4) and normal to a plane is

$$\frac{x-2}{4} = \frac{y+3}{2} = \frac{z-4}{-4} \quad \text{....(i)}$$

Any point on Eq. (i) is

$$M(4\lambda + 2, 2\lambda - 3, 4 - 4\lambda)$$

Since the point M lies on the plane, so

$$4(4\lambda + 2) + 2(2\lambda - 3) - 4(4 - 4\lambda) + 3 = 0$$

$\Rightarrow 36\lambda = 11$

$\Rightarrow \lambda = \dfrac{11}{36}$

Hence, the M point $= \left(\dfrac{29}{9}, -\dfrac{33}{18}, \dfrac{25}{9}\right)$

Let $Q(\alpha, \beta, \gamma)$ be the image of $P(2, -3, 4)$.

Here, M is the mid-point of P and Q.

Thus, $\dfrac{\alpha + 2}{2} = \dfrac{29}{9}, \dfrac{\beta - 3}{2} = \dfrac{-33}{18}, \dfrac{\gamma + 4}{2} = \dfrac{25}{9}$

$\Rightarrow (\alpha, \beta, \gamma) = \left(\dfrac{40}{9}, -\dfrac{2}{3}, \dfrac{14}{9}\right)$

which is the required image of $P(2, -3, 4)$.

82. Given line is

$$\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3} \quad \text{....(i)}$$

and the plane is

$$3x-3y+10z=26. \quad \text{...(ii)}$$

The direction ratios of the line are 9, –1, –3 and the direction ratios of the normal to the given plane are 3, –3 and 10.

Clearly line (i) is parallel to the plane (ii).

Let Q be the inage of the point $P(1, 2, -3)$.

Consider $Q = (\alpha, \beta, \gamma)$

Now,

$$\frac{\alpha-1}{3}=\frac{\beta-2}{-3}=\frac{\gamma+3}{10}=-2\left(\frac{3-6-30-26}{9+9+100}\right)$$

$$\frac{\alpha-1}{3}=\frac{\beta-2}{-3}=\frac{\gamma+3}{10}=1$$

$$\alpha=4,\ \beta=-1,\ \gamma=7$$

Hence, the required image of the line w.r.t. the given plane is

$$\frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}$$

83. Any plane containing the line

$$\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2} \text{ is}$$

$$a(x-1)+b(y+6)+c(z+1)=0 \quad \text{...(i)}$$

where $3a+4b+2c=0$ ...(ii)

Also, it is parallel to the second line. So,

$$2a-3b+2c=0 \quad \text{...(iii)}$$

Solving Eq. (ii) and (iii), we get

$$\frac{a}{26}=\frac{b}{-11}=\frac{c}{-17}$$

Hence, the equation of the plane is

$$26(x-1)-11(y+6)-17(z+1)=0$$

$$\Rightarrow \quad 26x-11y-17z=109$$

84. The equation of the line containg the givel two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 3 & 5 \end{vmatrix}=0$$

$$\Rightarrow \quad 4(x+1)-8(y+3)+4(z+5)=0$$

$$\Rightarrow \quad 4x-8y+4z=0$$

$$\Rightarrow \quad x-2y+z=0$$

Level II

1. The shortest distance of the point (a, b, c) from x-axis

$$=\sqrt{(a-a)^2+(b-0)^2+(c-0)^2}$$

$$=\sqrt{b^2+c^2}$$

2. Equation of x-axis is given by $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

3. Clearly, $\frac{1}{c^2}+\frac{1}{c^2}+\frac{1}{c^2}=1$

$$\Rightarrow \quad \frac{3}{c^2}=1$$

$$\Rightarrow \quad c^2=3$$

$$\Rightarrow \quad c=\pm\sqrt{3}$$

4. Ans. (c)

5. Ans. (d)

6. Let two diagonals of a cube are $a\mathbf{i}+a\mathbf{j}+a\mathbf{k}$ and $a\mathbf{i}+a\mathbf{j}-a\mathbf{k}$ and θ be the angle between them. So,

$$\cos(\theta)=\frac{a^2+a^2-a^2}{a\sqrt{3}\times a\sqrt{3}}=\frac{1}{3}$$

$$\Rightarrow \quad \tan(\theta)=2\sqrt{2}$$

$$\Rightarrow \quad \theta=\tan^{-1}(2\sqrt{2})$$

7. Let the diagonal of a cube and a diagonal of a face of a cube are $a\mathbf{i}+a\mathbf{j}+a\mathbf{k}$ and $a\mathbf{i}+a\mathbf{j}$, and θ be the angle between them.

$$\therefore \quad \cos\theta=\frac{a^2+a^2}{a\sqrt{3}a\sqrt{2}}=\frac{2}{\sqrt{3}\sqrt{2}}=\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad \theta=\cos^{-1}\sqrt{\frac{2}{3}}$$

8. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$

$$=1-\cos^2\alpha+1\ \cos^2\beta+1-\cos^2\gamma$$

$$=3-(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$$

$$=3-1=2$$

9. $\cos(2\alpha)+\cos(2\beta)+\cos(2\gamma)+3$

$$=2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)-3+3$$

$$=2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$$

$$=2$$

11. Clearly, $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (2, 2, 2)$

$\Rightarrow \quad a = 6, b = 6, c = 6$

Hence, the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\Rightarrow \quad \frac{x}{6} + \frac{y}{6} + \frac{z}{6} = 1$$

$$x + y + z = 6$$

12. The locus of the first degree equation in x, y, z represents a plane.

13. Let θ be the angle between the plane and the x-axis

$$\sin(\theta) = \frac{2.1}{\sqrt{4+9+36}} = \frac{2}{\sqrt{49}} = \frac{2}{7}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{2}{7}\right)$$

$$\Rightarrow \quad k = \frac{2}{7}$$

14. Hence, the distance between two parallel planes

$$= \frac{7-5}{\sqrt{4+9+36}} = \frac{2}{\sqrt{49}} = \frac{2}{7}$$

15. The equation of any plane parallel to $2x + y + 2z + 10 = 0$ is $2x + y + 2z + k = 0$ which is passing through (1, 2, 3). So,

$$2 + 2 + 6 + k = 0$$

$$\Rightarrow \quad k = -10$$

Hence, the equation of the plane is

$$2x + y + 2z - 10 = 0.$$

16. Let θ be the angle between them, then

$$\cos\theta = \frac{2-1+2}{\sqrt{4+1+1}\sqrt{1+1+4}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

17. (c)

18. (a)

19. The equation of any line passing through (2, 3, 4) and perpendicular to the plane $2x + 3y - z = 5$ is

$$\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{-1}$$

20. Clearly, $3.2 + 4.1 - 5.2 = 10 - 10 = 0$

21. Let θ be the angle between them.

Now, $\sin(\theta) = \frac{6-6-4}{\sqrt{9+36+4}\sqrt{4+1+4}} = -\frac{4}{21}$

$$\Rightarrow \quad \theta = \sin^{-1}\left(-\frac{4}{21}\right)$$

22. Any point on the given line can be considered as $M(6 - \lambda, -1, 4\lambda - 3)$.

Since the point M lies on the plane, so

$$6 - \lambda - 1 - 4\lambda + 3 = 3$$

$$\Rightarrow \quad 8 - 5\lambda = 3$$

$$\Rightarrow \quad -5\lambda = -5$$

$$\Rightarrow \quad \lambda = 1$$

Hence, the point of intersection is $M = (5, -1, 1)$.

23. Hence, the shortest distance between the lines

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u}\times\mathbf{v})}{|(\mathbf{u}\times\mathbf{v})|}$$

$$= \frac{-1+4-2}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

where $(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

and

$$(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\Rightarrow \quad |(\mathbf{u}\times\mathbf{v})| = \sqrt{1+4+1} = \sqrt{6}$$

24. The equation of any plane passing through (3, 2, 0) is

$$a(x-3) + b(y-2) + cz = 0$$

The line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$ containing the plane,

if $\quad a + 5b + 4c = 0$
$\quad 0 + 4b + 4c = 0$

Solving, we get

$$\frac{a}{4} = \frac{b}{-4} = \frac{c}{4}$$

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{-1} = \frac{c}{1}$$

Hence, the equation of the plane is

$$1(x-3) - (y-2) + z = 0$$

$$\Rightarrow \quad x - y + z = 1.$$

25. Clearly, $\mathbf{PQ} = 12\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$$|\mathbf{PQ}| = \sqrt{144+9+16} = \sqrt{169} = 13$$

26. The equation of any plane passing through (1, 1, 1) and perpendicular to $\frac{x-1}{3} = \frac{y-1}{0} = \frac{z-1}{4}$ is

$$3(x-1) + 0(y-1) + 4(z-1) = 0$$

$$3x + 4z = 7$$

Hence, the distance from the origin

$$= \left|\frac{0+0-7}{\sqrt{9+16}}\right| = \frac{7}{5}$$

27. The equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Clearly, the distance from the origin to the plane is 1

$$\left| \frac{0 + 0 + 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \right| = 1$$

$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1$

$\Rightarrow \quad \frac{1}{(3x)^2} + \frac{1}{(3y)^2} + \frac{1}{(3z)^2} = 1$

$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9$

$\Rightarrow \quad k = 9$

28. Given lines are

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{4-z}{k}$$

and $\quad \frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$

Since the lines are coplanar, so

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (1 + 2k) - (1 + k^2) + (2 - k) = 0$

$\Rightarrow \quad 1 + 2k - 1 - k^2 + 2 - k = 0$

$\Rightarrow \quad k^2 - k - 2 = 0$

$\Rightarrow \quad (k - 2)(k + 1) = 0$

$\Rightarrow \quad k = 2, -1$

29. Clearly, $a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 = 20 > 0$

Hence, the obtuse angle bisector

$$\frac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}} = \frac{2x - y + 2z + 3}{\sqrt{4 + 1 + 4}}$$

$\Rightarrow \quad \frac{3x - 2y + 6z + 8}{7} = \frac{2x - y + 2z + 3}{3}$

$\Rightarrow \quad 5x - y - 4z = 3$

30. The equation of any plane passing through the intersection of the given planes

$$(x - y - z - 4) + \lambda(x + y + 2z - 4) = 0$$

and $\quad (\lambda + 1)x + (\lambda - 1)y + (2\lambda - 1)x = 4\lambda + 4$

which is perpendicular with $x - y - z = 4$. So,

$$(\lambda + 1) - (\lambda - 1) - (2\lambda - 1) = 0$$

$\Rightarrow \quad \lambda = \frac{3}{2}$

Hence, the equation of the new position of the plane is

$$5x + y + 4z = 20.$$

31. Let a, b and c be the direction ratios of the line. So

$$3a - 2b + c = 0$$

and $\quad 4a - 3b + 4c = 0$

Solving, we get

$\Rightarrow \quad \frac{a}{5} = \frac{b}{8} = \frac{c}{1}$

Since the line is parallel to the plane, so

$$10 - 8 + m = 0$$

$\Rightarrow \quad m = -2$

32. Given striaght line can written as

$$\frac{x-1}{1} = \frac{y-0}{3} = \frac{z-1}{-1}$$

Since the lies in the plane, so

$1 + 3 - c = 0$ and $1 + 0 + c = d$

$\Rightarrow \quad c = 4$ and $d = 5$

Hence, the value of

$$(c + d - 5) = 4$$

33. Let the equation of the plane be

$$a(x - 2) + b(y - 1) + c(z + 1) = 0$$

Since the plane contains the line L_1 and parallel to L_2, so

$$a + b + c = 0$$

and $\quad a + 0 + 2c = 0$

$\Rightarrow \quad \frac{a}{2} = \frac{b}{-1} = \frac{c}{-1}$

Hence, the equation of the plane is

$$2(x - 2) - (y - 1) + (z + 1) = 0$$

$\Rightarrow \quad 2x - 4 - y + 1 + z + 1 = 0$

$\Rightarrow \quad 2x - y + z = 2$

Hence, the distance from the origin

$$= \frac{2}{\sqrt{4 + 1 + 1}} = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

34. Since the three planes pass through the same line, so

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow \quad (1 - a^2) - c^2 - abc - abc - b^2 = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc + 2 = 1 + 2 = 3$

35. The equation of any plane through (0, 2, 4) is $a(x-0)+b(y-2)+c(z-4)=0$ which containing the line

$$\frac{x+3}{3}=\frac{y-1}{4}=\frac{2-z}{2}$$

So, $3a+4b-2c=0$

and $-3a-b-2c=0$

Solving, we get

$$\frac{a}{-10}=\frac{b}{12}=\frac{c}{9}$$

Hence, the equation of the plane is

$$-10x+12(y-2)+9(z-4)=0$$

$$\Rightarrow \quad -10x+12y-24+9z-36=0$$

$$\Rightarrow \quad 10x-12y-9z+60=0$$

36. Ans. (d)

37. Since the line $\frac{x-2}{3}=\frac{1-y}{1}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$, so

$$3-3-2\alpha=0$$

and $2+3+2\alpha+\beta=0$

So, $\alpha=0,\ \beta=-5$

Hence, the value of

$$(\alpha+\beta+7)=2$$

38. Let the point of intersection is $(\lambda, 2\lambda, 3\lambda)$.

Clearly, $\lambda=3\mu+1,\ 2\lambda=2-\mu$

Solving, we get

$$\lambda=1,\ \mu=0$$

Hence, the point of intersection is (1, 2, 3).

Therefore, $\frac{1+k}{3}=\frac{2-1}{2}=\frac{3-2}{h}$

$$\Rightarrow \quad \frac{1+k}{3}=\frac{1}{2}=\frac{1}{h}$$

$$\Rightarrow \quad h=2,\ k=\frac{3}{2}-1=\frac{1}{2}$$

39. Given four points are

$$P(2-x, 2, 2),\ Q(2, 2-y, 2)$$

$$R(2, 2, 2, -z) \text{ and } S(1, 1, 1)$$

Now, $\mathbf{PQ}=(x, -y, 0)$

$\mathbf{PR}=(x, 0, -z)$

and $\mathbf{PS}=(x-1, -1, -1)$

It is given that, the vectors P, Q, R and S are coplanar, so the vectors **PQ**, **PR**, **PS** are coplanar.

$$\begin{vmatrix} x & -y & 0 \\ x & 0 & -z \\ x-1 & -1 & -1 \end{vmatrix}=0$$

$$\Rightarrow \quad x(0-z)+y(-x+xz-z)=0$$

$$\Rightarrow \quad -xz-xy+xyz-yz=0$$

$$\Rightarrow \quad xy+yz+zx=xyz$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

40. The equation of the line joining the points (2, 3, 4) and (6, 7, 8) is

$$\frac{x-2}{6-2}=\frac{y-3}{7-3}=\frac{z-4}{8-4}$$

$$\Rightarrow \quad \frac{x-2}{4}=\frac{y-3}{4}=\frac{z-4}{4}$$

$$\Rightarrow \quad \frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$$

The mid-point of the line segment joining the points (2, 3, 4) and (6, 7, 8) is (4, 5, 6).

Hence, the equation of the right bisector of the plane

$$1(x-4)+1(y-5)+1(z-6)=0$$

$$\Rightarrow \quad x+y+z=15$$

Comprehensive Link Passages

Passage I

1. Given,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

$$\Rightarrow \quad \cos^2\alpha+\cos^2(90°-\alpha)+\cos^2\gamma=1$$

$$\Rightarrow \quad \cos^2\alpha+\sin^2\alpha+\cos^2\gamma=1$$

$$\Rightarrow \quad 1+\cos^2\gamma=1$$

$$\Rightarrow \quad \cos^2\gamma=0$$

$$\Rightarrow \quad \gamma=90°$$

2. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\Rightarrow \quad \cos^2\alpha+\cos^2(90°+\alpha)+\cos^2\gamma=1$$

$$\Rightarrow \quad \cos^2\alpha+\sin^2\alpha+\cos^2\gamma=1$$

$$\Rightarrow \quad 1+\cos^2\gamma=1$$

$$\Rightarrow \quad \gamma=90°$$

3. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\Rightarrow \quad \cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$

$$\Rightarrow \quad 3\cos^2\alpha=1$$

$$\Rightarrow \quad \cos^2\alpha=\frac{1}{3}$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{\sqrt{3}}$$

Hence, the direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Passage II

1. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$
2. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta$

$$= 4 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta)$$
$$= 4 - \frac{4}{3}$$
$$= \frac{8}{3}$$

3. $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) + \cos(2\delta)$

$$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta) - 4$$
$$= 2\cdot\frac{4}{3}$$
$$= \frac{8}{3} - 4$$
$$= -\frac{4}{3}$$

Passage III

1. Volume of a tetrahedron

$$= \left|\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \end{vmatrix}\right|$$
$$= \left|-\frac{1}{6}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix}\right|$$
$$= \left|-\frac{1}{6} \times 24\right| = |-4| = 4 \text{ cu units.}$$

2. The volume of a tetrahedron

$$= \frac{1}{6}\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{vmatrix}$$
$$= \frac{1}{6}\left\{\begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix}\right\}$$
$$= \frac{1}{6}[12 - 2\{2(0-3) - 4(0-0)\}]$$
$$= \frac{1}{6}[12 + 12]$$
$$= 4$$

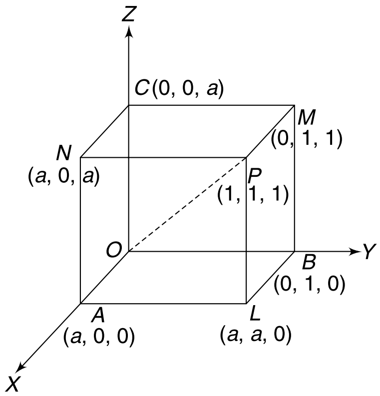
3. On solving, we get the vertices of a tetrahedron as (0, 0, 0, 0), (1, 1, –1), (1, –1, 1), (–1, 1, 1).
Volume of a tetrahedron

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix}$$
$$= -\frac{1}{6}\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$
$$= -\frac{1}{6}\begin{vmatrix} 0 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} \qquad \begin{pmatrix} C_1 \to C_1 + C_3 \\ C_2 \to C_2 + C_3 \end{pmatrix}$$
$$= \frac{4}{6} = \frac{2}{3} \text{ cu. units.}$$

4. On solving, we get the vertices of a tetrahedron as (0, 0, 0), $\left(\frac{1}{a}, \frac{1}{b}, -\frac{1}{c}\right)$, $\left(\frac{1}{a}, -\frac{1}{b}, \frac{1}{c}\right)$ and $\left(-\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ respectively.
Volume of a tetrahedron

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{a} & \frac{1}{b} & -\frac{1}{c} & 1 \\ \frac{1}{a} & -\frac{1}{b} & \frac{1}{c} & 1 \\ -\frac{1}{a} & \frac{1}{b} & \frac{1}{c} & 1 \end{vmatrix}$$
$$= -\frac{1}{6}\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & -\frac{1}{c} \\ \frac{1}{a} & -\frac{1}{b} & \frac{1}{c} \\ -\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$
$$= -\frac{1}{6}abc\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$
$$= \frac{4}{6}\,abc$$
$$= \frac{2}{3}\,abc$$

Passage IV



1. **OP** = **i** + **j** + **k**

   BN = **i** + **j** – **k**

   OP·**OL** = 1 + 1

   $\sqrt{3}\cdot\sqrt{3}\cos\theta = 1 + 1$

   $\cos\theta = \frac{1}{3}$

   $\theta = \cos^{-1}\left(\frac{1}{3}\right)$

   Hence, the angle between any two diagonals is $\theta = \cos^{-1}\left(\frac{1}{3}\right) = \tan^{-1}(2\sqrt{2})$

2. **OP** = **i** + **j** + **k**

   OL = **i** + **j**

   OP·**OL** = 1 + 1

   $\sqrt{3}\cdot\sqrt{2}\cos\theta = 1 + 1$

   $\sqrt{3}\cdot\sqrt{2}\cos\theta = 2$

   $\sqrt{3}\cos\theta = \sqrt{2}$

   $\cos\theta = \sqrt{\frac{2}{3}}$

   $\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$

   Hence, the angle between a diagonal and a diagonal of a face is

   $$\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right) = \cot^{-1}(\sqrt{2})$$

3. **a**·**k** = 2

   $|\mathbf{a}||\mathbf{i}|\cos\theta = 2$

   $\sqrt{6}\cos\theta = 2$

   $\cos\theta = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

   $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{2}{3}} = \frac{1}{\sqrt{3}}$

Passage V

1. Ans. (a)

2. $L_1: \frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}$

   $L_2: \frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}$

   Since L_1 and L_2 are perpendicular, so sum of the product of their sirection ratios is zero.

   Thus, $a\cdot a' + 1\cdot 1 + c\cdot c' = 0$

   $\Rightarrow aa' + cc' = -1.$

3. Given $L_1: \frac{1-x}{3} = \frac{7y-14}{25} = \frac{z-3}{2}$

   $\frac{x-1}{-3} = \frac{y-2}{25/7} = \frac{z-3}{2}$

   and $L_2: \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}.$

   $\frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$

   Since L_1 meets L_2 at right angles, so

   $$(-3)\left(-\frac{3p}{7}\right) + \frac{25}{7} - 10 = 0$$

   $\Rightarrow \frac{9p}{7} = 10 - \frac{25}{7}$

   $\Rightarrow \frac{9p}{7} = \frac{45}{7}$

   $\Rightarrow p = 5$

Passage VI

1. Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$

   and $\frac{x-4}{5} = \frac{y-1}{2} = z = \mu$

   Let two lines meet at P. So,

   $2\lambda + 1 = 5\mu + 4,\ 3\lambda + 2 = 2\lambda + 1$

   and $4\lambda + 3 = \mu$

   $\Rightarrow 3\lambda - 3\mu = -1,\ 4\lambda - \mu = -3$

   $\lambda = -1 = \mu$

   Hence, the point of intersection is (–1, –1, –1).

2. Since both the lines are coplaner, so

   $$\begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $2(3 - 8) - (k + 1)(2 - 4) - (4 - 3) = 0$

$\Rightarrow$ $-10 + 2(k + 1) - 1 = 0$

$\Rightarrow$ $2(k + 1) = 11$

$\Rightarrow$ $k = \frac{11}{2} - 1 = \frac{9}{2}$

3. Since the lines are coplaner, so

$$\begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $(2 - k) - (1 + k^2) - (1 + 2k) = 0$

$\Rightarrow$ $k^2 + 3k = 0$

$\Rightarrow$ $k(k + 3) = 0$

$\Rightarrow$ $k = 0, -3$

Passage VII

1. The shortest distance between two lines

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1).(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}$$

Now, $(\mathbf{u} \times \mathbf{v})$

$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$= -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

and $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Hence, the shortest distance

$$= \frac{(-1 + 4 - 2)}{\sqrt{1 + 4 + 1}} = \frac{1}{\sqrt{6}}$$

Let $L_1: \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$

and $L_2: \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{3}$

Also, let $A = (2, 3, 5)$.

2. Hence, the equation of the plane passing through A and containing the line L_1 is

$$\begin{vmatrix} x-2 & y-3 & z-5 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$\Rightarrow$ $-2(x - 2) + (z - 5) = 0$

$\Rightarrow$ $-2x + 4 + z - 5 = 0$

$\Rightarrow$ $-2x + z - 1 = 0$

$\Rightarrow$ $2x - z + 1 = 0$

3. Ans. (c)

Passage VIII

Let $A = (2, 3, 1)$, $P : 2x + y + z = 6$

and $L: \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$

1. Any point on the line is

$M(2\lambda + 1, \lambda + 2, 4\lambda + 3)$

The direction ratios of AM are

$(2\lambda - 1, \lambda - 1, 4\lambda + 2)$

Clearly, AM is perpendicular to the line L. So,

$2(2\lambda - 1) + (\lambda - 1) + 4(4\lambda + 2) = 0$

$\Rightarrow$ $21\lambda + 5 = 0$

$\Rightarrow$ $\lambda = -\frac{5}{21}$

Hence, the foot of the perpendicular is the

$\left(-\frac{11}{21}, \frac{37}{21}, -\frac{43}{21}\right)$

2. Hence, the image of the point (2, 3, 1) w.r.t. the plane $2x + y + z = 6$ is

$$\frac{\alpha - 2}{2} = \frac{\beta - 3}{1} = \frac{\gamma - 1}{1} = -\frac{2(4 + 3 + 1)}{(4 + 1 + 1)}$$

$$\Rightarrow \frac{\alpha - 2}{2} = \frac{\beta - 3}{1} = \frac{\gamma - 1}{1} = -\frac{8}{3}$$

$$\Rightarrow \alpha = 2 - \frac{16}{2}, \beta = 3 - \frac{8}{3}, \gamma = 1 - \frac{8}{3}$$

$$\Rightarrow \alpha = \frac{10}{3}, \beta = \frac{1}{3}, \gamma = -\frac{5}{3}$$

Hence the required image is $\left(\frac{10}{3}, \frac{1}{3}, -\frac{5}{3}\right)$.

3. Given plane is $P: 2x + y + z = 6$

and the line is $L: \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$

Let the image of the point $A(2, 3, 1)$ w.r.t. to the given plane is $Q(\alpha, \beta, \gamma)$.

So, the image of the given line w.r.t. to the given plane is $\frac{x-\alpha}{2} = \frac{y-\beta}{1} = \frac{z-\gamma}{4}$

Let $\frac{\alpha-2}{2} = \frac{\beta-3}{1} = \frac{\gamma-1}{1} = \gamma$

Let M is the mid-point of A and Q, i.e.

$$M = \left(\frac{\alpha+2}{2}, \frac{\beta+3}{2}, \frac{\gamma+1}{2}\right)$$

$$= \left(\frac{2\lambda+4}{2}, \frac{\lambda+3+3}{2}, \frac{\lambda+2}{2}\right)$$

Clearly, the point M lies on the plane

$$2\left(\frac{2\lambda+4}{2}\right) + \left(\frac{\lambda+6}{2}\right) + \frac{\lambda+2}{2} = 0$$

$\Rightarrow \quad 2\lambda + 4 + \frac{\lambda}{2} + 3 + \frac{\lambda}{2} + 1 = 0$

$\Rightarrow \quad 3\lambda = -8$

$\Rightarrow \quad \lambda = -\frac{8}{3}$

So, $\quad \alpha = 2\lambda + 2,\ \beta = 1 + 3,\ \gamma = \lambda + 1$

$\Rightarrow \quad \alpha = 2 - \frac{16}{3},\ \beta = 3 - \frac{8}{3},\ \gamma = 1 - \frac{8}{3}$

$\Rightarrow \quad \alpha = -\frac{10}{3},\ \beta = \frac{1}{3},\ \gamma = \frac{-5}{3}$

Hence, the required image of the line w.r.t. to the plane is

$$\frac{x+\frac{10}{3}}{2} = \frac{y-\frac{7}{3}}{1} = \frac{z+\frac{5}{3}}{4}$$

$\Rightarrow \quad \frac{3x+2}{2} = \frac{3y-7}{1} = \frac{3z-1}{4}$

$\Rightarrow \quad \frac{x+\frac{10}{3}}{2} = \frac{y-\frac{7}{3}}{1} = \frac{z+\frac{5}{3}}{4}$

$\Rightarrow \quad \frac{3x+10}{2} = \frac{3y-7}{1} = \frac{3z+5}{4}$

Match Matrix

1. (A) $\Sigma\cos^2\alpha = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(B) $\Sigma\sin^2\alpha = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

$= 3 - 1 = 2$

(C) $\Sigma\cos(2\alpha) = \cos(2\alpha) + \cos(2\beta) + \cos(2\gamma)$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$

$= 2 - 3 = -1$

(D) $\Sigma(\cos^2\alpha + \sin^2\alpha) = \Sigma 1 = 1 + 1 + 1 = 3$

2. (A) $\Sigma\cos^2\alpha = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= 4/3$

(B) $\Sigma\sin^2\alpha = \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta$

$= 4 - \frac{4}{3} = \frac{8}{3}$

(C) $\Sigma\cos(2\alpha)$

$= \cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) + \cos(2\delta)$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta) - 4$

$= 2\left(\frac{4}{3}\right) - 4 = \frac{8}{3} - 4 = -\frac{4}{3}$

(D) $\Sigma(\cos^2\alpha + \sin^2\alpha) = \Sigma 1$

$= 1 + 1 + 1 + 1$

$= 4$

3. (A) Let two diagonals of a cube are $a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$ and $-a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$

Let θ be the angle between them

$$\cos\theta = \frac{a\cdot -a + a\cdot a + a\cdot a}{a\sqrt{3}\cdot a\sqrt{3}} = \frac{a^2}{3a^2} = \frac{1}{3}$$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$

(B) Let the diagonal of the cube be $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and the diagonal of the face be $\mathbf{i} + \mathbf{j}$.

Let θ be the angle between them,

$$\cos(\theta) = \frac{1+1}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}}$$

$\Rightarrow \quad (\theta) = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right) = \cot^{-1}(\sqrt{2})$

(C) Let the diagonal of the cube be $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and the diagonal of the face intersecting it is $\mathbf{j} + \mathbf{k}$.

Let θ be the angle between them,

$$\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right) = \cot^{-1}(\sqrt{2})$$

(D) Let the diagonal of the faces of the same vertex are $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$.

Let θ be the angle between them. Then

$$\cos\theta = \frac{0+1+0}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$= \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = \cot^{-1}\frac{1}{\sqrt{3}}$$

4. (A) Given $l + m + n = 0,\ l^2 = m^2 + n^2$

$\Rightarrow \quad -(m+n) = m^2 + n^2$

$\Rightarrow \quad -2mn = 0$

$\Rightarrow \quad m = 0 = n$

When $m = 0$, then $1 = -n$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{0} = \frac{n}{-1} = \frac{1}{\sqrt{2}}$$

When $n = 0$, then $1 = -m$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{-1} = \frac{n}{0} = \frac{1}{\sqrt{2}}$$

Let θ be the angle between them. Then

$$\cos(\theta) = l_1l_2 + m_1m_2 + n_1n_2$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 0 + 0 = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

(B) Let θ be the angle between them. Then

$$\cos\theta = \frac{6+4-10}{\sqrt{9+16+25}\ \sqrt{4+1+4}} = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

(C) Let θ be the angle between them. Then

$$\cos(\theta) = \frac{2-1+2}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

(D) Let θ be the angle between them. Then

$$\sin\theta = \frac{2+1+0}{\sqrt{9}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

5. (A) Since the line lies in the plane, so the point (4, 2, k) lies in the plane.

Thus, $\quad 8 - 8 + k = 7$

$k = 7$

(B) Since the given lines are coplanar so

$$\begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & 1-k \\ k & k+2 & k+1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(k+1) + (k-1)(k+2) = 0$

$\Rightarrow \quad 2k + 2 + k^2 + k - 2 = 0$

$\Rightarrow \quad k^2 + 3k = 0$

$\Rightarrow \quad k = 0, -3.$

(C) Since the given lines are intersect, so they are coplanar.

Thus,

$$\begin{vmatrix} 3-1 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(3-8) - (k+1)(2-4) - (4-3) = 0$

$\Rightarrow \quad -10 + 2(k+1) - 1 = 0$

$\Rightarrow \quad 2k - 9 = 0$

$$\Rightarrow \quad k = \frac{9}{2}$$

(D) Given lines are

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-6}{5}$$

and $$\frac{x-5}{1} = \frac{y-2}{2} = \frac{z-1}{3}$$

Hence, the shortest distance

$$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u}\times\mathbf{v})}{|(\mathbf{u}\times\mathbf{v})|}$$

$= 0$

where, $(\mathbf{r}_2 - \mathbf{r}_1) =$

$(5-2,\ 2-3,\ 1-6) = (3, -1, -5)$

$$(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} i & j & k \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix} = 2i - 4j + 2k$$

$$\Rightarrow \quad |\mathbf{u}\times\mathbf{v}| = \sqrt{4+16+4} = 2\sqrt{6}$$

6. (A) Let any point on the given line be

$(\lambda + 1, 3\lambda - 2, 2 - 2\lambda)$

Clearly, this point lies on the plane so,

$3(\lambda + 1) + 4(3\lambda - 2) + 5(2 - 2\lambda) = 15$

$\Rightarrow$ $5\lambda + 5 = 15$

$\Rightarrow$ $5\lambda = 10$

$\Rightarrow$ $\lambda = 2$

Hence, the point of intersection is (3, 4, −2).

(B) Any point on the line $\frac{x}{2} = \frac{y-1}{2} = \frac{z-2}{3}$ can be considered as $M(2\lambda, 2\lambda + 1, 3\lambda + 2)$.

Let the point P be $P(2, 6, 3)$.

The direction ratios of PM are $(2\lambda - 2, 2\lambda - 5, 3\lambda - 1)$.

Clearly PM is perpendicular to the given line L. So,

$2(2\lambda - 2) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$

$\Rightarrow$ $17\lambda - 17 = 0$

$\Rightarrow$ $\lambda - 1 = 0$

$\Rightarrow$ $\lambda = 1$

Hence, the foot of the perpendicular is (2, 3, 5).

(C) Any point on the first line be $(\lambda + 2, 1 - 2\lambda, \lambda + 6)$.

Any point on the second line be

$(7\mu - 3, -6 - 3\mu, \mu - 3)$

Clearly, both the points are same. So,

$\lambda + 2 = 7\mu - 3, 1 + 6 = \mu - 3$

$1 - 7\mu = -5, 1 - \mu = -9$

Solving we get

$\lambda = -\frac{29}{3}, \mu = -\frac{2}{3}$

Hence, the point of intersection is

$\left(\frac{23}{3}, \frac{61}{3}, -\frac{11}{3}\right)$

(D) Clearly, $2\lambda - 3\mu = 4, 3\lambda - 4\mu = 4$

Solving we get

$\lambda = -2, \mu = -4$

Hence, the point of intersection is (−5, −8, −9)

7. Given $P(0, 3, -2)$, $Q(3, 7, -1)$ and $R(1, -3, -1)$.

Clearly, $L_1: \frac{x}{3} = \frac{y-3}{4} = \frac{z+2}{1}$

and $L_2: \frac{x-1}{1} = \frac{y+3}{0} = \frac{z+1}{1}$

The equation of the plane PQR is

$$\begin{vmatrix} x & y-3 & z+2 \\ 3 & 4 & 1 \\ 1 & -6 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $10x - 2(y - 3) - 22(z + 2) = 0$

$\Rightarrow$ $10x - 2y - 22z = 38$

$\Rightarrow$ $5x - y - 11z = 19$

(A) Let the foot of the perpendicular on $L2$ is M.

Thus, $M = (\lambda + 1, -3, \lambda - 1)$

The direction ratios of PM are $(\lambda + 1, -6, \lambda + 1)$

Since PM is perpendicular to $L2$, so

$1(\lambda + 1) + 0 + 1(\lambda + 1) = 0$

$\Rightarrow$ $\lambda = -2$

Hence, the foot of the perpendicular is (−1, −3, −3)

Thus, the length of the perpendicular

$= \sqrt{1 + 36 + 1} = \sqrt{38}$

(B) The shortest distance between L_1 and L_2

$= \frac{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}$

$= \frac{4 + 12 - 4}{6} = 2,$

where

$(\mathbf{r}_2 - \mathbf{r}_1) = (1, -6, 1) = i - 6j + k$

$\Rightarrow$ $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 4i - 2j - 4k$

$\Rightarrow$ $|\mathbf{u} \times \mathbf{v}| = \sqrt{16 + 4 + 16} = 6$

(C) Area of ΔPQR

$= \frac{1}{2} |\mathbf{PQ} \times \mathbf{PR}|$

$= \frac{1}{2} \left\|\begin{matrix} i & j & k \\ 3 & 4 & 1 \\ 1 & -6 & 1 \end{matrix}\right\|$

$= \frac{1}{2} |10i - 2j - 22k|$

$= \frac{1}{2}\sqrt{100 + 4 + 484} = \frac{\sqrt{588}}{2} = 7\sqrt{3}$

(D) Hence, the distance from the origin to the plane $5x - y - 11z = 19$

$= \frac{19}{\sqrt{25 + 1 + 121}} = \frac{19}{\sqrt{147}}$

8. (A) The shortest distance from the given point to the x-axis

$= \sqrt{4^2 + 5^2} = \sqrt{41}$

(B) The shortest distance from the given point to the y-axis $= \sqrt{3^2 + 4^2} = 5$

(C) Let θ be the angle between them. Then

$$\sin\theta = \frac{2}{7}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{2}{7}\right)$$

Clearly, $k = \frac{2}{7}$

$$\Rightarrow \quad 7k = 2$$

(D) Given line is

$$\mathbf{r} = (1 + t)\mathbf{i} + 5t\mathbf{j} + (1 - t)\mathbf{k}$$

$$= (\mathbf{i} + \mathbf{k}) + t(\mathbf{i} + 5\mathbf{j} - \mathbf{k})$$

Since the given line lies in the plane

so, $1 + 0 + c = d$
$1 + 5 - c = 0$

$\Rightarrow$ $1 + c = d$
$c = 6$

Thus, $c = 6, d = 7$

Hence, the value of $c + d$ is 13

9. (A) Hence, the volume of the tetrahedron

$$= \frac{1}{6}abc = \frac{1}{6} \times 2 \times 3 \times 4 = 4$$

(B) Let the points A, B, C and D are (0, 1, 2), (3, 0, 1), (4, 3, 6) and (2, 3, 2), respectively.

Now, $\mathbf{AB} = (3, -1, -1)$

$\mathbf{AC} = (4, 2, 4)$

$\mathbf{AD} = (2, 2, 0)$

Hence, the volume of the tetrahedron

$$= \frac{1}{6}[\mathbf{AB}, \mathbf{AC}, \mathbf{AD}]$$

$$= \frac{1}{6}\begin{vmatrix} 3 & -1 & -1 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{vmatrix}$$

$$= \left|\frac{1}{6}(-24 - 8 - 4)\right|$$

$$= 6 \text{ cu. units.}$$

(C) Volume of a tetrahedron = 2/3

(D) Volume of a tetrahedron $= \frac{2}{3}abc$

$$= \frac{2}{3} \times 2 \times 3 \times 4$$

$$= 16$$

10. As we know that the homogeneous equation of 2nd degree $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents two planes if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

(A) The given equation

$$2x^2 + 3y^2 + 4z^2 + 2kxy + 4yz = 0$$

represents two planes if

$$\begin{vmatrix} 2 & k & 0 \\ k & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow$ $2(12 - 4) - k(4k) = 0$

$\Rightarrow$ $16 - 4k^2 = 0$

$\Rightarrow$ $k^2 - 4 = 0$

$\Rightarrow$ $k^2 = 4$

(B) $x^2 + 5y^2 + 2kxy + 4xz = 0$ represents a pair of planes if

$$\begin{vmatrix} 1 & k & 0 \\ k & 5 & 2 \\ 0 & 2 & 5 \end{vmatrix} = 0$$

$\Rightarrow$ $21 - k(5k) = 0$

$\Rightarrow$ $k^2 = \frac{21}{5}$

$\Rightarrow$ $5k^2 = 21$

(C) $x^2 + y^2 + 4z^2 - 2kyz = 0$ represents a pair of planes if

$$\begin{vmatrix} 1 & 0 & -K \\ 0 & 1 & 0 \\ -K & 0 & 4 \end{vmatrix} = 0$$

$\Rightarrow$ $4 - k^2 = 0$

$\Rightarrow$ $k^2 = 4$

$\Rightarrow$ $k^2 + 2 = 6$

(D) $9x^2 + y^2 + z^2 - 6kxz = 0$ represents a pair of straight lines if

$$\begin{vmatrix} 9 & 0 & 0 \\ 0 & 1 & -3k \\ 0 & -3k & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $1 - 9k^2 = 0$

$\Rightarrow$ $9k^2 = 1$

Level III

1. We have,

$$al + bm + cn = 0$$

$$\Rightarrow \quad l = -\frac{(bm + cn)}{a}$$

Now,

$$ul^2 + vm^2 + wn^2 = 0.$$

$$\Rightarrow \quad u\left(\frac{bm + cn}{a}\right)^2 + vm^2 + wn^2 = 0$$

$$\Rightarrow \quad u(bm + cn)^2 + va^2m^2 + wa^2n^2 = 0$$

$$\Rightarrow \quad u(b^2m^2 + c^2n^2 + 2bcmn) + va^2m^2 + wa^2n^2 = 0$$

$$\Rightarrow \quad (ub^2 + va^2)m^2 + 2ubcmn + (uc^2 + wa^2)n^2 = 0$$

$$\Rightarrow \quad (ub^2 + va^2)\left(\frac{m}{n}\right)^2 + 2ubc\left(\frac{m}{n}\right) + (uc^2 + wa^2) = 0 \quad \text{...(i)}$$

Let its roots are $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$.

(a) If two straight lines are parallel, so the Eq. (i) will provide us equal roots.

Thus, $D = 0$

$$\Rightarrow \quad (2ubc)^2 - 4(b^2u + a^2v)(c^2u + a^2w) = 0$$

$$\Rightarrow \quad (ubc)^2 - (b^2u + a^2v)(c^2u + a^2w) = 0$$

$$\Rightarrow \quad (b^2u + a^2v)(c^2u + a^2w) - (ubc)^2 = 0$$

$$\Rightarrow \quad a^2b^2uw + a^2c^2uv + a^4vw = 0$$

$$\Rightarrow \quad b^2uw + c^2uv + a^2vw = 0$$

$$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$$

(b) When two straight lines are perpendicular. So, the product of the roots $= \frac{uc^2 + wa^2}{ub^2 + va^2}$

$$\Rightarrow \quad \frac{m_1}{n_1}\cdot\frac{m_2}{n_2} = \frac{uc^2 + wa^2}{ub^2 + va^2}$$

$$\Rightarrow \quad \frac{m_1m_2}{uc^2 + wa^2} = \frac{n_1n_2}{ub^2 + va^2}$$

Similarly, eliminating n, we get

$$\frac{l_1l_2}{wb^2 + vc^2} = \frac{m_1m_2}{uc^2 + wa^2}$$

Thus, $\frac{l_1l_2}{wb^2 + vc^2} = \frac{m_1m_2}{uc^2 + wa^2} = \frac{n_1n_2}{ub_2 + va^2}$

Two lines are perpendicular, if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

$$\Rightarrow \quad (wb^2 + vc^2) + (uc^2 + wa^2) + (ub^2 + va^2) = 0$$

$$\Rightarrow \quad a^2(u + v) + b^2(u + w) + c^2(u + v) = 0$$

Hence, the result.

2. Let $P = (\alpha, \beta, \gamma)$ and $Q = (x_1, y_1, z_1)$

Thus, the direction ratios of OP are (α, β, γ) and the direction ratios of OQ are (x_1, y_1, z_1).

Since O, Q, P are collinear, we have

$$\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} \quad \text{...(i)}$$

As $P(\alpha, \beta, \gamma)$ lies on the plane

$$lx + my + nz = p$$

so $\quad l\alpha + m\beta + n\gamma = p$

$$\Rightarrow \quad k(lx_1 + my_1 + nz_1) = p \quad \text{...(ii)}$$

Given $\quad OP \cdot OQ = p^2$

$$\Rightarrow \quad \sqrt{\alpha^2 + \beta^2 + \gamma^2}\cdot\sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$$

$$\Rightarrow \quad \sqrt{k^2(x_1^2 + y_1^2 + z_1^2)}\cdot\sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$$

$$\Rightarrow \quad \sqrt{k^2(x_1^2 + y_1^2 + z_1^2)^2} = p^2$$

$$\Rightarrow \quad k(x_1^2 + y_1^2 + z_1^2) = p^2 \quad \text{...(iii)}$$

On dividing Eq. (ii) and Eq. (iii), we get

$$\frac{lx_1 + my_1 + nz_1}{(x_1^2 + y_1^2 + z_1^2)} = \frac{1}{p}$$

$$\Rightarrow \quad p(lx_1 + my_1 + nz_1) = (x_1^2 + y_1^2 + z_1^2)$$

Hence, the locus of the point Q is

$$p(lx + my + nz) = (x^2 + y^2 + z^2)$$

3. Given plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$

Let $\quad P = (\alpha, \beta, \gamma)$

Then $\quad \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1 \quad \text{...(ii)}$

So, $\quad OP = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$

Therefore, the direction ratios of OP are

$$\frac{\alpha}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}, \frac{\beta}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}, \frac{\gamma}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

The equation of the normal through P and normal to OP is

$$\frac{\alpha}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}x + \frac{\beta}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}y + \frac{\gamma}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}z = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$$

$$\Rightarrow \quad \alpha x + \beta y + \gamma z = \sqrt{\alpha^2 + \beta^2 + \gamma^2} \quad \text{...(iii)}$$

Clearly, $\quad A = \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, 0, 0\right)$

$$B = \left(0, \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}, 0\right)$$

$$C = 0, 0, \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}\right)$$

Let $Q = (p, q, r)$

Then $p = \dfrac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, q = \dfrac{\alpha^2 + \beta^2 + \gamma^2}{\beta}$

and $r = \dfrac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}$ ...(iv)

Now, $\dfrac{1}{p^2} + \dfrac{1}{q^2} + \dfrac{1}{r^2} = \dfrac{\alpha^2 + \beta^2 + \gamma^2}{(\alpha^2 + \beta^2 + \gamma^2)^2}$

$\Rightarrow \dfrac{1}{p^2} + \dfrac{1}{q^2} + \dfrac{1}{r^2} = \dfrac{1}{(\alpha^2 + \beta^2 + \gamma^2)}$ ...(v)

From Eq. (iv), we get

$$\frac{p}{a} = \frac{\alpha^2 + \beta^2 + \gamma^2}{a\alpha}, \frac{q}{b} = \frac{\alpha^2 + \beta^2 + \gamma^2}{b\beta}$$

$$\frac{r}{c} = \frac{\alpha^2 + \beta^2 + \gamma^2}{c\gamma}$$

Therefore,

$$\frac{\alpha^2 + \beta^2 + \gamma^2}{a\alpha} + \frac{\alpha^2 + \beta^2 + \gamma^2}{b\beta} + \frac{\alpha^2 + \beta^2 + \gamma^2}{c\gamma}$$

$$= \frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1$$

$\Rightarrow \dfrac{1}{ap} + \dfrac{1}{bq} + \dfrac{1}{cr} = \dfrac{1}{\alpha^2 + \beta^2 + \gamma^2}$

$\Rightarrow = \dfrac{1}{p^2} + \dfrac{1}{q^2} + \dfrac{1}{r^2}$ (from Eq. v)

Hence, the locus of $Q(p, q, r)$ is

$$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

4. Let the equation of the plane be

$$Ax + By + Cz + D = 0 \quad ...(i)$$

which is passing through (a, b, c). So,

$$Aa + Bb + Cc + D = 0 \quad ...(ii)$$

Let $P(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the origin to the plane (i).

The direction ratios of OP are (α, β, γ).

Therefore, (α, β, γ) be the direction ratios of the same line OP. So,

$$\frac{\alpha}{A} = \frac{\beta}{B} = \frac{\gamma}{C} = \frac{1}{k}(\text{say}) \quad ...(iii)$$

From Eq. (ii) and (iii), we get

$$ka\alpha + kb\beta + kc\gamma + D = 0 \quad ...(iv)$$

Since (α, β, γ) lies in the plane (i), so

$$A\alpha + B\beta + C\gamma + D = 0 \quad ...(v)$$

From Eqs. (iii) and (v), we get

$$k\alpha^2 + k\beta^2 + k\gamma^2 + D = 0 \quad ...(vi)$$

From Eqs. (iv) and (vi), we get

$$k\alpha^2 + k\beta^2 + k\gamma^2 - k(a\alpha + b\beta + c\gamma) = 0$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 - (a\alpha + b\beta + c\gamma) = 0$$

Hence, the locus of the foot of the perpendicular P is

$$x^2 + y^2 + z^2 - (ax + by + cz) = 0.$$

5. Given planes are

$$x - y - z = 4 \quad ...(i)$$

and $x + y + 2z = 4$ ...(ii)

Since the required planes pass through the line of intersection of the planes (i) and (ii).

Thus, its equation will be

$$(x - y - z - 4) + \lambda(x + y + 2z - 4) = 0$$

$$\Rightarrow (1 + \lambda)x + (1 - \lambda)y + (2 - \lambda)z - 4 - 4\lambda = 0 \quad ...(iii)$$

Since Eq. (i) and (iii) are mutually perpendicular, so,

$$(1 + \lambda) - (1 - \lambda) - (2 - \lambda) = 0$$

$$\Rightarrow \lambda = \frac{2}{3}$$

Substituting the values of $\lambda = \dfrac{2}{3}$ in Eq. (iii), we get

$$5x + y + 4z = 20.$$

which is the required equation of the plane.

6. Given planes are

$$x - cy - bz = 0 \quad ...(i)$$

$$cx - y + az = 0 \quad ...(ii)$$

$$bx + ay - z = 0 \quad ...(iii)$$

The equation of any plane passing through the line of intersection of the planes (i) and (ii) may be taken as

$$(x - cy - bz) + \lambda(cx - y + az) = 0$$

$$\Rightarrow (1 + \lambda c)x - (c + \lambda)y + (a\lambda - b)z = 0 \quad ...(iv)$$

If planes (iii) and (iv) are the same, so the equations (iii) and (iv) are identical.

$$\frac{(1 + \lambda c)}{b} = -\frac{(c + \lambda)}{a} = \frac{(a\lambda - b)}{-1}$$

Thus, $\dfrac{(1 + \lambda c)}{b} = -\dfrac{(c + \lambda)}{a}$

$$\Rightarrow \lambda = -\frac{(a + bc)}{(ac + b)} \quad ...(v)$$

and $-\frac{(c+\lambda)}{a} = \frac{(a\lambda - b)}{-1}$

$\Rightarrow \quad \lambda = -\frac{(ab+c)}{(1-a^2)}$ ...(vi)

From Eqs (v) and (vi), we get

$$-\frac{(ac+b)}{(a+bc)} = -\frac{(ab+c)}{(1-a^2)}$$

$\Rightarrow \quad a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

$\Rightarrow \quad a^3 + ab^2 + ac^2 + 2a^2bc - a = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$

7. Any point on the given line is

$$P:(2\lambda + 1, 3\lambda - 2, 6\lambda + 3)$$

Let $Q:(1, -2, 3)$

Given $PQ = 3$

$\Rightarrow \quad PQ^2 = 9$

$\Rightarrow \quad (4\lambda^2 + 9\lambda^2 + 36\lambda^2) = 9$

$\Rightarrow \quad 49\lambda^2 = 9$

$\Rightarrow \quad \lambda^2 = \frac{9}{49} = \left(\frac{3}{7}\right)^2$

$\Rightarrow \quad \lambda = \pm\frac{3}{7}.$

Hence, the points are

$$\left(\pm 2\left(\frac{7}{3}\right) + 1, \pm 3\left(\frac{7}{3}\right) - 2, \pm 6\left(\frac{7}{3}\right) + 3\right)$$

$$= \left(\frac{17}{3}, 5, 17\right) \text{ and } \left(-\frac{11}{3}, -9, -11\right).$$

8. The equation of any line through (1, 0, –3) and parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{6-z}{6}$ is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6}.$$

Any point on the line (i) is $(2\lambda + 1, 3\lambda, -6 - 3\lambda)$ which is lies in the plane $x - y - z = 9$. So,

$$(2\lambda + 1) - (3\lambda) - (-6 - 3\lambda) = 9$$

$\Rightarrow \quad 2\lambda + 7 = 9$

$\Rightarrow \quad 2\lambda = 2 \Rightarrow \lambda = 1$

Thus, the point is (3, 3, –9).

Hence, the required distance

$$= \sqrt{(3-1)^2 + (3-0)^2 + (-9+3)^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49} = 7.$$

9. The equation of any plane passing through (1, 2, 0) is $a(x - 1) + b(y - 2) + cz = 0$ which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{2-z}{2}$.

So, $3a + 4b - 2c = 0$

and $-4a - b + 2c = 0$

Solving, we get

$$\frac{a}{8-1} = \frac{b}{8-6} = \frac{c}{-3+16}$$

$\Rightarrow \quad \frac{a}{7} = \frac{b}{12} = \frac{c}{13}$

Hence, the equation of the plane is

$$7(x - 1) + 2(y - 2) + 13z = 0$$

$\Rightarrow \quad 7x + 2y + 13z = 11$

10. Let the given line AB be

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{4} \quad \text{...(i)}$$

and the given plane is $x + 2y + z = 9$ ...(ii)

Let DC be the projection of AB on plane (ii).

Clearly the plane $ABCD$ is perpendicular to the plane (ii).

Equation of any plane through AB is

$$a(x - 1) + b(y + 1) + c(z - 3) = 0 \quad \text{...(iii)}$$

where $2a - b + 4c = 0$ ...(iv)

since the plane (iii) is perpendicular to the plane (ii), so

$$a + 2b + c = 0 \quad \text{...(v)}$$

Solving Eqs (iv) and (v), we get

$$\frac{a}{-9} = \frac{b}{2} = \frac{c}{3}$$

Putting these values of a, b and c in Eq. (iii), we get

$$9(x - 1) - 2(y + 1) - 5(z - 3) = 0$$

$\Rightarrow \quad 9x - 2y - 5z + 4 = 0$ ...(vi)

Since the projection DC of AB on the plane (ii) is the line of intersection of the plane $ABCD$ and the plane (ii).

Thus, the equation of DC will be

$$9x - 2y - 5z + 4 = 0$$

$\Rightarrow \quad x + 2y + z - 9 = 0$ ...(vii)

Let l, m, n be the direction ratios of the line of intersection of the planes (i) and (ii). So,

$$9l - 2m - 5n = 0$$

and $l + 2m + n = 0$

Solving, we get

$$\frac{l}{8} = \frac{m}{-14} = \frac{n}{20}$$

$\Rightarrow \quad \frac{l}{4} = \frac{m}{-7} = \frac{n}{10}$

Let the line of intersection of the planes (i) and (ii) meet xy-plane at $P(\alpha, \beta, 0)$. So, P lies on the planes (i) and (ii)

Therefore, $9\alpha - 2\beta + 4 = 0$

and $\alpha + 2\beta - 9 = 0$

So, $\alpha = \frac{1}{2}, \beta = \frac{17}{4}$

Hence, the equation of DC is

$$\frac{x - \frac{1}{2}}{4} = \frac{y - \frac{17}{4}}{-7} = \frac{z}{10}$$

$$\Rightarrow \frac{2x - 1}{8} = \frac{4y - 17}{-28} = \frac{z}{10}$$

$$\Rightarrow \frac{2x - 1}{4} = \frac{4y - 17}{-14} = \frac{z}{5}$$

11. Any plane passing through the first line is

$(2x + y + z - 1) + l(3x + y + 2z - 2) = 0$ ...(i)

which is parallel to the 2nd line $x = y = z$.

So,

$$(2 + 3\lambda)\cdot 1 + (1 + \lambda)\cdot 1 + (1 + 2\lambda)\cdot 1 = 0$$

$$\Rightarrow (2 + 3\lambda) + (1 + \lambda) + (1 + 2\lambda) = 0$$

$$\Rightarrow 6\lambda + 4 = 0$$

$$\Rightarrow \lambda = -\frac{2}{3}$$

Put $\lambda = -\frac{2}{3}$ in Eq. (i), we get

$$(2x + y + z - 1) - \frac{2}{3}(3x + y + 2z - 2) = 0$$

$\Rightarrow \quad y - z + 1 = 0$ ...(ii)

Thus, distance from (0, 0, 0) to the plane (ii) $= \frac{1}{\sqrt{2}}$.

12. We have,

$$4 + \lambda = 1 + 2\mu$$

$\Rightarrow \quad \lambda - 2\mu = -3$ ...(i)

Also, $\quad -3 - 4\lambda = -1 - 3\mu$

$\Rightarrow \quad 4\lambda - 3\mu = -2$ ...(ii)

From Eq. (i) and (ii), we get

$$\lambda = 2, \mu = 1$$

Thus, the point of intersection

$$= (4 + 1, -3 -4, -1 + 7)$$

$$= (5, -7, 6)$$

Hence, the required distance from (5, –7, 6) to (1, –4, 7)

$$= \sqrt{16 + 9 + 1}$$

$$= \sqrt{26}.$$

13. Let $\quad L_1 : x = ay + b, z = cy + d$

$$\Rightarrow \frac{x - b}{a} = \frac{y}{1} = \frac{z - d}{c}$$

and $\quad L_2 : x = a_1 y + b_1, z = c_1 y + d_1$

$$\Rightarrow \frac{x - b_1}{a_1} = \frac{y}{1} = \frac{z - d_1}{c_1}$$

Since L_1 and L_2 are perpendicular, so

$$aa_1 + 1.1 + c\cdot c_1 = 0$$

$$\Rightarrow aa_1 + cc_1 + 1 = 0$$

$$\Rightarrow k = 1$$

Now, $\quad k^2 + 4 = 1 + 4 = 5$

14. The equation of any line through (1, 2, 3) parallel to the given line is

$$\frac{x - 1}{1} = \frac{y - 2}{-2} = \frac{z - 3}{2}.$$

Any point on the above line can be considered as $(\lambda + 1, 2 - 2\lambda, 2\lambda + 3)$ which is also a common point of the plane.

Thus, $\quad \lambda + 1 + 2 - 2\lambda + 2\lambda + 3 = 11$

$$\Rightarrow \lambda = 5$$

Therefore, the point is (6, –8, 13).

Hence, the required distance

$$= \sqrt{(6 - 1)^2 + (-8 - 2)^2 + (13 - 3)^2}$$

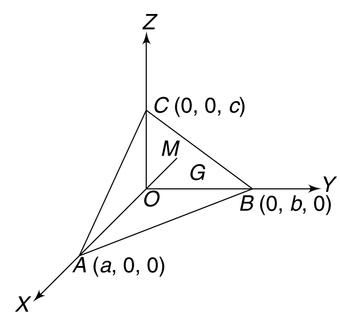
$$= \sqrt{25 + 100 + 100}$$

$$= \sqrt{225}$$

$$= 15$$

15. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



It is given that the volume of the tetrahedron $OABC$ $= 64k^3$

$$\Rightarrow \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 64k^3$$

$$\Rightarrow \frac{1}{6}(abc) = 64k^3$$

$$\Rightarrow (abc) = 384k^3$$

Let the centroid be $G(\alpha, \beta, \gamma)$.

Clearly, $\quad \frac{a}{4} = \alpha, \frac{b}{4} = \beta, \frac{c}{4} = \gamma$

$$\Rightarrow a = 4\alpha, b = 4\beta, c = 4\gamma$$

Therefore, $64(\alpha\ \beta\ \gamma) = 384k^3$

$$\Rightarrow (\alpha\ \beta\ \gamma) = 6k^3$$

Hence, the locus of $G(\alpha, \beta, \gamma)$ is $xyz = 6k^3$

16. Given lines $\frac{y}{b}+\frac{z}{c}=1$, $x=0$ and $\frac{x}{a}-\frac{z}{c}=1$, $y=0$ can be written as

$$L_1: \frac{x}{0}=\frac{y-b}{b}=\frac{z}{-c}$$

and $$L_2: \frac{x-a}{a}=\frac{y}{0}=\frac{z}{c}$$

Here, $\mathbf{r}_1 = b\mathbf{j}$

$\mathbf{r}_2 = a\mathbf{i}$

and $\mathbf{u} = b\mathbf{j} - c\mathbf{k}$

$\mathbf{v} = a\mathbf{i} + c\mathbf{k}$

Now, $(\mathbf{r}_2 - \mathbf{r}_1) = ai - b\mathbf{j}$

and $$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & b & -c \\ a & 0 & c \end{vmatrix}$$

$$= bc\mathbf{i} - ac\mathbf{j} - ab\mathbf{k}$$

$$\Rightarrow \quad |(\mathbf{u}\times\mathbf{v})| = \sqrt{(bc)^2+(ac)^2+(ab)^2}$$

Given shortest distance = $2d$

$$\Rightarrow \quad \frac{(\mathbf{r}_2-\mathbf{r}_1)\cdot(\mathbf{u}\times\mathbf{v})}{|(\mathbf{u}\times\mathbf{v})|} = 2d$$

$$\Rightarrow \quad \frac{abc+abc}{\sqrt{(bc)^2+(ac)^2+(ab)^2}} = 2d$$

$$\Rightarrow \quad \frac{2abc}{\sqrt{(bc)^2+(ac)^2+(ab)^2}} = 2d$$

$$\Rightarrow \quad \frac{abc}{\sqrt{(bc)^2+(ac)^2+(ab)^2}} = d$$

$$\Rightarrow \quad \frac{(abc)^2}{[(bc)^2+(ac)^2+(ab)^2]} = d^2$$

$$\Rightarrow \quad \frac{1}{d^2} = \frac{((bc)^2+(ac)^2+(ab)^2)}{(abc)^2}$$

$$\Rightarrow \quad \frac{1}{d^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

17. Let O be the origin and the direction ratios of OP are (a, b, c)

Hence, the equation of the plane is

$$a(x-a)+b(y-b)+c(z-c)=0$$

$$\Rightarrow \quad ax+by+cz = a^2+b^2+c^2$$

18. Let a point $(3\lambda+1, \lambda+2, 2\lambda+3)$ of the first line lies on the 2nd line. Thus,

$$\frac{3\lambda+1-3}{1} = \frac{\lambda+2-1}{2} = \frac{2\lambda+3-2}{3}$$

$$\Rightarrow \quad \frac{3\lambda-2}{1} = \frac{\lambda+1}{2} = \frac{2\lambda+1}{3}$$

$$\Rightarrow \quad \lambda = 1$$

Thus, the point of intersection of the two lines is $P(4, 3, 5)$.

Therefore, the equation of the plane perpendicular to OP, where O is $(0, 0, 0)$ and passing through P is $4x+3y+5z=50$.

19. The equation of the plane through the line be $A(x-1)+B(y+2)+C(z-0)=0$, then

$$2A-3B+5C=0$$

and $$A-B+C=0$$

Solving, we get

$$\frac{A}{2}=\frac{B}{3}=\frac{C}{1}$$

Hence, the equation of the plane is

$$2(x-1)+3(y+2)+z=0$$

$$\Rightarrow \quad 2x+3y+z+4=0$$

Thus, $a=2$, $b=-3$, $c=1$

Hence, the value of

$a+b+c+2.$

$=2-3+1+2$

$=2.$

20. The direction ratios of the normal to the plane is $(2, 3, 4)$.

The equation of any plane through $(2, 3, 4)$ and the maximum distance from the origin is

$$2(x-2)+3(y-3)+4(z-4)=0$$

$$\Rightarrow \quad 2x+3y+4z=29.$$

21. Let O be the origin $(0, 0, 0)$.

Let the co-ordinates of the points A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, respectively.

Thus, the equation of the sphere passing through O, A, B, and C is

$$x^2+y^2+z^2-ax-by-cz=0$$

But the radius of the sphere is given by r.

So, $$r = \sqrt{\frac{a^2+b^2+c^2}{4}}$$

$$\Rightarrow \quad (a^2+b^2+c^2) = 4r^2 \quad \text{...(i)}$$

Let (α, β, γ) be the centroid of the ΔABC

Then $$a = \frac{a+0+0}{3} = \frac{a}{3}$$

Similarly, $\beta = \frac{b}{3} = \gamma = \frac{c}{3}$

Putting the values of a, b and c in Eq. (i), we get

$$9(\alpha^2+\beta^2+\gamma^2) = 4r^2$$

Hence, the locus of (α, β, γ) is

$$9(x^2 + y^2 + z^2) = 4r^2$$

22. Let the co-ordinates of A, B and C be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, respectively

Hence, the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Since the given plane passes through (f, g, h), we get

$$\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 1 \quad \text{...(i)}$$

The equation of the sphere passing through 0, A, B and C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Let (α, β, γ) be its centre.

Thus, $\alpha = \frac{a}{2}, \beta = \frac{b}{2}, \gamma = \frac{c}{2}$

$\Rightarrow \quad a = 2\alpha, b = 2\beta, c = 2\gamma$

Putting the values of a, b and c in (i), we get

$$\frac{f}{2\alpha} + \frac{g}{2\beta} + \frac{h}{2\gamma} = 1$$

$$\Rightarrow \quad \frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} = 2$$

Hence, the locus of the centre (α, β, γ) is

$$\frac{f}{x} + \frac{g}{y} + \frac{h}{z} = 2$$

23. Let O be the origin (0, 0, 0). Let the co-ordinates of the points A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, respectively. Then the equation of the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

Also, the equation of the sphere passing through O, A, B, and C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

But the radius of the sphere is given by r.

So, $r = \sqrt{\frac{a^2 + b^2 + c^2}{4}}$

$$\Rightarrow \quad (a^2 + b^2 + c^2) = 4r^2 \quad \text{...(ii)}$$

Let (α, β, γ) be the foot of the perpendicular from the origin O to the plane (i)

Now, the equation of the line through O and the perpendicular to the plane (i) is

$$\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c} = r(\text{say})$$

Thus, the co-ordinates of any point on the line are

$$\left(\frac{r}{a}, \frac{r}{b}, \frac{r}{c}\right) \quad \text{...(iii)}$$

If it is the foot of the perpendicular from O to the plane (i), its co-ordinates will satisfy the equation of the plane.

Hence, $\frac{1}{a}\cdot\frac{r}{a} + \frac{1}{b}\cdot\frac{r}{b} + \frac{1}{c}\cdot\frac{r}{c} = 1$

$$\Rightarrow \quad r = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

Then from Eq. (iii), we get

$$\alpha = \frac{r}{a} = \frac{1/a}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

and $\beta = \frac{r}{b} = \frac{1/b}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$

and $\gamma = \frac{r}{c} = \frac{1/c}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$

Now, $\alpha^2 + \beta^2 + \gamma^2$

$$= \frac{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^2} = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

$$= a\alpha = b\beta = c\gamma$$

Thus, $a = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, b = \frac{\alpha^2 + \beta^2 + \gamma^2}{\beta}$

and $c = \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}$

Putting the values of a, b and c in (ii), we get

$$(\alpha^2 + \beta^2 + \gamma^2)\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) = 4r^2$$

Hence, the locus of the foot of the perpendicular (α, β, γ) is

$$(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4r^2$$

24. Given plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Let O be the origin.

Clearly, the plane intersects the axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, respectively.

Hence, the equation of the sphere passing through O, A, B and C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

25. The equation of any sphere through the given circle is $x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2\lambda\lambda + (7 + 3\lambda)y + (4\lambda - 2)z + (2 - 8\lambda) = 0$

The centre of the sphere is

$$\left(-\lambda, -\frac{7 + 3\lambda}{2}, -\frac{4\lambda - 2}{2}\right)$$

If the given circle is a great circle of the sphere, so the centre of the sphere lies on the plane

Thus, $2(-\lambda) + 3\left(-\frac{7 + 3\lambda}{2}\right) + 4(1 - 2\lambda) = 8$

$\Rightarrow \quad -4\lambda - 21 - 9\lambda + 8 - 16\lambda = 16$

$\Rightarrow \quad -29\lambda = 29$

$\Rightarrow \quad \lambda = -1$

Hence, the equation of the sphere is

$x^2 + y^2 + z^2 + 7y - 2z + 2 - 2x - 3y - 4z + 8 = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

Level (I)

(Problems For JEE-Advanced)

1. Given $al + bm + cn = 0$
and $fmn + gnl + hlm = 0$
Eliminating n, we get

$$fm\left(-\frac{al + bm}{c}\right) = gl\left(-\frac{al + bm}{c}\right) + hlm = 0$$

$\Rightarrow \quad -afm - bfm^2 - agl^2 - bglm + chlm = 0$

$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (af + bg - ch)\left(\frac{l}{m}\right) + bf = 0$

Let its roots are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

Now,

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{bf}{ag}$$

$$\Rightarrow \quad \frac{l_2 l_2}{bf} = \frac{m_1 m_2}{ag}$$

$$\Rightarrow \quad \frac{l_2 l_2}{bf} = \frac{m_1 m_2}{ag} = \frac{n_1 n_2}{ch}$$

$$\Rightarrow \quad \frac{l_2 l_2}{(f/a)} = \frac{m_1 m_2}{(g/b)} = \frac{n_1 n_2}{(h/c)}$$

(i) If the lines are perpendicular, so

$l_1 l_2 = m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow \quad \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

(ii) If the lines are parallel, so

$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

So, the roots of the Eq. (i) have equal roots, i.e. $D = 0$

$\Rightarrow \quad (af + bg - ch)^2 = 4abfg$

$\Rightarrow \quad (af + bg - ch) = \pm 2\sqrt{af}\ \sqrt{bg}$

$\Rightarrow \quad (af + bg \pm 2\sqrt{af}\sqrt{bg}) = ch$

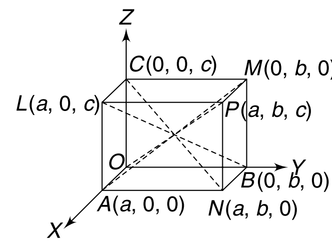
$\Rightarrow \quad (\sqrt{af} \pm \sqrt{bg})^2 = (\sqrt{ch})^2$

$\Rightarrow \quad (\sqrt{af} \pm \sqrt{bg}) = \pm(\sqrt{ch})$

$\Rightarrow \quad (\sqrt{af} \pm \sqrt{bg} + \sqrt{ch}) = 0$

which is the required condition.

2. Let $OA = a$, $OB = b$ and $OC = c$



Now, OP, CN, AM and BL are four diagonals.
Let θ be the angle between the diagonals OP and AM.
The direction ratios of OP and AM are
(a, b, c) and $(-a, b, c)$

Thus, $$\cos\theta = \frac{-a\cdot a + b\cdot b + c\cdot c}{\sqrt{a^2 + b^2 + c^2}\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{-a^2 + b^2 + c^2}{(a^2 + b^2 + c^2)}$$

Similarly, we can find the angle between other pairs of diagonals and we have six such pairs out of these four digonals and all these angles are given by

$$\cos\theta = \frac{\pm a^2 \pm b^2 + c^2}{(a^2 + b^2 + c^2)}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$$

3. The equation of any plane passing through P, Q, R is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \quad \text{...(i)}$$

where $P = (\alpha, 0, 0)$, $Q = (0, \beta, 0)$, $R = (0, 0, \gamma)$
which is passing through (a, b, c)

Thus, $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$

The equation of any plane passing through $Q(0, \beta, 0)$ and parallel to yz-plane is $x = \alpha$
The equation of any plane passing through $R(0, 0, \gamma)$ and parallel to zx-plane is $y = \beta$
The Equation of any plane passing through $P(\alpha, 0, 0)$ and parallel to xy-plane is $z = \beta$
Hence, the locus of the point (α, β, γ) is

$$\frac{x}{a} + \frac{b}{y} + \frac{c}{z} = 1$$

4. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad ...(i)$$

It is given that,

$$a^2 + b^2 + c^2 = k^2 \quad ...(ii)$$

Let $P(\alpha, \beta, \gamma)$ be the foot of the perpendicular drawn from the origin O to the plane (i).

Now the equation of the line through (0, 0, 0) and the perpendicular to the plane (i) is

$$\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c}$$

So, the co-ordinates of any point on the line are

$$\left(\frac{r}{a}, \frac{r}{b}, \frac{r}{c}\right) \quad ...(iii)$$

It is the foot of the perpendicular from O to the plane (i), its co-ordinates will satisfy the equation of the plane.

Hence, $\frac{1}{a}\cdot\frac{r}{a} + \frac{1}{b}\cdot\frac{r}{b} + \frac{1}{c}\cdot\frac{r}{c} = 1$

$\Rightarrow \quad r\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 1$

$$r = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

Then from (iii)

$$\alpha = \frac{r}{a} = \frac{1/a}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

$$\beta = \frac{r}{b} = \frac{1/b}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

and $\gamma = \frac{r}{c} = \frac{1/c}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$

Now, $\alpha^2 + \beta^2 + \gamma^2$

$$= \frac{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^2}$$

$$= \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} = a\alpha = b\gamma = c\gamma$$

Thus, $a = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, \; b = \frac{\alpha^2 + \beta^2 + \gamma^2}{\beta}$

and $c = \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}$

Putting the values of a, b and c in Eq. (ii), we get

$$(\alpha^2 + \beta^2 + \gamma^2)\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) = k^2$$

Hence, the locus of $P(\alpha, \beta, \gamma)$ is

$$(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = k^2$$

5. Clearly, the planes $x = 0$, $y = 0$ and $z = 0$ meet in (0, 0, 0)

Hence, the incentre lies on the perpendicular from (0, 0, 0) to the plane $x + y + z = a$ and divides it in the ratio 3 : 1, i.e. 3 from the vertex (0, 0, 0) and 1 from the plane $x + y + z = a$. The equation of the perpendicular from (0, 0, 0) to the plane $x + y + z = a$ is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = \lambda$(say)

Any point on the perpendicular is $(\lambda, \lambda, \lambda)$

If it lies on the plane $x + y + z = a$, so, $\lambda = \frac{a}{3}$

Thus, the perpendicular from (0, 0, 0) meets the plane $x + y + z = a$ in $(\lambda + \lambda + \lambda)$, i.e. $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$.

Also, the incentre divides the join of (0, 0, 0) and $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$ in the ratio 3 : 1.

Let (α, β, γ) be its incentre. Then

$$\alpha = \frac{3\cdot\frac{a}{3} + 1.0}{3 + 1} = \frac{a}{4}$$

Similarly, $\beta = \frac{a}{4} = \gamma$

Thus, the required incentre is $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4}\right)$.

6. The equation of any line passing through (3, 5, 7) and parallel to the normal to the plane is

$$\frac{x - 3}{2} = \frac{y - 5}{1} = \frac{z - 7}{1} \quad ...(i)$$

and the equation of any point on the line (i) is

$$M(2\lambda + 3, \lambda + 5, \lambda + 7)$$

which lies on the plane $2x + y + z = 16$.

So, $(2\lambda + 3) + (\lambda + 5) + (\lambda + 7) = 16$

$\Rightarrow \quad 4\lambda + 15 = 16$

$\Rightarrow \quad \lambda = \frac{1}{4}$

Thus, $M = \left(\frac{1}{2} + 3, \frac{1}{2} + 5, \frac{1}{2} + 7\right)$

$$= \left(\frac{7}{2}, \frac{11}{2}, \frac{15}{2}\right)$$

Let $Q(a, b, c)$ is the image of $P(3, 5, 7)$.

Clearly, M is the mid-point of P and Q.

Thus, $\frac{a+3}{2}=\frac{7}{2}, \frac{b+5}{2}=\frac{11}{2}, \frac{c+7}{2}=\frac{15}{2}$

$\Rightarrow \quad a=4, b=6, c=8$

Hence, the value of

$$a^2+b^2+c^2+18$$
$$=16+36+64+18$$
$$=134.$$

7. Given lines are

$$\frac{x-2}{a}=\frac{y+1}{b}=\frac{z}{c} \text{ and } \frac{x+1}{b}=\frac{y-3}{c}=\frac{z-1}{a}$$

Therefore, $\cos\theta=\frac{ab+bc+ca}{a^2+b^2+c^2}$ ...(i)

Since a, b and c are the roots of $x^3+x^2-4x-4=0$, so

$$a+b+c=-1$$
$$ab+bc+ca=4$$

and $\quad abc=4$

Now,

$$a^2+b^2+c^2$$
$$=(a+b+c)^2-2(ab+bc+ca)$$
$$=1+8=9$$

From Eq. (i), we get

$$\cos\theta=-\frac{4}{9}$$

Hence, the acute angle between the lines is

$$\theta=\cos^{-1}\left(\frac{4}{9}\right)$$

8. Since the planes $x=cy+bz$, $y=az+cx$ and $z=bx+ay$ meet in a line, so

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}$$

$\Rightarrow \quad 1(1-a^2)+c(-c-ab)-b(ca+b)=0$

$\Rightarrow \quad (1-a^2)+(-c^2-abc)+(-abc-b^2)=0$

$\Rightarrow \quad 1-a^2-b^2-c^2-2abc=0$

$\Rightarrow \quad a^2+b^2+c^2+2abc=1$

Let l, m, n be the direction cosines of the line of intersection. Therefore,

$$l-cm-bn=0$$

and $\quad cl-m+an=0$

Thus, $\frac{l}{ac+b}=\frac{m}{bc+a}=\frac{n}{1-c^2}$

Now,

$$(ac+b)^2=a^2c^2+b^2+2ab$$
$$=a^2c^2+b^2+1-a^2-b^2-c^2$$
$$=1-a^2-c^2+a^2c^2$$
$$=(1-a^2)(1-c^2)$$

Thus,

$$(ac+b)=\sqrt{(1-a^2)(1-c^2)}$$

Similarly, $(bc+a)=\sqrt{(1-b^2)(1-c^2)}$

Hence, $\frac{l}{\sqrt{1-a^2}}=\frac{m}{\sqrt{1-b^2}}=\frac{n}{\sqrt{1-c^2}}$

Since the three given planes pass through the origin, so the line of intersection of the planes also pass through the origin.

Thefore, the equation of the line is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

$\Rightarrow \quad \frac{x}{\sqrt{1-a^2}}=\frac{y}{\sqrt{1-b^2}}=\frac{z}{\sqrt{1-c^2}}$

9. Given planes are

$$x-y-z=2 \quad \text{...(i)}$$

and $\quad x+2y+z=2$ ...(ii)

The equation of any plane passing through the line of intersection of planes (i) and (ii) can be written as

$$(x-y-z-2)+\lambda(x+2y+z-2)=0 \quad \text{...(iii)}$$

The direction of the cosines of the normal to the plane (iii) are

$$\frac{1+\lambda}{\sqrt{6\lambda^2-4\lambda+3}}, \frac{2\lambda-1}{\sqrt{6\lambda^2-4\lambda+3}}, \frac{\lambda-1}{\sqrt{6\lambda^2-4\lambda+3}}$$

The direction cosines of the normal to the plane (i) are

$$\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$$

Since the angle between the planes (i) and (ii) is 90°, so,

$$\cos(90°)=\frac{(1+\lambda)}{p\sqrt{3}}-\frac{(2\lambda+1)}{p\sqrt{3}}-\frac{(\lambda-1)}{p\sqrt{3}}$$

where $\quad p=\frac{1}{\sqrt{6\lambda^2-4\lambda+3}}$

$\Rightarrow \quad (1+\lambda)-(2\lambda+1)-(\lambda-1)=0$

$\Rightarrow \quad \lambda=\frac{3}{2}$

Hence, the required equation of the plane is

$$(x-y-z-2)+\frac{3}{2}(x+2y+z-2)=0$$

$\Rightarrow \quad 2(x-y-z-2)+3(x+2y+z-2)=0$

$\Rightarrow \quad 5x+4y+z=10$

10. Given plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{...(i)}$$

Let $P(m, n, p)$ be a point on (i), so

$$\frac{m}{a}+\frac{n}{b}+\frac{p}{c}=1 \quad \text{...(ii)}$$

i.e $\quad OP = \sqrt{m^2 + n^2 + p^2}^2$

The direction cosines of OP are

$$\left(\frac{m}{\sqrt{m^2+n^2+p^2}^2}, \frac{n}{\sqrt{m^2+n^2+p}}, \frac{p}{\sqrt{m^2+n^2+p^2}^2}\right)$$

The equation of the plane through P and normal to OP is

$$\frac{mx}{\sqrt{m^2+n^2+p^2}^2} + \frac{ny}{\sqrt{m^2+n^2+p^2}^2} + \frac{pz}{\sqrt{m^2+n^2+p^2}^2} = \sqrt{m^2+n^2+p^2}^2$$

$$\Rightarrow \quad mx + ny + pz = (m^2+n^2+p^2)$$

Thus, $\quad A = \left(\frac{(m^2+n^2+p^2)}{m}, 0, 0\right)$

$$B = \left(0, \frac{(m^2+n^2+p^2)}{n}, 0\right)$$

and $\quad C = \left(0, 0, \frac{(m^2+n^2+p^2)}{p}\right)$

Let the point Q be (α, β, γ)

Thus, $\quad \alpha = \frac{m^2+n^2+p^2}{m}, \beta = \frac{m^2+n^2+p^2}{n}$

and $\quad \gamma = \frac{m^2+n^2+p^2}{p}$ ...(iii)

Now,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{(m^2+n^2+p^2)}{(m^2+n^2+p^2)^2}$$

$$= \frac{1}{(m^2+n^2+p^2)}$$

From Eqs. (iii), $m = \frac{(m^2+n^2+p^2)}{\alpha}$

$$\Rightarrow \quad \frac{m}{a} = \frac{(m^2+n^2+p^2)}{a\alpha}$$

Similarly, $\frac{n}{b} = \frac{(m^2+n^2+p^2)}{b\beta}$

and $\quad \frac{p}{c} = \frac{(m^2+n^2+p^2)}{c\gamma}$

Now, $\quad \frac{(m^2+n^2+p^2)}{a\alpha} + \frac{(m^2+n^2+p^2)}{b\beta}$

$$+ \frac{(m^2+n^2+p^2)}{c\gamma} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c} = 1$$

$$\Rightarrow \quad \frac{1}{a\alpha} + \frac{1}{b\beta} + \frac{1}{c\gamma} = \frac{1}{m^2+n^2+p^2}$$

$$= \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

Hence, the required locus of Q is

$$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

Integer Type Questions

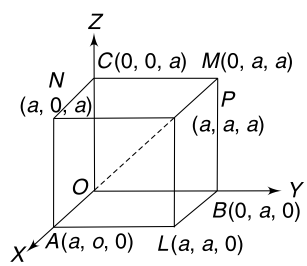
1. Let θ be angle between the lines.

$\therefore \quad \cos\theta = \frac{1}{3}$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$

Hence, $\quad (\alpha + \beta) = 1 + 3 = 4$

2. Four diagonals of a cube are OP, AM, CL, BN



Direction cosines of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Direction cosines of AM are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Direction cosines of CL are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

Direction cosines of BN are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Let l, m, n be the direction cosines of a line which is inclined at angles α, β, γ, δ.

Thus, $\quad \cos\alpha = \frac{l+m+n}{\sqrt{3}}$

Similarly, $\cos\beta = \frac{-l+m+n}{\sqrt{3}}, \cos\gamma = \frac{l+m-n}{\sqrt{3}}$

and $\quad \cos\delta = \frac{l-m+n}{\sqrt{3}}$

Thus, $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l+m+n)^2 + (l-m+n)^2]$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)]$$

$$= \frac{4}{3}$$

Thus,

$$3(\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta) = 4$$

3. The image of $P(1, 6, 3)$ w.r.t. the line

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} \text{ is } Q(1, 0, 7)$$

It is given that $Q(\alpha, \beta, \gamma) = Q(1, 0, 7)$

Thus, $\alpha + \beta + \gamma = 8$

4. Clearly, $3a + 2b + c = 7$

Thus, the minimum value of $(a^2 + b^2 + c^2)$ is

$$= \left|\frac{3.0 + 2.0 + 0 - 7}{\sqrt{3^2 + 2^2 + 1^2}}\right|^2$$

$$= \frac{49}{14} = \frac{7}{2}$$

$$\Rightarrow \quad (a^2 + b^2 + c^2) \geq \frac{7}{2}$$

$$\Rightarrow \quad 2(a^2 + b^2 + c^2) \geq 7$$

Hence, the least value of $(a^2 + b^2 + c^2)$ is 7.

5. The equation of any line passing through $(1, 0, -3)$ and parallel to $\frac{x-2}{2} = \frac{y+2}{3} = \frac{6-z}{6}$

is $$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} \quad \text{...(i)}$$

Any point on (i) can be considered as

$$P(2\lambda + 1, 3\lambda, -34 - 6\lambda)$$

which is also a point of the plane

$$2\lambda + 1 - 3\lambda + 3 + 6\lambda = 9$$

$$\Rightarrow \quad 5\lambda + 4 = 9$$

$$\Rightarrow \quad \lambda = 1$$

Thus, the point P is $(3, 3, -9)$.

Hence, the required distance

$$= \sqrt{(3-1)^2 + (3-0)^2 + (-9+3)^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

6. The equation of any plane containing the given line is $(x + y + 2z - 3) + \lambda(2x + 3y + 4z - 4) = 0$

$$\Rightarrow \quad (1+\lambda)x + (1+3\lambda)y + (2+4\lambda)z - (3+4\lambda) = 0 \quad \text{...(i)}$$

If the plane is parallel to z-axis, i.e $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$, the normal to the plane (i) is perpendicular to z-axis

$$(1 + \lambda(0) + (1 + 3\lambda)(0) + (2 + 4\lambda)(1) = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(x + y + 2z - 3) - \frac{1}{2}(2x + 3y + 4z - 4) = 0$$

$$\Rightarrow \quad y + 2 = 0 \quad \text{...(iii)}$$

Shortest Distance = Distance of any point on z-axis

$$= \frac{2}{\sqrt{1^2}} = 2$$

7. Let the points be A, B, C and D. The number of planes which have three points on one side and the fourth point on the othe side is 4.

The number of planes which have two points on each side of the plane is 3. Thus, the number of planes is 7.

8. Let the equation of the any plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

Since the plane (i) containing the lines, so

$$2a + 3b + 4c = 0$$

and $$3a + 4b + 4c = 0.$$

Therefore, $$\frac{a}{15-16} = \frac{b}{12-10} = \frac{c}{8-9}$$

$$\Rightarrow \quad \frac{a}{-1} = \frac{b}{2} = \frac{c}{-1}$$

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$$

Also, $$a(x-1) + b(y-2) + c(z-3) = 0$$

$$\Rightarrow \quad (x-1) - 2(y-2) + (z-3) = 0$$

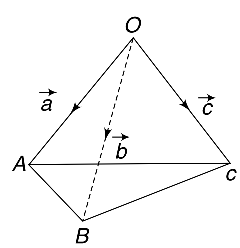
$$\Rightarrow \quad x - 2y + z = 0$$

Again distance between the plane is $\sqrt{6}$.

$$\Rightarrow \quad \left|\frac{d-0}{\sqrt{6}}\right| = \sqrt{6}$$

$$\Rightarrow \quad |d| = 6.$$

9. We shall find the angle between the faces OAB and OAC. The angle between the faces OAB and OAC is the angle between the normal to the faces. The normal to the face OAB is $\vec{a} \times \vec{b}$ and the normal to face OAC is $\vec{a} \times \vec{c}$.



Thus, $$\cos\theta = \frac{(\vec{a} \times \vec{b}) \cdot |(\vec{a} \times \vec{c})}{|(\vec{a} \times \vec{b})||(\vec{a} \times \vec{c})|}$$

$$= \frac{\vec{a} \cdot (\vec{b} \times (\vec{a} \times \vec{c}))}{\{1 \cdot 1 \cdot \sin(60°)\}^2}$$

$$= \frac{\vec{a}\cdot\{(\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{b})\vec{c}\}}{3/4}$$

$$= \frac{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})}{3/4}$$

$$= \frac{1(1\cdot 1\cdot\cos(60°)) - (1\cdot 1\cdot\cos(60°))^2}{3/4}$$

$$= \frac{\frac{1}{2} - \frac{1}{4}}{3/4} = \frac{1}{3}$$

Hence, the value of $(3\cos\theta + 2) = 1 + 2 = 3$.

10. Clearly, $l + \pm m = \pm n$

$\Rightarrow \quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad l = m = n = \pm\frac{1}{\sqrt{3}}$

Hence, 8 possible direction cosines and 4 lines are possible.

11. Since the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$

and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, so

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & 1-k \\ k & 2+k & 1+k \end{vmatrix} = 0$$

$\Rightarrow \quad -\{2(1+k) + (k-1)(k+2)\} = 0$

$\Rightarrow \quad -\{2 + 2k + k^2 + k - 2\} = 0$

$\Rightarrow \quad k^2 + 3k = 0$

$\Rightarrow \quad k = 0, -3$

Hence, the value of $(k + 5)$ is 5 or 2.

12. Let $\frac{x-2}{1} = \frac{y-3}{2} = \frac{\frac{9}{2} - z}{k} = \lambda$

and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z}{1} = \mu$

Thus, $x + \lambda + 2,\ y = 2\lambda + 3,\ z = \frac{9}{2} - k\lambda$

and $x = \mu + 1,\ y = 2\mu + 4,\ z = \mu$

Solving, we get

$$\lambda = 1,\ \mu = \frac{1}{2} \text{ and } k = 4$$

Hence, the value of k is 4.

13. Given planes are

$x = cy + bz$

$y = cx + az$

$z = bx + ay$

$\Rightarrow \quad x - cy - bz = 0$

$cx - y + az = 0$

$bx + ay - z = 0$

Since these three planes pass through a line, so

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(1 - a^2) + c(-c - ab) - b(ca + b) = 0$

$\Rightarrow \quad (1 - a^2) - (c^2 + abc) - (abc + b^2) = 0$

$\Rightarrow \quad 1 - a^2 - b^2 - c^2 - 2abc = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc + 2 = 1 + 2 = 3$

14. Since the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$

So, the point $(4, 2, k)$ lies in the plane

Thus, $2(4) - 4(2) + = 7$

$\Rightarrow \quad 8 - 8 + k = 7$

$\Rightarrow \quad k = 7$

15. Given plane is

$$ax - by + cz = 0 \qquad \text{....(i)}$$

and the line is

$$\frac{x-a}{a} = \frac{y-2d}{b} = \frac{z-c}{c} \qquad \text{...(ii)}$$

Since the plane (i), contains the line (ii), so,

$a(a) - b(b) + c(c) = 0$

$a(a) - b(2d) + c(c) = 0$

$\Rightarrow \quad a^2 + c^2 = b^2$

$a^2 + c^2 = 2bd$

Clearly, $b^2 = 2bd$

$= \frac{b}{d} = 2$

Questions asked in Past IIT-JEE Examinations

1. Since the line lies on the plane, so, the point $(4, 2, k)$ lies on the plane.

Thus, $8 - 8 + k = 7$

$\Rightarrow \quad k = 7.$

2. The equation of the plane passing through (2, 1, 0), and (4, 1, 1) is

$$\begin{vmatrix} x-2 & y-1 & z \\ 3 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)(-1-0)-(y-1)(3-2)+z(0+2)=0$

$\Rightarrow \quad x + y - 2z = 3$

3. The equation of the line PQ is

$$\frac{x-2}{1} = \frac{y-1}{2} = \frac{z-6}{-2}$$

Any point on the above line is

$$Q(\lambda + 2, \lambda + 1, 6 - 2\lambda)$$

The mid-point of PQ is

$$\left(\frac{\lambda}{2} + 2, \frac{\lambda}{2} + 1, 6 - \lambda\right)$$

Since the mid-point of PQ lies on the plane, so,

$$\left(\frac{\lambda}{2} + 2\right) + \left(\frac{\lambda}{2} + 1\right) - 2(6 - \lambda) = 3$$

$\Rightarrow \quad 3\lambda = 12$

$\Rightarrow \quad \lambda = 4.$

Thus, the co-ordinates of Q are (6, 5, −2).

4. Since the given lines intersect, so they are coplanar.

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad -10 - (k + 1)(2 - 4) - (4 - 3) = 0$

$\Rightarrow \quad -10 + 2(k + 1) - 1 = 0$

$\Rightarrow \quad 2k = 9$

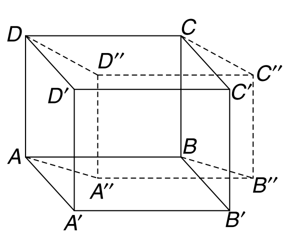
$\Rightarrow \quad k = \frac{9}{2}.$

5. Let $\mathbf{AA'} = \mathbf{a}$, $\mathbf{AB} = \mathbf{b}$, $\mathbf{AD} = \mathbf{d}$

Volume of the parallelopiped T,

$$V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Let $\quad \mathbf{AA''} = \mathbf{r}$



Let V_1, the volume of the new box = $\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c})$

It is given that $V_1 = \frac{90}{100} V_2 = \frac{9}{10} V_2$.

Thus, $\quad \mathbf{V_1} = (\mathbf{r} - 0 \cdot 9\mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c})$

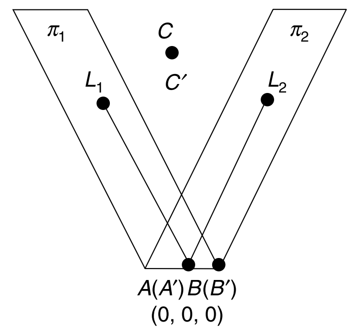
$\Rightarrow \quad$ **r** lies on a plane passing through $\mathbf{0 \cdot 9 a}$, and perpendicular to $(\mathbf{b} \times \mathbf{c})$.

6. We take $A = A'$ as the origin

and $B = B'$ = any point other than A on the line intersection of π_1 and π_2.

Now, consider $C = C'$ = any point niether on π_1 nor π_2.

Thus, in this case, both the conditions of (a) and (b) are fullfilled.

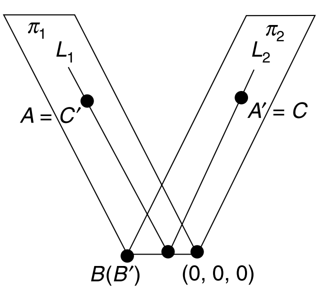


Similarly if we take,

A = non-origin point on L_1

B = non-origin point on the line of intersection of π_1 and π_2 and C = non-origin point on L_2.

If we take $A = C''$, $B = B'$, $C = A'$, both the conditions of (a) and (b) are fulfilled.



7. As the plane π parallel to $\mathbf{b}$ = (1, 0, −1) and $\mathbf{c}$ = (−1, 1, 0) normal to the plane is given by

$$\mathbf{b} \times \mathbf{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$\therefore$ The equation of the plane ABC is

$$1 \cdot (x - 1) + 1 \cdot (y - 1) + 1 \cdot (z - 1) = 0$$

$\Rightarrow \quad x + y + z - 3 = 0$

$\Rightarrow \quad x + y + z = 3$

$\Rightarrow \quad \frac{x}{3} + \frac{y}{3} + \frac{z}{3} = 1.$

This planes meets the axes in $A(3, 0, 0)$, $B(0, 3, 0)$, $C(0, 0, 3)$.

Thus, volume of the tetrahedron $OABC$

$$= \frac{1}{6}[\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC}]$$

$$= \frac{1}{6}[3\mathbf{i}\ 3\mathbf{j}\ 2\mathbf{k}]$$

$$= \frac{27}{6}[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]$$

$$= \frac{9}{2}$$

8. The equation of the plane passing through the line of

intersection $2x - y + z = 3$, $3x + y + z = 5$

is

$(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0 \quad$ ...(i)

$\Rightarrow \quad (2 + 3\lambda)x + (\lambda - 1)y + (\lambda + 1)z = (5\lambda + 3) = 0$

Also, it is given that, the distance from $(2, 1, -1)$ to the above plane is $\frac{1}{\sqrt{6}}$.

$$\left|\frac{2(2 + 3\lambda) + (\lambda - 1) - (\lambda + 1) - (5\lambda + 3)}{\sqrt{(2 + 3\lambda)^2 + (\lambda - 1)^2 + (\lambda + 1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$\Rightarrow \quad \left|\frac{\lambda - 1}{\sqrt{11\lambda^2 + 12\lambda + 6}}\right| = \frac{1}{\sqrt{6}}$

$\Rightarrow \quad 6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6$

$\Rightarrow \quad 5\lambda^2 + 24\lambda = 0$

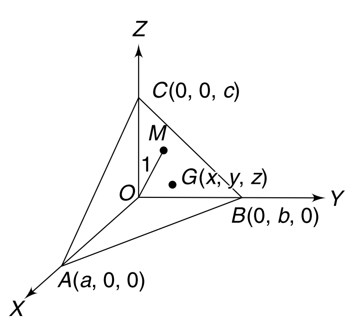
$\Rightarrow \quad \lambda = 0, -\frac{24}{5}.$

Putting the values of $\lambda = 0, -\frac{24}{5}$ in Eq. (i), we get the required equations of the planes are

$$2x - y + z = 3$$

or $\quad 62x + 29y + 19z - 105 = 0$

9.



Equation of any plane passing through A, B, C

is $\quad \frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} = 1$

Thus, centroid $= G(x, y, z)$

$$= \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right).$$

Also, it is given that $OM = 1$

$\Rightarrow \quad \left|\frac{0 + 0 + 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right| = 1$

$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1$

$\Rightarrow \quad \frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = 1$

$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9$

10. The equation of any plane passing through $(1, -2, 1)$ is

$$a(x - 1) + b(y + 2) + c(z - 1) = 0 \quad \text{...(i)}$$

The plane (i) is perpendicular to the planes

$$2x - 2y + z = 0 \text{ and } x - y + 2z = 4$$

So, $\quad 2a - 2b + c = 0$

and $\quad a - a + 2c = 0$

Thus, $\quad \frac{a}{-4 + 1} = \frac{b}{1 + 2} = \frac{c}{-2 + 2}$

$\Rightarrow \quad \frac{a}{-3} = \frac{b}{3} = \frac{c}{0}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{-1} = \frac{c}{0}$

Therefore, the equation of the plane is

$$(x - 1) - (y + 2) = 0$$

$\Rightarrow \quad x - y - 3 = 0.$

Thus, the distance of π from the point $(1, 2, 2)$

is $\quad \left|\frac{1 - 2 - 3}{\sqrt{1 + 1}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$

11. Given $\quad \mathbf{k} \times (\mathbf{k} \times \mathbf{a}) = \mathbf{0}$

$\Rightarrow \quad (\mathbf{k} \cdot \mathbf{a})\mathbf{k} - (\mathbf{k} \cdot \mathbf{k})\mathbf{a} = \mathbf{0}$

$\Rightarrow \quad \gamma\mathbf{k} - \mathbf{a} = \mathbf{0}$

$\Rightarrow \quad \gamma\mathbf{k} - (\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}) = \mathbf{0}$

$\Rightarrow \quad -(\alpha\mathbf{i} + \beta\mathbf{j}) = 0$

$\Rightarrow \quad -(\alpha\mathbf{i} + \beta\mathbf{j}) = 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j}$

$\Rightarrow \quad \alpha = 0, \beta = 0$

Since the point (α, β, γ) lies on the plane

$$x + y + z = 2, \text{ so}$$

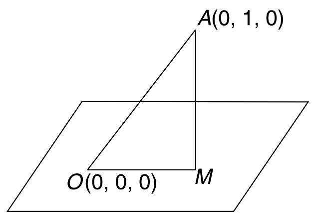
$$\alpha + \beta + \gamma = 2$$

$\Rightarrow \quad 0 + 0 + \gamma = 2$

$\Rightarrow \quad \gamma = 2$

12. The equation of any line passing through (0, 1, 0) and perpendicular to $x + 2y + 2z = 0$ is

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z}{2}.$$



Clearly $\quad OA = 1,$

$$AM = \frac{2}{\sqrt{1+4+4}} = \frac{2}{3}$$

Thus, $\quad OM^2 = OA^2 - AM^2 = 1 - \frac{4}{9} = \frac{5}{9}$

$\Rightarrow \quad OM = \frac{\sqrt{5}}{3}.$

13. The equation of any plane passing through (1, 2, 3) is

$$a(x-1) + b(y-2) + c(z-3) = 0$$

which is perpendicular to $x = 0$ and $y = 0$

So, $\quad a = 0, b = 0.$

Thus, the equation of the plane is

$$c(z-3) = 0$$

$\Rightarrow \quad (z-3) = 0$

Thus, the required distance from (0, –1, 0) to the plane $(z-3) = 0$ is

$$\left|\frac{0-3}{\sqrt{1}}\right| = 3.$$

14. Let a, b, c be the direction ratios of the line of intersection of $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Then $\quad 3a - 6b - 2c = 0$

and $\quad 2a + b - 2c = 0$

Thus, $\quad \frac{a}{12+2} = \frac{b}{-4+6} = \frac{c}{3+12}$

$\Rightarrow \quad \frac{a}{14} = \frac{b}{2} = \frac{c}{15}$

Clearly, the vector $14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ is parrallel to the line of intersections of the plane.

Now, we shall find the equation of the line put $z = 0$ in the given planes.

So $\quad 3x - 6y = 15$ and $2x + y = 5.$

On solving, we get

$$x = 3 \text{ and } y = -1$$

Hence, the equation of the line is

$$\frac{x-3}{14} = \frac{y+1}{2} = \frac{z}{15} = t \text{ (say)}$$

The parametric equations of the line are

$$x = 14t + 3,\ y = 2t - 1,\ z = 15t.$$

15. The given system of equations $ax + by + cz = 0$, $bx + cy + az = 0$, and $cx + ay + bz = 0$ can be written as

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$AX = O$$

Now, $\quad |A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= -(a^3 + b^3 + c^3 - 3abc)$$

$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

(a) Let $a + b + c = 0$

Then $(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \quad \frac{1}{2}((a-b)^2 + (a-b)^2 + (a-b)^2) = 0$

$\Rightarrow \quad (a-b) = 0 = (b-c) = (c-a)$

$\Rightarrow \quad a = b = c$

Thus, the three system of equations are identical planes.

i.e $\quad x + y + z = 0.$

(b) Suppose $a + b + c = 0$

and $\quad a^2 + b^2 + c^2 \neq ab + bc + ca$

In this case, at least one of the following is true.

$$a^2 \neq bc,\ b^2 \neq ca,\ c^2 = ab$$

Consider $b^2 \neq ac$

We can write the first two equations are

$$ax + by = (a+b)z$$

and $\quad bx + cy = (b+c)z$

Eliminating y, we get

$$(ac - b^2)x = (ac - b^2)z$$

$\Rightarrow \quad x = z$

Putting this in the first two equations, we get

$$ax + by + cx = 0$$

$\Rightarrow \quad by = -(a+c)x$

$by = bx$

$\Rightarrow \quad x = y$

Thus, $x = y = z$

(c) In this case $|A| \neq 0$.

So, the system of equations provide us a trivial solution.

Thus, $x = 0 = y = z$.

(d) Given

$a + b + c = 0,\ a^2 + b^2 + c^2 = ab + bc + ca$

$$\Rightarrow \quad a + b + c = 0,\ \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2) = 0$$

$\Rightarrow \quad a + b + c = 0,\ a = b = c$

$\Rightarrow \quad a = 0 = b = 0.$

Thus, the system of equations represent the whole three-dimensional space.

16. Let l, m, n be the direction ratios of the line L_1.

Then $\quad l + m - n = 0$

and $\quad l - 3m + 3n = 0$

Thus, $\quad \dfrac{l}{3-3} = \dfrac{m}{-1-3} = \dfrac{n}{-3-1}$

$\Rightarrow \quad \dfrac{l}{0} = \dfrac{m}{1} = \dfrac{n}{1}$

Thus, the direction ratios of L_1 are (0, 1, 1)

Similarly, the direction ratios of L_2 and L_3 are (0, 1, 1) and (0, 1, 1), respectively.

Therefore, the lines L_1, L_2 and L_3 are parallel.

So, Statement I is false.

Put $z = 0$ in

P_2: $x + y - z = -1$, P_3: $x - 3y + 3z = 2$

we get, $\quad x = -\dfrac{1}{4},\ y = -\dfrac{3}{4}$

Thus, any point on L_1 is $\left(-\dfrac{1}{4}, -\dfrac{3}{4}, 0\right)$.

So, the equation of the line L_1 is

$$\frac{x + 1/4}{0} = \frac{y + 3/4}{1} = \frac{z}{1}$$

which is parallel to the plane P_1.

Since $\left(-\dfrac{1}{4}, -\dfrac{3}{4}, 0\right)$ does not lie on P_1 , so no point of L_1 lies on P_1.

Therefore, the three planes do not have a common point.

Thus, the statement II is true.

17.

(i) The unit vector perpendicular to both L_1 and L_2 is $\dfrac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}$

Now, $\quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$

Thus, $\quad \dfrac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \dfrac{-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}}{5\sqrt{3}}.$

(ii) The shortest distance between L_1 and L_2

$$= \frac{(\mathbf{r}_2 \cdot \mathbf{r}_1) \cdot (\mathbf{u} \times \mathbf{v})}{|(\mathbf{u} \times \mathbf{v})|}$$

Here, $\quad \mathbf{r}_2 = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

and $\quad \mathbf{r}_1 = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\Rightarrow \quad \mathbf{r}_2 - \mathbf{r}_1 = 3\mathbf{i} + 4\mathbf{k}$

Thus, $\quad$ Shortest distance $= \left|\dfrac{-3 + 20}{5\sqrt{3}}\right| = \dfrac{17}{5\sqrt{3}}.$

(iii) The equation of any plane passing through (–1, –2, –1) is

$a(x + 1) + b(y + 2) + c(z + 1) = 0 \quad$ ...(i)

which is perpendiculat to the lines

$$L_1: \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$$

and $\quad L_2: \dfrac{x-2}{1} = \dfrac{y+2}{2} = \dfrac{z-3}{3}$

So, $\quad 3a + b + 2c = 0$

$\Rightarrow \quad a + 2b + 3c = 0$

Therefore, $\dfrac{a}{3-4} = \dfrac{b}{2-9} = \dfrac{c}{6-1}$

$\Rightarrow \quad \dfrac{a}{1} = \dfrac{b}{7} = \dfrac{c}{-5}.$

Hence, the equation of the plane is

$$(x + 1) + 7(y + 2) - 5(z + 1) = 0$$

$\Rightarrow \quad x + 7y - 5z + 10 = 0.$

Therefore, the required distance

$$= \left|\frac{1 + 7 - 5 + 10}{\sqrt{1 + 49 + 25}}\right| = \frac{13}{5\sqrt{3}}.$$

18. Let the equation of the line passing through $P(2, -1, 2)$ and makes equal angles with the co-ordinate axes is

$$\frac{x-2}{\frac{1}{\sqrt{3}}} = \frac{y+1}{\frac{1}{\sqrt{3}}} = \frac{z-2}{\frac{1}{\sqrt{3}}}$$

$\Rightarrow \quad \dfrac{x-2}{1} = \dfrac{y+1}{1} = \dfrac{z-2}{1}.$

Any point on the above line is

$$Q(\lambda + 2, + \lambda - 1, \lambda + 2)$$

Since the point Q lies on the plane, so

$\Rightarrow \quad 2(\lambda + 2) + (\lambda - 1) + (\lambda + 2) = 9$

$\Rightarrow \quad 4\lambda = 9 - 5 = 4$

$\Rightarrow \quad \lambda = 1.$

Thus, the point Q is (3, 0, 3).

Therefore, the length of PQ

$$= \sqrt{(3-2)^2 \; (0+1)^2 + (3-2)^2} = \sqrt{3}$$

19. The cartesian form of the given line is

$$\frac{x-1}{-3} = \frac{y+1}{1} = \frac{x-2}{5}$$

Let $\quad Q = (1 - 3\lambda, \lambda - 1, 5\lambda + 2)$

The direction ratios of PQ is $(-2 - 3\lambda, -\lambda - 3, 5\lambda - 4)$

Since **PQ** is parallel to the given plane.

So, $\quad (-2 - 3\lambda) - 4(\lambda - 3) + 3(5\lambda - 4) = 0.$

$\Rightarrow \quad \lambda = \frac{1}{4}$

Hence, the value of λ is $\frac{1}{4}$.

20. Let plane 1: $ax + by + cz = 0$ contains the line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

Thus, $\quad 2a + 3b + 4c = 0 \quad$ ...(i)

Let plane 2: $a; x + b'y + c'z = 0$ is perpendicular to the plane containing the lines

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{2} \text{ and } \frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

Thus, $\quad 3a' + 4b' + 2c' = 0$

$\Rightarrow \quad 4a' + 2b' + 2c' = 0$

Hence, $\quad \frac{a'}{12 - 4} = \frac{b'}{8 - 9} = \frac{c'}{6 - 16}$

$\Rightarrow \quad \frac{a'}{8} = \frac{b'}{-1} = \frac{c'}{-10}$

Also, Plane 1 is perpendicular to Plane 2

So, $\quad a \cdot a' + bb' \; c \cdot c' = 0$

$\Rightarrow \quad 8a - b - 10c = 0 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{a}{-30 + 4} = \frac{b}{32 + 20} = \frac{c}{-2 - 24}$$

$\Rightarrow \quad \frac{a}{-26} = \frac{b}{52} = \frac{c}{-26}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$

Hence, the euation of the required plane is

$$x - 2y + z = 0$$

21. Given $\quad PQ = 5$

$\Rightarrow \quad \left| \frac{1 - 4 - 2 - \alpha}{\sqrt{1 + 4 + 4}} \right| = 5$

$\Rightarrow \quad \left| \frac{\alpha + 5}{3} \right| = 5$

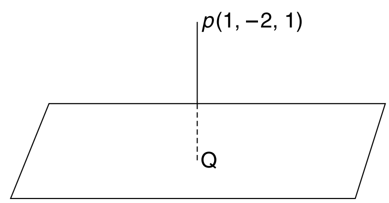
$\Rightarrow \quad \alpha + 5 = \pm 15$

$\Rightarrow \quad \alpha = -5 \pm 15 = 10, -15$

Since α is positive, so $\alpha = 10$.

The equation of the line PQ is

$$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2}.$$



Let $\quad Q = (\lambda + 1, 2\lambda - 2, 1 - 2\lambda)$

Also, $\quad PQ = 5$

$\Rightarrow \quad \lambda + 1 + 2(2\lambda - 2) - 2(1 - 2\lambda) = \alpha = 10$

$\Rightarrow \quad 9\lambda = 10 + 5$

$\Rightarrow \quad \lambda = \frac{15}{9} = \frac{5}{3}$

Thus the point Q is $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$

22. Let the equation of the any plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

Since Eqs. (i) containing the lines, so

$$2a + 3b + 4c = 0$$

and $\quad 3a + 4b + 5c = 0.$

Therefore, $\quad \frac{a}{15 - 16} = \frac{b}{12 - 10} = \frac{c}{8 - 9}$

$\Rightarrow \quad \frac{a}{-1} = \frac{b}{2} = \frac{c}{-1}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$

Also, $\quad a(x - 1) + b(y - 2) + c(z - 3) = 0$

$\Rightarrow \quad (x - 1) - 2(y - 2) + (z - 3) = 0$

$\Rightarrow \quad x - 2y + z = 0$

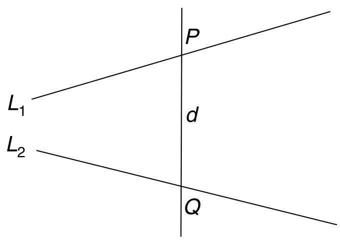
Again distance between the plane is $\sqrt{6}$.

$\Rightarrow \quad \left| \frac{d - 0}{\sqrt{6}} \right| = \sqrt{6}$

$\Rightarrow \quad |d| = 6.$

23. Let $L_1: \frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$

and $L_2: \frac{x-\frac{8}{3}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$



Let the line be

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \quad \text{...(i)}$$

intersects the lines L_1 and L_2.

So, the shortest distances between Eq. (i) and L_1, and Eq. (i) and L_2 are zero.

For PQ and L_1,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ 1 & -2 & 1 \end{vmatrix} = (b+2c)\mathbf{i} + (c-a)\mathbf{j} - (2a+b)\mathbf{k}$$

Since PQ is perpendicular to L_1

So, $2(b+2c) + (c-a) + (2a+b) = 0$

$\Rightarrow \quad a + 3b + 5c = 2$

For PQ and L_2

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ 2 & -1 & 1 \end{vmatrix} = (b+c)\mathbf{i} - (a-2c)\mathbf{j} - (a+2b)\mathbf{k}$$

Since PQ is perpendicular to L_2,

So, $\frac{8}{3}(b+c) + 3(a-2c) - (a+2b)\mathbf{k}$

$\Rightarrow \quad 3a + b - 5c = 0 \quad \text{...(ii)}$

Solving Eq. (i) and (ii), we get

$$\frac{a}{-15-5} = \frac{b}{15+5} = \frac{c}{1-9}$$

$\Rightarrow \quad \frac{a}{5} = \frac{b}{-5} = \frac{c}{2}$

Hence, the required equation of the line be

$$\frac{x}{5} = \frac{y}{-5} = \frac{z}{2}$$

Thus the point of intersections P and Q are

$$P = (5, -5, 2),\ Q = \left(\frac{10}{3}, -\frac{10}{3}, \frac{8}{3}\right)$$

Therefore, $d^2 = PQ^2 = 6$.

24. Given $[a\ b\ c]\begin{vmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{vmatrix} = [0\ 0\ 0]$

$\Rightarrow \quad a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

and $a + b + c = 0$

Solving, we get

$b = 6a,\ c = -7a$

Since the point P lies on the plane, so

$2a + b + c = 1$

$\Rightarrow \quad 2a + 6a - 7a = 1$

$\Rightarrow \quad a = 1$

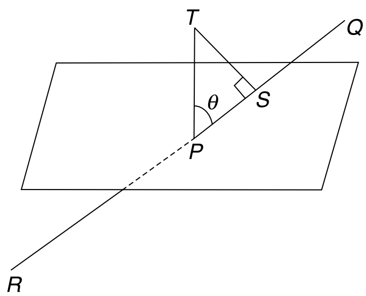
Thus, $b = 6$ and $c = -7$

Now the value of $7a + b + c$

$= 7 + 6 - 7$

$= 6.$

25.



The direction ratios of QR is $(1, 4, 1)$.

The co-rdinates of $P = \left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

The direction ratios of PT is $(2, 2, -1)$

The angle between OR and PT is

$$\cos\theta = \frac{2+8-1}{\sqrt{18}\sqrt{9}} = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

and the length of PT

$$= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$$

$\therefore \quad \cos\left(\frac{\pi}{4}\right) = \frac{PS}{1} \Rightarrow PS \Rightarrow \frac{1}{\sqrt{2}}$

$\sin\left(\frac{\pi}{4}\right) = \frac{TS}{1} \Rightarrow TS = \frac{1}{\sqrt{2}}$

Thus, $PS = TS = \frac{1}{\sqrt{2}}$.

26. The equation of the required plane is

$P: (x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$\Rightarrow\ (1+\lambda)x + (2-\lambda)y + (3+\lambda)z - (2+3\lambda) = 0$

Its distance from (3, 1, –1) is $\frac{2}{\sqrt{3}}$.

$$\Rightarrow \left|\frac{3(1+\lambda) + (2-\lambda) - (3+\lambda) - (2+3\lambda)}{\sqrt{(\lambda+1)^2 + (2-\lambda)^2 + (3+\lambda)^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{(-2\lambda)^2}{3\lambda^2 + 4\lambda + 14} = \frac{4}{3}$$

$\Rightarrow\ 4\lambda + 14 = 0$

$\Rightarrow\ \lambda = -\frac{7}{2}.$

Hence, the required equation of the plane is

$$-\frac{5}{2}x + \frac{11}{2}y - \frac{z}{2} + \frac{17}{2} = 0$$

$\Rightarrow\ -5x + 11y - z + 17 = 0$

$\Rightarrow\ 5x - 11y + z = 17$

27. Since given lines are coplanar, so we get

$$\begin{vmatrix} 2 & k & 2 \\ 5 & 2 & k \\ 2 & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow\ k^2 - 4 = 0$

$\Rightarrow\ k = \pm 2$

The equation of the plane containing these two lines is

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0$$

$\Rightarrow\ (x-1)(k^2-4) - (y+1)(2k-10) + z(4-5k) = 0$

$\Rightarrow\ -(y+1)(2k-10) + z(4-5k) = 0$

$\Rightarrow\ -(y+1)(\pm 4 - 10) + (4 - \pm 10) = 0$

Taking positive sign, we get

$(y+1) - z = 0$

$\Rightarrow\ y - z = -1$

Taking negative sign, we get

$-(y+1)(-14) + 14z = 0$

$\Rightarrow\ (y+1) + z = 0$

$\Rightarrow\ y + z = -1$

28. Any point P on the given line is

$(2\lambda - 2, -\lambda - 1, 3\lambda)$

The point P lies on the given plane for some λ

Thus, $(2\lambda - 2) + (-\lambda - 1) + 3\lambda = 3$

$\Rightarrow\ 4\lambda = 6$

$\Rightarrow\ \lambda = \frac{3}{2}$

So, the point P is $\left(1, -\frac{5}{2}, \frac{9}{2}\right)$.

The foot of the perpendicular from the point (–2, –1, 0) on the plane is the point $Q(0, 1, 2)$.

The direction ratio of $PQ = \left(1, -\frac{7}{2}, \frac{5}{2}\right) = (2, -7, 5)$

Hence, the equation of the line is

$$\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}.$$

29. The common perpendicular is along

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

Let $M = (2\lambda, -3\lambda, 2\lambda)$

So, $\frac{2\lambda - 3}{1} = \frac{-3\lambda + 1}{2} = \frac{2\lambda - 4}{2}$

$\Rightarrow\ \lambda = 1$

Thus, $M = (2, -3, 2)$

Let the required point be P.

Given,

$PM = \sqrt{17}$

$\Rightarrow\ (3 + 2s - 2)^2 + (3 + 2s + 3)^2 + (2 + s - 2)^2 = 17$

$\Rightarrow\ 9s^2 + 28s + 20 = 0$

$\Rightarrow\ (s+2)(9s+10) = 0$

$\Rightarrow\ s = -2, -\frac{10}{9}$

Thus, $P = (-1, -1, 0)$ or $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

30. Given lines are

$$L_1: \frac{x-5}{0} = \frac{-y}{\alpha - 3} = \frac{z}{-2}$$

and $$L_2: \frac{x-\alpha}{0} = \frac{y}{-1} = \frac{z}{2-\alpha}$$

Since L_1 and L_2 are coplanar, so

$$\begin{vmatrix} 5-\alpha & 0 & 0 \\ 0 & 3-\alpha & -2 \\ 0 & -1 & 2-\alpha \end{vmatrix} = 0$$

$\Rightarrow\ (5-\alpha)(3-\alpha)(2-\alpha) - 2) = 0$

$\Rightarrow\ (\alpha - 5)(\alpha^2 - 5\alpha + 4) = 0$

$\Rightarrow\ (\alpha - 5)(\alpha - 1)(\alpha - 4) = 0$

$\Rightarrow\ \alpha = 1, 4, 5.$

31. Ans. (A)

$$L_1: \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$$

$$L_2: \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$$

Normal of plane P: $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix}$

$= -16\mathbf{i} + 48\mathbf{j} + 32\mathbf{k}$

$\Rightarrow \quad \mathbf{n} = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

The point of intersection of L_1 and L_2 are

$2\lambda + 1 = \mu + 4, -\lambda = \mu - 3$

$1 = 3\mu - 2$

$\mu = 1$

Thus, the point of intersection is (5, −2, −1).

The equation of the plane passing through (5, −2, − 1) is

$a(x - 5) + b(y + 2) + c(z + 1) = 0$ ...(i)

Also, $ax + by + cz = d$ is perpendicular with

P_1: $7x + y + 2z = 3$, P_2: $3x + 5y - 6z = 4$

Thus, $7a + b + 2c = 0$

and $3a + 5b - 6c = 0$

Therefore, $\frac{a}{-6-10} = \frac{b}{6+42} = \frac{c}{35-3}$

$\Rightarrow \quad \frac{a}{-16} = \frac{b}{48} = \frac{c}{32}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{-3} = \frac{c}{-2}$

Now, from Eq. (i), we get

$(x - 5) - 3(y + 2) - 2(z + 1) = 0.$

$\Rightarrow \quad x - 3y - 2z = 13$

Thus, $a = 1, b = -3, c = -2, d = 13$

32. $L_1: \frac{x}{1} = \frac{y}{1} = \frac{z-1}{0} = r, Q(r, r, 1)$

$L_2: \frac{x}{1} = \frac{y}{-1} = \frac{z+1}{0} = s, Q(s, -s, -1,)$

$\mathbf{PQ} = (\lambda + r)\mathbf{i} + (\lambda - r)\mathbf{j} + (\lambda - 1)\mathbf{k}$

and $\lambda - r + \lambda - r = 0$, as **PQ** is $\perp t_0 L_1$

$\Rightarrow \quad 2\lambda = 2r$

$\Rightarrow \quad \lambda = r$

$\mathbf{PR} = (\lambda - s)\mathbf{i} + (\lambda + s)\mathbf{j} + (\lambda + 1)\mathbf{k}$

and $\lambda - s - \lambda - s = 0 \Rightarrow s = 0$

Since $\mathbf{PQ} \perp \mathbf{PR}$, so

$(\lambda - r)(\lambda - s) + (\lambda - r)(\lambda + s) + (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda + 1) = 0 \ (\because 1 = r)$

$\Rightarrow \quad \lambda = -1, 1$

For $\lambda = 1$, the point P and Q are concides

Hence, the value of $\lambda = -1$.

33. Let the required plane be $x + z + \lambda y - 1 = 0$

Now, $\left|\frac{\lambda - 1}{\sqrt{\lambda^2 + 2}}\right| = 1$

$\Rightarrow \quad (\lambda - 1)^2 = \lambda^2 + 2$

$\Rightarrow \quad -2\lambda + 1 = 2$

$\Rightarrow \quad \lambda = -\frac{1}{2}$

Thus, P_3: $2x - y + 2z - 2 = 0$

Distance of P_3 from (α, β, γ) is 2.

$\Rightarrow \quad \left|\frac{2\alpha - \beta + 2\gamma - 2}{\sqrt{4+1+4}}\right| = 2$

$\Rightarrow \quad 2\alpha - \beta + 2\gamma - 2 = \pm 6$

$\Rightarrow \quad 2\alpha - \beta + 2\gamma + 4 = 0$ and $2\alpha - \beta + 2\gamma - 8 = 0$

34. Line L will be parallel to the line of intersection of P_1 and P_2.

Let a, b and c be the direction ratios of line L.

Thus, $a + 2b - c = 0$ and $2a - b + c = 0$

$\Rightarrow \quad a : b : c :: 1 : -3, : -5$

The equation of the line L is

$\Rightarrow \quad \frac{x-0}{1} = \frac{y-0}{-3} = \frac{z-0}{-5}$

Again foot of the perpendicular from origin to plane P_1 is $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$

The equation of the projection of line L on plane P_1 is

$$\frac{x + \frac{1}{6}}{1} = \frac{y + \frac{1}{3}}{-3} = \frac{z - \frac{1}{6}}{-5} = k$$

Clearly points $\left(0, \frac{5}{6}, -\frac{2}{3}\right)$ and $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$ satisfy the line of projection, i.e. M.

35. Points O, P, Q, R, S are (0, 0, 0), (3, 0, 0), (3, 3, 0), (0, 3, 0), $\left(\frac{3}{2}, \frac{3}{2}, 0\right)$ respectively.

The angle between OQ nad OS is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

The equation of the plane containing the points O, Q and S is

$x - y = 0$

The perpendicular distance from $P(3, 0, 0)$ to the plane $x - y = 0$ is

$$\Rightarrow \quad \left|\frac{3-0}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

and the perpendicular distance from $O(0, 0, 0)$ to the line RS

$$\frac{x}{1} = \frac{y-3}{-1} = \frac{z}{2} \text{ is } \sqrt{\frac{15}{2}}.$$

36. The mirrior image of (3, 1, 7) w.r.t. the plane $x - y + z = 3$ is obtained by

$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = \frac{-2(3-1+7-3)}{3}$$

i.e. $\quad P = (-1, 5, 3)$

The equation of the plane passing through line and $P = (-1, 5, 3)$ is

$$\vec{n} = \begin{vmatrix} x & y & z \\ -1 & 5 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 0]$$

$$\Rightarrow \quad x - 4y + 7z = 0$$