CONTENTS

CHAPTER 1

REAL NUMBERS

1.1 INTRODUCTION

Since our childhood we have been using four fundamental operations of addition, subtraction, multiplication and division. We have applied these operations on natural numbers, integers, rational and irrational numbers. In this chapter, we will begin with a brief recall of divisibility on integers and will state some important properties of integers, namely, Euclid's division Lemma, Euclid's division algorithm and the Fundamental Theorem of Arithmetic which will be used in the remaining part of this chapter to learn more about integers and real numbers.

Euclid's division lemma tells us about divisibility of integers. It is quite easy to state and understand. It states that any positive integer a can be divided by any other positive integer b in such a way that it leaves a remainder r that is smaller than b. This is nothing but the usual long division process. Euclid's division lemma provides us a step-wise procedure to compute the HCF of two positive integers. This step-wise procedure is known as Euclid's algorithm. We will use the same for finding the HCF of positive integers.

The Fundamental Theorem of Arithmetic tells us about expressing positive integers as the product of prime integers. It states that every positive integer is either prime or it can be factorized (expressed) as a product of powers of prime integers. This theorem has many significant applications in mathematics and in other fields. We have learnt how to find the HCF and LCM of positive integers by using the Fundamental Theorem of Arithmetic in earlier classes. In this chapter, we will apply this theorem to prove the irrationality of many numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. We know that the decimal representation of a rational number is either terminating or if it is non-terminating then it is repeating. The prime factorisation of the denominator of a rational number completely reveals the nature of its decimal representation. In fact, by looking at the prime factorisation of the denominator of a rational number one can easily tell about its decimal representation whether it is terminating or non-terminating repeating. We will also use the Fundamental Theorem of Arithmetic to determine the nature of the decimal expansion of rational numbers.

Let us begin with the divisibility of integers.

1.2 DIVISIBILITY

We have been studying division of numbers for the last many years. Let us recall the same in a formal manner.

DEFINITION *A non-zero integer 'a' is said to divide an integer 'b' if there exists an integer c such that* $b = ac$.

The integer 'b' is called the *dividend*, integer 'a' is known as the divisor and integer 'c' is known as the quotient.

For example, 3 divides 36 because there is an integer 12 such that $36 = 3 \times 12$. However, 3 does not divide 35 because there do not exist an integer c such that $35 = 3 \times c$. In other words, $35 = 3 \times c$ is not true for any integer c.

If a non-zero integer 'a' divides an integer b, then we write $a \mid b$. This is read as "a divides b". When $a \mid b$, we say that 'b is divisible by a' or 'a is a factor of b' or 'b is a multiple of a' or 'a is a divisor of b'.

We write $a \nmid b$ to indicate that b is not divisible by a.

We observe that:

(i) $-4 \mid 20$, because there exists an integer -5 such that $20 = -4 \times (-5)$.

(ii) $4 \mid -20$, because there exists an integer -5 such that $-20 = 4 \times (-5)$.

(iii) $-4 \mid -20$, because there exists an integer 5 such that $-20 = -4 \times 5$.

ILLUSTRATION *State whether the following are true or not:*

(i) $3 \mid 93$	(ii) $6 \mid 28$	(iii) $0 \mid 4$	(iv) $5 \mid 0$	(v) $-2 \mid 8$
(vi) $-7 \mid -35$	(vii) $6 \mid 6$	(viii) $8 \mid -8$	(ix) $13 \mid -25$	(x) $1 \mid -1$

SOLUTION We observe that:

(i) $3 \mid 93$ is true, because $93 = 3 \times 31$

(ii) $6 \mid 28$ is not true, because $28 = 6c$ is not valid for any integer c.

(iii) $0 \mid 4$ is not true by definition.

(iv) $5 \mid 0$ is true, because $0 = 5 \times 0$

(v) $-2 \mid 8$ is true, because $8 = (-2) \times (-4)$

(vi) $-7 \mid -35$ is true, because $-35 = (-7) \times 5$

(vii) $6 \mid 6$ is true, because $6 = 6 \times 1$

(viii) $8 \mid -8$ is true, because $-8 = 8 \times -1$

(ix) $13 \mid -25$ is not true, because $-25 = 13 \times c$ is not valid for any integer c.

(x) $1 \mid -1$ is true, because $-1 = 1 \times -1$

Following are some properties of divisibility:

(i) ± 1 divides every non-zero integer.
i.e, $\pm 1 \mid a$ for every non-zero integer a.

(ii) 0 is divisible by every non-zero integer a.
i.e., $a \mid 0$ for every non-zero integer a.

(iii) 0 does not divide any integer.

(iv) If a is a non-zero integer and b is any integer, then

$$a \mid b \Rightarrow a \mid -b,\ -a \mid b \text{ and } -a \mid -b$$

(v) If a and b are non-zero integers, then

$$a \mid b \text{ and } b \mid a \Rightarrow a = \pm b$$

(vi) If a is a non-zero integer and b, c are any two integers, then

$$a \mid b \text{ and } a \mid c \Rightarrow \begin{cases} a \mid b \pm c \\ a \mid bc \\ a \mid bx \quad \text{for any integer } x. \end{cases}$$

(vii) If a and c are non-zero integers and b, d are any two integers, then

(i) $a \mid b$ and $c \mid d \Rightarrow ac \mid bd$

(ii) $ac \mid bc \Rightarrow a \mid b$

REMARK *If a divides b, then by property (iv) a divides $-b$ and $-a$ divides b. It is therefore enough if we consider positive divisors of positive integers only.*

Thus, whenever we will speak of divisors in this chapter we will mean positive divisors of positive integers.

1.3 EUCLID'S DIVISION LEMMA

Euclid was the first Greek Mathematician who initiated a new way of thinking the study of geometry. He is famous for his *Elements of Geometry* but few people are aware that he also made important contributions to the number theory. Among these is the Euclid's Lemma. A lemma is a proven statement which is used to prove other statements. This lemma was perhaps known for a long time, but was first recorded in Book VII of Euclid's Elements. Euclid's division algorithm is based on this lemma. This lemma is nothing but a restatement of the long division process we have been doing for the last many years.

Consider the division of one positive integer by another, say 58 by 9. The division can be carried out as follows:

$$\begin{array}{r|l} 9\,\overline{)\,58\,}\,(\,6 & \\ \underline{54} & \\ 4 & \end{array}$$

While carrying out this division, we had to think of about the largest multiple of 9 which does not exceed 58 so that after subtraction the remainder 4 is less than the divisor 9. The result of this division is that we get two integers, namely, 6 which is called the quotient and 4 which is called the remainder. We can write the result in the following form:

$$58 = 9 \times 6 + 4, \quad 0 \leq 4 < 9$$

Let us now apply the same procedure to other pairs of positive integers to see whether such a representation is always possible or not.

Pair of integers	*Representation*	
25,7	$25 = 7 \times 3 + 4, 0 \leq 4 < 7$	$\left[\because\right.$ 7 goes into 25 thrice and leaves remainder 4$\left.\right]$
20, 3	$20 = 3 \times 6 + 2,\ 0 \leq 2 < 3$	$\left[\because\right.$ 3 goes into 20 six times and leaves remainder 2$\left.\right]$
7,15	$7 = 15 \times 0 + 7,\ 0 \leq 7 < 15$	$\left[\because\right.$ 15 is larger than 7. So, this relation is always possible$\left.\right]$
35, 5	$35 = 5 \times 7 + 0,\ 0 \leq 0 < 5$	$\left[\because\right.$ 5 goes into 35 seven times and leaves no remainder$\left.\right]$

It is evident from the above discussion that the above representation also holds for other pairs of integers. We also observe that for each pair of positive integers a and b, we can find unique integers q and r satisfying the relation $a = bq + r, 0 \leq r < b$. In fact, this holds for every pair of positive integers as proved in the following lemma.

REMARK 1 *The above Lemma is nothing but a restatement of the long division process we have been doing all these years, and that the integers q and r are called the quotient and remainder, respectively.*

REMARK 2 *The above Lemma has been stated for positive integers only. But, it can be extended to all integers as stated below:*

Let a and b be any two integers with $b \neq 0$. Then, there exist unique integers q and r such that

$$a = bq + r, \text{ where } 0 \leq r < |b|$$

REMARK 3 *(i) When a positive integer is divided by 2, the remainder is either 0 or 1. So, any positive integer is of the form 2m, 2m + 1 for some interger m.*

(ii) When any positive integer is divided by 3, the remainder is 0 or 1 or 2. So, any positive integer can be written in the form 3m, 3m + 1, 3m + 2 for some integer m.

(iii) When a positive integer is divided by 4, the remainder can be 0 or 1 or 2 or 3. So, any positive integer is of the form 4q or, 4q + 1 or, 4q + 2 or, 4q + 3.

Let us now discuss some problems to illustrate the applications of Euclid's division Lemma.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Show that every positive even integer is of the form 2q, and that every positive odd integer is of the form $2q + 1$, where q is some integer.* [NCERT]

SOLUTION Let a be any positive integer and $b = 2$. Then, by Euclid's division Lemma there exist integers q and r such that

$$a = 2q + r, \text{ where } 0 \leq r < 2$$

Now, $0 \leq r < 2 \Rightarrow 0 \leq r \leq 1 \Rightarrow r = 0 \text{ or, } r = 1$ [$\because$ r is an integer]

$\therefore$ $a = 2q \text{ or, } a = 2q + 1$

If $a = 2q$, then a is an even integer.

We know that an integer can be either even or odd. Therefore, any odd integer is of the form $2q + 1$.

EXAMPLE 2 *Show that any positive integer is of the form $3q$ or, $3q + 1$ or, $3q + 2$ for some integer q.*

SOLUTION Let a be any positive integer and $b = 3$. Applying division Lemma with a and $b = 3$, we have

$$a = 3q + r, \text{ where } 0 \leq r < 3 \text{ and } q \text{ is some integer}$$

$\Rightarrow$ $a = 3q + 0$ or, $a = 3q + 1$ or, $a = 3q + 2$

$\Rightarrow$ $a = 3q$ or, $a = 3q + 1$ or, $a = 3q + 2$ for some integer q.

EXAMPLE 3 *Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.* [NCERT]

SOLUTION Let a be any odd positive integer and $b = 4$. By division Lemma there exists integers q and r such that

$$a = 4q + r, \textit{ where } 0 \leq r < 4$$

EUCLID'S DIVISION LEMMA *Let a and b be any two positive integers. Then, there exist unique integers q and r such that*

$$a = bq + r, \ 0 \le r < b$$

If $b \mid a$, then $r = 0$. Otherwise, r satisfies the stronger inequality $0 < r < b$.

PROOF Consider the following arithmetic progression

$$\ldots\ldots, a - 3b, a - 2b, a - b, a, a + b, a + 2b, a + 3b, \ldots\ldots$$

Clearly, it is an arithmetic progression with common difference 'b' and it extends indefinitely in both directions.

Let r be the smallest non-negative term of this arithmetic progression. Then, there exists a non-negative integer q such that

$$a - bq = r \Rightarrow a = bq + r$$

As, r is the smallest non-negative integer satisfying the above result. Therefore, $0 \le r < b$

Thus, we have

$$a = bq + r, \text{ where } 0 \le r < b$$

We shall now prove the uniqueness of q and r.

Uniqueness To prove the uniqueness of q and r, let us assume that there is another pair q_1 and r_1 of non-negative integers satisfying the same relation i.e.,

$$a = bq_1 + r_1, \ 0 \le r_1 < b$$

We shall now prove that $r_1 = r$ and $q_1 = q$

We have,

$$a = bq + r \text{ and } a = bq_1 + r_1$$

$\Rightarrow \quad bq + r = bq_1 + r_1$

$\Rightarrow \quad r_1 - r = bq - bq_1$

$\Rightarrow \quad r_1 - r = b(q - q_1)$

$\Rightarrow \quad b \mid r_1 - r$

$\Rightarrow \quad r_1 - r = 0 \qquad [\because 0 \le r < b \text{ and } 0 \le r_1 < b \Rightarrow 0 \le r_1 - r < b]$

$\Rightarrow \quad r_1 = r$

Now, $\quad r_1 = r$

$\Rightarrow \quad -r_1 = -r \qquad$ [Multiplying both sides by (–1)]

$\Rightarrow \quad a - r_1 = a - r \qquad$ [Adding a on both sides]

$\Rightarrow \quad bq_1 = bq \qquad \left[\begin{array}{l} \because a = bq + r \text{ and } a = bq_1 + r_1 \\ \because bq = a - r \text{ and } bq_1 = a - r_1 \end{array}\right]$

$\Rightarrow \quad q_1 = q$

Hence, the representation $a = bq + r, \ 0 \le r < b$ is unique.

Q.E.D.

$\Rightarrow \quad a = 4q$ or, $a = 4q + 1$ or, $a = 4q + 2$ or, $a = 4q + 3 \quad [\because 0 \le r < 4 \Rightarrow r = 0, 1, 2, 3]$

$\Rightarrow \quad a = 4q + 1$ or, $a = 4q + 3 \quad [\because a$ is an odd integer $\therefore a \neq 4q, a \neq 4q + 2]$

Hence, any odd integer is of the form $4q + 1$ or, $4q + 3$.

EXAMPLE 4 *Show that the square of an odd integer is of the form $4q + 1$ for the some integer q.* [NCERT EXEMPLAR]

SOLUTION Let x be any odd integer. Then, $x = 2m + 1$, for some integer m.

$\therefore \quad x^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4q + 1$, for some integer $q = m^2 + m$.

LEVEL-2

EXAMPLE 5 *Show that $n^2 - 1$ is divisible by 8, if n is an odd positive integer.* [NCERT EXEMPLAR]

SOLUTION We know that any odd positive integer is of the form $4q + 1$ or, $4q + 3$ for some integer q.

So, we have the following cases:

CASE I *When $n = 4q + 1$*: In this case, we have

$$n^2 - 1 = (4q + 1)^2 - 1 = 16q^2 + 8q + 1 - 1 = 16q^2 + 8q = 8q(2q + 1)$$

$\Rightarrow \quad n^2 - 1$ is divisible by 8 $\quad [\because 8q(2q + 1)$ is divisible by $8]$

CASE II *When $n = 4q + 3$*: In this case, we have

$$n^2 - 1 = (4q + 3)^2 - 1 = 16q^2 + 24q + 9 - 1 = 16q^2 + 24q + 8$$

$\Rightarrow \quad n^2 - 1 = 8(2q^2 + 3q + 1) = 8(2q + 1)(q + 1)$

$\Rightarrow \quad n^2 - 1$ is divisible by 8 $\quad [\because 8(2q + 1)(q + 1)$ is divisible by $8]$

Hence, $n^2 - 1$ is divisible by 8.

EXAMPLE 6 *Prove that if x and y are odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.*

SOLUTION We know that any odd positive integer is of the form $2q + 1$ for some integer q.

So, let $x = 2m + 1$ and $y = 2n + 1$ for some integers m and n.

$\therefore \quad x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

$\Rightarrow \quad x^2 + y^2 = 4(m^2 + n^2) + 4(m + n) + 2$

$\Rightarrow \quad x^2 + y^2 = 4\{(m^2 + n^2) + (m + n)\} + 2$

$\Rightarrow \quad x^2 + y^2 = 4q + 2$, where $q = (m^2 + n^2) + (m + n)$

$\Rightarrow \quad x^2 + y^2$ is even and leaves remainder 2 when divided by 4

$\Rightarrow \quad x^2 + y^2$ is even but not divisible by 4

EXAMPLE 7 *Prove that $n^2 - n$ is divisible by 2 for every positive integer n.*

SOLUTION We know that any positive integer is of the form $2q$ or $2q + 1$, for some integer q.

So, following cases arise:

CASE I *When* $n = 2q$: In this case, we have

$$n^2 - n = (2q)^2 - 2q = 4q^2 - 2q = 2q(2q-1)$$

$\Rightarrow$ $n^2 - n = 2r$, where $r = q(2q-1)$

$\Rightarrow$ $n^2 - n$ is divisible by 2

CASE II *When* $n = 2q+1$: In this case, we have

$$n^2 - n = (2q+1)^2 - (2q+1) = (2q+1)(2q+1-1) = 2q(2q+1)$$

$\Rightarrow$ $n^2 - n = 2r$, where $r = q(2q+1)$

$\Rightarrow$ $n^2 - n$ is divisible by 2.

Hence, $n^2 - n$ is divisible by 2 for every positive integer n.

EXAMPLE 8 *Show that the square of any positive integer is of the form 3m or, 3m + 1 for some integer m.* **[NCERT, CBSE 2008]**

SOLUTION Let a be any positive integer. Then, it is of the form $3q$ or, $3q+1$ or, $3q+2$. So, we have the following cases:

CASE I *When* $a = 3q$: In this case, we have

$$a^2 = (3q)^2 = 9q^2 = 3q(3q) = 3m, \text{ where } m = 3q$$

CASE II *When* $a = 3q+1$: In this case, we have

$$a^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3q(3q+2) + 1 = 3m+1, \text{ where } m = q(3q+2)$$

CASE III *When* $a = 3q+2$: In this case, we have

$$a^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1$$

$$= 3(3q^2 + 4q + 1) + 1 = 3m + 1, \text{ where } m = 3q^2 + 4q + 1$$

Hence, a is of the form $3m$ or $3m+1$.

EXAMPLE 9 *Use Euclid's division Lemma to show that the cube of any positive integer is either of the form 9m, 9m + 1 or, 9m + 8 for some integer m.* **[NCERT]**

SOLUTION Let x be any positive integer. Then, it is of the form $3q$ or, $3q+1$ or, $3q+2$. So, we have the following cases:

CASE I *When* $x = 3q$: In this case, we have

$$x^3 = (3q)^3 = 27q^3 = 9(3q^3) = 9m, \text{ where } m = 3q^3$$

CASE II *When* $x = 3q+1$: In this case, we have

$$x^3 = (3q+1)^3$$

$\Rightarrow$ $x^3 = 27q^3 + 27q^2 + 9q + 1$

$\Rightarrow$ $x^3 = 9q(3q^2 + 3q + 1) + 1$

$\Rightarrow$ $x^3 = 9m + 1$, where $m = q(3q^2 + 3q + 1)$

CASE III *When* $x = 3q+2$: In this case, we have

$$x^3 = (3q+2)^3$$

$\Rightarrow$ $x^3 = 27q^3 + 54q^2 + 36q + 8$

$\Rightarrow \quad x^3 = 9q\left(3q^2 + 6q + 4\right) + 8$

$\Rightarrow \quad x^3 = 9m + 8$, where $m = q\left(3q^2 + 6q + 4\right)$

Hence, x^3 is either of the form $9m$ or, $9m + 1$ or, $9m + 8$.

EXAMPLE 10 *Show that the cube of any positive integer is of the form $4m$, $4m + 1$, or $4m + 3$ for some integer m.*

SOLUTION Let n be any positive integer. Then, it is of the form $4q, 4q + 1, 4q + 2$, and $4q + 3$. So, we have the following cases:

CASE I *When* $n = 4q$: In this case, we have

$$n^3 = (4q)^3 = 64q^3 = 4(16q)^3 = 4m, \text{ where } m = 16q^3$$

CASE II *When* $n = 4q + 1$: In this case, we have

$$n^3 = (4q + 1)^3 = 64q^3 + 48q^2 + 12q + 1$$
$$= 4(16q^2 + 12q + 3q) + 1 = 4m + 1, \text{ where } m = 16q^2 + 12q + 3q$$

CASE III *When* $n = 4q + 2$: In this case, we have

$$n^3 = (4q + 2)^3 = 64q^3 + 96q^2 + 48q + 8$$
$$= 4(16q^3 + 24q^2 + 12q + 2) = 4m, \text{ where } m = 16q^3 + 24q^2 + 12q + 2$$

CASE IV *When* $n = 4q + 3$: In this case, we have

$$n^3 = (4q + 3)^3 = 64q^3 + 144q^2 + 108q + 27$$
$$= 64q^3 + 144q^2 + 108q + 24 + 3$$
$$= 4(16q^3 + 36q^2 + 27q + 6) + 3 = 4m + 3, \text{ where } m = 16q^3 + 36q^2 + 27q + 6$$

Hence, n is of the form $4m$, $4m + 1$ or $4m + 3$.

EXAMPLE 11 *Show that the square of any positive integer cannot be of the form $5q + 2$ or $5q + 3$ for any integer q.* [NCERT EXEMPLAR]

SOLUTION Let x be any positive integer. When we divide x by 5, the remainder is either 0 or 1 or 2 or 3 or 4. So, x can be written as $x = 5m$, or $x = 5m + 1$ or $x = 5m + 2$ or $x = 5m + 3$ or $x = 5m + 4$. Thus, we have the following cases:

CASE I *When* $x = 5m$: In this case,

$$x^2 = 25m^2 = 5(5m)^2 = 5q, \text{ where } q = 5m^2$$

CASE II *When* $x = 5m + 1$: In this case,

$$x^2 = (5m + 1)^2 = 25m^2 + 10m + 1 = 5(5m^2 + 2m) + 1 = 5q + 1, \text{ where } q = 5m^2 + 2m$$

CASE III *When* $x = 5m + 2$: In this case,

$$x^2 = (5m + 2)^2 = 25m^2 + 20m + 4 = 5(5m^2 + 4m) + 4 = 5q + 4, \text{ where } q = 5m^2 + 4m$$

CASE IV *When* $x = 5m + 3$: In this case,

$$x^2 = (5m + 3)^2 = 25m^2 + 30m + 9 = (5m^2 + 30m + 5) + 4$$
$$= 5(5m^2 + 6m + 1) + 4 = 5q + 4, \text{ where } q = 5m^2 + 6m + 1$$

CASE V *When* $x = 5m + 4$: In this case,

$$x^2 = (5m+4)^2 = 25m^2 + 40m + 16$$

$$= 5(5m^2 + 8m + 3) + 1 = 5q + 1 \text{ where } q = 5m^2 + 8m + 3$$

Hence, x is of the form $5q$ or $5q + 1$, $5q + 4$. So, it cannot be of the form $5q + 2$ or $5q + 3$.

EXAMPLE 12 *Show that one and only one out of* n, $n + 2$ *or*, $n + 4$ *is divisible by 3, where* n *is any positive integer.*

SOLUTION We know that any positive integer is of the form $3q$ or, $3q + 1$ or, $3q + 2$ for some integer q and one and only one of these possibilities can occur.
So, we have following cases:

CASE I *When* $n = 3q$: In this case, we have

$n = 3q$, which is divisible by 3

Now, $n = 3q$

$\Rightarrow$ $n + 2 = 3q + 2$,

$\Rightarrow$ $n + 2$ leaves remainder 2 when divided by 3

$\Rightarrow$ $n + 2$ is not divisible by 3

Again, $n = 3q$

$\Rightarrow$ $n + 4 = 3q + 4 = 3(q + 1) + 1$

$\Rightarrow$ $n + 4$ leaves remainder 1 when divided by 3

$\Rightarrow$ $n + 4$ is not divisible by 3

Thus, n is divisible by 3 but $n + 2$ and $n + 4$ are not divisible by 3.

CASE II *When* $n = 3q + 1$: In this case, we have

$n = 3q + 1$

$\Rightarrow$ n leaves remainder 1 when divided by 3

$\Rightarrow$ n is not divisible by 3

Now, $n = 3q + 1$

$\Rightarrow$ $n + 2 = (3q + 1) + 2 = 3(q + 1)$

$\Rightarrow$ $n + 2$ is divisible by 3

Again, $n = 3q + 1$

$\Rightarrow$ $n + 4 = 3q + 1 + 4 = 3q + 5 = 3(q + 1) + 2$

$\Rightarrow$ $n + 4$ leaves remainder 2 when divided by 3

$\Rightarrow$ $n + 4$ is not divisible by 3

Thus, $n + 2$ is divisible by 3 but n and $n + 4$ are not divisible by 3

CASE III *When* $n = 3q + 2$: In this case, we have

$n = 3q + 2$

$\Rightarrow$ n leaves remainder 2 when divided by 3

$\Rightarrow$ n is not divisible by 3

Now, $n = 3q + 2$

$\Rightarrow \quad n + 2 = 3q + 2 + 2 = 3(q + 1) + 1$

$\Rightarrow \quad n + 2$ leaves remainder 1 when divided by 3

$\Rightarrow \quad n + 2$ is not divisible by 3

Again, $\quad n = 3q + 2$

$\quad n + 4 = 3q + 2 + 4 = 3(q + 2)$

$\Rightarrow \quad n + 4$ is divisible by 3

Thus, $n + 4$ is divisible by 3 but n and $n + 2$ are not divisible by 3.

EXAMPLE 13 *Prove that one of every three consecutive positive integers is divisible by 3.*

SOLUTION Let n, $n + 1$, $n + 2$ be three consecutive positive integers. We know that n is of the form $3q$, $3q + 1$ or, $3q + 2$. So, we have the following cases:

CASE I *When* $n = 3q$: In this case,

n is divisible by 3 but $n + 1$ and $n + 2$ are not divisible by 3.

CASE II *When* $n = 3q + 1$: In this case,

$n + 2 = 3q + 1 + 2 = 3(q + 1)$ is divisible by 3 but n and $n + 1$ are not divisible by 3.

CASE III *When* $n = 3q + 2$: In this case,

$n + 1 = 3q + 1 + 2 = 3(q + 1)$ is divisible by 3 but n and $n + 2$ are not divisible by 3.

Hence, one of n, $n + 1$ and $n + 2$ is divisible by 3.

EXERCISE 1.1

LEVEL-1

1. If a and b are two odd positive integers such that $a > b$, then prove that one of the two numbers $\frac{a+b}{2}$ and $\frac{a-b}{2}$ is odd and the other is even.
2. Prove that the product of two consecutive positive integers is divisible by 2.
3. Prove that the product of three consecutive positive integer is divisible by 6.
4. For any positive integer n, prove that $n^3 - n$ divisible by 6. [NCERT EXEMPLAR]
5. Prove that if a positive integer is of the form $6q + 5$, then it is of the form $3q + 2$ for some integer q, but not conversely.
6. Prove that the square of any positive integer of the form $5q + 1$ is of the same form.
7. Prove that the square of any positive integer is of the form $3m$ or, $3m + 1$ but not of the form $3m + 2$.
8. Prove that the square of any positive integer is of the form $4q$ or $4q + 1$ for some integer q. [NCERT EXEMPLAR]
9. Prove that the square of any positive integer is of the form $5q$, $5q + 1$, $5q + 4$ for some integer q.
10. Show that the square of an odd positive integer is of the form $8q + 1$, for some integer q.

11. Show that any positive odd integer is of the form $6q+1$ or, $6q+3$ or, $6q+5$, where q is some integer. [NCERT]

LEVEL-2

12. Show that the square of any positive integer cannot be of the form $6m+2$ or $6m+5$ for any integer m. [NCERT EXEMPLAR]
13. Show that the cube of a positive integer is of the form $6q+r$, where q is an integer and $r = 0, 1, 2, 3, 4, 5$. [NCERT EXEMPLAR]
14. Show that one and only one out of n, $n+4$, $n+8$, $n+12$ and $n+16$ is divisible by 5, where n is any positive integer. [NCERT EXEMPLAR]
15. Show that the square of an odd positive integer can be of the form $6q+1$ or $6q+3$ for some integer q. [NCERT EXEMPLAR]
16. A positive integer is of the form $3q+1$, q being a natural number. Can you write its square in any form other than $3m+1$, $3m$ or $3m+2$ for some integer m? Justify your answer.
17. Show that the square of any positive integer cannot be of the form $3m+2$, where m is a natural number.

HINTS TO SELECTED PROBLEMS

2. Let $n-1$ and n be two consecutive positive integers. Then, their product is $(n-1)n = n^2 - n$. Now, proceed as in example 6.
3. Let n be any positive integer. Since any positive integer is of the form $6q$ or, $6q+1$ or, $6q+2$ or, $6q+3$ or, $6q+4$ or, $6q+5$.

   If $n = 6q$, then

   $n(n+1)(n+2) = 6q(6q+1)(6q+2)$, which is divisible by 6

   If $n = 6q+1$, then

   $n(n+1)(n+2) = (6q+1)(6q+2)(6q+3) = 6(6q+1)(3q+1)(2q+1)$, which is divisible by 6.

   If $n = 6q+2$, then

   $n(n+1)(n+2) = (6q+2)(6q+3)(6q+4) = 12(3q+1)(2q+1)(2q+3)$, which is divisible by 6.

   Similarly, $n(n+1)(n+2)$ is divisible by 6 if $n = 6q+3$ or, $6q+4$ or, $6q+5$.
4. We have,

   $n^3 - n = (n-1)(n)(n+1)$, which is product of three consecutive positive integers. So, proceed as in Q. No. 3.
5. Let $n = 6q+5$, where q is a positive integer. We know that any positive integer is of the form $3k$ or, $3k+1$ or, $3k+2$.

   $\therefore \quad q = 3k$ or, $3k+1$ or, $3k+2$

   If $q = 3k$, then

   $n = 6q+5 = 18k+5 = 3(6k+1)+2 = 3m+2$, where $m = 6k+1$

   If $q = 3k+1$, then

   $n = 6q+5 = 6(3k+1)+5 = 3(6k+3)+2 = 3m+2$, where $m = 6k+3$

If $q = 3k + 2$, then

$$n = 6q + 5 = 6(3k + 2) + 5 = 3(6k + 5) + 2 = 3m + 2, \text{ where } m = 6k + 5.$$

6. Let $n = 5q + 1$. Then,

$$n^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1 = 5m + 1, \text{ where } m = 5q^2 + 2q$$

$\Rightarrow$ n^2 is of the form $5m + 1$.

7. Any positive ingeger n is of the form $3q$, $3q + 1$ or, $3q + 2$.

If $n = 3q$, then

$$n^2 = 9q^2 = 3(3q^2) = 3m, \text{ where } m = 3q^2$$

If $n = 3q + 1$, then

$$n^2 = 9q^2 + 6q + 1 = 3q(3q + 2) + 1 = 3m + 1, \text{ where } m = q(3q + 2)$$

If $n = 3q + 2$, then

$$n^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1 = 3m + 1, \text{ where } m = 3q^2 + 4q + 1.$$

Hence, n^2 is of the form $3m$ or, $3m + 1$ but not of the form $3m + 2$.

8. Any positive integer n is of the form $2m$ or, $2m + 1$

If $n = 2m$, then

$$n^2 = 4m^2 = 4q, \text{ where } q = m^2$$

If $n = 2m + 1$, then

$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4m(m + 1) + 1 = 4q + 1, \text{ where } q = m(m + 1)$$

9. Any positive integer n is of the form $5m$ or $5m + 1$, or $5m + 2$ or $5m + 3$ or $5m + 4$.

If $n = 5m$, then

$$n^2 = 25m^2 = 5(5m^2) = 5q, \text{ where } q = 5m^2$$

If $n = 5m + 1$, then

$$n^2 = (5m + 1)^2 = 5m(5m + 2) + 1 = 5q + 1, \text{ where } q = m(5m + 2)$$

If $n = 5m + 2$, then

$$n^2 = (5m + 2)^2 = 5m(5m + 4) + 4 = 5q + 4, \text{ where } q = m(5m + 4)$$

If $n = 5m + 3$, then

$$n^2 = (5m + 3)^2 = 5(m^2 + 6m + 1) + 4 = 5q + 4, \text{ where } q = 5m^2 + 6m + 1$$

If $n = 5m + 4$, then

$$n^2 = 5(5m^2 + 8m + 3) + 1 = 5q + 1, \text{ where } q = 5m^2 + 8m + 3$$

Hence, n^2 is of the form $5q$ or, $5q + 1$ or, $5q + 4$.

10. Since, any odd positive integer n is of the form $4m + 1$ or $4m + 3$.

If $n = 4m + 1$, then

$$n^2 = (4m + 1)^2 = 16m^2 + 8m + 1 = 8m(m + 1) + 1 = 8q + 1 \text{ where } q = m(m + 1)$$

If $n = 4m + 3$, then

$$n^2 = (4m + 3)^2 = 16m^2 + 24m + 9 = 8(2m^2 + 3m + 1) + 1 = 8q + 1, \text{ where } q = 2m^2 + 3m + 1$$

Hence, n^2 is of the form $8q + 1$.

11. Let a be any odd positive integer and $b = 6$. Then, there exist integers q and r such that

$a = 6q + r, 0 \leq r < 6$ [By division algorithm]

$\Rightarrow \quad a = 6q$ or, $6q+1$ or, $6q+2$ or, $6q+3$ or, $6q+4$ or, $6q+5$

But, $6q, 6q+2$ and $6q+4$ are even positive integers.

$\therefore \quad a = 6q+1$ or, $6q+3$ or, $6q+5$

12. We know that any positive integer x can be of the form $6m$, $6m+1$, $6m+2$, $6m+3$, $6m+4$ or $6m+5$ *i.e.*, $x = 6m + r$, $r = 0, 1, 2, 3, 4, 5$.

    Now, show that x^2 is of the form $6m$, $6m+1$ or $6m+4$.
    Hence, x^2 cannot be of the form $6m+2$ or $6m+5$.

13. We know that any positive integer x can be of the form $6m$, $6m+1$, $6m+2$, $6m+3$, $6m+4$ or $6m+5$.

    CASE I *When* $x = 6q$: In this case,

    $$x^3 = (6q)^3 = 6(36q^3) = 6m, \text{ where } m = 36q^3$$

    CASE II *When* $x = 6q+1$: In this case,

    $$x^3 = (6q+1)^3 = 216q^3 + 108q^2 + 18q + 1 = 6(36q^3 + 18q^2 + 3q) + 1$$
    $$= 6m + 1, \text{ where } m = 36q^3 + 18q^2 + 3q \text{ and so on.}$$

14. We know that any positive integer can be of the form $5q$, $5q+1$, $5q+2$, $5q+3$ or $5q+4$.

15. We know that any positive integer can be of the form $6m$, $6m+1$, $6m+2$, $6m+3$, $6m+4$ or $6m+5$ for some integer m. So, an odd positive integer x is of the form $6m+1$ or $6m+3$.

    CASE I *When* $x = 6m+1$: In this case,

    $$x^2 = (6m+1)^2 = 36m^2 + 12m + 1 = 6(6m^2 + 2m) + 1 = 6q + 1, \text{ where q} = 6m^2 + m$$

    CASE II *When* $x = 6m+3$: In this case,

    $$x^2 = (6m+3)^2 = 36m^2 + 36m + 9 = (36m^2 + 36m + 6) + 3$$
    $$= 6(6m^2 + 6m + 1) + 3 = 6q + 3, \text{ where } q = 6m^2 + 6m + 3.$$

16. No. $(3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3m + 1$

17. Any positive integer x can be written as $3q$, $3q+1$, $3q+2$.

    CASE I *When* $x = 3q$: In this case,

    $$x^2 = (3q)^2 = 9q^2 = 3(3q^2) = 3m, \text{ where } m = 3q^2$$

    CASE II *When* $x = 3q+1$: In this case,

    $$x^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3m + 1, \text{ where } m = 3q^2 + 2q$$

    CASE III *When* $x = 3q+2$: In this case,

    $$x^2 = (3q+2)^2 = 9q^2 + 12q + 4 = (9q^2 + 12q + 3) + 1 = 3(3q^2 + 4q + 1) + 1 = 3m + 1,$$
    $$\text{where } m = 3q^2 + 4q + 1$$

1.4 EUCLID'S DIVISION ALGORITHM

In the previous section, we have learnt about Euclid's division lemma and its applications. We have seen that the said lemma is nothing but a restatement of the long division process which we have been using all these years. In this section, we will learn one more application of Euclid's division lemma known as Euclid's division algorithm. The word algorithm comes from the name of 9th century Persian mathematician al-Khwarizmi. An algorithm means a series of well defined steps which provide a procedure of calculation repeated successively on the results of earlier steps till the desired result is obtained. Euclid's division algorithm is also an algorithm to compute the highest common factor (HCF) of two given positive integers. So, let us first have a brief review of HCF of positive integers.

In section 1.2, we have recalled that if an integer c divides each one of several integers $x_1, x_2, \ldots x_n$, then it is called a common divisor of these integers. For example, 7 is a common divisor of 42 and 63 as it divides both the integers. Throughout this chapter we will discuss positive divisors of positive integers only and the word integer will mean positive integer. Note that 1 is a common divisor of all positive integers. Two or more integers may have many common divisors. For example, common divisors of 24 and 42 are 1, 2, 3, 4 and 6. The largest among these common divisors is 6. This is called the Greatest Common Divisor (GCD) or Highest Common Factor (HCF) of integers 24 and 42. Thus, the largest or greatest among the common divisors of two or more integers is called the Greatest Common Divisor (GCD) or Highest Common Factor (HCF) of the given integers. The HCF of two or more positive integers always exists and it is unique. The proof of the same is beyond the scope of this book.

Let a and b be two positive integers such that $a > b$. If b is not a divisor of a, then by Euclid's division lemma there exist positive integers q and r such that $a = bq + r$, where $0 < r < b$. Common divisors of a and b are closely associated with the common divisors of b and r. In fact, every common divisor of b and r is a common divisor of a and b and vice-versa as stated and proved in the following theorem.

THEOREM *If a and b are positive integers such that $a = bq + r$, then every common divisor of a and b is a common divisor of b and r, and vice-versa.*

PROOF Let c be a common divisor of a and b. Then,

$c \mid a \Rightarrow a = cq_1$ for some integer q_1 $\qquad$ $c \mid b \Rightarrow b = cq_2$ for some integer q_2

Now,

$$a = bq + r$$

$$\Rightarrow \quad r = a - bq$$

$$\Rightarrow \quad r = cq_1 - cq_2q$$

$$\Rightarrow \quad r = c(q_1 - q_2q)$$

$$\Rightarrow \quad c \mid r$$

$$\Rightarrow \quad c \mid r \text{ and } c \mid b \qquad [\because c \mid b \text{ (given)}]$$

$$\Rightarrow \quad c \text{ is a common divisor of } b \text{ and } r.$$

Hence, a common divisor of a and b is a common divisor of b and r.

Conversely, Let d be a common divisor of b and r. Then,

$$d \mid b \Rightarrow b = r_1 d \text{ for some integer } r_1$$

$$d \mid r \Rightarrow r = r_2 d \text{ for some integer } r_2$$

We will now show that d is a common divisor of a and b.
We have,

$$a = bq + r$$

$\Rightarrow$ $a = r_1 dq + r_2 d$

$\Rightarrow$ $a = (r_1 q + r_2)d$

$\Rightarrow$ $d \mid a$

$\Rightarrow$ $d \mid a$ and $d \mid b$ [$\because$ $d|b$ (given)]

$\Rightarrow$ d is a common divisor of a and b.

Q.E.D

Let us now discuss an application of this theorem and Euclid's division lemma. Consider integers 117 and 45.

Let $a = 117$ and $b = 45$. By Euclid's division lemma, we obtain

$$117 = 45 \times 2 + 27 \quad \ldots(\text{i}) \qquad \left[\because \begin{array}{r} 45\,)\,117\,(\,2 \\ 90 \\ \hline 27 \end{array}\right]$$

or, $a = bq_1 + r_1$, where $q_1 = 2$ and $r_1 = 27$

By using the above theorem, we observe that the common divisors of $a = 117$ and $b = 45$ are also the common divisors of $b = 45$ and $r_1 = 27$ and vice-versa.

Applying Euclid's division lemma on divisor $b = 45$ and remainder $r_1 = 27$, we get

$$45 = 27 \times 1 + 18 \quad \ldots(\text{ii}) \qquad \left[\because \begin{array}{r} 27\,)\,45\,(\,1 \\ 27 \\ \hline 18 \end{array}\right]$$

or, $b = q_2 r_1 + r_2$, where $q_2 = 1$ and $r_2 = 18$

Using the above theorem, we find that the common divisors of $r_1 = 27$ and $r_2 = 18$ are the common divisors of $b = 45$ and $r_1 = 27$ and vice-versa. But, common divisors of $b = 45$ and $r_1 = 27$ are the common divisors of $a = 117$ and $b = 45$ and vice-versa. Therefore, common divisors of $r_1 = 27$ and $r_2 = 18$ are the common divisors of $a = 117$ and $b = 45$ and vice-versa.

Applying Euclid's division lemma on $r_1 = 27$ and $r_2 = 18$, we get

$$27 = 18 \times 1 + 9 \quad \ldots(\text{iii}) \qquad \left[\because \begin{array}{r} 18\,)\,27\,(\,1 \\ 18 \\ \hline 9 \end{array}\right]$$

or, $r_1 = q_3 r_2 + r_3$, where $q_3 = 1$ and $r_3 = 9$

Again by using the above theorem, we find that common divisors of $r_2 = 18$ and $r_3 = 9$ are the common divisors of $a = 117$ and $b = 45$ and vice-versa.

Using Euclid's division lemma on $r_2 = 18$ and $r_3 = 9$, we get

$$18 = 9 \times 2 + 0 \qquad \text{...(iv)}$$

Therefore, $r_3 = 9$ is a divisor of $r_2 = 18$ and $r_3 = 9$. Also, it is the greatest common divisor (or HCF) of r_2 and r_3. Hence, $r_3 = 9$ is the greatest common divisor (or HCF) of $a = 117$ and $b = 45$. We also observe that $r_3 = 9$ is the last non-zero remainder in the above process of repeated application of Euclid's division lemma on the divisor and the remainder in the next step.

The set of equation (i) to (iv) is called Euclid's division algorithm for 117 and 45. The last divisor, or the last but one non-zero remained which is 9 is the HCF (or GCD) of 117 and 45. The above process of finding HCF can also be carried out by successive divisons as follows:

$$
\begin{array}{llll}
45\,)\,117\,(\,2 & & & \\
\quad\;\underline{90} & & & \\
\quad\;27 &)\,45\,(\,1 & & \\
 & \;\;\underline{27} & & \\
 & \;\;18 &)\,27\,(\,1 & \\
 & & \;\;\underline{18} & \\
 & & \;\;9 &)\,18\,(\,2 \\
 & & & \;\;\underline{18} \\
 & & & \;\;00
\end{array}
$$

OR,

1	45 27	117 90	2
2	18 18	27 18	1
	00	9	

In the general form Euclid's division algorithm may be described as follows:

Let a, b be positive integers such that $a > b$.

Applying Euclid's division lemma in succession as discussed above, we obtain.

$$a = bq_1 + r_1 \qquad 0 < r_1 < b \qquad \text{... (i)}$$

$$b = r_1q_2 + r_2 \qquad 0 < r_2 < r_1 \qquad \text{... (ii)}$$

$$r_1 = r_2q_3 + r_3 \qquad 0 < r_3 < r_2 \qquad \text{... (iii)}$$

$$r_2 = r_3q_4 + r_4 \qquad 0 < r_4 < r_3 \qquad \text{... (iv)}$$

$$\vdots$$

$$r_{n-3} = r_{n-2}\, q_{n-1} + r_{n-1} \quad 0 < r_{n-1} < r_{n-2} \qquad \ldots (n-1)$$

$$r_{n-2} = r_{n-1}\, q_n + r_n \qquad 0 = r_n \qquad \ldots (n)$$

Clearly, $a > b > r_1 > r_2 > r_3 > \ldots > r_{n-2} > r_{n-1} > 0 = r_n$

Thus, the remainders are positive and decreasing. Therefore, for some natural number n, r_n must be zero and the process of applying division lemma ends there.

The set of equations (i) to (n) is called Euclid's division algorithm for numbers a and b.

This algorithm is not only useful for computing the HCF of very large positive integers, but also because it is one of the earliest examples of an algorithm that a computer had been programmed to carry out. Although we have stated Euclid's division algorithm for only positive integers, but it can be extended for all integers except zero, i.e. $b \neq 0$. However, we shall not discuss this aspect in this chapter.

REMARK *The HCF of numbers is a common divisor of the numbers which are divisors of their LCM. Consequently, HCF is a divisor of LCM.*

In order to compute the HCF of two positive integers, say a and b, with $a > b$ we may follow the following steps:

STEP I *Apply Euclid's division lemma to a and b and obtain whole numbers q_1 and r_1 such that $a = bq_1 + r_1, 0 \leq r_1 < b$.*

STEP II *If $r_1 = 0$, b is the HCF of a and b*

STEP III *If $r_1 \neq 0$, apply Euclid's division lemma to b and r_1 and obtain two whole numbers q_1 and r_2 such that $b = q_1 r_1 + r_2$.*

STEP IV *If $r_2 = 0$, then r_1 is the HCF of a and b.*

STEP V *If $r_2 \neq 0$, then apply Euclid's division lemma to r_1 and r_2 and continue the above process till the remainder r_n is zero. The divisor at this stage i.e. r_{n-1}, or the non-zero remainder at the previous stage, is the HCF of a and b.*

Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON FINDING THE HCF OF TWO POSITIVE INTEGERS

EXAMPLE 1 *Use Euclid's division algorithm to find the HCF of 210 and 55.*

SOLUTION Given integers are 210 and 55. Clearly, 210 > 55. Applying Euclid's division lemma to 210 and 55, we get

$$210 = 55 \times 3 + 45 \qquad \ldots \text{(i)} \qquad \left[\because\ 55\overline{)\,210\,}(3 \atop {\underline{165} \atop 45}\right]$$

Since the remainder $45 \neq 0$. So, we apply the division lemma to the divisor 55 and remainder 45 to get

$$55 = 45 \times 1 + 10 \qquad \ldots\text{(ii)} \qquad \left[\begin{array}{r} \because\ 45\,\overline{)\,55\,}(\,1 \\ 45 \\ \hline 10 \end{array}\right]$$

Now, we apply division lemma to the new divisor 45 and new remainder 10 to get

$$45 = 10 \times 4 + 5 \qquad \ldots\text{(iii)} \qquad \left[\begin{array}{r} \because\ 10\,\overline{)\,45\,}(\,4 \\ 40 \\ \hline 5 \end{array}\right]$$

We now consider the new divisor 10 and the new remainder 5, and apply division lemma to get

$$10 = 5 \times 2 + 0 \qquad \ldots\text{(iv)}$$

The remainder at this stage is zero. So, the divisor at this stage or the remainder at the previous stage i.e. 5 is the HCF of 210 and 55.

EXAMPLE 2 *Use Euclid's division algorithm to find the HCF of 4052 and 12576.*

SOLUTION Given integers are 4052 and 12576 such that 12576 > 4052. Applying Euclid's division lemma to 12576 and 4052, we get

$$12576 = 4052 \times 3 + 420 \qquad \ldots\text{(i)} \qquad \left[\begin{array}{r} \because\ 4052\,\overline{)\,12576\,}(\,3 \\ 12156 \\ \hline 420 \end{array}\right]$$

Since the remainder $420 \neq 0$. So, we apply the division lemma to 4052 and 420, to get

$$4052 = 420 \times 9 + 272 \qquad \ldots\text{(ii)} \qquad \left[\begin{array}{r} \because\ 420\,\overline{)\,4052\,}(\,9 \\ 3780 \\ \hline 272 \end{array}\right]$$

We consider the new divisor 420 and the new remainder 272 and apply division lemma to get

$$420 = 272 \times 1 + 148 \qquad \ldots\text{(iii)} \qquad \left[\begin{array}{r} \because\ 272\,\overline{)\,420\,}(\,1 \\ 272 \\ \hline 148 \end{array}\right]$$

Let us now consider the new divisor 272 and the new remainder 148 and apply division lemma to get

$$272 = 148 \times 1 + 124 \qquad \ldots\text{(iv)} \qquad \left[\begin{array}{r} \because\ 148\,\overline{)\,272\,}(\,1 \\ 148 \\ \hline 124 \end{array}\right]$$

We consider now the new divisor 148 and the new remainder 124 and apply division lemma to get

$$148 = 124 \times 1 + 24 \qquad \ldots\text{(v)} \qquad \left[\begin{array}{r} \because\ 124\,\overline{)\,148\,}(\,1 \\ 124 \\ \hline 24 \end{array}\right]$$

We consider now the new divisor 124 and the new remainder 24 and apply division lemma to get

$$124 = 24 \times 5 + 4 \qquad \ldots(vi) \qquad \left[\because\ 24\overline{)124}(5 \atop {120 \atop \overline{4}}\right]$$

We consider the new divisor 24 and the new remainder 4 and apply division lemma to get

$$24 = 4 \times 6 + 0 \qquad \ldots(vii) \qquad \left[\because\ 4\overline{)24}(6 \atop {24 \atop \overline{0}}\right]$$

We observe that the remainder at this stage is zero. Therefore, the divisor at this stage i.e. 4 (or the remainder at the earlier stage) is the HCF of 4052 and 12576.

Type II ON FINDING THE HCF OF THREE NUMBERS

To find the hcf of three numbers, we use the following steps:

STEP I *Find the HCF of any two of the given numbers.*

STEP II *Find the HCF of the third given number and the HCF obtained in step I.*

STEP III *The HCF obtained in step II is the HCF of three given numbers.*

EXAMPLE 3 *Use Euclid's division algorithm to find the HCF of 441, 567 and 693.*

[NCERT EXEMPLAR]

SOLUTION Let us first find the HCF of 441 and 567 by using Euclid's lemma. Applying Euclid's division lemma to 441 and 567, we obtain

$$567 = 441 \times 1 + 126$$

We find that the remainder is 126 which is *a* non-zero number. So, we apply Euclid's division lemma to 441 (divisor) and 126 (remainder) to get

$$441 = 126 \times 3 + 63$$

Now, we apply Euclid's division lemma to the divisor 126 and the remainder 63, to get

$$126 = 63 \times 2 + 0$$

The remainder at this stage is 0. So, the divisor at the previous stage *i.e.*, 63 is the HCF of 441 and 567.

Now, we use Euclid's division lemma to find the HCF of 63 and 693. We observe that

$$693 = 63 \times 11 + 0$$

So, the HCF of the third number 693 and 63 (the HCF of first two numbers 441 and 567) is 63.

Hence, the HCF of 441, 567 and 693 is 63.

Type III ON SOME APPLICATIONS OF HCF

EXAMPLE 4 *A sweet seller has 420 Kaju burfis and 130 Badam burfis she wants to stack them in such a way that each stack has the same number, and they take up the least area of the tray. What is the number of burfis that can be placed in each stack for this purpose?* **[NCERT]**

SOLUTION The area of the tray that is used up in stacking the burfis will be least if the sweet seller stacks maximum number of burfis in each stack. Since each stack must have the same number of burfis. Therefore, the number of stacks will be least if the number of burfis in each stack is equal to the HCF of 420 and 130.

In order to find the HCF of 420 and 130, let us apply Euclid's division lemma to 420 and 130 to get

$$420 = 130 \times 3 + 30 \qquad \ldots(i)$$

Let us now consider the divisor 130 and the remainder 30 and apply division lemma to get

$$130 = 30 \times 4 + 10 \qquad \ldots(ii)$$

Considering now divisor 30 and the remainder 10 and apply division lemma, we get

$$30 = 3 \times 10 + 0 \qquad \ldots(iii)$$

Since, the remainder at this stage is zero. Therefore, last divisor 10 is the HCF of 420 and 130.

Hence, the sweet seller can make stacks of 10 burfis of each kind to cover the least area of the tray.

EXAMPLE 5 *Any contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?* [NCERT]

SOLUTION The maximum number of columns is the HCF of 616 and 32. In order to find the HCF of 616 and 32, let us apply Euclid's division lemma to 616 and 32 to get

$$616 = 32 \times 19 + 8$$

Let us now take the divisor 32 as dividend and remainder 8 as divisor and apply Euclid's division lemma to get

$$32 = 8 \times 4 + 0$$

Since, the remainder at this stage is 0. Therefore, the last divisor i.e. 8 is the HCF of 616 and 32.

Hence, the maximum number of columns in which they can march is 8.

EXAMPLE 6 *Two tankers contain 850 litres and 680 litres of petrol respectively. Find the maximum capacity of a container which can measure the petrol of either tanker in exact number of times.*

SOLUTION Clearly, the maximum capacity of the container is the HCF of 850 and 680 in litres. So, Let us find the HCF of 850 and 680 by Euclid's algorithm.

	680	850	
4	680	680	1
	0 (Remainder)	170 (HCF)	

Clearly, HCF of 850 and 680 is 170.

Hence, capacity of the container must be 170 litres.

EXAMPLE 7 *Find the largest number which divides 245 and 1029 leaving remainder 5 in each case.*

SOLUTION It is given that the required number when divides 245 and 1029, the remainder is 5 in each case. This means that $245 - 5 = 240$ and $1029 - 5 = 1024$ are completely divisible by the required number.

It follows from this that the required number is a common factor of 240 and 1024. It is also given that the required number is the largest number satisfying the given property. Therefore, it is the HCF of 240 and 1024.

Let us now find the HCF of 240 and 1024 by Euclid's algorithm.

3	240 192	1024 960	4
3	48 48	64 48	1
	0 (Remainder)	16 (HCF)	

Clearly, HCF of 240 and 1024 is the last divisor i.e. 16. Hence, required number = 16.

EXAMPLE 8 *Find the largest number that divides 2053 and 967 and leaves a remainder of 5 and 7 respectively.*

SOLUTION It is given that on dividing 2053 by the required number, there is a remainder of 5. This means that $2053 - 5 = 2048$ is exactly divisible by the required number.

Similarly, $967 - 7 = 960$ is also exactly divisible by the required number.

Also, the required number is the largest number satisfying the above property. Therefore, it is the HCF of 2048 and 960.

Let us now find the HCF of 2048 and 960 by Euclid's algorithm.

7	960 896	2048 1920	2
	64 (HCF)	128 128	2
		0 (Remainder)	

Clearly, HCF of 960 and 2048 is the last divisor i.e. 64. Hence, required number = 64.

EXAMPLE 9 *Find the largest number that will divide 398, 436 and 542 leaving remainders 7, 11 and 15 respectively.*

SOLUTION Clearly, the required number is the HCF of the numbers

$$398 - 7 = 391,\ 436 - 11 = 425,\ \text{and},\ 542 - 15 = 527.$$

First we find the HCF of 391 and 425 by Euclid's algorithm as given below.

11	391 374	425 391	1
	17 (HCF)	34 34	2
		0 (Remainder)	

Clearly, H.C.F. of 391 and 425 is 17.

Let us now the HCF of 17 and the third number 527 by Euclid's algorithm:

17 (HCF)	527 51	3
	17 17	1
	0 (Remainder)	

The HCF of 17 and 527 is 17. Hence, HCF of 391, 4250 and 527 is 17.
Hence, the required number is 17.

EXAMPLE 10 *Can two numbers have 18 as their HCF and 380 as their LCM? Justify your answer.*

SOLUTION We know that HCF of two numbers is a divisor of their LCM. Here, 18 is not a divisor of 380. So, 18 and 380 cannot be respectively HCF and LCM of two numbers.

EXAMPLE 11 *The numbers 525 and 3000 are both divisible only by 3, 5, 15, 25 and 75. What is HCF (525, 3000)?*

SOLUTION It is given that 3, 5, 15, 25 and 75 are the only common factors of 525 and 3000. The highest of these common factors is 75. Hence, HCF (525, 3000) = 75.

LEVEL-2

Type IV ON EXPRESSING THE HCF OF TWO NUMBERS a AND b IN THE FORM $xa + xy$

EXAMPLE 12 *If the HCF of 210 and 55 is expressible in the form* $210 \times 5 + 55y$, *find* y.

SOLUTION Let us first find the HCF of 210 and 55.

Applying Euclid's division lemma on 210 and 55, we get

$$210 = 55 \times 3 + 45 \qquad \ldots \text{(i)} \qquad \left[\begin{array}{r} \because\ 55\,\overline{)\,210\,}(\,3 \\ 165 \\ \hline 45 \end{array}\right]$$

Since, the remainder $45 \neq 0$. So, we now apply division lemma on the divisor 55 and the remainder 45 to get

$$55 = 45 \times 1 + 10 \qquad \ldots \text{(ii)} \qquad \left[\begin{array}{r} \because\ 45\,\overline{)\,55\,}(\,1 \\ 45 \\ \hline 10 \end{array}\right]$$

We consider the divisor 45 and the remainder 10 and apply division lemma to get

$$45 = 4 \times 10 + 5 \qquad \ldots \text{(iii)} \qquad \left[\begin{array}{r} \because\ 10\,\overline{)\,45\,}(\,4 \\ 40 \\ \hline 5 \end{array}\right]$$

We consider the divisor 10 and the remainder 5 and apply division lemma to get

$$10 = 5 \times 2 + 0 \qquad \ldots \text{(iv)}$$

We observe that the remainder at this stage is zero. So, the last divisor i.e. 5 is the HCF of 210 and 55.

$$5 = 210 \times 5 + 55y$$

$$\Rightarrow \qquad 55y = 5 - 210 \times 5 = 5 - 1050$$

$$\Rightarrow \qquad 55y = -1045$$

$$\Rightarrow \qquad y = \frac{-1045}{55} = -19$$

EXAMPLE 13 *In a seminar, the number of participants in Hindi, English and Mathematics are 60, 84 and 108, respectively. Find the minimum number of rooms required if in each room the same number of participants are to be seated and all of them being in the same subject.*

SOLUTION The number of participants in each room must be the HCF of 60, 84 and 108. In order to find the HCF of 60, 84 and 108, we first find the HCF of 60 and 84 by Euclid's division algorithm:

2	60 48	84 60	1
	12 (HCF)	24 24	2
		0 (Remainder)	

Clearly, HCF of 60 and 84 is 12

Now, we find the HCF of 12 and 108

12 (HCF)	108 108	9
	0 (Remainder)	

Clearly, HCF of 12 and 108 is 12. Hence, the HCF of 60, 84 and 108 is 12.
Therefore, in each room maximum 12 participants can be seated.
We have,

$$\text{Total number of participants} = 60 + 84 + 108 = 252$$

$$\therefore \qquad \text{Number of rooms required} = \frac{252}{12} = 21.$$

EXAMPLE 14 *Three sets of English, Hindi and Mathematics books have to be stacked in such a way that all the books are stored topic-wise and the height of each stack is the same. The number of English books is 96, the number of Hindi books is 240 and the number of Mathematics books is 336. Assuming that the books are of the same thickness, determine the number of stacks of English, Hindi and Mathematics books.*

SOLUTION In order to arrange the books as required, we have to find the largest number that divides 96, 240 and 336 exactly. Clearly, such a number is their HCF Computation of HCF of 96 and 240:

2	96 96	240 192	2
	0 (Remainder)	48 (HCF)	

Clearly, HCF of 96 and 240 is 48.
Computation of HCF of 48 and 336:

48 (HCF)	336 336	7
	0 (Remainder)	

Clearly, HCF of 48 and 336 is 48. Thus, HCF of 96, 240 and 336 is 48.
Hence, there must be 48 books in each stack.

Now, Number of stacks of English books $= \dfrac{\text{Number of English books}}{\text{Number of books in each stack}} = \dfrac{96}{48} = 2$

Number of stacks of Hindi books $= \dfrac{\text{Number of Hindi books}}{\text{Number of books in each stack}} = \dfrac{240}{48} = 5$

and, Number of stacks of Mathematics books $= \dfrac{\text{No. of Mathematics books}}{\text{No. of books in each stack}} = \dfrac{336}{48} = 7$

LEVEL-3

Type V ON EXPRESSING THE HCF OF TWO NUMBERS a AND b IN THE FORM $xa + yb$

EXAMPLE 15 *Find the HCF of 81 and 237 and express it as a linear combination of 81 and 237.*

SOLUTION Given integers are 81 and 237 such that $81 < 237$.
Applying division lemma to 81 and 237, we get

$$237 = 81 \times 2 + 75 \qquad \ldots(\text{i}) \qquad \left[\because 81\overline{)237}(2 ;\ 162 ;\ 75\right]$$

Since the remainder $75 \neq 0$. So, consider the divisor 81 and the remainder 75 and apply division lemma to get

$$81 = 75 \times 1 + 6 \qquad \ldots(\text{ii}) \qquad \left[\because 75\overline{)81}(1 ;\ 75 ;\ 6\right]$$

We consider the new divisor 75 and the new remainder 6 and apply division lemma to get

$$75 = 6 \times 12 + 3 \quad \ldots\text{(iii)} \qquad \left[\because \begin{array}{r} 6\overline{)75}(12 \\ 72 \\ \hline 3 \end{array}\right]$$

We consider the new divisor 6 and the new remainder 3 and apply division lemma to get

$$6 = 3 \times 2 + 0 \quad \ldots\text{(iv)} \qquad \left[\because \begin{array}{r} 3\overline{)6}(2 \\ 6 \\ \hline 0 \end{array}\right]$$

The remainder at this stage is zero. So, the divisor at this stage or the remainder at the earlier stage i.e. 3 is the HCF of 81 and 237.

To represent the HCF as a linear combination of the given two numbers, we start from the last but one step and successively eliminate the previous remainders as follows:

From (iii), we have

$$3 = 75 - 6 \times 12$$

$$\Rightarrow \quad 3 = 75 - (81 - 75 \times 1) \times 12 \qquad \left[\begin{array}{l}\text{Substituting } 6 = 81 - 75 \times 1 \\ \text{obtained from (ii)}\end{array}\right]$$

$$\Rightarrow \quad 3 = 75 - 12 \times 81 + 12 \times 75$$

$$\Rightarrow \quad 3 = 13 \times 75 - 12 \times 81 \qquad \left[\begin{array}{l}\text{Substituting } 75 = 237 - 81 \times 2 \\ \text{obtained from (i)}\end{array}\right]$$

$$\Rightarrow \quad 3 = 13 \times (237 - 81 \times 2) - 12 \times 81$$

$$\Rightarrow \quad 3 = 13 \times 237 - 26 \times 81 - 12 \times 81$$

$$\Rightarrow \quad 3 = 13 \times 237 - 38 \times 81$$

$$\Rightarrow \quad 3 = 237x + 81y, \text{ where } x = 13 \text{ and } y = -38.$$

REMARK *It follows from the above example that the HCF (say d) of two positive integers a and b can be expressed as a linear combination of a and b i.e., $d = xa + yb$ for some integers x and y. Also, this representation is not unique. Because,*

$$d = xa + yb$$

$$\Rightarrow \quad d = xa + yb + ab - ab$$

$$\Rightarrow \quad d = (x + b)\,a + (y - a)\,b$$

In the above example, we had

$$3 = 13 \times 237 - 38 \times 81$$

$$\Rightarrow \quad 3 = 13 \times 237 - 38 \times 81 + 237 \times 81 - 237 \times 81$$

$$\Rightarrow \quad 3 = (13 \times 237 + 237 \times 81) + (-38 \times 81 - 237 \times 81)$$

$$\Rightarrow \quad 3 = (13 + 81) \times 237 + (-38 - 237) \times 81$$

$$\Rightarrow \quad 3 = 94 \times 237 - 275 \times 81$$

$$\Rightarrow \quad 3 = 94 \times 237 + (-275) \times 81$$

EXAMPLE 16 *Find the HCF of 65 and 117 and express it in the form $65m + 117n$.*

SOLUTION Given integers are 65 and 117 such that $117 > 65$.

Applying division lemma to 65 and 117, we get

$$117 = 65 \times 1 + 52 \qquad \ldots\text{(i)} \qquad \left[\because\ 65\overline{)117}(1 \quad \begin{array}{r}65\\\hline 52\end{array}\right]$$

Since the remainder $52 \neq 0$. So, we apply the division lemma to the divisor 65 and the remainder 52 to get

$$65 = 52 \times 1 + 13 \qquad \ldots\text{(ii)} \qquad \left[\because\ 52\overline{)65}(1 \quad \begin{array}{r}52\\\hline 13\end{array}\right]$$

We consider the new divisor 52 and the new remainder 13 and apply division lemma, to get

$$52 = 13 \times 4 + 0 \qquad \ldots\text{(iii)}$$

At this stage the remainder is zero. So, the last divisor or the non-zero remainder at the earlier stage i.e. 13 is the HCF of 65 and 117.

From (ii), we have

$$13 = 65 - 52 \times 1$$

$$\Rightarrow \quad 13 = 65 - (117 - 65 \times 1) \qquad [\text{Substituting } 52 = 117 - 65 \times 1 \text{ obtain from (i)}]$$

$$\Rightarrow \quad 13 = 65 - 117 + 65 \times 1$$

$$\Rightarrow \quad 13 = 65 \times 2 + 117 \times (-1)$$

$$\Rightarrow \quad 13 = 65 - 117 + 65 \times 1$$

$$\Rightarrow \quad 13 = 65m + 117n, \text{ where } m = 2 \text{ and } n = -1.$$

EXAMPLE 17 *If d is the HCF of 56 and 72, find x, y satisfying $d = 56x + 72y$. Also, show that x and y are not unique.*

SOLUTION Applying Euclid's division lemma to 56 and 72, we get

$$72 = 56 \times 1 + 16 \qquad \ldots\text{(i)} \qquad \left[\because\ 56\overline{)72}(1 \quad \begin{array}{r}56\\\hline 16\end{array}\right]$$

Since, the remainder $16 \neq 0$. So, we consider the divisor 56 and the remainder 16 and apply division lemma to get

$$56 = 16 \times 3 + 8 \qquad \ldots\text{(ii)} \qquad \left[\because\ 16\overline{)56}(3 \quad \begin{array}{r}48\\\hline 8\end{array}\right]$$

We consider the divisor 16 and the remainder 8 and apply division algorithm to get

$$16 = 8 \times 2 + 0 \qquad \ldots\text{(iii)} \qquad \left[\because\ 8\overline{)16}(2 \quad \begin{array}{r}16\\\hline 0\end{array}\right]$$

We observe that the remainder at this stage is zero. Therefore, last divisor 8 (or the remainder at the earlier stage) is the HCF of 56 and 72.
From (ii), we get

$$8 = 56 - 16 \times 3$$

$\Rightarrow \quad 8 = 56 - (72 - 56 \times 1) \times 3 \qquad [\because\ 16 = 72 - 56 \times 1 \text{ (from (i))}]$

$\Rightarrow \quad 8 = 56 - 3 \times 72 + 56 \times 3$

$\Rightarrow \quad 8 = 56 \times 4 + (-3) \times 72$

$\therefore \quad x = 4$ and $y = -3$.

Now, $\quad 8 = 56 \times 4 + (-3) \times 72$

$\quad 8 = 56 \times 4 + (-3) \times 72 - 56 \times 72 + 56 \times 72$

$\Rightarrow \quad 8 = 56 \times 4 - 56 \times 72 + (-3) \times 72 + 56 \times 72$

$\Rightarrow \quad 8 = 56 \times (4 - 72) + \{(-3) + 56\} \times 72$

$\Rightarrow \quad 8 = 56 \times (-68) + (53) \times 72$

$\therefore \quad x = -68$ and $y = 53$.

Hence, x and y are not unique.

EXERCISE 1.2

LEVEL-1

1. Define HCF of two positive integers and find the HCF of the following pairs of numbers:
   (i) 32 and 54 (ii) 18 and 24 (iii) 70 and 30
   (iv) 56 and 88 (v) 475 and 495 (vi) 75 and 243.
   (vii) 240 and 6552 (viii) 155 and 1385 (ix) 100 and 190 [CBSE 2009]
   (x) 105 and 120 [CBSE 2009]
2. Use Euclid's division algorithm to find the HCF of
   (i) 135 and 225 (ii) 196 and 38220 (iii) 867 and 255. [NCERT]
   (iv) 184, 230 and 276 (v) 136, 170 and 255
3. Find the HCF of the following pairs of integers and express it as a linear combination of them.
   (i) 963 and 657 (ii) 592 and 252 (iii) 506 and 1155
   (iv) 1288 and 575
4. Find the largest number which divides 615 and 963 leaving remainder 6 in each case.
5. If the HCF of 408 and 1032 is expressible in the form $1032\, m - 408 \times 5$, find m.
6. If the HCF of 657 and 963 is expressible in the form $657\, x + 963 \times -15$, find x.
7. An army contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?
8. A merchant has 120 litres of oil of one kind, 180 litres of another kind and 240 litres of third kind. He wants to sell the oil by filling the three kinds of oil in tins of equal capacity. What should be the greatest capacity of such a tin?
9. During a sale, colour pencils were being sold in packs of 24 each and crayons in packs of 32 each. If you want full packs of both and the same number of pencils and crayons, how many of each would you need to buy?

10. 144 cartons of Coke Cans and 90 cartons of Pepsi Cans are to be stacked in a Canteen. If each stack is of the same height and is to contain cartons of the same drink, what would be the greatest number of cartons each stack would have?
11. Find the greatest number which divides 285 and 1249 leaving remainders 9 and 7 respectively.
12. Find the largest number which exactly divides 280 and 1245 leaving remainders 4 and 3, respectively.
13. What is the largest number that divides 626, 3127 and 15628 and leaves remainders of 1, 2 and 3 respectively?
14. Find the greatest number that will divide 445, 572 and 699 leaving remainders 4, 5 and 6 respectively.
15. Find the greatest number which divides 2011 and 2623 leaving remainders 9 and 5 respectively.
16. Using Euclid's division algorithm, find the largest number that divides 1251, 9377 and 15628 leaving remainders 1, 2 and 3 respectively. [NCERT EXEMPLAR]

LEVEL-2

17. Two brands of chocolates are available in packs of 24 and 15 respectively. If I need to buy an equal number of chocolates of both kinds, what is the least number of boxes of each kind I would need to buy?
18. A mason has to fit a bathroom with square marble tiles of the largest possible size. The size of the bathroom is 10 ft. by 8 ft. What would be the size in inches of the tile required that has to be cut and how many such tiles are required?
19. 15 pastries and 12 biscuit packets have been donated for a school fete. These are to be packed in several smaller identical boxes with the same number of pastries and biscuit packets in each. How many biscuit packets and how many pastries will each box contain?
20. 105 goats, 140 donkeys and 175 cows have to be taken across a river. There is only one boat which will have to make many trips in order to do so. The lazy boatman has his own conditions for transporting them. He insists that he will take the same number of animals in every trip and they have to be of the same kind. He will naturally like to take the largest possible number each time. Can you tell how many animals went in each trip?
21. The length, breadth and height of a room are 8 m 25 cm, 6 m 75 cm and 4 m 50 cm, respectively. Determine the longest rod which can measure the three dimensions of the room exactly.
22. Express the HCF of 468 and 222 as $468x + 222y$ where x, y are integers in two different ways.

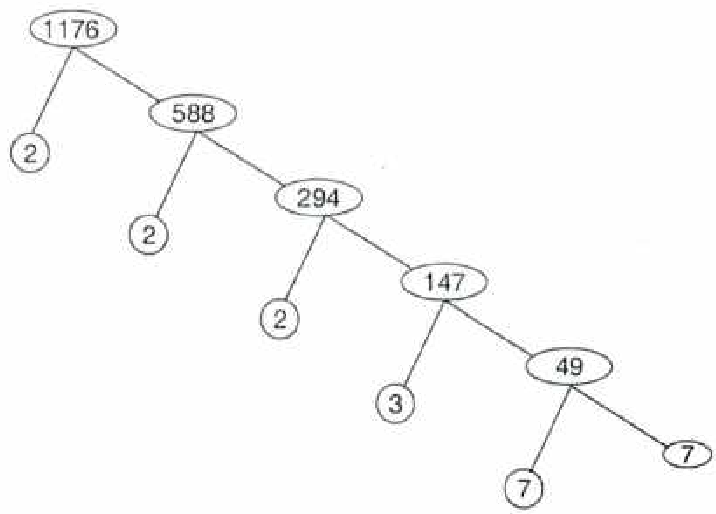
ANSWERS

1. (i) 2 (ii) 6 (iii) 10 (iv) 8 (v) 5 (vi) 3 (vii) 24 (viii) 5 (ix) 10 (x) 15
2. (i) 45, $45 = (-1)\,225 + 2 \times 135$ (ii) 196, $196 = 38220 \times 1 + (-194) \times 196$
   (iii) $51 = 51 = (-2)\,867 + 7 \times 255$ (iv) 46 (v) 17
3. (i) $9 = (-15) \times 963 + 22 \times 657$ (ii) $4 = 77 \times 252 + (-20)\,592$
   (iii) $11 = 16 \times 506 + (-7) \times 1155$ (iv) $23 = (-4) \times 1288 + 9 \times 575$
4. 87 5. 2 6. 22 7. 8 columns 8. 60 litres

9. 4 packets of colour pencils, 3 packets of crayons 10. 18 11. 138 12. 138
13. 625 14. 63 15. 154 16. 625 17. 5 of first kind, 8 of second kind
18. 24 inches, 20 tiles 19.4 biscuit packets, 5 pastries
20. 35 21. 75 cm 22. $6 = 468 \times -9 + 222 \times 19, 6 = 468 \times 213 + 222 \times (-449)$

1.5 THE FUNDAMENTAL THEOREM ARITHMETIC

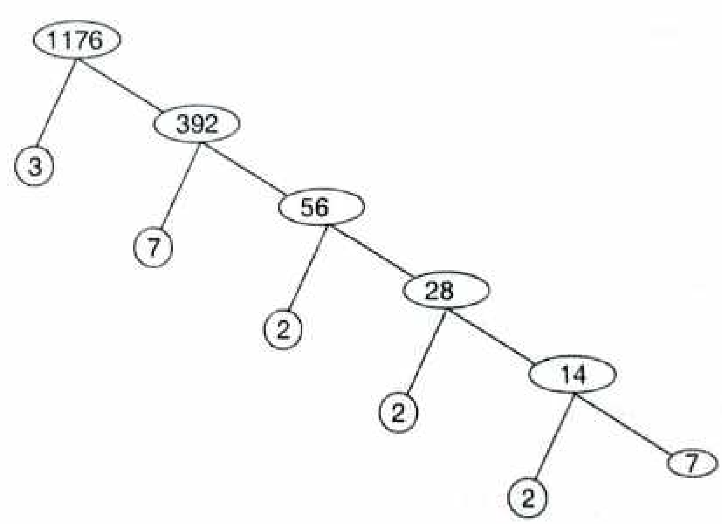
In earlier classes, we have learnt about prime and composite numbers. Let us recall that a positive integer p is prime if $p \neq 1$ and the only positive divisors of p are 1 and p. For example, 2, 3, 5, 7, 11, 13, 19, 23, 29, 31, 37, ... are the first few primes. We have learnt that every positive integer, other than 1, is either prime or composite. If a given positive integer is a composite number, it can be written as the product of two of its factors. These factors in turn are also either prime or composite. If composite, the factors can be split up further. If we keep on doing this factorization, ultimately we will arrive at a stage when all the factors are prime numbers as shown below for the positive integer 1176.



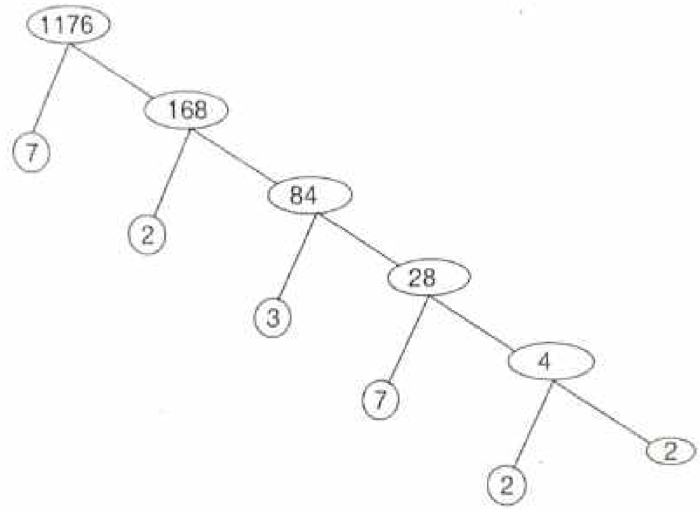
Thus, we have

$$1176 = 2 \times 2 \times 2 \times 3 \times 7 \times 7$$

Also, we have



and,



$\therefore \quad 1176 = 3 \times 7 \times 2 \times 2 \times 2 \times 7$

and, $\quad 1176 = 7 \times 2 \times 3 \times 7 \times 2 \times 2$ etc.

We observe that in all these prime factorizations of 1176, the prime numbers appearing are same, although the order in which they appear are different. Thus, the prime factorization of 1176 is unique except for the order in which the primes occur.

Let us now try another positive integer, say, 32760. This can be written as

$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$

i.e., $\quad 32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$

$\quad = 3 \times 2 \times 2 \times 3 \times 5 \times 7 \times 13 \times 2$ etc.

We observe that the above observation is also true for the positive integer 32760. This leads us to a conjecture that every positive integer is either prime or it can be expressed as the product of primes. In fact, this statement is true, and is called the *Fundamental Theorem of Arithmetic* because of its basic importance in the development of number theory. Let us now formally state this theorem.

THEOREM 1: **(FUNDAMENTAL THEOREM OF ARITHMETIC)** *Every composite number can be expressed (factorised) as a product of primes, and this factorization is unique except for the order in which the prime factors occur.*

While writing a positive integer as the product of primes, if we decide to write the prime factors in ascending order and we combine the same primes, then the integer is expressed as the product of powers of primes and the representation is unique. So, we can say that every composite number can be expressed as the products of powers distinct primes in ascending or descending order in a unique way.

Following theorem is a direct consequence of the Fundamental Theorem of Arithmetic.

THEOREM 2 *Let p be a prime number and a be a positive integer. If p divides a^2, then p divides a.* **[NCERT]**

PROOF From the Fundamental Theorem of Arithmetic integer a can be factorised as the product of primes. Let $a = p_1 p_2 p_3 \cdots p_n$ be the prime factorisation of a, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Now,

$$a = p_1 p_2 p_3 \cdots p_n$$

$$\Rightarrow \quad a^2 = (p_1 p_2 p_3 \ldots p_n)(p_1 p_2 p_3 \ldots p_n)$$

$$\Rightarrow \quad a^2 = p_1^2 p_2^2 p_3^2 \ldots p_n^2$$

It is given that p is prime and it divides a^2. Therefore, p is a prime factor of a^2. From the uniqueness part of the Fundamental Theorem of Arithmetic it follows that the only prime factors of a^2 are $p_1, p_2, p_3, \ldots, p_n$. Therefore, p is one of $p_1, p_2, p_3, \ldots, p_n$. This implies that

$$p \mid p_1 p_2 p_3 \cdots p_n \Rightarrow p \mid a.$$ **Q.E.D.**

ILLUSTRATIVE EXAMPLES

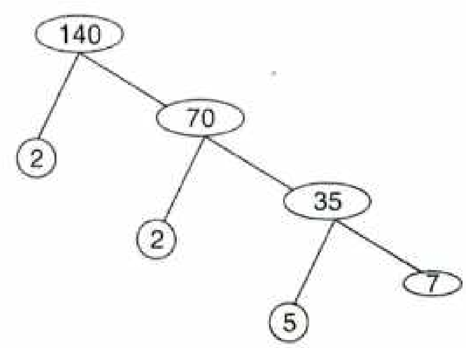
LEVEL-1

Type I **ON EXPRESSING A POSITIVE INTEGER AS THE PRODUCT OF ITS PRIME FACTORS**

EXAMPLE 1 *Express each of the following positive integers as the product of its prime factors:*

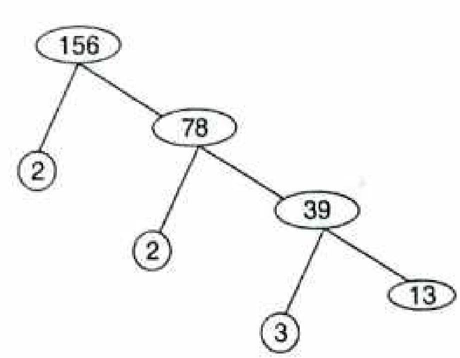
(i) 140 *(ii)* 156 *(iii)* 234

SOLUTION (i) Using the factor tree for prime factorization, we have



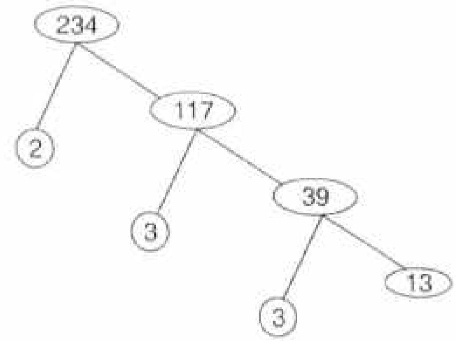
$$\therefore \quad 140 = 2 \times 2 \times 5 \times 7 = 2^2 \times 5 \times 7$$

(ii) Using the factor tree for prime factorisation, we have



$$\therefore \quad 156 = 2 \times 2 \times 3 \times 13$$

(iii) Using the factor tree for prime factorization, we have

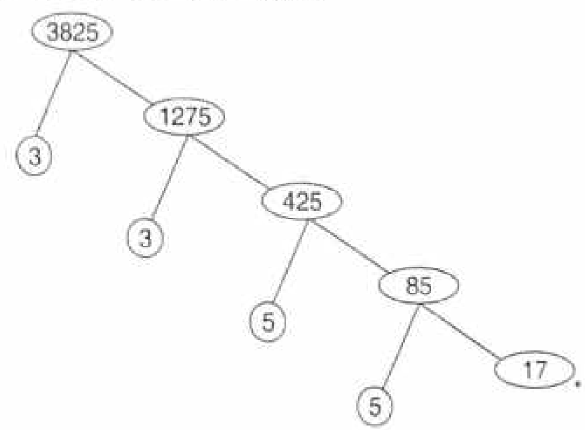


$\therefore$ $\quad 234 = 2 \times 3 \times 3 \times 13 = 2 \times 3^2 \times 13$

EXAMPLE 2 *Express each of the following positive integers as the product of its prime factors:*

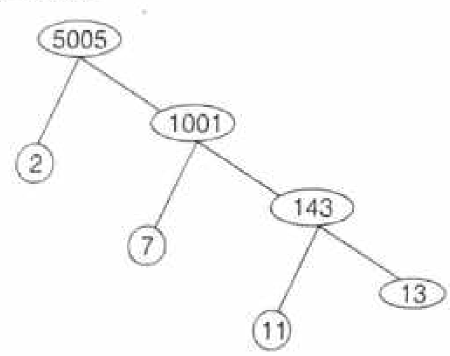
(i) 3825 *(ii) 5005* *(iii) 7429*

SOLUTION (i) Using the factor tree, we have



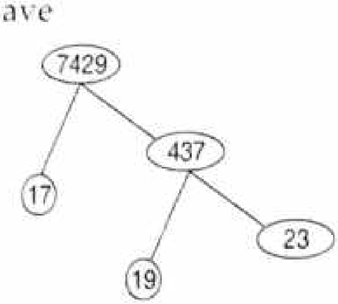
$\therefore$ $\quad 3825 = 3 \times 3 \times 5 \times 5 \times 17 = 3^2 \times 5^2 \times 17$

(ii) Using the factor tree, we have



$\therefore$ $\quad 5005 = 5 \times 7 \times 11 \times 13$

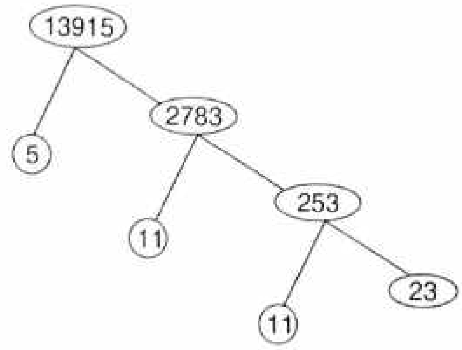
(iii) Using the factor tree, we have



$\therefore$ $\quad 7429 = 17 \times 19 \times 23$

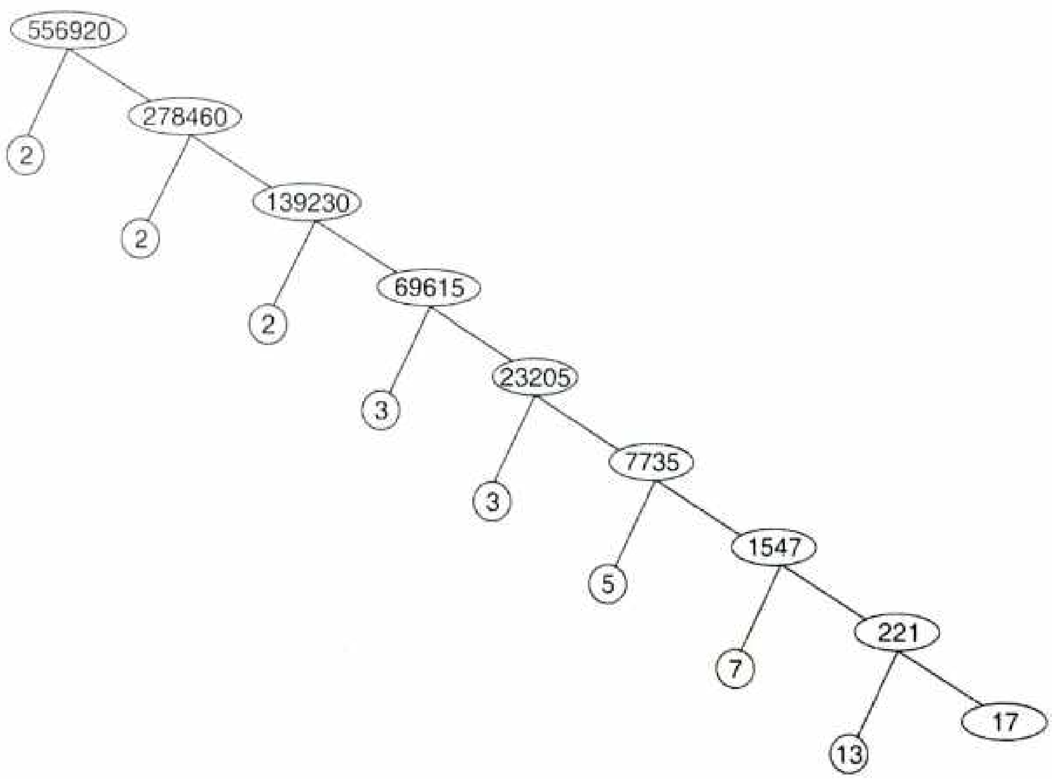
EXAMPLE 3 *Determine the prime factorization of each of the following numbers:*
(i) 13915 *(ii) 556920*

SOLUTION (i) Using the prime factorization tree, we have



$\therefore$ $$13915 = 5 \times 11 \times 11 \times 23 = 5 \times 11^2 \times 23$$

(ii) Using the prime factorisation tree, we have



$\therefore$ $$556920 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 \times 17 = 2^3 \times 3^2 \times 5 \times 7 \times 13 \times 17$$

Type II **ON MORE APPLICATIONS OF THE FUNDAMENTAL THEOREM OF ARITHMETIC**

EXAMPLE 4 *Prove that there is no natural number for which 4^n ends with the digit zero.* [NCERT]

SOLUTION We know that any positive integer ending with the digit zero is divisible by 5 and so its prime factorization must contain the prime 5.

We have,

$$4^n = (2^2)^n = 2^{2n}$$

$\Rightarrow$ The only prime in the factorization of 4^n is 2.

$\Rightarrow$ There is no other primes in the factorization of $4^n = 2^{2n}$

[By uniqueness of the Fundamental Theorem of Arithmetic]

$\Rightarrow$ 5 does not occur in the prime factorization of 4^n for any n.

$\Rightarrow$ 4^n does not end with the digit zero for any natural number n.

EXAMPLE 5 *Show that 12^n cannot end with digit 0 or 5 for any natural number n.* [NCERT EXEMPLAR]

SOLUTION Expressing 12 as the product of primes, we obtain

$$12 = 2^2 \times 3$$

$$\Rightarrow \quad 12^n = (2^2 \times 3)^n = (2^2)^n \times 3^n = (2)^{2n} \times 3^n$$

So, only primes in the factorisation of 12^n are 2 and 3 and, not 5.

Hence, 12^n cannot end with digit 0 or 5.

LEVEL-2

EXAMPLE 6 *Show that there are infinitely many positive primes.*

SOLUTION If possible, let there be finite number of positive primes $p_1, p_2, \dots, p_n$. Such that $p_1 < p_2 < p_3 < \dots < p_n$.

Let $x = 1 + p_1\, p_2\, p_3 \cdots p_n$. Clearly, $p_1\, p_2 \dots p_n$ is divisible by each of $p_1, p_2, p_3, \dots, p_n$.

$\therefore$ $x = 1 + p_1\, p_2\, p_3 \cdots p_n$ is not divisible by any one of $p_1, p_2, \dots, p_n$

$\Rightarrow$ x is a prime or it has prime divisors other than $p_1, p_2, \dots, p_n$

There exists a positive prime different from $p_1, p_2, \dots, p_n$

This contradicts that there are finite number of positive primes.

Hence, the number of positive primes is infinite.

EXAMPLE 7 *Prove that every positive integer different from 1 can be expressed as a product of a non-negative power of 2 and an odd number.*

SOLUTION Let n be a positive integer other than 1. By the fundamental theorem of Arithmetic n can be uniquely expressed as powers of primes in ascending order. So, let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the unique factorisation of n into primes with $p_1 < p_2 < p_3 < \dots < p_k$. Clearly, either $p_1 = 2$ and $p_2, p_3, \dots, p_k$ are odd positive integers or each of $p_1, p_2, \dots, p_k$ is an odd positive integer.

Therefore, we have the following cases:

CASE I *When $p_1 = 2$ and $p_2, p_3, \dots, p_k$ are odd positive integers:*

In this case, we have

$$n = 2^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$$

$$\Rightarrow \quad n = 2^{a_1} \times (p_2^{a_2}\ p_3^{a_3} \dots p_k^{a_k})$$

$$\Rightarrow \quad n = 2^{a_1} \times \text{An odd positive integer.}$$

$$\Rightarrow \quad n = (\text{A non-negative power of 2}) \times (\text{An odd positive integer})$$

CASE II *When each of $p_1, p_2, p_3, \dots, p_k$ is an odd positive integer:*

In this case, we have

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_k^{a_k}$$

$\Rightarrow \quad n = 2^0 \times (p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_k^{a_k})$

$\Rightarrow \quad n =$ (A non-negative power of 2) $\times$ (An odd positive integer)

Hence, in either case n is expressible as the product of a non-negative power of 2 and an odd positive integer.

EXAMPLE 8 *Prove that a positive integer n is prime number, if no prime p less than or equal to $\sqrt{n}$ divides n.*

SOLUTION Let n be a positive integer such that no prime less than or equal to $\sqrt{n}$ divides n. Then, we have to prove that n is prime. Suppose n is not a prime integer. Then, we may write

$n = ab$ where $1 < a \le b$

$\Rightarrow \quad a \le \sqrt{n}$ and $b \ge \sqrt{n}$

Let p be a prime factor of a. Then, $p \le a \le \sqrt{n}$ and $p \mid a$

$\Rightarrow \quad p \mid ab$

$\Rightarrow \quad p \mid n$

$\Rightarrow$ a prime less than $\sqrt{n}$ divides n.

This contradicts our assumption that no prime less than $\sqrt{n}$ divides n. So, our assumption is is wrong. Hence, n is a prime.

EXERCISE 1.3

LEVEL-1

1. Express each of the following integers as a product of its prime factors:
   (i) 420 (ii) 468 (iii) 945 (iv) 7325
2. Determine the prime factorisation of each of the following positive integer:
   (i) 20570 (ii) 58500 (iii) 45470971
3. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
4. Check whether 6^n can end with the digit 0 for any natural number n. [NCERT]
5. Explain why $3 \times 5 \times 7 + 7$ is a composite number. [NCERT EXEMPLAR]

ANSWERS

1. (i) $2^2 \times 3 \times 5 \times 7$ (ii) $2^2 \times 3^2 \times 13$ (iii) $3^3 \times 5 \times 7$ (iv) $5^2 \times 293$
2. (i) $2 \times 5 \times 11^2 \times 17$ (ii) $2^2 \times 3^2 \times 5^3 \times 13$ (iii) $7^2 \times 13^2 \times 17^2 \times 19$
3. Since $7 \times 11 \times 13 + 13 = (7 \times 11 + 1) \times 13$

   and, $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5 = (7 \times 6 \times 4 \times 3 \times 2 \times 1 + 1) \times 5$
4. No

HINTS TO SELECTED PROBLEMS

4. We have, $6^n = (2 \times 3)^n = 2^n \times 3^n$. Therefore, prime factorisation of 6^n does not contain 5 as a factor. Hence, 6^n can never end with the digit 0 for any natural number.

5. Since $3 \times 5 \times 7 + 7 = (3 \times 5 + 1) \times 7 = (15 + 1) \times 7 = 16 \times 7$. Hence, it is a composite number.

1.6 SOME APPLICATIONS OF THE FUNDAMENTAL THEOREM OF ARITHMETIC

In this section, we will learn about various applications of the Fundamental Theorem of Arithmetic. In fact, we have studied about some of these applications in earlier classes even without realising their dependence on the Fundamental Theorem of Arithmetic. For example, we have have used prime factorisation method to find the HCF and LCM of positive integers. In this method, we use the Fundamental Theorem of Arithmetic in expressing the given integers as the product of primes. We will also discuss some other applications of the Fundamental Theorem of Arithmetic.

1.6.1 FINDING HCF AND LCM OF POSITIVE INTEGERS

In order to find the HCF and LCM of two or more positive integers, we may use the following algorithm.

ALGORITHM

STEP I *Factorize each of the given positive integers and express them as a product of powers of primes in ascending order of magnitudes of primes.*

STEP II *To find the HCF, identify common prime factors and find the smallest (least) exponent of these common factors. Now raise these common prime factors to their smallest exponents and multiply them to get the HCF.*

To find the LCM, list all prime factors (once only) occuring in the prime factorisation of the given positive integers.

For each of these factors, find the greatest exponent and raise each prime factor to the greatest exponent and multiply them to get the LCM.

REMARK *To find the LCM of two positive integers a and b, we can also use the following result, if we have already found the HCF.*

$$HCF \times LCM = a \times b.$$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE HCF AND LCM BY PRIME FACTORISATION**

EXAMPLE 1 *Find the HCF and LCM of 90 and 144 by the prime factorisation method.*

SOLUTION Using the factor tree for the prime factorisation of 90 and 144, we have

$$90 = 2 \times 3^2 \times 5 \text{ and } 144 = 2^4 \times 3^2$$

To find the HCF, we list the common prime factors and their smallest exponents in 90 and 144 as under:

Common prime factors	Least exponents
2	1
3	2

$\therefore \qquad \text{HCF} = 2^1 \times 3^2 = 2 \times 9 = 18$

To find the LCM, we list all prime factors of 90 and 144 and their greatest exponents as follows:

Prime factors of 90 and 144	Greatest exponents
2	4
3	2
5	1

$\therefore \qquad \text{LCM} = 2^4 \times 3^2 \times 5^1 = 16 \times 9 \times 5 = 720$

EXAMPLE 2 *Find the HCF and LCM of 144, 180 and 192 by prime factorisation method.*

SOLUTION Using the factor tree for the prime factorisation of 144, 180 and 192, we have

$$144 = 2^4 \times 3^2, 180 = 2^2 \times 3^2 \times 5 \text{ and } 192 = 2^6 \times 3$$

To find the HCF, we list the common prime factors and their smallest exponents in 144, 180 and 192 as follows:

Common prime factors	Least exponents
2	2
3	1

$\therefore \qquad \text{HCF} = 2^2 \times 3^1 = 12$

To find the LCM, we list all prime factors of 144, 180, 192 and their greatest exponents as follows:

Prime factors of 144, 180 and 192	Greatest exponents
2	6
3	2
5	1

$\therefore \qquad \text{LCM} = 2^6 \times 3^2 \times 5^1 = 64 \times 9 \times 5 = 2880$

EXAMPLE 3 *Find the HCF of 96 and 404 by prime factorisation method. Hence, find their LCM.* [NCERT]

SOLUTION We have,

$$96 = 2^5 \times 3 \text{ and } 404 = 2^2 \times 101$$

$\therefore \qquad \text{HCF} = 2^2 = 4$

Now, $\qquad \text{HCF} \times \text{LCM} = 96 \times 404$

$$\Rightarrow \qquad \text{LCM} = \frac{96 \times 404}{\text{HCF}} = \frac{96 \times 404}{4} = 96 \times 101 = 9696$$

REMARK *The product of two positive integers is equal to the product of their HCF and LCM, but the same is not true for three or more positive integers.*

Type II ON APPLICATIONS OF HCF AND LCM

EXAMPLE 4 *Find the largest positive integer that will divide 398, 436 and 542 leaving remainders 7, 11 and 15 respectively.*

SOLUTION It is given that on dividing 398 by the required number, there is a remainder of 7. This means that $398 - 7 = 391$ is exactly divisible by the required number. In other words required number is a factor of 391.

Similarly, required positive integer is a factor of $436 - 11 = 425$ and $542 - 15 = 527$.

Clearly, required number is the HCF of 391, 425 and 527.

Using the factor tree the prime factorisations of 391, 425 and 527 are as follows:

$$391 = 17 \times 23,\ 425 = 5^2 \times 17 \text{ and } 527 = 17 \times 31$$

$\therefore$ HCF of 391, 425 and 527 is 17

Hence, required number $= 17$

EXAMPLE 5 *There is a circular path around a sports field. Priya takes 18 minutes to drive one round of the field, while Ravish takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?* [NCERT]

SOLUTION Required number of minutes is the LCM of 18 and 12.

We have,

$$18 = 2 \times 3^2 \text{ and } 12 = 2^2 \times 3$$

$\therefore$ LCM of 18 and 12 is $2^2 \times 3^2 = 36$

Hence, Ravish and Priya will meet again at the starting point after 36 minutes.

EXAMPLE 6 *In a school there are two sections – section A and section B of classX. There are 32 students in section A and 36 students in section B. Determine the minimum number of books required for their class library so that they can be distributed equally among students of section A or section B.*

SOLUTION Since the books are to be distributed equally among the students of section *A* or section *B*. Therefore, number of books must be a multiple of 32 as well as 36. Hence, required number of books is the LCM of 32 and 36.

We have,

$$32 = 2^5 \text{ and } 36 = 2^2 \times 3^2$$

$\therefore$ LCM of 32 and 36 is $2^5 \times 3^2 = 288$

Hence, required number of books is 288.

EXAMPLE 7 *On a morning walk, three persons step off together and their steps measure 40 cm, 42 cm and 45 cm respectively. What is the minimum distance each should walk so that each can cover the same distance and complete steps?* [NCERT EXEMPLAR]

SOLUTION Each person will cover the same distance in complete steps if the distance covered in cm is the LCM of 40, 42 and 45.

Now,

$$40 = 2^3 \times 5,\ 42 = 2 \times 3 \times \times 7 \text{ and } 45 = 3^2 \times 5$$

$\therefore$ LCM of 40, 42 and 45 is $2^3 \times 3^2 \times 5 \times 7 = 8 \times 9 \times 5 \times 7 = 2520$

Hence, minimum distance each should walk = 2520 cm.

LEVEL-2

EXAMPLE 8 *In a seminar, the number of participants in Hindi, English and Mathematics are 60, 84 and 108 respectively. Find the minimum number of rooms required if in each room the same number of participants are to be seated and all of them being in the same subject.*

SOLUTION The Number of room will be minimum if each room accomodates maximum number of participants. Since in each room the same number of participants are to be seated and all of them must be of the same subject. Therefore, the number of participants in each room must be the HCF of 60, 84 and 108. The prime factorisations of 60, 84 and 108 are as under:

$$60 = 2^2 \times 3 \times 5,\ 84 = 2^2 \times 3 \times 7 \text{ and } 108 = 2^2 \times 3^3$$

$\therefore$ HCF of 60, 84 and 108 is $2^2 \times 3 = 12$

Therefore, in each room 12 participants can be seated.

$$\therefore \quad \text{Number of rooms required} = \frac{\text{Total number of participants}}{12}$$

$$= \frac{60 + 84 + 108}{12} = \frac{252}{12} = 21$$

EXAMPLE 9 *Three sets of English, Hindi and Mathematics books have to be stacked in such a way that all the books are stored topic wise and the height of each stack is the same. The number of English books is 96, the number of Hindi books is 240 and the number of Mathematics books is 336. Assuming that the books are of the same thickness, determine the number of stacks of English, Hindi and Mathematics books.*

SOLUTION In order to arrange the books as required, we have to find the largest number that divides 96, 240 and 336 exactly. Clearly, such a number is their HCF.

We have,

$$96 = 2^5 \times 3,\ 240 = 2^4 \times 3 \times 5 \text{ and } 336 = 2^4 \times 3 \times 7$$

$\therefore$ HCF of 96, 240 and 336 is $2^4 \times 3 = 48$

So, there must be 48 books in each stack.

$$\therefore \quad \text{Number of stacks of English books} = \frac{96}{48} = 2$$

$$\text{Number of stacks of Hindi books} = \frac{240}{48} = 5$$

$$\text{Number of stacks of Mathematics books} = \frac{336}{48} = 7$$

EXERCISE 1.4

LEVEL-1

1. Find the LCM and HCF of the following pairs of integers and verify that LCM × HCF = Product of the integers:

   (i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54 (iv) 404 and 96 [CBSE 2018]

2. Find the LCM and HCF of the following integers by applying the prime factorisation method:

   (i) 12, 15 and 21 [NCERT] (ii) 17, 23 and 29 [NCERT] (iii) 8, 9 and 25 [NCERT]

   (iv) 40, 36 and 126 (v) 84, 90 and 120 (vi) 24, 15 and 36

3. Given that HCF (306, 657) = 9, find LCM (306, 657). [NCERT]

4. Can two numbers have 16 as their HCF and 380 as their LCM? Give reason.
5. The HCF of two numbers is 145 and their LCM is 2175. If one number is 725, find the other.
6. The HCF of two numbers is 16 and their product is 3072. Find their LCM.
7. The LCM and HCF of two numbers are 180 and 6 respectively. If one of the numbers is 30, find the other number.

LEVEL-2

8. Find the smallest number which when increased by 17 is exactly divisible by both 520 and 468.
9. Find the smallest number which leaves remainders 8 and 12 when divided by 28 and 32 respectively.
10. What is the smallest number that, when divided by 35, 56 and 91 leaves remainders of 7 in each case?
11. A rectangular courtyard is 18 m 72 cm long and 13 m 20 cm broad. It is to be paved with square tiles of the same size. Find the least possible number of such tiles.
12. Find the greatest number of 6 digits exactly divisible by 24, 15 and 36.
13. Determine the number nearest to 110000 but greater than 100000 which is exactly divisible by each of 8, 15 and 21.
14. Find the least number that is divisible by all the numbers between 1 and 10 (both inclusive).
15. A circular field has a circumference of 360 km. Three cyclists start together and can cycle 48, 60 and 72 km a day, round the field. When will they meet again?
16. In a morning walk three persons step off together, their steps measure 80 cm, 85 cm and 90 cm respectively. What is the minimum distance each should walk so that he can cover the distance in complete steps?

ANSWERS

1. (i) LCM = 182, HCF = 13, (ii) LCM = 23 460, HCF = 2, (iii) LCM = 3024, HCF = 6

(iv) LCM = 9696, HCF = 4

2.	LCM	HCF
(i)	420	3
(ii)	1139	1
(iii)	1800	1
(iv)	2520	2
(v)	2520	6
(vi)	360	3

3. 22338
4. No
5. 435
6. 192
7. 36
8. 4663
9. 204
10. 3647
11. 4290
12. 999720
13. 109200
14. 2520
15. 30 days
16. 122 m 40 cm

HINT TO SELECTED PROBLEM

12. Greatest number of 6 digits is 999999. Required number must be divisible by the LCM of 24, 15 and 36 i.e., by 360.

Hence, required number = 999999 – Remainder when 999999 is divided by 360

1.6.2 PROVING IRRATIONALITY OF NUMBERS

In class IX, we have learnt about irrational numbers and their properties. We have also learnt about the existence of irrational numbers and their representation on the number line. Recall that a number is an irrational number if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\sqrt{2}, \sqrt{3}, \sqrt{7}, \frac{\sqrt{2}}{\sqrt{5}}, \pi$ etc. are irrational numbers. In this section, we will prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. are irrational numbers by using the Fundamental Theorem of Arithmetic. In fact, for any prime number p, $\sqrt{p}$ is an irrational number. In proving the irrationality of these numbers, we will use the result that if a prime p divides a^2, then it divides a also (see Theorem 2 on page 1.30).We will prove the irrationality of numbers by using the method of contradiction. In class IX, we have also learnt that the sum or difference of a rational and an irrational number is an irrational number. Also, the product and quotient of a non-zero rational number and an irrational number is an irrational number. We will also prove these results in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that $\sqrt{2}$ is an irrational number.* [NCERT, CBSE 2010]

SOLUTION Let us assume on the contrary that $\sqrt{2}$ is a rational number. Then, there exist positive integers a and b such that

$\sqrt{2} = \frac{a}{b}$ where, a and b, are co-prime i.e. their HCF is 1

$\Rightarrow \quad (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$

$\Rightarrow \quad 2 = \frac{a^2}{b^2}$

$\Rightarrow \quad 2b^2 = a^2$

$\Rightarrow \quad 2 | a^2$ $\quad [\because 2 | 2b^2 \text{ and } 2b^2 = a^2]$

$\Rightarrow \quad 2 | a$ $\quad$ [Using Theorem 2 on page 1.29] ...(i)

$\Rightarrow \quad a = 2c$ for some integer c

$\Rightarrow \quad a^2 = 4c^2$

$\Rightarrow \quad 2b^2 = 4c^2$ $\quad [\because 2b^2 = a^2]$

$\Rightarrow \quad b^2 = 2c^2$

$\Rightarrow \quad 2 | b^2$ $\quad [\because 2 | 2c^2]$

$\Rightarrow \quad 2 | b$ ...(ii)

From (i) and (ii), we obtain that 2 is a common factor of a and b. But, this contradicts the fact that a and b have no common factor other than 1. This means that our supposition is wrong.

Hence, $\sqrt{2}$ is an irrational number.

EXAMPLE 2 *Prove that $\sqrt{3}$ is an irrational number.* [NCERT, CBSE 2009, 2010]

SOLUTION Let us assume on the contrary that $\sqrt{3}$ is a rational number. Then, there exist positive integers a and b such that

$$\sqrt{3} = \frac{a}{b}, \text{ where } a \text{ and } b \text{ are co-prime i.e. their HCF is 1.}$$

Now,

$$\sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \quad 3 = \frac{a^2}{b^2}$$

$$\Rightarrow \quad 3b^2 = a^2$$

$$\Rightarrow \quad 3 \mid a^2 \qquad [\because 3 \mid 3b^2]$$

$$\Rightarrow \quad 3 \mid a \qquad \text{[By Theorem 2 on page 1.29]} \quad \ldots \text{(i)}$$

$$\Rightarrow \quad a = 3c \text{ for some integer } c$$

$$\Rightarrow \quad a^2 = 9c^2$$

$$\Rightarrow \quad 3b^2 = 9c^2 \qquad [\because a^2 = 3b^2]$$

$$\Rightarrow \quad b^2 = 3c^2$$

$$\Rightarrow \quad 3 \mid b^2 \qquad [\because 3 \mid 3c^2]$$

$$\Rightarrow \quad 3 \mid b \qquad \text{[By Theorem 2 on page 1.29]} \quad \ldots \text{(ii)}$$

From (i) and (ii), we observe that a and b have at least 3 as a common factor. But this contradicts the fact that a and b are co-prime. This means that our assumption is not correct.

Hence, $\sqrt{3}$ is an irrational number.

EXAMPLE 3 *Prove that $3\sqrt{2}$ is irrational.* [NCERT]

SOLUTION Let us assume, to the contrary, that $3\sqrt{2}$ is rational. Then, there exist co-prime positive integers a and b such that

$$3\sqrt{2} = \frac{a}{b}$$

$$\Rightarrow \quad \sqrt{2} = \frac{a}{3b}$$

$\Rightarrow \quad \sqrt{2}$ is rational $\quad \left[\because 3, a \text{ and } b \text{ are integers} \therefore \dfrac{a}{3b} \text{ is a rational number}\right]$

This contradicts the fact that $\sqrt{2}$ is irrational. So, our assumption is not correct.

Hence, $3\sqrt{2}$ is an irrational number.

EXAMPLE 4 *Prove that $\sqrt{5}$ is an irrational number.* [NCERT, CBSE 2009, 2010]

SOLUTION Let us assume on the contrary that $\sqrt{5}$ is a rational number. Then, there exist co-prime positive integers a and b such that

$$\sqrt{5} = \frac{a}{b}$$

$$\Rightarrow \quad 5b^2 = a^2$$

$$\Rightarrow \quad 5 \mid a^2 \qquad [\because 5 \mid 5b^2]$$

$$\Rightarrow \quad 5 \mid a \qquad \text{[See Theorem 2 on page 1.29]} \qquad \ldots\text{(i)}$$

$$\Rightarrow \quad a = 5c \text{ for some positive integer } c$$

$$\Rightarrow \quad a^2 = 25c^2$$

$$\Rightarrow \quad 5b^2 = 25c^2 \qquad [\because a^2 = 5b^2]$$

$$\Rightarrow \quad b^2 = 5c^2$$

$$\Rightarrow \quad 5 \mid b^2 \qquad [\because 5 \mid 5c^2]$$

$$\Rightarrow \quad 5 \mid b \qquad \text{[See Theorem 2 on page 1.29]} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we find that a and b have at least 5 as a common factor. This contradicts the fact that a and b are co-prime.

Hence, $\sqrt{5}$ is an irrational number.

EXAMPLE 5 *Prove that $5 - \sqrt{3}$ is an irrational number.* [NCERT]

SOLUTION Let us assume on the contrary that $5 - \sqrt{3}$ is rational. Then, there exist co-prime positive integers a and b such that

$$5 - \sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \quad 5 - \frac{a}{b} = \sqrt{3}$$

$$\Rightarrow \quad \frac{5b - a}{b} = \sqrt{3}$$

$\Rightarrow \quad \sqrt{3}$ is rational $\quad \left[\because a, b \text{ are integers } \therefore \dfrac{5b - a}{b} \text{ is a rational number}\right]$

This contradicts the fact that $\sqrt{3}$ is irrational. So, our assumption is incorrect. Hence, $5 - \sqrt{3}$ is an irrational number.

EXAMPLE 6 *Prove that* $3+2\sqrt{5}$ *is irrational.* [NCERT]

SOLUTION Let us assume on the contrary that $3+2\sqrt{5}$ is rational. Then there exist co-prime positive integers a and b such that

$$3+2\sqrt{5}=\frac{a}{b}$$

$$\Rightarrow \quad 2\sqrt{5}=\frac{a}{b}-3$$

$$\Rightarrow \quad \sqrt{5}=\frac{a-3b}{2b}$$

$\Rightarrow$ $\sqrt{5}$ is rational $\left[\because a, b \text{ are integers } \therefore \frac{a-3b}{2b} \text{ is a rational}\right]$

This contradicts the fact that $\sqrt{5}$ is irrational. So, our supposition is incorrect. Hence, $3+2\sqrt{5}$ is an irrational number.

EXAMPLE 7 *Prove that* $\sqrt{2}+\sqrt{5}$ *is irrational.*

SOLUTION Let us assume on the contrary that $\sqrt{2}+\sqrt{5}$ is a rational number. Then, there exist co-prime positive integers a and b such that

$$\sqrt{2}+\sqrt{5}=\frac{a}{b}$$

$$\Rightarrow \quad \frac{a}{b}-\sqrt{2}=\sqrt{5}$$

$$\Rightarrow \quad \left(\frac{a}{b}-\sqrt{2}\right)^2=(\sqrt{5})^2 \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad \frac{a^2}{b^2}-\frac{2a}{b}\sqrt{2}+2=5$$

$$\Rightarrow \quad \frac{a^2}{b^2}-3=\frac{2a}{b}\sqrt{2}$$

$$\Rightarrow \quad \frac{a^2-3b^2}{2ab}=\sqrt{2}$$

$\Rightarrow$ $\sqrt{2}$ is a rational number $\left[\because a, b \text{ are integers } \because \frac{a^2-3b^2}{2ab} \text{ is rational}\right]$

This contradicts the fact that $\sqrt{2}$ is irrational. So, our assumption is wrong. Hence, $\sqrt{2}+\sqrt{5}$ is irrational.

LEVEL-2

EXAMPLE 8 *Show that there is no positive integer n for which* $\sqrt{n-1}+\sqrt{n+1}$ *is rational.*

SOLUTION If possible, let there be a positive integer n for which $\sqrt{n-1}+\sqrt{n+1}$ is rational equal to $\frac{a}{b}$ (say), where a, b are positive integers. Then,

$$\frac{a}{b} = \sqrt{n-1} + \sqrt{n+1} \qquad \ldots \text{(i)}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{1}{\sqrt{n-1} + \sqrt{n+1}}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{\sqrt{n+1} - \sqrt{n-1}}{\{\sqrt{n+1} + \sqrt{n-1}\}\{\sqrt{n+1} - \sqrt{n-1}\}} = \frac{\sqrt{n+1} - \sqrt{n-1}}{(n+1)-(n-1)} = \frac{\sqrt{n+1} - \sqrt{n-1}}{2}$$

$$\Rightarrow \quad \frac{2b}{a} = \sqrt{n+1} - \sqrt{n-1} \qquad \ldots \text{(ii)}$$

Adding (i) and (ii) and subtracting (ii) from (i), we get

$$2\sqrt{n+1} = \frac{a}{b} + \frac{2b}{a} \text{ and } 2\sqrt{n-1} = \frac{a}{b} - \frac{2b}{a}$$

$$\Rightarrow \quad \sqrt{n+1} = \frac{a^2 + 2b^2}{2ab} \text{ and } \sqrt{n-1} = \frac{a^2 - 2b^2}{2ab}$$

$$\Rightarrow \quad \sqrt{n+1} \text{ and } \sqrt{n-1} \text{ are rationals} \qquad \left[\begin{array}{l} \because a, b \text{ are integers } \therefore \dfrac{a^2 + 2b^2}{2ab} \text{ and } \dfrac{a^2 - 2b^2}{2ab} \\ \text{are rationals.} \end{array}\right]$$

$\Rightarrow$ $(n + 1)$ and $(n - 1)$ are perfect squares of positive integers.

This is not possible as any two perfect squares differ at least by 3.

Hence, there is no positive integer n for which $(\sqrt{n-1} + \sqrt{n+1})$ is rational.

EXAMPLE 9 *Let a and b be positive integers. Show that $\sqrt{2}$ always lies between $\frac{a}{b}$ and $\frac{a+2b}{a+b}$.*

SOLUTION We do not know whether $\frac{a}{b} < \frac{a+2b}{a+b}$ or, $\frac{a}{b} > \frac{a+2b}{a+b}$.

Therefore, to compare these two numbers, let us compute $\frac{a}{b} - \frac{a+2b}{a+b}$

We have,

$$\frac{a}{b} - \frac{a+2b}{a+b} = \frac{a(a+b) - b(a+2b)}{b(a+b)} = \frac{a^2 + ab - ab - 2b^2}{b(a+b)} = \frac{a^2 - 2b^2}{b(a+b)}$$

$$\therefore \quad \frac{a}{b} - \frac{a+2b}{a+b} > 0$$

$$\Rightarrow \quad \frac{a^2 - 2b^2}{b(a+b)} > 0$$

$$\Rightarrow \quad a^2 - 2b^2 > 0$$

$$\Rightarrow \quad a^2 > 2b^2$$

$$\Rightarrow \quad a > \sqrt{2}\, b$$

and, $\frac{a}{b} - \frac{a+2b}{a+b} < 0$

$\Rightarrow \quad \frac{a^2 - 2b^2}{b(a+b)} < 0$

$\Rightarrow \quad a^2 - 2b^2 < 0$

$\Rightarrow \quad a^2 < 2b^2$

$\Rightarrow \quad a < \sqrt{2}\, b$

Thus, $\frac{a}{b} > \frac{a+2b}{a+b}$, if $a > \sqrt{2}\, b$ and $\frac{a}{b} < \frac{a+2b}{a+b}$, if $a < \sqrt{2}\, b$.

So, we have the following cases:

CASE I *When* $a > \sqrt{2}\, b$

In this case, we have

$$\frac{a}{b} > \frac{a+2b}{a+b} \text{ i.e., } \frac{a+2b}{a+b} < \frac{a}{b}$$

We have to prove that

$$\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$$

We have,

$a > \sqrt{2}b$

$\Rightarrow \quad a^2 > 2b^2$

$\Rightarrow \quad a^2 + a^2 > a^2 + 2b^2$ [Adding a^2 on both sides]

$\Rightarrow \quad 2a^2 + 2b^2 > (a^2 + 2b^2) + 2b^2$ [Adding $2b^2$ on both sides]

$\Rightarrow \quad 2(a^2 + b^2) + 4ab > a^2 + 4b^2 + 4ab$ [Adding $4ab$ on both sides]

$\Rightarrow \quad 2(a^2 + 2ab + b^2) > a^2 + 4ab + 4b^2$

$\Rightarrow \quad 2(a+b)^2 > (a+2b)^2$

$\Rightarrow \quad \sqrt{2}(a+b) > a + 2b$

$\Rightarrow \quad \sqrt{2} > \frac{a+2b}{a+b}$ ... (i)

Again,

$a > \sqrt{2}\, b \Rightarrow \frac{a}{b} > \sqrt{2}$ ... (ii)

From (i) and (ii), we get

$$\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$$

CASE II *When* $a < \sqrt{2}\, b$

In this case, we have

$$\frac{a}{b} < \frac{a+2b}{a+b}$$

We have to show that $\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$

We have,

$$a < \sqrt{2}\, b$$

$$\Rightarrow \quad a^2 < 2b^2$$

$$\Rightarrow \quad a^2 + a^2 < a^2 + 2b^2 \qquad \text{[Adding } a^2 \text{ on both sides]}$$

$$\Rightarrow \quad 2a^2 + 2b^2 < a^2 + 4b^2 \qquad \text{[Adding } 2b^2 \text{ on both sides]}$$

$$\Rightarrow \quad 2a^2 + 4ab + 2b^2 < a^2 + 4ab + 4b^2$$

$$\Rightarrow \quad 2(a+b)^2 < (a+2b)^2$$

$$\Rightarrow \quad \sqrt{2}\,(a+b) < a+2b$$

$$\Rightarrow \quad \sqrt{2} < \frac{a+2b}{a+b} \qquad \text{... (iii)}$$

$$\Rightarrow \quad a < \sqrt{2}\, b \Rightarrow \frac{a}{b} < \sqrt{2} \qquad \text{... (iv)}$$

From (iii) and (iv), we get

$$\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$$

Hence, $\sqrt{2}$ lies between $\frac{a}{b}$ and $\frac{a+2b}{a+b}$.

EXAMPLE 10 *Let a, b, c, d be positive rationals such that $a + \sqrt{b} = c + \sqrt{d}$, then either $a = c$ and $b = d$ or b and d are squares of rationals.*

SOLUTION If $a = c$, then

$$a + \sqrt{b} = c + \sqrt{d} \Rightarrow \sqrt{b} = \sqrt{d} \Rightarrow b = d.$$

So, let $a \neq c$. Then, there exists a positive rational number x such that $a = c + x$.

Now,

$$a + \sqrt{b} = c + \sqrt{d}$$

$$\Rightarrow \quad c + x + \sqrt{b} = c + \sqrt{d} \qquad [\because a = c + x]$$

$$\Rightarrow \quad x + \sqrt{b} = \sqrt{d} \qquad \text{...(i)}$$

$$\Rightarrow \quad \left(x + \sqrt{b}\right)^2 = \left(\sqrt{d}\right)^2$$

$$\Rightarrow \quad x^2 + 2\sqrt{b}\, x + b = d$$

$$\Rightarrow \quad d - x^2 - b = 2x\sqrt{b}$$

$$\Rightarrow \quad \sqrt{b} = \frac{d - x^2 - b}{2x}$$

$\Rightarrow \quad \sqrt{b}$ is rational $\left[\because d, x, b \text{ are rationals } \therefore \dfrac{d - x^2 - b^2}{2x} \text{ is rational}\right]$

$\Rightarrow \quad b$ is the square of a rational number.

From (i), we have

$$\sqrt{d} = x + \sqrt{b}$$

$\Rightarrow \quad \sqrt{d}$ is rational $[\because \sqrt{b} \text{ is rational}]$

$\Rightarrow \quad d$ is the square of a rational number.

Hence, either $a = c$ and $b = d$ or b and d are the squares of rationals.

EXAMPLE 11 *Let a, b, c, p be rational numbers such that p is not a perfect cube. If $a + b\,p^{\frac{1}{3}} + c\,p^{\frac{2}{3}} = 0$, then prove that $a = b = c = 0$.*

SOLUTION We have,

$$a + b\,p^{\frac{1}{3}} + c\,p^{\frac{2}{3}} = 0 \quad \text{... (i)}$$

Multiplying both sides by $p^{\frac{1}{3}}$, we get

$$a\,p^{\frac{1}{3}} + b\,p^{\frac{2}{3}} + cp = 0 \quad \text{... (ii)}$$

Multiplying (i) by b and (ii) by c and subtracting, we get

$$(ab + b^2 p^{1/3} + bcp^{2/3}) - (acp^{1/3} + bcp^{2/3} + c^2 p) = 0$$

$$\Rightarrow \quad (b^2 - ac)\,p^{1/3} + ab - c^2 p = 0$$

$$\Rightarrow \quad b^2 - ac = 0 \text{ and } ab - c^2 p = 0 \quad [\because p^{1/3} \text{ is irrational}]$$

$$\Rightarrow \quad b^2 = ac \text{ and } ab = c^2 p$$

$$\Rightarrow \quad b^2 = ac \text{ and } a^2 b^2 = c^4 p^2$$

$$\Rightarrow \quad a^2(ac) = c^4 p^2 \quad [\text{Putting } b^2 = ac \text{ in } a^2 b^2 = c^4 p^2]$$

$$\Rightarrow \quad a^3 c - p^2 c^4 = 0$$

$$\Rightarrow \quad (a^3 - p^2 c^3)\,c = 0$$

$$\Rightarrow \quad a^3 - p^2 c^3 = 0, \text{ or } c = 0$$

Now, $\quad a^3 - p^2 c^3 = 0$

$$\Rightarrow \quad p^2 = \frac{a^3}{c^3} \Rightarrow (p^2)^{1/3} = \left(\frac{a^3}{c^3}\right)^{1/3} \Rightarrow (p^{1/3})^2 = \left\{\left(\frac{a}{c}\right)^3\right\}^{1/3} \Rightarrow (p^{1/3})^2 = \frac{a}{c}$$

This is not possible as $p^{1/3}$ is irrational and $\dfrac{a}{c}$ is rational.

$\therefore \qquad a^3 - p^2c^3 \neq 0$

Hence, $\quad c = 0$

Putting $c = 0$ in $b^2 - ac = 0$, we get $b = 0$

Putting $b = 0$ and $c = 0$ in $a + b\,p^{1|3} + cp^{2|3} = 0$, we get $a = 0$

Hence, $\quad a = b = c = 0.$

EXAMPLE 12 *For any positive real number x, prove that there exists an irrational number y such that* $0 < y < x$.

SOLUTION If x is irrational, then $y = \frac{x}{2}$ is also an irrational number such that $0 < y < x$.

If x is rational, then $\frac{x}{\sqrt{2}}$ is an irrational number such that $\frac{x}{\sqrt{2}} < x$ as $\sqrt{2} > 1$.

$\therefore \qquad y = \frac{x}{\sqrt{2}}$ is an irrational number such that $0 < y < x$.

EXERCISE 1.5

LEVEL-1

1. Show that the following numbers are irrational.

   (i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$ (iv) $3 - \sqrt{5}$

2. Prove that following numbers are irrationals:

   (i) $\frac{2}{\sqrt{7}}$ (ii) $\frac{3}{2\sqrt{5}}$ (iii) $4 + \sqrt{2}$ (iv) $5\sqrt{2}$

3. Show that $2 - \sqrt{3}$ is an irrational number. [CBSE 2008]
4. Show that $3 + \sqrt{2}$ is an irrational number. [CBSE 2009]
5. Prove that $4 - 5\sqrt{2}$ is an irrational number. [CBSE 2010]
6. Show that $5 - 2\sqrt{3}$ is an irrational number. [CBSE 2009]
7. Prove that $2\sqrt{3} - 1$ is an irrational number. [CBSE 2010]
8. Prove that $2 - 3\sqrt{5}$ is an irrational number. [CBSE 2010]
9. Prove that $\sqrt{5} + \sqrt{3}$ is irrational. [NCERT EXEMPLAR]
10. Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number. [NCERT EXEMPLAR]
11. Given that $\sqrt{2}$ is irrational, prove that $(5 + 3\sqrt{2})$ is an irrational number. [CBSE 2018]

LEVEL-2

12. Prove that for any prime positive integer p, $\sqrt{p}$ is an irrational number.
13. If p, q are prime positive integers, prove that $\sqrt{p} + \sqrt{q}$ is an irrational number. [NCERT EXEMPLAR]

HINTS TO SELECTED PROBLEMS

1. (i) If possible, let $\frac{1}{\sqrt{2}}$ be rational. Then, there exist positive co-primes a and b such that

$$\frac{1}{\sqrt{2}} = \frac{a}{b}$$

$\Rightarrow \quad 2a^2 = b^2$

$\Rightarrow \quad 2 \mid b^2 \qquad [\because 2 \mid 2a^2]$

$\Rightarrow \quad 2 \mid b$

$\Rightarrow \quad b = 2c$ for some positive integer c

$\therefore \quad 2a^2 = b^2 \Rightarrow 2a^2 = 4c^2 \Rightarrow a^2 = 2c^2 \Rightarrow 2 \mid a^2 \qquad [\because 2 \mid 2c^2]$

$\Rightarrow \quad 2 \mid a$

This is a contradiction to the fact that a, b are co-primes.

Hence, $\frac{1}{\sqrt{2}}$ is irrational.

(ii) Let $7\sqrt{5}$ be rational. Then,

$$7\sqrt{5} = \frac{a}{b} \Rightarrow \sqrt{5} = \frac{a}{7b} \Rightarrow \sqrt{5} \text{ is rational, a contradiction.}$$

$\therefore \quad 7\sqrt{5}$ is irrational.

(iii) Let $6 + \sqrt{2}$ be a rational number equal to $\frac{a}{b}$, where a, b are positive co-primes. Then,

$$6 + \sqrt{2} = \frac{a}{b}$$

$\Rightarrow \quad \sqrt{2} = \frac{a}{b} - 6$

$\Rightarrow \quad \sqrt{2} = \frac{a - 6b}{b}$

$\Rightarrow \quad \sqrt{2}$ is rational.

This is a contradiction.

Hence, $6 + \sqrt{2}$ is irrational

(iv) Let $3 - \sqrt{5}$ be a rational equal to $\frac{a}{b}$ Then,

$$3 - \sqrt{5} = \frac{a}{b}$$

$\Rightarrow \quad \sqrt{5} = 3 - \frac{a}{b}$

$$\Rightarrow \quad \sqrt{5} = \frac{3b - a}{b}$$

$\Rightarrow \quad \sqrt{5}$ is rational.

This is a contradiction.

Hence, $3 - \sqrt{5}$ is irrational.

9. Let $\sqrt{5} + \sqrt{3}$ be rational equal to $\frac{a}{b}$. Then,

$$\sqrt{5} + \sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \quad \sqrt{5} = \frac{a}{b} - \sqrt{3}$$

$$\Rightarrow \quad (\sqrt{5})^2 = \left(\frac{a}{b} - \sqrt{3}\right)^2 \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad 5 = \frac{a^2}{b^2} - \frac{2a\sqrt{3}}{b} + 3$$

$$\Rightarrow \quad 2 = \frac{a^2}{b^2} - 2\sqrt{3}\,\frac{a}{b}$$

$$\Rightarrow \quad 2\sqrt{3}\,\frac{a}{b} = \frac{a^2 - 2b^2}{b^2}$$

$\Rightarrow \quad \sqrt{3} = \dfrac{a^2 - 2b^2}{2ab} \Rightarrow \sqrt{3}$ is rational, a contradiction.

Hence, $\sqrt{5} + \sqrt{3}$ is irrational.

12. Let us assume on the contrary that $\sqrt{p}$ is rational. Then, there exist positive co-primes a and b such that

$$\sqrt{p} = \frac{a}{b}$$

$$\Rightarrow \quad p = \frac{a^2}{b^2}$$

$$\Rightarrow \quad b^2 p = a^2$$

$$\Rightarrow \quad p \mid a^2 \qquad [\because\ p \mid b^2 p]$$

$$\Rightarrow \quad p \mid a$$

$\Rightarrow \quad a = pc$ for some positive integer c.

Now, $b^2 p = a^2$

$$\Rightarrow \quad b^2 p = p^2 c^2 \qquad [\because\ a = pc]$$

$$\Rightarrow \quad b^2 = pc^2$$

$\Rightarrow \quad p \mid b^2$

$\Rightarrow \quad p \mid b$

$\therefore \quad p \mid a$ and $p \mid b$

This contradicts that a and b are co-primes.

Hence, $\sqrt{p}$ is irrational.

13. Let us assume that $\sqrt{p} + \sqrt{q}$ is a rational number equal to $\frac{a}{b}$, where a and b are integers having no common factor.

Now, $\sqrt{p} + \sqrt{q} = \frac{a}{b}$

$$\Rightarrow \quad \sqrt{p} = \frac{a}{b} - \sqrt{q}$$

$$\Rightarrow \quad \left(\sqrt{p}\right)^2 = \left(\frac{a}{b} - \sqrt{q}\right)^2$$

$$\Rightarrow \quad p = \frac{a^2}{b^2} - 2\left(\frac{a}{b}\right)\sqrt{q} + q$$

$$\Rightarrow \quad 2\left(\frac{a}{b}\right)\sqrt{q} = \frac{a^2}{b^2} + q - p$$

$$\Rightarrow \quad 2\frac{a}{b}\sqrt{q} = \frac{a^2 + b^2(q-p)}{b^2}$$

$$\Rightarrow \quad \sqrt{q} = \frac{a^2 + b^2(q-p)}{2ab}$$

$\Rightarrow \quad \sqrt{q}$ is a rational number.

This is a contradiction as $\sqrt{q}$ is an irrational number.

Hence, $\sqrt{p} + \sqrt{q}$ is an irrational number.

1.6.3 DETERMINING THE NATURE OF THE DECIMAL EXPANSIONS OF RATIONAL NUMBERS

In class IX, we have studied that the decimal expansion of a rational number is either terminating or non-terminating repeating (or recurring) without knowing when it is terminating and when it is non-terminating repeating. In this section, we will explore exactly when the decimal expansion of a rational number is terminating and when it is non-terminating repeating. In earlier classes, we have also learnt that any rational number having terminating decimal expansion can be written as a rational number whose denominator is some power of 10. For example,

(i) $0.875 = \frac{875}{1000} = \frac{875}{10^3}$

(ii) $1.512 = \frac{1512}{1000} = \frac{1512}{10^3}$

(iii) $0.01764 = \frac{1764}{100000} = \frac{1764}{10^5}$

(iv) $26.7624 = \frac{267624}{10000} = \frac{267624}{10^4}$ etc.

As we know that 2 and 5 are the only prime factors of 10. Therefore, any positive integral power of 10, say 10^n, is expressible in the form $(2 \times 5)^n = 2^n \times 5^n$. For example,

$$10 = 2 \times 5, \quad 100 = 10^2 = 2^2 \times 5^2, \quad 1000 = 10^3 = 2^3 \times 5^3, \quad 10000 = 10^4 = 2^4 \times 5^4 \text{ etc.}$$

Therefore, denominators of rational numbers having terminating decimal expansions are of the form $2^n \times 5^n$. If we express the numerators of such rational numbers as products of primes and cancel out the common factors between the numerators and the corresponding denominators, we find that the prime factorisations of their denominators are of the form $2^m \times 5^n$, where m and n are non-negative integers. For example,

(i) $0.875 = \dfrac{875}{10^3} = \dfrac{5^3 \times 7}{2^3 \times 5^3} = \dfrac{7}{2^3} = \dfrac{7}{2^3 \times 5^0}$

(ii) $1.512 = \dfrac{1512}{10^3} = \dfrac{2^3 \times 3^3 \times 7}{2^3 \times 5^3} = \dfrac{3^3 \times 7}{5^3} = \dfrac{189}{2^0 \times 5^3}$

(iii) $0.01764 = \dfrac{1764}{10^5} = \dfrac{2^2 \times 3^2 \times 7^2}{2^5 \times 5^5} = \dfrac{7^2}{2^3 \times 5^3} = \dfrac{49}{2^3 \times 5^3}$

(iv) $27.7624 = \dfrac{277624}{10^4} = \dfrac{2^3 \times 3^2 \times 7 \times 531}{2^4 \times 5^4} = \dfrac{3^2 \times 7 \times 531}{2 \times 5^4} = \dfrac{33453}{2^1 \times 5^4}$

It follows from the above discussion that the denominators of the rational numbers having terminating decimal expansions are expressible in the form $2^m \times 5^n$, where m, n are non-negative integers.

This result can be stated formally as a theorem as follows:

THEOREM 1 *Let x be a rational number whose decimal expansion terminates. Then, x can expressed in the form $\frac{p}{q}$, where p and q are co-primes, and the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative integers.*

Let us now see whether the converse of this theorem is also true or not. That is, if we have a rational number of the form $\frac{p}{q}$, and the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative integers, then does $\frac{p}{q}$ have a terminating decimal?

Let $\frac{a}{b}$ be a rational number in the lowest form such that the prime factorisation of b is of the form $2^m \times 5^n$, where m, n are non-negative integers.

We have the following cases:

CASE I *When* $m = n$:

In this case, we have

$$\frac{a}{b} = \frac{a}{2^m \times 5^n} = \frac{a}{2^m \times 5^m} = \frac{a}{(10)^m}$$

CASE II When $m > n$:

In this case, we have

$$m = n + p, \text{ where } p \text{ is a positive integer.}$$

$$\therefore \quad \frac{a}{b} = \frac{a}{2^m \times 5^n} = \frac{a \times 5^p}{2^m \times 5^{n+p}} = \frac{a \times 5^p}{2^m \times 5^m} = \frac{a \times 5^p}{(2 \times 5)^m} = \frac{c}{10^m}, \text{ where } c = a \times 5^p$$

CASE III When $m < n$:

In this case, we have

$$n = m + p, \text{ where } p \text{ is a positive integer.}$$

$$\therefore \quad \frac{a}{b} = \frac{a}{2^m \times 5^n} = \frac{a \times 2^p}{2^{m+p} \times 5^n} = \frac{a \times 2^p}{2^n \times 5^n} = \frac{a \times 2^p}{(2 \times 5)^n} = \frac{c}{10^n}, \text{ where } c = a \times 2^p$$

Thus, a rational number whose denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers, can be converted to an equivalent rational number of the form $\frac{c}{d}$, where d is a power of 10.

For example,

(i) $\frac{7}{8} = \frac{7}{2^3} = \frac{7 \times 5^3}{2^3 \times 5^3} = \frac{7 \times 125}{(2 \times 5)^3} = \frac{875}{10^3}$

(ii) $\frac{189}{125} = \frac{189}{5^3} = \frac{2^3 \times 189}{2^3 \times 5^3} = \frac{8 \times 189}{(2 \times 5)^3} = \frac{1512}{10^3}$

(iii) $\frac{49}{500} = \frac{49}{2^2 \times 5^3} = \frac{49 \times 2}{2^3 \times 5^3} = \frac{98}{(2 \times 5)^3} = \frac{98}{10^3}$

(iv) $\frac{2139}{1250} = \frac{2139}{2^1 \times 5^4} = \frac{2139 \times 2^3}{2^4 \times 5^4} = \frac{2139 \times 8}{(2 \times 5)^4} = \frac{17112}{10^4}$

$$\therefore \quad \frac{7}{8} = \frac{875}{10^3} = 0.875$$

$$\frac{189}{125} = \frac{1512}{10^3} = 1.512$$

$$\frac{49}{500} = \frac{98}{10^3} = 0.098$$

and, $$\frac{2139}{1250} = \frac{17112}{10^4} = 1.7112$$

This shows that the decimal expansion of a rational number whose denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers, is terminating. Also, it terminates after k places of decimals, where k is the larger of m and n.

This result can be stated formally as a theorem as follows:

THEOREM 2 *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has a decimal expansion which terminates after k places of decimals, where k is the larger of m and n.*

Let us now consider rational numbers whose decimal expansions are non-terminating and repeating. For example,

(i) $\frac{5}{3} = 1.66666\cdots\cdots$ (ii) $\frac{17}{6} = 2.83333\cdots\cdots$ (iii) $\frac{1}{7} = 0.142857142857\cdots\cdots$

We observe that the prime factorisation of the denominators of these rational numbers are not of the form $2^m \times 5^n$, where m, n are non-negative integers.

So, we arrive at the following conclusion.

THEOREM 3 *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating.*

Let us now discuss some examples to determine the nature of the decimal expansions of rational numbers by using the above theorems.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Without actually performing the long division, state whether the following rational numbers will have terminating decimal expansion or a non-terminating repeating decimal expansion. Also, find the number of places of decimals after which the decimal expansion terminates.*

(i) $\frac{17}{8}$ [NCERT] (ii) $\frac{64}{455}$ [NCERT] (iii) $\frac{29}{343}$ [NCERT]

(iv) $\frac{15}{1600}$ [NCERT] (v) $\frac{13}{3125}$ [NCERT] (vi) $\frac{23}{2^3 5^2}$ [NCERT]

SOLUTION (i) We have, $\frac{17}{8} = \frac{17}{2^3 \times 5^0}$

So, the denominator 8 of $\frac{17}{8}$ is of the form $2^m \times 5^n$, where m, n are non-negative integers.

Hence, $\frac{17}{8}$ has terminating decimal expansion. The decimal expansion of $\frac{17}{8}$ terminates after three places of decimals.

(ii) We have,

$$\frac{64}{455} = \frac{64}{5 \times 7 \times 13}$$

Clearly, 455 is not of the form $2^m \times 5^n$. So, the decimal expansion of $\frac{64}{455}$ is non-terminating repeating.

(iii) We have,

$$\frac{29}{343} = \frac{29}{3^5}$$

Clearly, 343 is not of the form $2^m \times 5^n$.

Hence, the decimal expansion of $\frac{29}{343}$ is non-terminating repeating.

(iv) We have,

$$\frac{15}{1600}=\frac{3}{320}=\frac{3}{2^6\times 5}$$

This means that the prime factorisation of the denominator of $\frac{15}{1600}$ is of the form $2^m \times 5^n$. Hence, it has terminating decimal expansion which terminates after 6 places of decimals.

(v) We have,

$$\frac{13}{3125}=\frac{13}{2^0\times 5^5}$$

This shows that the prime factorisation of the denominator of $\frac{13}{3125}$ is of the form $2^m \times 5^n$. Hence, it has terminating decimal expansion which terminates after 5 places of decimals.

(vi) Clearly, the prime factorisation of the denominator of $\frac{23}{2^3\times 5^2}$ is of the form $2^m \times 5^n$. So, it has terminating decimal expansion which terminates after 3 places of decimals.

LEVEL-2

EXAMPLE 2 *What can you say about the prime factorisations of the denominators of the following rationals:*

(i) 34.12345 (ii) $34.\overline{5678}$

SOLUTION (i) Since 34.12345 has terminating decimal expansion. So, its denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers.

(ii) Since $34.\overline{5678}$ has non-terminating repeating decimal expansion. So, its denominator has factors other than 2 or 5.

EXERCISE 1.6

LEVEL-1

1. Without actually performing the long division, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating repeating decimal expansion.

(i) $\frac{23}{8}$ (ii) $\frac{125}{441}$ (iii) $\frac{35}{50}$ [NCERT]

(iv) $\frac{77}{210}$ [NCERT] (v) $\frac{129}{2^2\times 5^7\times 7^{17}}$ (vi) $\frac{987}{10500}$ [NCERT EXEMPLAR]

2. Write down the decimal expansions of the following rational numbers by writing their denominators in the form $2^m \times 5^n$, where m, n are non-negative integers.

(i) $\frac{3}{8}$ (ii) $\frac{13}{125}$ (iii) $\frac{7}{80}$

(iv) $\frac{14588}{625}$ (v) $\frac{129}{2^2 \times 5^7}$ [NCERT]

3. Write the denominator of the rational number $\frac{257}{5000}$ in the form $2^m \times 5^n$, where m, n are non-negative integers. Hence, write the decimal expansion, without actual division.

LEVEL-2

4. What can you say about the prime factorisations of the denominators of the following rationals:

(i) 43.123456789 (ii) $43.\overline{123456789}$

(iii) $27.\overline{142857}$ [CBSE 2010] (iv) 0.120120012000120000.... [NCERT]

5. A rational number in its decimal expansion is 327.7081. What can you say about the prime factors of q, when this number is expressed in the form $\frac{p}{q}$? Give reasons. [NCERT EXEMPLAR]

ANSWERS

1. (i) Terminating (ii) Non-terminating repeating (iii) Terminating (iv) Non-terminating repeating (v) Non-terminating repeating. (vi) Terminating
2. (i) 0.375 (ii) 0.104 (iii) 0.0875 (iv) 23.3408 (v) 0.0004128
3. $2^3 \times 5^4$; 0.0514
4. (i) Prime factorisation of the denominator is of the form $2^m \times 5^n$, where m, n are non-negative integers.
   (ii) Prime factorisation of the denominator contains factors other than 2 or 5.
   (iii) Prime factorisation of the denominator contains factors other than 2 or 5.
   (iv) Prime factorisation of the denominator contains factors other than 2 or 5.
5. Since 327.7081 is a terminating decimal number, so, q must be of the form $2^m \times 5^n$; m, n are natural numbers.

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

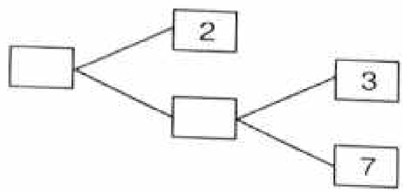
Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. State Euclid's division lemma.
2. State Fundamental Theorem of Arithmetic.
3. Write 98 as product of its prime factors.
4. Write the exponent of 2 in the prime factorization of 144.
5. Write the sum of the exponents of prime factors in the prime factorization of 98.
6. If the prime factorization of a natural number n is $2^3 \times 3^2 \times 5^2 \times 7$, write the number of consecutive zeros in n.
7. If the product of two numbers is 1080 and their HCF is 30, find their LCM.

8. Write the condition to be satisfied by q so that a rational number $\frac{p}{q}$ has a terminating decimal expansion. [CBSE 2008]

9. Write the condition to be satisfied by q so that a rational number $\frac{p}{q}$ has a non-terminating decimal expansion.

10. Complete the missing entries in the following factor tree.



[CBSE 2008]

11. The decimal expansion of the rational number $\frac{43}{2^4 \times 5^3}$ will terminate after how many places of decimals? [CBSE 2009]

12. Has the rational number $\frac{441}{2^2 \times 5^7 \times 7^2}$ a terminating or a nonterminating decimal representation? [CBSE 2010]

13. Write whether $\frac{2\sqrt{45} + 3\sqrt{20}}{2\sqrt{5}}$ on simplification gives a rational or an irrational number. [CBSE 2010]

14. What is an algorithm?

15. What is a lemma?

16. If p and q are two prime numbers, then what is their HCF?

17. If p and q are two prime numbers, then what is their LCM?

18. What is the total number of factors of a prime number?

19. What is a composite number?

20. What is the HCF of the smallest composite number and the smallest prime number? [CBSE 2018]

21. HCF of two numbers is always a factor of their LCM (True/False).

22. π is an irrational number (True/False).

23. The sum of two prime numbers is always a prime number (True/False).

24. The product of any three consecutive natural numbers is divisible by 6 (True/False).

25. Every even integer is of the form $2m$, where m is an integer (True/False).

26. Every odd integer is of the form $2m - 1$, where m is an integer (True/False).

27. The product of two irrational numbers is an irrational number (True/False).

28. The sum of two irrational numbers is an irrational number (True/False).

29. For what value of n, $2^n \times 5^n$ ends in 5.

30. If a and b are relatively prime numbers, then what is their HCF?

31. If a and b are relatively prime numbers, then what is their LCM?

32. Two numbers have 12 as their HCF and 350 as their LCM (True/False).

ANSWERS

1. See text 2. See text 3. 2×7^2 4. 4

5. 3 6. 2 7. 36

8. The prime factorization of q must be of the form $2^m \times 5^n$, where m, n are non-negative integers.

9. The prime factorization of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. 10. 42, 21 11. 4

12. Non-terminating 13. Rational Number 16. 1

17. $p \times q$ 18. 2 20. 2 21. True

22. True 23. False 24. True 25. True

26. True 27. False 28. False 29. No value of n

30. 1 31. ab 32. False

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following questions.

1. The exponent of 2 in the prime factorisation of 144, is
   (a) 4 (b) 5 (c) 6 (d) 3

2. The LCM of two numbers is 1200. Which of the following cannot be their HCF?
   (a) 600 (b) 500 (c) 400 (d) 200

3. If $n = 2^3 \times 3^4 \times 5^4 \times 7$, then the number of consecutive zeros in n, where n is a natural number, is
   (a) 2 (b) 3 (c) 4 (d) 7

4. The sum of the exponents of the prime factors in the prime factorisation of 196, is
   (a) 1 (b) 2 (c) 4 (d) 6

5. The number of decimal places after which the decimal expansion of the rational number $\frac{23}{2^2 \times 5}$ will terminate, is
   (a) 1 (b) 2 (c) 3 (d) 4

6. If p_1 and p_2 are two odd prime numbers such that $p_1 > p_2$, then $p_1^2 - p_2^2$ is
   (a) an even number (b) an odd number
   (c) an odd prime number (d) a prime number

7. If two positive ingeters a and b are expressible in the form $a = pq^2$ and $b = p^3q$; p, q being prime numbers, then LCM (a, b) is
   (a) pq (b) p^3q^3 (c) p^3q^2 (d) p^2q^2

8. In Q. No. 7, HCF (a, b) is

(a) pq (b) p^3q^3 (c) p^3q^2 (d) p^2q^2

9. If two positive integers m and n are expressible in the form $m = pq^3$ and $n = p^3q^2$, where p, q are prime numbers, then HCF $(m, n) =$

(a) pq (b) pq^2 (c) p^3q^3 (d) p^2q^3

10. If the LCM of a and 18 is 36 and the HCF of a and 18 is 2, then $a =$

(a) 2 (b) 3 (c) 4 (d) 1

11. The HCF of 95 and 152, is

(a) 57 (b) 1 (c) 19 (d) 38

12. If HCF (26, 169) = 13, then LCM (26, 169) =

(a) 26 (b) 52 (c) 338 (d) 13

13. If $a = 2^3 \times 3$, $b = 2 \times 3 \times 5$, $c = 3^n \times 5$ and LCM $(a, b, c) = 2^3 \times 3^2 \times 5$, then $n =$

(a) 1 (b) 2 (c) 3 (d) 4

14. The decimal expansion of the rational number $\frac{14587}{1250}$ will terminate after

(a) one decimal place (b) two decimal place
(c) three decimal place (d) four decimal place

15. If p and q are co-prime numbers, then p^2 and q^2 are

(a) coprime (b) not coprime (c) even (d) odd

16. Which of the following rational numbers have terminating decimal?

(i) $\frac{16}{225}$ (ii) $\frac{5}{18}$ (iii) $\frac{2}{21}$ (iv) $\frac{7}{250}$

(a) (i) and (ii) (b) (ii) and (iii) (c) (i) and (iii) (d) (i) and (iv)

17. If 3 is the least prime factor of number a and 7 is the least prime factor of number b, then the least prime factor of $a + b$, is

(a) 2 (b) 3 (c) 5 (d) 10

18. $3.\overline{27}$ is

(a) an integer (b) a rational number
(c) a natural number (d) an irrational number

19. The smallest number by which $\sqrt{27}$ should be multiplied so as to get a rational number is

(a) $\sqrt{27}$ (b) $3\sqrt{3}$ (c) $\sqrt{3}$ (d) 3

20. The smallest rational number by which $\frac{1}{3}$ should be multiplied so that its decimal expansion terminates after one place of decimal, is

(a) $\frac{3}{10}$ (b) $\frac{1}{10}$ (c) 3 (d) $\frac{3}{100}$

21. If n is a natural number, then $9^{2n} - 4^{2n}$ is always divisible by

(a) 5 (b) 13 (c) both 5 and 13 (d) None of these

[**Hint:** $9^{2n} - 4^{2n}$ is of the form $a^{2n} - b^{2n}$ which is divisible by both $a - b$ and $a + b$. So, $9^{2n} - 4^{2n}$ is divisible by both $9 - 4 = 5$ and $9 + 4 = 13$.]

22. If n is any natural number, then $6^n - 5^n$ always ends with

(a) 1 (b) 3 (c) 5 (d) 7

[**Hint:** For any $n \in N$, 6^n and 5^n end with 6 and 5 respectively. Therefore, $6^n - 5^n$ always ends with $6 - 5 = 1$.]

23. The LCM and HCF of two rational numbers are equal, then the numbers must be

(a) prime (b) co-prime (c) composite (d) equal

24. If the sum of LCM and HCF of two numbers is 1260 and their LCM is 900 more than their HCF, then the product of two numbers is

(a) 203400 (b) 194400 (c) 198400 (d) 205400

25. The remainder when the square of any prime number greater than 3 is divided by 6, is

(a) 1 (b) 3 (c) 2 (d) 4

[**Hint:** Any prime number greater than 3 is of the form $6k \pm 1$, where k is a natural number and $(6k \pm 1)^2 = 36k^2 \pm 12k + 1 = 6k(6k \pm 2) + 1$]

26. For some integer m, every even integer is of the form

(a) m (b) $m + 1$ (c) $2m$ (d) $2m + 1$

27. For some integer q, every odd integer is of the form

(a) q (b) $q + 1$ (c) $2q$ (d) $2q + 1$

28. $n^2 - 1$ is divisible by 8, if n is

(a) an integer (b) a natural number

(c) an odd integer (d) an even integer

29. The decimal expansion of the rational number $\dfrac{33}{2^2 \times 5}$ will terminate after

(a) one decimal place (b) two decimal places

(c) three decimal places(d)more than 3 decimal places

30. If two positive integers a and b are written as $a = x^3y^2$ and $b = xy^3$; x, y are prime numbers, then HCF (a, b) is

(a) xy (b) xy^2 (c) x^3y^3 (d) x^2y^2

31. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(a) 10 (b) 100 (c) 504 (d) 2520

32. The largest number which divides 70 and 125, leaving remainders 5 and 8, respectively, is

(a) 13 (b) 65 (c) 875 (d) 1750

33. If the HCF of 65 and 117 is expressible in the form $65m - 117$, then the value of m is

(a) 4 (b) 2 (c) 1 (d) 3

34. The decimal expansion of the rational number $\frac{14587}{1250}$ will terminate after:

(a) one decimal place (b) two decimal places

(c) three decimal places (d) four decimal places

35. Euclid's division lemma states that for two positive integers a and b, there exist unique integers q and r such that $a = bq + r$, where r must satisfy

(a) $1 < r < b$ (b) $0 < r \le b$ (c) $0 \le r < b$ (d) $0 < r < b$

ANSWERS

1. (a)	2. (b)	3. (b)	4. (c)	5. (b)
6. (a)	7. (c)	8. (a)	9. (b)	10. (c)
11. (c)	12. (c)	13. (b)	14. (d)	15. (a)
16. (d)	17. (a)	18. (b)	19. (c)	20. (a)
21. (c)	22. (a)	23. (d)	24. (b)	25. (a)
26. (c)	27. (d)	28. (c)	29. (b)	30. (b)
31. (d)	32. (a)	33. (b)	34. (d)	35. (c)

SUMMARY

1. *Euclid's division lemma*: Given positive integers a and b there exist whole numbers q and r satisfying $a = bq + r, 0 \leq r < b$.

2. *Euclid's division algorithm*: In order to compute the HCF of two positive integers, say a and b, with $a > b$ by using Euclid's algorithm we follow the following steps:

   STEP I *Apply Euclid's division lemma to a and b and obtain whole numbers q_1 and r_1 such that $a = bq_1 + r_1, 0 \leq r_1 < b$.*

   STEP II *If $r_1 = 0$, b is the HCF of a and b*

   STEP III *If $r_1 \neq 0$, apply Euclid's division lemma to b and r_1 and obtain two whole numbers q_1 and r_2 such that $b = q_1 r_1 + r_2$.*

   STEP IV *If $r_2 = 0$, then r_1 is the HCF of a and b.*

   STEP V *If $r_2 \neq 0$, then apply Euclid's division lemma to r_1 and r_2 and continue the above process till the remainder r_n is zero. The divisor at this stage i.e. r_{n-1}, or the non-zero remainder at the previous stage, is the HCF of a and b.*

3. *The Fundamental Theorem of Arithmetic*: Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique except for the order in which the prime factors occur.

4. Every composite number can be uniquely expressed as the product of powers of primes in ascending or descending order.

5. Let a be a positive integer and p be a prime number such that $p \mid a^2$, then $p \mid a$.

6. There are infinitely many positive primes.

7. Every positive integer different from 1 can be expressed as a product of non-negative power of 2 and an odd number.

8. A positive integer n is prime, if it is not divisible by any prime less than or equal to $\sqrt{n}$.

9. If p is a positive prime, then $\sqrt{p}$ is an irrational number. For example, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}$ etc. are irrational numbers.

10. Let x be a rational number whose decimal expansion terminates. Then, x can be expressed in the form $\frac{p}{q}$, where p and q are co-prime, and the prime factorization of q is of the form $2^m \times 5^n$, where m, n are non-negative integers.

11. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form $2^m \times 5^n$ where m, n are non-negative integers. Then, x has a terminating decimal expansion which terminates after k places of decimals, where k is the larger of m and n.

12. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has non-terminating repeating decimal expansion.

NOTE: Formative assessment also includes lab activities, projects, assignments (Home work), oral and visual testing.

CHAPTER 2

POLYNOMIALS

2.1 INTRODUCTION

In earlier classes, we have learnt about polynomials in one variable, their degrees, factors, multiples and zeros (or roots). In this chapter, we will study about the geometrical representation of linear and quadrat $p \mid b$ ic polynomials and geometrical meaning of their zeros. We will also study about the relationship between the zeros and coefficients of a polynomial. Let us first recall some useful definitions and results which we have studied in class IX.

2.2 RECAPITULATION

POLYNOMIAL *Let x be a variable, n be a positive integer and as, $a_1, a_2, \ldots, a_n$ be constants (real numbers). Then, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is called a polynomial in variable x.*

In the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, $a_n x^n, a_{n-1} x^{n-1}, \ldots, a_1 x$ and a_0 are known as the terms of the polynomial and $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$ and a_0 are their coefficients.

For example,

(i) $p(x) = 3x - 2$ is a polynomial in variable x.

(ii) $q(y) = 3y^2 - 2y + 4$ is a polynomial in variable y.

(iii) $f(u) = \frac{1}{2} u^3 - 3u^2 + 2u - 4$ is a polynomial in variable u.

Note that the expressions like $2x^2 - 3\sqrt{x} + 5, \frac{1}{x^2 - 2x + 5}, 2x^3 - \frac{3}{x} + 4$ etc. are not polynomials.

DEGREE OF A POLYNOMIAL *The exponent of the highest degree term in a polynomial is known as its degree.*

In other words, the highest power of x in a polynomial $f(x)$ is called the degree of the polynomial $f(x)$.

For example,

(i) $f(x) = 3x + \frac{1}{2}$ is a polynomial of degree 1 in the variable x.

(ii) $g(y) = 2y^2 - \frac{3}{2} y + 7$ is a polynomial of degree 2 in the variable y.

(iii) $p(x) = 5x^3 - 3x^2 + x - \frac{1}{\sqrt{2}}$ is a polynomial of degree 3 in the variable x.

(iv) $q(u) = 9u^5 - \frac{2}{3}u^4 + u^2 - \frac{1}{2}$ is a polynomial of degree 5 in the variable u.

CONSTANT POLYNOMIAL *A polynomial of degree zero is called a constant polynomial.*

For example, $f(x) = 7, g(x) = -\frac{3}{2}, h(y) = 2, p(t) = 1$ etc are constant polynomial.

The constant polynomial 0 or $f(x) = 0$ is called the zero polynomial. The degree of the zero polynomial is not defined, because

$$f(x) = 0, g(x) = 0x, h(x) = 0x^2, p(x) = 0x^3, q(x) = 0x^{12}$$

etc. are all equal to the zero polynomial.

LINEAR POLYNOMIAL *A polynomial of degree 1 is called a linear polynomial.*

For example, $p(x) = 4x - 3, q(y) = 3y, f(t) = \sqrt{3}\,t + 5$ and $g(u) = \frac{2}{3}u - \frac{5}{2}$ etc are all linear polynomials.

Polynomials such as $f(x) = 2x^2 + 3, g(x) = 3 - x^2$ etc are not linear polynomials.

More generally, any linear polynomial in variable x with real coefficients is of the form $f(x) = ax + b$, where a, b are real numbers and $a \neq 0$.

REMARK 1 *A linear polynomial may be a monomial or a binomial. Because, linear polynomial $f(x) = \frac{2}{3}x - \frac{5}{2}$ is a binomial whereas the linear polynomial $g(x) = \frac{2}{5}x$ is a monomial.*

QUADRATIC POLYNOMIAL *polynomial of degree 2 is called a quadratic polynomial.*

The name 'quadratic' has been derived from 'quadrate', which means 'square'.

For example,

$$f(x) = 2x^2 + 3x - \frac{4}{5}, g(y) = 2y^2 - 3, h(u) = 2 - u^2 + \sqrt{3}\,u,\ p(v) = \sqrt{3}\,v^2 - \frac{4}{3}v + \frac{1}{2},$$

$q(\alpha) = \frac{2}{3}\alpha^2 + 4\alpha$ etc. are quadratic polynomials with real coefficients.

More generally, any quadratic polynomial in variable x with real coefficients is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

REMARK 2 *A quadratic polynomial may be a monomial or a binomial or a trinomial, because $f(x) = \frac{1}{5}x^2$ is a monomial, $g(x) = 3x^2 - 5$ is a binomial and $h(x) = 3x^2 - 2x + 5$ is a trinomial.*

CUBIC POLYNOMIAL *A polynomial of degree 3 is called a cubic polynomial.*

For example,

(i) $f(x) = \frac{9}{5}x^3 - 2x^2 + \frac{7}{3}x - \frac{1}{5}$ is a cubic polynomial in variable x.

(ii) $g(y) = 2y^3 + 5y - 7$ is a cubic polynomial in variable y.

(iii) $p(u) = \frac{\sqrt{2}}{3}u^3 + 1$ is a cubic polynomial in variable u.

The most general form of a cubic polynomial with coefficients as real numbers is

$$f(x) = ax^3 + bx^2 + cx + d, \text{ where } a \neq 0, b, c, d \text{ are real numbers.}$$

BI-QUADRATIC POLYNOMIAL *A fourth degree polynomial is called a biquadratic polynomial.*

For example,

(i) $f(x)=\frac{3}{5}x^4-2x^3+\frac{3}{2}x^2-\sqrt{2}x+\frac{1}{5}$ is a biquadratic polynomial with real coefficients in variable x.

(ii) $g(y)=2y^4+3$ is a biquadratic polynomial in variable y.

(iii) $h(u)=3u^4-5u^2+2$ is a biquadratic polynomial in variable u.

The most general form of a biquadratic polynomial with real coefficients in variable x is

$$f(x)=ax^4+bx^3+cx^2+dx+e, \text{ where } a\neq 0, b, c, d, e \text{ are real numbers.}$$

REMARK 3 *Throughout this chapter, we shall be studying polynomials with real coefficients.*

VALUE OF A POLYNOMIAL *If $f(x)$ is a polynomial and α is any real number, then the real number obtained by replacing x by α in $f(x)$, is called the value of $f(x)$ at $x=\alpha$ and is denoted by $f(\alpha)$.*

The values of the quadratic polynomial $f(x)=2x^2-3x-2$ at $x=1$ and $x=-2$ are given by

$$f(1)=2\times(1)^2-3\times 1-2=2-3-2=-3$$

and, $$f(-2)=2\times(-2)^2-3\times(-2)-2=8+6-2=12$$

If $f(x)=2x^3-13x^2+17x+12$, then its value at $x=-\frac{1}{2}$ is

$$f\left(-\frac{1}{2}\right)=2\times\left(-\frac{1}{2}\right)^3-13\times\left(-\frac{1}{2}\right)^2+17\times\left(-\frac{1}{2}\right)+12=-\frac{1}{4}-\frac{13}{4}-\frac{17}{2}+12=0$$

Consider the cubic polynomial $f(x)=x^3-6x^2+11x-6$. The value of this polynomial at $x=2$ is given by

$$f(2)=2^3-6\times 2^2+11\times 2-6=8-24+22-6=0$$

Also, $$f(1)=1^3-6\times 1^2+11\times 1-6=1-6+11-6=0$$

and, $$f(3)=3^3-6\times 3^2+11\times 3-6=27-54+33-6=0$$

Thus, we find that the values of $f(x)$ at $x=1, 2,$ and 3 are each equal to zero. So, 1, 2 and 3 are called zeros of the cubic polynomial $f(x)=x^3-6x^2+11x-6$.

Thus, we may define zeros of a polynomial as follows:

ZERO OF A POLYNOMIAL *A real number α is a zero of a polynomial $f(x)$, if $f(\alpha)=0$.*

Finding a zero of a polynomial $f(x)$ means solving the polynomial equation $f(x)=0$.

In class IX, we have learnt how to find the zero of a linear polynomial. We have studied that the linear polynomial $f(x)=ax+b, a\neq 0$ has only one zero α given by

$$\alpha=-\frac{b}{a}=-\frac{\text{Constant term}}{\text{Coefficient of } x}$$

We observe that the zero of a linear polynomial is related to its coefficients. In fact, zeros of any polynomial are related to its coefficients. We will study this in the susequent sections. Let us first discuss about the graphs of polynomials of degree 1, 2 and 3.

2.3 GRAPHS OF POLYNOMIALS

In algebraic or in set theoretic language the graph of a polynomial $f(x)$ is the collection (or set) of all points (x, y), where $y = f(x)$. In geometrical or in graphical language the graph of a polynomial $f(x)$ is a smooth free hand curve passing through points (x_1, y_1), (x_2, y_2), (x_3, y_3), ... etc, where $y_1, y_2, y_3, \ldots$ are the values of the polynomial $f(x)$ at x_1, x_2, $x_3 \ldots$ respectively.

In this section, we will learn about the construction of graphs of linear, quadratic and cubic polynomials.

In order to draw the graph of a polynomial $f(x)$, we may follow the following algorithm.

ALGORITHM

STEP I *Find the values $y_1, y_2, \ldots, y_n, \ldots$ of polynomial $f(x)$ at different points x_1 $x_2, \ldots, x_n, \ldots$ and prepare a table that gives values of y or $f(x)$ for various values of x.*

x:	x_1	x_2	...	x_n	x_{n+1}	...
$y = f(x)$:	$y_1 = f(x_1)$	$y_2 = f(x_2)$	...	$y_n = f(x_n)$	$y_{n+1} = f(x_{n+1})$	...

STEP II *Plot the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n), \ldots$ on rectangular coordinate system. In plotting these points you may use different scales on the x and y-axes.*

STEP III *Draw a free hand smooth curve passing through points plotted in step II to get the graph of the polynomial $f(x)$.*

2.3.1 GRAPH OF A LINEAR POLYNOMIAL

Consider a linear polynomial $f(x) = ax + b, a \neq 0$. In class IX, we have learnt that the graph of $y = ax + b$ is a straight line. That is why $f(x) = ax + b$ is called a linear polynomial. Since two points determine a straight line, so only two points need to be plotted to draw the line $y = ax + b$. The line represented by $y = ax + b$ crosses the x-axis at exactly one point, namely $\left(-\frac{b}{a}, 0\right)$.

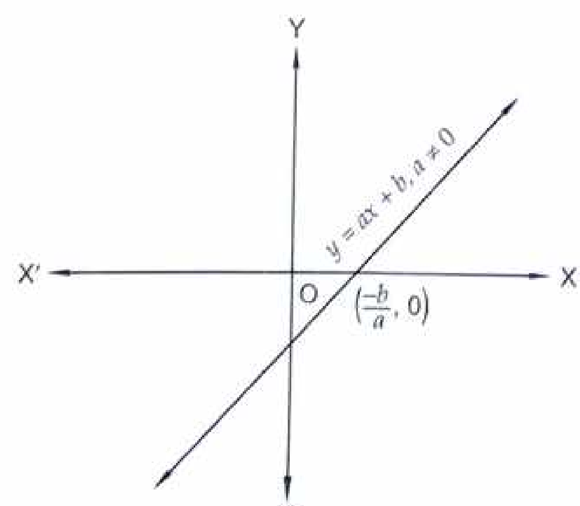


Fig. 2.1

ILLUSTRATION 1 *Draw the graph of the polynomial* $f(x) = 2x - 5$. *Also, find the coordinates of the point where it crosses X-axis.*

SOLUTION Let $y = 2x - 5$.

The following table lists the values of y corresponding to different values of x.

x	1	4
y	-3	3

The points $A(1, -3)$ and $B(4, 3)$ are plotted on the graph paper on a suitable scale. A line is drawn passing through these points to obtain the graph of the given polynomial.

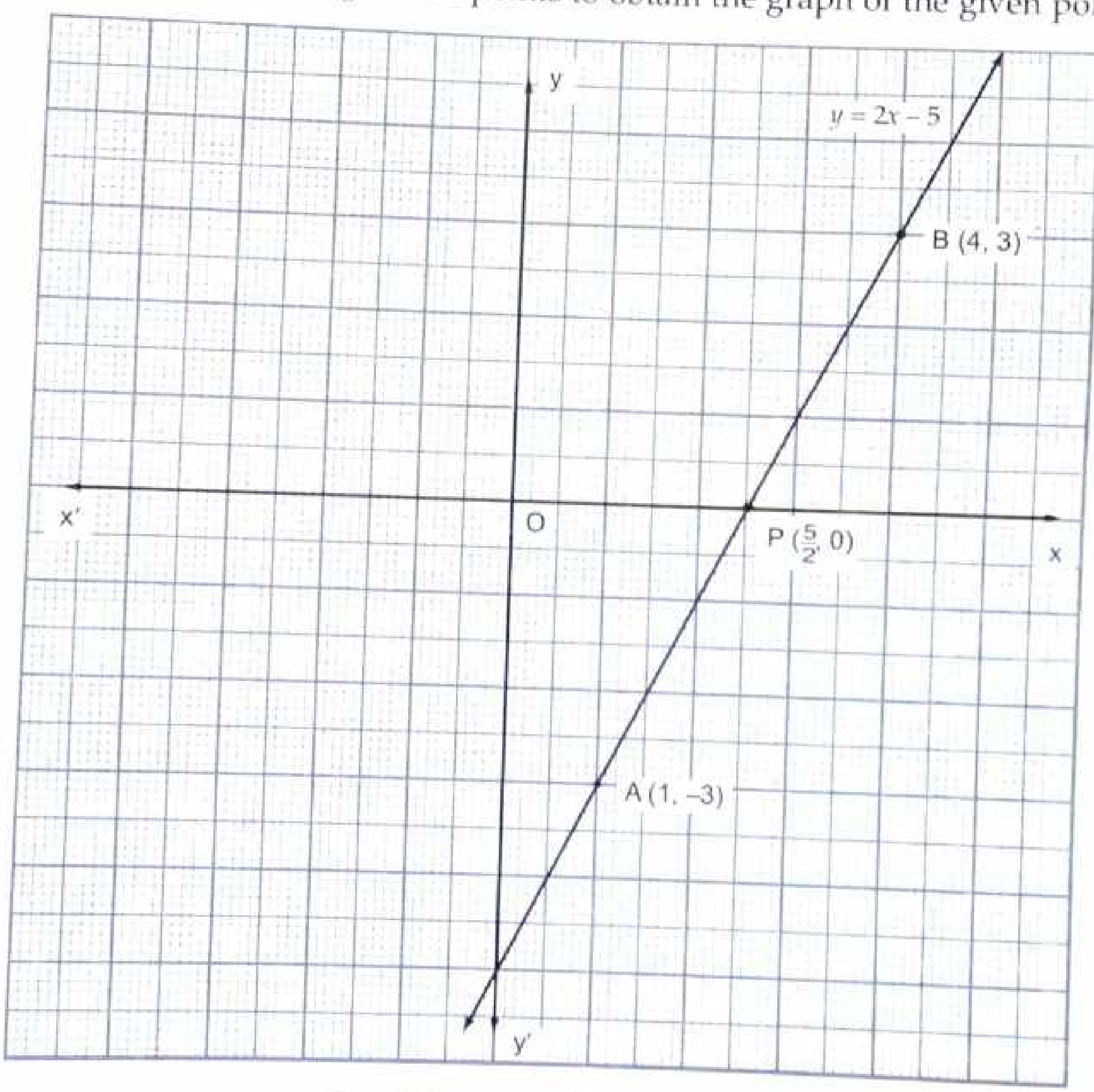


Fig. 2.2 Graph of $f(x) = 2x - 5$

2.3.2 GRAPH OF A QUADRATIC POLYNOMIAL

In this section, we will be interested to see what the graph of a quadratic polynomial $ax^2 + bx + c, a \neq 0$ looks like. We will also learn the construction of the graph of a quadratic polynomial without plotting many points on the graph paper.

ILLUSTRATION 1 *Draw the graph of the polynomial* $f(x) = x^2 - 2x - 8$.

SOLUTION Let $y = x^2 - 2x - 8$.

The following table gives the values of y or $f(x)$ for various values of x.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
$y = x^2 - 2x - 8$	16	7	0	-5	-8	-9	-8	-5	0	7	16

Let us now plot the points $(-4, 16)$, $(-3, 7)$, $(-2, 0)$, $(-1, -5)$, $(0, -8)$, $(1, -9)$, $(2, -8)$, $(3, -5)$, $(4, 0)$, $(5, 7)$ and $(6, 16)$ on a graph paper and draw a smooth free hand curve passing through these points. The curve thus obtained represents the graph of the polynomial $f(x) = x^2 - 2x - 8$. This is called a parabola. The lowest point P, called a minimum point, is the vertex of the parabola.

Vertical line passing through P is called the axis of the parabola. Parabola is symmetric about the axis. So, it is also called the line of symmetry.

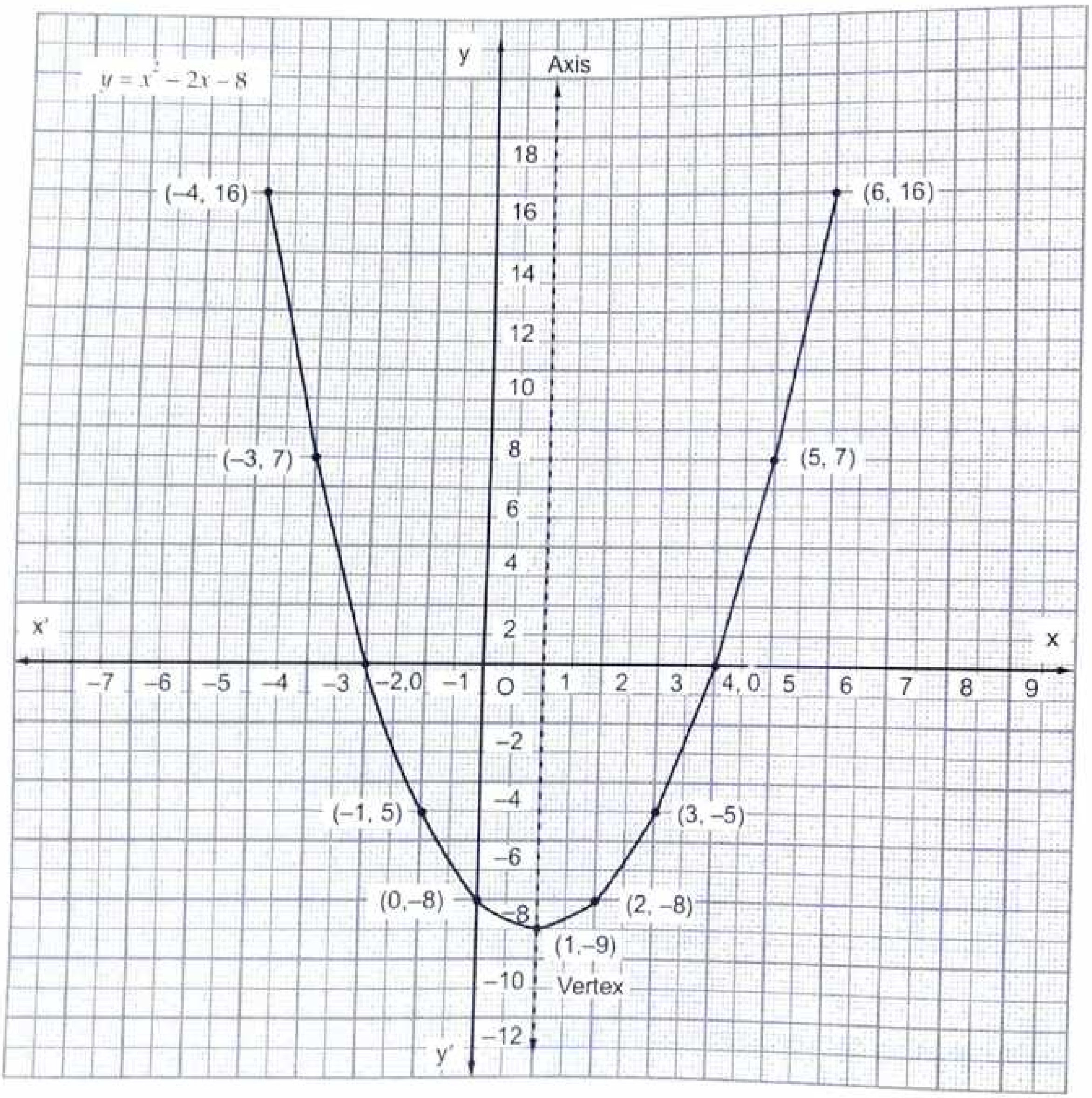


Fig. 2.3 Graph of $f(x) = x^2 - 2x - 8$

Observations: From the graph of the polynomial $f(x) = x^2 - 2x - 8$, we make the following observations:

(i) The coefficient of x^2 in $f(x) = x^2 - 2x - 8$ is 1 (a positive real number) and so the parabola opens upwards.

(ii) The polynomial $f(x) = x^2 - 2x - 8 = (x-4)(x+2)$ is factorizable into two distinct linear factors $(x-4)$ and $(x+2)$. So, the parabola cuts X-axis at two distinct points $(4, 0)$ and $(-2, 0)$. The x-coordinates of these points are zeros of $f(x)$.

(iii) The polynomial $f(x) = x^2 - 2x - 8$ has two distinct zeros namely 4 and -2. So, the parabola cuts X-axis at $(4, 0)$ and $(-2, 0)$.

(iv) On comparing the polynomial $x^2 - 2x - 8$ with $ax^2 + bx + c$, we get $a = 1$, $b = -2$ and $c = -8$. The vertex of the parabola has coordinates $(1, -9)$ i.e. $(-b/2a, -D/4a)$, where $D = b^2 - 4ac$.

(v) $D = b^2 - 4ac = 4 + 32 = 36 > 0$. So, the parabola cuts X-axis at two distinct points.

ILLUSTRATION 2 *Draw the graph of the quadratic polynomial* $f(x) = 3 - 2x - x^2$.

SOLUTION Let $y = f(x)$ or, $y = 3 - 2x - x^2$.

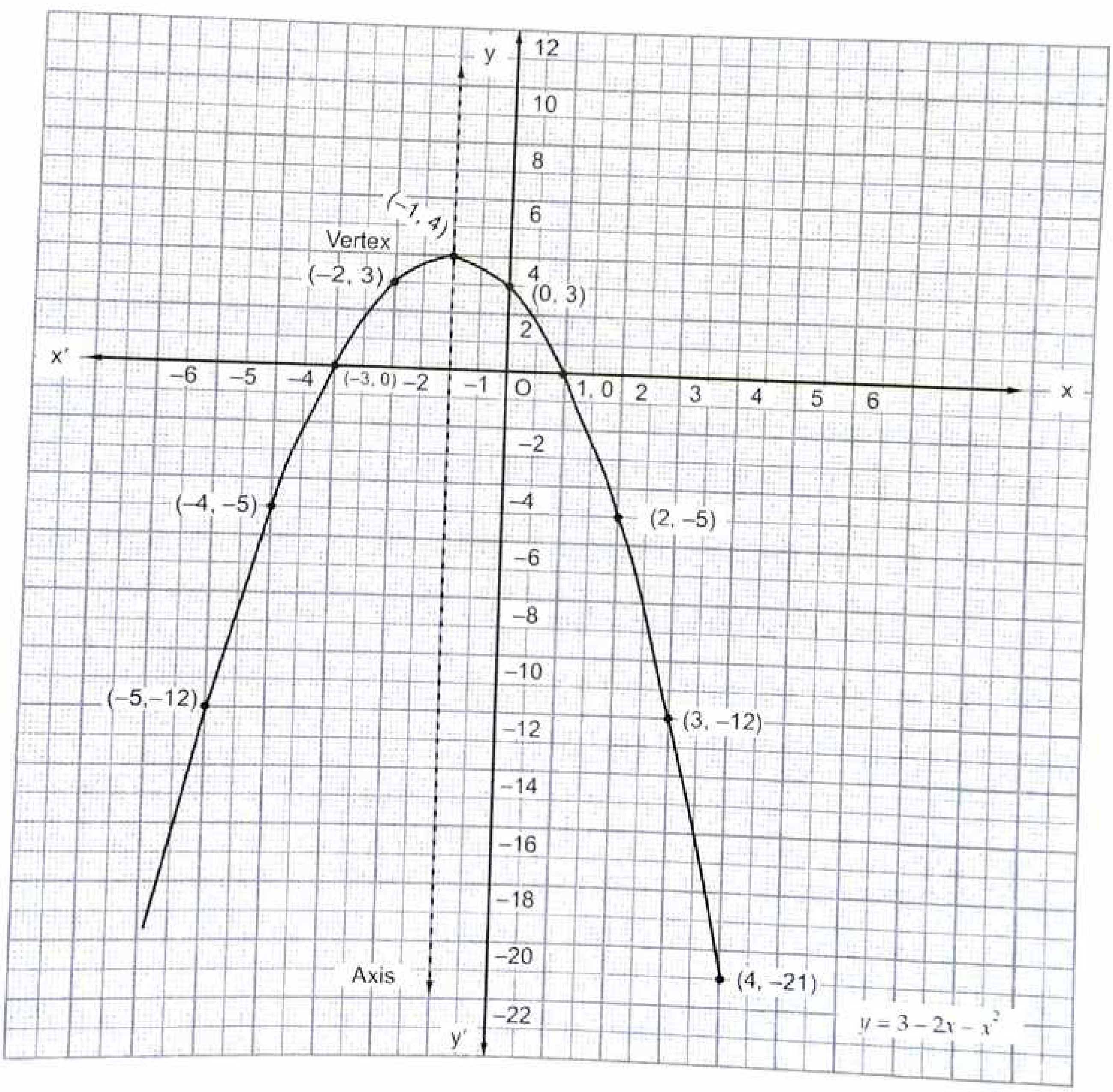


Fig. 2.4 Graph of $f(x) = 3 - 2x - x^2$

Let us list a few values of $y = 3 - 2x - x^2$ corresponding to a few values of x as follows:

x	-5	-4	-3	-2	-1	0	1	2	3	4
$y = 3 - 2x - x^2$	-12	-5	0	3	4	3	0	-5	-12	-21

Thus, the following points lie on the graph of the polynomial $y = 3 - 2x - x^2$:

$(-5, -12), (-4, -5), (-3, 0), (-2, 3), (-1, 4), (0, 3), (1, 0), (2, -5), (3, -12)$ and $(4, -21)$.

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graph of $y = 3 - 2x - x^2$. The curve thus obtained represents a parabola, as shown in Fig. 2.4. The highest point $P(-1, 4)$, is called a maximum point, is the vertex of the parabola. Vertical line through P is the axis of the parabola. Clearly, parabola is symmetric about the axis.

Observations: We make the following observations from the graph of the polynomial $f(x) = 3 - 2x - x^2$.

(i) The coefficient of x^2 in $f(x) = 3 - 2x - x^2$ is -1 i.e. a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = 3 - 2x - x^2 = (1 - x)(x + 3)$ is factorizable into two distinct linear factors $(1 - x)$ and $(x + 3)$. So, the parabola cuts X-axis at two distinct points $(1, 0)$ and $(-3, 0)$. The x-coordinates of these points are zeros of $f(x)$.

(iii) The polynomial $f(x) = 3 - 2x - x^2$ has two distinct roots namely -3 and 1. So, the parabola $y = 3 - 2x - x^2$ cuts X-axis at two distinct points.

(iv) On comparing the polynomial $3 - 2x - x^2$ with $ax^2 + bx + c$, we have $a = -1$, $b = -2$ and $c = 3$. The vertex of the parabola is at the point $(-1, 4)$ i.e. at $\left(-b/2a, -D/4a\right)$, where $D = b^2 - 4ac$.

(v) $D = b^2 - 4ac = 4 + 12 = 16 > 0$. So, the parabola cuts x-axis at two distinct points.

ILLUSTRATION 3 *Draw the graph of the polynomial* $f(x) = x^2 - 6x + 9$.

SOLUTION Let $y = f(x)$ or, $y = x^2 - 6x + 9$.

The following table gives the values of y or $f(x)$ for various values of x.

x	-2	-1	0	1	2	3	4	5	6	7	8
$y = x^2 - 6x + 9$	25	16	9	4	1	0	1	4	9	16	25

Thus, the graph of $y = x^2 - 6x - 9$ passes through the points $(-2, 25)$, $(-1, 16)$, $(0, 9)$, $(1, 4), (2, 1), (3, 0), (4, 1), (5, 4), (6, 9), (7, 16)$ and $(8, 25)$.

Let us plot these points on the graph and draw a free hand smooth curve passing through these points. We observe that the vertex of the parabola is at point $P(3, 0)$ as shown in Fig. 2.5.

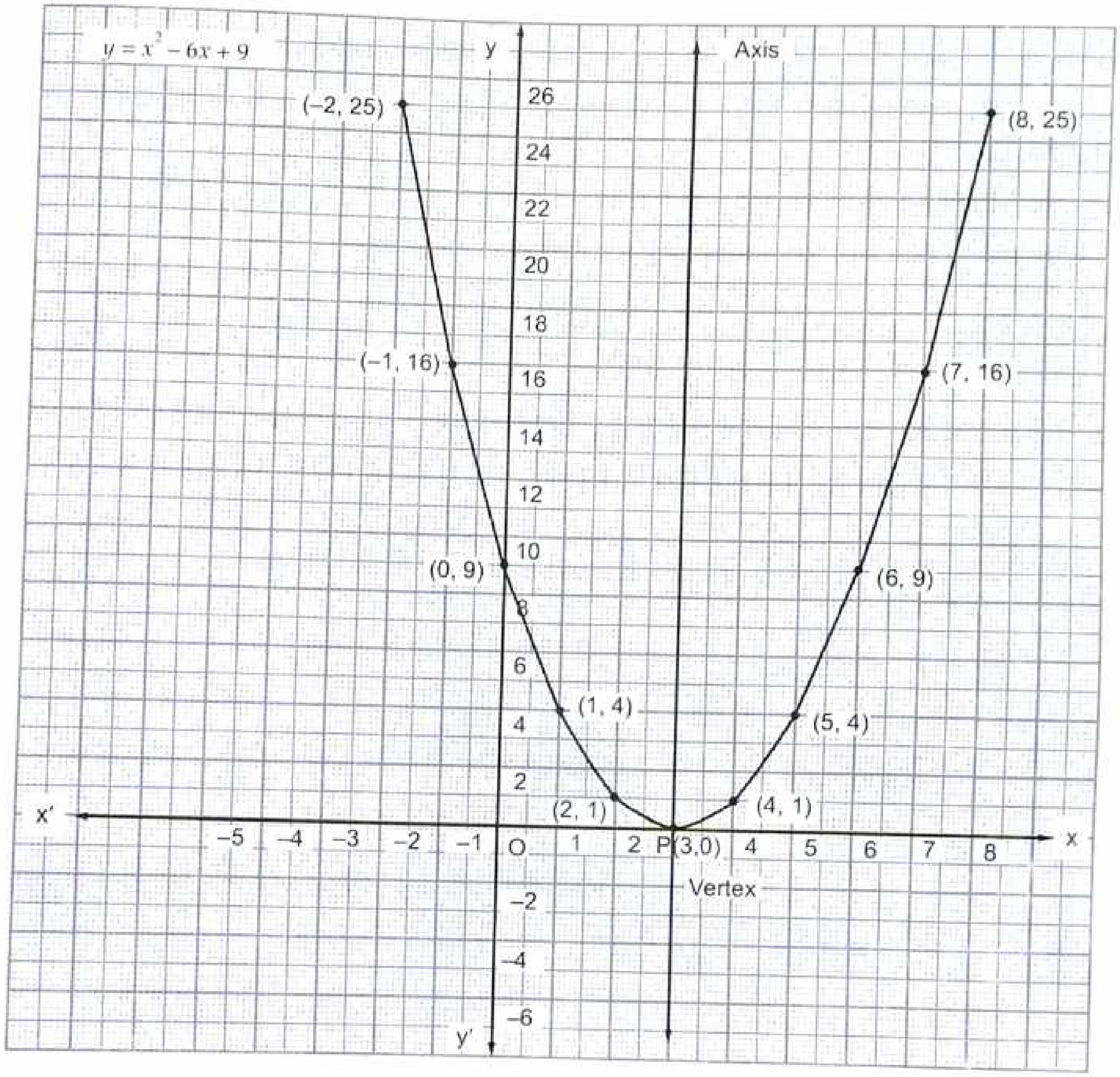


Fig. 2.5 Graph of $f(x) = x^2 - 6x + 9$

Observations: From the graph of the polynomial $f(x) = x^2 - 6x + 9$, we make the following observations:

(i) The coefficient of x^2 in $f(x) = x^2 - 6x + 9$ is 1, a positive real number, and so the parabola opens upwards.

(ii) The polynomial $f(x) = x^2 - 6x + 9 = (x - 3)^2$ is factorizable into two equal factors each equal to $(x - 3)$. So, the parabola $y = x^2 - 6x + 9$ touches X-axis at one point $(3, 0)$. In other words, $y = x^2 - 6x + 9$ touches X-axis at one point $(3, 0)$. In other words, $y = x^2 - 6x + 9$ cuts X-axis at coincident points. The x-coordinate of this point gives two equal roots of the polynomial.

(iii) The polynomial $f(x) = x^2 - 6x + 9$ has two equal roots each equal to 3. So, the parabola $y = x^2 - 6x + 9$ touches X-axis at $(3, 0)$ i.e. it cuts X-axis at coincident points.

(iv) On comparing the polynomial $x^2 - 6x + 9$ with $ax^2 + bx + c$, we get $a = 1$, $b = -6$ and $c = 9$. The vertex of the parabola is at $(3, 0)$ i.e., at $\left(-b/2a, -D/4a\right)$, where $D = b^2 - 4ac$.

(v) $D = b^2 - 4ac = 36 - 36 = 0$. So, the parabola touches X-axis.

ILLUSTRATION 4 *Draw the graph of the polynomial* $f(x) = -4x^2 + 4x - 1$. *Also, find the vertex of this parabola.*

SOLUTION Let $y = f(x)$ or, $y = -4x^2 + 4x - 1$

The following table gives the values of y for various values of x.

x	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$y = -4x^2 + 4x - 1$	-25	-16	-9	-4	-1	0	-1	-4	-9	-16	-25

Thus, the following points lie on the graph of $y = -4x^2 + 4x - 1$: $(-2, -25)$, $(-3/2, -16)$, $(-1, -9)$, $(-1/2, -4)$, $(0, -1)$, $(1/2, 0)$, $(1, -1)$, $(3/2, -4)$, $(2, -9)$, $(5/2, -16)$, $(3, -25)$ etc.

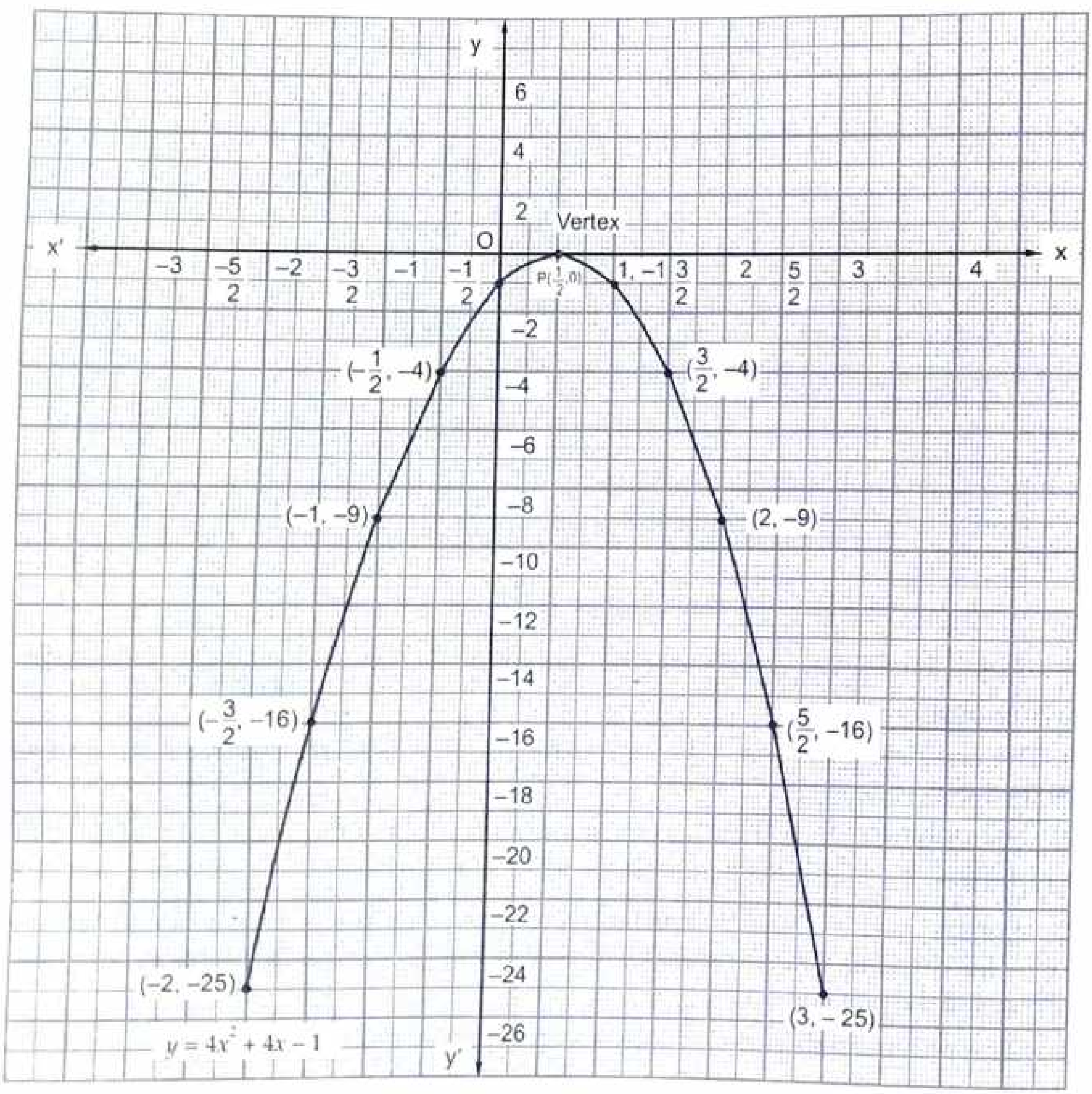


Fig. 2.6 Graph of $f(x) = -4x^2 + 4x - 1$

Let us plot these points on a graph paper and draw a free hand smooth curve passing through these points. The shape of the curve is shown in Fig. 2.6. It is a parabola opening downward having its vertex at $(1/2, 0)$.

Observations: From the graph of the polynomial $f(x) = -4x^2 + 4x - 1$, we make the following observations:

(i) The coefficient of x^2 in $f(x) = -4x^2 + 4x - 1$ is -4, a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = -4x^2 + 4x - 1 = -(2x-1)^2$ is factorizable into two equal factors each equal to $2x - 1$. So, the parabola cuts X-axis at two coincident points having coordinates $(1/2, 0)$.

(iii) The polynomial $f(x) = -4x^2 + 4x - 1$ has two equal roots each equal to $1/2$. So, the parabola touches X-axis at one point $(1/2, 0)$ only i.e. it cuts X-axis at coincident points.

(iv) On comparing the polynomial $-4x^2 + 4x - 1$ with $ax^2 + bx + c$, we get $a = -4$, $a = -4, b = 4$ and $c = -1$. The vertex of the parabola has the coordinates $(1/2, 0)$ i.e. $(-b/2a, -D/4a)$, where $D = b^2 - 4ac$.

(v) $d = b^2 - 4ac = 4 - 4 = 0$. So, the parabola touches X-axis.

ILLUSTRATION 5 *Draw the graph of the polynomial* $f(x) = 2x^2 - 4x + 5$.

SOLUTION Let $y = f(x)$ or, $y = 2x^2 - 4x + 5$.

The following table gives the values of y for various values of x:

x	-3	-2	-1	0	1	2	3	4	5
$y = 2x^2 - 4x + 5$	35	21	11	5	3	5	11	21	35

Thus, the graph of $y = 2x^2 - 4x + 5$ passes through the following points:

$$(-3, 35), (-2, 21), (-1, 11), (0, 5), (1, 3), (2, 5), (3, 11), (4, 21), (5, 35) \text{ etc.}$$

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graph of $y = 2x^2 - 4x + 5$ as shown in Fig. 2.7.

Observations: From the graph of the polynomial $f(x) = 2x^2 - 4x + 5$, we make following observations:

(i) The coefficient of x^2 in $f(x) = 2x^2 - 4x + 5$ is 2 i.e. a positive real number and so the parabola opens upwards.

(ii) The polynomial $f(x) = 2x^2 - 4x + 5$ is not factorizable into linear factors and so the parabola $y = 2x^2 - 4x + 5$ does not cross or touch X-axis.

(iii) The polynomial $f(x) = 2x^2 - 4x + 5$ does not have any real zero and so the parabola does not cut X-axis.

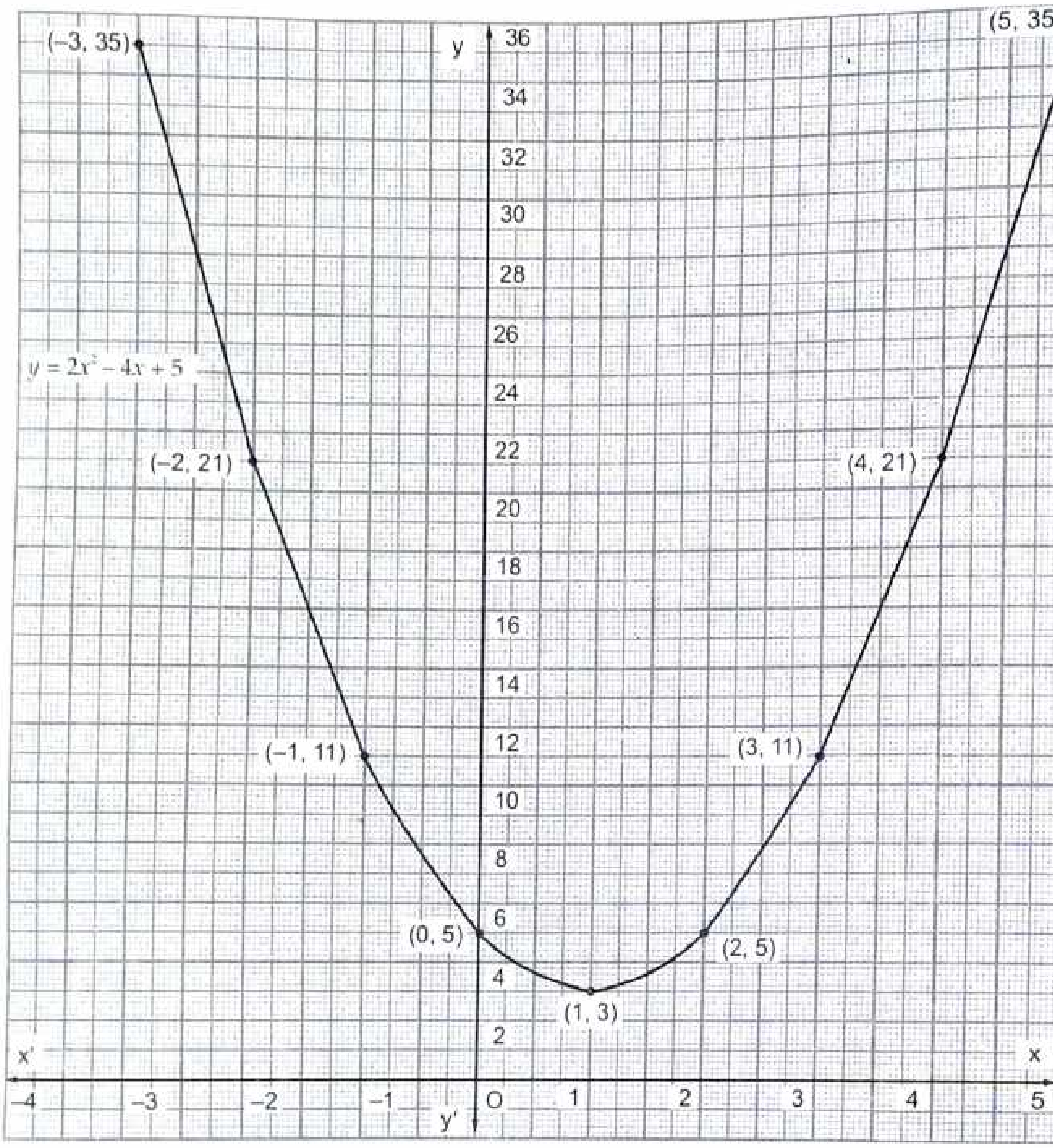


Fig. 2.7 Graph of $f(x) = 2x^2 - 4x + 5$

(iv) On comparing the polynomial $f(x) = 2x^2 - 4x + 5$ with $ax^2 + bx + c$, we get $a = 2, b = -4$ and $c = 5$. The vertex of the parabola is at $(-b/2a, -D/4a)$ i.e. at $(1, 3)$, where $D = b^2 - 4ac$.

(v) All values of $f(x)$ are positive as the parabola remains above X-axis.

(vi) $D = b^2 - 4ac = 16 - 40 < 0$. So, the parabola does not cross X-axis.

ILLUSTRATION 6 *Draw the graph of the polynomial* $f(x) = -3x^2 + 2x - 1$.

SOLUTION Let $y = f(x)$ or, $y = -3x^2 + 2x - 1$.

The values of y for various values of x are listed in the following table:

x	-4	-3	-2	-1	0	1	2	3	4
$y = 3x^2 + 2x - 1$	-57	-34	-17	-6	-1	-2	-9	-26	-41

Thus, the graph of $y = -3x^2 + 2x - 1$ passes through the points: $(-4, -57)$, $(-3, -34)$, $(-2, -17)$, $(-1, -6)$, $(0, -1)$, $(1, -2)$, $(2, -9)$, $(3, -26)$, $(4, -41)$ etc.

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points. The curve thus obtained is shown in Fig. 2.8.

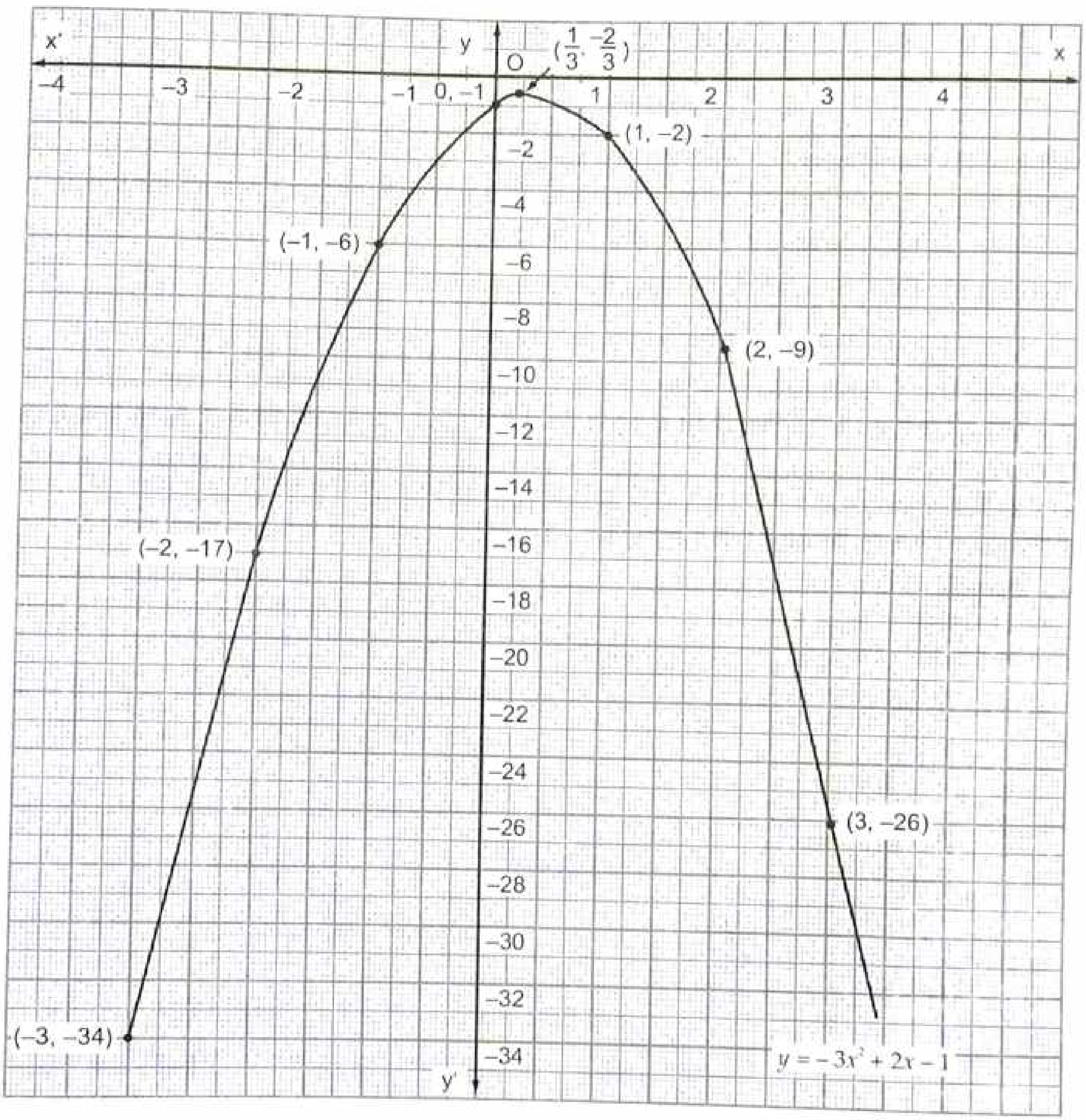


Fig. 2.8 Graph of $f(x) = -3x^2 + 2x - 1$

Observations: We make the following observations from the graph of the polynomial $f(x) = -3x^2 + 2x - 1$:

(i) The coefficient of x^2 in $f(x) = -3x^2 + 2x - 1$ is -3 i.e. a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = -3x^2 + 2x - 1$ is not factorizable into real factors and so the parabola does not cross or touch x-axis.

(iii) The polynomial $f(x) = -3x^2 + 2x - 1$ does not have any real zero and the parabola does not cross or touch x-axis.

(iv) All values of $f(x)$ are negative as the parabola opens downwards and remains below x-axis.

(v) On comparing the polynomial $f(x) = -3x^2 + 2x - 1$ with $ax^2 + bx + c$, we get $a = -3, b = +2$ and $c = -1$. The vertex of the parabola is at $(-b/2a, -D/4a)$, i.e. at $(1/3, -2/3)$, where $D = b^2 - 4ac$.

It follows from the above discussion on the graph of a quadratic polynomial that the graph of the quadratic polynomial $ax^2 + bx + c, a \neq 0$ is a parabola which opens upwards ($\cup$) or downwards ($\cap$) according as $a > 0$ or $a < 0$.

We also observe that there are following three possibilities:

CASE I *When polynomial $ax^2 + bx + c$ is factorizable into two distinct linear factors:*

In this case, the graph of $ax^2 + bx + c$ or the curve $y = ax^2 + bx + c$ cuts x-axis at two distinct points A and A'. The x-coordinates of these points are the two zeros of the polynomial $ax^2 + bx + c$. The coordinates of the vertex of the parabola $y = ax^2 + bx + c$ are $(-b/2a, -D/4a)$, where $D = b^2 - 4ac$.

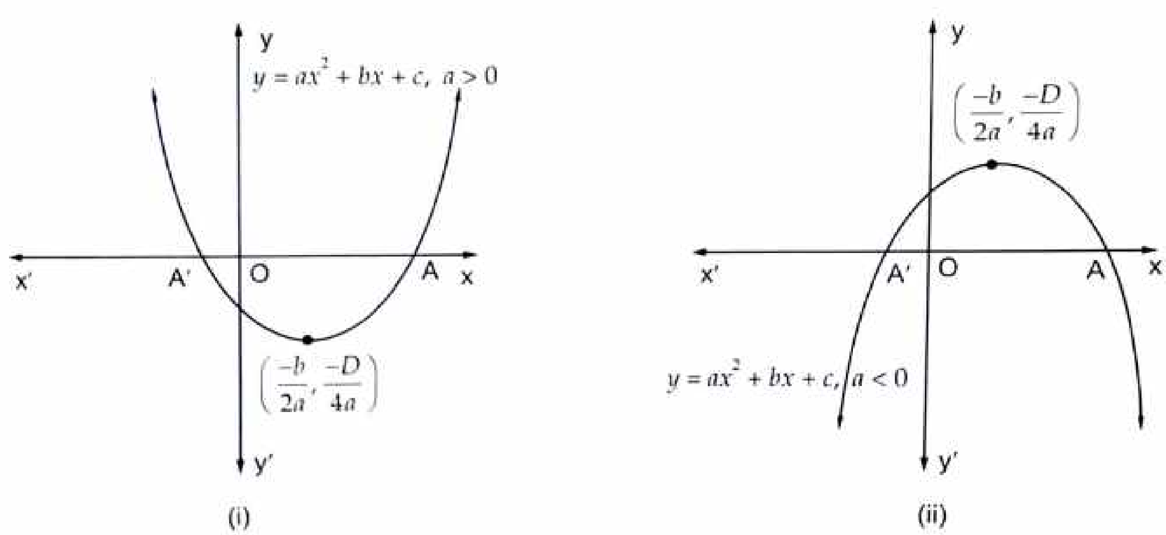


Fig. 2.9

CASE II *When polynomial $ax^2 + bx + c$ is factorizable into two equal factors:*

In this case, the graph of the polynomial $ax^2 + bx + c$ or the curve $y = ax^2 + bx + c$ touches x-axis at point $(-b/2a, 0)$. The x-coordinate of this point gives two equal zeros of the polynomial.

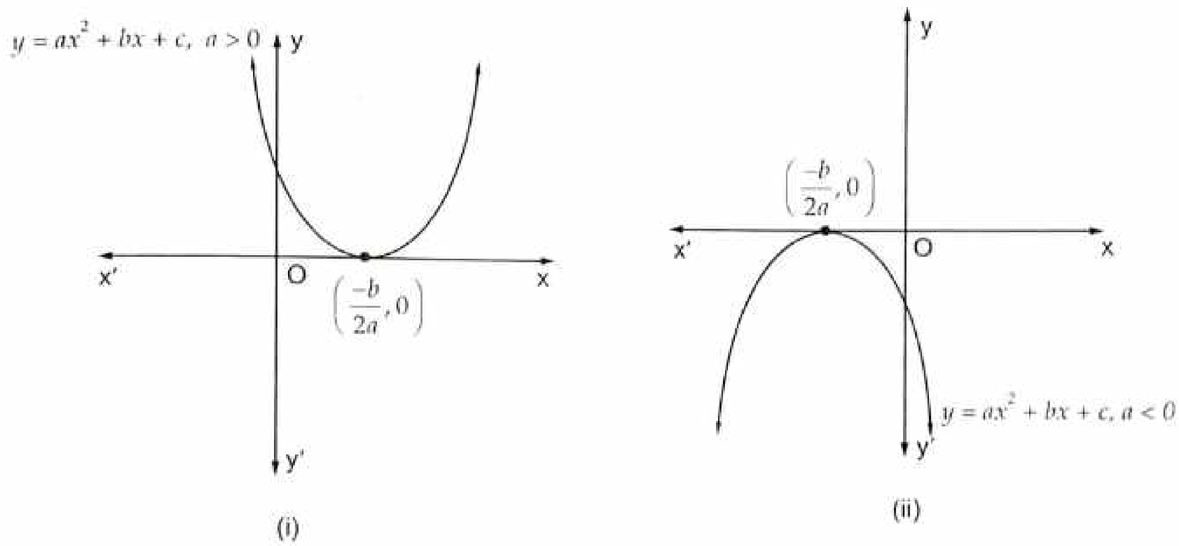


Fig. 2.10 Graph of $y = ax^2 + bx + c$

CASE III *When the polynomial $ax^2 + bx + c$ is not factorizable:*

In this case, the graph of the polynomial $ax^2 + bx + c$ or the curve $y = ax^2 + bx + c$ does not cut or touch x-axis. The parabola $y = ax^2 + bx + c$ opens upwards and remains completely above x-axis, if $a > 0$. The parabola opens downward and remains completely below x-axis, if $a < 0$.

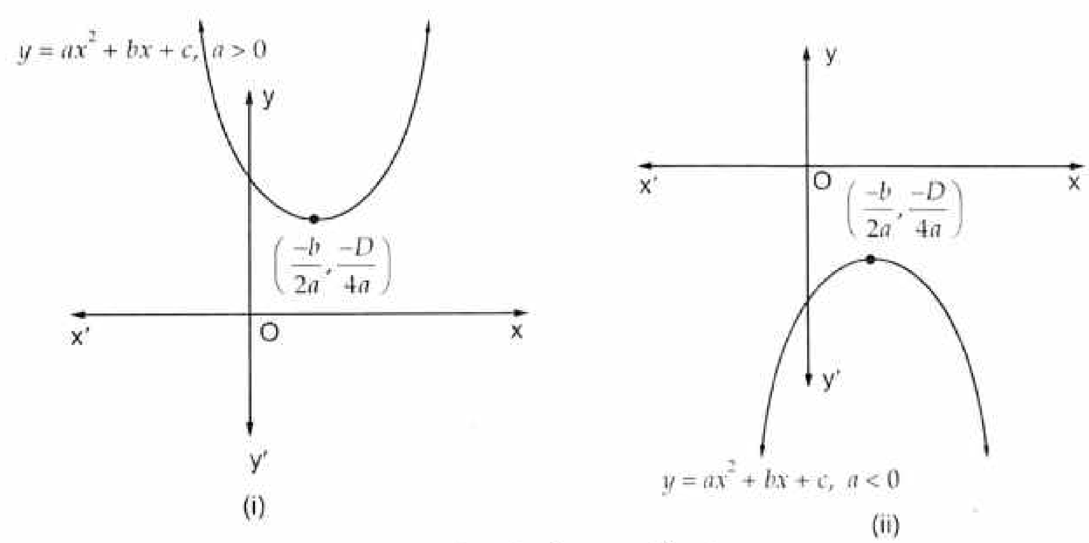


Fig. 2.11 Graph of $y = ax^2 + bx + c$

2.3.3 GRAPH OF A CUBIC POLYNOMIAL

In the previous section, we have seen that the graph of a quadratic polynomial is always a parabola either opening upwards or opening downwards. In this section, we will see that the graph of a cubic polynomial does not have a fixed standard shape. We have also seen that the graph of a quadratic polynomial may or may not cut or touch x-axis. But, in case of a cubic polynomial the graph will always cross x-axis at least once and at most thrice.

ILLUSTRATION 1 *Draw the graph of the polynomial $f(x) = x^3 - 4x$.*

SOLUTION Let $y = f(x)$ or, $y = x^3 - 4x$.

The values of y for various values of x are listed in the following table:

x	–4	–3	–2	–1	0	1	2	3	4
$y = x^3 - 4x$	–48	–15	0	3	0	–3	0	15	48

Thus, the curve $y = x^3 - 4x$ passes through the points $(-4, -48), (-3, -15), (-2, 0)$, $(-1, 3)$ $(0, 0), (1, -3), (2, 0), (3, 15), (4, 48)$ etc.

By plotting these points on a graph paper and drawing a free hand smooth curve through these points, we obtain the graph of the given polynomial as shown in Fig. 2.12.

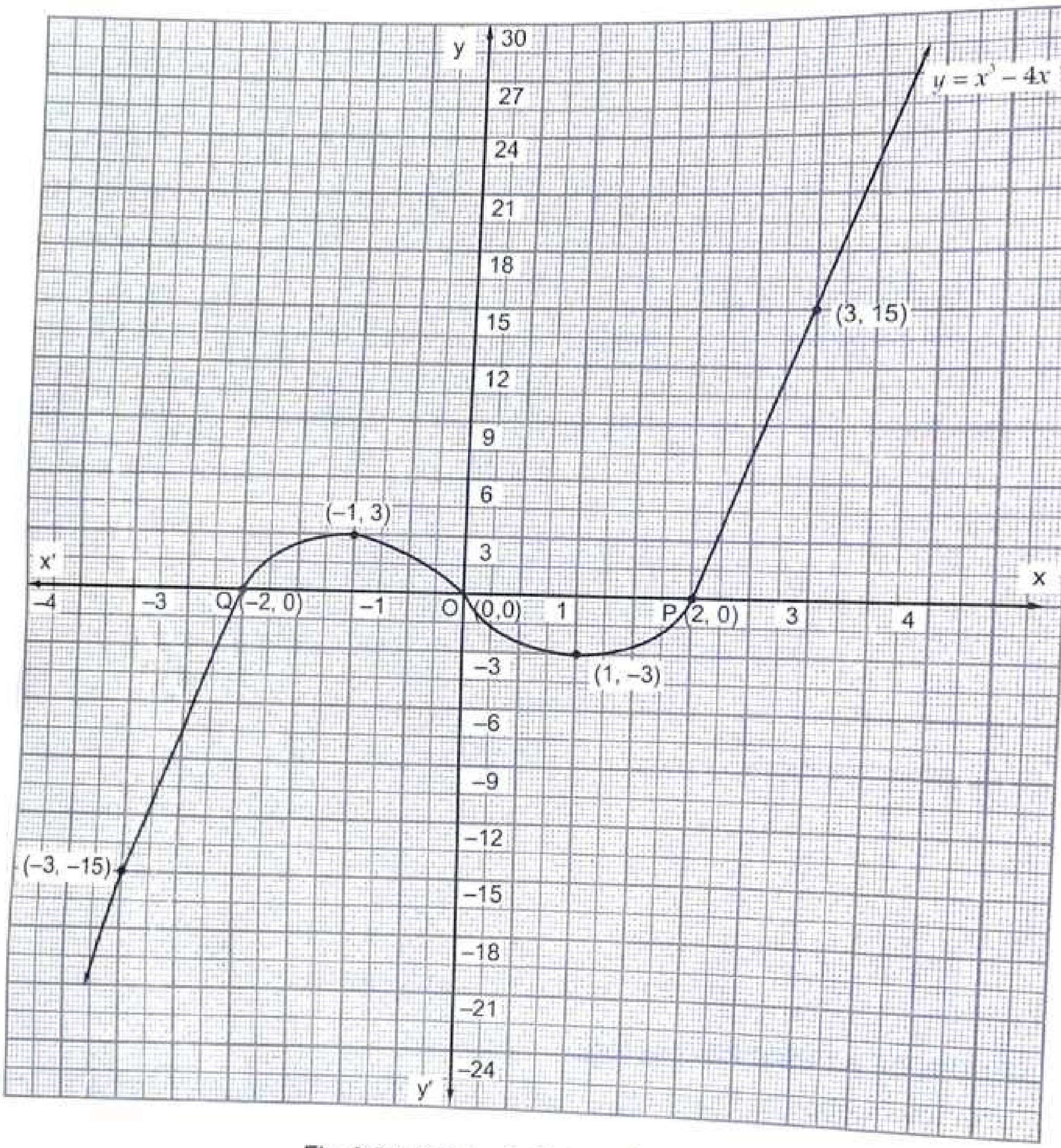


Fig. 2.12 Graph of $f(x) = x^3 - 4x$

Observations: From the graph of the polynomial $f(x) = x^3 - 4x$, we make the following observations:

(i) The polynomial $f(x) = x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$ is factorizable into three distinct linear factors. The curve $y = f(x)$ also cuts x-axis at three distinct ponits.

(ii) We have, $f(x) = x(x-2)(x+2)$

Therefore, 0, 2 and -2 are three zeros of $f(x)$. The curve $y = f(x)$ cuts x-axis at three points O (0, 0), P (2, 0) and Q (-2, 0) whose x-coordinates are the zeros of the polynomial $f(x)$.

ILLUSTRATION 2 *Draw the graph of the cubic polynomial* $f(x) = x^3 - 2x^2$.

SOLUTION Let $y = f(x)$ or, $y = x^3 - 2x^2$.

The following table gives values of y for various values of x:

x	-3	-2	-1	0	1	2	3	4
$y = x^3 - 2x^2$	-45	-16	-3	0	-1	0	9	32

Thus, the curve $y = x^3 - 2x^2$ passes through the points $(-3, -45), (-2, -16), (-1, -3), (0, 0), (1, -1), (2, 0), (3, 9), (4, 32)$ etc. By plotting these points on a graph paper and drawing a free hand smooth curve passing through these points, we obtain the graph of the polynomial as shown in Fig. 2.13.

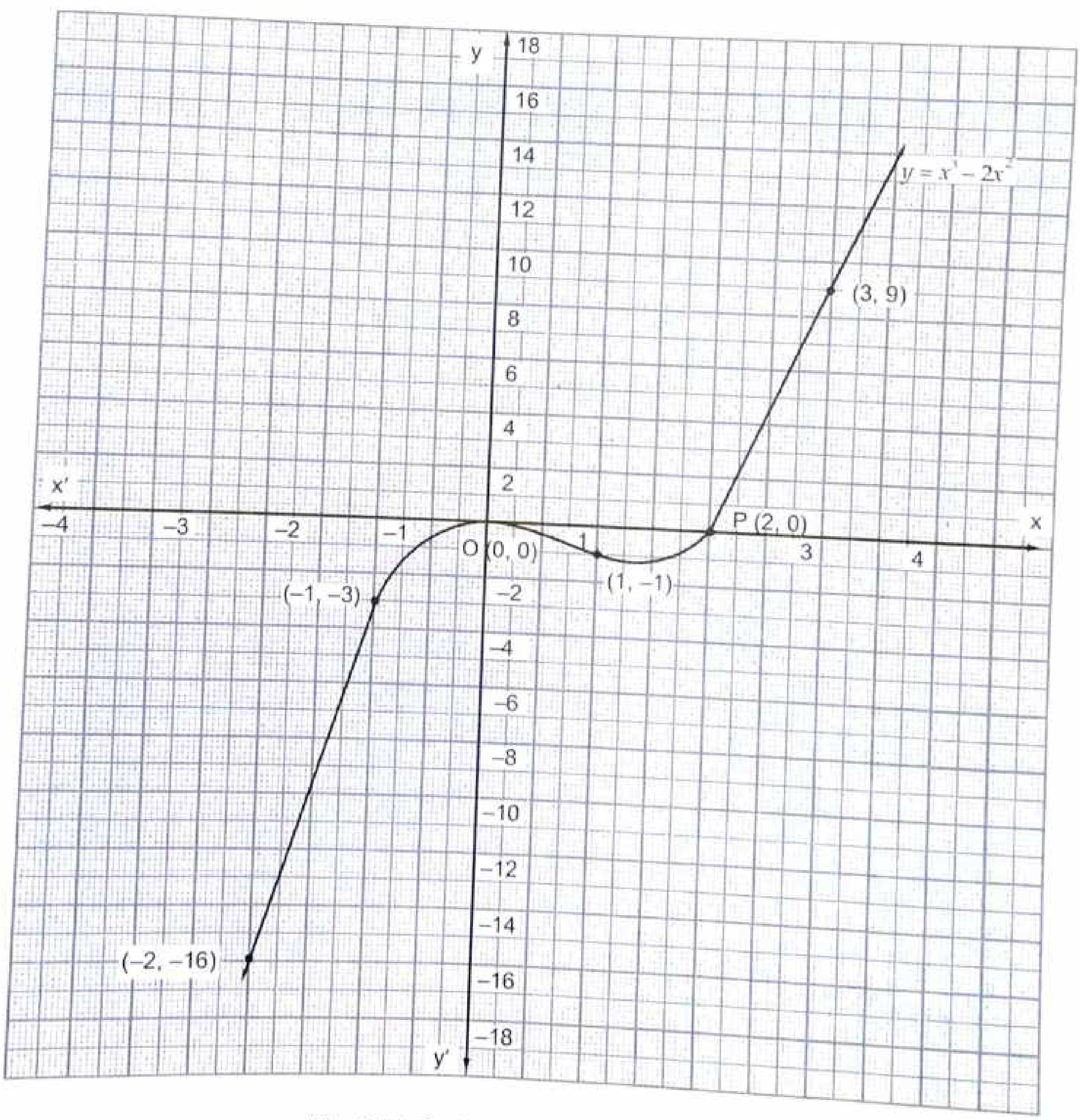


Fig. 2.13 Grahp of $f(x) = x^3 - 2x^2$

Observations: We make the following observations from the graph of the polynomial

$f(x) = x^3 - 2x^2$.

(i) The polynomial $f(x) = x^3 - 2x^2 = x^2(x-2) = (x-0)(x-0)(x-2)$ is factorizable into two identical factors each equal to x and a linear factor $(x-2)$. The curve $y = x^3 - 2x^2$ cuts x-axis at two points.

(ii) We have, $f(x) = (x-0)(x-0)(x-2)$

So, 0 and 2 are two zeros of $f(x)$. The curve cuts x-axis at two points $O(0, 0)$, $P(2, 0)$ whose x-coordinates are the zeros of the polynomial $f(x)$.

ILLUSTRATION 3 *Draw the graph of the polynomial* $f(x) = x^3$.

SOLUTION Let $y = f(x)$ or, $y = x^3$.

The values of y for various values of x are given in the following table:

x	−4	−3	−2	−1	0	1	2	3	4
$y = x^3$	−64	−27	−8	−1	0	1	8	27	64

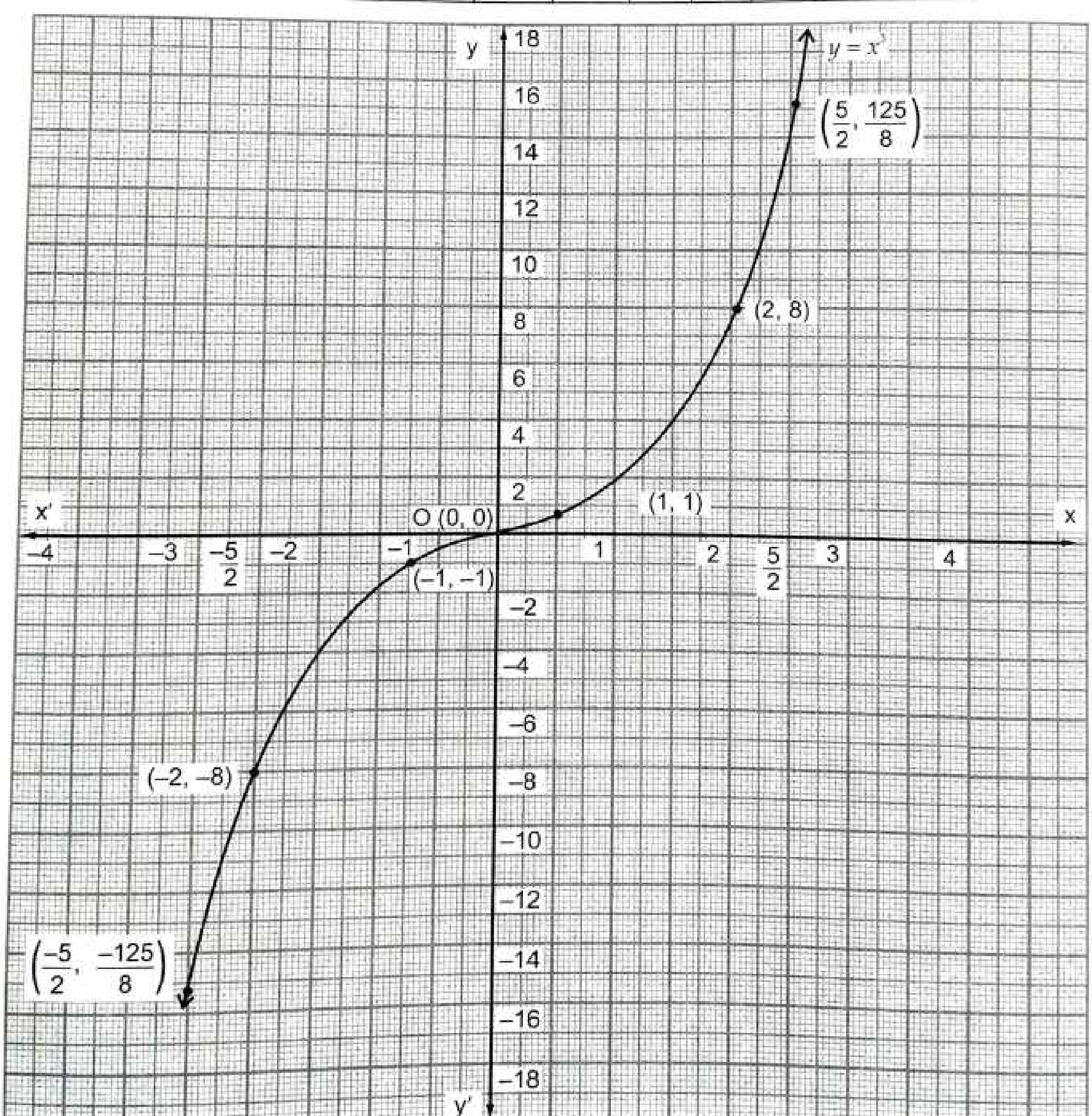


Fig. 2.14 Graph of $f(x) = x^3$.

Thus, the curve $y = x^3$ passes through the points $(-4,-64)$, $(-3,-27)$, $(-2,-8)$, $(-1,-1)$, $(0, 0)$, $(1, 1)$, $(2, 8)$, $(3, 27)$, $(4, 64)$ etc.

Plotting these points on a graph paper and drawing a free hand curve passing through these points, we obtain the graph of the given polynomial as shown in Fig. 2.14.

Observations: We make the following observation from the graph of the polynomial $f(x) = x^3$.

(i) The polynomial $f(x) = x^3 = (x-0)(x-0)(x-0)$ has three identical factors. The curve $y = x^3$ cuts x-axis at three coincident points i.e. at exactly one point.

(ii) The polynomial $f(x) = x^3$ has exactly one zero equal to 0. The curve $y = x^3$ cuts x-axis at the point $O(0, 0)$ whose x-coordinate is equal to zero of the polynomial.

From the above discussion we infer that the graph of a linear polynomial crosses x-axis at one point only and the graph of a quadratic polynomial crosses x-axis at atmost two points. Also, the graph of a cubic polynomial crosses x-axis at atmost three points. In general, graph of an nth degree polynomial crosses the x-axis at atmost n points.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If each one of the following graphs is the graph of a polynomial, then identify which one corresponds to a linear polynomial and which one corresponds to a quadratic polynomial?*

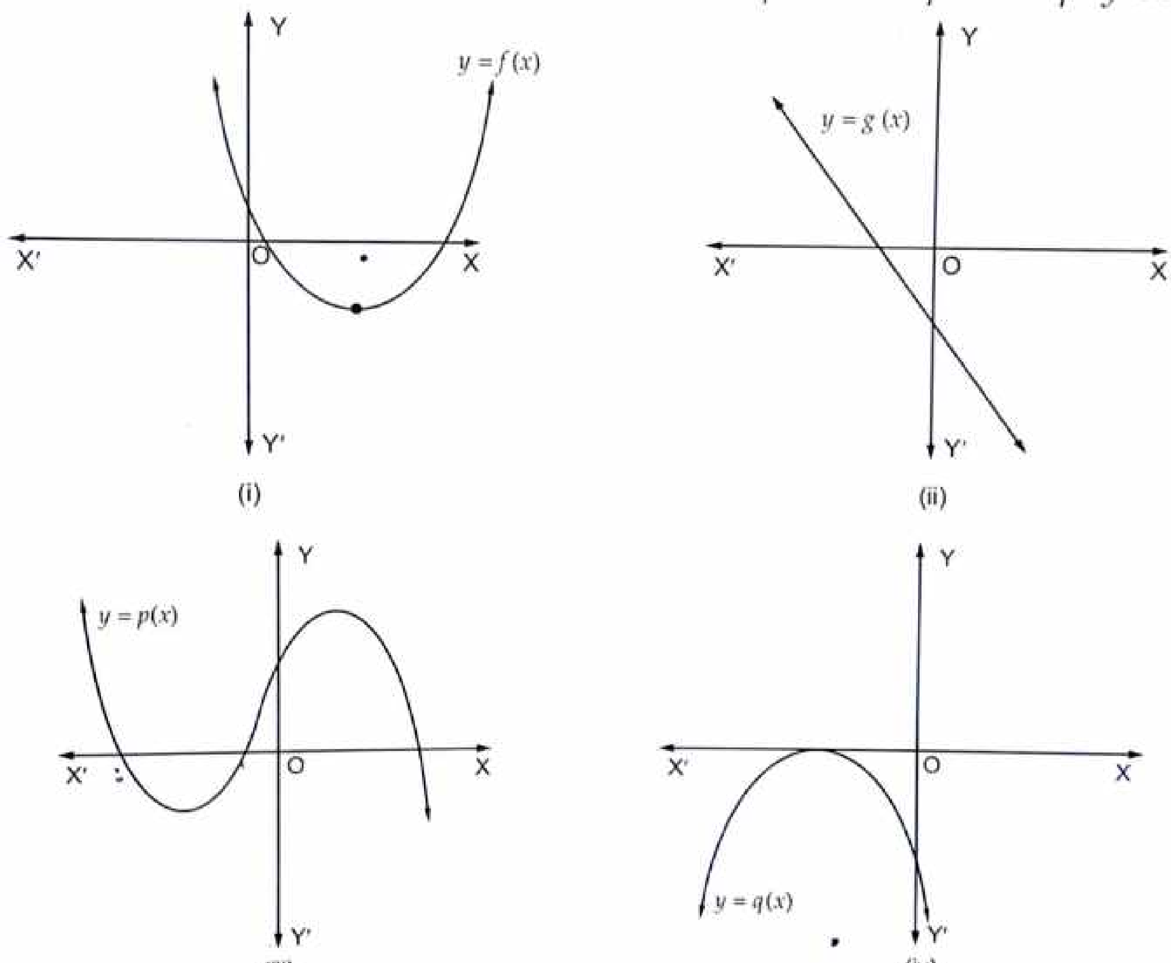


Fig. 2.15

SOLUTION (i) We observe that the graph $y = f(x)$ is a parabola opening upwards. Therefore, $f(x)$ is a quadratic polynomial in which coefficient of x^2 is positive.

(ii) We find that the graph $y = g(x)$ is a straight line. So, $g(x)$ is a linear polynomial.

(iii) Here, $p(x)$ is neither linear nor quadratic.

(iv) Here, $q(x)$ is a quadratic polynomial in which coefficient of x^2 is negative because the graph is a parabola opening downwards.

LEVEL-2

EXAMPLE 2 *The graphs of $y = ax^2 + bx + c$ are given in Fig. 2.16. Identify the signs of a, b and c in each of the following:*

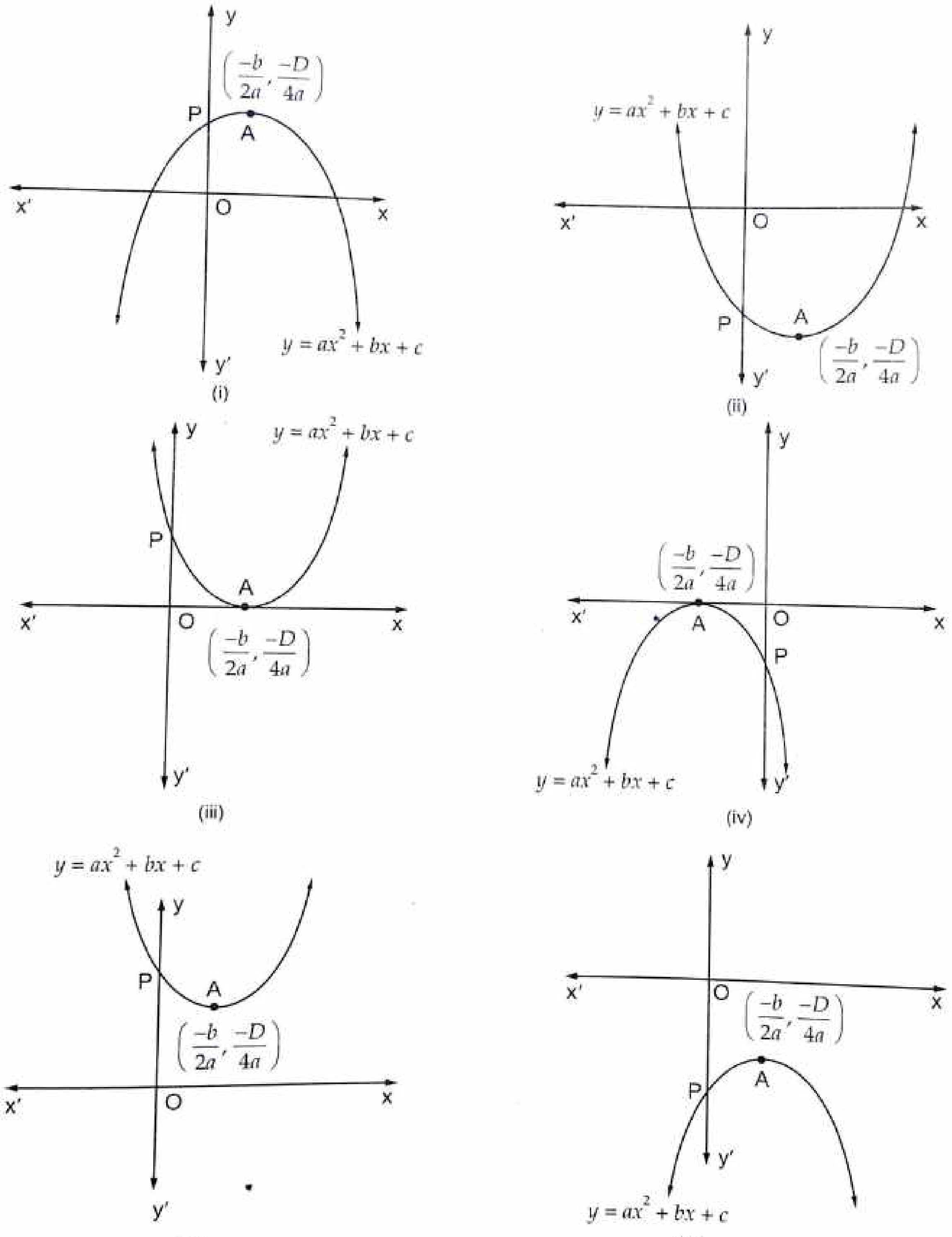


Fig. 2.16

SOLUTION (i) We observe that $y = ax^2 + bx + c$ represents a parabola opening downwards. Therefore, $a < 0$. We also observe that the vertex of the parabola is in first quadrant.

$$\therefore \quad -\frac{b}{2a} > 0 \Rightarrow -b < 0 \Rightarrow b > 0 \qquad [\because a < 0]$$

Parabola $y = ax^2 + bx + c$ cuts y-axis at P. On y-axis, we have $x = 0$. Putting $x = 0$ in $y = ax^2 + bx + c$, we get $y = c$.

So, the coordinates of P are $(0, c)$. As P lies on the positive direction of y-axis. Therefore, $c > 0$. Hence, $a < 0$, $b > 0$ and $c > 0$.

(ii) We find that $y = ax^2 + bx + c$ represents a parabola opening upwards. Therefore, $a > 0$. The vertex of the parabola is in fourth quadrant.

$$\therefore \quad \frac{-b}{2a} > 0 \Rightarrow -b > 0 \Rightarrow b < 0. \qquad [\because a > 0]$$

Parabola $y = ax^2 + bx + c$ cuts y-axis at P and on y-axis. We have $x = 0$. Therefore, on putting $x = 0$ in $y = ax^2 + bx + c$, we get $y = c$. So, the coordinates of P are $(0, c)$. As P lies on OY'. Therefore, $c < 0$. Hence, $a > 0$, $b < 0$ and $c < 0$.

(iii) Clearly, $y = ax^2 + bx + c$ represents a parabola opening upwards. Therefore, $a > 0$. The vertex of the parabola lies on OX.

$$\therefore \quad \frac{-b}{2a} > 0 \Rightarrow -b > 0 \Rightarrow b < 0 \qquad [\because a > 0]$$

The parabola $y = ax^2 + bx + c$ cuts y-axis at P which lies on OY. Putting $x = 0$ in $y = ax^2 + bx + c$, we get $y = c$. So, the coordinates of P are $(0, c)$. Clearly, P lies on OY. Therefore, $c > 0$. Hence, $a > 0$, $b < 0$, and $c > 0$.

(iv) The parabola $y = ax^2 + bx + c$ opens downwards. Therefore, $a < 0$. The vertex $(-b/2a, \ -D/4a)$ of the parabola is on OX'.

$$\therefore \quad \frac{-b}{2a} < 0 \Rightarrow b < 0 \qquad [\because a < 0]$$

Parabola $y = ax^2 + bx + c$ cuts y-axis at $P(0, c)$ which lies on OY'. Therefore, $c < 0$. Hence, $a < 0$, $b < 0$ and $c < 0$.

(v) We notice that the parabola $y = ax^2 + bx + c$ opens upwards. Therefore, $a > 0$. The vertex $(-b/2a, \ -D/4a)$ of the parabola lies in the first quadrant.

$$\therefore \quad \frac{-b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow b < 0 \qquad [\because a > 0]$$

As $P(0, c)$ lies on OY. Therefore, $c > 0$. Hence, $a > 0$, $b < 0$ and $c > 0$.

(vi) Clearly, $a < 0$. As $(-b/2a, \ -D/4a)$ lies in the fourth quadrant.

$$\therefore \quad \frac{-b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow b > 0 \qquad [\because a < 0]$$

As $P(0, c)$ lies on OY'. Therefore, $c < 0$. Hence, $a < 0$, $b > 0$ and $c < 0$.

2.4 GEOMETRICAL MEANING OF THE ZEROS OF A POLYNOMIAL

In the previous section, we have seen that the graph of a linear polynomial is a straight line and it cuts x-axis at exactly one point. The graph of a quadratic polynomial is a parabola which cuts x-axis at atmost two points. We have also seen that the graph of a cubic polynomial cuts x-axis at atmost 3 points. In general, the graph of an nth degree polynomial crosses x-axis at atmost n points. Also, x-coordinates of these points are the zeros of the polynomial. Thus, geometrically zeros of a polynomial are the x-coordinates of the points where its graph crosses or touches x-axis.

It follows from the above discussion that an nth degree polynomial can have at most n real zeros. That is the number of real zeros of a polynomial is less than or equal to the degree of the polynomial. In higher classes, we will study that the total number of zeros (real or imaginary) of an nth degree polynomial is exactly n.

2.5 RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS OF A POLYNOMIAL

In class IX, we have learnt about the factorization of polynomials. In the previous section, we have studied that a polynomial of degree n has exactly n zeros (real or imaginary). The zeros of a polynomial are closely connected to its coefficients. In this section, we will find out relationship between the zeros and coefficients of a polynomial.

Consider the quadratic polynomial $f(x) = 6x^2 - x - 2$. By the method of splitting the middle term, we obtain

$$f(x) = 6x^2 - x - 2 = 6x^2 - 4x + 3x - 2 = 2x(3x-2) + 1(3x-2) = (3x-2)(2x+1)$$

$\therefore \quad f(x) = 0$

$\Rightarrow \quad (3x-2)(2x+1) = 0 \Rightarrow 3x - 2 = 0$ or, $2x + 1 = 0 \Rightarrow x = \frac{2}{3}$ or, $x = -\frac{1}{2}$

Hence, the zeros of $6x^2 - x - 2$ are $\alpha = \frac{2}{3}$ and $\beta = -\frac{1}{2}$.

We observe that

$$\text{Sum of its zeros} = \alpha + \beta = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} = \frac{-(-1)}{6} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

Product of its zeros

Let us now consider a cubic polynomial $p(x)$ given by

$p(x) = 6x^3 + 5x^2 - 12x + 4$

$\Rightarrow \quad p(x) = (x+2)(2x-1)(3x-2)$ [By using factorization]

$\therefore \quad p(x) = 0$

$\Rightarrow \quad (x+2)(2x-1)(3x-2) = 0$

$\Rightarrow \quad x = -2, \frac{1}{2}, \frac{2}{3}$

Hence, the zeros of $p(x) = 6x^3 + 5x^2 - 12x + 4$ are $\alpha = -2, \beta = \frac{1}{2}$ and $\gamma = \frac{2}{3}$.

Also, $p(x)$ is a cubic polynomial. So, $p(x)$ can have atmost three real zeros.

Now,

$$\text{Sum of the zeros} = \alpha + \beta + \gamma = -2 + \frac{1}{2} + \frac{2}{3} = -\frac{5}{6} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\text{Sum of the products of zeros taken two at a time} = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$= (-2) \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} + \frac{2}{3} \times (-2)$$

$$= -1 + \frac{1}{3} - \frac{4}{3} = -2$$

$$= \frac{-12}{6} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\text{Product of all zeros} = \alpha\beta\gamma = (-2) \times \frac{1}{2} \times \frac{2}{3} = -\frac{2}{3} = -\frac{4}{6} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Let us now find a formal relation between zeros and coefficients of a polynomial.

2.5.1 RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS OF A QUADRATIC POLYNOMIAL

Let α and β be the zeros of a quadratic polynomial $f(x) = ax^2 + bx + c$. By factor theorem $(x - \alpha)$ and $(x - \beta)$ are the factors of $f(x)$.

$\therefore \quad f(x) = k\,(x - \alpha)(x - \beta)$, where k is a constant

$\Rightarrow \quad ax^2 + bx + c = k\left\{x^2 - (\alpha + \beta)\,x + \alpha\beta\right\}$

$\Rightarrow \quad ax^2 + bx + c = kx^2 - k\,(\alpha + \beta)\,x + k\alpha\beta$

Comparing the coefficients of x^2, x and constant terms on both sides, we get

$$a = k,\ b = -k\,(\alpha + \beta) \text{ and } c = k\alpha\beta$$

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\Rightarrow \quad \alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and, } \alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence,

$$\text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}, \text{Product of the zeros} = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

REMARK *If α and β are the zeros of a quadratic polynomial $f(x)$. Then, the polynomial $f(x)$ is given by*

$$f(x) = k\,\{x^2 - (\alpha + \beta)\,x + \alpha\beta\}$$

or, $\quad f(x) = k\left\{x^2 - (\text{Sum of the zeros})\,x + \text{Product of the zeros}\right\}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON VERIFYING THE RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS

EXAMPLE 1 *Find the zeros of the quadratic polynomial $x^2 + 7x + 12$, and verify the relation between the zeros and its coefficients.*

SOLUTION We have,

$$f(x) = x^2 + 7x + 12 = x^2 + 4x + 3x + 12 = x(x+4) + 3(x+4) = (x+4)(x+3)$$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $f(x) = 0$

$\Rightarrow \quad x^2 + 7x + 12 = 0$

$\Rightarrow \quad (x+4)(x+3) = 0$

$\Rightarrow \quad x + 4 = 0$ or, $x + 3 = 0$

$\Rightarrow \quad x = -4$ or, $x = -3$.

Thus, the zeros of $f(x) = x^2 + 7x + 12$ are $\alpha = -4$ and $\beta = -3$.

Now,

$$\text{Sum of the zeros} = \alpha + \beta = (-4) + (-3) = -7 \text{ and, } -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{7}{1} = -7$$

$$\therefore \quad \text{Sum of the zeros} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of the zeros} = \alpha\beta = (-4) \times (-3) = 12 \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{12}{1} = 12$$

$$\therefore \quad \text{Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

EXAMPLE 2 *Find the zeros of the quadratic polynomial $f(x) = 6x^2 - 3$, and verify the relationship between the zeros and its coefficients:*

SOLUTION We have,

$$f(x) = 6x^2 - 3$$

$\Rightarrow \quad f(x) = (\sqrt{6}x)^2 - (\sqrt{3})^2$

$\Rightarrow \quad f(x) = (\sqrt{6}x - \sqrt{3})(\sqrt{6}x + \sqrt{3})$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $f(x) = 0$

$\Rightarrow \quad (\sqrt{6}x - \sqrt{3})(\sqrt{6}x + \sqrt{3}) = 0$

$\Rightarrow \quad \sqrt{6}x - \sqrt{3} = 0$ or, $\sqrt{6}x + \sqrt{3} = 0$

$\Rightarrow \quad x = \dfrac{\sqrt{3}}{\sqrt{6}}$ or, $x = \dfrac{-\sqrt{3}}{\sqrt{6}}$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$ or, $x = -\frac{1}{\sqrt{2}}$

Hence, the zeros of $f(x) = 6x^2 - 3$ are: $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$.

Now,

$$\text{Sum of the zeros} = \alpha + \beta = \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = 0 \text{ and, } -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{0}{6} = 0$$

$$\therefore \quad \text{Sum of the zeros} = -\frac{\text{Coefficient of } x}{\text{coefficient of } x^2}$$

Also,

$$\text{Product of the zeros} = \alpha\beta = \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = \frac{-1}{2} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-3}{6} = \frac{-1}{2}$$

$$\therefore \quad \text{Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

EXAMPLE 3 *Find the zeros of the polynomial $f(u) = 4u^2 + 8u$, and verify the relationship between the zeros and its coefficients.* [NCERT]

SOLUTION We have,

$$f(u) = 4u^2 + 8u$$

$$\Rightarrow \quad f(u) = 4u(u+2)$$

The zeros of $f(u)$ are given by $f(u) = 0$.

Now, $\quad f(u) = 0$

$\Rightarrow \quad 4u(u+2) = 0$

$\Rightarrow \quad u = 0$ or, $u + 2 = 0$

$\Rightarrow \quad u = 0$ or, $u = -2$

Hence, the zeros of $f(u)$ are: $\alpha = 0$ and $\beta = -2$

Now,

$$\alpha + \beta = 0 + (-2) = -2 \text{ and } \alpha\beta = 0 \times -2 = 0$$

Also, $\quad -\frac{\text{Coefficient of } u}{\text{Coefficient of } u^2} = -\frac{8}{4} = -2$ and, $\frac{\text{Constant term}}{\text{Coefficient of } u^2} = \frac{0}{4} = 0$

$$\therefore \quad \text{Sum of the zeros} = -\frac{\text{Coefficient of } u}{\text{Coefficient of } u^2} \text{ and, Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } u^2}$$

EXAMPLE 4 *Find the zeros of the polynomial $f(x) = 4\sqrt{3}x^2 + 5x - 2\sqrt{3}$, and verify the relationship between the zeros and its coefficients.*

SOLUTION We have,

$$f(x) = 4\sqrt{3}x^2 + 5x - 2\sqrt{3}$$

$\Rightarrow \quad f(x) = 4\sqrt{3}\,x^2 + 8x - 3x - 2\sqrt{3}$ [Splitting the middle term]

$\Rightarrow \quad f(x) = 4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2)$

$\Rightarrow \quad f(x) = (\sqrt{3}x + 2)(4x - \sqrt{3})$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $\quad f(x) = 0$

$\Rightarrow \quad (\sqrt{3}x + 2)(4x - \sqrt{3}) = 0$

$\Rightarrow \quad \sqrt{3}x + 2 = 0$ or, $4x - \sqrt{3} = 0$

$\Rightarrow \quad x = -\frac{2}{\sqrt{3}}$ or, $x = \frac{\sqrt{3}}{4}$

Hence, the zeros of $f(x)$ are: $\alpha = -\frac{2}{\sqrt{3}}$ and $\beta = \frac{\sqrt{3}}{4}$

Now,

$$\alpha + \beta = -\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{4} = \frac{-8+3}{4\sqrt{3}} = -\frac{5}{4\sqrt{3}} \text{ and, } \alpha\beta = \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{4} = -\frac{1}{2}$$

Also,

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{5}{4\sqrt{3}} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-2\sqrt{3}}{4\sqrt{3}} = -\frac{1}{2}$$

Hence, Sum of the roots $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ and, Prodcut of the roots $= \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

EXAMPLE 5 *Find the zeros of the quadratic polynomial $f(x) = abx^2 + (b^2 - ac)x - bc$, and verify the relationship between the zeros and its coefficients.*

SOLUTION We have,

$f(x) = abx^2 + (b^2 - ac)x - bc$

$\Rightarrow \quad f(x) = abx^2 + b^2x - acx - bc$

$\Rightarrow \quad f(x) = bx(ax + b) - c(ax + b)$

$\Rightarrow \quad f(x) = (ax + b)(bx - c)$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $\quad f(x) = 0$

$\Rightarrow \quad (ax + b)(bx - c) = 0$

$\Rightarrow \quad ax + b = 0$ or, $bx - c = 0$

$\Rightarrow \quad x = -\frac{b}{a}$ or, $x = \frac{c}{b}$

Thus, the zeros of $f(x)$ are: $\alpha = -\frac{b}{a}$ and, $\beta = \frac{c}{b}$.

Now,

$$\alpha + \beta = -\frac{b}{a} + \frac{c}{b} = \frac{ac - b^2}{ab} \text{ and, } \alpha\beta = -\frac{b}{a} \times \frac{c}{b} = -\frac{c}{a}$$

Also,

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\left(\frac{b^2 - ac}{ab}\right) = \frac{ac - b^2}{ab} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = -\frac{bc}{ab} = -\frac{c}{a}$$

Hence,

$$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and, Product of the zeros} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

EXAMPLE 6 *Find the zeros of the polynomial* $x^2 + \frac{1}{6}x - 2$, *and verify the relation between the coefficients and zeros of the polynomial.* **[NCERT EXEMPLAR]**

SOLUTION Let $f(x) = x^2 + \frac{1}{6}x - 2$. Then,

$$f(x) = \frac{1}{6}(6x^2 + x - 12) = \frac{1}{6}(6x^2 + 9x - 8x - 12)$$

$$\Rightarrow \quad f(x) = \frac{1}{6}\{(6x^2 + 9x) - (8x + 12)\} = \frac{1}{6}\{3x(2x + 3) - 4(2x + 3)\} = \frac{1}{6}(2x + 3)(3x - 4)$$

The zeros of $f(x)$ are given by $f(x) = 0$.

Now, $f(x) = 0 \Rightarrow \frac{1}{6}(2x + 3)(3x - 4) = 0 \Rightarrow 2x + 3 = 0$ or, $3x - 4 = 0 \Rightarrow x = \frac{-3}{2}$ or, $x = \frac{4}{3}$

Hence, $\alpha = \frac{-3}{2}$ and $\beta = \frac{4}{3}$ are the zeros of the given polynomial.

Now,

$$\alpha + \beta = \left(-\frac{3}{2}\right) + \frac{4}{3} = -\frac{1}{6} \text{ and, } \alpha\beta = \left(\frac{-3}{2}\right)\left(\frac{4}{3}\right) = -2$$

The given polynomial is $f(x) = x^2 + \frac{1}{6}x - 2$.

$$\therefore \quad -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = \left(\frac{-1/6}{1}\right) = \frac{-1}{6} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-2}{1} = -2$$

Clearly,

$$\alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \text{ and, } \alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence, the relation between the coefficients and zeros is verified.

Type II ON FINDING THE VALUES OF SYMMETRIC EXPRESSIONS INVOLVING ZEROS OF A QUADRATIC POLYNOMIAL

EXAMPLE 7 *If* α *and* β *are the zeros of the quadratic polynomial* $f(x) = x^2 - px + q$, *then find the values of (i)* $\alpha^2 + \beta^2$ *(ii)* $\frac{1}{\alpha} + \frac{1}{\beta}$

SOLUTION It is given that α and β are the zero of the polynomial $f(x) = x^2 - px + q$.

$\therefore \qquad \alpha + \beta = -\left(\dfrac{-p}{1}\right) = p$ and, $\alpha\beta = \dfrac{q}{1} = q$

(i) $\qquad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q \qquad [\because \alpha + \beta = p \text{ and } \alpha\beta = q]$

(ii) $\qquad \dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{p}{q}$

Type III ON FINDING A QUADRATIC POLYNOMIAL WHEN THE SUM AND PRODUCT OF ITS ZEROES ARE GIVEN

EXAMPLE 8 *Find a quadratic polynomial each with the given numbers as the sum and product of its zeros respectively*

(i) $\dfrac{1}{4}, -1$ [NCERT] (ii) $\sqrt{2}, \dfrac{1}{3}$ [NCERT] (iii) $0, \sqrt{5}$ [NCERT]

SOLUTION We know that a quadratic polynomial when the sum and product of its zeros are given by

$f(x) = k\left\{x^2 - \text{(Sum of the zeros)}\,x + \text{Product of the zeros}\right\}$, where k is a constant.

(i) We have, Sum $= \dfrac{1}{4}$ and, Product $= -1$. So, required quadratic polynomial $f(x)$ is given by

$$f(x) = k\left(x^2 - \frac{1}{4}x - 1\right)$$

(ii) We have, Sum $= \sqrt{2}$ and, Product $= \dfrac{1}{3}$. So, required quadratic polynomial $f(x)$ is given by

$$f(x) = k\left(x^2 - \sqrt{2}x + \frac{1}{3}\right)$$

(iii) We have, Sum $= 0$ and, Product $= \sqrt{5}$. So, required quadratic polynomial $f(x)$ is given by

$$f(x) = k\,(x^2 - 0x + \sqrt{5}) = k\,(x^2 + \sqrt{5})$$

EXAMPLE 9 *Find a quadratic polynomial, the sum and product of whose zeroes are* $\sqrt{2}$ *and* $-\dfrac{3}{2}$ *respectively. Also, find its zeroes.* [NCERT EXEMPLAR]

SOLUTION Let α, β be the zeros of required polynomial. It is given that $\alpha + \beta = \sqrt{2}$ and $\alpha\beta = -\dfrac{3}{2}$.

The quadratic polynomial is $f(x) = x^2 - (\alpha + \beta)\,x + \alpha\beta$ or, $f(x) = x^2 - \sqrt{2}\,x - \dfrac{3}{2}$

Now, $\qquad f(x) = x^2 - \sqrt{2}x - \dfrac{3}{2}$

$\Rightarrow \qquad f(x) = \dfrac{1}{2}(2x^2 - 2\sqrt{2}x - 3)$

$\Rightarrow \qquad f(x) = \dfrac{1}{2}(2x^2 - 3\sqrt{2}x + \sqrt{2}x - 3)$

$$\Rightarrow \qquad f(x) = \frac{1}{2}\{\sqrt{2}x(\sqrt{2}x - 3) + (\sqrt{2}x - 3)\}$$

$$\Rightarrow \qquad f(x) = \frac{1}{2}(\sqrt{2}x - 3)(\sqrt{2}x + 1)$$

The zeroes of $f(x)$ are given by $f(x) = 0$.

Now, $\quad f(x) = 0$

$$\Rightarrow \quad \frac{1}{2}(\sqrt{2}x - 3)(\sqrt{2}x + 1) = 0 \Rightarrow \sqrt{2}x - 3 = 0 \text{ or, } \sqrt{2}x + 1 = 0 \Rightarrow x = \frac{3}{\sqrt{2}} \text{ or, } x = -\frac{1}{\sqrt{2}}$$

Hence, the zeroes of $f(x)$ are $\frac{3}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$.

EXAMPLE 10 *If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - x - 2$, find a polynomial whose zeros are $2\alpha + 1$ and $2\beta + 1$.*

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = x^2 - x - 2$.

$$\therefore \qquad \alpha + \beta = -\left(-\frac{1}{1}\right) = 1 \text{ and, } \alpha\beta = -\frac{2}{1} = -2$$

Let S and P denote respectively the sum and the product of zeros of the required polynomial. Then,

$$S = (2\alpha + 1) + (2\beta + 1) = 2(\alpha + \beta) + 2 = 2 \times 1 + 2 = 4 \qquad [\because \alpha + \beta = 1]$$

and, $\quad P = (2\alpha + 1)(2\beta + 1) = 4\alpha\beta + 2\alpha + 2\beta + 1 = 4\alpha\beta + 2(\alpha + \beta) + 1$

$$= 4 \times -2 + 2 \times 1 + 1 = -8 + 2 + 1 = -5 \qquad [\because \alpha + \beta = 1 \text{ and } \alpha\beta = -2]$$

Hence, required polynomial $g(x)$ is

$g(x) = k\{x^2 - Sx + P\} = k(x^2 - 4x - 5)$, where k is any non-zero constant.

LEVEL-2

Type IV ON FINDING THE VALUES OF SYMMETRIC EXPRESSIONS INVOLVING ZEROES OF A QUADRATIC POLYNOMIAL

EXAMPLE 11 *If α and β are the zeros of the quadratic polynomial $f(x) = ax^2 + bx + c$, then evaluate:*

(i) $\alpha^2 + \beta^2$ (ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (iii) $\alpha^3 + \beta^3$

(iv) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ (v) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x) = ax^2 + bx + c$.

$$\therefore \qquad \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

(i) We know that

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\therefore \qquad \alpha^2 + \beta^2 = \left(\frac{-b}{a}\right)^2 - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

(ii) $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\frac{c}{a}}=\frac{b^2-2ac}{ac}$$

(iii) We know that $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta\,(\alpha+\beta)$

$$\therefore \quad \alpha^3+\beta^3=\left(\frac{-b}{a}\right)^3-3\frac{c}{a}\left(\frac{-b}{a}\right)=\frac{-b^3}{a^3}+\frac{3bc}{a^2}=\frac{-b^3+3abc}{a^3}=\frac{3abc-b^3}{a^3}$$

(iv) $$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{(\alpha\beta)^3}=\frac{\frac{3abc-b^3}{a^3}}{\left(\frac{c}{a}\right)^3}=\frac{3abc-b^3}{c^3}$$ [Using (iii)]

(v) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta\,(\alpha+\beta)}{\alpha\beta}=\frac{\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{\frac{c}{a}}=\frac{3abc-b^3}{a^2c}$$

EXAMPLE 12 *If α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$, then evaluate:*

(i) $\alpha^4+\beta^4$ *(ii)* $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}$

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$.

$$\therefore \quad \alpha+\beta=-\frac{b}{a} \text{ and, } \alpha\beta=\frac{c}{a}$$

(i) We have,

$$\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\,\alpha^2\beta^2$$

$$\Rightarrow \quad \alpha^4+\beta^4=\left\{(\alpha+\beta)^2-2\alpha\beta\right\}^2-2\,(\alpha\beta)^2$$

$$\Rightarrow \quad \alpha^4+\beta^4=\left\{\left(-\frac{b}{a}\right)^2-2\frac{c}{a}\right\}^2-2\left(\frac{c}{a}\right)^2 \qquad \left[\because \alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}\right]$$

$$\Rightarrow \quad \alpha^4+\beta^4=\left(\frac{b^2-2ac}{a^2}\right)^2-\frac{2c^2}{a^2}=\frac{(b^2-2ac)^2-2a^2c^2}{a^4}$$

(ii) We have,

$$\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(b^2-2ac)^2-2a^2c^2}{a^4\times\left(\frac{c}{a}\right)^2}$$ [Using (i)]

$$=\frac{(b^2-2ac)^2-2\,a^2\,c^2}{a^2c^2}$$

Type V ON FINDING AN UNKNOWN WHEN A RELATION BETWEEN ZEROS AND COEFFICIENTS IS GIVEN

EXAMPLE 13 *If α and β are the zeros of the polynomial $f(x) = x^2 - 5x + k$ such that $\alpha - \beta = 1$, find the value of k.*

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = x^2 - 5x + k$.

$\therefore \quad \alpha + \beta = -\left(\frac{-5}{1}\right) = 5$ and, $\alpha\beta = \frac{k}{1} = k$

Now, $\quad \alpha - \beta = 1$ [Given]

$\Rightarrow \quad (\alpha - \beta)^2 = 1$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow \quad 25 - 4k = 1 \Rightarrow 24 = 4k \Rightarrow k = 6$

Hence, the value of k is 6.

EXAMPLE 14 *If α and β are the zeros of the quadratic polynomial $f(x) = kx^2 + 4x + 4$ such that $\alpha^2 + \beta^2 = 24$, find the values of k.*

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x) = kx^2 + 4x + 4$

$\therefore \quad \alpha + \beta = -\frac{4}{k}$ and, $\alpha\beta = \frac{4}{k}$

We have,

$\alpha^2 + \beta^2 = 24$

$\Rightarrow \quad (\alpha + \beta)^2 - 2\alpha\beta = 24$ $\quad [\because \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta]$

$\Rightarrow \quad \left(-\frac{4}{k}\right)^2 - 2 \times \frac{4}{k} = 24$

$\Rightarrow \quad \frac{16}{k^2} - \frac{8}{k} = 24$

$\Rightarrow \quad 16 - 8k = 24k^2$

$\Rightarrow \quad 3k^2 + k - 2 = 0$

$\Rightarrow \quad 3k^2 + 3k - 2k - 2 = 0$

$\Rightarrow \quad 3k(k + 1) - 2(k + 1) = 0$

$\Rightarrow \quad (k + 1)(3k - 2) = 0 \Rightarrow k + 1 = 0$ or, $3k - 2 = 0 \Rightarrow k = -1$ or, $k = \frac{2}{3}$

Hence, $\quad k = -1$ or, $k = \frac{2}{3}$

EXAMPLE 15 *If α, β are the zeros of the polynomial $f(x) = 2x^2 + 5x + k$ satisfying the relation $\alpha^2 + \beta^2 + \alpha\beta = \frac{21}{4}$, then find the value of k for this to be possible.*

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = 2x^2 + 5x + k$.

$\therefore \quad \alpha + \beta = -\frac{5}{2}$ and, $\alpha\beta = \frac{k}{2}$

We have,

$$\alpha^2 + \beta^2 + \alpha\beta = \frac{21}{4}$$

$$\Rightarrow \quad (\alpha^2 + \beta^2 + 2\alpha\beta) - \alpha\beta = \frac{21}{4}$$

$$\Rightarrow \quad (\alpha + \beta)^2 - \alpha\beta = \frac{21}{4}$$

$$\Rightarrow \quad \frac{25}{4} - \frac{k}{2} = \frac{21}{4} \qquad \left[\because \alpha + \beta = -\frac{5}{2} \text{ and, } \alpha\beta = \frac{k}{2}\right]$$

$$\Rightarrow \quad -\frac{k}{2} = -1 \Rightarrow k = 2$$

EXAMPLE 16 *If sum of the squares of zeros of the quadratic polynomial* $f(x) = x^2 - 8x + k$ *is 40, find the value of k.*

SOLUTION Let α, β be the zeros of the polynomial $f(x) = x^2 - 8x + k$. Then,

$$\alpha + \beta = -\left(\frac{-8}{1}\right) = 8 \text{ and, } \alpha\beta = \frac{k}{1} = k$$

It is given that

$$\alpha^2 + \beta^2 = 40$$

$$\Rightarrow \quad (\alpha + \beta)^2 - 2\alpha\beta = 40$$

$$\Rightarrow \quad 8^2 - 2k = 40 \qquad [\because \alpha + \beta = 8 \text{ and } \alpha\beta = k]$$

$$\Rightarrow \quad 2k = 64 - 40 \Rightarrow 2k = 24 \Rightarrow k = 12$$

Type VI ON FINDING A QUADRATIC POLYNOMIAL WHEN THE SUM AND PRODUCT OF ITS ZEROS ARE GIVEN

EXAMPLE 17 *If* α *and* β *are the zeros of the quadratic polynomial* $f(x) = 2x^2 - 5x + 7$, *find a polynomial whose zeros are* $2\alpha + 3\beta$ *and* $3\alpha + 2\beta$.

SOLUTION It is given that α and β are the zeros of the quadratic polynomial $f(x) = 2x^2 - 5x + 7$.

$$\therefore \quad \alpha + \beta = -\left(-\frac{5}{2}\right) = \frac{5}{2} \text{ and, } \alpha\beta = \frac{7}{2}$$

Let S and P denote respectively the sum and product of zeros of the required polynomial. Then,

$$S = (2\alpha + 3\beta) + (3\alpha + 2\beta) = 5(\alpha + \beta) = 5 \times \frac{5}{2} = \frac{25}{2}$$

and,

$$P = (2\alpha + 3\beta)(3\alpha + 2\beta) = 6(\alpha^2 + \beta^2) + 13\alpha\beta = 6\alpha^2 + 6\beta^2 + 12\alpha\beta + \alpha\beta$$

$$= 6(\alpha + \beta)^2 + \alpha\beta = 6 \times \left(\frac{5}{2}\right)^2 + \frac{7}{2} = \frac{75}{2} + \frac{7}{2} = 41$$

Hence, the required polynomial $g(x)$ is given by

$g(x) = k(x^2 - Sx + P)$ or, $g(x) = k\left(x^2 - \frac{25}{2}x + 41\right)$, where k is any non-zero real number.

EXAMPLE 18 *If α and β are the zeros of the quadratic polynomial $f(x) = 3x^2 - 4x + 1$, find a quadratic polynomial whose zeros are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$.*

SOLUTION It is given that α and β are the zeros of the polynomial $f(x) = 3x^2 - 4x + 1$.

$$\therefore \quad \alpha + \beta = -\left(-\frac{4}{3}\right) = \frac{4}{3} \text{ and, } \alpha\beta = \frac{1}{3}$$

Let S and P denote respectively the sum and product of the zeros of the polynomial whose zeros are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$. Then,

$$S = \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{\left(\frac{4}{3}\right)^3 - 3 \times \frac{1}{3} \times \frac{4}{3}}{\frac{1}{3}} = \frac{28}{9}$$

and, $$P = \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \alpha\beta = \frac{1}{3}$$

Hence, the required polynomial $g(x)$ is given by

$g(x) = k(x^2 - Sx + P)$ or, $g(x) = k\left(x^2 - \frac{28}{9}x + \frac{1}{3}\right)$, where k is any non-zero real number.

EXAMPLE 19 *Find a quadratic polynomial whose zeros are reciprocals of the zeros of the polynomial $f(x) = ax^2 + bx + c$, $a \neq 0$, $c \neq 0$.*

SOLUTION Let α and β be the zeros of the polynomial $f(x) = ax^2 + bx + c$. Then,

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Let S and P denote respectively the sum and product of the zeros of a polynomial whose zeros are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. Then,

$$S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c} \text{ and } P = \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$$

Hence, the required polynomial $g(x)$ is given by

$$g(x) = k(x^2 - Sx + P) = k\left(x^2 + \frac{bx}{c} + \frac{a}{c}\right), \text{ where } k \text{ is any non-zero constant.}$$

EXERCISE 2.1

LEVEL-1

1. Find the zeros of each of the following quadratic polynomials and verify the relationship between the zeros and their coefficients:
   (i) $f(x) = x^2 - 2x - 8$ [NCERT] (ii) $g(s) = 4s^2 - 4s + 1$ [NCERT]
   (iii) $h(t) = t^2 - 15$ [NCERT] (iv) $f(x) = 6x^2 - 3 - 7x$ [NCERT]
   (v) $p(x) = x^2 + 2\sqrt{2}x - 6$ (vi) $q(x) = \sqrt{3}x^2 + 10x + 7\sqrt{3}$

(vii) $f(x) = x^2 - (\sqrt{3} + 1)x + \sqrt{3}$ (viii) $g(x) = a(x^2 + 1) - x(a^2 + 1)$

(ix) $h(s) = 2s^2 - (1 + 2\sqrt{2})s + \sqrt{2}$ [NCERT EXEMPLAR]

(x) $f(v) = v^2 + 4\sqrt{3}v - 15$ [NCERT EXEMPLAR]

(xi) $p(y) = y^2 + \frac{3\sqrt{5}}{2}y - 5$ [NCERT EXEMPLAR]

(xii) $q(y) = 7y^2 - \frac{11}{3}y - \frac{2}{3}$ [NCERT EXEMPLAR]

2. For each of the following, find a quadratic polynomial whose sum and product respectively of the zeroes are as given. Also, find the zeroes of these polynomials by factorization.

(i) $-\frac{8}{3}, \frac{4}{3}$ (ii) $\frac{21}{8}, \frac{5}{16}$ (iii) $-2\sqrt{3}, -9$ (iv) $\frac{-3}{2\sqrt{5}}, -\frac{1}{2}$

[NCERT EXEMPLAR]

3. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 5x + 4$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} - 2\alpha\beta$.

4. If α and β are the zeros of the quadratic polynomial $p(y) = 5y^2 - 7y + 1$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

5. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - x - 4$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} - \alpha\beta$.

6. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 + x - 2$, find the value of $\frac{1}{\alpha} - \frac{1}{\beta}$.

7. If one zero of the quadratic polynomial $f(x) = 4x^2 - 8kx - 9$ is negative of the other, find the value of k.

8. If the sum of the zeros of the quadratic polynomial $f(t) = kt^2 + 2t + 3k$ is equal to their product, find the value of k.

LEVEL-2

9. If α and β are the zeros of the quadratic polynomial $p(x) = 4x^2 - 5x - 1$, find the value of $\alpha^2\beta + \alpha\beta^2$.

10. If α and β are the zeros of the quadratic polynomial $f(t) = t^2 - 4t + 3$, find the value of $\alpha^4\beta^3 + \alpha^3\beta^4$.

11. If α and β are the zeros of the quadratic polynomial $f(x) = 6x^2 + x - 2$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

12. If α and β are the zeros of the quadratic polynomial $p(s) = 3s^2 - 6s + 4$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta$.

13. If the squared difference of the zeros of the quadratic polynomial $f(x) = x^2 + px + 45$ is equal to 144, find the value of p.

14. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - px + q$, prove that $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - \frac{4p^2}{q} + 2.$

15. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - p(x+1) - c$, show that $(\alpha + 1)(\beta + 1) = 1 - c$.

16. If α and β are the zeros of a quadratic polynomial such that $\alpha + \beta = 24$ and $\alpha - \beta = 8$, find a quadratic polynomial having α and β as its zeros.

17. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 1$, find a quadratic polynomial whose zeros are $\frac{2\alpha}{\beta}$ and $\frac{2\beta}{\alpha}$.

18. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 3x - 2$, find a quadratic polynomial whose zeros are $\frac{1}{2\alpha + \beta}$ and $\frac{1}{2\beta + \alpha}$.

19. If α and β are the zeros of the polynomial $f(x) = x^2 + px + q$, form a polynomial whose zeros are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$.

20. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - 2x + 3$, find a polynomial whose roots are (i) $\alpha + 2, \beta + 2$ (ii) $\frac{\alpha - 1}{\alpha + 1}, \frac{\beta - 1}{\beta + 1}$.

21. If α and β are the zeros of the quadratic polynomial $f(x) = ax^2 + bx + c$, then evaluate:

(i) $\alpha - \beta$ (ii) $\frac{1}{\alpha} - \frac{1}{\beta}$ (iii) $\frac{1}{\alpha} + \frac{1}{\beta} - 2\alpha\beta$

(iv) $\alpha^2\beta + \alpha\beta^2$ (v) $\alpha^4 + \beta^4$ (vi) $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$

(vii) $\frac{\beta}{a\alpha + b} + \frac{\alpha}{a\beta + b}$ (viii) $a\left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right) + b\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$

ANSWERS

1. (i) $4, -2$ (ii) $\frac{1}{2}, \frac{1}{2}$ (iii) $\sqrt{15}, -\sqrt{15}$ (iv) $\frac{3}{2}, -\frac{1}{3}$

(v) $-3\sqrt{2}, \sqrt{2}$ (vi) $-\sqrt{3}, -\frac{7}{\sqrt{3}}$ (vii) $\sqrt{3}, 1$ (viii) $a, \frac{1}{a}$

(ix) $\sqrt{2}, \frac{1}{2}$ (x) $\sqrt{3}, -5\sqrt{3}$ (xi) $-2\sqrt{5}, \frac{\sqrt{5}}{2}$ (xii) $-\frac{1}{7}, \frac{2}{3}$

2. (i) $f(x) = k\left(x^2 + \frac{8}{3}x + \frac{4}{3}\right)$ (ii) $f(x) = k\left(x^2 - \frac{21}{8}x + \frac{5}{16}\right)$

(iii) $f(x) = k\left(x^2 + 2\sqrt{3}x - 9\right)$ (iv) $f(x) = k\left(x^2 + \frac{3}{2\sqrt{5}}x - \frac{1}{2}\right)$, where k is any constant

3. $-\frac{27}{4}$ **4.** 7 **5.** $\frac{15}{4}$ **6.** $-\frac{3}{2}$

7. 0 **8.** $-\frac{2}{3}$ **9.** $\frac{-5}{16}$ **10.** 108

11. $-\frac{25}{12}$ **12.** 8 **13.** ± 18

16. $f(x) = k(x^2 - 24x + 128)$ **17.** $f(x) = k(x^2 + 4x + 4)$

18. $f(x) = k\left(x^2 - \frac{9}{16}x + \frac{1}{16}\right)$ **19.** $f(x) = k\left\{x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q)\right\}$

20. (i) $f(x) = k(x^2 - 6x + 11)$ (ii) $f(x) = k\left\{x^2 - \frac{2}{3}x + \frac{1}{3}\right\}$

21. (i) $\frac{\sqrt{b^2 - 4ac}}{a}$ (ii) $\frac{\sqrt{b^2 - 4ac}}{c}$ (iii) $-\left(\frac{b}{c} + \frac{2c}{a}\right)$ (iv) $-\frac{bc}{a^2}$

(v) $\frac{(b^2 - 2ac)^2 - 2a^2c^2}{a^4}$ (vi) $\frac{b}{ac}$ (vii) $\frac{-2}{a}$ (viii) b

2.5.2 RELATIONSHIP BETWEEN ZEROS AND COEFFICIENTS OF A CUBIC POLYNOMIAL

Let α, β, γ be the zeros of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$. Then, by factor theorem, $x - \alpha, x - \beta$ and $x - \gamma$ are factors of $f(x)$. Also, $f(x)$, being a cubic polynomial, cannot have more than three linear factors.

$\therefore \quad f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$

$\Rightarrow \quad ax^3 + bx^2 + cx + d = k(x - \alpha)(x - \beta)(x - \gamma)$

$\Rightarrow \quad ax^3 + bx^2 + cx + d = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\}$

$\Rightarrow \quad ax^3 + bx^2 + cx + d = kx^3 - k(\alpha + \beta + \gamma)x^2 + k(\alpha\beta + \beta\gamma + \gamma\alpha)x - k\alpha\beta\gamma$

Comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$$a = k, b = -k(\alpha + \beta + \gamma), c = k(\alpha\beta + \beta\gamma + \gamma\alpha) \text{ and } d = -k(\alpha\beta\gamma)$$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\frac{b}{a}, \ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \text{ and, } \alpha\beta\gamma = -\frac{d}{a}$$

$$\Rightarrow \quad \text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\text{Sum of the products of the zeros taken two at a time} = \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\text{Product of the zeros} = -\frac{d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

REMARK 1 *It follows from the above discussion that a cubic polynomial having α, β and γ as its zeros is given by*

$$f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$$

or, $f(x) = k\left\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\right\}$, where k is any non-zero real number.

REMARK 2 *If* $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ *is a polynomial of degree 4 having* α, β, γ *and* δ *as its zeros, then*

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \alpha\gamma + \beta\delta = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

or, $$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

or, $$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^4}$$

OR

$$\text{Sum of the zeros} = -\frac{b}{a} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$\text{Sum of the products of the zeros taken two at a time} = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$\text{Sum of the products of the zeros taken three at a time} = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

$$\text{Product of the zeros} = \frac{e}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^4}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON VERIFYING THE RELATIONSHIP BETWEEN THE ZEROS AND COEFFICIENTS OF A POLYNOMIAL**

EXAMPLE 1 *Verify that* $3, -1$ *and* $-\frac{1}{3}$ *are the zeros of the cubic polynomial* $p(x) = 3x^3 - 5x^2 - 11x - 3$ *and then verify the relationship between the zeros and its coefficients.* [NCERT]

SOLUTION Given polynomial is $p(x) = 3x^3 - 5x^2 - 11x - 3$

$\therefore$ $$p(3) = 3 \times 3^3 - 5 \times 3^2 - 11 \times 3 - 3 = 81 - 45 - 33 - 3 = 0$$

$$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0$$

and, $$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3 = -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = 0$$

So, 3, -1 and $-\frac{1}{3}$ are the zeros of polynomial $p(x)$.

Let $\alpha = 3, \beta = -1$ and $\gamma = -\frac{1}{3}$. Then,

$$\alpha + \beta + \gamma = 3 - 1 - \frac{1}{3} = \frac{5}{3} \text{ and, } -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3} = -\left(\frac{-5}{3}\right) = \frac{5}{3}$$

$$\therefore \quad \alpha + \beta + \gamma = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = -\frac{11}{3}$$

and, $$\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = -\frac{11}{3}$$

$$\therefore \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 \text{ and, } -\frac{\text{Constant term}}{\text{Coefficient of } x^3} = -\left(\frac{-3}{3}\right) = 1$$

$$\therefore \quad \alpha\beta\gamma = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Type II **ON FINDING A CUBIC POLYNOMIAL WHEN SUM, SUM OF THE PRODUCTS OF ITS ZEROS TAKEN TWO AT A TIME, AND PRODUCT OF ITS ZEROS ARE GIVEN**

EXAMPLE 2 *Find a cubic polynomial with the sum, sum of the products of its zeros taken two at a time, and product of its zeros as 2, – 7, – 14 respectively.* [NCERT]

SOLUTION If α, β and γ are the zeros of a cubic polynomial $f(x)$, then

$$f(x) = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\},$$

where k is any non-zero real number.

Here, $\alpha + \beta + \gamma = 2, \alpha\beta + \beta\gamma + \gamma\alpha = -7$ and $\alpha\beta\gamma = -14$

$\therefore$ $f(x) = k(x^3 - 2x^2 - 7x + 14)$, where k is any non-zero real number.

Type III **ON FINDING THE ZEROS OF A CUBIC POLYNOMIAL**

EXAMPLE 3 *If two zeros of the polynomial* $f(x) = x^3 - 4x^2 - 3x + 12$ *are* $\sqrt{3}$ *and* $-\sqrt{3}$, *then find its third zero.* [CBSE 2010]

SOLUTION Let $\alpha = \sqrt{3}, \beta = -\sqrt{3}$ be the given zeros and γ be the third zero of $f(x)$. Then,

$$\alpha + \beta + \gamma = -\left(\frac{-4}{1}\right) \qquad \left[\text{Using } \alpha + \beta + \gamma = -\frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^3}\right]$$

$$\Rightarrow \quad \sqrt{3} - \sqrt{3} + \gamma = 4$$

$$\Rightarrow \quad \gamma = 4$$

Hence, third zero is 4.

EXAMPLE 4 *Find the zeros of the polynomial* $f(x) = x^3 - 5x^2 - 16x + 80$, *if its two zeros are equal in magnitude but opposite in sign.*

SOLUTION Let α, β, γ be the zeros of polynomial $f(x)$ such that $\alpha + \beta = 0$. Then,

$$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$\Rightarrow \quad \alpha + \beta + \gamma = -\left(-\frac{5}{1}\right)$

$\Rightarrow \quad 0 + \gamma = 5 \qquad [\because \alpha + \beta = 0]$

$\Rightarrow \quad \gamma = 5$

$$\text{Product of the zeros} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

$\Rightarrow \quad \alpha\beta\gamma = -\frac{80}{1}$

$\Rightarrow \quad 5\,\alpha\beta = -80 \qquad [\because \gamma = 5]$

$\Rightarrow \quad \alpha\beta = -16$

$\Rightarrow \quad -\alpha^2 = -16 \qquad [\because \alpha + \beta = 0 \therefore \beta = -\alpha]$

$\Rightarrow \quad \alpha = \pm 4$

CASE I *When* $\alpha = 4$: In this case,

$\alpha + \beta = 0 \Rightarrow 4 + \beta = 0 \Rightarrow \beta = -4$

So, the zeros are $\alpha = 4, \beta = -4$ and $\gamma = 5$

CASE II *When* $\alpha = -4$: In this case,

$\alpha + \beta = 0 \Rightarrow -4 + \beta = 0 \Rightarrow \beta = 4$

So, the zeros are $\alpha = -4, \beta = 4$ and $\gamma = 5$

Hence, in either case the zeros are 4, – 4 and 5.

EXAMPLE 5 *If the zeros of the polynomial* $f(x) = x^3 - 3x^2 + x + 1$ *are* $a - b$, a, $a + b$, *find* a *and* b. [NCERT]

SOLUTION It is given that $a - b$, a and $a + b$ are the zeros of $f(x)$.

$$\text{Now, Sum of the zeros} = -\frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^3}$$

$\Rightarrow \quad a - b + a + a + b = -\frac{-3}{1} \Rightarrow 3a = 3 \Rightarrow a = 1$

and, $\quad \text{Product of zeros} = -\dfrac{\text{Constant term}}{\text{Coeff. of } x^3}$

$\Rightarrow \quad (a - b)\,a(a + b) = -\frac{1}{1}$

$\Rightarrow \quad a(a^2 - b^2) = -1$

$\Rightarrow \quad 1 - b^2 = -1 \qquad [\because a = 1]$

$\Rightarrow \quad b^2 = 2 \Rightarrow b = \pm\sqrt{2}$

LEVEL-2

Type IV ON FINDING THE RELATIONSHIP BETWEEN THE COEFFICIENTS OF A POLYNOMIAL WHEN ITS ZEROS SATISFY CERTAIN RELATIONSHIP

EXAMPLE 6 *Given than the zeroes of the cubic polynomial* $f(x) = x^3 - 6x^2 + 3x + 10$ *are of the form* $a, a+b, a+2b$ *for some real numbers a and b, find the values of a and b as well as the zeros of the given polynomial.* [NCERT EXEMPLAR]

SOLUTION It is given that $a, a+b$, and $a+2b$ are zeros of $f(x) = x^3 - 6x^2 + 3x + 10$.

$\therefore$ Sum of the zeros $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

$\Rightarrow$ $a + (a+b) + (a+2b) = -\left(\dfrac{-6}{1}\right) = 6$ ...(i)

$\Rightarrow$ $3a + 3b = 6 \Rightarrow a + b = 2 \Rightarrow b = 2 - a$ ...(ii)

and, Product of the zeros $= -\dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

$\Rightarrow$ $a(a+b)(a+2b) = \dfrac{-10}{1}$

$\Rightarrow$ $a(a+b)(a+2b) = -10$ ...(iii)

$\Rightarrow$ $a \times 2 \times (a + 4 - 2a) = -10$ [From (ii) $b = 2 - a$]

$\Rightarrow$ $2a(4-a) = -10$

$\Rightarrow$ $a(4-a) = -5$

$\Rightarrow$ $4a - a^2 = -5 \Rightarrow a^2 - 4a - 5 = 0 \Rightarrow (a-5)(a+1) = 0 \Rightarrow a = 5, -1$

CASE I *When* $a = 5$: In this case,

$b = 2 - a \Rightarrow b = 2 - 5 = -3$

So, the roots are $a = 5, a + b = 2$ and $a + 2b = 5 - 6 = -1$

CASE II *When* $a = -1$: In this case,

$b = 2 - a \Rightarrow b = 2 + 1 = 3$

So, the roots are $a = -1, a+b = -1+3 = 2$ and $a + 2b = -1 + 6 = 5$

Hence, either $a = -1$ and $b = 3$ or, $a = 5$ and $b = -3$

In either case, the zeros of the polynomial are 5, 2 and –1.

ALITER *It is given that the roots are* $a, a+b, a+2b$ *which are in A.P.*

Let $a = \alpha - \beta, a + b = \alpha$ and $a + 2b = \alpha + \beta$.

Now, proceed as in Example 10 on page 2.43.

EXAMPLE 7 *Find the condition which must be satisfied by the coefficients of the polynomial* $f(x) = x^3 - px^2 + qx - r$ *when the sum of its two zeros is zero.*

SOLUTION Let α, β and γ be the zeros of the polynomial $f(x)$ such that $\alpha + \beta = 0$.

Now, Sum of the zeros $= -\dfrac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

$$\alpha + \beta + \gamma = -\left(\frac{-p}{1}\right)$$

$\Rightarrow \quad \alpha + \beta + \gamma = p$

$\Rightarrow \quad 0 + \gamma = p \qquad [\because \alpha + \beta = 0]$

$\Rightarrow \quad \gamma = p$

Since γ is a zero of the polynomial $f(x)$. Therefore,

$f(\gamma) = 0$

$\Rightarrow \quad \gamma^3 - p\gamma^2 + q\gamma - r = 0$

$\Rightarrow \quad p^3 - p^3 + qp - r = 0 \qquad [\because \gamma = p]$

$\Rightarrow \quad pq = r$, which is the required condition.

EXAMPLE 8 *Find the condition that the zeros of the polynomial $f(x) = x^3 - px^2 + qx - r$ may be in arithmetic progression.*

SOLUTION Let $a - d$, a and $a + d$ be the zeros of the polynomial $f(x)$. Then,

$$\text{Sum of the zeros} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\Rightarrow \quad (a - d) + a + (a + d) = -\frac{(-p)}{1}$$

$$\Rightarrow \quad 3a = p \Rightarrow a = \frac{p}{3}$$

Since a is a zero of the polynomial $f(x)$. Therefore,

$f(a) = 0$

$\Rightarrow \quad a^3 - pa^2 + qa - r = 0$

$$\Rightarrow \quad \left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0 \qquad \left[\because a = \frac{p}{3}\right]$$

$\Rightarrow \quad p^3 - 3p^3 + 9pq - 27r = 0$

$\Rightarrow \quad 2p^3 - 9pq + 27r = 0$, which is the required condition.

EXAMPLE 9 *Find the zeros of the polynomial $f(x) = x^3 - 5x^2 - 2x + 24$, if it is given that the product of its two zeros is 12.*

SOLUTION Let α, β, γ be the zeros of polynomial $f(x)$ such that $\alpha\beta = 12$. Then,

$$\alpha + \beta + \gamma = -\left(-\frac{5}{1}\right) = 5 \qquad \text{...(i)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-2}{1} = -2 \qquad \text{...(ii)}$$

and, $$\alpha\beta\gamma = -\frac{24}{1} = -24 \qquad \text{...(iii)}$$

Putting $\alpha\beta = 12$ in $\alpha\beta\gamma = -24$, we get

$$12\gamma = -24 \Rightarrow \gamma = -\frac{24}{12} = -2$$

Putting $\gamma = -2$ in (i), we get

$$\alpha + \beta - 2 = 5$$

$\Rightarrow \quad \alpha + \beta = 7$

Now, $\quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow \quad (\alpha - \beta)^2 = 7^2 - 4 \times 12 \quad [\because \alpha + \beta = 7 \text{ and } \alpha\beta = 12]$

$\Rightarrow \quad (\alpha - \beta)^2 = 1$

$\Rightarrow \quad \alpha - \beta = \pm 1$

Thus, we have

$$\alpha + \beta = 7 \text{ and } \alpha - \beta = 1 \text{ or, } \alpha + \beta = 7 \text{ and } \alpha - \beta = -1$$

CASE I *When* $\alpha + \beta = 7$ *and* $\alpha - \beta = 1$

Solving $\alpha + \beta = 7$ and $\alpha - \beta = 1$, we get $\alpha = 4$ and $\beta = 3$

CASE II *When* $\alpha + \beta = 7$ *and* $\alpha - \beta = -1$

Solving $\alpha + \beta = 7$ and $\alpha - \beta = -1$, we get $\alpha = 3$ and $\beta = 4$.

Hence, the zeros of the given polynomial are 3, 4 and – 2.

EXAMPLE 10 *Find the zeros of the polynomial* $f(x) = x^3 - 12x^2 + 39x - 28$, *if it is given that the zeros are in A.P.*

SOLUTION Let $\alpha = a - d, \beta = a$ and $\gamma = a + d$ be the zeros of the polynomial

$$f(x) = x^3 - 12x^2 + 39x - 28.$$

$\therefore \quad \alpha + \beta + \gamma = -\left(\frac{-12}{1}\right) = 12$ and, $\alpha\beta\gamma = -\left(\frac{-28}{1}\right) = 28$

$\Rightarrow \quad (a - d) + a + (a + d) = 12$ and $(a - d)\, a\, (a + d) = 28$

$\Rightarrow \quad 3a = 12$ and $a(a^2 - d^2) = 28$

$\Rightarrow \quad a = 4$ and $4(16 - d^2) = 28$

$\Rightarrow \quad a = 4$ and $16 - d^2 = 7$

$\Rightarrow \quad a = 4$ and $d^2 = 9$

$\Rightarrow \quad a = 4$ and $d = \pm 3$

CASE I *When* $a = 4$ *and* $d = 3$: In this case,

$$\alpha = a - d = 4 - 3 = 1, \beta = a = 4 \text{ and } \gamma = a + d = 7$$

CASE II *When* $a = 4$ *and* $d = -3$: In this case,

$$\alpha = a - d = 4 + 3 = 7, \beta = a = 4 \text{ and } \gamma = a + d = 4 - 3 = 1$$

Hence, in either case the zeros of the given polynomial are 1, 4 and 7.

EXERCISE 2.2

LEVEL-1

1. Verify that the numbers given along side of the cubic polynomials below are their zeros. Also, verify the relationship between the zeros and coefficients in each case: [NCERT]

   (i) $f(x) = 2x^3 + x^2 - 5x + 2;\ \frac{1}{2}, 1, -2$

   (ii) $g(x) = x^3 - 4x^2 + 5x - 2;\ 2, 1, 1$

2. Find a cubic polynomial with the sum, sum of the product of its zeros taken two at a time, and product of its zeros as 3, – 1 and – 3 respectively.

LEVEL-2

3. If the zeros of the polynomial $f(x) = 2x^3 - 15x^2 + 37x - 30$ are in A.P., find them.
4. Find the condition that the zeros of the polynomial $f(x) = x^3 + 3px^2 + 3qx + r$ may be in A.P.
5. If the zeros of the polynomial $f(x) = ax^3 + 3bx^2 + 3cx + d$ are in A.P., prove that $2b^3 - 3abc + a^2d = 0$.
6. If the zeros of the polynomial $f(x) = x^3 - 12x^2 + 39x + k$ are in A.P., find the value of k.

ANSWERS

2. $f(x) = k\,(x^3 - 3x^2 - x + 3), k$ is any non-zero real number.
3. $2, 3, \frac{5}{2}$
4. $2p^2 - 3pq + r = 0$
6. $k = -28$

2.6 DIVISION ALGORITHM FOR POLYNOMIALS

In earlier classes, we have studied division of integers. We have seen that on division of an integer (called dividend) by a non-zero integer (called divisor), we obtain the quotient and the remainder which is either zero or less than the divisor. Also, dividend, divisor, quotient and the remainder always satisfy the following relation.

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

This is known as Euclid's division lemma which we have studied in chapter 1.

In earlier classes, we have studied about division of polynomials. In this section, we shall show that the division of polynomials also follows the similar rule which is known as the division algorithm for polynomials. We will also discuss problems on finding zeros of cubic and biquadratic polynomials when some of its zeros are given.

Let us first refresh the method of dividing one polynomial by another polynomial through following illustrations.

ILLUSTRATION 1 *Divide the polynomial* $f(x) = 14x^3 - 5x^2 + 9x - 1$ *by the polynomial* $g(x) = 2x - 1$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$
\begin{array}{r|l}
2x-1 & 14x^3 - 5x^2 + 9x - 1 \quad (7x^2 + x + 5 \\
 & \underline{\underset{-}{14x^3} \underset{+}{-7x^2}} \\
 & \qquad 2x^2 + 9x - 1 \\
 & \qquad \underline{\underset{-}{2x^2} \underset{+}{-x}} \\
 & \qquad\qquad 10x - 1 \\
 & \qquad\qquad \underline{\underset{-}{10x} \underset{+}{-5}} \\
 & \qquad\qquad\qquad 4
\end{array}
$$

Clearly, quotient $q(x) = 7x^2 + x + 5$ and remainder $r(x) = 4$

Now,

$$
\begin{aligned}
q(x)\, g(x) + r(x) &= (7x^2 + x + 5)(2x - 1) + 4 \\
&= 14x^3 + 2x^2 + 10x - 7x^2 - x - 5 + 4 \\
&= 14x^3 - 5x^2 + 9x - 1 \\
&= f(x)
\end{aligned}
$$

i.e. $f(x) = g(x)\, q(x) + r(x)$ or, Dividend = Quotient × Divisor+ Remainder

ILLUSTRATION 2 *Divide the polynomial* $f(x) = 6x^3 + 11x^2 - 39x - 65$ *by the polynomial* $g(x) = x^2 - 1 + x$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$
\begin{array}{r|l}
x^2 + x - 1 & 6x^3 + 11x^2 - 39x - 65 \quad (6x + 5 \\
 & \underline{\underset{-}{6x^3} + \underset{-}{6x^2} \underset{+}{-\ 6x}} \\
 & \qquad 5x^2 - 33x - 65 \\
 & \qquad \underline{\underset{-}{5x^2} + \underset{-}{5x} \underset{+}{-\ 5}} \\
 & \qquad\qquad -38x - 60
\end{array}
$$

Clearly, quotient $q(x) = 6x + 5$ and remainder $r(x) = -38x - 60$.

Also,

$$
\begin{aligned}
g(x)\, q(x) + r(x) &= (x^2 + x - 1)(6x + 5) + (-38x - 60) \\
&= 6x^3 + 6x^2 - 6x + 5x^2 + 5x - 5 - 38x - 60
\end{aligned}
$$

i.e. $f(x) = g(x)\, q(x) + r(x) = 6x^3 + 11x^2 - 39x - 65 = f(x)$

or, Dividend = Quotient × Divisor + Remainder

ILLUSTRATION 3 *Divide the polynomial* $u(x) = 9x^4 - 4x^2 + 4$ *by the polynomial* $v(x) = 3x^2 + x - 1$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$
\begin{array}{r|l}
3x^2+x-1 & 9x^4+0x^3-4x^2+0x+4 \;\big(3x^2-x \\
 & 9x^4+3x^3-3x^2 \\
 & \underline{-\quad -\quad +} \\
 & -3x^3-x^2+0x+4 \\
 & -3x^3-x^2+x \\
 & \underline{+\quad +\quad -} \\
 & -x+4
\end{array}
$$

Cleraly, quotient $q(x) = 3x^2 - x$ and remainder $r(x) = -x + 4$.

Also,

$$v(x)\,q(x) + r(x) = (3x^2 + x - 1)(3x^2 - x) + (-x + 4)$$
$$= 9x^4 + 3x^3 - 3x^2 - 3x^3 - x^2 + x - x + 4$$

i.e. $\quad u(x) = v(x)\,q(x) + r(x) = 9x^4 + 0x^3 - 4x^2 + 0x + 4 = u(x)$

or, $\quad$ Dividend = Quotient × Divisor + Remainder

ILLUSTRATION 4 *Divide the polynomial* $f(x) = 30x^4 + 11x^3 - 82x^2 - 12x + 48$ *by* $3x^2 + 2x - 4$. *Also, find the quotient and remainder.*

SOLUTION Using long division method, we obtain

$$
\begin{array}{r|l}
3x^2+2x-4 & 30x^4+11x^3-82x^2-12x+48 \;\big(10x^2-3x-12 \\
 & 30x^4+20x^3-40x^2 \\
 & \underline{-\quad -\quad +} \\
 & -9x^3-42x^2-12x+48 \\
 & -9x^3-6x^2+12x \\
 & \underline{+\quad +\quad -} \\
 & -36x^2-24x+48 \\
 & -36x^2-24x+48 \\
 & \underline{+\quad +\quad -} \\
 & 0
\end{array}
$$

Clearly, quotient $q(x) = 10x^2 - 3x - 12$ and remainder $r(x) = 0$.

Also,

$$g(x)\,q(x) + r(x) = (3x^2 + 2x - 4)(10x^2 - 3x - 12) + 0$$
$$= 30x^4 - 9x^3 - 36x^2 + 20x^3 - 6x^2 - 24x - 40x^2 + 12x + 48 + 0$$

i.e. $\quad f(x) = g(x)\,q(x) + r(x) = 30x^4 + 11x^3 - 82x^2 - 12x + 48 = f(x)$

or, $\quad$ Dividend = Quotient × Divisor + Remainder

In the above illustrations, we observe that the division process is stoped when either the remainder is zero or its degree is less than the degree of divisor. Also, divident, divisor, quotient and remainder satisfy the relation

Dividend = Quotient × Divisor + Remainder

This is an algorithm similar to Euclid's division algorithm for integers and is known as the division algorithm for polynomials as defined below.

DIVISION ALGORITHM *If $f(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can always find polynomials $q(x)$ and $r(x)$ such that*

$$f(x) = q(x)\, g(x) + r(x), \text{ where } r(x) = 0 \text{ or degree } r(x) < \text{degree } g(x).$$

REMARK *If $r(x) = 0$, then polynomial $g(x)$ is a factor of polynomial $f(x)$.*

Following examples will illustrate various applications of division algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON VERIFYING THE DIVISION ALGORITHM FOR POLYNOMIALS

EXAMPLE 1 *Divide the polynomial $f(x) = 3x^2 - x^3 - 3x + 5$ by the polynomial $g(x) = x - 1 - x^2$ and verify the division algorithm.*

SOLUTION Writing the given polynomials in standard form, we get

$$f(x) = -x^3 + 3x^2 - 3x + 5 \text{ and } g(x) = -x^2 + x - 1$$

Using long division method, we obtain

$$\begin{array}{r|l}
-x^2 + x - 1 & -x^3 + 3x^2 - 3x + 5 \;\big(\, x - 2 \\
 & -x^3 + x^2 - x \\
 & + \quad - \quad + \\
\hline
 & 2x^2 - 2x + 5 \\
 & 2x^2 - 2x + 2 \\
 & - \quad + \quad - \\
\hline
 & 3
\end{array}$$

$\therefore$ Quotient $q(x) = x - 2$ and, Remainder $r(x) = 3$

Now,

$$\begin{aligned}
\text{Quotient} \times \text{Divisor} + \text{Remainder} &= (x-2)(-x^2 + x - 1) + 3 \\
&= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3 \\
&= -x^3 + 3x^2 - 3x + 5 = \text{Dividend}
\end{aligned}$$

Hence, the division algorithm is verified.

Type II ON FINDING THE REMAINING ZEROS OF A POLYNOMIAL WHEN SOME OF ITS ZEROS ARE GIVEN

EXAMPLE 2 *Find all the zeros of the polynomial $f(x) = 2x^4 - 3x^3 - 3x^2 + 6x - 2$, if two of its zeros are $\sqrt{2}$ and $-\sqrt{2}$.* [NCERT]

SOLUTION We know that, if $x = \alpha$ is a zero of a polynomial, then $x - \alpha$ is a factor of $f(x)$. Since $\sqrt{2}$ and $-\sqrt{2}$ are zeros of $f(x)$. Therefore, $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$ is a factor of $f(x)$. Let us now divide $f(x) = 2x^4 - 3x^3 - 3x^2 + 6x - 2$ by $g(x) = x^2 - 2$ to find the other zeros of $f(x)$.

Using long division method, we obtain

$$
\begin{array}{r|l}
x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \;\big(\, 2x^2 - 3x + 1 \\
 & 2x^4 \qquad\quad - 4x^2 \\
 & - \qquad\qquad + \\
\hline
 & -3x^3 + x^2 + 6x - 2 \\
 & -3x^3 \qquad + 6x \\
 & + \qquad\qquad - \\
\hline
 & \qquad\qquad x^2 - 2 \\
 & \qquad\qquad x^2 - 2 \\
 & \qquad\qquad - \quad + \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

Thus, Quotient = $2x^2 - 3x + 1$ and Remainder = 0.

By division algorithm, we have

$$f(x) = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

$$\Rightarrow \quad f(x) = (x^2 - 2)(2x^2 - 3x + 1) + 0$$

$$\Rightarrow \quad f(x) = (x - \sqrt{2})(x + \sqrt{2})(2x^2 - 2x - x + 1)$$

$$\Rightarrow \quad f(x) = (x - \sqrt{2})(x + \sqrt{2})\{2x(x-1) - (x-1)\}$$

$$\Rightarrow \quad f(x) = (x - \sqrt{2})(x + \sqrt{2})(x - 1)(2x - 1)$$

On equating factors $x - \sqrt{2}, x + \sqrt{2}, x - 1$ and $2x - 1$ to zero, we get $x = \sqrt{2}, -\sqrt{2}, 1, \frac{1}{2}$.

Hence, the zeros of the given polynomial are $\sqrt{2}, -\sqrt{2}, 1$ and $\frac{1}{2}$.

EXAMPLE 3 *Obtain all the zeros of the polynomial* $f(x) = 3x^4 + 6x^3 - 2x^2 - 10x - 5$, *if two of its zeros are* $\sqrt{\frac{5}{3}}$ *and* $-\sqrt{\frac{5}{3}}$. [NCERT]

SOLUTION It is given that $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$ are two zeros of $f(x)$. Therefore, $\left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right)$ is a factor of $f(x)$. But, $\left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right) = \left(x^2 - \frac{5}{3}\right) = \frac{1}{3}(3x^2 - 5)$.

Therefore, $3x^2 - 5$ is a factor of $f(x)$. Let us now divide $f(x)$ by $3x^2 - 5$.

Using long division method, we obtain

$$
\begin{array}{r|l}
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \; (x^2 + 2x + 1 \\
 & 3x^4 + 0x^3 - 5x^2 \\
 & - \quad - \quad + \\
\hline
 & 6x^3 + 3x^2 - 10x - 5 \\
 & 6x^3 + 0x^2 - 10x \\
 & + \quad - \quad + \\
\hline
 & 3x^2 - 5 \\
 & 3x^2 - 5 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

Clearly, Quotient = $x^2 + 2x + 1$ and Remainder = 0.
By division algorithm, we obtain

$$f(x) = (3x^2 - 5)(x^2 + 2x + 1) + 0$$

$$\Rightarrow \qquad f(x) = (\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5})(x + 1)^2$$

Thus, the factors of $f(x)$ are $\sqrt{3}x + \sqrt{5}, \sqrt{3}x - \sqrt{5}, x + 1$ and $x + 1$. Equating each factor to zero, we obtain $x = -\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}, -1, -1$. Hence, the zeros of $f(x)$ are $-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}, -1$ and -1.

EXAMPLE 4 *If two zeros of the polynomial* $f(x) = x^4 - 6x^3 - 26x^2 + 138x - 35$ *are* $2 \pm \sqrt{3}$, *find other zeros.* **[NCERT]**

SOLUTION It is given that $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are two zeros of $f(x)$. Therefore, $\left(x - (2 + \sqrt{3})\right)$ and $\left(x - (2 - \sqrt{3})\right)$ are factors of $f(x)$.

But, $\{x - (2\sqrt{3})\}\{x - (2 - \sqrt{3})\} = (x - 2 - \sqrt{3})(x - 2 + \sqrt{3}) = (x - 2)^2 - (\sqrt{3})^2 = x^2 - 4x + 1$. Therefore, $x^2 - 4x + 1$ is a factor of $f(x)$. Let us now divide $f(x)$ by $x^2 - 4x + 1$.

Using long division method, we obtain

$$
\begin{array}{r|l}
x^2 - 4x + 1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \; (x^2 - 2x - 35 \\
 & x^4 - 4x^3 + x^2 \\
 & - \quad + \quad - \\
\hline
 & -2x^3 - 27x^2 + 138x \\
 & -2x^3 + 8x^2 - 2x \\
 & + \quad - \quad + \\
\hline
 & -35x^2 + 140x - 35 \\
 & -35x^2 + 140x - 35 \\
 & + \quad - \quad + \\
\hline
 & 0
\end{array}
$$

Thus, Quotient $q(x) = x^2 - 2x - 35$ and Remainder $= 0$.

By division algorithm, we obtain

$$f(x) = (x^2 - 4x + 1)(x^2 - 2x - 35)$$

Hence, other two zeros of $f(x)$ are the zeros of the polynomial $x^2 - 2x - 35$.

Now,

$$x^2 - 2x - 35 = x^2 - 7x + 5x - 35 = (x - 7)(x + 5)$$

On equating each factor to zero, we obtain $x = 7, -5$.

Hence, other two zeros of $f(x)$ are 7 and -5.

LEVEL-2

Type III ON FINDING THE QUOTIENT AND REMAINDER USING DIVISION ALGORITHM

EXAMPLE 5 *Apply the division algorithm to find the quotient and remainder on dividing $f(x)$ by $g(x)$ as given below:*

(i) $f(x) = x^3 - 6x^2 + 11x - 6,\ g(x) = x + 2$

(ii) $f(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$ [NCERT]

(iii) $f(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$ [NCERT]

(iv) $f(x) = x^4 - 5x + 6,\ g(x) = 2 - x^2$ [NCERT]

SOLUTION (i) We have,

$$f(x) = x^3 - 6x^2 + 11x - 6 \text{ and } g(x) = x + 2$$

We find that degree $f(x) = 3$ and degree $g(x) = 1$. Therefore, quotient $q(x)$ is of degree $3 - 1 = 2$ and the remainder $r(x)$ is of degree zero. Let $q(x) = ax^2 + bx + c$ and $r(x) = k$.

By using division algorithm, we obtain

$$f(x) = q(x) \times g(x) + r(x)$$

$\Rightarrow \quad x^3 - 6x^2 + 11x - 6 = (ax^2 + bx + c)(x + 2) + k$

$\Rightarrow \quad x^3 - 6x^2 + 11x - 6 = ax^3 + (2a + b)x^2 + (2b + c)x + 2c + k$

Equating the coefficients of like powers of x on both sides, we get

$1 = a$ [On equating the coefficients of x^3]

$-6 = 2a + b$ [On equating the coefficients of x^2]

$11 = 2b + c$ [On equating the coefficients of x]

and, $-6 = 2c + k$ [On equating the constant terms]

Solving these equations, we get

$$a = 1, b = -8, c = 27 \text{ and } k = -60$$

$\therefore$ Quotient $q(x) = ax^2 + bx + c = x^2 - 8x + 27$ and, Remainder $r(x) = k = -60$.

(ii) We have,

$$f(x) = x^3 - 3x^2 + 5x - 3 \text{ and } g(x) = x^2 - 2.$$

We find that degree $(f(x)) = 3$ and degree $(g(x)) = 2$. Therefore, quotient $q(x)$ is of degree 1 and the remainder $r(x)$ is of degree less than 2. Let $q(x) = ax + b$ and $r(x) = cx + d$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^3 - 3x^2 + 5x - 3 = (x^2 - 2)(ax + b) + (cx + d)$$

$$\Rightarrow \quad x^3 - 3x^2 + 5x - 3 = ax^3 + bx^2 + (c - 2a)x - 2b + d$$

On equating the coefficients of various powers of x on both sides, we get

$1 = a$ [On equating the coefficients of x^3]

$-3 = b$ [On equating the coefficients of x^2]

$5 = c - 2a$ [On equating the coefficients of x]

$-3 = -2b + d$ [On equating the constant terms]

Solving these equations, we get : $a = 1, b = -3, c = 7$ and $d = -9$

$\therefore$ Quotient $q(x) = ax + b = x - 3$ and Remainder $r(x) = 7x - 9$.

(iii) We have,

$$f(x) = x^4 - 3x^2 + 4x + 5 \text{ and } g(x) = x^2 - x + 1$$

We find that degree $(f(x)) = 4$ and degree $(g(x)) = 2$. Therefore, quotient $q(x)$ is of degree $2 (= 4 - 2)$ and remainder $r(x)$ is of degree less than 2 $(= \text{degree } (g(x)))$. So, let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^4 + 0x^3 - 3x^2 + 4x + 5 = (x^2 - x + 1)(ax^2 + bx + c) + px + q$$

$$\Rightarrow \quad x^4 + 0x^3 - 3x^2 + 4x + 5 = ax^4 + (b - a)x^3 + (c - b + a)x^2 + (b - c + p)x + c + q$$

On equating the coefficients of various powers of x on both sides, we get

$a = 1$ [On equating the coefficients of x^2]

$b - a = 0$ [On equating the coefficients of x^3]

$c - b + a = -3$ [On equating the coefficients of x^2]

$b - c + p = 4$ [On equating the coefficient of x]

and, $c + q = 5$ [On equating the constant terms]

Solving these equations, we get

$$a = 1, b = 1, c = -3, p = 0 \text{ and } q = 8$$

$\therefore$ Quotient $q(x) = x^2 + x - 3$ and Remainder $r(x) = 8$

(iv) We have,

$$f(x) = x^4 + 0x^3 + 0x^2 - 5x + 6 \text{ and } g(x) = -x^2 + 2$$

We find that degree $(f(x)) = 4$ and degree $g(x) = 2$. Therefore, quotient $q(x)$ and remainder $r(x)$ are of degree 2 and less than 2 respectively.

Let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$

By division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^4 + 0x^3 + 0x^2 - 5x + 6 = (-x^2 + 2)(ax^2 + bx + c) + px + q$$

$$\Rightarrow \quad x^4 + 0x^3 + 0x^2 - 5x + 6 = -ax^4 - bx^3 + (2a - c)x^2 + (2b + p)x + 2c + q$$

Equating the coefficients of various powers of x, we get

$-a = 1$ [On equating the coefficients of x^4]

$-b = 0$ [On equating the coefficients of x^3]

$2a - c = 0$ [On equating the coefficients of x^2]

$2b + p = -5$ [On equating the coefficient of x]

and, $2c + q = 6$ [On equating the constant terms]

Solving these equations, we get

$a = -1, b = 0, c = -2, p = -5$ and $q = 10$

$\therefore$ Quotient $q(x) = -x^2 - 2$ and Remainder $r(x) = -5x + 10$.

Type IV **ON CHECKING WHETHER A GIVEN POLYNOMIAL IS A FACTOR OF THE OTHER POLYNOMIAL BY APPLYING THE DIVISION ALGORITHM**

EXAMPLE 6 *By applying division algorithm prove that the polynomial* $g(x) = x^2 + 3x + 1$ *is a factor of the polynomial* $f(x) = 3x^4 + 5x^3 - 7x^2 + 2x + 2$.

SOLUTION We find that degree $(f(x)) = 4$ and degree $(g(x)) = 2$. Therefore, quotient $q(x)$ is of degree 2 (= 4 − 2) and the remainder $r(x)$ is of degree 1 or less. Let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad 3x^4 + 5x^3 - 7x^2 + 2x + 2 = (ax^2 + bx + c)(x^2 + 3x + 1) + (px + q)$$

$$\Rightarrow \quad 3x^4 + 5x^3 - 7x^2 + 2x + 2 = ax^4 + (3a + b)x^3 + (a + 3b + c)x^2 + (b + 3c + p)x + c + q$$

Equating coefficients of various powers of x, we get

$a = 3$ [On equating the coefficients of x^4]

$3a + b = 5$ [On equating the coefficients of x^3]

$a + c + 3b = -7$ [On equating the coefficients of x^2]

$b + 3c + p = 2$ [On equating the coefficient of x]

and, $c + q = 2$ [On equating the constant terms]

Solving these equations, we get

$a = 3, b = -4, c = 2, p = 0$ and $q = 0$

$\therefore$ Quotient $q(x) = 3x^2 - 4x + 2$ and, Remainder $r(x) = 0x + 0 = 0$

Clearly, $r(x) = 0$. Hence, $g(x)$ is a factor of $f(x)$.

Type V **MISCELLANEOUS APPLICATIONS OF DIVISION ALGORITHM**

EXAMPLE 7 *On dividing the polynomial* $f(x) = x^3 - 3x^2 + x + 2$ *by a polynomial* $g(x)$, *the quotient* $q(x)$ *and remainder* $r(x)$ *where* $q(x) = x - 2$ *and* $r(x) = -2x + 4$ *respectively. Find the polynomial* $g(x)$. [NCERT]

SOLUTION By division algorithm, we obtain

$$f(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad g(x) \times q(x) = f(x) - r(x)$$

$$\Rightarrow \quad g(x)(x - 2) = x^3 - 3x^2 + x + 2 - (-2x + 4)$$

$$\Rightarrow \quad g(x)(x-2) = x^3 - 3x^2 + 3x - 2$$

Thus, $g(x)$ is a factor of $x^3 - 3x^2 + 3x - 2$ other than the factor $(x-2)$. So, to get $g(x)$, we divide $x^3 - 3x^2 + 3x - 2$ by $(x-2)$ as follows:

$$\begin{array}{r|l}
x-2 & x^3 - 3x^2 + 3x - 2 \;\big(\, x^2 - x + 1 \\
 & x^3 - 2x^2 \\
 & \underline{-\quad +} \\
 & \quad -x^2 + 3x - 2 \\
 & \quad -x^2 + 2x \\
 & \quad \underline{+\quad -} \\
 & \qquad x - 2 \\
 & \qquad x - 2 \\
 & \qquad \underline{-\quad +} \\
 & \qquad\quad 0
\end{array}$$

Hence, $g(x) = x^2 - x + 1$.

EXAMPLE 8 *What must be subtracted from* $8x^4 + 14x^3 - 2x^2 + 7x - 8$ *so that the resulting polynomial is exactly divisible by* $4x^2 + 3x - 2$.

SOLUTION We know that

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

$$\Rightarrow \quad \text{Dividend} - \text{Remainder} = \text{Quotient} \times \text{Divisor}$$

Clearly, RHS of the above result is divisible by the divisor. Therefore, LHS is also divisible by the divisor. Thus, if we subtract remainder from the dividend, then it will be exactly divisible by the divisor.

Let us now divide $8x^4 + 14x^3 - 2x^2 + 7x - 8$ by $4x^2 + 3x - 2$ long division method.

$$\begin{array}{r|l}
4x^2 + 3x - 2 & 8x^4 + 14x^3 - 2x^2 + 7x - 8 \;\big(\, 2x^2 + 2x - 1 \\
 & 8x^4 + 6x^3 - 4x^2 \\
 & \underline{-\quad -\quad +} \\
 & \quad 8x^3 + 2x^2 + 7x - 8 \\
 & \quad 8x^3 + 6x^2 - 4x \\
 & \quad \underline{-\quad -\quad +} \\
 & \qquad -4x^2 + 11x - 8 \\
 & \qquad -4x^2 - 3x + 2 \\
 & \qquad \underline{+\quad +\quad -} \\
 & \qquad\quad 14x - 10
\end{array}$$

$$\therefore \quad \text{Quotient} = 2x^2 + 2x - 1 \text{ and Remainder} = 14x - 10$$

Thus, if we subtract the remainder $14x - 10$ from $8x^4 + 14x^3 - 2x^2 + 7x - 8$, it will be exactly divisible by $4x^2 + 3x - 2$.

EXAMPLE 9 *Find the values of a and b so that* $x^4 + x^3 + 8x^2 + ax + b$ *is divisible by* $x^2 + 1$.

SOLUTION If $x^4 + x^3 + 8x^2 + ax + b$ is exactly divisible by $x^2 + 1$, then the remainder should be zero. Let us now divide $x^4 + x^3 + 8x^2 + ax + b$ by $x^2 + 1$ by long division method.

$$
\begin{array}{r|l}
x^2+1 & x^4+x^3+8x^2+ax+b \;\big(\, x^2+x+7 \\
 & x^4 \qquad\quad + x^2 \\
 & - \qquad\quad\; - \\ \hline
 & \quad x^3+7x^2+ax+b \\
 & \quad x^3 \qquad\quad + x \\
 & \quad - \qquad\quad\; - \\ \hline
 & \qquad 7x^2+x(a-1)+b \\
 & \qquad 7x^2 \qquad\quad +7 \\
 & \qquad - \qquad\qquad - \\ \hline
 & \qquad\quad x(a-1)+b-7
\end{array}
$$

$\therefore$ Quotient $= x^2+x+7$ and, Remainder $= x(a-1)+(b-7)$

Now,

Remainder $= 0$

$\Rightarrow \quad x(a-1)+(b-7)=0$

$\Rightarrow \quad x(a-1)+(b-7)=0x+0$

$\Rightarrow \quad a-1=0$ and $b-7=0$ [On equating the coefficients of like powers of x]

$\Rightarrow \quad a=1$ and $b=7$

EXAMPLE 10 *What must be added to* $f(x)=4x^4+2x^3-2x^2+x-1$ *so that the resulting polynomial is divisible by* $g(x)=x^2+2x-3$?

SOLUTION By division algorithm, we have

$$f(x)=g(x)\times q(x)+r(x)$$

$\Rightarrow \quad f(x)-r(x)=g(x)\times q(x)$

$\Rightarrow \quad f(x)+\{-r(x)\}=g(x)\times q(x)$

Clearly, RHS is divisible by $g(x)$. Therefore, LHS is also divisible by $g(x)$. Thus, if we add $-r(x)$ to $f(x)$, then the resulting polynomial is divisible by $g(x)$. Let us now find the remainder when $f(x)$ is divided by $g(x)$. Using long division method, we obtain

$$
\begin{array}{r|l}
x^2+2x-3 & 4x^4+2x^3-2x^2+x-1 \;\big(\, 4x^2-6x+22 \\
 & 4x^4+8x^3-12x^2 \\
 & - \quad - \quad\;\; + \\ \hline
 & \quad -6x^3+10x^2+x-1 \\
 & \quad -6x^3-12x^2+18x \\
 & \quad + \qquad + \qquad - \\ \hline
 & \qquad\quad 22x^2-17x-1 \\
 & \qquad\quad 22x^2+44x-66 \\
 & \qquad\quad - \qquad - \qquad + \\ \hline
 & \qquad\qquad\quad -61x+65
\end{array}
$$

$\therefore \quad r(x)=-61x+65$

Hence, we should add $-r(x)=61x-65$ to $f(x)$ so that the resulting polynomial is divisible by $g(x)$.

EXAMPLE 11 *If the polynomial $f(x) = x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.* [NCERT]

SOLUTION By division algorithm, we have

Dividend = Divisor × Quotient + Remainder

⇒ Dividend − Remainder = Divisor × Quotient

⇒ (Dividend − Remainder) is always divisible by the divisor.

It is given that $f(x) = x^4 - 6x^3 + 16x^2 - 25x + 10$ when divided by $x^2 - 2x + k$ leaves $x + a$ as remainder. Therefore,

$f(x) - (x + a) = x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ is exactly divisible by $x^2 - 2x + k$.

Let us now divide $x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ by $x^2 - 2x + k$. Using long division method:

$$
\begin{array}{r|l|l}
x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 26x + 10 - a & x^2 - 4x + (8 - k) \\
 & x^4 - 2x^3 + kx^2 & \\
 & - \quad + \quad - & \\
\hline
 & -4x^3 + (16 - k)x^2 - 26x + 10 - a & \\
 & -4x^3 + 8x^2 \qquad - 4kx & \\
 & + \quad - \qquad + & \\
\hline
 & (8 - k)x^2 - (26 - 4k)x + 10 - a & \\
 & (8 - k)x^2 - (16 - 2k)x + (8k - k^2) & \\
 & - \quad + \quad - & \\
\hline
 & (-10 + 2k)x + (10 - a - 8k + k^2) &
\end{array}
$$

For $f(x) - (x + a) = x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ to be exactly divisible by $x^2 - 2x + k$, we must have

Remainder = 0

⇒ $(-10 + 2k)x + (10 - a - 8k + k^2) = 0$ for all x

⇒ $-10 + 2k = 0, 10 - a - 8k + k^2 = 0$ [On equating the coefficients of like powers of x]

⇒ $k = 5, 10 - a - 40 + 25 = 0$

⇒ $k = 5$ and $a = -5$.

EXAMPLE 12 *If the polynomial $6x^4 + 8x^3 + 17x^2 + 21x + 7$ is divided by another polynomial $3x^2 + 4x + 1$, the remainder comes out to be $ax + b$, find a and b.* [CBSE 2009]

SOLUTION Let us divide the polynomial $f(x) = 6x^4 + 8x^3 + 17x^2 + 21x + 7$ by the polynomial $g(x) = 3x^2 + 4x + 1$ to find the remainder by long division method as shown below:

$$
\begin{array}{r|l|l}
3x^2 + 4x + 1 & 6x^4 + 8x^3 + 17x^2 + 21x + 7 & 2x^2 + 5 \\
 & 6x^4 + 8x^3 + 2x^2 & \\
 & - \quad - \quad - & \\
\hline
 & 15x^2 + 21x + 7 & \\
 & 15x^2 + 20x + 5 & \\
 & - \quad - \quad - & \\
\hline
 & x + 2 &
\end{array}
$$

Clearly, remainder $= x + 2$. It is given that the remainder is $ax + b$.

$\therefore \quad ax + b = x + 2 \Rightarrow a = 1, b = 2.$ [On compairing the coefficients of like powers of x]

EXAMPLE 13 *Find k so that* $x^2 + 2x + k$ *is a factor of* $2x^4 + x^3 - 14x^2 + 5x + 6$. *Also, find all the zeroes of the two polynomials.* **[NCERT EXEMPLAR]**

SOLUTION It is given that $x^2 + 2x + k$ is a factor of the polynomial $f(x) = 2x^4 + x^3 - 14x^2 + 5x + 6$ when divided by $x^2 + 2x + k$, the remainder is zero.

Let us now divide $f(x) = 2x^4 + x^3 - 14x^2 + 5x + 6$ by $x^2 + 2x + k$ using by long division method.

$$
\begin{array}{l}
x^2 + 2x + k \,\big)\, \overline{2x^4 + x^3 - 14x^2 + 5x + 6} \;\big(2x^2 - 3x - 2(k+4) \\
\qquad\qquad\quad 2x^4 + 4x^3 + 2kx^2 \\
\qquad\qquad\quad - \quad\; - \quad\;\; - \\
\hline
\qquad\qquad\qquad - 3x^3 - 2x^2(k+7) + 5x + 6 \\
\qquad\qquad\qquad - 3x^3 - 6x^2 \qquad\quad - 3kx \\
\qquad\qquad\qquad + \qquad + \qquad\qquad + \\
\hline
\qquad\qquad\qquad\qquad - 2x^2(k+4) + x(5+3k) + 6 \\
\qquad\qquad\qquad\qquad - 2x^2(k+4) - 4x(k+4) - 2x(k+4) \\
\qquad\qquad\qquad\qquad + \qquad\qquad + \qquad\qquad + \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad x(7k+21) + (2k^2 + 8k + 6)
\end{array}
$$

Thus, Remainder $= x(7k + 21) + (2k^2 + 8k + 6)$ and Quotient $= 2x^2 - 3x - 2(k + 4)$.

$\therefore \quad$ Remainder $= 0$.

$\Rightarrow \quad x(7k + 21) + 2(k^2 + 4k + 3) = 0$ for all x.

$\Rightarrow \quad 7k + 21 = 0$ and $k^2 + 4k + 3 = 0$

$\Rightarrow \quad 7(k + 3) = 0$ and $(k + 1)(k + 3) = 0$

$\Rightarrow \quad k + 3 = 0 \Rightarrow k = -3$

Substituting the value of k in $x^2 + 2x + k$, we obtain: $x^2 + 2x - 3 = (x + 3)(x - 1)$ as the divisor. Clearly, its zeros are -3 and 1. Consequently, two zeros of $f(x)$ are -3 and 1.

For $k = -3$, we obtain

Quotient $= 2x^2 - 3x - 2 = 2x^2 - 4x + x - 2 = 2x(x - 2) + 1(x - 2) = (x - 2)(2x + 1)$

and, Divisor $= x^2 + 2x - 3 = x^2 + 3x - x - 3 = x(x + 3) - 1(x + 3) = (x - 1)(x + 3)$

$\therefore \quad f(x) = (\text{Quotient}) \times (\text{Divisor})$

$\Rightarrow \quad f(x) = 2x^4 + x^3 - 14x^2 + 5x + 6 = (x - 2)(2x + 1)(x - 1)(x + 3)$

Hence, zeros of $f(x)$ are $2, -1/2, 1$ and -3.

EXAMPLE 14 *If the remainder on division of* $x^3 + 2x^2 + kx + 3$ *by* $x - 3$ *is 21, find the quotient and the value of k. Hence, find the zeroes of the cubic polynomial* $x^3 + 2x^2 + kx - 18$.

SOLUTION Let $f(x) = x^3 + 2x^2 + kx + 3$. It is given that $f(x)$ when divided by $x - 3$ gives 21 as remainder.

$\therefore \quad f(3) = 21$

$\Rightarrow \quad 3^3 + 2 \times 3^2 + 3k + 3 = 21$

$\Rightarrow \quad 27 + 18 + 3k + 3 = 21$

$\Rightarrow \quad 3k + 48 = 21$

$\Rightarrow \quad 3k = -27 \Rightarrow k = -9$

Hence, the given polynomial is $f(x) = x^3 + 2x^2 - 9x + 3$.

Let us now divide $f(x)$ by $(x - 3)$ to find the quotient. Using long division method, we obtain

$$
\begin{array}{r|l}
x - 3 & x^3 + 2x^2 - 9x + 3 \;\big(\, x^2 + 5x + 6 \\
 & \underline{\overset{-}{x^3} \overset{+}{-} 3x^2} \\
 & \quad 5x^2 - 9x + 3 \\
 & \quad \underline{\overset{-}{5x^2} \overset{+}{-} 15x} \\
 & \qquad 6x + 3 \\
 & \qquad \underline{\overset{-}{6x} \overset{+}{-} 18} \\
 & \qquad\quad 21
\end{array}
$$

So, Quotient $= x^2 + 5x + 6$

Thus, when $f(x)$ is divided by $x - 3$ the quotient and the remainder are $x^2 + 5x + 6$ and 21 respectively. Therefore, using division algorithm, we obtain

$$f(x) = (x^2 + 5x + 6)(x - 3) + 21$$

$\Rightarrow \quad x^3 + 2x^2 - 9x + 3 - 21 = (x + 2)(x + 3)(x - 3)$

$\Rightarrow \quad x^3 + 2x^2 - 9x - 18 = (x + 2)(x + 3)(x - 3)$

Hence, the zeros of $x^3 + 2x^2 - 9x - 18$ *i.e.* $x^3 + 2x^2 + kx - 18$ are $-2, -3$ and 3.

EXAMPLE 15 *For which values of a and b are the zeros of $q(x) = x^3 + 2x^2 + a$ also the zeros of the polynomial $p(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x + b$? Which zeros of $p(x)$ are not the zeros of $q(x)$?*

SOLUTION If zeros of $q(x)$ are also the zeroes of $p(x)$, then $p(x)$ is divisible by $q(x)$. In otherwords, when $p(x)$ is divided by $q(x)$, the remainder is zero. Let us now divide $p(x)$ by $q(x)$ to obtain the remainder.

$$
\begin{array}{r|l}
x^3 + 2x^2 + a & x^5 - x^4 - 4x^3 + 3x^2 + 3x + b \;\big(\, x^2 - 3x + 2 \\
 & \underline{\overset{-}{x^5} \overset{-}{+} 2x^4 \overset{-}{+} ax^2} \\
 & \quad -3x^4 - 4x^3 + (3 - a)x^2 + 3x + b \\
 & \quad \underline{\overset{+}{-} 3x^4 \overset{+}{-} 6x^3 \overset{+}{-} 3ax} \\
 & \qquad 2x^3 + (3 - a)x^2 + 3x(1 + a) + b \\
 & \qquad \underline{\overset{-}{2x^3} \overset{-}{+} 4x^2 \qquad \overset{-}{+} 2a} \\
 & \qquad\quad (-1 - a)x^2 + 3x(1 + a) + b - 2a
\end{array}
$$

If $p(x)$ is divisible by $q(x)$, then remainder must be zero.

Now, Remainder = 0

$\Rightarrow \quad (-1-a)x^2 + 3x(1+a) + b - 2a = 0$ for all x.

$\Rightarrow \quad -1 - a = 0,\ 3(1+a) = 0$ and $b - 2a = 0$ $\quad \left[\begin{array}{l}\text{On equating the coefficients} \\ \text{of like powers of } x\end{array}\right]$

$\Rightarrow \quad a = -1$ and $b - 2a = 0$

$\Rightarrow \quad a = -1$ and $b = -2$

Substituting $a = -1$ in $q(x) = x^3 + 2x^2 + a$, we obtain $q(x) = x^3 + 2x^2 - 1$.

Now, $\quad x^2 - 3x + 2 = (x-1)(x-2)$

So, zeros of $x^2 - 3x + 2$ are 1 and 2. We find that $q(1) = 1 + 2 - 1 = 2 \neq 0$ and $q(2) = 8 + 8 - 1 = 15 \neq 0$. So, the zeros of the quotient $x^2 - 3x + 2$ are not the zeros of $q(x)$. Hence, 1 and 2 are zeros of $p(x)$ which are not zeros of $q(x)$.

EXERCISE 2.3

LEVEL-1

1. Apply division algorithm to find the quotient $q(x)$ and remainder $r(x)$ on dividing $f(x)$ by $g(x)$ in each of the following:
   (i) $f(x) = x^3 - 6x^2 + 11x - 6,\ g(x) = x^2 + x + 1$
   (ii) $f(x) = 10x^4 + 17x^3 - 62x^2 + 30x - 3,\ g(x) = 2x^2 + 7x + 1$
   (iii) $f(x) = 4x^3 + 8x + 8x^2 + 7,\ g(x) = 2x^2 - x + 1$
   (iv) $f(x) = 15x^3 - 20x^2 + 13x - 12,\ g(x) = 2 - 2x + x^2$
2. Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm:
   (i) $g(t) = t^2 - 3,\ f(t) = 2t^4 + 3t^3 - 2t^2 - 9t - 12$ [NCERT]
   (ii) $g(x) = x^3 - 3x + 1,\ f(x) = x^5 - 4x^3 + x^2 + 3x + 1$ [NCERT]
   (iii) $g(x) = 2x^2 - x + 3,\ f(x) = 6x^5 - x^4 + 4x^3 - 5x^2 - x - 15$
3. Obtain all zeros of the polynomial $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$, if two of its zeros are -2 and -1.
4. Obtain all zeros of $f(x) = x^3 + 13x^2 + 32x + 20$, if one of its zeros is -2.
5. Obtain all zeros of the polynomial $f(x) = x^4 - 3x^3 - x^2 + 9x - 6$, if two of its zeros are $-\sqrt{3}$ and $\sqrt{3}$.
6. Find all zeros of the polynomial $f(x) = 2x^4 - 2x^3 - 7x^2 + 3x + 6$, if its two zeros are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$.
7. Find all the zeros of the polynomial $x^4 + x^3 - 34x^2 - 4x + 120$, if two of its zeros are 2 and -2. [CBSE 2008]
8. Find all zeros of the polynomial $2x^4 + 7x^3 - 19x^2 - 14x + 30$, if two of its zeros are $\sqrt{2}$ and $-\sqrt{2}$. [CBSE 2008]

9. Find all the zeros of the polynomial $2x^3 + x^2 - 6x - 3$, if two of its zeros are $-\sqrt{3}$ and $\sqrt{3}$. [CBSE 2009]

10. Find all the zeros of the polynomial $x^3 + 3x^2 - 2x - 6$, if two of its zeros are $-\sqrt{2}$ and $\sqrt{2}$. [CBSE 2009]

11. Find all zeros of the polynomial $2x^4 - 9x^3 + 5x^2 + 3x - 1$, if two of its zeros are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. [CBSE 2018]

LEVEL-2

12. What must be added to the polynomial $f(x) = x^4 + 2x^3 - 2x^2 + x - 1$ so that the resulting polynomial is exactly divisible by $x^2 + 2x - 3$?

13. What must be subtracted from the polynomial $f(x) = x^4 + 2x^3 - 13x^2 - 12x + 21$ so that the resulting polynomial is exactly divisible by $x^2 - 4x + 3$?

14. Given that $\sqrt{2}$ is a zero of the cumbic polynomial $6x^3 + \sqrt{2}x^2 - 10x - 4\sqrt{2}$, find its other two zeroes. [NCERT EXEMPLAR]

15. Given that $x - \sqrt{5}$ is a factor of the cubic polynomial $x^3 - 3\sqrt{5}x^2 + 13x - 3\sqrt{5}$, find all the zeroes of the polynomial. [NCERT EXEMPLAR]

ANSWERS

3. $-\frac{1}{2}, 3, -2, -1$ 4. $-10, -1, -2$ 5. $-\sqrt{3}, \sqrt{3}, 1, 2$ 6. $2, -1, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$

7. $2, -2, 5, -6$ 8. $\sqrt{2}, -\sqrt{2}, -5, \frac{3}{2}$ 9. $-\sqrt{3}, \sqrt{3}, -\frac{1}{2}$ 10. $-\sqrt{2}, \sqrt{2}, -3$

11. $1, -\frac{1}{2}, 2 + \sqrt{3}, 2 - \sqrt{3}$ 12. $x - 2$ 13. $2x - 3$

14. $\frac{-\sqrt{2}}{2}, \frac{-2\sqrt{2}}{3}$ 15. $\sqrt{5}, \sqrt{5} + \sqrt{2}, \sqrt{5} - \sqrt{2}$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per the exact requirement of the questions:

1. Define a polynomial with real coefficients.
2. Define degree of a polynomial.
3. Write the standard form of a linear polynomial with real coefficients.
4. Write the standard form of a quadratic polynomial with real coefficients.
5. Write the standard form of a cubic polynomial with real coefficients.
6. Define value of a polynomial at a point.
7. Define zero of a polynomial.
8. The sum and product of the zeros of a quadratic polynomial are $-\frac{1}{2}$ and -3 respectively. What is the quadratic polynomial.
9. Write the family of quadratic polynomials having $-\frac{1}{4}$ and 1 as its zeros.
10. If the product of zeros of the quadratic polynomial $f(x) = x^2 - 4x + k$ is 3, find the value of k.
11. If the sum of the zeros of the quadratic polynomial $f(x) = kx^2 - 3x + 5$ is 1, write the value of k.

12. In Fig. 2.17, the graph of a polynomial $p(x)$ is given. Find the zeros of the polynomial.

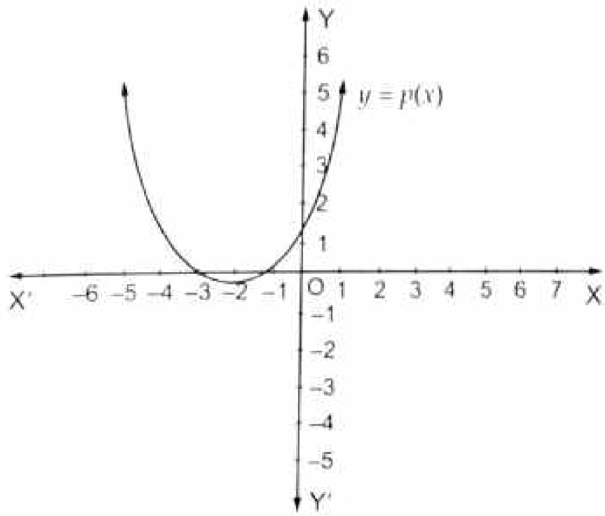


Fig. 2.17

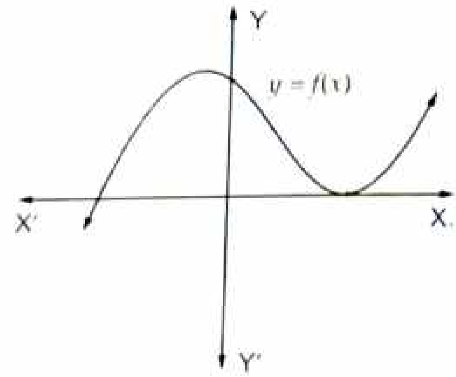


Fig. 2.18

13. The graph of a polynomial $y = f(x)$, shown in Fig. 2.18. Find the number of real zeros of $f(x)$.

14. The graph of the polynomial $f(x) = ax^2 + bx + c$ is as shown below (Fig. 2.19). Write the signs of 'a' and $b^2 - 4ac$.

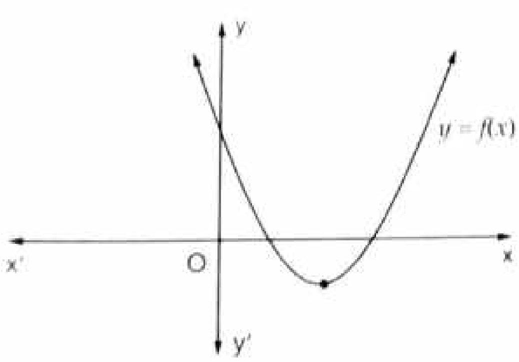


Fig. 2.19

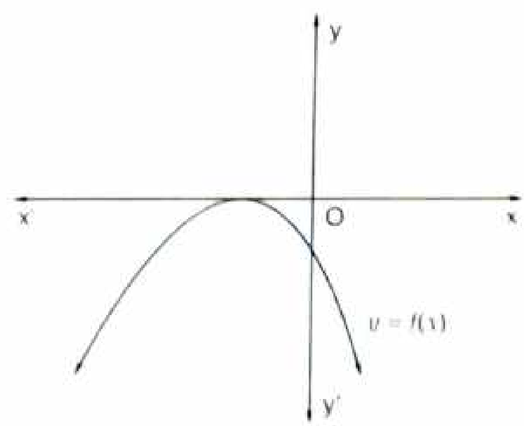


Fig. 2.20

15. The graph of the polynomial $f(x) = ax^2 + bx + c$ is as shown in Fig. 2.20. Write the value of $b^2 - 4ac$ and the number of real zeros of $f(x)$.

16. In Q. No. 14, write the sign of c.

17. In Q. No. 15, write the sign of c.

18. The graph of a polynomial $f(x)$ is as shown in Fig. 2.21. Write the number of real zeros of $f(x)$.

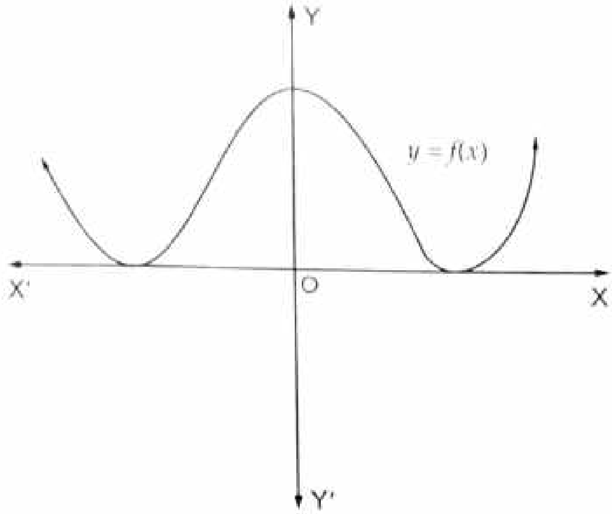


Fig. 2.21

19. If $x = 1$ is a zero of the polynomial $f(x) = x^3 - 2x^2 + 4x + k$, write the value of k.
20. State division algorithm for polynomials.
21. Give an example of polynomials $f(x), g(x), q(x)$ and $r(x)$ satisfying $f(x) = g(x)$ $q(x) + r(x)$, where degree $r(x) = 0$.
22. Write a quadratic polynomial, sum of whose zeros is $2\sqrt{3}$ and their product is 2.
23. If fourth degree polynomial is divided by a quadratic polynomial, write the degree of the remainder.
24. If $f(x) = x^3 + x^2 - ax + b$ is divisible by $x^2 - x$ write the values of a and b.
25. If $a - b$, a and $a + b$ are zeros of the polynomial $f(x) = 2x^3 - 6x^2 + 5x - 7$, write the value of a.
26. Write the coefficients of the polynomial $p(z) = z^5 - 2z^2 + 4$.
27. Write the zeros of the polynomial $x^2 - x - 6$. **[CBSE 2008]**
28. If $(x + a)$ is a factor of $2x^2 + 2ax + 5x + 10$, find a. **[CBSE 2008]**
29. For what value of k, -4 is a zero of the polynomial $x^2 - x - (2k + 2)$? **[CBSE 2009]**
30. If 1 is a zero of the polynomial $p(x) = ax^2 - 3(a - 1)x - 1$, then find the value of a.
31. If α, β are the zeros of a polynomial such that $\alpha + \beta = -6$ and $\alpha\beta = -4$, then write the polynomial. **[CBSE 2010]**
32. If α, β are the zeros of the polynomial $2y^2 + 7y + 5$, write the value of $\alpha + \beta + \alpha\beta$. **[CBSE 2010]**
33. For what value of k, is 3 a zero of the polynomial $2x^2 + x + k$? **[CBSE 2010]**
34. For what value of k, is -3 a zero of the polynomial $x^2 + 11x + k$? **[CBSE 2010]**
35. For what value of k, is -2 a zero of the polynomial $3x^2 + 4x + 2k$? **[CBSE 2010]**
36. If a quadratic polynomial $f(x)$ is factorizable into linear distinct factors, then what is the total number of real and distinct zeros of $f(x)$?

37. If a quadratic polynomial $f(x)$ is a square of a linear polynomial, then its two zeroes are coincident. (True/False)

38. If a quadratic polynomial $f(x)$ is not factorizable into linear factors, then it has no real zero. (True/False)

39. If $f(x)$ is a polynomial such that $f(a)\,f(b) < 0$, then what is the number of zeros lying between a and b?

40. If graph of quadratic polynomial ax^2+bx+c cuts positive direction of y-axis, then what is the sign of c?

41. If the graph of quadratic polynomial ax^2+bx+c cuts negative direction of y-axis, then what is the sign of c?

ANSWERS

3. $f(x) = ax + b, a \neq 0$ 4. $f(x) = ax^2 + bx + c, a \neq 0$ 5. $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$

8. $f(x) = k\left(x^2 + \frac{x}{2} - 3\right)$, where k is any non-zero real number.

9. $f(x) = k\left(x^2 - \frac{3}{4}x - \frac{1}{4}\right)$, where k is any non-zero real number. 10. $k = 3$ 11. 3

12. -3 and -1 13. 3 14. $a > 0, b^2 - 4ac > 0$ 15. $b^2 - 4ac = 0$, Two

16. $c > 0$ 17. $c < 0$ 18. 4 19. $k = -3$

21. $f(x) = x^3 + x^2 + x + 1$, $g(x) = x + 2, q(x) = x^2 - x + 3, r(x) = -5$

22. $f(x) = x^2 - 2\sqrt{3}x + 2$ 23. Less than or equal to 1 24. $a = 2, b = 0$

25. 1 26. 1, 0, 0, −2, 0, 4 27. 3, −2 28. 2 29. 9 30. 1

31. $f(x) = x^2 + 6x - 4$ 32. −1 33. −21 34. 24 35. −2 36. 2

37. True 38. True 39. At least one 40. Positive 41. Negative

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If α, β are the zeros of the polynomial $f(x) = x^2 + x + 1$, then $\frac{1}{\alpha} + \frac{1}{\beta} =$

   (a) 1 (b) −1 (c) 0 (d) None of these

2. If α, β are the zeros of the polynomial $p(x) = 4x^2 + 3x + 7$, then $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to

   (a) $\frac{7}{3}$ (b) $-\frac{7}{3}$ (c) $\frac{3}{7}$ (d) $-\frac{3}{7}$

3. If one zero of the polynomial $f(x) = (k^2 + 4)x^2 + 13x + 4k$ is reciprocal of the other, then $k =$

   (a) 2 (b) −2 (c) 1 (d) −1

4. If the sum of the zeros of the polynomial $f(x) = 2x^3 - 3kx^2 + 4x - 5$ is 6, then the value of k is

(a) 2 (b) 4 (c) -2 (d) -4

5. If α and β are the zeros of the polynomial $f(x) = x^2 + px + q$, then a polynomial having $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is its zeros is

(a) $x^2 + qx + p$ (b) $x^2 - px + q$ (c) $qx^2 + px + 1$ (d) $px^2 + qx + 1$

6. If α, β are the zeros of polynomial $f(x) = x^2 - p(x + 1) - c$, then $(\alpha + 1)(\beta + 1) =$

(a) $c - 1$ (b) $1 - c$ (c) c (d) $1 + c$

7. If α, β are the zeros of the polynomial $f(x) = x^2 - p(x + 1) - c$ such that $(\alpha + 1)(\beta + 1) = 0$, then $c =$

(a) 1 (b) 0 (c) -1 (d) 2

8. If $f(x) = ax^2 + bx + c$ has no real zeros and $a + b + c < 0$, then

(a) $c = 0$ (b) $c > 0$ (c) $c < 0$ (d) None of these

9. If the diagram in Fig. 2.22 shows the graph of the polynomial $f(x) = ax^2 + bx + c$, then

(a) $a > 0, b < 0$ and $c > 0$ (b) $a < 0, b < 0$ and $c < 0$

(c) $a < 0, b > 0$ and $c > 0$ (d) $a < 0, b > 0$ and $c < 0$

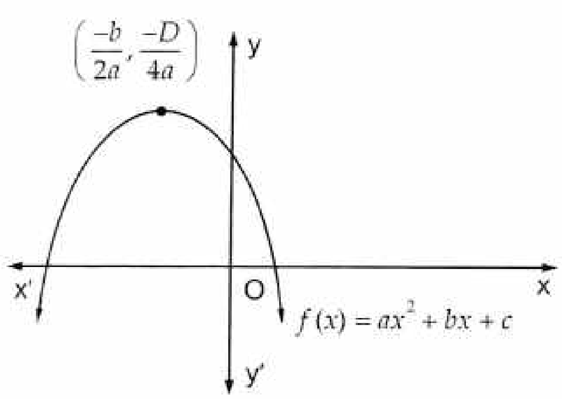


Fig. 2.22

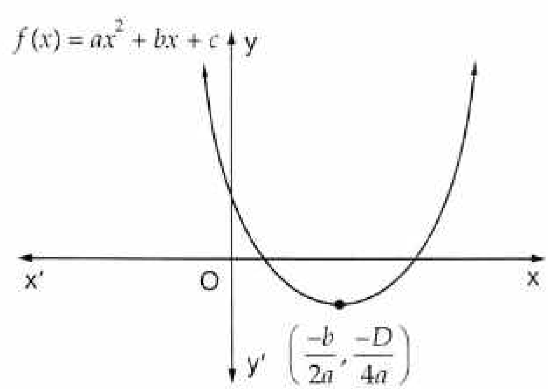


Fig. 2.23

10. Figure 2.23 shows the graph of the polynomial $f(x) = ax^2 + bx + c$ for which

(a) $a < 0, b > 0$ and $c > 0$ (b) $a < 0, b < 0$ and $c > 0$

(c) $a < 0, b < 0$ and $c < 0$ (d) $a > 0, b > 0$ and $c < 0$

11. If the product of zeros of the polynomial $f(x) = ax^3 - 6x^2 + 11x - 6$ is 4, then $a =$

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

12. If zeros of the polynomial $f(x) = x^3 - 3px^2 + qx - r$ are in A.P., then

(a) $2p^3 = pq - r$ (b) $2p^3 = pq + r$ (c) $p^3 = pq - r$ (d) None of these

13. If the product of two zeros of the polynomial $f(x) = 2x^3 + 6x^2 - 4x + 9$ is 3, then its third zero is

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{9}{2}$ (d) $-\frac{9}{2}$

14. If the polynomial $f(x) = ax^3 + bx - c$ is divisible by the polynomial $g(x) = x^2 + bx + c$, then $ab =$

(a) 1 (b) $\frac{1}{c}$ (c) -1 (d) $-\frac{1}{c}$

15. In Q. No. 14, $c =$
(a) b (b) $2b$ (c) $2b^2$ (d) $-2b$

16. If one root of the polynomial $f(x) = 5x^2 + 13x + k$ is reciprocal of the other, then the value of k is

(a) 0 (b) 5 (c) $\frac{1}{6}$ (d) 6

17. If α, β, γ are the zeros of the polynomial $f(x) = ax^3 + bx^2 + cx + d$, then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} =$

(a) $-\frac{b}{d}$ (b) $\frac{c}{d}$ (c) $-\frac{c}{d}$ (d) $-\frac{c}{a}$

18. If α, β, γ are the zeros of the polynomial $f(x) = ax^3 + bx^2 + cx + d$, then $\alpha^2 + \beta^2 + \gamma^2 =$

(a) $\frac{b^2 - ac}{a^2}$ (b) $\frac{b^2 - 2ac}{a}$ (c) $\frac{b^2 + 2ac}{b^2}$ (d) $\frac{b^2 - 2ac}{a^2}$

19. If α, β, γ are the zeros of the polynomial $f(x) = x^3 - px^2 + qx - r$, then $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} =$

(a) $\frac{r}{p}$ (b) $\frac{p}{r}$ (c) $-\frac{p}{r}$ (d) $-\frac{r}{p}$

20. If α, β are the zeros of the polynomial $f(x) = ax^2 + bx + c$, then $\frac{1}{\alpha^2} + \frac{1}{\beta^2} =$

(a) $\frac{b^2 - 2ac}{a^2}$ (b) $\frac{b^2 - 2ac}{c^2}$ (c) $\frac{b^2 + 2ac}{a^2}$ (d) $\frac{b^2 + 2ac}{c^2}$

21. If two of the zeros of the cubic polynomial $ax^3 + bx^2 + cx + d$ are each equal to zero, then the third zero is

(a) $\frac{-d}{a}$ (b) $\frac{c}{a}$ (c) $\frac{-b}{a}$ (d) $\frac{b}{a}$

22. If two zeros of $x^3 + x^2 - 5x - 5$ are $\sqrt{5}$ and $-\sqrt{5}$, then its third zero is
(a) 1 (b) -1 (c) 2 (d) -2

23. The product of the zeros of $x^3 + 4x^2 + x - 6$ is
(a) -4 (b) 4 (c) 6 (d) -6

24. What should be added to the polynomial $x^2 - 5x + 4$, so that 3 is the zero of the resulting polynomial?
   (a) 1 (b) 2 (c) 4 (d) 5

25. What should be subtracted to the polynomial $x^2 - 16x + 30$, so that 15 is the zero of the resulting polynomial?
   (a) 30 (b) 14 (c) 15 (d) 16

26. A quadratic polynomial, the sum of whose zeroes is 0 and one zero is 3, is
   (a) $x^2 - 9$ (b) $x^2 + 9$ (c) $x^2 + 3$ (d) $x^2 - 3$

27. If two zeroes of the polynomial $x^3 + x^2 - 9x - 9$ are 3 and –3, then its third zero is
   (a) –1 (b) 1 (c) –9 (d) 9

28. If $\sqrt{5}$ and $-\sqrt{5}$ are two zeroes of the polynomial $x^3 + 3x^2 - 5x - 15$, then its third zero is
   (a) 3 (b) –3 (c) 5 (d) –5

29. If $x + 2$ is a factor of $x^2 + ax + 2b$ and $a + b = 4$, then
   (a) $a = 1, b = 3$ (b) $a = 3, b = 1$ (c) $a = -1, b = 5$ (d) $a = 5, b = -1$

30. The polynomial which when divided by $-x^2 + x - 1$ gives a quotient $x - 2$ and remainder 3, is
   (a) $x^3 - 3x^2 + 3x - 5$ (b) $-x^3 - 3x^2 - 3x - 5$
   (c) $-x^3 + 3x^2 - 3x + 5$ (d) $x^3 - 3x^2 - 3x + 5$

31. The number of polynomials having zeroes –2 and 5 is
   (a) 1 (b) 2 (c) 3 (d) more than 3.

32. If one of the zeroes of the quadratic polynomial $(k - 1)x^2 + kx + 1$ is –3, then the value of k is
   (a) $\frac{4}{3}$ (b) $-\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

33. The zeroes of the quadratic polynomial $x^2 + 99x + 127$ are
   (a) both positive (b) both negative
   (c) both equal (d) one positive and one negative

34. If the zeroes of the quadratic polynomial $x^2 + (a + 1)x + b$ are 2 and –3, then
   (a) $a = -7, b = -1$ (b) $a = 5, b = -1$ (c) $a = 2, b = -6$ (d) $a = 0, b = -6$

35. Given that one of the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$ is zero, the product of the other two zeroes is
   (a) $-\frac{c}{a}$ (b) $\frac{c}{a}$ (c) 0 (d) $-\frac{b}{a}$

36. The zeroes of the quadratic polynomial $x^2 + ax + a, a \neq 0$,
   (a) cannot both be positive (b) cannot both be negative
   (c) are always unequal (d) are always equal

37. If one of the zeros of the cubic polynomial $x^3 + ax^2 + bx + c$ is –1, then the product of other two zeros is
   (a) $b - a + 1$ (b) $b - a - 1$ (c) $a - b + 1$ (d) $a - b - 1$

38. Given that two of the zeros of the cubic polynomial $ax^3 + bx^2 + cx + d$ are 0, the third zero is

(a) $-\frac{b}{a}$ (b) $\frac{b}{a}$ (c) $\frac{c}{a}$ (d) $-\frac{d}{a}$

39. If one zero of the quadratic polynomial $x^2 + 3x + k$ is 2, then the value of k is

(a) 10 (b) -10 (c) 5 (d) -5

40. If the zeros of the quadratic polynomial $ax^2 + bx + c, c \neq 0$ are equal, then

(a) c and a have opposite signs (b) c and b have opposite signs

(c) c and a have the same sign (d) c and b have the same sign

41. If one of the zeros of a quadratic polynomial of the form $x^2 + ax + b$ is the negative of the other, then it

(a) has no linear term and constant term is negative.

(b) has no linear term and the constant term is positive.

(c) can have a linear term but the constant term is negative.

(d) can have a linear term but the constant term is positive.

42. Which of the following is not the graph of a quadratic polynomial?

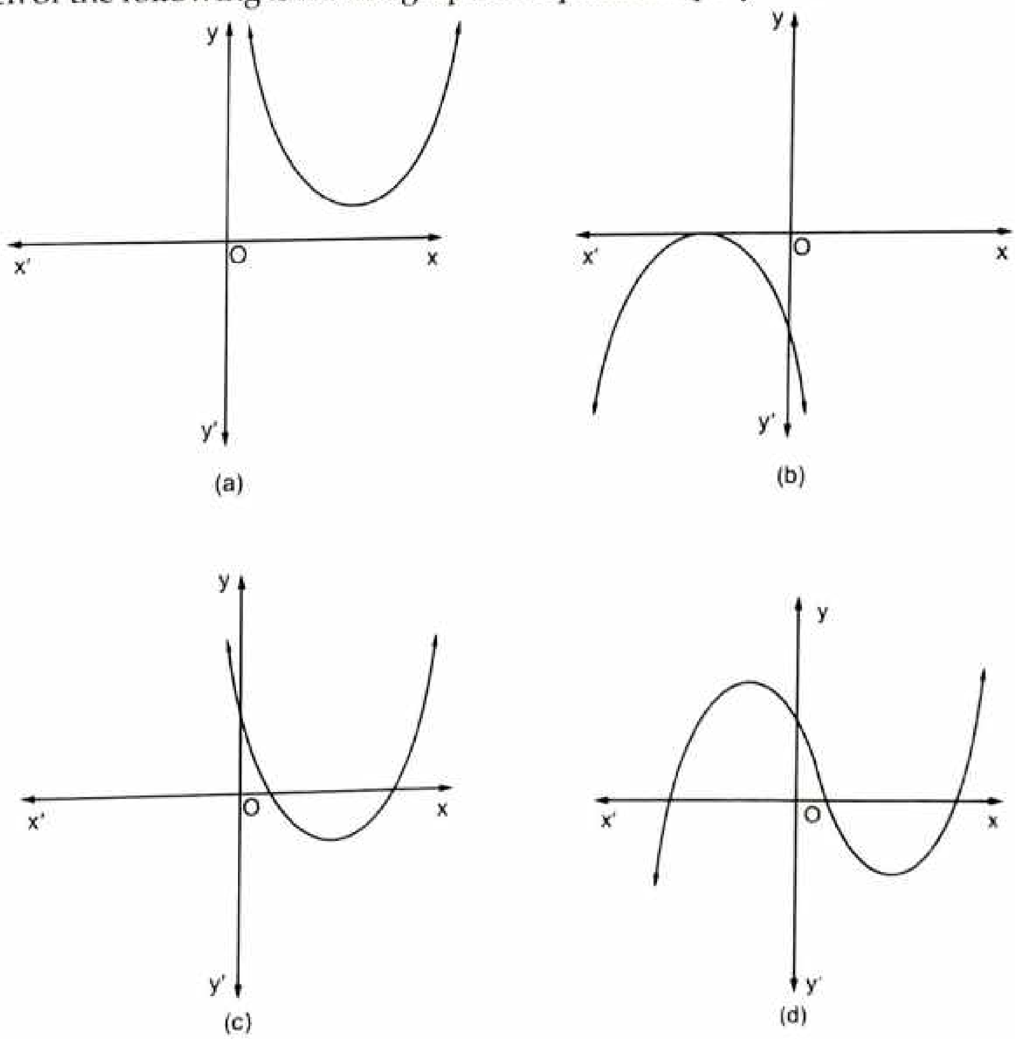


Fig. 2.24

ANSWERS

1. (b)	2. (d)	3. (a)	4. (b)	5. (c)	6. (b)	7. (a)
8. (c)	9. (a)	10. (b)	11. (a)	12. (a)	13. (b)	14. (a)
15. (b)	16. (b)	17. (c)	18. (d)	19. (b)	20. (b)	21. (c)
22. (b)	23. (c)	24. (b)	25. (c)	26. (a)	27. (a)	28. (b)
29. (b)	30. (c)	31. (d)	32. (a)	33. (b)	34. (d)	35. (b)
36. (a)	37. (a)	38. (a)	39. (b)	40. (c)	41. (a)	42. (d)

SUMMARY

1. Let x be a variable, n be a positive integer and $a_0, a_1, a_2, \dots, a_n$ be constants (real numbers). Then, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, is called a polynomial in variable x.
2. The exponent of the highest degree term in a polynomial is known as its degree.

   A polynomial of degree 0 is called a constant polynomial.

   A polynomial of degree 1, 2 or 3 is called a linear polynomial, a quadratic polynomial or a cubic polynomial respectively.

   Following are the forms of various degree polynomials.

Degree	Name of the polynomial	Form of the polynomial
0	Constant polynomial	$f(x) = a$, a is a constant
1	Linear polynomial	$f(x) = ax + b, a \neq 0$
2	Quadratic polynomial	$f(x) = ax^2 + bx + c, a \neq 0$
3	Cubic polynomial	$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$
4	Biquadratic polynomial	$f(x) = ax^4 + bx^3 + cx^2 + dx + e,\ a \neq 0$

3. If $f(x)$ is a polynomial and α is any real number, then the real number obtained by replacing x by α in $f(x)$ is known as the value of $f(x)$ at $x = \alpha$ and is denoted by $f(\alpha)$.
4. A real number α is a zero of a polynomial $f(x)$, if $f(\alpha) = 0$.
5. A polynomial of degree n can have at most n real zeros.
6. Geometrically the zeros of a polynomial $f(x)$ are the x-coordinates of the points where the graph $y = f(x)$ intersects x-axis.
7. If α and β are the zeros of a quadratic polynomial $f(x) = ax^2 + bx + c$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2},\quad \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

8. If α, β, γ are the zeros of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3},\quad \alpha\beta\gamma = -\frac{d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

9. If $\alpha, \beta, \gamma, \delta$ are the zeros of a biquadratic polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, then

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$(\alpha + \beta)\gamma\delta + \alpha\beta(\gamma + \delta) = -\frac{d}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}, \quad \alpha\beta\gamma\delta = \frac{e}{a} = \frac{\text{Constant terms}}{\text{Coefficient of } x^4}$$

10. If $f(x)$ is a polynomial and $g(x)$ is a non-zero polynomial, then there exist two polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x) \times q(x) + r(x)$, where $r(x) = 0$ or degree $r(x) <$ degree $g(x)$. This is known as the division algorithm.

NOTE: Formative assessment also includes lab activities, projects, assignments (Home work), oral and visual testings.

CHAPTER 3

PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

3.1 INTRODUCTION

In the middle school mathematics, we have learnt about linear equations in one variable and their applications in solving word problems. If a and b are two real numbers such that $x \neq 0$ and x is a variable, then as we have learnt that an equation of the form $ax = b$ or, $ax + b = 0$ is called a linear equation in one variable. Recall that a value of the variable which satisfies a given linear equation in one variable is known as its solution.

In class IX, we have learnt about linear equations in two variables. The general form of a linear equation in two variables is $ax + by + c = 0$ or, $ax + by = c$ where a, b, c are real numbers such that $a \neq 0, b \neq 0$ and x, y are variables (we often denote the condition a and b are not both zero by $a^2 + b^2 \neq 0$). Any pair of values of x and y which satisfies the equation $ax + by + c = 0$ or $ax + by = c$ is called its solution. For example, $x = 2$ and $y = 1$ is a solution of the equation $4x - 3y = 5$. We have also learnt about the graph of a linear equation. The graph of a linear equation in one variable is a straight line parallel to x-axis or y-axis according as the equation is of the form $ay = b$ or $ax = b$, where $a \neq 0$. The graph of a linear equation in two variables is also a straight line. The coordinates of every point on the line representing a linear equation determine a solution of the equation and every solution of linear equation is represented by a point on the line represented by it. Thus, there is one-to-one correspondence between the solutions of a linear equation and points lying on the straight line represented by it.

In this chapter, we shall study about systems of linear equations in two variables, solution of a system of linear equations in two variables and graphical and algebraic methods of solving a system of linear equations in two variables. In the end of the chapter, we shall be discussing some applications of linear equations in two variables in solving simple problems from different areas.

3.2 SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES

In earlier class, we have studies about a linear equation in two variables. In this section, we shall introduce the notion of system of simultaneous linear equations as defined below.

DEFINITION *A pair of linear equations in two variables is said to form a system of simultaneous linear equations.*

Each of the following pairs of linear equations forms a system of two simultaneous linear equations in two variables:

(i) $x + 2y = 3$
$2x - y = 5$

(ii) $2u + 5v + 1 = 0$
$u - 2v + 9 = 0$

(iii) $\frac{3}{x} + \frac{2}{y} = 9$

$\frac{1}{x} - \frac{1}{y} = 5$

(iv) $2a + b - 1 = 0$

$a + b + 5 = 0.$

The general form of a pair of linear equations in two variables x and y is

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0,$$

where $a_1, b_1, c_1, a_2, b_2\ c_2$ are all real numbers and $a_1^2 + b_1^2 \neq 0, a_2^2 + b_2^2 \neq 0$. This is known as the algebraic representation of a system of simultaneous linear equations in two variables.

SOLUTION *A pair of values of the variables x and y satisfying each one of the equations in a given system of two simultaneous linear equations in x and y is called a solution of the system.*

Clearly, $x = 2, y = -1$ is a solution of the system of simultaneous linear equations

$$x + y = 1$$
$$2x - 3y = 7.$$

ILLUSTRATION 1 *Show that $x = 2, y = 1$ is a solution of the system of simultaneous linear equations*

$$3x - 2y = 4$$
$$2x + y = 5.$$

SOLUTION The given system of equations is

$$3x - 2y = 4 \quad \ldots(i)$$
$$2x + y = 5 \quad \ldots(ii)$$

Putting $x = 2$ and $y = 1$ in equation (i), we have

$$\text{LHS} = 3 \times 2 - 2 \times 1 = 4 = \text{RHS}$$

Putting $x = 2$ and $y = 1$ in equation (ii), we have

$$\text{LHS} = 2 \times 2 + 1 \times 1 = 5 = \text{RHS}$$

Thus, $x = 2$ and $y = 1$ satisfy both the equations of the given system.

Hence, $x = 2, y = 1$ is a solution of the given system.

ILLUSTRATION 2 *Show that $x = 2, y = 1$ is not a solution of the system of simultaneous linear equations*

$$2x + 7y = 11$$
$$x - 3y = 5$$

SOLUTION The given system of equations is

$$2x + 7y = 11 \quad \ldots(i)$$
$$x - 3y = 5. \quad \ldots(ii)$$

Putting $x = 2, y = 1$ in equation (i), we have

$$\text{LHS} = 2 \times 2 + 7 \times 1 = 11 = \text{RHS}$$

So, $x = 2$ and $y = 1$ satisfy equation (i)

Putting $x = 2, y = 1$ in equation (i), we have,

$$\text{LHS} = 2 \times 1 - 3 \times 1 = -1 \neq \text{RHS}$$

So, $x = 2$ and $y = 1$ do not satisfy equation (ii)

Hence, $x = 2, y = 1$ is not a solution of the given system of equations.

ILLUSTRATION 3 *Show that* $x = 2, y = 1$ *and* $x = 4, y = 4$ *are solutions of the system of equations*

$$3x - 2y = 4$$
$$6x - 4y = 8.$$

SOLUTION The given system of equations is

$$3x - 2y = 4 \quad \ldots\text{(i)}$$
$$6x - 4y = 8 \quad \ldots\text{(ii)}$$

Putting $x = 2$ and $y = 1$ in equation (i) and (ii) respectively, we get

$$\text{LHS} = 3 \times 2 - 2 \times 1 = 4 = \text{RHS}$$
$$\text{LHS} = 6 \times 2 - 4 \times 1 = 8 = \text{RHS}$$

So, $x = 2, y = 1$ is a solution of the given system of equations.

Similarly, it can be checked that $x = 4, y = 4$ is also a solution of the given system.

Hence $x = 2, y = 1$ and $x = 4, y = 4$ are solutions of the given system of equations.

In the above discussion, we have seen that a system of linear equations will have either a unique solution or an infinitely many solutions or no solution. If a system of simultaneous linear equations has a solution (either unique or infinitely many), then the system is said to be consistent of other wise it is said to be an in-consistent system as defined below.

CONSISTENT SYSTEM *A system of simultaneous linear equations is said to be consistent, if it has at least one solution.*

IN-CONSISTENT SYSTEM *A system of simultaneous linear equations is said to be in-consistent, if it has no solution.*

Clearly, systems of equations discussed in illustrations 1, 2, and 3 are consistent whereas the system of equations $x - 2y = 1, 2x - 4y = 3$ is in-consistent because there is no pair of values of x and y which satisfies the two equations simultaneously.

3.3 GRAPHICAL REPRESENTATION OF LINEAR EQUATIONS

In the previous section, we have seen what a pair of linear equations in two variables look like algebraically? In class IX, we have learnt that the graphical (i.e. geometric) representation of a linear equation in two variables is a straight line such that every point on the line represents a solution of the equation and every solution of the equation is represented by a point on the line. Let us now see what a pair of linear equations in two variables will look like, graphically? Since a linear equation in two variables represents a straight line. Therefore, a pair of linear equations in two variables will be represented by two straight lines, both to be considered together. We know that given two lines in a plane, only one of the following three possibilities can happen:

(i) The two lines intersect at one point.

(ii) The two lines are parallel i.e. they do not intersect however far they are extended.

(iii) The two lines are coincident lines i.e. one line overlaps the other line.

Thus, the graphical representation of a pair of simultaneous linear equations in two variables will be in one of the following forms.

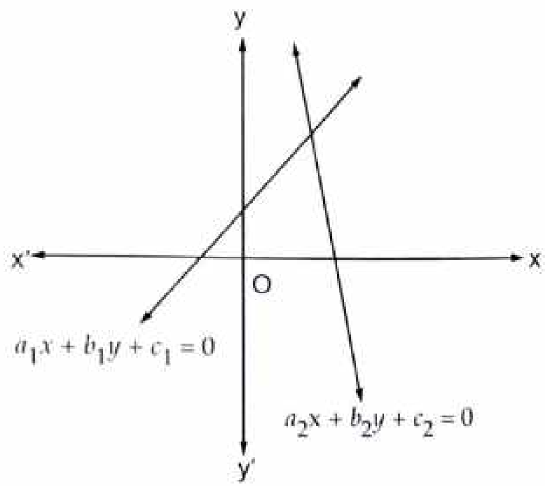


Fig. 3.1 (Intersecting lines)

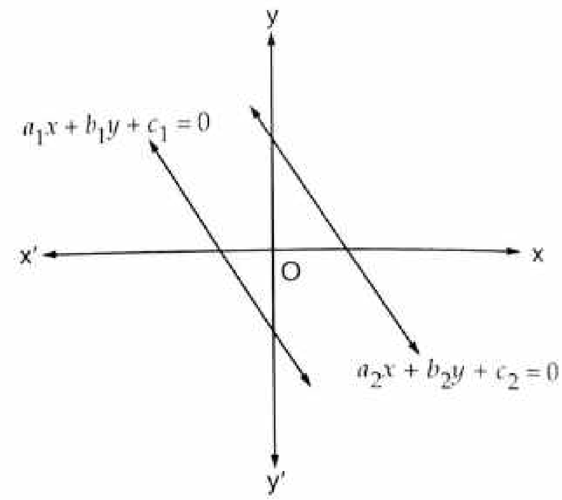


Fig. 3.2 (Parallel lines)

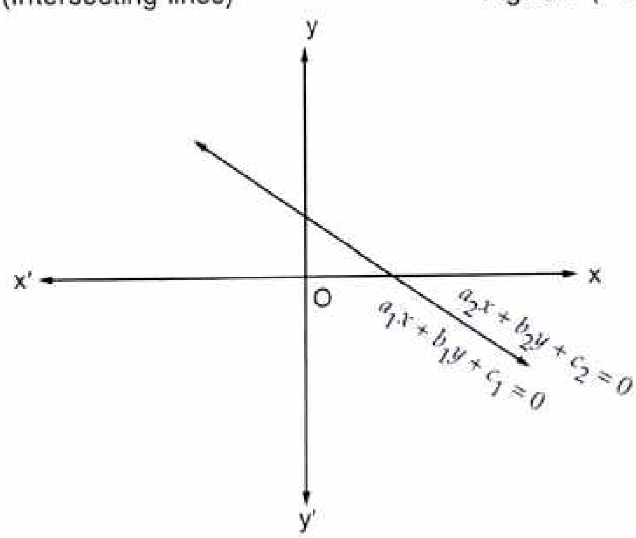


Fig. 3.3 (Coincident lines)

Let us now consider some examples on formulation, algrbraic and graphical representation of a pair of linear equations in two variables.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Ten students of class X took part in Mathematics quiz. If the number of girls is 4 more than the number of boys. Represent this situation algebraically and graphically.*

SOLUTION *Formulation:* Let the number of girls be x and the number of boys be y.

It is given that total ten students took part in the quiz.

$\therefore$ Number of girls + Number of boys = 10

i.e. $x + y = 10$

It is also given that the number of girls is 4 more than the number of boys.

$\therefore$ Number of girls = Number of boys + 4

i.e. $x = y + 4$

or, $x - y = 4$

Algebraic Representation: Thus, the algebraic representation of the given situation is

$$x + y = 10 \quad \text{...(i)}$$

$$x - y = 4 \quad \text{...(ii)}$$

Graphical Representation: In order to represent the above pair of linear equations graphically, we will have to find two points on the line represented by each equation. That is, we will have to find two solutions of each equation. As we have in class IX that there are infinitely many solutions of each linear equation. So, we can choose any two solutions of each equation. We know that it is always convenient to plot points having integral coordinates on the graph paper in comparison to points with fractional coordinates. So, we choose solutions having integral values. For this, we give such an integral value to one of the variables that the value of the other variable is also an integer. The most convenient integer value is zero. So, putting $y = 0$ in $x + y = 10$, we get $x = 10$. Similarly, by putting $x = 0$ in $x + y = 10$, we get $y = 10$.

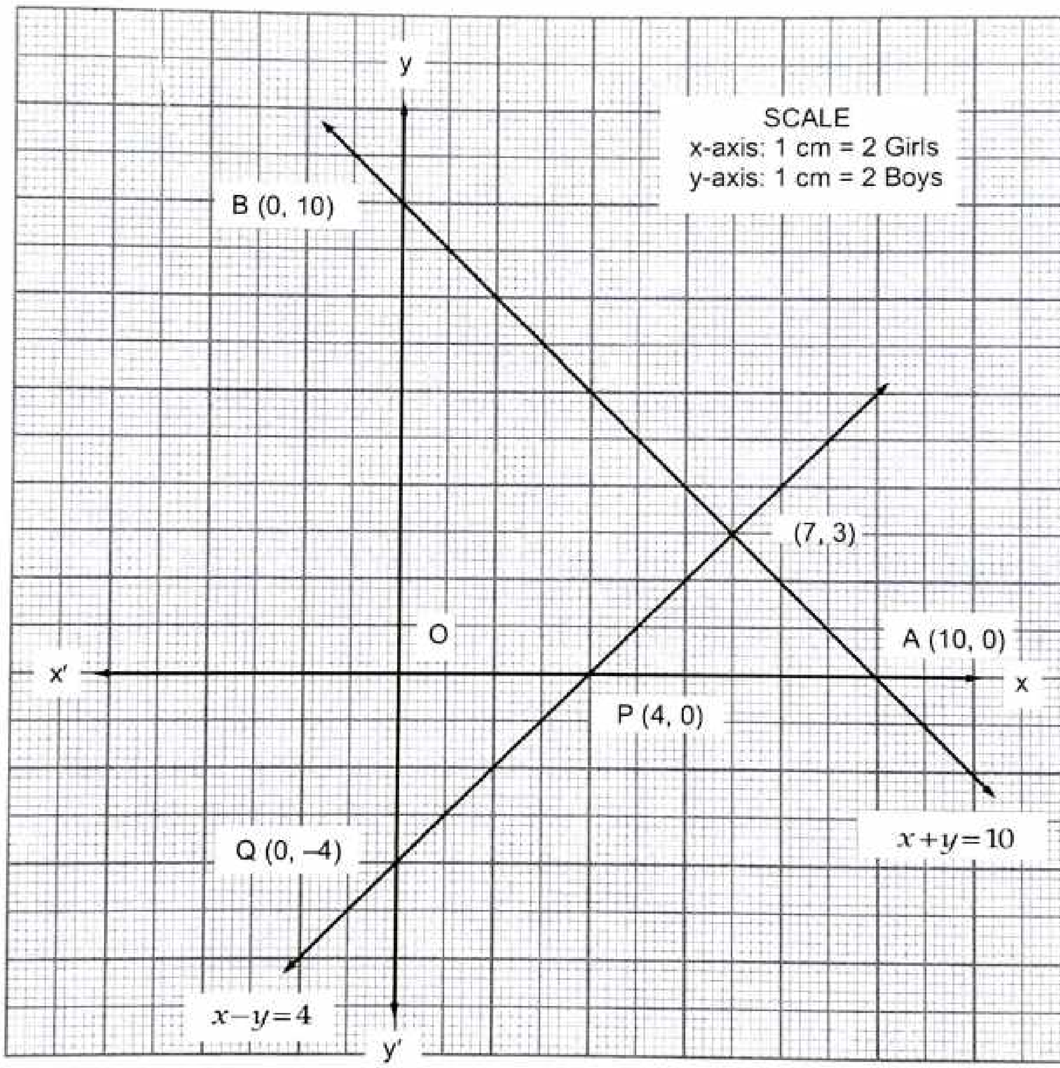


Fig. 3.4

Thus, two solutions of equation (i) are:

x	10	0
y	0	10

Similarly, two solutions of equation (ii) are:

x	4	0
y	0	-4

Now, we plot the points A (10, 0), B (0, 10), P (4, 0) and Q (0, –4) corresponding to these solutions on the graph paper and draw the lines AB and PQ representing the equations $x + y = 10$ and $x - y = 4$ as shown in Fig. 3.4.

We observe that the two lines representing the two equations are intersecting at the point (7, 3).

EXAMPLE 2 *The coach of a cricket team buys 3 bats and 6 balls for ₹3900. Later, he buys another bat and 3 more balls of the same kind for ₹ 1300. Represent this situation algebraically and geometrically.* **[NCERT]**

SOLUTION *Formulation:* Let the price of a bat be ₹ x and that of a ball be ₹ y.
It is given that 3 bats and 6 balls are bought for ₹ 3900.

$$\therefore \quad 3x + 6y = 3900$$

It is also given that one bat and 3 balls of the same kind cost ₹ 1300.

$$\therefore \quad x + 3y = 1300$$

Algebraic Representation: The algebraic representation of the given situation is

$$3x + 6y = 3900 \qquad \dots(i)$$

$$x + 3y = 1300 \qquad \dots(ii)$$

Graphical Representation: In order to obtain the equivalent graphical representation, we find two points on the line representing each equation. That is, we find two solutions of each equation.

We have,

$$3x + 6y = 3900$$

When $y = 0$, we have

$$3x + 0 = 3900 \Rightarrow x = \frac{3900}{3} = 1300$$

When $x = 0$, we have

$$0 + 6y = 3900 \Rightarrow y = \frac{3900}{6} = 650$$

Thus, two solutions of equation (i) are:

x	1300	0
y	0	650

We have,

$$x + 3y = 1300$$

When $y = 100$, we have

$$x + 300 = 1300 \Rightarrow x = 1000$$

When $x = 100$, we have

$$100 + 3y = 1300 \Rightarrow 3y = 1200 \Rightarrow y = 400$$

Thus, two solutions of equation (ii) are:

x	1000	100
y	100	400

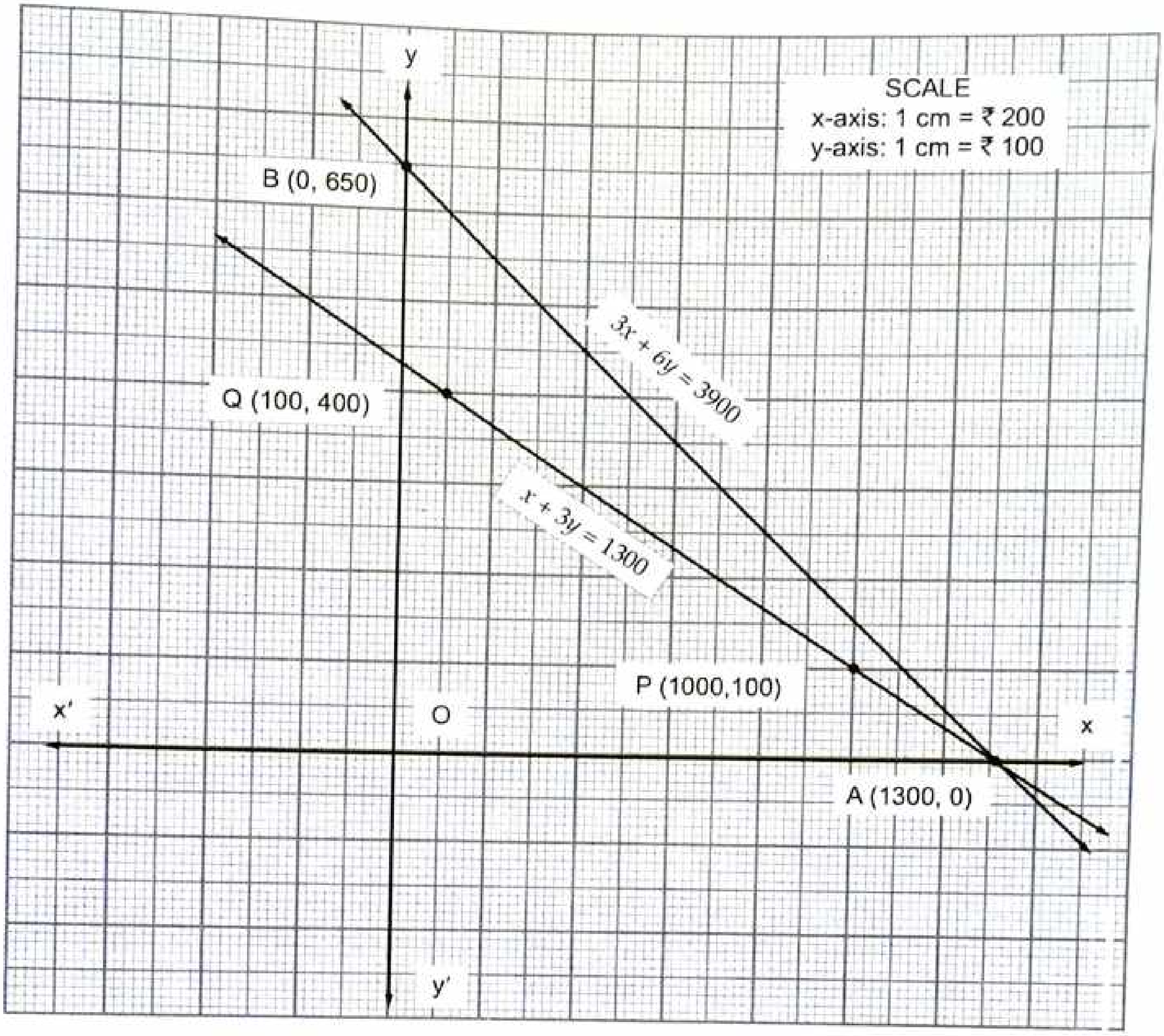


Fig. 3.5

Now, we plot the points A (1300, 0) and B (0, 650) and draw the line AB passing through these two points to represent equation $3x + 6y = 3900$ as shown in Fig. 3.5. To represent the equation $x + 3y = 1300$, we plot the points P (1000, 100) and Q (100, 400) and the line passing through these points is as shown in Fig. 3.5.

We observe that the two lines representing the two equations are intersecting at the point A (1300, 0).

REMARK *If we look at the graphical (geometrical) representation of the pair of linear equations in the above examples, we find that each pair represents intersecting lines. The pair of linear equations in Example 2 is*

$$3x + 6y - 3900 = 0$$

$$x + 3y - 1300 = 0$$

or,

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 3, b_2 = 6, c_1 = -3900, \ a_2 = 1, b_2 = 3, c_2 = -1300$

We have,

$$\frac{a_1}{a_2} = \frac{3}{1} = 3 \text{ and } \frac{b_1}{b_2} = \frac{6}{3} = 2$$

$\therefore \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Thus, the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

will represent intersecting lines, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. *The converse is also true for any pair of linear equations.*

EXAMPLE 3 *Romila went to a stationary stall and purchased 2 pencils and 3 erasers for ₹ 9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹ 18. Represent this situation algebraically and graphically.* **[NCERT]**

SOLUTION *Formulation:* Let the cost of 1 pencil be ₹ x and that of one eraser be ₹ y.

It is given that Romila purchased 2 pencils and 3 erasers for ₹ 9.

$\therefore \quad 2x + 3y = 9$

It is also given that Sonali purchased 4 pencils and 6 erasers for ₹ 18.

$\therefore \quad 4x + 6y = 18$

Algebraic Representation: The algebraic representation of the given situation is

$$2x + 3y = 9 \quad \ldots(i)$$
$$4x + 6y = 18 \quad \ldots(ii)$$

Graphical Representation: In order to obtain the graphical representation of the above pair of linear equations, we find two points on the line representing each equation. That is, we find two solutions of each equation. Let us find these solutions. We will try to find solutions having integral values.

We have,

$$2x + 3y = 9$$

Putting $x = -3$, we get

$$-6 + 3y = 9 \Rightarrow 3y = 15 \Rightarrow y = 5$$

Putting $x = 0$, we get

$$0 + 3y = 9 \Rightarrow y = 3$$

Thus, two solutions of $2x + 3y = 9$ are:

x	-3	0
y	5	3

We have,

$$4x + 6y = 18$$

Putting $x = 3$, we get

$$12 + 6y = 18 \Rightarrow 6y = 6 \Rightarrow y = 1$$

Putting $x = -6$, we get

$$-24 + 6y = 18 \Rightarrow 6y = 42 \quad y = 7$$

Thus, two solutions of $4x + 6y = 18$ are

x	3	-6
y	1	7

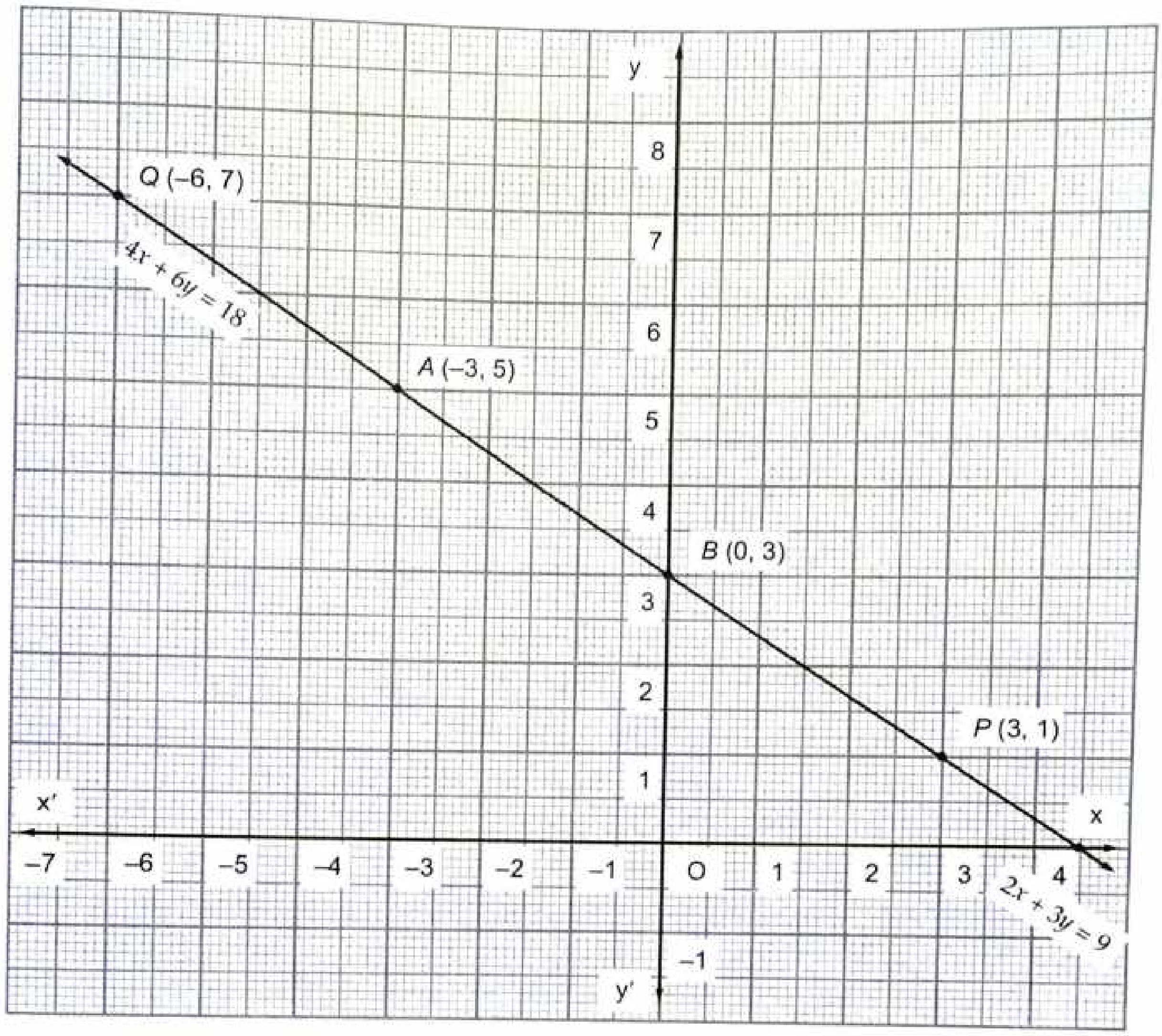


Fig. 3.6

Now, we plot the points $A(-3, 5)$ and $B(0, 3)$ and draw the line passing through these points to obtain the graph of the line $2x + 3y = 9$. Points $P(3, 1)$ and $Q(-6, 7)$ are plotted on the graph paper and we join them to obtain the graph of the line $4x + 6y = 18$. We find that both the lines AB and PQ coincide.

REMARK *Graphical representation of the pair of linear equations in the above example provides us coincident lines. Let us write the above pair of linear equations as*

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

where $a_1 = 2, b_1 = 3, c_1 = -9, a_2 = 4, b_2 = 6, c_2 = -18$

We observe that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$$

Thus, the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

will represent coincident lines, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

The converse is also true for any pair of linear equations.

EXAMPLE 4 *The path of a train A is given by the equation $x + 2y - 4 = 0$ and the path of another train B is given by the equation $2x + 4y - 12 = 0$. Represent this situation graphically.*

[NCERT]

SOLUTION The paths of two trains are given by the following pair of linear equations.

$$x + 2y - 4 = 0 \quad \ldots(i)$$

$$2x + 4y - 12 = 0 \quad \ldots(ii)$$

In order to represent the above pair of linear equations graphically, we need two points on the line representing each equation. That is, we find two solutions of each equation as given below:

We have,

$$x + 2y - 4 = 0$$

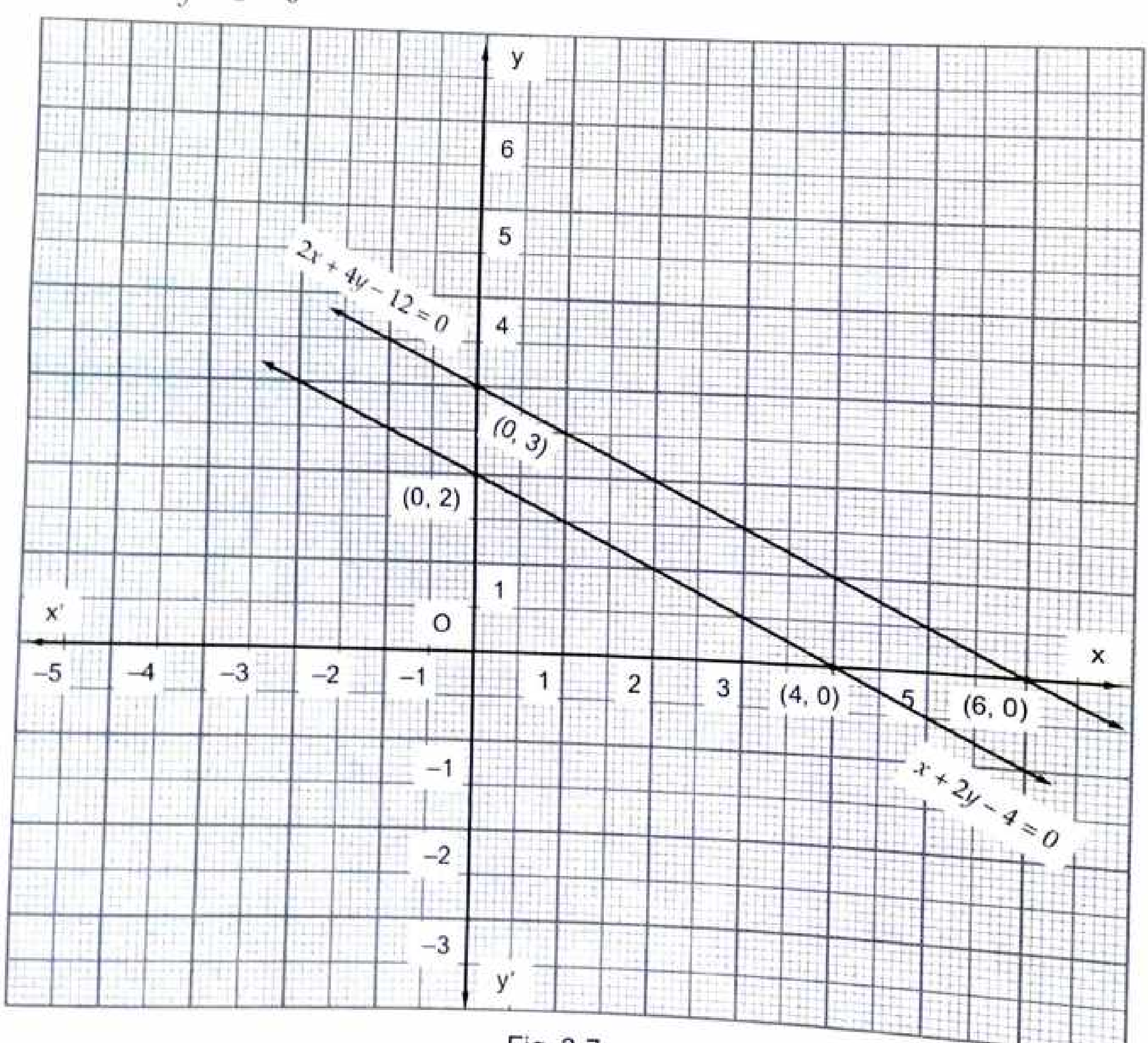


Fig. 3.7

Putting $y = 0$, we get

$$x + 0 - 4 = 0 \Rightarrow x = 4$$

Putting $x = 0$, we get

$$0 + 2y - 4 = 0 \Rightarrow y = 2$$

Thus, two solutions of equation $x + 2y - 4 = 0$ are:

x	4	0
y	0	2

We have,

$$2x + 4y - 12 = 0$$

Putting $x = 0$, we get

$$0 + 4y - 12 = 0 \Rightarrow y = 3$$

Putting $y = 0$, we get

$$2x + 0.12 = 0 \Rightarrow x = 6$$

Thus, two solutions of equation $2x + 4y - 12 = 0$ are:

x	0	6
y	3	0

Now, we plot the points A (4, 0) and B (0, 2) and draw a line passing through these two points to get the graph of the line represented by the equations (i).

We also plot the points P (0, 3) and Q (6, 0) and draw a line passing through these two points to get the graph of the line represented by the equation (ii).

We observe that the lines are parallel and they do not intersect any where.

REMARK *The graphical representation of the above pair of linear equations provides us a pair of parallel lines.*

Let us write the pair of linear equations.

$$x + 2y - 4 = 0$$
$$2x + 4y - 12 = 0$$

as

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

where $a_1 = 1, b_1 = 2, c_1 = -4, a_2 = 2, b_2 = 4$ *and* $c_2 = -12$.

We have,

$$\frac{a_1}{a_2} = \frac{1}{2}, \frac{b_1}{b_2} = \frac{2}{4} = \frac{1}{2} \text{ and } \frac{c_1}{c_2} = \frac{-4}{-12} = \frac{1}{3}$$

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Thus, the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

will represent parallel lines, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

The converse is also true for any pair of linear equations.

It follows from the above examples that the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

will represent:

(i) intersecting lines, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(ii) coincident lines, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) parallel lines, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

EXERCISE 3.1

LEVEL-1

1. Akhila went to a fair in her village. She wanted to enjoy rides on the Giant Wheel and play Hoopla (a game in which you throw a rig on the items kept in the stall, and if the ring covers any object completely you get it). The number of times she played Hoopla is half the number of rides she had on the Giant Wheel. Each ride costs ₹ 3, and a game of Hoopla costs ₹ 4. If she spent ₹ in the fair, represent this situation algebraically and graphically. [NCERT]
2. Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." Is not this interesting? Represent this situation algebraically and graphically. [NCERT]
3. The path of a train A is given by the equation $3x + 4y - 12 = 0$ and the path of another train B is given by the equation $6x + 8y - 48 = 0$. Represent this situation graphically.
4. Gloria is walking along the path joining (– 2, 3) and (2, – 2), while Suresh is walking along the path joining (0, 5) and (4, 0). Represent this situation graphically.
5. On comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, and without drawing them, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincide:

   (i) $5x - 4y + 8 = 0$, $7x + 6y - 9 = 0$
   (ii) $9x + 3y + 12 = 0$, $18x + 6y + 24 = 0$
   (iii) $6x - 3y + 10 = 0$, $2x - y + 9 = 0$
6. Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:
   (i) intersecting lines (ii) parallel lines (iii) coincident lines.
7. The cost of 2kg of apples and 1 kg of grapes on a day was found to be ₹ 160. After a month, the cost of 4kg of apples and 2kg of grapes is ₹ 300. Represent the situation algebraically and geometrically. [NCERT]

ANSWERS

1. $x - 2y = 0$, $3x + 4y = 20$
2. $x - 7y + 42 = 0$, $x - 3y - 6 = 0$
5. (i) Intersecting lines (ii) Coincident lines (iii) Parallel lines
6. (i) $x + 2y - 4 = 0$ (ii) $4x + 6y - 12 = 0$ (iii) $4x + 6y - 16 = 0$
7. $2x + y = 160, 4x + 2y = 300$.

3.4 GRAPHICAL METHOD OF SOLVING SIMULTANEOUS LINEAR EQUATIONS

In this section, we shall use the knowledge of construction of graphs of linear equations in solving systems of simultaneous linear equations in two variables. We have learnt that the coordinates of every point on the line representing a linear equation in two variables determine a solution of the equation and every solution of the equation is represented by a point on the line. Thus, if there is a system of simultaneous linear equations in two variables such that the lines representing the equations intersect at a point $P(\alpha, \beta)$. Clearly, point P lies on both the lines, so its coordinates will satisfy both the equations in the system. Thus, $x = \alpha$, $y = \beta$ is the solution of the given system of equations. If the lines represented by the two equations are coincident, then they have infinitely many common points. Therefore, every point on the line provides a solution of the given system of equations and hence it has infinitely many solutions. If the lines represented by the two equations are parallel, then they do not have a common point and so the system has no solution i.e. it is in-consistent.

The procedure of solving a system of simultaneous linear equations in two variables by drawing their graphs is known as the graphical method.

We may use the following algorithm to solve a system of simultaneous linear equations in two variables by graphical method:

ALGORITHM

STEP I *Obtain the given system of simultaneous linear equations in x and y.*
Let the system of simultaneous linear equations be

$$a_1x + b_1y = c_1 \quad \ldots\text{(i)}$$
$$a_2x + b_2y = c_2 \quad \ldots\text{(ii)}$$

STEP II *Draw the graphs of the equations (i) and (ii) in step I.*
Let the lines l_1 and l_2 represent the graphs of (i) and (ii) respectively.

STEP III *If the lines l_1 and l_2 intersect at a point and (α, β) are the coordinates of this point, then the given system has a unique solution given by $x = \alpha$, $y = \beta$. Otherwise go to step IV.*

STEP IV *If the lines l_1 and l_2 are coincident, then the system is consistent and has infinitely many solutions. In this case, every solution of one of the equations is a solution of the system. Otherwise go step V.*

STEP V *If the lines l_1 and l_2 are parallel, then the given system of equations is in-consistent i.e. it has no solution.*

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve graphically the system of equations:*

$$x + y = 3$$
$$3x - 2y = 4$$

SOLUTION Graph of the equation $x + y = 3$:

$$x + y = 3 \Rightarrow y = 3 - x$$

When $x = 1$, we have

$$y = 3 - 1 = 2$$

When $x = 2$, we have

$$y = 3 - 2 = 1$$

Thus, we have the following table:

x	1	2
y	2	1

Plotting the points $A(1, 2)$ and $B(2, 1)$ and drawing a line joining them, we get the graph of the equation $x + y = 3$ as shown in Fig. 3.8.

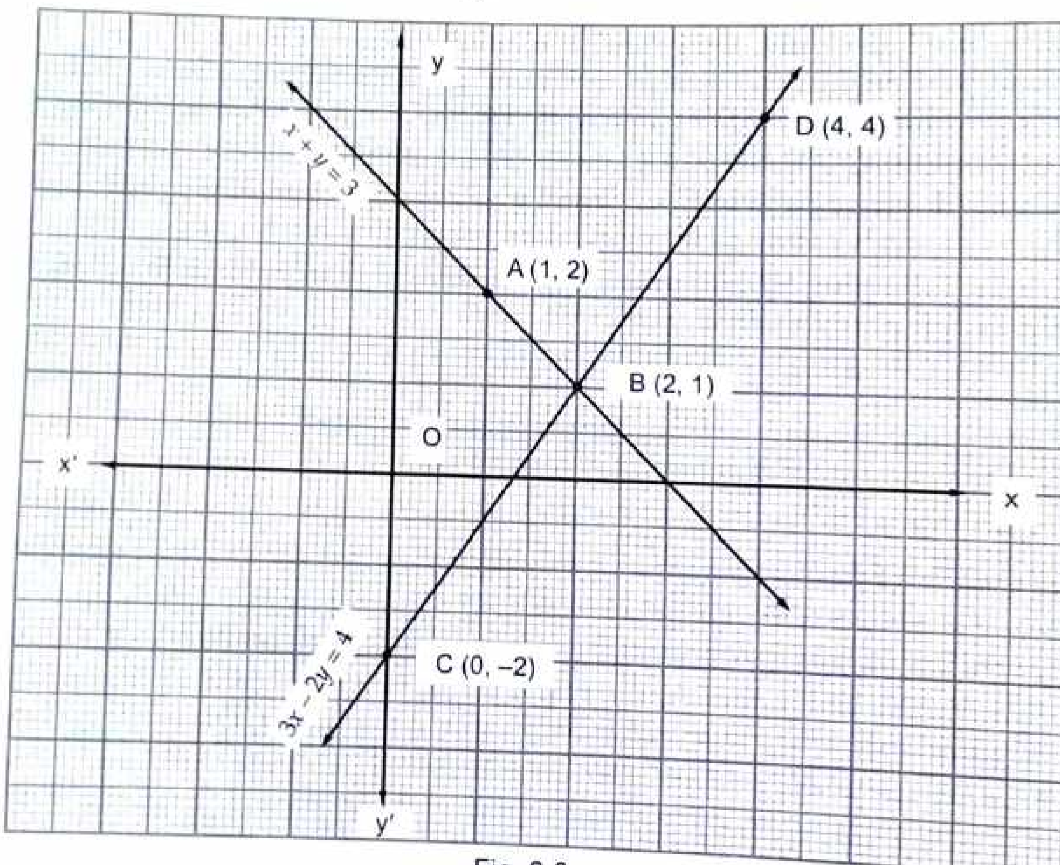


Fig. 3.8

Graph of the equation $3x - 2y = 4$:

We have,

$$3x - 2y = 4 \Rightarrow 2y = 3x - 4 \Rightarrow y = \frac{3x - 4}{2}$$

When $x = 0$, we have

$$y = \frac{3 \times 0 - 4}{2} = -2$$

When $x = 4$, we have

$$y = \frac{3 \times 4 - 4}{2} = 4$$

Thus, we have the following table:

x	0	4
y	-2	4

Plotting the point $C(0, -2)$ and $D(4, 4)$ on the same graph paper and drawing a line joining them, we obtain the graph of the equation $3x - 2y = 4$.

Clearly, the two lines intersect at point $P(2, 1)$.
Hence, $x = 2, y = 1$ is the solution of the given system.

EXAMPLE 2 *Show graphically that the system of equations*

$$2x + 4y = 10$$
$$3x + 6y = 12$$

has no solution.

SOLUTION Graph of $2x + 4y = 10$:
We have,

$$2x + 4y = 10 \Rightarrow 4y = 10 - 2x \Rightarrow y = \frac{5 - x}{2}$$

When $x = 1$, we have

$$y = \frac{5 - 1}{2} = 2$$

When $x = 3$, we have

$$y = \frac{5 - 3}{2} = 1$$

Thus, we have the following table:

x	1	3
y	2	1

Plot the points $A(1, 2)$ and $B(3, 1)$ on a graph paper. Join A and B and extend it on both sides to obtain the graph of $2x + 4y = 10$ as shown in Fig. 3.9.

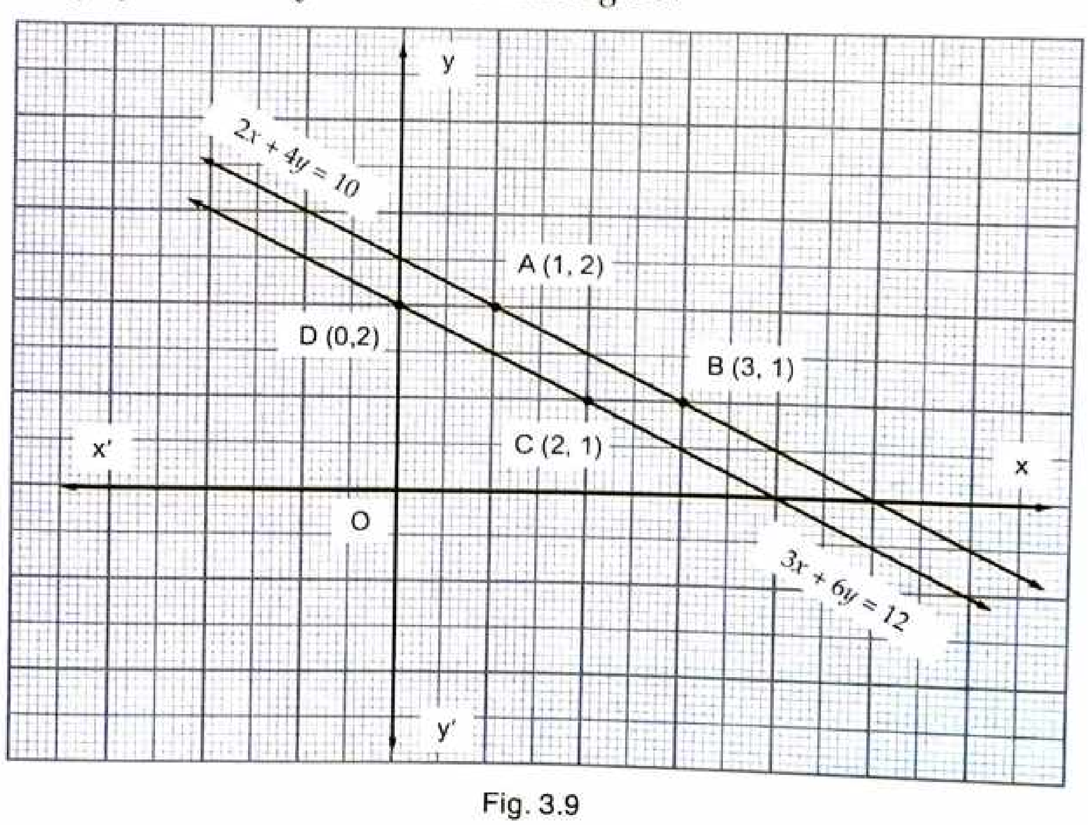


Fig. 3.9

Graph of $3x + 6y = 12$:

We have, $3x + 6y = 12 \Rightarrow 6y = 12 - 3x \Rightarrow y = \dfrac{4 - x}{2}$

When $x = 2$, we have

$$y = \frac{4-2}{2} = 1$$

When $x = 0$, we have

$$y = \frac{4-0}{2} = 2.$$

Thus, we have the following table:

x	2	0
y	1	2

Plot the points $C(2, 1)$ and $D(0, 2)$ on the same graph paper. Join C and D and extend it on both sides to obtain the graph of $3x + 6y = 12$ as shown in Fig. 3.9.

We find the lines represented by equations $2x + 4y = 10$ and $3x + 6y = 12$ are parallel. So, the two lines have no common point. Hence, the given system of equations has no solution.

EXAMPLE 3 *Show graphically that the system of equations*

$$3x - y = 2$$
$$9x - 3y = 6$$

has infinitely many solutions.

SOLUTION Graph of $3x - y = 2$:

We have, $3x - y = 2 \Rightarrow y = 3x - 2$

When $x = 2$, we have

$$y = 3 \times 2 - 2 = 4$$

When $x = 1$, we have

$$y = 3 \times 1 - 2 = 1$$

Thus, we have the following table:

x	2	1
y	4	1

Plotting the points $A(2, 4)$ and $B(1, 1)$ on the graph paper and drawing a line passing through A and B, we obtain the graph of $3x - y = 2$ as shown in Fig. 3.10.

Graph of $9x - 3y = 6$:

We have, $9x - 3y = 6$

$$\Rightarrow \quad y = 9x - 6$$

$$\Rightarrow \quad y = \frac{9x-6}{3}$$

When $x = 0$, we have

$$y = \frac{9 \times 0 - 6}{3} = -2$$

When $x = -1$, we have

$$y = \frac{9 \times -1 - 6}{3} = -5$$

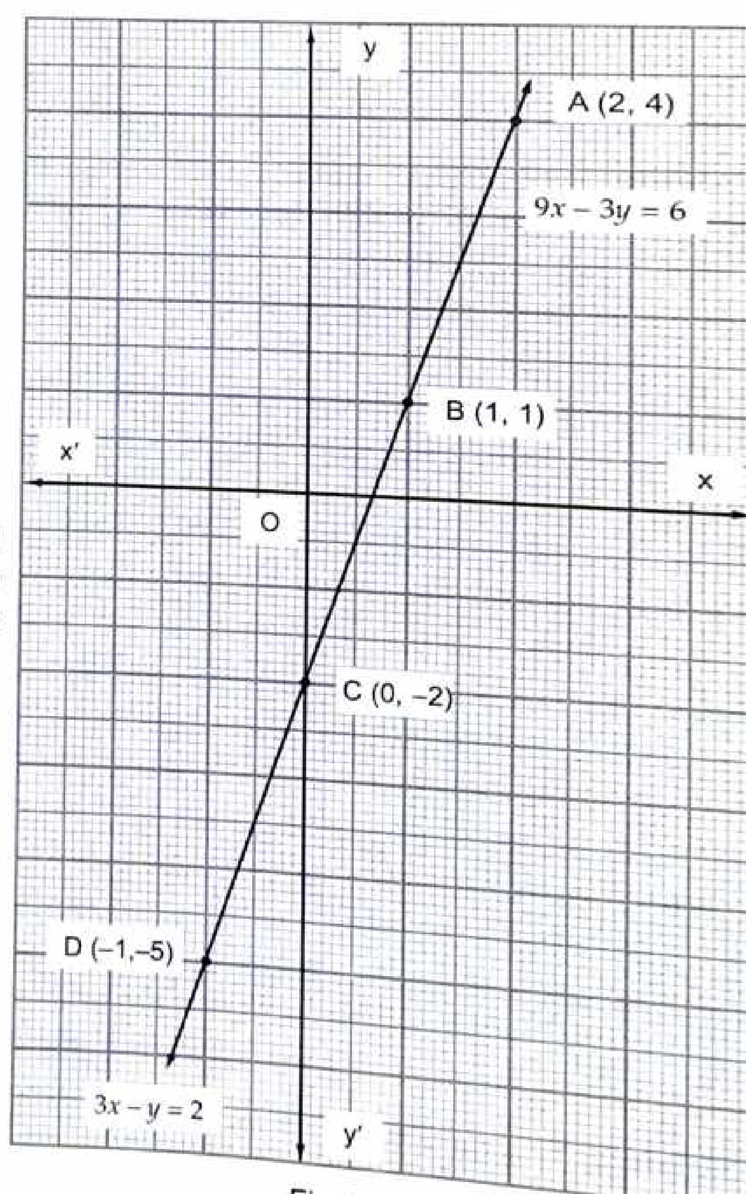


Fig. 3.10

Thus, we have the following table:

x	0	-1
y	-2	-5

Plotting the points $C(0,-2)$ and $D(-1,-5)$ on the graph paper and drawing a line passing through these two points on the same graph paper we obtain the graph of $9x-3y=6$. We find the C and D both lie on the graphy of $3x-y=2$. Thus, the graphs of the two equations are coincident. Consequently, every solution of one equation is a solution of the other. Hence, the system of equations has infinitely many solutions.

EXAMPLE 4 *Use a single graph paper and draw the graph of the following equations:*

$$2y-x=8;\ 5y-x=14,\ y-2x=1.$$

Obtain the vertices of the triangle so obtained.

SOLUTION Graph of $2y-x=8$:

We have, $2y-x=8 \Rightarrow x=2y-8$

When $y=2$, we have

$$x=2\times 2-8=-4$$

When $y=3$, we have

$$x=2\times 3-8=-2.$$

Thus, we have the following table:

x	-4	-2
y	2	3

Plot the points $A_1(-4, 2)$ and $B_1(-2, 3)$ on the graph paper. Join A_1 and B_1 and extend it on both sides to obtain the graph of $2y-x=8$ as shown in Fig. 3.11.

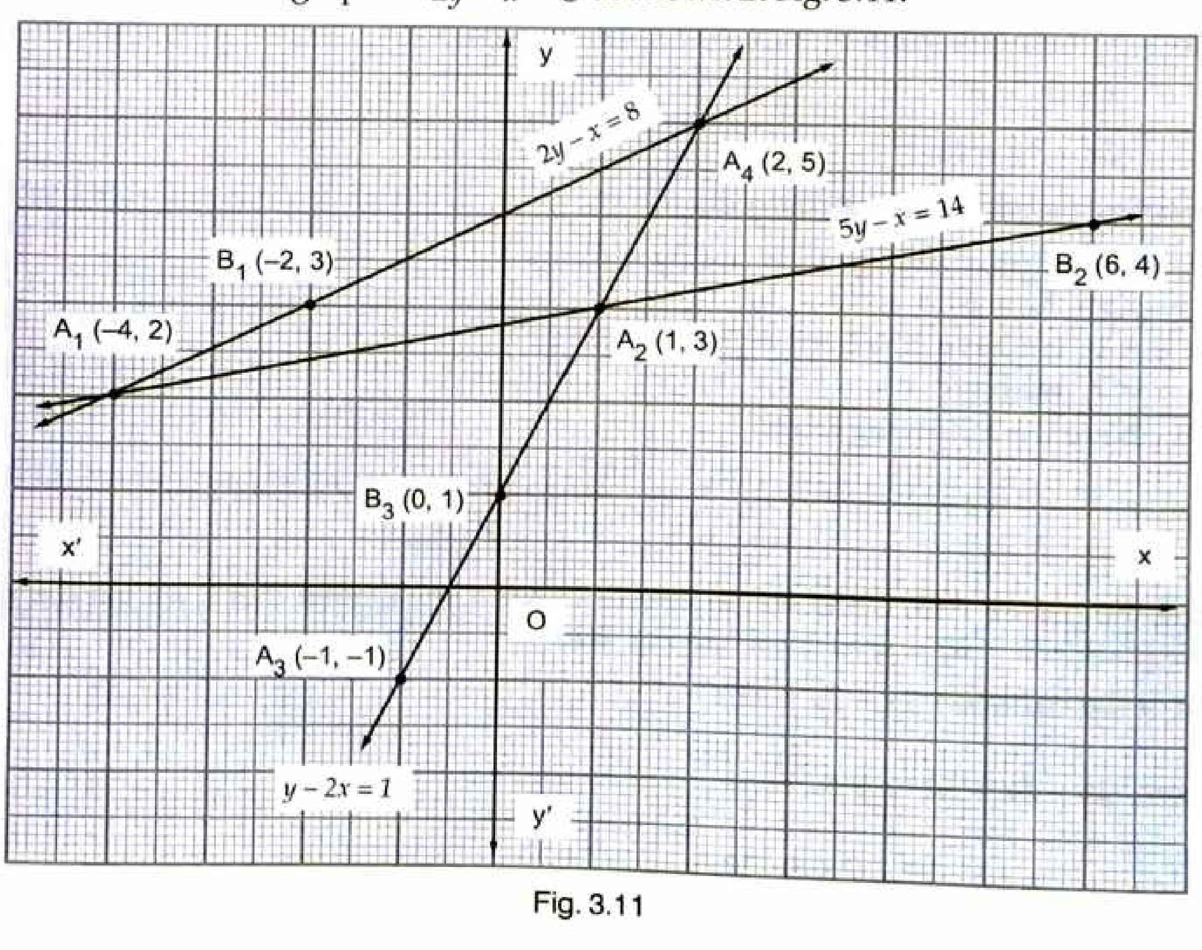


Fig. 3.11

Graph of $5y - x = 14$:

We have, $5y - x = 14 \Rightarrow x = 5y - 14$

When $y = 3$, we have $x = 5 \times 3 - 14 = 1$

When $y = 4$, we have $x = 5 \times 4 - 14 = 6$

Thus, we have the following table:

x	1	6
y	3	4

Plot the points $A_2(1, 3)$ and $B_2(6, 4)$ on a graph paper. Join A_2 and B_2 and extend it on both sides to obtain the graph of $5y - x = 14$ as shown in Fig. 3.11.

Graph of $y - 2x = 1$:

We have, $y - 2x = 1 \Rightarrow y = 2x + 1$

When $x = -1$, we have $y = 2 \times -1 + 1 = -1$

When $x = 0$, we have $y = 2 \times 0 + 1 = 1$

Thus, we have the following table:

x	-1	0
y	-1	1

Plot the points $A_3(-1 - 1)$ and $B_3(0, 1)$ on the same graph paper. Join A_3 and B_3 and extend it on both sides to obtain the graph of $y - 2x = 1$ as shown in Fig. 3.11.

From the graph of the three equations, we find that the three lines taken in pairs intersect each other at points $A_1(-4, 2)$, $A_2(1, 3)$ and $A_4(2, 5)$.

Hence, the vertices of the required triangle are $(-4, 2)$, $(1, 3)$ and $(2, 5)$.

EXAMPLE 5 *Solve the following system of equations graphically*

$$x + 3y = 6$$

$$2x - 3y = 12$$

and hence find the value of a, if $4x + 3y = a$. **[CBSE 2008]**

SOLUTION Graph of the equation $x + 3y = 6$:

We have, $x + 3y = 6 \Rightarrow x = 6 - 3y$

When $y = 1$, we have $x = 6 - 3 = 3$

When $y = 2$, we have $x = 6 - 6 = 0$

Thus we have the following table:

x	3	0
y	1	2

Plotting the points $A(3, 1)$ and $B(0, 2)$ and drawing a line joining them, we get the graph of the equation $x + 3y = 6$ as shown in Fig. 3.12.

Graph of the equation $2x - 3y = 12$:

We have, $2x - 3y = 12 \Rightarrow y = \dfrac{2x - 12}{3}$

When $x = 3$, we have $y = \dfrac{2 \times 3 - 12}{3} = -2$

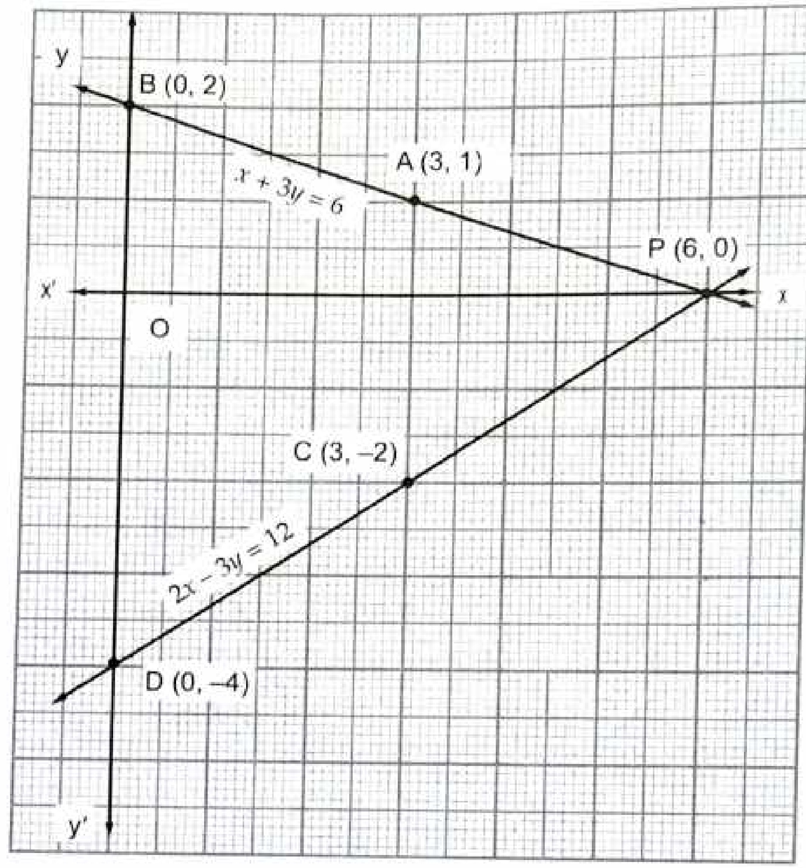


Fig. 3.12

When $x = 0$, we have $y = \dfrac{0 - 12}{3} = -4$

Thus, we have the following table:

x	3	0
y	-2	-4

Plotting the points $C(3, -2)$ and $D(0, -4)$ on the same graph paper and drawing a l joining them, we obtain the graph of the equation $2x - 3y = 12$ as shown in Fig. 3.12. Clearly, two lines intersect at $P(6, 0)$.

Hence, $x = 6, y = 0$ is the solution of the given system of equations.

Putting $x = 6, y = 0$ in $a = 4x + 3y$, we get

$$a = (4 \times 6) + (3 \times 0) = 24$$

EXAMPLE 6 *Solve the following system of linear equations graphically:*

$$2x - y - 4 = 0$$
$$x + y + 1 = 0$$

Find the points where the lines meet y-axis. **[CBSE 200**

SOLUTION We have,

$$2x - y - 4 = 0$$
$$x + y + 1 = 0$$

Graph of the equation $2x - y - 4 = 0$:

We have,

$$2x - y - 4 = 0$$

$\Rightarrow \quad y = 2x - 4$

When $x = 0$, we have $y = -4$

When $x = 2$, we have $y = 0$

Thus, we have the following table giving points on the line $2x - y - 4 = 0$.

x	0	2
y	-4	0

Plotting the points $A(0,-4)$ and $B(2,0)$ on the graph paper on a suitable scale and drawing a line passing through these two points we obtain the graph of the line given by the equation $2x - y - 4 = 0$ as shown in Fig. 3.13.

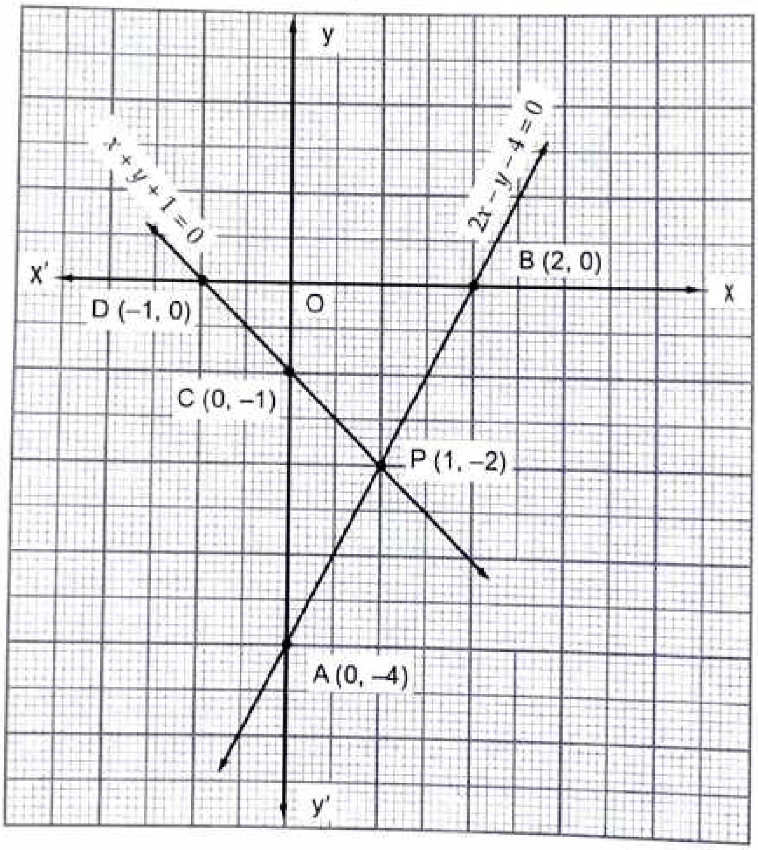


Fig. 3.13

Graph of the equation $x + y + 1 = 0$:

We have,

$$x + y + 1 = 0 \Rightarrow y = -x - 1 \text{ and } x = -y - 1.$$

When $x = 0$, we have $y = -1$

When $x = -1$, we have $y = 0$

Thus we have the following table giving points on the line $x + y + 1 = 0$

x	0	-1
y	-1	0

Plotting the points $C(0,-1)$ and $D(-1,0)$ on the same graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $x + y + 1 = 0$ as shown in Fig. 3.13.

Clearly, the two lines intersect at $P(1,-2)$. Hence, $x = 1, y = -2$ is the solution of the given system of equations.

From Fig. 3.13, we observe that the lines represented by the equations $2x - y - 4 = 0$ and $x + y + 1 = 0$ meet y-axis at $A(0,-4)$ and $C(0,-1)$ respectively.

EXAMPLE 7 *Draw the graphs of $2x + y = 6$ and $2x - y + 2 = 0$. Shade the region bounded by these lines and x-axis. Find the area of the shaded region.* **[CBSE 2002]**

SOLUTION We have,

$$2x + y = 6 \quad \ldots(i)$$
$$2x - y + 2 = 0 \quad \ldots(ii)$$

Graph of the equation $2x + y = 6$:

We have,

$$2x + y = 6 \Rightarrow y = 6 - 2x$$

When $x = 0$, we have $y = 6$

When $x = 2$, we have $y = 2$

Thus, we have the following table giving two points on the line represented by the equation $2x + y = 6$

x	0	3
y	6	0

Plotting the points $A(0, 6)$ and $B(3, 0)$ on the graph paper on a suitable scale and drawing a line joining them, we obtain the graph of the line represented by the equation $2x + y = 6$ as shown in Fig. 3.14.

Graph of the equation $2x - y + 2 = 0$:

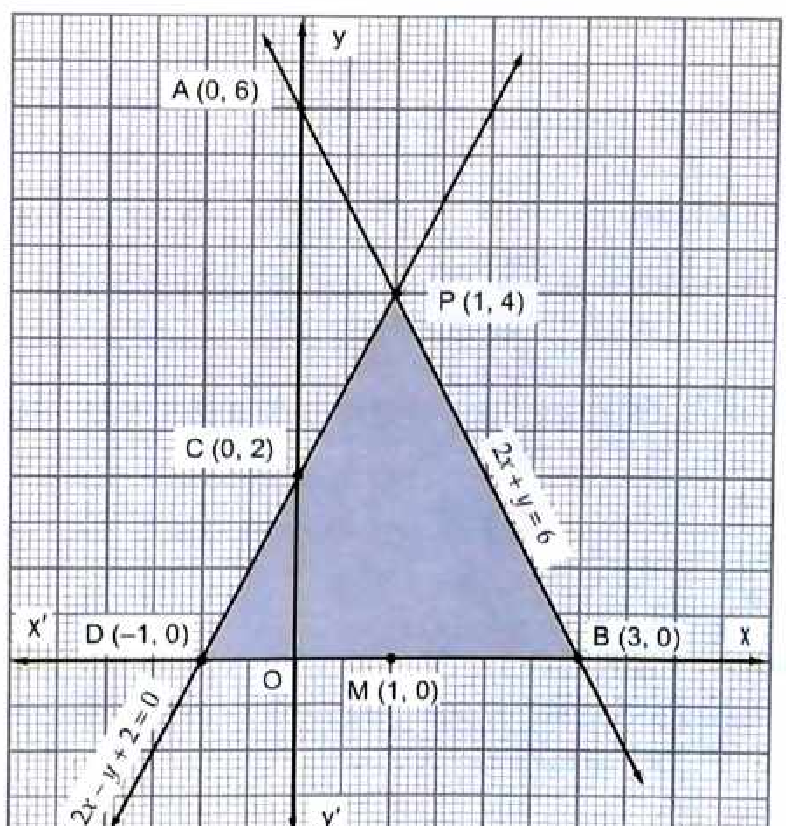


Fig. 3.14

We have,

$$2x - y + 2 = 0 \Rightarrow y = 2x + 2$$

When $x = 0$, we have $y = 2$

When $x = -1$, we have $y = 0$

Thus, we have the following table giving two points on the line representing the given equation

x	0	-1
y	2	0

Plotting the points $C(0, 2)$ and $D(-1, 0)$ on the same graph paper and joining them, we obtain the graph of the line represented by the equation $2x - y + 2 = 0$ as shown in Fig. 3.14.

It is evident from the graph that the two lines intersect at point $P(1, 4)$. The area enclosed the lines and x-axis is shown in Fig. 3.14.

Thus, $x = 1, y = 4$ is the solution of the given system of equations. Draw PM perpendicular from P on x-axis

Clearly, we have

$$PM = y\text{-coordinate of point } P(1, 4)$$

$\Rightarrow$ $PM = 4$

and, $DB = 4$

$\therefore$ Area of the shaded region = Area of ΔPBD

$\Rightarrow$ Area of the shaded region $= \frac{1}{2}(\text{Base} \times \text{Height})$

$\Rightarrow$ Area of the shaded region $= \frac{1}{2}(DB \times PM)$

$\Rightarrow$ Area of the shaded region $= \left(\frac{1}{2} \times 4 \times 4\right)$ sq. units = 8 sq. units

EXAMPLE 8 *Solve the following system of linear equations graphically:*

$$x - y = 1$$
$$2x + y = 8.$$

Shade the area bounded by these two lines and y-axis. Also, determine this area.

[CBSE 2001]

SOLUTION We have,

$$x - y = 1$$
$$2x + y = 8$$

Graph of the equation $x - y = 1$:

We have,

$$x - y = 1 \Rightarrow y = x - 1 \text{ and } x = y + 1$$

Putting $x = 0$, we get $y = -1$

Putting $y = 0$, we get $x = 1$

Thus, we have the following table for the points on the line $x - y = 1$:

x	0	1
y	-1	0

Plotting points $A(0, -1)$, $B(1, 0)$ on the graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $x - y = 1$ as shown in Fig. 3.15.

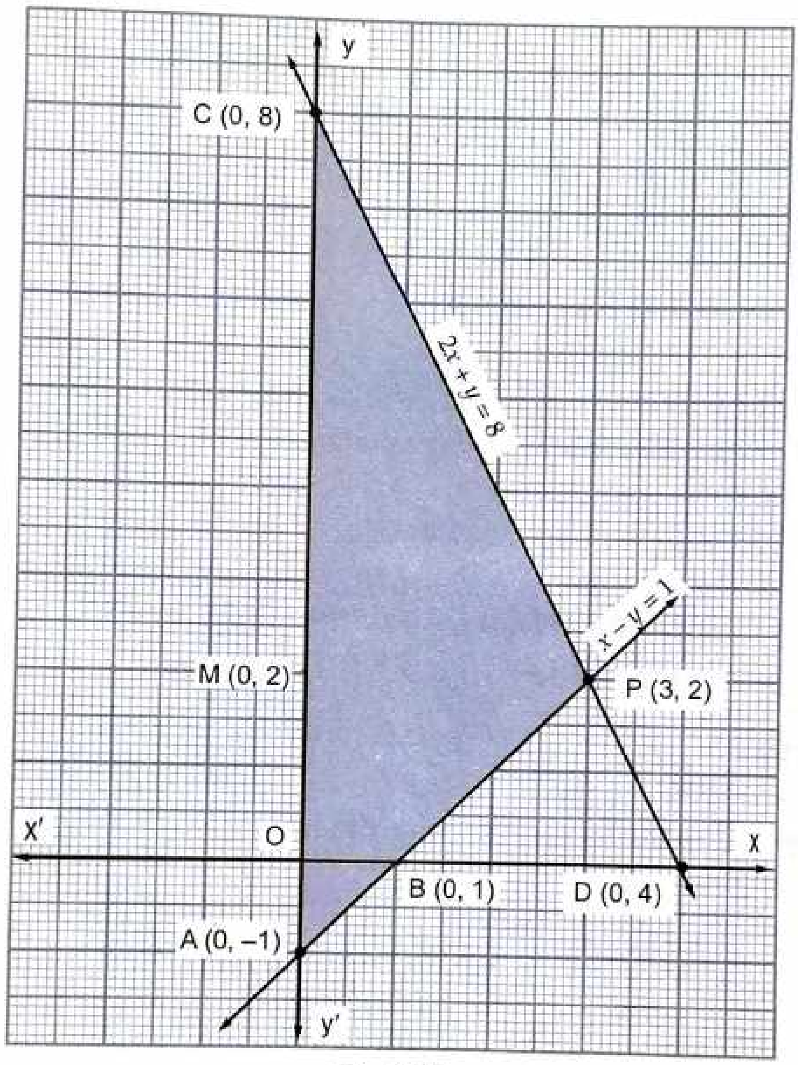


Fig. 3.15

Graph of the equation $2x + y = 8$:
We have,

$$2x + y = 8 \Rightarrow y = 8 - 2x \text{ and } x = \frac{8 - y}{2}$$

Putting $x = 0$, we get $y = 8$

Putting $y = 0$, we get $x = 4$

Thus, we have the following table giving two points on the line represented by the equation $2x + y = 8$.

x	0	4
y	8	0

Plotting points $C(0, 8)$ and $D(4, 0)$ on the same graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $2x + y = 8$ as shown in Fig. 3.15.

Clearly, the two lines intersect at $P(3, 2)$. The area enclosed by the lines represented by the given equations and the y-axis is shaded in Fig. 3.15.

Now, Required area = Area of the shaded region

$\Rightarrow$ Required area = Area of ΔPAC

$\Rightarrow$ Required area $= \frac{1}{2}(\text{Base} \times \text{Height})$

$\Rightarrow$ Required area $= \frac{1}{2}(AC \times PM)$

$\Rightarrow$ Required area $= \frac{1}{2}(9 \times 3)$ sq. units [$\because$ PM = x-coordinate of $P = 3$]

$= 13.5$ sq. units.

LEVEL-2

EXAMPLE 9 *Draw the graphs of the following equations:*

$2x - y - 2 = 0$

$4x + 3y - 24 = 0$

$y + 4 = 0.$

Obtain the vertices of the triangle so obtained. Also, determine its area.

SOLUTION Graph of the equation $2x - y - 2 = 0$:

We have, $2x - y - 2 = 0$

When $y = 0$, we have $x = 1$.

When $x = 0$, we have $y = -2$.

Thus, we obtain the following table giving coordinates of two poitns on the line represented by the equation $2x - y - 2 = 0$.

x	1	0
y	0	-2

Plotting points $A(1, 0)$ and $B(0, -2)$ on the graph paper on a suitable scale and drawing a line passing through them, we obtain the graph of the line represented by the equation $2x - y - 2 = 0$ as shown in Fig. 3.16.

Graph of the equation $4x + 3y - 24 = 0$:

We have, $4x + 3y - 24 = 0 \Rightarrow y = \frac{24 - 4x}{3}$ and $x = \frac{24 - 3y}{4}$

When $y = 0$, we have $x = 6$.

When $x = 0$, we have $y = 8$.

Thus, we obtain the following table giving coordinates of two points on the line represented by the equation $4x + 3y - 24 = 0$.

x	6	0
y	0	8

Plotting points $C(6, 0)$ and $D(0, 8)$ Plotting points $C(6, 0)$ and $D(0, 8)$ on the same graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $4x + 3y - 24 = 0$ as shown in Fig. 3.16.

Graph of the equation $y + 4 = 0$:

Clearly, $y = -4$ for every value of x. So, let $E(2, -4)$ and $F(0, -4)$ be two points on the line represented by $y + 4 = 0$. Plotting these points on the same graph and drawing a line passing through them, we obtain the graph of the line represented by the equation $y + 4 = 0$ as shown in Fig. 3.16.

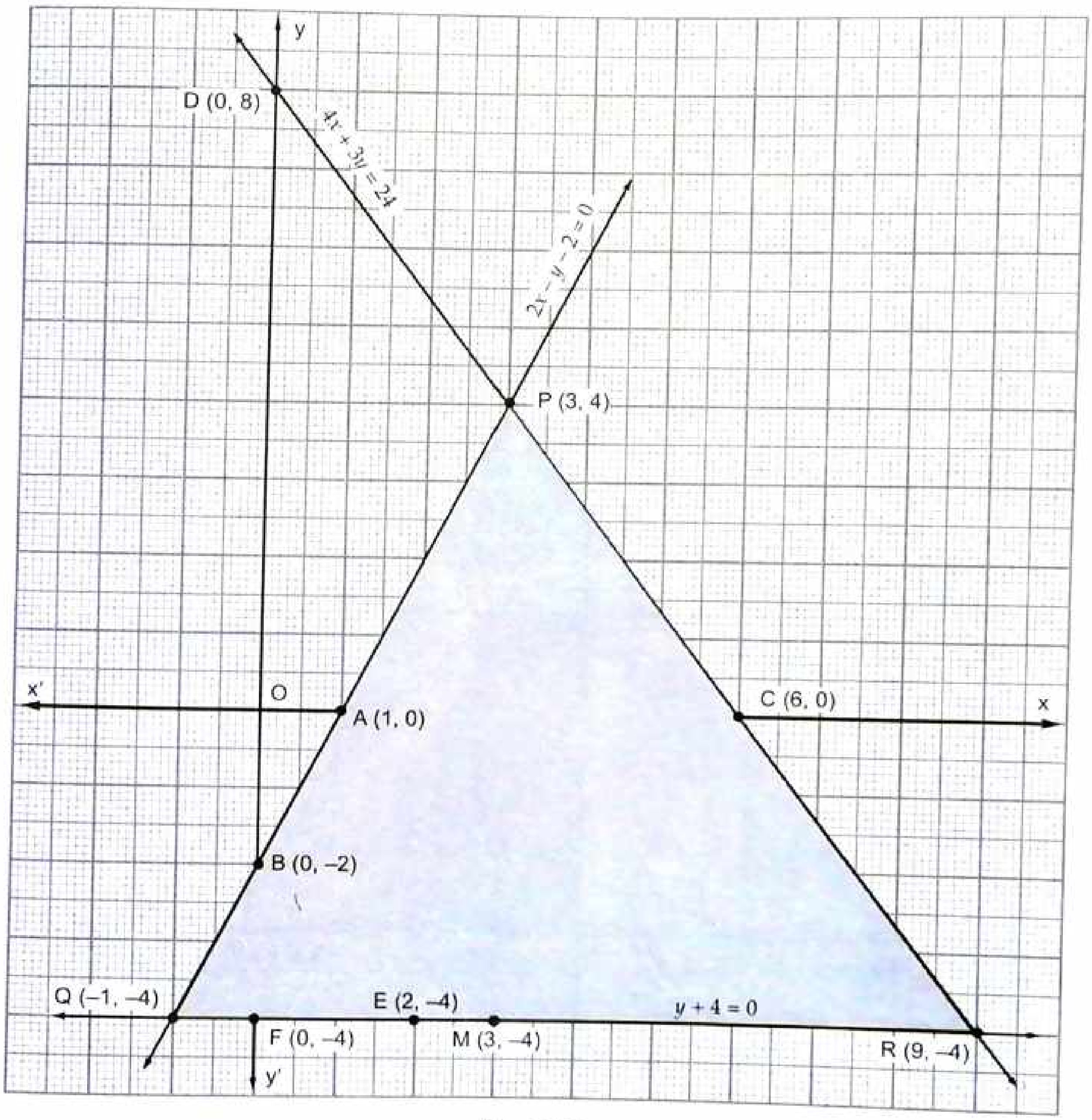


Fig. 3.16

From the graph of the lines represented by the given equations, we observe that the lines taken in pairs intersect each other at points $P(3, 4)$, $Q(-1, -4)$ and $R(9, -4)$ as shown in Fig. 3.16.

From Fig. 3.16, we have

$$PM = 8 \text{ and } QR = 10.$$

$$\therefore \quad \text{Area of } \Delta PQR = \frac{1}{2}(\text{Base} \times \text{Height})$$

$\Rightarrow$ Area of $\Delta PQR = \frac{1}{2}(QR \times PM) = \frac{1}{2}(10 \times 8)$ sq. units

$\Rightarrow$ Area of $\Delta PQR = 40$ sq. units.

EXAMPLE 10 *Determine graphically the vertices of a trapezium, the equations of whose sides are: $x = 0$, $y = 0$, $y = 4$ and $2x + y = 6$ Also, determine its area.*

SOLUTION Clearly, $x = 0$ represents y-axis and $y = 0$ represents x-axis.

Graph of the equation $y = 4$:

Clearly, $y = 4$ for every value of x. So, let A (3, 4) and B (0, 4) be two points on the line represented by $y = 4$. Plotting these points on the same graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $y = 4$ as shown in Fig. 3.17. It is a line parallel to x-axis at a distance of 4 units from it.

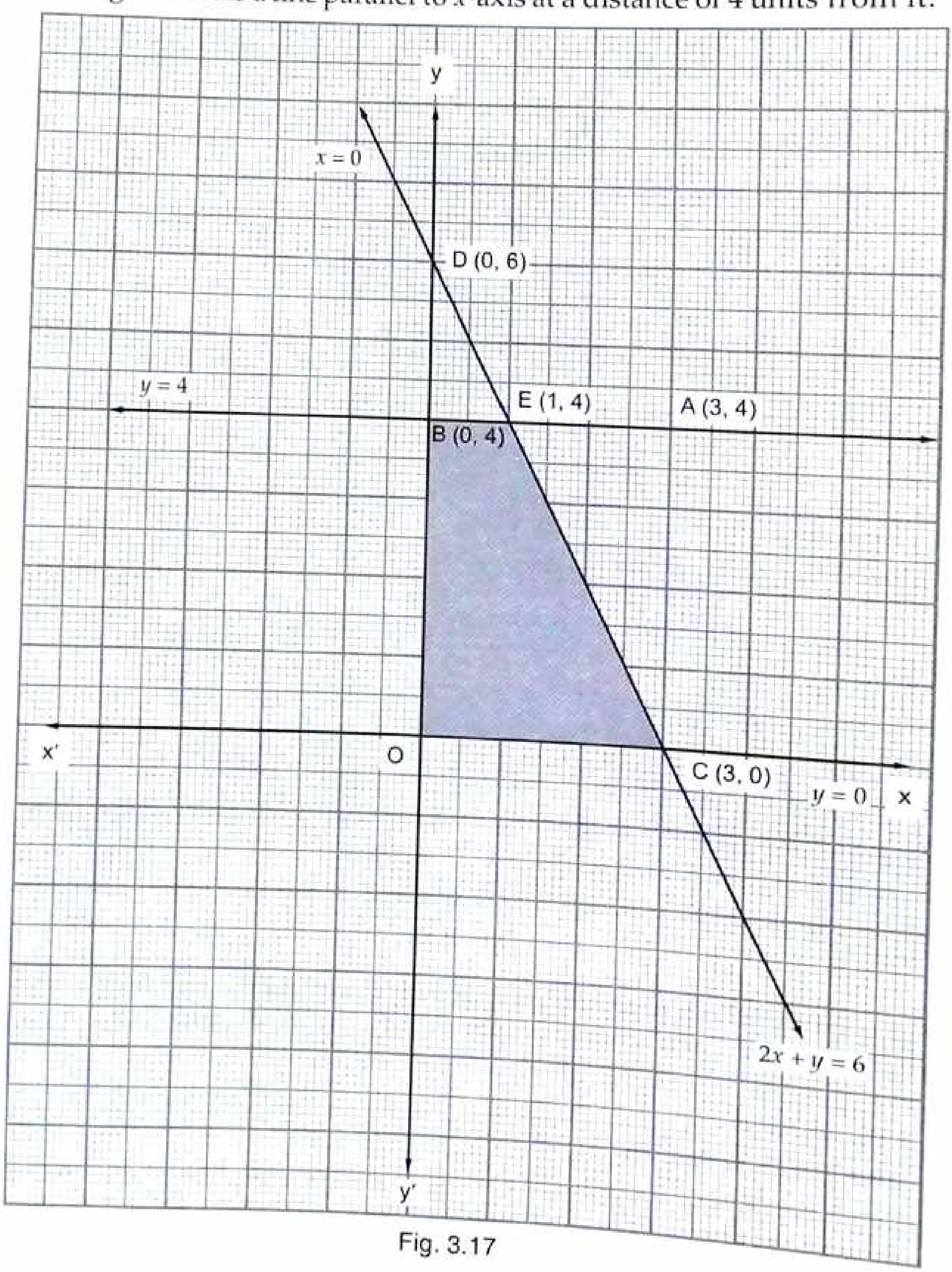


Fig. 3.17

Graph of the equation $2x + y = 6$:

We have $2x + y = 6$

When $y = 0$, we get $x = 3$ and $x = 0$ gives $y = 6$.

Thus, we obtain the following table giving coordinates of two points on the line represented by the equation $2x + y = 6$.

x	3	0
y	0	6

Plotting point C (3, 0) and D (0, 6) on the same graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $2x + y = 6$ as shown in Fig. 3.17.

We find that the lines represented by the given equations form the trapezium OCEB as shown in Fig. 3.17. The coordinates of its vertices are O (0, 0), C (3, 0), E (1, 4) and B (0, 4).

Area of trapezium $OCEB = \frac{1}{2}(OC + BE) \times OB = \frac{1}{2}(3 + 1) \times 4 = 8$ sq. units

EXAMPLE 11 *Draw the graphs of the following equations on the same graph paper*

$$2x + y = 2;\ 2x + y = 6$$

Find the coordinates of the vertices of the trapezium formed by these lines. Also, find the area of the trapezium so formed.

SOLUTION *Graph of the equation $2x + y = 2$:*

We have, $2x + y = 2$

When $y = 0$, we have $x = 1$

When $x = 0$, we have $y = 2$

Thus, we obtain the following table giving coordinates of two points on the line represented by the equation $2x + y = 2$.

x	1	0
y	0	2

Plotting points A (1, 0) and B (0, 2) on the graph paper on a suitable scale and drawing a line passing through them, we obtain the graph of the line represented by the equation $2x + y = 2$ as shown in Fig. 3.18.

Graph of the equation $2x + y = 6$:

We have, $2x + y = 6$

When $y = 0$, we get $x = 3$

When $x = 0$, we get $y = 6$

Thus, we obtain the following table giving coordinates of two points on the line represented by the equation $2x + y = 6$.

x	3	0
y	0	6

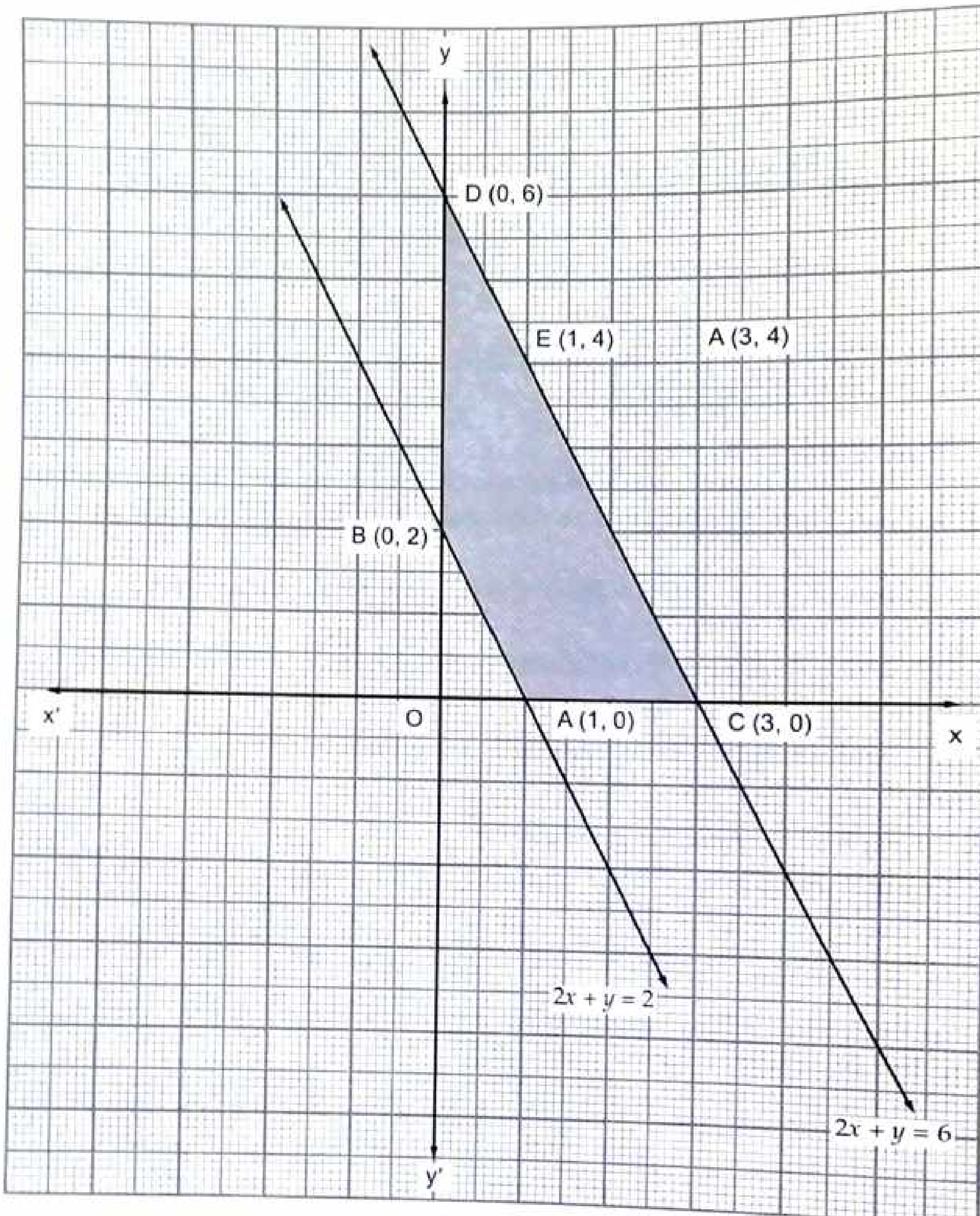


Fig. 3.18

Plotting point C (3, 0) and D (0, 6) on the same graph paper and drawing a line passing through them, we obtain the graph of the line represented by the equation $2x + y = 6$ as shown in Fig. 3.18.

Clearly, lines AB and CD form trapezium $ABCD$.

Also,

Area of trapezium $ACDB$ = Area of ΔOCD – Area of ΔOAB

$$= \frac{1}{2}(OC \times OD) - \frac{1}{2}(OA \times OB)$$

$$= \frac{1}{2}(3 \times 6) - \frac{1}{2}(1 \times 2)$$

$$= 8 \text{ sq. units}$$

EXERCISE 3.2

LEVEL-1

Solve the following systems of equations graphically:

1. $x + y = 3$
   $2x + 5y = 12$
2. $x - 2y = 5$
   $2x + 3y = 10$
3. $3x + y + 1 = 0$
   $2x - 3y + 8 = 0$
4. $2x + y - 3 = 0$
   $2x - 3y - 7 = 0$
5. $x + y = 6$
   $x - y = 2$
6. $x - 2y = 6$
   $3x - 6y = 0$
7. $x + y = 4$
   $2x - 3y = 3$
8. $2x + 3y = 4$
   $x - y + 3 = 0$
9. $2x - 3y + 13 = 0$
   $3x - 2y + 12 = 0$ [CBSE 2001C]
10. $2x + 3y + 5 = 0$
    $3x - 2y - 12 = 0$ [CBSE 2001C]

Show graphically that each one of the following systems of equations has infinitely many solutions:

11. $2x + 3y = 6$ [CBSE 2010]
    $4x + 6y = 12$
12. $x - 2y = 5$
    $3x - 6y = 15$
13. $3x + y = 8$
    $6x + 2y = 16$
14. $x - 2y + 11 = 0$
    $3x - 6y + 33 = 0$

Show graphically that each one of the following systems of equations is in-consistent (i.e. has no solution):

15. $3x - 5y = 20$
    $6x - 10y = -40$
16. $x - 2y = 6$
    $3x - 6y = 0$
17. $2y - x = 9$
    $6y - 3x = 21$
18. $3x - 4y - 1 = 0$
    $2x - \frac{8}{3}y + 5 = 0$
19. Determine graphically the vertices of the triangle, the equations of whose sides are given below:
    (i) $2y - x = 8$, $5y - x = 14$ and $y - 2x = 1$
    (ii) $y = x$, $y = 0$ and $3x + 3y = 10$
20. Determine, graphically whether the system of equations $x - 2y = 2$, $4x - 2y = 5$ is consistent or in-consistent.
21. Determine, by drawing graphs, whether the following system of linear equations has a unique solution or not:
    (i) $2x - 3y = 6$, $x + y = 1$
    (ii) $2y = 4x - 6$, $2x = y + 3$
22. Solve graphically each of the following systems of linear equations. Also find the coordinates of the points where the lines meet axis of y.
    (i) $2x - 5y + 4 = 0$,
    $2x + y - 8 = 0$ [CBSE 2005]
    (ii) $3x + 2y = 12$
    $5x - 2y = 4$ [CBSE 2006C]
    (iii) $2x + y - 11 = 0$,
    $x - y - 1 = 0$ [CBSE 2000C]
    (iv) $x + 2y - 7 = 0$,
    $2x - y - 4 = 0$ [CBSE 2000C]

(v) $3x + y - 5 = 0,$
$2x - y - 5 = 0$ **[CBSE 2002C]**

(vi) $2x - y - 5 = 0,$
$x - y - 3 = 0$ **[CBSE 2002C]**

23. Solve the following system of linear equations graphically and shade the region between the two lines and x-axis:

(i) $2x + 3y = 12,$
$x - y = 1$ **[CBSE 2001]**

(ii) $3x + 2y - 4 = 0,$
$2x - 3y - 7 = 0$ **[CBSE 2006C]**

(iii) $3x + 2y - 11 = 0$
$2x - 3y + 10 = 0$ **[CBSE 2006C]**

24. Draw the graphs of the following equations on the same graph paper:

$2x + 3y = 12$

$x - y = 1.$

Find the coordinates of the vertices of the triangle formed by the two straight lines and the y-axis. **[CBSE 2001]**

25. Draw the graphs of $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and x- axis and shade the triangular area. Calculate the area bounded by these lines and x-axis. **[CBSE 2002]**

26. Solve graphically the system of linear equations:

$4x - 3y + 4 = 0$

$4x + 3y - 20 = 0$

Find the area bounded by these lines and x-axis. **[CBSE 2002]**

27. Solve the following system of linear equations graphically:
$3x + y - 11 = 0,\ x - y - 1 = 0.$
Shade the region bounded by these lines and y -axis. Also, find the area of the region bounded by the these lines and y-axis. **[CBSE 2002C]**

28. Solve graphically each of the following systems of linear equations. Also, find the coordinates of the points where the lines meet the axis of x in each system:

(i) $2x + y = 6$
$x - 2y = -2$

(ii) $2x - y = 2$
$4x - y = 8$

(iii) $x + 2y = 5$
$2x - 3y = -4$ **[CBSE 2005]**

(iv) $2x + 3y = 8$
$x - 2y = -3$ **[CBSE 2005]**

29. Draw the graphs of the following equations:

$2x - 3y + 6 = 0$

$2x + 3y - 18 = 0$

$y - 2 = 0$

Find the vertices of the triangle so obtained. Also, find the area of the triangle.

30. Solve the following system of equations graphically:

$2x - 3y + 6 = 0$

$2x + 3y - 18 = 0$

Also, find the area of the region bounded by these two lines and y-axis.

31. Solve the following system of linear equations graphically:

$4x - 5y - 20 = 0$

$3x + 5y - 15 = 0$

Determine the vertices of the triangle formed by the lines representing the above equation and the y-axis. **[CBSE 2004]**

32. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and y-axis. Calculate the area of the triangle so formed. [NCERT]

33. Form the pair of linear equations in the following problems, and find their solution graphically:
   (i) 10 students of class X took part in Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz. [NCERT]
   (ii) 5 pencil and 7 pens together cost ` 50, whereas 7 pencils and 5 pens together cost ` 46. Find the cost of one pencil and a pen. [NCERT]
   (iii) Champa went to a 'sale' to purchase some pants and skirts. When her friends asked her how many of each she had bought, she answered, "The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased." Help her friends to find how many pants and skirts Champa bought. [NCERT]

34. Solve the following system of equations graphically:
   Shade the region between the lines and the y-axis
   (i) $3x - 4y = 7$
   $5x + 2y = 3$ [CBSE 2006C]
   (ii) $4x - y = 4$
   $3x + 2y = 14$ [CBSE 2006C]

35. Represent the following pair of equations graphically and write the coordinates of points where the lines intersects y-axis
   $x + 3y = 6$
   $2x - 3y = 12$ [CBSE 2008]

36. Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is
   (i) intersecting lines (ii) Parallel lines (iii) coincident lines [NCERT]

LEVEL-2

37. Determine graphically the coordinates of the vertices of a triangle, the equations of whose sides are:
   (i) $y = x, y = 2x$ and $y + x = 6$ [CBSE 2000]
   (ii) $y = x,\ 3y = x,\ x + y = 8$ [CBSE 2000]

38. Graphically, solve the following pair of equations:
   $2x + y = 6$
   $2x - y + 2 = 0$
   Find the ratio of the areas of the two triangles formed by the lines representing these equations with the x-axis and the lines with the y-axis. [NCERT EXEMPLAR]

39. Determine, graphically, the vertices of the triangle formed by the lines $y = x, 3y = x, x + y = 8$. [NCERT EXEMPLAR]

40. Draw the graph of the equations $x = 3, x = 5$ and $2x - y - 4 = 0$. Also, find the area of the quadrilateral formed by the lines and the x-axis. [NCERT EXEMPLAR]

41. Draw the graphs of the lines $x = -2$, and $y = 3$. Write the vertices of the figure formed by these lines, the x-axis and the y-axis. Also, find the area of the figure. [NCERT EXEMPLAR]

42. Draw the graphs of the pair of linear equations $x - y + 2 = 0$ and $4x - y - 4 = 0$. Calculate the area of the triangle formed by the lines so drawn and the x-axis. [NCERT EXEMPLAR]

ANSWERS

1. $x = 1, y = 2$
2. $x = 5, y = 0$
3. $x = -1, y = 2$
4. $x = 2, y = -1$
5. $x = 4, y = 2$
6. No solution
7. $x = 3, y = 1$
8. $x = -1, y = 2$
9. $x = -2, y = 3$
10. $x = 2, y = -3$
19. (i) $(-4, 2)$ $(1, 3)$, $(2, 5)$ (ii) $(0, 0)$ $(10/3, 0)$, $(5/3, 5/3)$
20. Consistent
21. (i) Unique solution (ii) Infinitely many solutions.
22. (i) $x = 3, y = 2;$ $(0, 4/5), (0, 8)$ (ii) $x = 2, y = 3;$ $(0, 6)$ and $(0, -2)$ (iii) $x = 4, y = 3;$ $(0, 11)$ and $(0, -1)$ (iv) $x = 3, y = 2$, $(0, 3.5), (0, -4)$ (v) $x = 2, y = -1$, $(0, 5)$ $(0, -5)$ (vi) $x = 2, y = -1$, $(0, -5), (0, -3)$
23. (i) $x = 3, y = 2$ (ii) $x = 2, y = -1$ (iii) $x = 1, y = 4$
24. $x = 3, y = 2$, $A(0, 4)$, $B(0, -1)$, $C(3, 2)$
25. 7.5 sq. units
26. $x = 2, y = 4$, 12 sq. units
27. $x = 3, y = 2$, 18 sq. units
28. (i) $x = 2, y = 2$, $(3, 0)$ $(-2, 0)$ (ii) $x = 3, y = 4$, $(1, 0), (2, 0)$ (iii) $x = 1, y = 2$, $(5, 0), (-2, 0)$ (iv) $x = 1, y = 2$, $(4, 0), (-3, 0)$
29. $(3, 4), (0, 2), (6, 2)$, Area = 6 sq. units.
30. $x = 3, y = 4$, Area = 6 sq. units
31. $x = 5, y = 0;$ $(5, 0)$ $(0, 3)$ $(0, -4)$.
32. $(1, 0), (0, -3), (0, -5)$, and 1 sq. unit.
33. (i) Girls = 7, Boys = 3 (ii) Pencil: ₹ 3, Pen: ₹ 5 (iii) Pant = 1, Skirt = 0.
34. (i) $x = 1, y = -1.$ (ii) $x = 2, y = 4$
35. $(0, 2), (0, -4)$
36. (i) $3x + 2y - 6 = 0$ (ii) $4x + 6y = 15$ (iii) $4x + 6y = 16$
37. (i) $A(0, 0)$, $B(2, 4)$, $C(3, 3)$ (ii) $A(0, 0)$, $B(4, 4)$ $C(6, 2)$
38. 4 : 1
39. $(0, 0), (4, 4), (6, 2)$
40. 8 sq. units
41. $(0, 0)$, $A(0, 3)$, $B(-2, 3)$, $C(-2, 0)$; 6 sq. units
42. 6 sq. units

3.5 ALGEBRAIC METHODS OF SOLVING SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES

In the previous sections, we have discussed graphical method of solving two simultaneous linear equations in two variables. Most of the times the graphical method is not convenient, particularly when the point of intersection of the lines represented by two given equations has coordinates as rational numbers. In such situations, graphical method does not give an accurate answer. For example, if the solution to a system of two linear equations is $x = -11/13, y = 5/7$, then by the graphical method, the point of intersection would be $(-11/13, 5/7)$. This point is so close to $(-0.8, 0.7)$ that on the graph paper it is not convenient to distinguish these two points and while reading the coordinates of the point of intersection we are likely to make an error. We may read $x = -0.8, y = 0.7$ as the solution of the given system of equations whereas the correct solution is $x = -11/13, y = 5/7$. Hence, it is necessary to use some precise method to obtain an accurate answer. The algebraic methods described below determine the accurate answer.

The most commonly used algebraic methods of solving simultaneous linear equations in two variables are:

(i) Method of elimination by substitution.
(ii) Method of elimination by equating the coefficients.
(iii) Method of cross-multiplication.

3.5.1 METHOD OF ELIMINATION BY SUBSTITUTION

In this method, we express one of the variables in terms of the other variable from either of the two equations and then this expression is put in the other equation to obtain an equation in one variable as explained in the following algorithm.

ALGORITHM

STEP I *Obtain the two equations. Let the equations be*

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

and, $$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

STEP II *Choose either of the two equations, say (i), and find the value of one variable, say y, in terms of the other, i.e. x.*

STEP III *Substitute the value of y, obtained in step II, in the other equation i.e. (ii) to get an equation in x.*

STEP IV *Solve the equation obtained in step III to get the value of x.*

STEP V *Substitute the value of x obtained in step IV in the expression for y in terms of x obtained in step II to get the value of y.*

STEP VI *The values of x and y obtained in steps IV and V respectively constitute the solution of the given system of two linear equations.*

Following solved examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following systems of equations by using the method of substitution:*

(i) $3x - 5y = -1$
$x - y = -1$

(ii) $x + 2y = -1$
$2x - 3y = 12$

SOLUTION (i) The given system of equations is

$$3x - 5y = -1 \quad \ldots(i)$$
$$x - y = -1 \quad \ldots(ii)$$

From (ii), we get

$$y = x + 1$$

Substituting $y = x + 1$ in (i), we get

$$3x - 5(x + 1) = -1$$
$$\Rightarrow \quad -2x - 5 = -1$$
$$\Rightarrow \quad -2x = 4 \Rightarrow x = -2$$

Putting $x = -2$ in $y = x + 1$ we get $y = -1$.

Hence, the solution of the given system of equations is $x = -2, y = -1$.

(ii) The given system of equations is

$$x + 2y = -1 \quad \ldots(i)$$

$$2x - 3y = 12 \quad \ldots(ii)$$

From equation (i), we get

$$x = -1 - 2y$$

Substituting $x = -1 - 2y$ in equation (ii) we get

$\Rightarrow$ $2(-1 - 2y) - 3y = 12$

$\Rightarrow$ $-2 - 4y - 3y = 12$

$\Rightarrow$ $-7y = 14$

$\Rightarrow$ $y = -2$

Putting $y = -2$ in $x = -1 - 2y$, we get

$$x = -1 - 2 \times (-2) = 3$$

Hence, the solution of the given system of equations is $x = 3, y = -2$.

EXAMPLE 2 *Solve the following systems of equations by using the method of substitution:*

(i) $2x + 3y = 9$

$3x + 4y = 5$

(ii) $\dfrac{2x}{a} + \dfrac{y}{b} = 2$

$\dfrac{x}{a} - \dfrac{y}{b} = 4$

SOLUTION (i) The given system of equations is

$$2x + 3y = 9 \quad \ldots(i)$$

$$3x + 4y = 5 \quad \ldots(ii)$$

From equation (i), we get

$$3y = 9 - 2x \Rightarrow y = \frac{9 - 2x}{3}$$

Substituting $y = \dfrac{9 - 2x}{3}$ in equation (ii), we get

$$3x + 4\left(\frac{9 - 2x}{3}\right) = 5$$

$\Rightarrow$ $\dfrac{9x + 36 - 8x}{3} = 5$

$\Rightarrow$ $x + 36 = 15$

$\Rightarrow$ $x = -21$

Putting $x = -21$ in $y = \dfrac{9 - 2x}{3}$, we get

$$y = \frac{9 + 42}{3} = 17$$

Hence, the solution of the given system of equations is $x = -21, y = 17$

(ii) The given system of equation is

$$\frac{2x}{a} + \frac{y}{b} = 2 \quad \ldots(i)$$

$$\frac{x}{a} - \frac{y}{b} = 4 \quad \ldots(ii)$$

From equation (i), we get

$$\frac{y}{b} = 2 - \frac{2x}{a} \Rightarrow y = b\left(2 - \frac{2x}{a}\right)$$

Substituting $y = b\left(2 - \frac{2x}{a}\right)$ in equation (ii), we get

$$\frac{x}{a} - \frac{b}{b}\left(2 - \frac{2x}{a}\right) = 4$$

$$\Rightarrow \quad \frac{x}{a} - 2 + \frac{2x}{a} = 4$$

$$\Rightarrow \quad \frac{3x}{a} = 6$$

$$\Rightarrow \quad 3x = 6a$$

$$\Rightarrow \quad x = 2a$$

Putting $x = 2a$ in equation (i), we get

$$4 + \frac{y}{b} = 2 \Rightarrow \frac{y}{b} = -2 \Rightarrow y = -2b$$

Hence, the solution of the given system of equations is $x = 2a, y = -2b$.

3.5.2 METHOD OF ELIMINATION BY EQUATING THE COEFFICIENTS

In this method, we eliminate one of the two variables to obtain an equation in one variable which can easily be solved. Putting the value of this variable in any one of the given equations, the value of the other variable can be obtained.

Following algorithm explains the procedure.

ALGORITHM

STEP I *Obtain the two equations.*

STEP II *Multiply the equations so as to make the coefficients of the variable to be eliminated equal.*

STEP III *Add or subtract the equations obtained in step II according as the terms having the same coefficients are of opposite or of the same sign.*

STEP IV *Solve equation in one variable obtained in step III.*

STEP V *Substitute the value found in step IV in any one of the given equations and find the value of the other variable.*

The values of the variables in steps IV and V constitute the solution of the given system of equations.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **SOLVING SIMULTANEOUS LINEAR EQUATION IN TWO VARIABLES**

EXAMPLE 1 *Solve the following systems of linear equations by using the method of elimination by equating the coefficients:*

(i) $3x + 2y = 11$

$2x + 3y = 4$

(ii) $8x + 5y = 9$

$3x + 2y = 4$

SOLUTION (i) The given systems of equations is

$$3x + 2y = 11 \qquad \ldots(i)$$

$$2x + 3y = 4 \qquad \ldots(ii)$$

Let us eliminate y from the given equations. The coefficients of y in the given equations are 2 and 3 respectively. The l.c.m. of 2 and 3 is 6. So, we make the coefficients of y equal to 6 in the two equations.

Multiplying (i) by 3 and (ii) by 2, we get

$$9x + 6y = 33 \qquad \ldots(iii)$$

$$4x + 6y = 8 \qquad \ldots(iv)$$

Subtracting (iv) from (iii), we get

$$5x = 25 \Rightarrow x = 5$$

Substituting $x = 5$ equation in (i), we get

$$15 + 2y = 11 \Rightarrow 2y = -4 \Rightarrow y = -2$$

Hence, the solution of the given system of equations is $x = 5, y = -2$

(ii) The given system of equation is

$$8x + 5y = 9 \qquad \ldots(i)$$

$$3x + 2y = 4 \qquad \ldots(ii)$$

Let us eliminate x from the given equations. The coefficients of x in the given equations are 8 and 3 respectively. The l.c.m. of 8 and 3 is 24. So, we make both the coefficients equal to 24.

Multiplying both sides of equation (i) by 3 and equation (ii) by 8, we get

$$24x + 15y = 27$$

$$24 + 16y = 32$$

Subtracting (iv) from (iii), we get

$$-y = -5 \Rightarrow y = 5$$

Putting $y = 5$ in (i), we get

$$8x + 25 = 9 \Rightarrow 8x = -16 \Rightarrow x = -2$$

Hence, the solution of the given system of equations is $x = -2, y = 5$.

EXAMPLE 2 *Solve the following system of equations by using the method of elimination by equating the coefficients:*

$$\frac{x}{10} + \frac{y}{5} + 1 = 15$$

$$\frac{x}{8} + \frac{y}{6} = 15$$

SOLUTION (i) The given system of equations is

$$\frac{x}{10} + \frac{y}{5} = 14$$

$$\frac{x}{8} + \frac{y}{6} = 15$$

This system of equations can be re-written as

$$x + 2y = 140 \qquad \ldots(i)$$

$$3x + 4y = 360 \qquad \ldots(ii)$$

Let us eliminate y from the equations (i) and (ii). The coefficients of y in the given equations are 2 and 4 respectively. The l.c.m. of 2 and 4 is 4.

Multiplying (i) by 2, we get

$$2x + 4y = 280 \qquad \ldots(\text{iii})$$

$$3x + 4y = 360 \qquad \ldots(\text{iv})$$

Subtracting (iv) from (iii), we get

$$-x = -80 \Rightarrow x = 80$$

Putting $x = 80$ in equation (i), we get

$$80 + 2y = 140 \Rightarrow 2y = 60 \Rightarrow y = 30$$

Hence, the solution of the given system of equations is $x = 80, y = 30$.

Type II SOLVING A SYSTEM OF EQUATIONS WHICH IS REDUCIBLE TO A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

EXAMPLE 3 *Solve the following system of equations:*

$$\frac{1}{2x} - \frac{1}{y} = -1$$

$$\frac{1}{x} + \frac{1}{2y} = 8, \textit{ where } x \neq 0, y \neq 0.$$

SOLUTION Taking $\frac{1}{x} = u$ and $\frac{1}{y} = v$, the given equations become

$$\frac{u}{2} - v = -1 \Rightarrow u - 2v = -2 \qquad \ldots(\text{i})$$

and, $$u + \frac{v}{2} = 8 \Rightarrow 2u + v = 16 \qquad \ldots(\text{ii})$$

Let us eliminate u from equations (i) and (ii). Multiplying equation (i) by 2, we get

$$2u - 4v = -4 \qquad \ldots(\text{iii})$$

$$2u + v = 16 \qquad \ldots(\text{iv})$$

Subtracting (iv) from (iii), we get

$$-5v = -20 \Rightarrow v = 4$$

Putting $v = 4$ in equation (i), we get

$$u - 8 = -2 \Rightarrow u = 6$$

Hence, $x = \frac{1}{u} = \frac{1}{6}$ and $y = \frac{1}{v} = \frac{1}{4}$

So, the solution of the given system of equation is $x = \frac{1}{6}, y = \frac{1}{4}$.

EXAMPLE 4 *Solve:* $$\frac{2}{x} + \frac{2}{3y} = \frac{1}{6}$$

$$\frac{3}{x} + \frac{2}{y} = 0$$

and hence find 'a' for which $y = ax - 4$.

SOLUTION Taking $\frac{1}{x} = u$ and $\frac{1}{y} = v$. The given system of equations become

$$2u + \frac{2}{3}v = \frac{1}{6}$$

$\Rightarrow \qquad 12u + 4v = 1$ ...(i)

and, $\qquad 3u + 2v = 0$ ...(ii)

Multiplying equation (ii) by 2 and subtracting from equation (i), we get

$$6u = 1 \Rightarrow u = \frac{1}{6}$$

Putting $u = \frac{1}{6}$ in (i), we get

$$2 + 4v = 1 \Rightarrow v = -\frac{1}{4}$$

Hence, $x = \frac{1}{u} = 6$ and $y = \frac{1}{v} = -4$

So, the solution of the given system of equations is $x = 6, y = -4$

Putting $x = 6,\ y = -4$ in $y = ax - 4$, we get

$$-4 = 6a - 4 \Rightarrow a = 0$$

EXAMPLE 5 *Solve:* $4x + \frac{6}{y} = 15$

$$6x - \frac{8}{y} = 14$$

and hence find 'p' if $y = px - 2$

SOLUTION The given system of equation is

$$4x + \frac{6}{y} = 15 \qquad \text{...(i)}$$

$$6x - \frac{8}{y} = 14 \qquad \text{...(ii)}$$

Multiplying equation (i) by 4 and equation (ii) by 3, we get

$$16x + \frac{24}{y} = 60 \qquad \text{...(iii)}$$

$$18x - \frac{24}{y} = 42 \qquad \text{...(iv)}$$

Adding (iv) and (iii), we get

$$34x = 102 \Rightarrow x = 3$$

Putting $x = 3$ in equation (i), we get

$$12 + \frac{6}{y} = 15 \Rightarrow y = 2$$

Hence, the solution of the given system of equations is $x = 3, y = 2$.

Putting $x = 3, y = 2$ in $y = px - 2$, we get

$$2 = 3p - 2 \Rightarrow p = 4/3$$

EXAMPLE 6 *Solve:* $\dfrac{1}{2(2x+3y)}+\dfrac{12}{7(3x-2y)}=\dfrac{1}{2}$

$$\frac{7}{2x+3y}+\frac{4}{3x-2y}=2,$$

where $2x+3y\neq 0$ *and* $3x-2y\neq 0$.

SOLUTION Let $\dfrac{1}{2x+3y}=u$ and $\dfrac{1}{3x-2y}=v$. Then, the given system of equations becomes

$$\frac{1}{2}u+\frac{12}{7}v=\frac{1}{2}\Rightarrow 7u+24v=7 \qquad \ldots(\text{i})$$

and, $$7u+4v=2 \qquad \ldots(\text{ii})$$

Subtracting equation (ii) from equation (i), we get

$$20v=5\Rightarrow v=\frac{1}{4}$$

Putting $v=\dfrac{1}{4}$ in equation (i), we get

$$7u+6=7\Rightarrow u=\frac{1}{7}$$

Now, $$u=\frac{1}{7}\Rightarrow \frac{1}{2x+3y}=\frac{1}{7}\Rightarrow 2x+3y=7 \qquad \ldots(\text{iii})$$

and, $$u=\frac{1}{4}\Rightarrow \frac{1}{3x-2y}=\frac{1}{4}\Rightarrow 3x-2y=4 \qquad \ldots(\text{iv})$$

Multiplying equation (iii) by 2 and equation (iv) by 3, we get

$$4x+6y=14 \qquad \ldots(\text{v})$$

$$9x-6y=12 \qquad \ldots(\text{vi})$$

Adding equations (v) and (vi), we get

$$13x=26\Rightarrow x=2$$

Putting $x=2$ in equation (v), we get

$$8+6y=14\Rightarrow y=1$$

Hence, $x=2, y=1$ is the solution of the given system of equations.

EXAMPLE 7 *Solve:* $\dfrac{5}{x+y}-\dfrac{2}{x-y}=-1$

$$\frac{15}{x+y}+\frac{7}{x-y}=10,$$

where $x+y\neq 0$ *and* $x-y\neq 0$.

SOLUTION Let $\dfrac{1}{x+y}=u$ and $\dfrac{1}{x-y}=v$. Then, the given system of equations becomes

$$5u-2v=-1 \qquad \ldots(\text{i})$$

$$15u+7v=10 \qquad \ldots(\text{ii})$$

Multiplying equation (i) by 3, this system of equations becomes

$$15u - 6v = -3 \qquad \text{...(iii)}$$

$$15u + 7v = 10 \qquad \text{...(iv)}$$

Subtracting equation (iv) from equation (iii), we get

$$-13v = -13 \Rightarrow v = 1$$

Putting $v = 1$ in equation (i), we get

$$5u - 2 = -1 \Rightarrow u = \frac{1}{5}$$

Now, $u = \frac{1}{5} \Rightarrow \frac{1}{x+y} = \frac{1}{5} \Rightarrow x + y = 5$ ...(v)

and, $v = 1 \Rightarrow \frac{1}{x-y} = 1 \Rightarrow x - y = 1$ ...(vi)

Adding equations (vi) and (v), we get $2x = 6 \Rightarrow x = 3$.
Putting $x = 3$ in equation (v), we get $y = 2$.
Hence, $x = 3, y = 2$ is the solution of the given system of equations.

LEVEL-2

Type III **ON SOLVING SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES**

EXAMPLE 8 *Solve the following system of equations:*

$$8v - 3u = 5uv$$

$$6v - 5u = -2uv$$

SOLUTION Clearly the given equations are not linear in the variables u and v but can be reduced into linear equations by the an appropriate substitution.
If we put $u = 0$ in either of the two equations, we get $v = 0$.
Thus, $u = 0, v = 0$ form one solution of the given system of equations.
To find the other solutions, we assume that $u \neq 0, v \neq 0$.
Since $u \neq 0, v \neq 0$. Therefore, $uv \neq 0$.
On dividing each of the given equations by uv, we get

$$\frac{8}{u} - \frac{3}{v} = 5 \qquad \text{...(i)}$$

$$\frac{6}{u} - \frac{5}{v} = -2 \qquad \text{...(ii)}$$

Taking $\frac{1}{u} = x$ and $\frac{1}{v} = y$, the above equations become

$$8x - 3y = 5 \qquad \text{...(iii)}$$

$$6x - 5y = -2 \qquad \text{...(iv)}$$

Multiplying equation (i) by 3 and equation (ii) by 4, we get

$$24x - 9y = 15 \qquad \text{...(v)}$$

$$24x - 20y = -8 \qquad \text{...(vi)}$$

Subtracting equation (vi) from equation (v), we get

$$11y = 23 \Rightarrow y = \frac{23}{11}$$

Putting $y = \frac{23}{11}$ in equation (iii), we get

$$8x - \frac{69}{11} = 5 \Rightarrow 8x = \frac{69}{11} + 5 \Rightarrow 8x = \frac{124}{11} \Rightarrow x = \frac{31}{22}$$

Now, $x = \frac{31}{22} \Rightarrow \frac{1}{u} = \frac{31}{22} \Rightarrow u = \frac{22}{31}$

and, $y = \frac{23}{11} \Rightarrow \frac{1}{v} = \frac{23}{11} \Rightarrow v = \frac{11}{23}$

Hence, the given system of equations has two solutions given by

(i) $u = 0, v = 0$ (ii) $u = \frac{22}{31}, v = \frac{11}{23}$

EXAMPLE 9 *Solve:* $3(2u + v) = 7uv$
$3(u + 3v) = 11uv$

SOLUTION Clearly, the given equations are not linear equations in the variables u and v but can be reduced to linear equations by an appropriate substitution.

If we put $u = 0$ in either of the two equations, we get $v = 0$.

So, $u = 0, v = 0$ form a solution of the given system of equations.

To find the other solutions, we assume that $u \neq 0, v \neq 0$.

Now, $u \neq 0, v \neq 0 \Rightarrow uv \neq 0$.

On dividing each one of the given equations by uv, we get

$$\frac{6}{v} + \frac{3}{u} = 7 \quad \ldots(i)$$

$$\frac{3}{v} + \frac{9}{u} = 11 \quad \ldots(ii)$$

Taking $\frac{1}{u} = x$ and $\frac{1}{v} = y$, the above equations become

$$3x + 6y = 7 \quad \ldots(iii)$$

$$9x + 3y = 11 \quad \ldots(iv)$$

Multiplying equation (iv) by 2, the above system of equations becomes

$$3x + 6y = 7 \quad \ldots(v)$$

$$18x + 6y = 22 \quad \ldots(vi)$$

Substracting equation (vi) from equation (v), we get

$$-15x = -15 \Rightarrow x = 1$$

Putting $x = 1$ in equation (iii), we get

$$3 + 6y = 7 \Rightarrow y = \frac{4}{6} = \frac{2}{3}$$

Now, $x = 1 \Rightarrow \frac{1}{u} = 1 \Rightarrow u = 1$

and, $y = \frac{2}{3} \Rightarrow \frac{1}{v} = \frac{2}{3} \Rightarrow v = \frac{3}{2}$

Hence, the given system of equations has two solutions given by

(i) $u = 0, v = 0$ (ii) $u = 1, v = 3/2$

Type IV EQUATIONS OF THE FORM $ax + by = c$ AND $bx + ay = d$ WHERE $a \neq b$

To solve the above type of equations, following algorithm may be used.

ALGORITHM

STEP I *Obtain the two equations.*

Let the equation be $ax + by = c$ *and* $bx + ay = d$

STEP II *Adding and subtracting the two equations, we obtain*

$$(a+b)x + (a+b)y = c + d \Rightarrow x + y = \frac{c+d}{a+b} \quad \ldots(i)$$

$$(a-b)\,x - (a-b)\,y = c - d \Rightarrow x - y = \frac{c-d}{a-b} \quad \ldots(ii)$$

STEP III *Add and subtract equations (i) and (ii) to get the values of x and y.*

EXAMPLE 10 *Solve:* $217x + 131y = 913$

$131x + 217y = 827$

SOLUTION We have,

$$217x + 131y = 913 \quad \ldots(i)$$

$$131x + 217y = 827 \quad \ldots(ii)$$

Adding equations (i) and (ii), we get

$$348x + 348y = 1740 \Rightarrow x + y = 5 \quad \ldots(iii)$$

Subtracting equation (ii) from equation (i), we get

$$86x - 86y = 86 \Rightarrow x - y = 1 \quad \ldots(iv)$$

Adding equation (iii) and (iv), we get

$$2x = 6 \Rightarrow x = 3$$

Putting $x = 3$ in equation (iii), we get $y = 2$.

Hence, $x = 3$ and $y = 2$ is the solution of the given system of equations.

EXAMPLE 11 *Solve:* $37x + 41y = 70$

$41x + 37y = 86$

SOLUTION We have,

$$37x + 41y = 70 \quad \ldots(i)$$

$$41x + 37y = 86 \quad \ldots(ii)$$

Adding equation (i) and (ii), we get

$$78x + 78y = 156 \Rightarrow x + y = 2 \quad \ldots(iii)$$

Subtracting equation (i) from equation (ii), we get

$$4x - 4y = 16 \Rightarrow x - y = 4 \quad \ldots(iv)$$

Adding equation (iii) and (iv), we get

$$2x = 6 \Rightarrow x = 3$$

Putting $x = 3$ in equation (iii), we get $y = -1$.

Hence, $x = 3$ and $y = -1$ is the solution of the given system of equations.

Type IV EQUATIONS OF THE FORM

$$a_1x + b_1y + c_1 = d_1$$
$$a_2x + b_2y + c_2 = d_2$$
$$a_3x + b_3y + c_3 = d_3$$

To solve the above type of equations, following algorithm may be used.

ALGORITHM

STEP I *Take any one of the three equations.*

STEP II *Obtain the value of one of the variable, say z from it.*

STEP III *Substitute the value of z obtained in Step II in the remaining two equations to obtain two linear equations in x, y.*

STEP IV *Solve the equations in x, y obtained in Step III by elimination method.*

STEP V *Substitute the values of x, y obtained in Step IV and step II to get the value of z.*

Following examples illustrate the above procedure.

EXAMPLE 12 *Solve:* $2x - y = 4$

$$y - z = 6$$
$$x - z = 10$$

SOLUTION We have,

$$2x - y = 4 \quad \ldots(i)$$
$$y - z = 6 \quad \ldots(ii)$$
$$x - z = 10 \quad \ldots(iii)$$

From equation (iii), we get $z = x - 10$

Substituting the value of z in equation (ii), we get

$$y - (x - 10) = 6$$
$$\Rightarrow \quad -x + y = -4 \quad \ldots(iv)$$

Adding equations (i) and (iv), we get

$$x = 0$$

Putting $x = 0$ equation in (i) and (iii) we get

$$y = -4 \text{ and } z = -10$$

Hence, $x = 0, y = -4, z = -10$ is the solution of the given system of equations.

EXAMPLE 13 *Solve:* $x + 2y + z = 7$

$$x + 3z = 11$$
$$2x - 3y = 1$$

SOLUTION We have,

$$x + 2y + z = 7 \quad \ldots(i)$$
$$x + 3z = 11 \quad \ldots(ii)$$
$$2x - 3y = 1 \quad \ldots(iii)$$

From equation (i), we get

$$z = 7 - x - 2y$$

Substituting $z = 7 - x - 2y$ in equation (ii), we get

$$x + 3(7 - x - 2y) = 11$$

$\Rightarrow \quad x + 21 - 3x - 6y = 11$

$\Rightarrow \quad -2x - 6y = -10$ ...(iv)

Adding equations (iii) and (iv), we get

$-9y = -9 \Rightarrow y = 1$

Putting $y = 1$ in equation (iii), we get $x = 2$.

Putting $x = 2, y = 1$ in equation (i), we get

$2 + 2 + z = 7 \Rightarrow z = 3$

Hence, $x = 2, y = 1, z = 3$

EXERCISE 3.3

LEVEL-1

Solve the following systems of equations:

1. $11x + 15y + 23 = 0$
   $7x - 2y - 20 = 0$

2. $3x - 7y + 10 = 0$
   $y - 2x - 3 = 0$

3. $0.4x + 0.3y = 1.7$
   $0.7x - 0.2y = 0.8$

4. $\frac{x}{2} + y = 0.8$
   $\frac{7}{x + \frac{y}{2}} = 10$

5. $7(y + 3) - 2(x + 2) = 14$
   $4(y - 2) + 3(x - 3) = 2$

6. $\frac{x}{7} + \frac{y}{3} = 5$
   $\frac{x}{2} - \frac{y}{9} = 6$

7. $\frac{x}{3} + \frac{y}{4} = 11$
   $\frac{5x}{6} - \frac{y}{3} = -7$

8. $\frac{4}{x} + 3y = 8$
   $\frac{6}{x} - 4y = -5$ [CBSE 2010]

9. $x + \frac{y}{2} = 4$
   $\frac{x}{3} + 2y = 5$

10. $x + 2y = \frac{3}{2}$
    $2x + y = \frac{3}{2}$

11. $\sqrt{2}x - \sqrt{3}y = 0$
    $\sqrt{3}x - \sqrt{8}y = 0$ [NCERT]

12. $3x - \frac{y + 7}{11} + 2 = 10$
    $2y + \frac{x + 11}{7} = 10$

13. $2x - \frac{3}{y} = 9$
    $3x + \frac{7}{y} = 2, y \neq 0$

14. $0.5x + 0.7y = 0.74$
    $0.3x + 0.5y = 0.5$

15. $\dfrac{1}{7x}+\dfrac{1}{6y}=3$

$\dfrac{1}{2x}-\dfrac{1}{3y}=5$

16. $\dfrac{1}{2x}+\dfrac{1}{3y}=2$

$\dfrac{1}{3x}+\dfrac{1}{2y}=\dfrac{13}{6}$

17. $\dfrac{15}{u}+\dfrac{2}{v}=17$

$\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{36}{5}$

18. $\dfrac{3}{x}-\dfrac{1}{y}=-9$

$\dfrac{2}{x}+\dfrac{3}{y}=5$

19. $\dfrac{2}{x}+\dfrac{5}{y}=1$

$\dfrac{60}{x}+\dfrac{40}{y}=19$

20. $\dfrac{1}{5x}+\dfrac{1}{6y}=12$

$\dfrac{1}{3x}-\dfrac{3}{7y}=8$

21. $\dfrac{4}{x}+3y=14$

$\dfrac{3}{x}-4y=23$ **[NCERT]**

22. $\dfrac{4}{x}+5y=7$ **[CBSE 2003]**

$\dfrac{3}{x}+4y=5$

23. $\dfrac{2}{x}+\dfrac{3}{y}=13$

$\dfrac{5}{x}-\dfrac{4}{y}=-2$ **[NCERT]**

24. $\dfrac{2}{\sqrt{x}}+\dfrac{3}{\sqrt{y}}=2$

$\dfrac{4}{\sqrt{x}}-\dfrac{9}{\sqrt{y}}=-1$ **[NCERT]**

LEVEL-2

25. $\dfrac{x+y}{xy}=2$

$\dfrac{x-y}{xy}=6$

26. $\dfrac{2}{x}+\dfrac{3}{y}=\dfrac{9}{xy}$

$\dfrac{4}{x}+\dfrac{9}{y}=\dfrac{21}{xy}$

27. $\dfrac{6}{x+y}=\dfrac{7}{x-y}+3$

$\dfrac{1}{2(x+y)}=\dfrac{1}{3(x-y)}$

28. $\dfrac{xy}{x+y}=\dfrac{6}{5}$

$\dfrac{xy}{y-x}=6$

29. $\dfrac{22}{x+y}+\dfrac{15}{x-y}=5$

$\dfrac{55}{x+y}+\dfrac{45}{x-y}=14$

30. $\dfrac{5}{x+y}-\dfrac{2}{x-y}=-1$

$\dfrac{15}{x+y}+\dfrac{7}{x-y}=10$

31. $\dfrac{3}{x+y}+\dfrac{2}{x-y}=2$

$\dfrac{9}{x+y}-\dfrac{4}{x-y}=1$

32. $\dfrac{1}{2(x+2y)}+\dfrac{5}{3(3x-2y)}=\dfrac{-3}{2}$

$\dfrac{5}{4(x+2y)}-\dfrac{3}{5(3x-2y)}=\dfrac{61}{60}$

33. $\frac{5}{x+1} - \frac{2}{y-1} = \frac{1}{2}$

$\frac{10}{x+1} + \frac{2}{y-1} = \frac{5}{2}$, where $x \neq -1$ and $y \neq 1$

34. $x + y = 5xy$

$3x + 2y = 13xy$,

35. $x + y = 2xy$

$\frac{x-y}{xy} = 6$

36. $2(3u - v) = 5uv$

$2(u + 3v) = 5uv$

37. $\frac{2}{3x+2y} + \frac{3}{3x-2y} = \frac{17}{5}$

$\frac{5}{3x+2y} + \frac{1}{3x-2y} = 2$

38. $\frac{44}{x+y} + \frac{30}{x-y} = 10$

$\frac{55}{x+y} + \frac{40}{x-y} = 13$ **[CBSE 2002C]**

39. $\frac{5}{x-1} + \frac{1}{y-2} = 2$ **[NCERT, CBSE 09]**

$\frac{6}{x-1} - \frac{3}{y-2} = 1$

40. $\frac{10}{x+y} + \frac{2}{x-y} = 4$

$\frac{15}{x+y} - \frac{9}{x-y} = -2$ **[NCERT]**

41. $\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$

$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = -\frac{1}{8}$ **[NCERT]**

42. $\frac{7x-2y}{xy} = 5$

$\frac{8x+7y}{xy} = 15$ **[NCERT]**

43. $152x - 378y = -74$

$-378x + 152y = -604$ **[NCERT]**

44. $99x + 101y = 499$

$101x + 99y = 501$

45. $23x - 29y = 98$

$29x - 23y = 110$

46. $x - y + z = 4$

$x - 2y - 2z = 9$

$2x + y + 3z = 1$

47. $x - y + z = 4$

$x + y + z = 2$

$2x + y - 3z = 0$

48. $21x + 47y = 110$

$47x + 21y = 162$ **[NCERT EXEMPLAR]**

49. If $x + 1$ is a factor of $2x^3 + ax^2 + 2bx + 1$, then find the values of a and b given that $2a - 3b = 4$. **[NCERT EXEMPLAR]**

50. Find the solution of the pair of equations $\frac{x}{10} + \frac{y}{5} - 1 = 0$ and $\frac{x}{8} + \frac{y}{6} = 15$. Hence, find λ, if $y = \lambda x + 5$. **[NCERT EXEMPLAR]**

51. Find the values of x and y in the following rectangle. [NCERT EXEMPLAR]

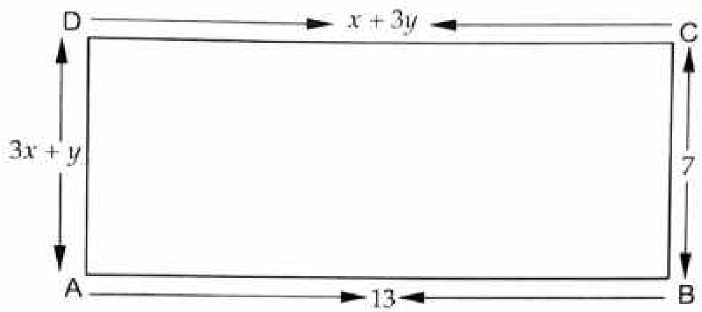


Fig. 3.19

52. Write an equation of a line passing through the point representing solution of the pair of linear equations $x + y = 2$ and $2x - y = 1$. How many such lines can we find?. [NCERT EXEMPLAR]

53. Write a pair of linear equations which has the unique solution $x = -1, y = 3$. How many such pairs can you write? [NCERT EXEMPLAR]

ANSWERS

1. C$x = 2, y = -3$
2. $x = -1, y = 1$
3. $x = 2, y = 3$
4. $x = 0.4, y = 0.6$
5. $x = 5, y = 1$
6. $x = 14, y = 9$
7. $x = 6, y = 36$
8. $x = 2, y = 2$
9. $x = 3, y = 2$
10. $x = \frac{1}{2}, y = \frac{1}{2}$
11. $x = 0, y = 0$
12. $x = 3, y = 4$
13. $x = 3, y = -1$
14. $x = 0.5, y = 0.7$
15. $x = \frac{1}{14}, y = \frac{1}{6}$
16. $x = \frac{1}{2}, y = \frac{1}{3}$
17. $u = 5, v = \frac{1}{7}$
18. $x = -\frac{1}{2}, y = \frac{1}{3}$
19. $x = 4, y = 10$
20. $x = \frac{89}{4080}, y = \frac{89}{1512}$
21. $x = \frac{1}{5}, y = -2$
22. $x = \frac{1}{3}, y = -1.$
23. $x = \frac{1}{2}, y = \frac{1}{3}$
24. $x = 4, y = 9$
25. $x = -\frac{1}{2}, y = \frac{1}{4}$
26. $x = 1, y = 3$
27. $x = -\frac{5}{4}, y = -\frac{1}{4}$
28. $x = 2, y = 3$
29. $x = 8, y = 3$
30. $x = 3, y = 2$
31. $x = \frac{5}{2}, y = \frac{1}{2}$
32. $x = \frac{1}{2}, y = \frac{5}{4}$
33. $x = 4, y = 5$
34. $x = \frac{1}{2}, y = \frac{1}{3}$
35. $x = \frac{-1}{2}, y = \frac{1}{4}$
36. $u = 2, v = 1$
37. $x = 1, y = 1$
38. $x = 8, y = 3$
39. $x = 4, y = 5$
40. $x = \frac{21}{8}, y = \frac{9}{8}$
41. $x = 1, y = 1$
42. $x = 1, y = 1$

43. $x = 2, y = 1$ 44. $x = 3, y = 2$ 45. $x = 3, y = -1$

46. $x = 3, y = -2, z = -1$ 47. $x = 2, y = -1, z = 1$ 48. $x = 3, y = 1$

49. $a = 5, b = 2$ 50. $x = 340, y = -165, \lambda = -\frac{1}{2}$ 51. $x=1, y=4$

52. $3x + 2y = 5$, Infinitely many 53. $12x + 5y = 3$, Infinitely many

3.5.3 METHOD OF CROSS-MULTIPLICATION

THEOREM *Let* $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

be a system of simultaneous linear equations in two variables x and y such that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ *i.e.* $a_1b_2 - a_2b_1 \neq 0$. *Then the system has a unique solution given by*

$$x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)} \text{ and } y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}$$

PROOF The given system of equations is

$$a_1x + b_1y + c_1 = 0 \qquad \ldots\text{(i)}$$

$$a_2x + b_2y + c_2 = 0 \qquad \ldots\text{(ii)}$$

Multiplying equation (i) by b_2, (ii) by b_1 and subtracting, we get

$$b_2(a_1x + b_1y + c_1) - b_1(a_2x + b_2y + c_2) = 0$$

$$\Rightarrow \quad x(a_1b_2 - a_2b_1) = (b_1c_2 - b_2c_1)$$

$$\Rightarrow \quad x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)} \qquad [\because (a_1b_2 - a_2b_1) \neq 0]$$

Multiplying equation (i) by a_2, (ii) by a_1, and subtracting, we get

$$a_2(a_1x + b_1y + c_1) - a_1(a_2x + b_2y + c_2) = 0$$

$$\Rightarrow \quad y(a_2b_1 - a_1b_2) + (c_1a_2 - c_2a_1) = 0$$

$$\Rightarrow \quad y(a_1b_2 - a_2b_2) = (c_1a_2 - c_2a_1)$$

$$\Rightarrow \quad y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)} \qquad [\because (a_1b_2 - c_2b_1) \neq 0]$$

Hence, $x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)}$ and $y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}$

REMARK 1 *The above solution is generally written as*

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

or,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

REMARK 2 *The following procedure is very helpful in determining the solution without remembering the above formula:*

STEP 1 *Obtain the two equations.*

STEP II *Shift all terms on LHS in the two equations to introduce zeros on RHS i.e., write the two equations in the following form:*

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

STEP III *In the above system of equations there are three columns viz. column containing x i.e.* $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, *column containing y i.e.* $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *and column containing constant terms i.e.* $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

To obtain the solution, write x, –y and 1 separated by equality signs as shown below:

$$\frac{x}{\begin{array}{cc} b_1 & c_1 \\ b_2 & c_2 \end{array}} = \frac{-y}{\begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array}} = \frac{1}{\begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array}}$$

In the denominator of x leave column containing x and write remaining two columns in the same order, in the denominator of – y leave column containing y and write the remaining two columns. Similarly, in the denominator of one write columns containing x and y. Mark crossed-arrows pointing downward from top to bottom and pointing upward from bottom to top as shown above.

The arrows between two numbers indicate that the numbers are to be multiplied.

STEP IV *To obtain the denominators of x, – y and 1, multiply the numbers with downward arrow and from their product subtract the product of the numbers with upward arrow. Applying this, we get*

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

STEP V *Obtain the value of x by equating first and third expression in step IV. The value of y is obtained by equating second and third expressions in step IV.*

Following examples illustrate the above procedure

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the each of the following systems of equations by using the method of cross-multiplication:*

(i) $x + y = 7$
$5x + 12y = 7$

(ii) $2x + 3y = 17$
$3x - 2y = 6$

(iii) $2x - y - 3 = 0$
$4x + y - 3 = 0$

(iv) $2x + y - 35 = 0$
$3x + 4y - 65 = 0$

SOLUTION (i) The given system of equations is

$$x + y - 7 = 0$$
$$5x + 12y - 7 = 0$$

By cross-multiplication, we get

$$\frac{x}{\begin{array}{cc} 1 & -7 \\ 12 & -7 \end{array}} = \frac{-y}{\begin{array}{cc} 1 & -7 \\ 1 & -7 \end{array}} = \frac{1}{\begin{array}{cc} 1 & 1 \\ 5 & 12 \end{array}}$$

$$\Rightarrow \qquad \frac{x}{1 \times -7 - 12 \times -7} = \frac{-y}{1 \times -7 - 5 \times -7} = \frac{1}{1 \times 12 - 5 \times 1}$$

$$\Rightarrow \qquad \frac{x}{-7 + 84} = \frac{-y}{-7 + 35} = \frac{1}{12 - 5}$$

$$\Rightarrow \qquad \frac{x}{77} = \frac{-y}{28} = \frac{1}{7}$$

$$\Rightarrow \qquad x = \frac{77}{7} \text{ and } y = -\frac{28}{7} \Rightarrow x = 11 \text{ and } y = -4$$

Hence, the solution of the given system of equations is $x = 11$, $y = -4$.

(ii) The given system of equations is

$$2x + 3y - 17 = 0$$
$$3x - 2y - 6 = 0$$

By cross-multiplication, we have

$$\frac{x}{\begin{array}{cc} 3 & -17 \\ -2 & -6 \end{array}} = \frac{-y}{\begin{array}{cc} 2 & -17 \\ 3 & -6 \end{array}} = \frac{1}{\begin{array}{cc} 2 & 3 \\ 2 & -2 \end{array}}$$

$$\Rightarrow \qquad \frac{x}{3 \times -6 - (-2) \times -17} = \frac{-y}{2 \times -6 - 3 \times -17} = \frac{1}{2 \times -2 - 3 \times 3}$$

$$\Rightarrow \qquad \frac{x}{-18 - 34} = \frac{-y}{-12 + 51} = \frac{1}{-4 - 9}$$

$$\Rightarrow \qquad \frac{x}{-52} = \frac{-y}{39} = \frac{1}{-13}$$

$$\Rightarrow \qquad x = \frac{-52}{-13} \text{ and } y = \frac{-39}{-13} \Rightarrow x = 4 \text{ and } y = 3$$

Hence, $x = 4$, $y = 3$ is the solution of the given system of equations.

(iii) The given system of equations is

$$2x - y - 3 = 0$$
$$4x + y - 3 = 0$$

By cross-multiplication, we have

$$\frac{x}{\begin{array}{cc} -1 & -3 \\ 1 & -3 \end{array}} = \frac{-y}{\begin{array}{cc} 2 & -3 \\ 4 & -3 \end{array}} = \frac{1}{\begin{array}{cc} 2 & -1 \\ 4 & 1 \end{array}}$$

$$\Rightarrow \qquad \frac{x}{-1 \times -3 - 1 \times -3} = \frac{-y}{2 \times -3 - 4 \times -3} = \frac{1}{2 \times 1 - 4 \times -1}$$

$$\Rightarrow \qquad \frac{x}{3 + 3} = \frac{-y}{-6 + 12} = \frac{1}{2 + 4}$$

$$\Rightarrow \qquad \frac{x}{6} = \frac{y}{-6} = \frac{1}{6}$$

$$\Rightarrow \qquad x = \frac{6}{6} = 1 \text{ and } y = -\frac{6}{6} = -1$$

Hence, the solution of the given system of equations is $x = 1, y = -1$

(iv) The given system of equations is

$$2x + y - 35 = 0$$

$$3x + 4y - 65 = 0$$

$$\frac{x}{\begin{array}{cc} 1 & -35 \\ 4 & -65 \end{array}} = \frac{-y}{\begin{array}{cc} 2 & -35 \\ 3 & -65 \end{array}} = \frac{1}{\begin{array}{cc} 2 & 1 \\ 3 & 4 \end{array}}$$

$$\Rightarrow \qquad \frac{x}{1 \times -65 - 4 \times -35} = \frac{-y}{2 \times -65 - 3 \times -35} = \frac{1}{2 \times 4 - 3 \times 1}$$

$$\Rightarrow \qquad \frac{x}{-65 + 140} = \frac{-y}{-130 + 105} = \frac{1}{8 - 3}$$

$$\Rightarrow \qquad \frac{x}{75} = \frac{y}{25} = \frac{1}{5} \Rightarrow x = \frac{75}{5} = 15 \text{ and } y = \frac{25}{5} = 5$$

Hence, the solution of the given system of equations is $x = 15, y = 5$.

EXAMPLE 2 *Solve* : $\dfrac{x}{a} + \dfrac{y}{b} = a + b$

$$\frac{x}{a^2} + \frac{y}{b^2} = 2$$

SOLUTION The given system of equations may be written as

$$\frac{1}{a} \cdot x + \frac{1}{b} \cdot y - (a + b) = 0$$

$$\frac{1}{a^2} \cdot x + \frac{1}{b^2} \cdot y - 2 = 0$$

By cross-multiplication, we have

$$\frac{x}{\begin{array}{cc} 1/b & -(a+b) \\ 1/b^2 & -2 \end{array}} = \frac{-y}{\begin{array}{cc} 1/a & -(a+b) \\ 1/a^2 & -2 \end{array}} = \frac{1}{\begin{array}{cc} 1/a & 1/b \\ 1/a^2 & 1/b^2 \end{array}}$$

$$\Rightarrow \qquad \frac{x}{\frac{1}{b} \times (-2) - \frac{1}{b^2} \times -(a + b)} = \frac{-y}{\frac{1}{a} \times -2 - \frac{1}{a^2} \times -(a + b)} = \frac{1}{\frac{1}{a} \times \frac{1}{b^2} - \frac{1}{a^2} \times \frac{1}{b}}$$

$$\Rightarrow \qquad \frac{x}{-\frac{2}{b} + \frac{a}{b^2} + \frac{1}{b}} = \frac{-y}{-\frac{2}{a} + \frac{1}{a} + \frac{b}{a^2}} = \frac{1}{\frac{1}{ab^2} - \frac{1}{a^2 b}}$$

$$\Rightarrow \qquad \frac{x}{\frac{a}{b^2} - \frac{1}{b}} = \frac{-y}{-\frac{1}{a} + \frac{b}{a^2}} = \frac{1}{\frac{1}{ab^2} - \frac{1}{a^2 b}}$$

$$\Rightarrow \quad \frac{x}{\frac{a-b}{b^2}} = \frac{y}{\frac{a-b}{a^2}} = \frac{1}{\frac{a-b}{a^2b^2}}$$

$$\Rightarrow \quad x = \frac{a-b}{b^2} \times \frac{1}{\frac{a-b}{a^2b^2}} = a^2 \text{ and } y = \frac{a-b}{a^2} \times \frac{1}{\frac{a-b}{a^2b^2}} = b^2$$

Hence, $x = a^2, y = b^2$ is the solution of the given system of equations.

EXAMPLE 3 *Solve:* $ax + by = a - b$ **[NCERT, CBSE 2000,**

$bx - ay = a + b$

SOLUTION The given system of equations may be written as

$$ax + by - (a - b) = 0$$
$$bx - ay - (a + b) = 0$$

By cross-multiplication, we get

$$\frac{x}{\begin{array}{cc} b & -(a-b) \\ -a & -(a+b) \end{array}} = \frac{-y}{\begin{array}{cc} a & -(a-b) \\ b & -(a+b) \end{array}} = \frac{1}{\begin{array}{cc} a & b \\ b & -a \end{array}}$$

$$\Rightarrow \quad \frac{x}{b \times -(a+b) - (-a) \times -(a-b)} = \frac{-y}{a \times -(a+b) - b \times -(a-b)} = \frac{1}{-a^2 - b^2}$$

$$\Rightarrow \quad \frac{x}{-b\,(a+b) - a(a-b)} = \frac{-y}{-a\,(a+b) + b\,(a-b)} = \frac{1}{-(a^2 + b^2)}$$

$$\Rightarrow \quad \frac{x}{-b^2 - a^2} = \frac{-y}{-a^2 - b^2} = \frac{1}{-(a^2 + b^2)}$$

$$\Rightarrow \quad \frac{x}{-(a^2 + b^2)} = \frac{y}{a^2 + b^2} = \frac{1}{-(a^2 + b^2)}$$

$$\Rightarrow \quad x = -\frac{(a^2 + b^2)}{-(a^2 + b^2)} = 1 \text{ and } y = \frac{(a^2 + b^2)}{-(a^2 + b^2)} = -1$$

Hence, the solution of the given system of equations is $x = 1, y = -1$.

EXAMPLE 4 *Solve:* $x + y = a + b$

$ax - by = a^2 - b^2$

SOLUTION The given system of equations may be written as

$$x + y - (a + b) = 0$$
$$ax - by - (a^2 - b^2) = 0$$

By cross-multiplication, we get

$$\frac{x}{-(a^2 - b^2) - (-b) \times -(a+b)} = \frac{-y}{-(a^2 - b^2) - a \times -(a+b)} = \frac{1}{1 \times -b - a \times 1}$$

$$\Rightarrow \quad \frac{x}{-a^2 + b^2 - ab - b^2} = \frac{-y}{-a^2 + b^2 + a^2 + ab} = \frac{1}{-b - a}$$

$$\Rightarrow \quad \frac{x}{-a(a+b)} = \frac{-y}{b(a+b)} = \frac{1}{-(a+b)}$$

$$\Rightarrow \quad \frac{x}{-a(a+b)} = \frac{y}{-b(a+b)} = \frac{1}{-(a+b)}$$

$$\Rightarrow \quad x = \frac{-a(a+b)}{-(a+b)} = a \text{ and } y = \frac{-b(a+b)}{-(a+b)} = b$$

Hence, the solution of the given system of equations is $x = a, y = b$

EXAMPLE 5 *Solve:* $\frac{x}{a} + \frac{y}{b} = 2$

$ax - by = a^2 - b^2$ **[CBSE 2005]**

SOLUTION The given system of equations may be writen as

$$bx + ay - 2ab = 0$$
$$ax - by - (a^2 - b^2) = 0$$

By cross-multiplication, we have

$$\frac{x}{-a(a^2-b^2)-(-b)(-2ab)} = \frac{-y}{-b(a^2-b^2)-a(-2ab)} = \frac{1}{b \times -b - a \times a}$$

$$\Rightarrow \quad \frac{x}{-a(a^2-b^2)-2ab^2} = \frac{-y}{-b(a^2-b^2)+2a^2b} = \frac{1}{-b^2-a^2}$$

$$\Rightarrow \quad \frac{x}{-a(a^2-b^2+2b^2)} = \frac{-y}{-(a^2-b^2-2a^2)} = \frac{1}{-(a^2+b^2)}$$

$$\Rightarrow \quad \frac{x}{-(a^2+b^2)} = \frac{-y}{-b(-a^2-b^2)} = \frac{1}{-(a^2+b^2)}$$

$$\Rightarrow \quad x = \frac{-a(a^2+b^2)}{-(a^2+b^2)} = a \text{ and } y = \frac{-b(a^2+b^2)}{-(a^2+b^2)} = b$$

Hence, the solution of the given system of equations is $x = a, y = b$.

LEVEL-2

EXAMPLE 6 *Solve the following system of equations in x and y*

$$(a-b)x + (a+b)y = a^2 - 2ab - b^2$$
$$(a+b)(x+y) = a^2 + b^2$$

[NCERT]

SOLUTION The given system of equations may be written as

$$(a-b)x + (a+b)y - (a^2 - 2ab - b^2) = 0$$
$$(a+b)x + (a+b)y - (a^2 + b^2) = 0$$

By cross-multiplication, we have

$$\frac{x}{(a+b)\times-(a^2+b^2)-(a+b)\times-(a^2-2ab-b^2)} = \frac{-y}{(a-b)\times-(a^2+b^2)-(a+b)\times-(a^2-2ab-b^2)}$$

$$= \frac{1}{(a-b)(a+b)-(a+b)^2}$$

$$\Rightarrow \quad \frac{x}{-(a+b)(a^2+b^2)+(a+b)(a^2-2ab-b^2)} = \frac{-y}{-(a-b)(a^2+b^2)+(a+b)(a^2-2ab-b^2)}$$
$$= \frac{1}{(a-b)(a+b)-(a+b)^2}$$

$$\Rightarrow \quad \frac{x}{(a+b)\{-(a^2+b^2)+(a^2-2ab-b^2)\}} = \frac{-y}{(a+b)(a^2-2ab-b^2)-(a-b)(a^2+b^2)}$$
$$= \frac{1}{(a+b)(a-b-a-b)}$$

$$\Rightarrow \quad \frac{x}{(a+b)(-2ab-2b^2)} = \frac{-y}{a^3-a^2b-3ab^2-b^3-a^3-ab^2+a^2b+b^3} = \frac{1}{-(a+b)\,2b}$$

$$\Rightarrow \quad \frac{x}{-2b\,(a+b)^2} = \frac{-y}{-4ab^2} = \frac{1}{-2b\,(a+b)}$$

$$\Rightarrow \quad x = \frac{-2b\,(a+b)^2}{-2b\,(a+b)} = a+b \text{ and } y = \frac{4ab^2}{-2b\,(a+b)} = \frac{-2ab}{a+b}$$

Hence, the solution of the given system of equations is $x = a+b,\ y = -\dfrac{2ab}{a+b}$.

EXAMPLE 7 *Solve the following system of equations in x and y:*

$$\frac{a}{x} - \frac{b}{y} = 0$$
$$\frac{ab^2}{x} + \frac{a^2b}{y} = a^2+b^2, \text{ where } x, y \neq 0.$$

SOLUTION Taking $\dfrac{1}{x} = u$ and $\dfrac{1}{y} = v$, the above system of equations becomes

$$au - bv + 0 = 0$$
$$ab^2u + a^2bv - (a^2+b^2) = 0$$

By cross-multiplication, we have

$$\frac{u}{-b\times-(a^2+b^2)-a^2b\times 0} = \frac{-v}{a\times-(a^2+b^2)-ab^2\times 0} = \frac{1}{a\times a^2b-ab^2\times -b}$$

$$\Rightarrow \quad \frac{u}{b\,(a^2+b^2)} = \frac{-v}{-a\,(a^2+b^2)} = \frac{1}{a^3b+ab^3}$$

$$\Rightarrow \quad \frac{u}{b\,(a^2+b^2)} = \frac{v}{a\,(a^2+b^2)} = \frac{1}{ab\,(a^2+b^2)}$$

$$\Rightarrow \quad u = \frac{b\,(a^2+b^2)}{ab\,(a^2+b^2)} = \frac{1}{a} \text{ and } v = \frac{a\,(a^2+b^2)}{ab\,(a^2+b^2)} = \frac{1}{b}$$

Now, $\quad u = \dfrac{1}{a} \Rightarrow \dfrac{1}{x} = \dfrac{1}{a} \Rightarrow x = a$ and $v = \dfrac{1}{b} \Rightarrow \dfrac{1}{y} = \dfrac{1}{b} \Rightarrow y = b$

Hence, the solution of the given system of equation is $x = a,\ y = b$

EXAMPLE 8 *Solve the following system of equations in x and y*

$$ax + by = 1$$

$$bx + ay = \frac{(a+b)^2}{a^2+b^2} - 1 \text{ or, } bx + ay = \frac{2ab}{a^2+b^2}$$

SOLUTION The given system of equations may be written as

$$ax + by - 1 = 0$$

$$bx + ay - \frac{2ab}{a^2+b^2} = 0$$

By cross-multiplication, we have

$$\frac{x}{b \times -\frac{2ab}{a^2+b^2} - a \times -1} = \frac{-y}{a \times -\frac{2ab}{a^2+b^2} - b \times -1} = \frac{1}{a \times a - b \times b}$$

$$\Rightarrow \quad \frac{x}{\frac{-2ab^2}{a^2+b^2} + a} = \frac{-y}{\frac{-2a^2b}{a^2+b^2} + b} = \frac{1}{a^2 - b^2}$$

$$\Rightarrow \quad \frac{x}{\frac{-2ab^2 + a^3 + ab^2}{a^2+b^2}} = \frac{-y}{\frac{-2a^2b + a^2b + b^3}{a^2+b^2}} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \quad \frac{x}{\frac{a^3 - ab^2}{a^2+b^2}} = \frac{-y}{\frac{-a^2b + b^3}{a^2+b^2}} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \quad \frac{x}{\frac{a(a^2-b^2)}{a^2+b^2}} = \frac{-y}{\frac{b(a^2-b^2)}{a^2+b^2}} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \quad x = \frac{a(a^2-b^2)}{a^2+b^2} \times \frac{1}{a^2-b^2} \text{ and } y = \frac{b(a^2-b^2)}{a^2+b^2} \times \frac{1}{a^2-b^2}$$

$$\Rightarrow \quad x = \frac{a}{a^2+b^2} \text{ and } y = \frac{b}{a^2+b^2}$$

Hence, the solution of the given system of equations is $x = \frac{a}{a^2+b^2}, y = \frac{b}{a^2+b^2}$

EXAMPLE 9 *Solve:* $a(x+y) + b(x-y) = a^2 - ab + b^2$

$$a(x+y) - b(x-y) = a^2 + ab + b^2$$

SOLUTION Taking $x + y = u$ and $x - y = v$ the given system of equations becomes

$$au + bu - (a^2 - ab + b^2) = 0$$

$$au - bv - (a^2 + ab + b^2) = 0$$

By cross-multiplication, we have

$$\Rightarrow \quad \frac{u}{b \times -(a^2+ab+b^2) - (-b) \times -(a^2-ab+b^2)} = \frac{-v}{a \times -(a^2+ab+b^2) + a(a^2-ab+b^2)}$$

$$= \frac{1}{a \times -b - a \times b}$$

$$\Rightarrow \quad \frac{u}{-b(a^2+ab+b^2)-b(a^2-ab+b^2)} = \frac{-v}{-a(a^2+ab+b^2)+a(a^2-ab+b^2)} = \frac{1}{-ab-ab}$$

$$\Rightarrow \quad \frac{u}{-b(a^2+ab+b^2+a^2-ab+b^2)} = \frac{-v}{-a(a^2+ab+b^2-a^2+ab-b^2)} = \frac{1}{-2ab}$$

$$\Rightarrow \quad \frac{u}{-2b(a^2+b^2)} = \frac{-v}{-a(2ab)} = \frac{1}{-2ab}$$

$$\Rightarrow \quad u = \frac{-2b(a^2+b^2)}{-2ab}, v = \frac{2a^2b}{-2ab} \Rightarrow u = \frac{a^2+b^2}{a}, v = -a$$

Now, $u = \dfrac{a^2+b^2}{a} \Rightarrow x + y = \dfrac{a^2+b^2}{a}$ ...(i)

and, $v = -a \Rightarrow x - y = -a$ ...(ii)

Adding equations (i) and (ii), we get

$$2x = \frac{a^2+b^2}{a} - a \Rightarrow 2x = \frac{a^2+b^2-a^2}{a} \Rightarrow 2x = \frac{b^2}{a} \Rightarrow x = \frac{b^2}{2a}$$

Subtracting equation (ii) from equation (i), we get

$$2y = \frac{a^2+b^2}{a} + a \Rightarrow 2y = \frac{a^2+b^2+a^2}{a} \Rightarrow y = \frac{2a^2+b^2}{2a}$$

Hence, the solution of the given system of equations is $x = \dfrac{b^2}{2a}, y = \dfrac{2a^2+b^2}{2a}$

EXAMPLE 10 *Solve:* $ax + by = c$

$bx + ay = 1 + c$ **[NCERT]**

SOLUTION The given system of equations may be written as

$$ax + by - c = 0$$
$$bx + ay - (1+c) = 0$$

By cross-multiplication, we have

$$\frac{x}{b \times -(1+c) - a \times -c} = \frac{-y}{a \times -(1+c) - b \times -c} = \frac{1}{a \times a - b \times b}$$

$$\Rightarrow \quad \frac{x}{-b(1+c)+ac} = \frac{-y}{-a(1+c)+bc} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \quad \frac{x}{ac-bc-b} = \frac{y}{ac-bc+a} = \frac{1}{a^2-b^2}$$

$$\Rightarrow \quad \frac{x}{c(a-b)-b} = \frac{y}{c(a-b)+a} = \frac{1}{(a-b)(a+b)}$$

$$\Rightarrow \quad x = \frac{c(a-b)-b}{(a-b)(a+b)} \text{ and } y = \frac{c(a-b)+a}{(a-b)(a+b)}$$

$$\Rightarrow \quad x = \frac{c}{a+b} - \frac{b}{(a-b)(a+b)} \text{ and } y = \frac{c}{a+b} + \frac{a}{(a-b)(a+b)}$$

Hence, the solution of the given system of equations is

$$x = \frac{c}{a+b} - \frac{b}{a^2-b^2}, \; y = \frac{c}{a+b} + \frac{a}{a^2-b^2}$$

EXAMPLE 11 *Solve the following system of equations:*

$$x + y = a - b$$
$$ax - by = a^2 + b^2$$

SOLUTION The given system of equations may be written as

$$x + y - (a - b) = 0$$
$$ax - by - (a^2 + b^2) = 0$$

By cross-multiplication, we have

$$\frac{x}{-(a^2 + b^2) - b(a - b)} = \frac{y}{-a(a - b) + (a^2 + b^2)} = \frac{1}{-b - a}$$

$$\Rightarrow \quad \frac{x}{-a^2 - ab} = \frac{y}{ab + b^2} = \frac{1}{-b - a}$$

$$\Rightarrow \quad \frac{x}{-a(a + b)} = \frac{y}{b(a + b)} = \frac{1}{-(a + b)}$$

$$\Rightarrow \quad x = \frac{-a(a + b)}{-(a + b)} = a \text{ and } y = \frac{b(a + b)}{-(a + b)} = -b$$

Hence, $x = a$, $y = -b$ is the solution of the given system of equations.

EXERCISE 3.4

LEVEL-1

Solve each of the following systems of equations by the method of cross-multiplication:

1. $x + 2y + 1 = 0$
   $2x - 3y - 12 = 0$
2. $3x + 2y + 25 = 0$
   $2x + y + 10 = 0$
3. $2x + y = 35$
   $3x + 4y = 65$
4. $2x - y = 6$
   $x - y = 2$
5. $\frac{x + y}{xy} = 2, \; \frac{x - y}{xy} = 6$
6. $ax + by = a - b$
   $bx - ay = a + b$
7. $x + ay = b$
   $ax - by = c$
8. $ax + by = a^2$
   $bx + ay = b^2$
9. $\frac{5}{x + y} - \frac{2}{x - y} = -1$
   $\frac{15}{x + y} + \frac{7}{x - y} = 10$
10. $\frac{2}{x} + \frac{3}{y} = 13$
    $\frac{5}{x} - \frac{4}{y} = -2$
11. $\frac{57}{x + y} + \frac{6}{x - y} = 5$
    $\frac{38}{x + y} + \frac{21}{x - y} = 9$ [CBSE 2002C]

LEVEL-2

12. $\frac{x}{a} + \frac{y}{b} = 2$
    $ax - by = a^2 - b^2$
13. $\frac{x}{a} + \frac{y}{b} = a + b$ [NCERT EXEMPLAR]
    $\frac{x}{a^2} + \frac{y}{b^2} = 2$

14. $\frac{x}{a} = \frac{y}{b}$

$ax + by = a^2 + b^2$ [NCERT]

15. $2ax + 3by = a + 2b$

$3ax + 2by = 2a + b$

16. $5ax + 6by = 28$

$3ax + 4by = 18$

17. $(a + 2b)x + (2a - b)y = 2$

$(a - 2b)x + (2a + b)y = 3$

18. $x\left(a - b + \frac{ab}{a-b}\right) = y\left(a + b - \frac{ab}{a+b}\right)$

$x + y = 2a^2$

19. $bx + cy = a + b$

$ax\left(\frac{1}{a-b} - \frac{1}{a+b}\right) + cy\left(\frac{1}{b-a} - \frac{1}{b+a}\right) = \frac{2a}{a+b}$

20. $(a - b)x + (a + b)y = 2a^2 - 2b^2$

$(a + b)(x + y) = 4ab$

21. $a^2x + b^2y = c^2$

$b^2x + a^2y = d^2$

22. $ax + by = \frac{a+b}{2}$

$3x + 5y = 4$

23. $2(ax - by) + a + 4b = 0$

$2(bx + ay) + b - 4a = 0$ [CBSE 2004]

24. $6(ax + by) = 3a + 2b$

$6(bx - ay) = 3b - 2a$

[CBSE 2004]

25. $\frac{a^2}{x} - \frac{b^2}{y} = 0$

$\frac{a^2b}{x} + \frac{b^2a}{y} = a + b, x, y \neq 0$

[CBSE 2006C]

26. $mx - ny = m^2 + n^2$

$x + y = 2m$ [CBSE 2006C]

27. $\frac{ax}{b} - \frac{by}{a} = a + b$

$ax - by = 2ab$ [CBSE 2009]

28. $\frac{b}{a}x + \frac{a}{b}y = a^2 + b^2$

$x + y = 2ab$ [CBSE 2010]

ANSWERS

1. $x = 3, y = -2$
2. $x = 5, y = -20$
3. $x = 15, y = 5$
4. $x = 4, y = 2$
5. $x = \frac{-1}{2}, y = \frac{1}{4}$
6. $x = 1, y = -1$
7. $x = \frac{ac + b^2}{a^2 + b}, y = \frac{ab - c}{a^2 + b}$
8. $x = \frac{a^2 + ab + b^2}{a + b}, y = \frac{-ab}{a + b}$
9. $x = 3, y = 2$
10. $x = \frac{1}{2}, y = \frac{1}{3}$
11. $x = 11, y = 8$
12. $x = a, y = b$
13. $x = a^2, y = b^2$
14. $x = a, y = b$
15. $x = \frac{4a - b}{5a}, y = \frac{-a + 4b}{5b}$
16. $x = \frac{2}{a}, y = \frac{3}{b}$
17. $x = \frac{5b - 2a}{10ab}, y = \frac{a + 10b}{10ab}$
18. $x = \frac{a^3 - b^3}{a}, y = \frac{a^3 + b^3}{a}$

19. $x = \frac{a}{b}, y = \frac{b}{c}$ 20. $x = \frac{2ab - a^2 + b^2}{b}, y = \frac{(a-b)(a^2 + b^2)}{b(a+b)}$

21. $x = \frac{a^2c^2 - b^2d^2}{a^4 - b^4}, y = \frac{a^2d^2 - b^2c^2}{a^4 - b^4}$ 22. $x = \frac{1}{2}, y = \frac{1}{2}$

23. $x = \frac{-1}{2}, y = 2$ 24. $x = \frac{1}{2}, y = \frac{1}{3}$. 25. $x = a^2, y = b^2$

26. $x = m + n, y = m - n$ 27. $x = b, y = -a$ 28. $x = y = ab$

3.6 CONDITIONS FOR SOLVABILITY (OR CONSISTENCY)

Uptill now we have been discussing various methods of solving a system of simultaneous linear equations in two variables with the assumption that the system has a unique solution. In this section, we shall be discussing the conditions for solvability of a system of simultaneous linear equations in two variables.

Consider the following three systems of simultaneous linear equations:

(i) $2x + y = 5$, $3x - 2y = 4$ (ii) $3x + 4y = 2$, $6x + 8y = 4$ (iii) $2x - 3y = 5$, $4x - 6y = 9$

Clearly, $x = 2, y = 1$ satisfies the first system of equations. So, it is a solution of this system. Also, no other set of values of x and y satisfy this system of equations. So, we say that the system is consistent (solvable) with a unique solution. Graphically, it is due to the reason that the lines represented by the two equations intersect at only one point.

Now, consider the second system of equations. It can easily be checked that $x = 2, y = -1$ is a solution of this system of equations. So, we say that the system is consistent i.e. it possesses a solution. Also, $x = 1, y = -1/4; x = -2, y = 2;\ x = 6,\ y = -4$ etc. are solutions of this system. If follows from this that the second system of equations is consistent with infinitely many solutions. Graphically, it is due to the reason that the lines represented by the two equations are coincident.

For the third system of equations there is no set of values of x and y which satisfy the two equations simultaneously. This is because the lines represented by the two equations are parallel. So, we say that the system is inconsistent or not solvable.

It follows from the above discussion that a given system of equations is either inconsistent (does not have a solution) or it is consistent with infinitely many solutions or it is consistent with a unique solution.

In the following discussion, we shall find the conditions for consistency of a system of simultaneous linear equations.

Consider the system of equations

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$
$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

Multiplying equation (i) by b_2, equation (ii) by b_1 and subtracting, we get

$$x(a_1b_2 - a_2b_1) = (b_1c_2 - b_2c_1) \quad \ldots(iii)$$

Multiplying equation (ii) by a_1, equation (i) by a_2 and subtracting, we get

$$y(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1) \quad \ldots(iv)$$

Now, the following cases arise:

CASE I *When* $a_1b_2 - a_2b_1 \neq 0$, *i.e.* $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

If $a_1b_2 - a_2b_1 \neq 0$, *then from equations (iii) and (iv), we have*

$$x = \frac{(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)} \text{ and } y = \frac{(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)}$$

It follows from this that the system of equations $a_1x + b_1y + c_1 = 0$ *and*, $a_2x + b_2y + c_2 = 0$ *is consistent with unique solution, if*

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

REMARK 1 *If* $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

is a system of simultaneous linear equations with $a_1b_2 - a_2b_1 \neq 0$, *then the lines represented by the equations* $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$ *intersect at exactly one point having coordinates* $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$ *as shown in Fig. 3.20. Hence, the given system of equations has unique solution.*

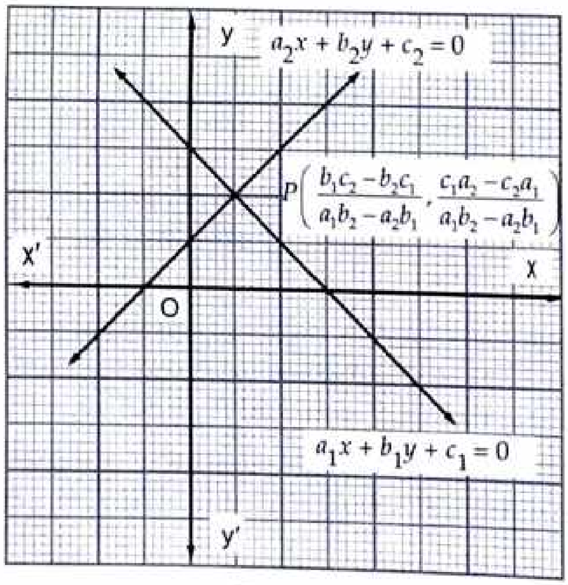


Fig. 3.20

CASE II *When* $a_1b_2 - a_2b_1 = 0$ *i.e.*, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$. *Then*, $a_1 = ka_2$ and $b_1 = kb_2$

Putting $a_1 = ka_2$ and $b_1 = kb_2$ *in equation (i), we get*

$$k(a_2x + b_2y) + c_1 = 0 \quad \ldots(v)$$

From equation (ii), we have

$$a_2x + b_2y = -c_2$$

Putting $a_2x + b_2y = -c_2$ *in equation (v), we get*

$$-kc_2 + c_1 = 0 \Rightarrow c_1 = kc_2$$

Thus, we have

$$a_1 = ka_2, b_1 = kb_2 \text{ and } c_1 = kc_2$$

Substituting the values of a_1, b_1, c_1 *in equation (i), we obtain*

$$k(a_2x + b_2y + c_2) = 0$$

Thus, the system of equations reduces to

$$k(a_2x + b_2y + c_2) = 0$$

$$a_2x + b_2y + c_2 = 0$$

Clearly, every solution of first of these two equations is a solution of the other and vice-versa.

Thus, in this case, the system has infintely many solutions.

It follows from this that the system of equations $a_1x + b_1y + c_1 = 0$ *and,* $a_2x + b_2y + c_2 = 0$ *is consistent with infinitely many solutions, if*

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

REMARK 2 *If* $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

is a system of simultaneous linear equations with $a_1b_2 - a_2b_1 = 0$ *and* $a_1c_2 - a_2c_1 = 0$ *i.e.* $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, *then the lines represented by the equations* $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$ *are coincident i.e. the two equations represent the same line. Consequently, every point on the line determines a solution of the system. Hence, the system of equations has infinitely many solutions.*

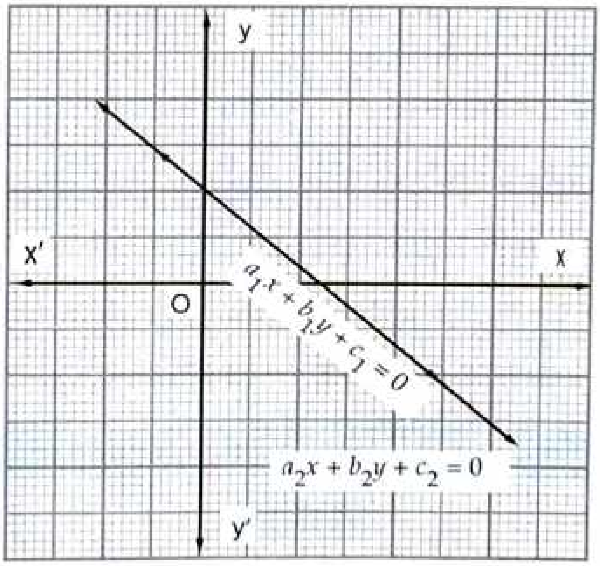


Fig. 3.21

CASE III When $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

In this case, it is clear that the given system of equations is inconsistent i.e. it has no solution.

REMARK 3 *If* $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

is a system of simultaneous linear equations with $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, *then the lines represented by the equations* $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$ *are parallel and non-coincident. Consequently, the system has no solution or it is inconsistent.*

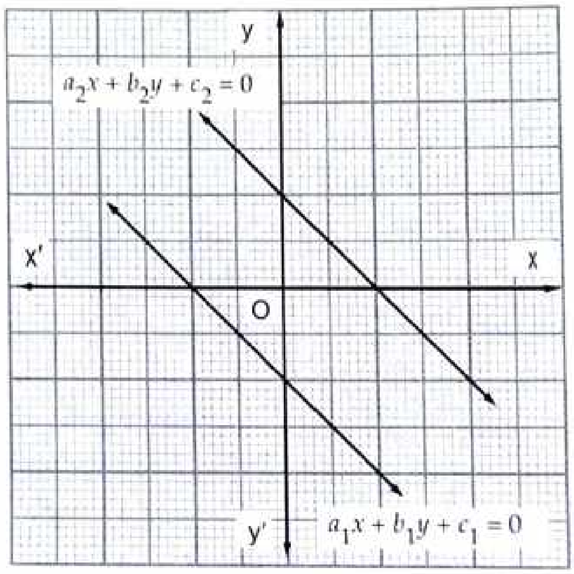


Fig. 3.22

SUMMARY *The above results can be summarised as under:*

The system of equations

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$
$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

(i) *is consistent with unique solution, if* $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

i.e., lines represented by equations (i) and (ii) are not parallel

(ii) *is consistent with infinitely many solutions, if* $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

i.e., line represented by equation (i) and (ii) are coincident.

(iii) *is inconsistent, if* $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

i.e., lines represented by equations (i) and (ii) are parallel and non-coincident.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In each of the following systems of equations determine whether the system has a unique solution, no solution or infinitely many solutions. In case there is a unique solution, find it.*

(i) $2x + 3y = 7$
$6x + 5y = 11$

(ii) $6x + 5y = 11$
$9x + \frac{15}{2}y = 21$

(iii) $-3x + 4y = 5$
$\frac{9}{2}x - 6y + \frac{15}{2} = 0$

SOLUTION (i) The given system of equations may be written as

$$2x + 3y - 7 = 0$$
$$6x + 5y - 11 = 0$$

The given system of equations is of the form

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 2, b_1 = 3, c_1 = -7$ and $a_2 = 6, b_2 = 5, c_2 = -11$

We have, $\dfrac{a_1}{a_2} = \dfrac{2}{6} = \dfrac{1}{3}$ and $\dfrac{b_1}{b_2} = \dfrac{3}{5}$

Clearly, $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

So, the given system of equations has unique solution.

To find the solution, we use the cross-multiplication method.

By cross-multiplication, we have

$$\frac{x}{3 \times -11 - 5 \times -7} = \frac{-y}{2 \times -11 - 6 \times -7} = \frac{1}{2 \times 5 - 6 \times 3}$$

$$\Rightarrow \quad \frac{x}{-33 + 35} = \frac{-y}{-22 + 42} = \frac{1}{10 - 18}$$

$$\Rightarrow \quad \frac{x}{2} = \frac{y}{-20} = \frac{1}{-8}$$

$$\Rightarrow \quad x = \frac{-2}{8} = -\frac{1}{4} \text{ and } y = \frac{-20}{-8} = \frac{5}{2}$$

Hence, the given system of equations has unique solution given by $x = -\dfrac{1}{4}, y = \dfrac{5}{2}$.

(ii) The given system of equations may be written as

$$6x + 5y - 11 = 0$$
$$9x + \frac{15}{2}y - 21 = 0$$

The given system of equations is of the form $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 6, b_1 = 5, c_1 = -11, a_2 = 9, b_2 = \dfrac{15}{2}, c_2 = -21$

We have, $\dfrac{a_1}{a_2} = \dfrac{6}{9} = \dfrac{2}{3}, \dfrac{b_1}{b_2} = \dfrac{5}{15/2} = \dfrac{2}{3}$ and $\dfrac{c_1}{c_2} = \dfrac{-11}{-21} = \dfrac{11}{21}$

Clearly, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

So, the given system of equations has no solution i.e. it is in-consistent.

(iii) The given system of equations may be written as

$$-3x + 4y - 5 = 0$$
$$\frac{9}{2}x - 6y + \frac{15}{2} = 0$$

The given system of equations is of the form

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = -3, b_1 = 4, c_1 = -5$ and $a_2 = \frac{9}{2}, b_2 = -6, c_2 = \frac{15}{2}$

We have, $\frac{a_1}{a_2} = \frac{-3}{9/2} = \frac{-2}{3}, \frac{b_1}{b_2} = \frac{4}{-6} = \frac{-2}{3}$ and $\frac{c_1}{c_2} = \frac{-5}{15/2} = \frac{-2}{3}$

Clearly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

So, the given system of equations has infinitely many solutions.

EXAMPLE 2 *For each of the following systems of equations determine the value of k for which the given system of equations has a unique solution:*

(i) $x - ky = 2$
$3x + 2y = -5$

(ii) $2x - 3y = 1$
$kx + 5y = 7$

(iii) $2x + 3y - 5 = 0$
$kx - 6y - 8 = 0$

(iv) $2x + ky = 1$
$5x - 7y = 5$

SOLUTION (i) The given system of equations is

$$x - ky - 2 = 0$$
$$3x + 2y + 5 = 0$$

This system of equation is of the form

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 1, b_1 = -k, c_1 = -2$ and $a_2 = 3, b_2 = 2, c_2 = 5$

For a unique solution, we must have

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ i.e., } \frac{1}{3} \neq \frac{-k}{2} \Rightarrow k \neq \frac{-2}{3}$$

So, the given system of equations will have unique solution for all real values of k other than $-2/3$.

(ii) The given system of equations is

$$2x - 3y - 1 = 0$$
$$kx + 5y - 7 = 0$$

It is of the form $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 2, b_1 = -3, c_1 = -1$ and $a_2 = k, b_2 = 5, c_2 = -7$

For a unique solution, we must have

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ i.e., } \frac{2}{k} \neq \frac{-3}{5} \Rightarrow k \neq \frac{-10}{3}$$

So, the given system of equations is consistent with unique solution for all values of k other than $-10/3$.

(iii) The given system of equations is

$$2x + 3y - 5 = 0$$
$$kx - 6y - 8 = 0$$

It is of the form $a_1x + b_1y + c_1 = 0$

and, $a_2x + b_2y + c_2 = 0$

where, $a_1 = 2, b_1 = 3, c_1 = -5$ and $a_2 = k,\ b_2 = -6$ and $c_2 = -8$

For a unique solution, we must have

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ i.e., } \frac{2}{k} \neq \frac{3}{-6} \Rightarrow k \neq -4$$

So, the given system of equations will have unique solution for all real values of k other than -4.

(iv) The given system of equations is

$$2x + ky - 1 = 0$$
$$5x - 7y - 5 = 0$$

It is of the form $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 2, b_1 = k, c_1 = -1$ and $a_2 = 5, b_2 = -7$ and $c_2 = -5$

For a unique solution, we must have

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ i.e., } \frac{2}{5} \neq \frac{k}{-7} \Rightarrow k \neq \frac{-14}{5}$$

So, the given system of equations will have unique solution for all real values of k other than $\frac{-14}{5}$.

EXAMPLE 3 *For each of the following systems of equations determine the value of k for which the given system of equations has infinitely many solutions.*

(i) $5x + 2y = k$
$10x + 4y = 3$

(ii) $(k-3)x + 3y = k$
$kx + ky = 12$

(iii) $kx + 3y = k - 3$
$12x + ky = k$

[NCERT]

SOLUTION (i) The given system of equations is

$$5x + 2y - k = 0$$
$$10x + 4y - 3 = 0$$

This system of equation is of the form

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 5, b_1 = 2, c_1 = -k$ and $a_2 = 10, b_2 = 4$ and $c_2 = -3$

For infinitely many solutions, we must have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \text{ i.e., } \frac{5}{10} = \frac{2}{4} = \frac{-k}{-3} \Rightarrow \frac{1}{2} = \frac{k}{3} \Rightarrow k = \frac{3}{2}$$

Hence, the given system of equations will have infinitely many solutions, if $k = \frac{3}{2}$.

(ii) The given system of equations is

$$(k-3)x + 3y - k = 0$$
$$kx + ky - 12 = 0$$

For infinitely many solutions, we must have

$$\frac{k-3}{k}=\frac{3}{k}=\frac{-k}{-12}$$

$$\Rightarrow \quad \frac{k-3}{k}=\frac{3}{k} \text{ and } \frac{3}{k}=\frac{k}{12}$$

$$\Rightarrow \quad k^2-3k=3k \text{ and } k^2=36$$

$$\Rightarrow \quad k^2-6k=0 \text{ and } k^2=36$$

$$\Rightarrow \quad (k=0 \text{ or, } k=6) \text{ and } (k=\pm 6)$$

$$\Rightarrow \quad k=6$$

Hence, the given system has infinitely many solutions, if $k = 6$.

(iii) The given system of equations is

$$kx+3y-(k-3)=0$$

$$12x+ky-k=0$$

For infinitely many solutions, we must have

$$\frac{k}{12}=\frac{3}{k}=\frac{-(k-3)}{-k}$$

$$\Rightarrow \quad \frac{k}{12}=\frac{3}{k} \text{ and } \frac{3}{k}=\frac{k-3}{k}$$

$$\Rightarrow \quad k^2=36 \text{ and } k^2-3k=3k$$

$$\Rightarrow \quad k^2=36 \text{ and } k^2-6k=0$$

$$\Rightarrow \quad (k=\pm 6) \text{ and } (k=0 \textit{ or, } k=6)$$

$$\Rightarrow \quad k=6$$

Hence, the given system of equations has infinitely many solutions, if $k = 6$.

EXAMPLE 4 *For each of the following system of equations determine the values of k for which the given system has no solution:*

(i) $3x-4y+7=0$
$kx+3y-5=0$

(ii) $2x-ky+3=0$
$3x+2y-1=0$

SOLUTION (i) The given system of equations is

$$3x-4y+7=0$$

$$kx+3y-5=0$$

This is of the form $a_1x+b_1y+c_1=0$

$$a_2x+b_2y+c_2=0,$$

where, $a_1=3, b_1=-4, c_1=7$ and $a_2=k, b_2=3, c_2=-5$

For no solution, we must have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$$

We have, $\dfrac{b_1}{b_2}=\dfrac{-4}{3}$ and $\dfrac{c_1}{c_2}=\dfrac{-7}{5}$

Clearly, $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

So, the given system will have no solution.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{3}{k} = \frac{-4}{3} \Rightarrow k = \frac{-9}{4}$

Clearly, for this value of k, we have $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Hence, the given system of equations has no solution, when $k = \frac{-9}{4}$

(ii) The given system of equations is

$$2x - ky + 3 = 0$$
$$3x + 2y - 1 = 0$$

This is of the form $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

where, $a_1 = 2, b_1 = -k, c_1 = 3$ and $a_2 = 3, b_2 = 2, c_2 = -1$

For no solution, we must have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

We have, $\frac{a_1}{a_2} = \frac{2}{3}$ and $\frac{c_1}{c_2} = \frac{3}{-1}$

Clearly, $\frac{a_1}{a_2} \neq \frac{c_1}{c_2}$

So, the given system of equations will have no solution, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \text{ i.e., } \frac{2}{3} = \frac{-k}{2} \Rightarrow k = \frac{-4}{3}$$

Hence, the given system of equations will have no solution, if $k = \frac{-4}{3}$.

EXAMPLE 5 *Find the value(s) of k for which the system of equations*

$$kx - y = 2$$
$$6x - 2y = 3$$

has (i) a unique solution (ii) no solution.

Is there a value of k for which the system has infinitely many solutions?

SOLUTION The given system of equations is

$$kx - y - 2 = 0$$
$$6x - 2y - 3 = 0$$

It is of the form $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0,$$

where $a_1 = k, b_1 = -1, c_1 = -2$ and $a_2 = 6, b_2 = -2, c_2 = -3$

(i) The given system will have a unique solution, if

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ i.e., if } \frac{k}{6} \neq \frac{-1}{-2} \text{ i.e., } k \neq 3.$$

So, the given of equations system will have a unique solution, if $k \neq 3$.

(ii) The given system will have no solution, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

We have, $\frac{b_1}{b_2} = \frac{-1}{-2} = \frac{1}{2}$ and $\frac{c_1}{c_2} = \frac{-2}{-3} = \frac{2}{3}$

Clearly, $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

So, the system of equations will have no solution, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{k}{6} = \frac{-1}{-2} \Rightarrow k = 3$$

Hence, the given system will have no solution, if $k = 3$.

For the given system to have infinite number of solutions, we must have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

We have, $\frac{a_1}{a_2} = \frac{k}{6}, \frac{b_1}{b_2} = \frac{-1}{-2} = \frac{1}{2}$ and $\frac{c_1}{c_2} = \frac{-2}{-3} = \frac{2}{3}$

Clearly, $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

So, whatever be the value of k, we cannot have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Hence, there is no value of k, for which the given system of equations has infinitely many solutions.

EXAMPLE 6 *For what value of k will the equations $x + 2y + 7 = 0, 2x + ky + 14 = 0$ represent coincident lines?*

SOLUTION The given equations are of the form

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0,$$

where $a_1 = 1, b_1 = 2, c_1 = 7$ and $a_2 = 2, b_2 = k, c_2 = 14$

The given equations will represent coincident lines if they have infinitely many solutions. The condition for which is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{1}{2} = \frac{2}{k} = \frac{7}{14} \Rightarrow k = 4$$

Hence, the given system of equations will represent coincident lines, if $k = 4$.

EXAMPLE 7 *For what value of k, will the following system of equations have infinitely many solutions?*

$$2x + 3y = 4$$
$$(k + 2)x + 6y = 3k + 2$$

SOLUTION We know that the system of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

has infinitely many solutions, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given system of equations will have infinitely many solutions, if

$$\frac{2}{k+2} = \frac{3}{6} = \frac{4}{3k+2}$$

$$\Rightarrow \quad \frac{2}{k+2} = \frac{3}{6} \text{ and } \frac{3}{6} = \frac{4}{3k+2}$$

$$\Rightarrow \quad \frac{2}{k+2} = \frac{1}{2} \text{ and } \frac{1}{2} = \frac{4}{3k+2}$$

$$\Rightarrow \quad k+2 = 4 \text{ and } 3k+2 = 8$$

$$\Rightarrow \quad k = 2 \text{ and } k = 2$$

$$\Rightarrow \quad k = 2$$

Hence, the given system of equations will have infinitely many solutions, if $k = 2$.

EXAMPLE 8 *Determine the values of a and b for which the following system of linear equations has infinite solutions:*

$$2x - (a-4)\,y = 2b + 1$$
$$4x - (a-1)\,y = 5b - 1$$

SOLUTION We know that the system of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

has infinite number of solutions, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given system of equations will have infinite number of solutions, if

$$\frac{2}{4} = \frac{-(a-4)}{-(a-1)} = \frac{2b+1}{5b-1}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{a-4}{a-1} = \frac{2b+1}{5b-1}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{a-4}{a-1} \text{ and } \frac{1}{2} = \frac{2b+1}{5b-1}$$

$$\Rightarrow \quad a - 1 = 2a - 8 \text{ and } 5b - 1 = 4b + 2$$

$$\Rightarrow \quad a = 7 \text{ and } b = 3$$

Hence, the given system of equations will have infinitely many solutions, if $a = 7$ and $b = 3$.

EXAMPLE 9 *For what value of k will the following system of linear equations has no solution?*

$$3x + y = 1$$

$$(2k - 1)x + (k - 1)y = 2k + 1$$

[NCERT, CBSE 2000]

SOLUTION We know that the system of equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

has no solution, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

So, the given system of equations will have no solution, if

$$\frac{3}{2k-1} = \frac{1}{k-1} \neq \frac{1}{2k+1}$$

$$\frac{3}{2k-1} = \frac{1}{k-1} \text{ and } \frac{1}{k-1} \neq \frac{1}{2k+1}$$

Now,

$$\frac{3}{2k-1} = \frac{1}{k-1} \Rightarrow 3k - 3 = 2k - 1 \Rightarrow k = 2$$

Clearly, for $k = 2$ we have $\frac{1}{k-1} \neq \frac{1}{2k+1}$

Hence, the given system of equations will have no solution, if $k = 2$.

EXAMPLE 10 *Find the value of k for which the following system of linear equations has infinite solutions:*

$$x + (k + 1)y = 5$$

$$(k + 1)x + 9y = 8k - 1$$

[CBSE 2002C]

SOLUTION The given system of equations will have infinite solutions, if

$$\frac{1}{k+1} = \frac{k+1}{9} = \frac{5}{8k-1}$$

$$\Rightarrow \quad \frac{1}{k+1} = \frac{k+1}{9} \text{ and } \frac{k+1}{9} = \frac{5}{8k-1}$$

$$\Rightarrow \quad (k+1)^2 = 9 \text{ and } (k+1)(8k-1) = 45$$

Now, $(k+1)^2 = 9$

$$\Rightarrow \quad k + 1 = \pm 3 \Rightarrow k = 2, -4$$

We observe that $k = 2$ satisfies the equation $(k+1)(8k-1) = 45$ but, $k = -4$ does not satisfy it.

Hence, the given system of equations will have infinitely many solutions, if $k = 2$.

EXAMPLE 11 *Find the values of p and q for which the following system of equations has infinite number of solutions:*

$$2x + 3y = 7$$

$$(p + q)x + (2p - q)y = 21$$

[CBSE 2001]

SOLUTION The given system of equations will have infinite number of solutions, if

$$\frac{2}{p+q} = \frac{3}{2p-q} = \frac{7}{21}$$

$\Rightarrow \quad \frac{2}{p+q} = \frac{3}{2p-q} = \frac{1}{3}$

$\Rightarrow \quad \frac{2}{p+q} = \frac{1}{3}$ and $\frac{3}{2p-q} = \frac{1}{3}$

$\Rightarrow \quad p+q = 6$ and $2p-q = 9$

$\Rightarrow \quad (p+q)+(2p-q) = 6+9$ [On adding]

$\Rightarrow \quad 3p = 15$

$\Rightarrow \quad p = 15$

Putting $p = 5$ in $p+q = 6$ or, $2p-q = 9$, we get $q = 1$.

Hence, the given system of equations will have infinitely many solutions, if $p = 5$ and $q = 1$.

EXAMPLE 12 *For what value of k, will the system of equations*

$x + 2y = 5$

$3x + ky - 15 = 0.$

has (i) a unique solution? (ii) no solution? [CBSE 2001]

SOLUTION The given system of equations can be written as

$x + 2y = 5$

$3x + ky = 15$

(i) The above system of equations will have a unique solution, if

$\frac{1}{3} \neq \frac{2}{k} \Rightarrow k \neq 6$

(ii) The above system of equations will have no solution, if

$\frac{1}{3} = \frac{2}{k} \neq \frac{5}{15}$ $\left[\text{Using: } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}\right]$

$\Rightarrow \quad \frac{1}{3} = \frac{2}{k}$ and $\frac{2}{k} \neq \left(\frac{1}{3}\right)$

$\Rightarrow \quad k = 6$ and $k \neq 6$, which is not possible.

Hence, there is no value of k for which the given system of equations has no solution.

EXAMPLE 13 *Find the values of α and β for which the following system of linear equations has infinite number of solutions:*

$2x + 3y = 7$

$2\alpha x + (\alpha + \beta) y = 28$

has (i) a unique solution? (ii) no solution? [CBSE 2001]

SOLUTION The given system of equations will have infinite number of solutions, if

$\frac{2}{2\alpha} = \frac{3}{\alpha+\beta} = \frac{7}{28}$

$\Rightarrow \quad \frac{1}{\alpha} = \frac{3}{\alpha+\beta} = \frac{1}{4}$

$\Rightarrow \quad \frac{1}{\alpha} = \frac{1}{4}$ and $\frac{3}{\alpha+\beta} = \frac{1}{4}$

$\Rightarrow \quad \alpha = 4$ and $\alpha + \beta = 12$

$\Rightarrow \quad \alpha = 4$ and $\beta = 8$

Hence, the given system of equations will have infinitely many solutions, if $\alpha = 4$ and $\beta = 8$.

EXAMPLE 14 *Determine the values of m and n so that the following system of linear equations have infinite number of solutions:*

$$(2m - 1)x + 3y - 5 = 0$$
$$3x + (n - 1)y - 2 = 0$$

SOLUTION The given system of equations will have infinite number of solutions, if

$$\frac{2m-1}{3} = \frac{3}{n-1} = \frac{-5}{-2}$$

$$\Rightarrow \quad \frac{2m-1}{3} = \frac{3}{n-1} = \frac{5}{2}$$

$$\Rightarrow \quad \frac{2m-1}{3} = \frac{5}{2} \text{ and } \frac{3}{n-1} = \frac{5}{2}$$

$$\Rightarrow \quad 4m - 2 = 15 \text{ and } 6 = 5n - 5$$

$$\Rightarrow \quad 4m = 17 \text{ and } 5n = 11$$

$$\Rightarrow \quad m = \frac{17}{4} \text{ and } n = \frac{11}{5}$$

Hence, the given system of equations will have infinite number of solutions, if $m = \frac{17}{4}$ and $n = \frac{11}{5}$.

EXAMPLE 15 *Determine the value of k so that the following linear equations have no solution:*

$$(3k + 1)x + 3y - 2 = 0$$
$$(k^2 + 1)x + (k - 2)y - 5 = 0$$

[CBSE 2001C]

SOLUTION The given system of equations will have no solution, if

$$\frac{3k+1}{k^2+1} = \frac{3}{k-2} \neq \frac{-2}{-5}$$

$$\Rightarrow \quad \frac{3k+1}{k^2+1} = \frac{3}{k-2} \text{ and } \frac{3}{k-2} \neq \frac{2}{5}$$

Now, $\quad \frac{3k+1}{k^2+1} = \frac{3}{k-2}$

$$\Rightarrow \quad (3k + 1)(k - 2) = 3(k^2 + 1)$$

$$\Rightarrow \quad 3k^2 - 5k - 2 = 3k^2 + 3$$

$$\Rightarrow \quad -5k - 2 = 3$$

$$\Rightarrow \quad -5k = 5$$

$$\Rightarrow \quad k = -1$$

Clearly, $\quad \frac{3}{k-2} \neq \frac{2}{5}$ for $k = -1$.

Hence, the given system of equations will have no solution for $k = -1$.

EXERCISE 3.5

LEVEL-1

In each of the following systems of equations determine whether the system has a unique solution, no solution or inifinitely many solutions. In case there is a unique solution, find it: (1-4)

1. $x - 3y = 3$
   $3x - 9y = 2$

2. $2x + y = 5$
   $4x + 2y = 10$

3. $3x - 5y = 20$
   $6x - 10y = 40$

4. $x - 2y = 8$
   $5x - 10y = 10$

Find the value of k for which the following system of equations has a unique solution: (5-8)

5. $kx + 2y = 5$
   $3x + y = 1$

6. $4x + ky + 8 = 0$
   $2x + 2y + 2 = 0$ [NCERT]

7. $4x - 5y = k$
   $2x - 3y = 12$

8. $x + 2y = 3$
   $5x + ky + 7 = 0$

Find the value of k for which each of the following systems of equations have infinitely many solution: (9-19)

9. $2x + 3y - 5 = 0$
   $6x + ky - 15 = 0$

10. $4x + 5y = 3$
    $kx + 15y = 9$

11. $kx - 2y + 6 = 0$
    $4x - 3y + 9 = 0$

12. $8x + 5y = 9$
    $kx + 10y = 18$

13. $2x - 3y = 7$
    $(k + 2)x - (2k + 1)y = 3(2k - 1)$

14. $2x + 3y = 2$
    $(k + 2)x + (2k + 1)y = 2(k - 1)$ [CBSE 2000, 2003]

15. $x + (k + 1)y = 4$
    $(k + 1)x + 9y = 5k + 2$ [CBSE 2000C]

16. $kx + 3y = 2k + 1$
    $2(k + 1)x + 9y = 7k + 1$ [CBSE 2000C]

17. $2x + (k - 2)y = k$
    $6x + (2k - 1)y = 2k + 5$ [CBSE 2000C]

18. $2x + 3y = 7$
    $(k + 1)x + (2k - 1)y = 4k + 1$ [CBSE 2001]

19. $2x + 3y = k$
    $(k - 1)x + (k + 2)y = 3k$ [CBSE 2001]

Find the value of k for which the following system of equations has no solution: (20-25):

20. $kx - 5y = 2$
    $6x + 2y = 7$

21. $x + 2y = 0$
    $2x + ky = 5$

22. $3x - 4y + 7 = 0$

$kx + 3y - 5 = 0$

23. $2x - ky + 3 = 0$

$3x + 2y - 1 = 0$

24. $2x + ky = 11$

$5x - 7y = 5$

25. $kx + 3y = k - 3$

$12x + ky = 6$ **[NCERT EXEMPLAR]**

26. For what value of k the following system of equations will be inconsistent?

$$4x + 6y = 11$$
$$2x + ky = 7$$

27. For what value of α, the system of equations

$$\alpha x + 3y = \alpha - 3$$
$$12x + \alpha y = \alpha$$

[CBSE 2003, 2009]

will have no solution?

28. Find the value of k for which the system

$$kx + 2y = 5$$
$$3x + y = 1$$

has (i) a unique solution, and (ii) no solution.

29. Prove that there is a value of $c\ (\neq 0)$ for which the system

$$6x + 3y = c - 3$$
$$12x + cy = c$$

has infinitely many solutions. Find this value.

30. Find the values of k for which the system

$$2x + ky = 1$$
$$3x - 5y = 7$$

will have (i) a unique solution, and (ii) no solution. Is there a value of k for which the system has infinitely many solutions?

31. For what value of k, the following system of equations will represent the coincident lines?

$$x + 2y + 7 = 0$$
$$2x + ky + 14 = 0$$

32. Obtain the condition for the following system of linear equations to have a unique solution

$$ax + by = c$$
$$lx + my = n$$

33. Determine the values of a and b so that the following system of linear equations have infinitely many solutions:

$$(2a - 1)x + 3y - 5 = 0$$
$$3x + (b - 1)y - 2 = 0$$

[CBSE 2001C]

34. Find the values of a and b for which the following system of linear equations has infinite number of solutions:

$2x - 3y = 7$

$(a + b)\,x - (a + b - 3)\,y = 4a + b$ [CBSE 2002]

35. Find the values of p and q for which the following system of linear equations has infinite number of solutions:

$2x + 3y = 9$

$(p + q)\,x + (2p - q)\,y = 3\,(p + q + 1)$ [CBSE 2002]

36. Find the values of a and b for which the following system of equations has infinitely many solutions:

(i) $(2a - 1)\,x - 3y = 5$
$3x + (b - 2)\,y = 3$
[CBSE 2002C]

(ii) $2x - (2a + 5)\,y = 5$
$(2b + 1)\,x - 9y = 15$
[CBSE 2002C]

(iii) $(a - 1)\,x + 3y = 2$
$6x + (1 - 2b)\,y = 6$
[CBSE 2002C]

(iv) $3x + 4y = 12$
$(a + b)\,x + 2\,(a - b)\,y = 5a - 1$
[CBSE 2002C]

(v) $2x + 3y = 7$
$(a - b)\,x + (a + b)\,y = 3a + b - 2$
[NCERT]

(vi) $2x + 3y - 7 = 0$
$(a - 1)x + (a + 1)\,y = (3a - 1)$
[CBSE 2010]

(vii) $2x + 3y = 7$
$(a - 1)x + (a + 2)y = 3a$
[CBSE 2010]

(viii) $x + 2y = 1$
$(a - b)\,x + (a + b)\,y = a + b - 2$
[NCERT EXEMPLAR]

(ix) $2x + 3y = 7$
$2ax + ay = 28 - by$
[NCERT EXEMPLAR]

37. For which value(s) of λ, do the pair of linear equations $\lambda x + y = \lambda^2$ and $x + \lambda y = 1$ have

(i) no solution?
(ii) infinitely many solutions?
(iii) a unique solution? [NCERT EXEMPLAR]

ANSWERS

1. No solution
2. Infinitely many solutions
3. Infinitely many solutions
4. No solution
5. $k \neq 6$
6. $k \neq 4$
7. k is any real number
8. $k \neq 10$
9. $k = 9$
10. $k = 12$
11. $k = 8/3$
12. $k = 16$
13. $k = 4$
14. $k = 4$
15. $k = 2$
16. $k = 2$
17. $k = 5$
18. $k = 5$
19. $k = 7$
20. $k = -15$
21. $k = 4$
22. $k = -\dfrac{9}{4}$
23. $k = -\dfrac{4}{3}$
24. $k = -\dfrac{14}{5}$
25. $k = -6$
26. $k = 3$

27. $\alpha = -6$

28. (i) $k \neq 6$ (ii) $k = 6$

29. $c = 6$

30. (i) $k \neq \frac{-10}{3}$ (ii) $k = \frac{-10}{3}$, No

31. $k = 4$

32. $am \neq bl$

33. $a = \frac{17}{4}, b = \frac{11}{5}$

34. $a = -5, b = -1$

35. $p = 5/3, q = 1/3$

36. (i) $a = 3, b = \frac{1}{5}$ (ii) $a = -1, b = \frac{5}{2}$ (iii) $a = 3, b = -4$ (iv) $a = 5, b = 1$

(v) $a = 5, b = 1$ (vi) $a = 5$ (vii) $a = 7$ (viii) $a = 3, b = 1$ (ix) $a = 4, b = 8$

37. (i) $\lambda = -1$ (ii) $\lambda = 1$ (iii) All real values of λ except ± 1.

3.7 APPLICATIONS TO WORD PROBLEMS

In this section, we shall learn about some applications of simultaneous linear equations in solving problems related to our day-to-day life. There is a wide variety of such problems which are generally called 'word problems'. In solving such problems, we may use the following algorithm.

ALGORITHM

STEP I *Read the problem carefully and identify the unknown quantities. Give these quantities a variable name like x, y, u, v, w etc.*

STEP II *Identify the variables to be determined.*

STEP III *Read the problem carefully and formulate the equations in terms of the variables to be determined.*

STEP IV *Solve the equations obtained in step III using any one of the methods learnt earlier.*

3.7.1 APPLICATIONS TO PROBLEMS BASED ON ARTICLES AND THEIR COSTS

Following examples will illustrate the applications of linear equations in solving word problems based on articles and their costs.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *4 chairs and 3 tables cost ₹2100 and 5 chairs and 2 tables cost ₹1750. Find the cost of a chair and a table separately.*

SOLUTION Let the cost of a chair be ₹ x and that of a table be ₹ y. Then,

$$4x + 3y = 2100$$

and, $$5x + 2y = 1750$$

This system of equations can be written as

$$4x + 3y - 2100 = 0$$

$$5x + 2y - 1750 = 0$$

By using cross-multiplication, we have

$$\frac{x}{3 \times -1750 - 2 \times -2100} = \frac{-y}{4 \times -1750 - 5 \times -2100} = \frac{1}{4 \times 2 - 3 \times 5}$$

$$\Rightarrow \quad \frac{x}{-5250+4200} = \frac{-y}{-7000+10500} = \frac{1}{8-15}$$

$$\Rightarrow \quad \frac{x}{-1050} = \frac{y}{-3500} = \frac{1}{-7}$$

$$\Rightarrow \quad x = \frac{-1050}{-7} = 150 \text{ and } y = \frac{-3500}{-7} = 500.$$

EXAMPLE 2 *37 pens and 53 pencils together cost ₹320, while 53 pens and 37 pencils together cost ₹400. Find the cost of a pen and that of a pencil.*

SOLUTION Let the cost of a pen be ₹ x and that of a pencil be ₹ y. Then,

$$37x + 53y = 320 \qquad \ldots(i)$$

and, $$53x + 37y = 400 \qquad \ldots(ii)$$

Adding equations (i) and (ii), we get

$$90x + 90y = 720 \Rightarrow x + y = 8 \qquad \ldots(iii)$$

Subtracting equation (i) and (ii), we get

$$16x - 16y = 80 \Rightarrow x - y = 5 \qquad \ldots(iv)$$

Adding equations (iii) and (iv), we get

$$2x = 13 \Rightarrow x = 6.5$$

Substituting $x = 6.5$ in equation (iii), we get

$$y = (8 - 6.5) = 1.5$$

Hence, cost of one pen = ₹ 6.50 and cost of one pencil = ₹ 1.50.

EXAMPLE 3 *2 tables and 3 chairs together cost ₹2000 whereas 3 tables and 2 chairs together cost ₹2500. Find the total cost of 1 table and 5 chairs*

SOLUTION Let the cost of a table be ₹ x and that of a chair be ₹ y. Then,

$$2x + 3y = 2000$$

and, $$3x + 2y = 2500$$

This system of equations may be written as

$$2x + 3y - 2000 = 0$$

$$3x + 2y - 2500 = 0$$

By using cross-multiplication, we get

$$\frac{x}{3 \times -2500 - 2 \times -2000} = \frac{-y}{2 \times -2500 - 3 \times -2000} = \frac{1}{2 \times 2 - 3 \times 3}$$

$$\Rightarrow \quad \frac{x}{-7500+4000} = \frac{-y}{-5000+6000} = \frac{1}{4-9}$$

$$\Rightarrow \quad \frac{x}{-3500} = \frac{y}{-1000} = \frac{1}{-5}$$

$$\Rightarrow \quad x = \frac{-3500}{-5} = 700 \text{ and } y = \frac{-1000}{-5} = 200$$

$\therefore$ Cost of a table = ₹ 700 and, cost of a chair = ₹ 200

Hence, cost of one table and 5 chairs = ₹ $(x + 5y)$ = ₹ 1700.

LEVEL-2

EXAMPLE 4 *A and B each have certain number of oranges. A says to B, "if you give me 10 of your oranges, I will have twice the number of oranges left with you." B replies, "if you give me 10 of your oranges, I will have the same number of oranges as left with you." Find the number of oranges with A and B separately.*

SOLUTION Suppose A has x oranges and B has y oranges.

According to the given conditions, we have

$$x + 10 = 2(y - 10) \Rightarrow x - 2y + 30 = 0 \quad \text{...(i)}$$

and, $$y + 10 = x - 10 \Rightarrow x - y - 20 = 0 \quad \text{...(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$-y + 50 = 0 \Rightarrow y = 50$$

Putting $y = 50$ in equation (i), we get $x = 70$

Hence, A has 70 oranges and B has 50 oranges.

EXAMPLE 5 *A man has only 20 paisa coins and 25 paisa coins in his purse. If he has 50 coins in all totalling ₹11.25, how many coins of each kind does he have?*

SOLUTION Let the number of 20 paisa coins be x and that of 25 paisa coins be y. Then,

$$x + y = 50 \quad \text{...(i)}$$

Total value of 20 paisa coins = $20x$ paisa

Total value of 25 paisa coins = $25y$ paisa

$\therefore$ $$20x + 25y = 1125$$

$\Rightarrow$ $$4x + 5y = 225 \quad \text{...(ii)}$$

Thus, we get the following system of linear equations

$$x + y - 50 = 0$$

$$4x + 5y - 225 = 0$$

By using cross-multiplication, we have

$$\frac{x}{-225 + 250} = \frac{-y}{-225 + 200} = \frac{1}{5 - 4}$$

$\Rightarrow$ $$\frac{x}{25} = \frac{y}{25} = \frac{1}{1} \Rightarrow x = 25 \text{ and } y = 25$$

Hence, there are 25 coins of each kind.

EXERCISE 3.6

LEVEL-1

1. 5 pens and 6 pencils together cost ₹ 9 and 3 pens and 2 pencils cost ₹ 5. Find the cost of 1 pen and 1 pencil.
2. 7 audio cassettes and 3 video cassettes cost ₹ 1110, while 5 audio cassettes and 4 video cassettes cost ₹ 1350. Find the cost of an audio cassette and a video cassette.
3. Reena has pens and pencils which together are 40 in number. If she has 5 more pencils and 5 less pens, then nuimber of pencils would become 4 times the number of pens. Find the original number of pens and pencils.
4. 4 tables and 3 chairs, together, cost ₹ 2,250 and 3 tables and 4 chairs cost ₹ 1950. Find the cost of 2 chairs and 1 table.

5. 3 bags and 4 pens together cost ₹ 257 whereas 4 bags and 3 pens together cost ₹ 324. Find the total cost of 1 bag and 10 pens.

6. 5 books and 7 pens together cost ₹ 79 whereas 7 books and 5 pens together cost ₹ 77. Find the total cost of 1 book and 2 pens.

7. Jamila sold a table and a chair for ₹ 1050, thereby making a profit of 10% on a table and 25% on the chair. If she had taken a profit of 25% on the table and 10% on the chair she would have got ₹ 1065. Find the cost price of each. [NCERT EXEMPLAR]

8. Susan invested certain amount of money in two schemes A and B, which offer interest at the rate of 8% per annum and 9% per annum, respectively. She received ₹ 1860 as annual interest. However, had she interchanged the amount of investment in the two schemes, she would have received ₹ 20 more as annual interest. How much money did she invest in each scheme? [NCERT EXEMPLAR]

9. The coach of a cricket team buys 7 bats and 6 balls for ₹ 3800. Later, he buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball. [NCERT]

10. A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹ 27 for a book kept for seven days, while Susy paid ₹ 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day. [NCERT]

11. The cost of 4 pens and 4 pencil boxes is ₹ 100. Three times the cost of a pen is ₹ 15 more than the cost of a pencil box. Form the pair of linear equations for the above situation. Find the cost of a pen and a pencil box. [NCERT EXEMPLAR]

LEVEL-2

12. One says, "Give me a hundred, friend! I shall then become twice as rich as you." The other replies, "If you give me ten, I shall be six times as rich as you." Tell me what is the amount of their respective capital? [NCERT]

13. A and B each have a certain number of mangoes. A says to B, "if you give 30 of your mangoes, I will have twice as many as left with you." B replies, "if you give me 10, I will have thrice as many as left with you." How many mangoes does each have?

14. Vijay had some bananas, and he divided them into two lots A and B. He sold first lot at the rate of ₹ 2 for 3 bananas and the second lot at the rate of ₹ 1 per banana and got a total of ₹ 400. If he had sold the first lot at the rate of ₹ 1 per banana and the second lot at the rate of ₹ 4 per five bananas, his total collection would have been ₹ 460. Find the total number of bananas he had. [NCERT EXEMPLAR]

15. On selling a T.V. at 5% gain and a fridge at 10% gain, a shopkeeper gains ₹ 2000. But if he sells the T.V. at 10% gain and the fridge at 5% loss. He gains ₹ 1500 on the transaction. Find the actual prices of T.V. and fridge.

ANSWERS

1. Cost of one pen = ₹ 3/2, Cost of one pencil = ₹ 1/4
2. ₹ 30, ₹ 300
3. No. of pens = 13, No. of pencils = 27
4. ₹ 750
5. ₹ 155
6. ₹ 20
7. Table ₹ 500, chair ₹ 400
8. ₹ 10000 in scheme A, ₹ 12000 in scheme B
9. Bat ₹ 500, ball ₹ 50

10. ₹ 15, ₹ 3
11. ₹ 10, ₹ 15
12. ₹ 40, ₹ 170
13. $A:34, B:62$
14. 500
15. ₹ 20,000, ₹ 10,000

HINTS TO SELECTED PROBLEMS

1. Let the cost of each pen be x and that of each pencil be ₹ y. Then, we have $5x + 6y = 9$ and $3x + 2y = 5$.
2. Let the cost of an audio cassette be ₹ x and that of a video cassette be ₹ y. Then we have $7x + 3y = 1110$ and $5x + 4y = 1350$.
7. Let the cost price of a table and that of a chair be x and y respectively. Then,

CASE I S.P. *of a table* $= ₹\left(x + \frac{10}{100}x\right) = ₹\,\frac{110}{100}x$

S.P. *of a chair* $= ₹\left(y + \frac{25}{100}y\right) = ₹\,\frac{125}{100}y$

$$\therefore \quad \frac{110}{100}x + \frac{125}{100}y = 1050 \Rightarrow 110x + 125y = 105000 \quad \text{...(i)}$$

CASE II S.P. *of a table* $= ₹\left(x + \frac{25}{100}x\right) = ₹\,\frac{125}{100}x$

S.P. *of a chair* $= ₹\left(y + \frac{10}{100}y\right) = ₹\,\frac{110}{100}y$

$$\therefore \quad \frac{125}{100}x + \frac{110}{100}y = 1065 \Rightarrow 125x + 110y = 106500 \quad \text{...(ii)}$$

Adding (i) and (ii), we obtain

$$235(x + y) = 211500 \Rightarrow x + y = 900 \quad \text{...(iii)}$$

Subtracting (i) from (ii), we obtain

$$15(x - y) = 1500 \Rightarrow x - y = 100 \quad \text{...(iv)}$$

Solving these two equations, we get

$$x = 500, y = 400$$

8. Suppose she invested ₹ x in Scheme A and ₹ y in Scheme B. Then,

$$\frac{8x}{100} + \frac{9y}{100} = 1860 \text{ and } \frac{8y}{100} + \frac{9x}{100} = 1880 \Rightarrow 8x + 9y = 186000 \text{ and } 9x + 8y = 188000$$

Adding and subtracting these two equations, we obtain

$$17(x + y) = 374000 \text{ and } -x + y = 2000$$

$$\Rightarrow x + y = 22000 \text{ and } -x + y = 2000$$

$$\Rightarrow x = 10000, y = 12000$$

10. Let the fixed charge be of ₹ x and the extra charge for each day be ₹ y. Then,

$x + 4y = 27$ and $x + 2y = 21$

14. Let there be x bananas in lot A and y bananas in lot B. Then,

$$\frac{2}{3}x + y = 400 \text{ and } x + \frac{4}{5}y = 460 \Rightarrow 2x + 3y - 1200 = 0 \text{ and } 5x + 4y = 2300$$

Solving these two equations, we get $x = 300$ and $y = 200$

15. Let the price of a T.V. be ₹ x and that of a fridge be ₹ y. Then, we have

$$\frac{5x}{100} + \frac{10y}{100} = 2000 \text{ and, } \frac{10x}{100} - \frac{5y}{100} = 1500$$

3.7.2 APPLICATIONS TO PROBLEMS BASED ON NUMBERS

Following examples, will illustrate the applications of linear equations in solving word problems based on numbers. Recall that the two digit number having a and b as units and ten's digits respectively is equal to $10b + a$ and the number obtained by reversing the order of digits is $10a + b$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Sum of two numbers is 35 and their difference is 13. Find the numbers.*

SOLUTION Let the two numbers be x and y. Then,

$$x + y = 35 \quad \ldots(i)$$

and, $$x - y = 13 \quad \ldots(ii)$$

Adding equations (i) and (ii), we get

$$2x = 48 \Rightarrow x = 24$$

Subtracting equation (ii) from equation (i), we get

$$2y = 22 \Rightarrow y = 11$$

Hence, the two numbers are 24 and 11.

EXAMPLE 2 *In a two digit number, the unit's digit is twice the ten's digit. If 27 is added to the number, the digits interchange their places. Find the number*

SOLUTION (i) Let the digit in the unit's place be x and digit in the ten's place be y. Then,

$$x = 2y \quad \text{[Given]} \ldots(i)$$

and, Number $= 10y + x$

Number obained by reversing the digits $= 10x + y$

It is given that the digits interchange their places if 27 is added to the number.

i.e., Number + 27 = Number obtained by interchanging the digits

$\therefore$ $10y + x + 27 = 10x + y$

$\Rightarrow$ $9x - 9y = 27$

$\Rightarrow$ $x - y = 3 \quad \ldots(ii)$

Putting $x = 2y$ in equation (ii), we get

$$2y - y = 3 \Rightarrow y = 3$$

Putting $y = 3$ in equation (ii), we get

$$x = 6$$

Hence, the number is $10y + x = 10 \times 3 + 6 = 36$

EXAMPLE 3 *In a two digit number, the ten's digit is three times the unit's digit. When the number is decreased by 54, the digits are reversed. Find the number.*

SOLUTION Let the digit in the unit's place be x and the digit in the ten's place be y. Then,

$$\text{Number} = 10y + x$$

According to the given condition, we have

$$y = 3x \quad \text{...(i)}$$

Number obtained by reversing the digits $= 10x + y$

If the number is decreased by 54, the digits are reversed.

$\therefore$ Number $- 54 =$ Number obtained by reversing the digits

$\Rightarrow \quad 10y + x - 54 = 10x + y$

$\Rightarrow \quad 9x - 9y = -54 \Rightarrow x - y = -6 \quad$ ...(ii)

Putting $y = 3x$ in equation (ii), we get

$$x - 3x = -6 \Rightarrow x = 3$$

Putting $x = 3$ in $y = 3x$, we get $y = 9$

Hence, number $= 10y + x = 10 \times 9 + 3 = 93$

EXAMPLE 4 *The sum of the digits of a two digit number is 8 and the difference between the number and that formed by reversing the digits is 18. Find the number.*

SOLUTION Let the digit at unit's place be x and the digit at ten's place be y. Then,

Number $= 10y + x$

Number formed by reversing the digits $= 10y + x$

According to the given conditions, we have

$x + y = 8 \quad$ ...(i)

and, $\quad (10y + x) - (10x + y) = 18$

$\Rightarrow \quad 9(y - x) = 18$

$\Rightarrow \quad y - x = 2 \quad$ ...(ii)

On solving equations (i) and (ii), we get $x = 3, y = 5$

Hence, number $= 10y + x = 10 \times 5 + 3 = 53$

EXAMPLE 5 *The sum of a two digit number and the number obtained by reversing the order of its digits is 121, and the two digits differ by 3. Find the number.*

SOLUTION Let the digit in the unit's place be x and the digit at the ten's place be y. Then,

$$\text{Number} = 10y + x$$

The number obtained by reversing the order of the digits is $10x + y$.

According to the given conditions, we have

$(10y + x) + (10x + y) = 121$

$\Rightarrow \quad 11(x + y) = 121$

$\Rightarrow \quad x + y = 11$

and, $\quad x - y = \pm 3 \quad$ [$\because$ Difference of digits is 3]

Thus, we have the following sets of simultaneous equations

$$\left.\begin{aligned} x + y &= 11 \quad \ldots(i) \\ \text{and,} \quad x - y &= 3 \quad \ldots(ii) \end{aligned}\right\} \quad \text{or,} \quad \left\{\begin{aligned} x + y &= 11 \quad \ldots(iii) \\ x - y &= -3 \quad \ldots(iv) \end{aligned}\right.$$

On solving equation (i) and (ii), we get $x = 7, y = 4$

On solving equations (iii) and (iv), we get $x = 4, y = 7$

When $x = 7, y = 4$, we have

$$\text{Number} = 10y + x = 10 \times 4 + 7 = 47$$

When $x = 4, y = 7$, we have

$$\text{Number} = 10y + x = 10 \times 7 + 4 = 74$$

Hence, the required number is either 47 or, 74.

EXAMPLE 6 *The sum of a two digit number and the number formed by interchanging its digits is 110. If 10 is subtracted from the first number, the new number is 4 more than 5 times the sum of the digits in the first number. Find the first number.* **[CBSE 2002C]**

SOLUTION Let the digits at units and tens place in the given number be x and y respectively. Then,

$$\text{Number} = 10y + x \quad \ldots(i)$$

Number formed by interchanging the digits $= 10x + y$

According to the given conditions, we have

$$(10y + x) + (10x + y) = 110$$

and, $(10y + x) - 10 = 5(x + y) + 4$

$\Rightarrow$ $11x + 11y = 110$

and, $4x - 5y + 14 = 0$

$\Rightarrow$ $x + y - 10 = 0$

and, $4x - 5y + 14 = 0$

By using cross-multiplication, we have

$$\frac{x}{14 - 50} = \frac{y}{-40 - 14} = \frac{1}{-5 - 4}$$

$$\Rightarrow \quad \frac{x}{-36} = \frac{y}{-54} = \frac{1}{-9}$$

$$\Rightarrow \quad x = \frac{-36}{-9} \text{ and } y = \frac{-54}{-9}$$

$$\Rightarrow \quad x = 4 \text{ and } y = 6.$$

Putting the values of x and y in equation (i), we get

$$\text{Number} = 10 \times 6 + 4 = 64$$

EXAMPLE 7 *The sum of a two digit number and the number formed by interchanging the digit is 132. If 12 is added to the number, the new number becomes 5 times the sum of the digits. Find the number.* **[CBSE 2002C]**

SOLUTION Let the digits at units and tens place in the given number be x and y respectively. Then,

$$\text{Number} = 10y + x \quad \ldots(i)$$

Number formed by interchanging the digits $= 10x + y$

According to the given conditions, we have

$$(10y + x) + (10x + y) = 132$$

and, $\quad (10y + x) + 12 = 5(x + y)$

$\Rightarrow \quad 11x + 11y = 132$

and, $\quad 4x - 5y = 12$

$\Rightarrow \quad x + y - 12 = 0$

and, $\quad 4x - 5y - 12 = 0$

Solving these two equations by cross-multiplication, we have

$$\frac{x}{-12-60} = \frac{y}{-48+12} = \frac{1}{-5-4}$$

$$\Rightarrow \quad \frac{x}{-72} = \frac{y}{-36} = \frac{1}{-9}$$

$$\Rightarrow \quad x = \frac{-72}{-9} \text{ and } y = \frac{-36}{-9}$$

$$\Rightarrow \quad x = 8 \text{ and } y = 4$$

Substituting the values of x and y in equation (i), we have

$$\text{Number} = 10 \times 4 + 8 = 48.$$

EXAMPLE 8 *The sum of a two-digit number and the number obtained by reversing the order of its digits is 165. If the digits differ by 3, find the number.* **[CBSE 2002]**

SOLUTION Let the digits at units and tens place of the given number be x and y respectively. Then,

$$\text{Number} = 10y + x \qquad \ldots\text{(i)}$$

Number obtained by reversing the order of the digits $= 10x + y$

According to the given conditions, we have

$$(10y + x) + (10x + y) = 165$$

and, $\quad x - y = 3$ or, $y - x = 3$

$\Rightarrow \quad 11x + 11y = 165$

and, $\quad x - y = 3$ or, $y - x = 3$

$\Rightarrow \quad x + y = 15$

and, $\quad x - y = 3$ or, $y - x = 3$

Thus, we obtain the following systems of linear equations.

(i) $\quad x + y = 15$

$\quad x - y = 3$

(ii) $\quad x + y = 15$

$\quad y - x = 3$

Solving first system of equations, we get

$$x = 9, y = 6$$

Solving second system of equations, we get

$$x = 6, y = 9$$

Substituting the values of x and y in equation (i), we have

$$\text{Number} = 69 \text{ or, } 96.$$

LEVEL-2

EXAMPLE 9 *A two digit number is obtained by either multiplying sum of the digits by 8 and adding 1 or by multiplying the difference of the digits by 13 and adding 2. Find the number.*

SOLUTION Let the digit at units place be x and the digit at ten's place be y. Then,

$$\text{Number} = 10y + x$$

According to the given conditions, we have

$$10y + x = 8(x + y) + 1 \Rightarrow 7x - 2y + 1 = 0$$

and, $$10y + x = 13(y - x) + 2 \Rightarrow 14x - 3y - 2 = 0$$

By using cross-multiplication, we have

$$\frac{x}{-2 \times -2 - (-3) \times 1} = \frac{-y}{7 \times -2 - 14 \times 1} = \frac{1}{7 \times -3 - 14 \times -2}$$

$$\Rightarrow \quad \frac{x}{4 + 3} = \frac{-y}{-14 - 14} = \frac{1}{-21 + 28}$$

$$\Rightarrow \quad \frac{x}{7} = \frac{y}{28} = \frac{1}{7}$$

$$\Rightarrow \quad x = \frac{7}{7} = 1 \text{ and } y = \frac{28}{7} = 4$$

Hence, the number $= 10y + x = 10 \times 4 + 1 = 41$.

REMARK *In the above example, if we take the difference of the digits as $x - y$, then we get fractional values of x and y which are not admissible.*

EXAMPLE 10 *If three times the larger of the two numbers is divided by the smaller one, we get 4 as quotient and 3 as the remainder. Also, if seven times the smaller number is divided by the larger one, we get 5 as quotient and 1 as remainder. Find the numbers.*

SOLUTION Let the larger number be x and smaller one be y. We know that

$$\text{Dividend} = (\text{Divisor} \times \text{Quotient}) + \text{Remainder} \quad \ldots(\text{i})$$

When $3x$ is divided by y, we get 4 as quotient and 3 as remainder. Therefore, by using (i), we get

$$3x = 4y + 3 \Rightarrow 3x - 4y - 3 = 0 \quad \ldots(\text{ii})$$

When $7y$ is divided by x, we get 5 as quotient and 1 as remainder. Therefore, by using (i), we get

$$7y = 5x + 1 \Rightarrow 5x - 7y + 1 = 0 \quad \ldots(\text{iii})$$

Solving equations (ii) and (iii), by cross-multiplication, we get

$$\frac{x}{-4 - 21} = \frac{-y}{3 + 15} = \frac{1}{-21 + 20} \Rightarrow x = 25 \text{ and } y = 18$$

Hence, the required numbers are 25 and 18.

EXERCISE 3.7

LEVEL-1

1. The sum of two numbers is 8. If their sum is four times their difference, find the numbers.
2. The sum of digits of a two digit number is 13. If the number is subtracted from the one obtained by interchanging the digits, the result is 45. What is the number?

3. A number consists of two digits whose sum is five. When the digits are reversed, the number becomes greater by nine. Find the number.
4. The sum of digits of a two digit number is 15. The number obtained by reversing the order of digits of the given number exceeds the given number by 9. Find the given number. [CBSE 2004]
5. The sum of a two-digit number and the number formed by reversing the order of digits is 66. If the two digits differ by 2, find the number. How many such numbers are there? [NCERT]
6. The sum of two numbers is 1000 and the difference between their squares is 256000. Find the numbers.
7. The sum of a two digit number and the number obtained by reversing the order of its digits is 99. If the digits differ by 3, find the number. [CBSE 2002]
8. A two-digit number is 4 times the sum of its digits. If 18 is added to the number, the digits are reversed. Find the number. [CBSE 2001C]
9. A two-digit number is 3 more than 4 times the sum of its digits. If 18 is added to the number, the digits are reversed. Find the number. [CBSE 2001C]
10. A two-digit number is 4 more than 6 times the sum of its digits. If 18 is subtracted from the number, the digits are reversed. Find the number. [CBSE 2001C]
11. A two-digit number is 4 times the sum of its digits and twice the product of the digits. Find the number. [CBSE 2005]
12. A two-digit number is such that the product of its digits is 20. If 9 is added to the number, the digits interchange their places. Find the number. [CBSE 2005]
13. The difference between two numbers is 26 and one number is three times the other. Find them.
14. The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number. [NCERT]

LEVEL-2

15. Seven times a two-digit number is equal to four times the number obtained by reversing the digits. If the difference between the digits is 3. Find the number.
16. Two numbers are in the ratio 5 : 6. If 8 is subtracted from each of the numbers, the ratio becomes 4 : 5. Find the numbers. [NCERT EXEMPLAR]
17. A two-digit number is obtained by either multiplying the sum of the digits by 8 and then subtracting 5 or by multiplying the difference of the digits by 16 and then adding 3. Find the number. [NCERT EXEMPLAR]

ANSWERS

1. 5, 3	2. 49	3. 23	4. 78	5. 42 or, 24	6. 628, 372
7. 63 or, 36	8. 24	9. 35	10. 64	11. 36	12. 45
13. 39, 13	14. 18	15. 36	16. 40, 48	17. 83	

HINTS TO SELECTED PROBLEMS

1. Let the numbers be x and y. Then, we have

$x + y = 8$ and $x + y = 4(x - y)$.

6. Let, the large number be x, and, the smaller number be y. Then,

$x + y = 1000$ and $x^2 - y^2 = 256000$.

Now, $x^2 - y^2 = 256000$

$$\Rightarrow (x + y)(x - y) = 256000 \Rightarrow x - y = \frac{256000}{x + y} \Rightarrow x - y = \frac{256000}{1000} = 256.$$

3.7.3 APPLICATION TO PROBLEMS BASED ON FRACTIONS

Folloging examples will illustrate applications of simultaneous linear equations in solving word problems on fractions.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A fraction becomes 4/5, if 1 is added to both numerator and denominator. If, however, 5 is subtracted from both numerator and denominator, the fraction becomes 1/2. What is the fraction?*

SOLUTION Let the fraction be $\frac{x}{y}$.

Then, according to the given conditions, we have

$$\frac{x + 1}{y + 1} = \frac{4}{5} \text{ and } \frac{x - 5}{y - 5} = \frac{1}{2}$$

$$\Rightarrow \quad 5x + 5 = 4y + 4 \text{ and } 2x - 10 = y - 5$$

$$\Rightarrow \quad 5x - 4y + 1 = 0 \text{ and } 2x - y - 5 = 0$$

By using cross-multiplication, we have

$$\frac{x}{-4 \times -5 - (-1) \times 1} = \frac{-y}{5 \times -5 - 2 \times 1} = \frac{1}{5 \times -1 - 2 \times -4}$$

$$\Rightarrow \quad \frac{x}{20 + 1} = \frac{y}{25 + 2} = \frac{1}{-5 + 8}$$

$$\Rightarrow \quad \frac{x}{21} = \frac{y}{27} = \frac{1}{3} \Rightarrow x = \frac{21}{3} = 7 \text{ and } y = \frac{27}{3} = 9$$

Hence, the given fraction is 7/9.

LEVEL-2

EXAMPLE 2 *A fraction is such that if the numerator is multiplied by 3 and the denominator is reduced by 3, we get 18/11, but if the numerator is increased by 8 and the denominator is doubled, we get 2/5. Find the fraction.*

SOLUTION Let the fraction be $\frac{x}{y}$.

Then, according to the given conditions, we have

$$\frac{3x}{y - 3} = \frac{18}{11} \text{ and } \frac{x + 8}{2y} = \frac{2}{5}$$

$$\Leftrightarrow \quad 11x = 6y - 18 \text{ and } 5x + 40 = 4y$$

$$\Leftrightarrow \quad 11x - 6y + 18 = 0 \text{ and } 5x - 4y + 40 = 0$$

By cross-multiplication, we have

$$\frac{x}{(-6)\times 40-(-4)\times 18}=\frac{-y}{11\times 40-5\times 18}=\frac{1}{11\times(-4)-5\times(-6)}$$

$$\Rightarrow \quad \frac{x}{-240+72}=\frac{-y}{440-90}=\frac{1}{-44+30}$$

$$\Rightarrow \quad \frac{x}{-168}=\frac{y}{-350}=\frac{1}{-14}$$

$$\Rightarrow \quad x=\frac{-168}{-14} \text{ and } y=\frac{-350}{-14}$$

$$\Rightarrow \quad x=12 \text{ and } y=25$$

Hence, the fraction is $\frac{12}{25}$.

EXAMPLE 3 *The denominator of a fraction is 4 more than twice the numerator. When both the numerator and denominator are decreased by 6, then the denominator becomes 12 times the numerator. Determine the fraction.* [CBSE 2001C]

SOLUTION Let the numerator and denominator of the fraction be x and y respectively. Then,

$$\text{Fraction}=\frac{x}{y}$$

It is given that

$$\text{Denominator} = 2\,(\text{Numerator})+4$$

$$\Rightarrow \quad y=2x+4$$

$$\Rightarrow \quad 2x-y+4=0$$

According to the given condition, we have

$$y-6=12(x-6)$$

$$\Rightarrow \quad y-6=12x-72$$

$$\Rightarrow \quad 12x-y-66=0$$

Thus, we have the following system of equations

$$2x-y+4=0 \quad \ldots(i)$$

$$12x-y-66=0 \quad \ldots(ii)$$

Subtracting equation (i) from equation (ii), we get

$$10x-70=0 \Rightarrow x=7$$

Putting $x=7$ in equation (i), we get

$$14-y+4=0 \Rightarrow y=18$$

Hence, required fraction $=\frac{7}{18}$.

EXERCISE 3.8

LEVEL-1

1. The numerator of a fraction is 4 less than the denominator. If the numerator is decreased by 2 and denominator is increased by 1, then the denominator is eight times the numerator. Find the fraction.

2. A fraction becomes 9/11 if 2 is added to both numerator and the denominator. If 3 is added to both the numerator and the denominator it becomes 5/6. Find the fraction. [NCERT]
3. A fraction becomes 1/3 if 1 is subtracted from both its numerator and denominator. If 1 is added to both the numerator and denominator, it becomes 1/2. Find the fraction.
4. If we add 1 to the numerator and subtract 1 from the denominator, a fraction becomes 1. It also becomes 1/2 if we only add 1 to the denominator. What is the fraction? [NCERT]
5. The sum of the numerator and denominator of a fraction is 12. If the denominator is increased by 3, the fraction becomes 1/2. Find the fraction. [CBSE 2006C]
6. When 3 is added to the denominator and 2 is subtracted from the numerator a fraction becomes 1/4. And, when 6 is added to numerator and the denominator is multiplied by 3, it becomes 2/3. Find the fraction.
7. The sum of a numerator and denominator of a fraction is 18. If the denominator is increased by 2, the fraction reduces to 1/3. Find the fraction.
8. If 2 is added to the numerator of a fraction, it reduces to 1/2 and if 1 is subtracted from the denominator, it reduces to 1/3. Find the fraction.

LEVEL-2

9. The sum of the numerator and denominator of a fraction is 4 more than twice the numerator. If the numerator and denominator are increased by 3, they are in the ratio 2 : 3. Determine the fraction. [CBSE 2001C, 2010]
10. If the numerator of a fraction is multiplied by 2 and the denominator is reduced by 5 the fraction becomes 6/5. And, if the denominator is doubled and the numerator is increased by 8, the fraction becomes 2/5. Find the fraction.
11. The sum of the numerator and denominator of a fraction is 3 less than twice the denominator. If the numerator and denominator are decreased by 1, the numerator becomes half the denominator. Determine the fraction. [CBSE 2001C, 2010]

ANSWERS

1. $\frac{3}{7}$ 2. $\frac{7}{9}$ 3. $\frac{3}{7}$ 4. $\frac{3}{5}$ 5. $\frac{5}{7}$ 6. $\frac{4}{5}$
7. $\frac{5}{13}$ 8. $\frac{3}{10}$ 9. $\frac{5}{9}$ 10. $\frac{12}{25}$ 11. $\frac{4}{7}$

3.7.4 APPLICATIONS TO PROBLEMS ON AGES

Following examples will illustrate the use of solutions of simultaneous linear equations in solving word problems on ages.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If twice the son's age in years is added to the father's age, the sum is 70. But if twice the father's age is added to the son's age, the sum is 95. Find the ages of father and son.*

SOLUTION Suppose father's age (in years) be x and that of son's be y. Then,

$$x + 2y = 70$$

and, $$2x + y = 95$$

This system of equations may be written as

$$x + 2y - 70 = 0$$

$$2x + y - 95 = 0$$

By cross-multiplication, we get

$$\frac{x}{2 \times -95 - (-70)} = \frac{-y}{1 \times -95 - 2 \times -70} = \frac{1}{1 \times 1 - 2 \times 2}$$

$$\Rightarrow \quad \frac{x}{-190 + 70} = \frac{-y}{-95 + 140} = \frac{1}{-3}$$

$$\Rightarrow \quad \frac{x}{-120} = \frac{y}{-45} = \frac{1}{-3} \Rightarrow x = \frac{-120}{-3} = 40 \text{ and } y = \frac{-45}{-3} = 15$$

Hence, father's age is 40 years and the son's age is 15 years.

EXAMPLE 2 *I am three times as old as my son. Five years later, I shall be two and a half times as old as my son. How old am I and how old is my son?*

SOLUTION Suppose my age is x years and my son's age is y years. Then,

$$x = 3y \quad \ldots(i)$$

Five years later, my age will be $(x + 5)$ years and my son's age will be $(y + 5)$ years.

$$\therefore \quad x + 5 = \frac{5}{2}(y + 5) \quad \text{[Given]}$$

$$\Rightarrow \quad 2x - 5y - 15 = 0 \quad \ldots(ii)$$

Putting $x = 3y$ in equation (ii), we get

$$6y - 5y - 15 = 0 \Rightarrow y = 15$$

Putting $y = 15$ in equation (i), we get

$$x = 45$$

Hence, my present age is 45 years and my son's present age is 15 years.

EXAMPLE 3 *Ten years ago, father was twelve times as old as his son and ten years hence, he will be twice as old as his son will be. Find their present ages.*

SOLUTION Let the present ages of father and son be x years and y years respectively.

Ten years ago, Father's age $= (x - 10)$ years

Son's age $= (y - 10)$ years

$$\therefore \quad x - 10 = 12(y - 10) \Rightarrow x - 12y + 110 = 0 \quad \ldots(i)$$

Ten years later, Father's age $= (x + 10)$ years.

Son's age $= (y + 10)$

$$\therefore \quad x + 10 = 2(y + 10) \Rightarrow x - 2y - 10 = 0 \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$-10y + 120 = 0 \Rightarrow 10y = 120 \Rightarrow y = 12$$

Putting $y = 12$ in (i), we get

$$x - 144 + 110 = 0 \Rightarrow x = 34$$

Thus, present age of father is 34 years and the present age of son is 12 years.

EXAMPLE 4 *Five years hence, father's age will be three times the age of his son. Five years ago, father was seven times as old as his son. Find their present ages.* [NCERT]

SOLUTION Let the present age of father be x years and the present age of son be y years.

Five years hence, Father's age $= (x + 5)$ years
Son's age $= (y + 5)$ years

Using the given information, we have

$$x + 5 = 3(y + 5) \Rightarrow x - 3y - 10 = 0 \quad \ldots(i)$$

Five years ago, Father's age $= (x - 5)$ years
Son's age $= (y - 5)$ years

Using the given information, we get

$$(x - 5) = 7(y - 5) \Rightarrow x - 7y + 30 = 0 \quad \ldots(ii)$$

Subtracting equation (ii) from equation (i), we get

$$4y - 40 = 0 \Rightarrow y = 10$$

Putting $y = 10$ in equation (i), we get

$$x - 30 - 10 = 0 \Rightarrow x = 40$$

Hence, present age of father is 40 years and present age of son is 10 years.

LEVEL-2

EXAMPLE 5 *A and B are friends and their ages differ by 2 years. A's father D is twice as old as A and B is twice as old as his sister C. The age of D and C differ by 40 years. Find the ages of A and B.*

SOLUTION Let the ages of A and B be x and y years respectively. Then,

$$x - y = \pm 2 \quad \text{[Given]}$$

D's age $= 2x$ years. and, C's age $= \frac{y}{2}$ years.

Clearly, D is older than C

$$\therefore \quad 2x - \frac{y}{2} = 40 \Rightarrow 4x - y = 80$$

Thus, we have the following two systems of linear equations

$$x - y = 2 \quad \ldots(i)$$

and, $$4x - y = 80 \quad \ldots(ii)$$

$$x - y = -2 \quad \ldots(iii)$$

and, $$4x - y = 80 \quad \ldots(iv)$$

Subtracting equation (i) from equation (ii), we get

$$3x = 78 \Rightarrow x = 26$$

Putting $x = 26$ in equation (i), we get $y = 24$

Subtracting equation (iv) from equation (iii), we get

$$-3x = -82 \Rightarrow x = \frac{82}{3} = 27\frac{1}{3}$$

Putting $x = \frac{82}{3}$ in equation (iii), we get

$$y = \frac{82}{3} + 2 = \frac{88}{3} = 29\frac{1}{3}$$

Hence, A's age = 26 years and B's age = 24 years

or,

A's age $= 27\frac{1}{3}$ years and B's age $= 29\frac{1}{3}$ years.

EXERCISE 3.9

LEVEL-1

1. A father is three times as old as his son. After twelve years, his age will be twice as that of his son then. Find their present ages.
2. Ten years later, A will be twice as old as B and five years ago, A was three times as old as B. What are the present ages of A and B ?
3. Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu? [NCERT]
4. Six years hence a man's age will be three times the age of his son and three years ago he was nine times as old as his son. Find their present ages.
5. Ten years ago, a father was twelve times as old as his son and ten years hence, he will be twice as old as his son will be then. Find their present ages.

LEVEL-2

6. The present age of a father is three years more than three times the age of the son. Three years hence father's age will be 10 years more than twice the age of the son. Determine their present ages.
7. A father is three times as old as his son. In 12 years time, he will be twice as old as his son. Find the present ages of father and the son.
8. Father's age is three times the sum of ages of his two children. After 5 years his age will be twice the sum of ages of two children. Find the age of father. [CBSE 2003]
9. Two years ago, a father was five times as old as his son. Two years later, his age will be 8 more than three times the age of the son. Find the present ages of father and son. [CBSE 2004]
10. A is elder to B by 2 years. A's father F is twice as old as A and B is twice as old as his sister S. If the ages of the father and sister differ by 40 years, find the age of A.
11. The ages of two friends Ani and Biju differ by 3 years. Ani's father Dharam is twice as old as Ani and Biju as twice as old as his sister Cathy. The ages of Cathy and Dharam differ by 30 years. Find the ages of Ani and Biju. [NCERT]
12. Two years ago, Salim was thrice as old as his daughter and six years later, he will be four years older than twice her age. How old are they now? [NCERT EXEMPLAR]
13. The age of the father is twice the sum of the ages of his two children. After 20 years, his age will be equal to the sum of the ages of his children. Find the age of the father. [NCERT EXEMPLAR]

ANSWERS

1. Father's age = 36 years, Son's age = 12 years
2. A's present age = 50 years, B's present age = 20 years.
3. Nuri's age = 50 years, Sonu's age = 20 years

4. Man's age = 30 years, Son's age = 6 years
5. Father's age = 34 years, Son's age = 12 years
6. Father's age = 33 years, Son's age = 10 years
7. Father's age = 36 years, Son's age = 12 years
8. 45 years
9. Father's age = 42 years, Son's age = 10 years
10. 26 years
11. Ani's age = 19 years, Biju's age = 16 years
12. Salim's age = 38 years, Daughter's age = 14 years
13. 40 years

3.7.5 APPLICATION TO PROBLEMS BASED ON TIME DISTANCE AND SPEED

In solving problems based on time, distance and speed, we use the following formulae:

$$\text{Distance} = \text{Speed} \times \text{Time}; \ \text{Time} = \frac{\text{Distance}}{\text{Speed}} \text{ and, Speed} = \frac{\text{Distance}}{\text{Time}}$$

Also, if

Speed of a boat in still water = u km/hr and, Speed of the current = v km/hr

then,

Speed upstream $= (u - v)$ km/hr

Speed downstream $= (u + v)$ km/hr

Following examples will illustrate the use of these formulae.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Points A and B are 90 km apart from each other on a highway. A car starts from A and another from B at the same time. If they go in the same direction they meet in 9 hours and if they go in opposite directions they meet in 9/7 hours. Find their speeds.*

SOLUTION Let X and Y be two cars starting from points A and B respectively. Let the speed of car X be x km/hr and that of car Y be y km/hr.

CASE I *When two cars move in the same directions:*

Suppose two cars meet at point Q. Then,

Distance travelled by car $X = AQ$,

Distance travelled by car $Y = BQ$.

It is given that two cars meet in 9 hours.

$\therefore$ Distance travelled by car X in 9 hours $= 9x$ km.

$\Rightarrow \quad AQ = 9x$

Distance travelled by car y in 9 hours $= 9y$ km.

$\Rightarrow \quad BQ = 9y$

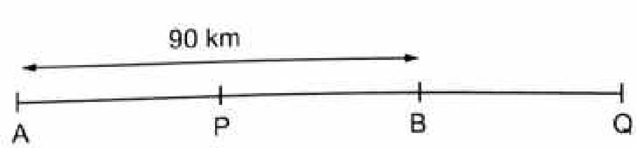


Fig. 3.23

Clearly, $\quad AQ - BQ = AB$

$\Rightarrow \quad 9x - 9y = 90$ $\quad [\because AB = 90 \text{ km}]$

$\Rightarrow \quad x - y = 10$ ...(i)

CASE II *When two cars move in opposite directions:*

Suppose two cars meet at point P. Then,

Distance travelled by car $X = AP$,

Distance travelled by car $Y = BP$.

In this case, two cars meet in 9/7 hours.

$\therefore$ Distance travelled by car X in $\frac{9}{7}$ hours $= \frac{9}{7}x$ km

$\Rightarrow \quad BP = \frac{9}{7}y$

Clearly, $AP + BP = AB$

$\Rightarrow \quad \frac{9}{7}x + \frac{9}{7}y = 90$

$\Rightarrow \quad \frac{9}{7}(x + y) = 90$

$\Rightarrow \quad x + y = 70$ ...(ii)

Solving equations (i) and (ii), we get

$x = 40$ and $y = 30$.

Hence, speed of car X is 40 km/hr and speed of car Y is 30 km/hr.

EXAMPLE 2 *Ved travels 600 km to his home partly by train and partly by car. He takes 8 hours if he travels 120 km by train and the rest by car. He takes 20 minutes longer if he travels 200 km by train and the rest by car. Find the speed of the train and the car.* **[NCERT]**

SOLUTION Let the speed of the train be x km/hr and the speed of the car be y km/hr.

CASE I *When he travels 120 km by train and the rest by car.*

If Ved travels 120 km by train, then

Distance covered by car is $(600 - 120)$ km $= 480$ km.

Now, Time taken to cover 120 km by train $= \frac{120}{x}$ hrs $\quad \left[\because \text{Time} = \frac{\text{Distance}}{\text{Speed}}\right]$

Time taken to cover 480 km by car $= \frac{480}{y}$ hrs

It is given that the total time of the journey is 8 hours.

$\therefore \quad \frac{120}{x} + \frac{480}{y} = 8$

$\Rightarrow \quad 8\left(\frac{15}{x} + \frac{60}{y}\right) = 8$

$\Rightarrow \quad \frac{15}{x} + \frac{60}{y} = 1$

$\Rightarrow \quad \frac{15}{x} + \frac{60}{y} - 1 = 0$ ...(i)

CASE II *When he travels 200 km by train and the rest by car*

If Ved travels 200 km by train, then

Distance travelled by car is (600 – 200) km = 400 km

Now, Time taken to cover 200 km by train $= \dfrac{200}{x}$ hrs

Time taken to cover 400 km by train $= \dfrac{400}{y}$ hrs

In this case the total time of journey is 8 hour 20 minutes

$$\therefore \quad \frac{200}{x} + \frac{400}{y} = 8 \text{ hrs } 20 \text{ minutes}$$

$$\Rightarrow \quad \frac{200}{x} + \frac{400}{y} = 8\frac{1}{3} \qquad \left[\because 8 \text{ hrs } 20 \text{ minutes} = 8\frac{20}{60} \text{ hrs} = 8\frac{1}{3} \text{ hrs}\right]$$

$$\Rightarrow \quad \frac{200}{x} + \frac{400}{y} = \frac{25}{3}$$

$$\Rightarrow \quad 25\left(\frac{8}{x} + \frac{16}{y}\right) = \frac{25}{3}$$

$$\Rightarrow \quad \frac{8}{x} + \frac{16}{y} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{24}{x} + \frac{48}{y} = 1$$

$$\Rightarrow \quad \frac{24}{x} + \frac{48}{y} - 1 = 0 \qquad \text{...(ii)}$$

Putting $\dfrac{1}{x} = u$ and $\dfrac{1}{y} = v$ in equations (i) and (ii), we get

$$15u + 60v - 1 = 0 \qquad \text{...(iii)}$$

$$24u + 48v - 1 = 0 \qquad \text{...(iv)}$$

By using cross-multiplication, we have

$$\frac{u}{60 \times -1 - 48 \times -1} = \frac{-v}{15 \times -1 - 24 \times -1} = \frac{1}{15 \times 48 - 24 \times 60}$$

$$\Rightarrow \quad \frac{u}{-60 + 48} = \frac{-v}{-15 + 24} = \frac{1}{720 - 1440}$$

$$\Rightarrow \quad \frac{u}{-12} = \frac{v}{-9} = \frac{1}{-720}$$

$$\Rightarrow \quad u = \frac{-12}{-720} = \frac{1}{60} \text{ and } v = \frac{-9}{-720} = \frac{1}{80}$$

Now, $u = \dfrac{1}{x} \Rightarrow \dfrac{1}{60} = \dfrac{1}{x} \Rightarrow x = 60$

and, $v = \dfrac{1}{y} \Rightarrow \dfrac{1}{80} = \dfrac{1}{y} \Rightarrow y = 80$

Hence, speed of train = 60 km/hr and speed of car = 80 km/hr.

EXAMPLE 3 *A man travels 370 km partly by train and partly by car. If he covers 250 km by train and the rest by car, it takes him 4 hours. But, if he travels 130 km by train and the rest by car, he takes 18 minutes longer. Find the speed of the train and that of the car.* **[CBSE 2001]**

SOLUTION Let the speed of the train be x km/hr and that of the car be y km/hr. We have following cases:

CASE I *When he travels 250 km by train and the rest by car:*

In this case, we have

$$\text{Time taken by the man to travel 250 km by train} = \frac{250}{x} \text{ hrs}$$

$$\text{Time taken by the man to travel } (370 - 250) = 120 \text{ km by car} = \frac{120}{y} \text{ hrs}$$

$$\therefore \quad \text{Total time taken by the man to cover 370 km} = \frac{250}{x} + \frac{120}{y}$$

It is given that the total time taken is 4 hours

$$\therefore \quad \frac{250}{x} + \frac{120}{y} = 4$$

$$\Rightarrow \quad \frac{125}{x} + \frac{60}{y} = 2 \qquad \ldots\text{(i)}$$

CASE II *When he travels 130 km by train and the rest by car:*

In this case, we have

$$\text{Time taken by the man to travel 130 km by train} = \frac{130}{x} \text{ hrs}$$

$$\text{Time taken by the man to travel } (370 - 130) = 240 \text{ km by car} = \frac{240}{y} \text{ hrs.}$$

In this case, total time of the journey is 4 hrs 18 minutes.

$$\therefore \quad \frac{130}{x} + \frac{240}{y} = 4 \text{ hrs 18 minutes}$$

$$\Rightarrow \quad \frac{130}{x} + \frac{240}{y} = 4\frac{18}{60}$$

$$\Rightarrow \quad \frac{130}{x} + \frac{240}{y} = \frac{43}{10} \qquad \ldots\text{(ii)}$$

Thus, we obtain the following system of equations:

$$\frac{125}{x} + \frac{60}{y} = 2$$

$$\frac{130}{x} + \frac{240}{y} = \frac{43}{10}$$

Putting $\frac{1}{x} = u$ and $\frac{1}{y} = v$, the above system reduces to

$$125u + 60v = 2 \qquad \ldots\text{(iii)}$$

$$130u + 240v = \frac{43}{10} \qquad \ldots\text{(iv)}$$

Multiplying equation (iii) by 4 the above system of equations becomes

$$500u + 240v = 8 \quad \text{...(v)}$$

$$130u + 240v = \frac{43}{10} \quad \text{...(vi)}$$

Subtracting equation (vi) from equation (v), we get

$$370u = 8 - \frac{43}{10} \Rightarrow 370u = \frac{37}{10} \Rightarrow u = \frac{1}{100}$$

Putting $u = \frac{1}{100}$ in equation (v), we get

$$5 + 240v = 8 \Rightarrow 240v = 3 \Rightarrow v = \frac{1}{80}$$

Now, $u = \frac{1}{100}$ and $v = \frac{1}{80}$

$\Rightarrow$ $\frac{1}{x} = \frac{1}{100}$ and $\frac{1}{y} = \frac{1}{80}$

$\Rightarrow$ $x = 100$ and $y = 80$.

Hence, Speed of the train = 100 km/hr

Speed of the car = 80 km/hr.

EXAMPLE 4 *A boat covers 32 km upstream and 36 km downstream in 7 hours. Also, it covers 40 km upstream and 48 km downstream in 9 hours. Find the speed of the boat in still water and that of the stream.*

SOLUTION Let the speed of the boat in still water be x km/hr and the speed of the stream be y km/hr. Then,

Speed upstream = $(x - y)$ km/hr

Speed downstream = $(x + y)$ km/hr

Now, Time taken to cover 32 km upstream $= \frac{32}{x - y}$ hrs

Time taken to cover 36 km downstream $= \frac{36}{x + y}$ hrs

But, total time of journey is 7 hours.

$$\therefore \quad \frac{32}{x - y} + \frac{36}{x + y} = 7 \quad \text{...(i)}$$

Time taken to cover 40 km upstream $= \frac{40}{x - y}$

Time taken to cover 48 km downstream $= \frac{48}{x + y}$

In this case, total time of journey is given to be 9 hours.

$$\therefore \quad \frac{40}{x - y} + \frac{48}{x + y} = 9 \quad \text{...(ii)}$$

Putting $\frac{1}{x - y} = u$ and $\frac{1}{x + y} = v$ in equations (i) and (ii), we get

$$32u + 36v = 7 \Rightarrow 32u + 36v - 7 = 0 \quad \text{...(iii)}$$

$$40u + 48v = 9 \Rightarrow 40u + 48v - 9 = 0 \quad \text{...(iv)}$$

Solving these equations by cross-multiplication, we get

$$\frac{u}{36 \times -9 - 48 \times -7} = \frac{-v}{32 \times -9 - 40 \times -7} = \frac{1}{32 \times 48 - 40 \times 36}$$

$$\Rightarrow \quad \frac{u}{-324 + 336} = \frac{-v}{-288 + 280} = \frac{1}{1536 - 1440}$$

$$\Rightarrow \quad \frac{u}{12} = \frac{v}{8} = \frac{1}{96}$$

$$\Rightarrow \quad u = \frac{12}{96} \text{ and } v = \frac{8}{96}$$

$$\Rightarrow \quad u = \frac{1}{8} \text{ and } v = \frac{1}{12}$$

Now, $\quad u = \frac{1}{8} \Rightarrow \frac{1}{x - y} = \frac{1}{8} \Rightarrow x - y = 8 \quad \text{... (v)}$

and, $\quad v = \frac{1}{12} \Rightarrow \frac{1}{x + y} = \frac{1}{12} \Rightarrow x + y = 12 \quad \text{... (vi)}$

Solving equations (v) and (vi), we get $x = 10$ and $y = 2$.
Hence, Speed of the boat in still water = 10 km/hr
Speed of the stream = 2 km/hr.

LEVEL-2

EXAMPLE 5 *X takes 3 hours more than Y to walk 30 km. But, if X doubles his pace, he is ahead of Y by $1\frac{1}{2}$ hours. Find their speed of walking.*

SOLUTION Let the speed of X and Y be x km/hr and y km/hr respectively. Then,

Time taken by X to cover 30 km $= \frac{30}{x}$ hrs,

and, Time taken by Y to cover 30 km $= \frac{30}{y}$ hrs

By the given conditions, we have

$$\frac{30}{x} - \frac{30}{y} = 3 \quad \Rightarrow \quad \frac{10}{x} - \frac{10}{y} = 1 \quad \text{... (i)}$$

If X doubles his pace, then speed of X is $2x$ km/hr

$\therefore$ Times taken by X to cover 30 km $= \frac{30}{2x}$ hrs.

Times taken by Y to cover 30 km $= \frac{30}{y}$ hrs

According to the given conditions, we have

$$\frac{30}{y} - \frac{30}{2x} = 1\frac{1}{2}$$

$$\Rightarrow \quad \frac{30}{y} - \frac{30}{2x} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{10}{y} - \frac{5}{x} = \frac{1}{2}$$

$$\Rightarrow \quad -\frac{10}{x} + \frac{20}{y} = 1 \qquad \ldots(ii)$$

Putting $\frac{1}{x} = u$ and $\frac{1}{y} = v$, in equations (i) and (ii) we get

$$10u - 10v = 1 \Rightarrow 10u - 10v - 1 = 0 \qquad \ldots(iii)$$

$$-10u + 20v = 1 \Rightarrow -10u + 20v - 1 = 0 \qquad \ldots(iv)$$

Adding equations (iii) and (iv), we get

$$10v - 2 = 0 \Rightarrow v = \frac{1}{5}$$

Putting $v = \frac{1}{5}$ in equation (iii), we get

$$10u - 3 = 0 \Rightarrow u = \frac{3}{10}$$

Now, $u = \frac{3}{10} \Rightarrow \frac{1}{x} = \frac{3}{10} \Rightarrow x = \frac{10}{3}$ and, $v = \frac{1}{5} \Rightarrow \frac{1}{y} = \frac{1}{5} \Rightarrow y = 5$

Hence, X's speed $= \frac{10}{3}$ km/hr and, Y's speed = 5 km/hr.

EXAMPLE 6 *After covering a distance of 30 km with a uniform speed there is some defect in a train engine and therefore, its speed is reduced to 4/5 of its original speed. Consequently, the train reaches its destination late by 45 minutes. Had it happened after covering 18 kilometres more, the train would have reached 9 minutes earlier. Find the speed of the train and the distance of journey.*

SOLUTION Let the original speed of the train be x km/hr and the length of the journey be y km. Then,

Time taken $= (y/x)$ hrs.

CASE I *When defect in the engine occurs after covering a distance of 30 km.*

We have,

Speed for first 30 km $= x$ km/hr

and, Speed for the remaining $(y - 30)$ km $= \frac{4}{5}x$ km/hrs

$\therefore$ Time taken to cover 30 km $= \frac{30}{x}$ hrs

Time taken to cover $(y - 30)$ km $= \frac{y - 30}{(4x/5)}$ hrs. $= \frac{5}{4x}(y - 30)$ hrs.

According to the given condition, we have,

$$\frac{30}{x} + \frac{5}{4x}(y - 30) = \frac{y}{x} + \frac{45}{60}$$

$$\Rightarrow \quad \frac{30}{x} + \frac{5y - 150}{4x} = \frac{y}{x} + \frac{3}{4}$$

$\Rightarrow \quad \dfrac{120+5y-150}{4x}=\dfrac{4y+3x}{4x}$

$\Rightarrow \quad 5y-30=4y+3x$

$\Rightarrow \quad 3x-y+30=0$ ...(i)

CASE II *When defect in the engine occurs after covering a distance of 48 km.*

Speed for first 48 km = x km/hr.

Speed for the remaining $(y-48)$ km $=\dfrac{4x}{5}$ km/hr

$\therefore$ Time taken to cover 48 km $=\dfrac{48}{x}$ hrs.

Time taken to cover $(y-48)$ km $=\left(\dfrac{y-48}{4x/5}\right)$ hr $=\left\{\dfrac{5(y-48)}{4x}\right\}$ hr

According to the given condition, the train now reaches 9 minutes earlier i.e., 36 minutes later.

$$\frac{48}{x}+\frac{5(y-48)}{4x}=\frac{y}{x}+\frac{36}{60}$$

$\Rightarrow \quad \dfrac{48}{x}+\dfrac{5y-240}{4x}=\dfrac{y}{x}+\dfrac{3}{5}$

$\Rightarrow \quad \dfrac{192+5y-240}{4x}=\dfrac{5y+3x}{5x}$

$\Rightarrow \quad \dfrac{5y-48}{4}=\dfrac{5y+3x}{5}$

$\Rightarrow \quad 25y-240=20y+12x$

$\Rightarrow \quad 12x-5y+240=0$ ...(ii)

Thus, we have the following system of simultaneous equations:

$$3x-y+30=0$$

$$12x-5y+240=0$$

By using cross-multiplication, we have

$$\frac{x}{-240+150}=\frac{-y}{720-360}=\frac{1}{-15+12}$$

$\Rightarrow \quad \dfrac{x}{-90}=\dfrac{-y}{360}=\dfrac{1}{-3}$

$\Rightarrow \quad x=\dfrac{-90}{-3}=30$ and $y=\dfrac{-360}{-3}=120$

Hence, the original speed of the train is 30 km/hr and the length of the journey is 120 km.

EXAMPLE 7 *A train covered a certain distance at a uniform speed. If the train would have been 6 km/h faster, it would have taken 4 hours less than the scheduled time. And, if the train were slower by 6 km/h, it would have taken 6 hours more than the scheduled time. Find the length of the journey.*

SOLUTION Let the actual speed of the train be x km/hr and the actual time taken be y hours. Then,

$$\text{Distance covered} = (xy) \text{ km} \quad \dots \text{(i)} \quad [\because \text{Distance} = \text{Speed} \times \text{Time}]$$

If the speed is increased by 6 km/hr, then time of journey is reduced by 4 hours i.e., when speed is $(x+6)$ km/hr, time of journey is $(y-4)$ hours.

$\therefore$ Distance covered $= (x+6)(y-4)$

$\Rightarrow$ $xy = (x+6)(y-4)$ [Using (i)]

$\Rightarrow$ $-4x + 6y - 24 = 0$

$\Rightarrow$ $-2x + 3y - 12 = 0$ ... (ii)

When the speed is reduced by 6 km/hr, then the time of journey is increased by 6 hours i.e., when speed is $(x-6)$ km/hr, time of journey is $(y-6)$ hours.

$\therefore$ Distance covered $= (x-6)(y+6)$

$\Rightarrow$ $xy = (x-6)(y+6)$ [Using (i)]

$\Rightarrow$ $6x - 6y - 36 = 0$

$\Rightarrow$ $x - y - 6 = 0$ ... (iii)

Thus, we obtain the following system of equations:

$$-2x + 3y - 12 = 0$$
$$x - y - 6 = 0$$

By using cross-multiplication, we have,

$$\frac{x}{3 \times -6 - (-1) \times -12} = \frac{-y}{-2 \times -6 - 1 \times -12} = \frac{1}{-2 \times -1 - 1 \times 3}$$

$$\Rightarrow \quad \frac{x}{-30} = \frac{-y}{24} = \frac{1}{-1}$$

$$\Rightarrow \quad x = 30 \text{ and } y = 24.$$

Putting the values of x and y in equation (i), we obtain

$$\text{Distance} = (30 \times 24) \text{ km} = 720 \text{ km}.$$

Hence, the length of the journey is 720 km.

EXERCISE 3.10

LEVEL-1

1. Points A and B are 70 km. a part on a highway. A car starts from A and another car starts from B simulataneously. If they travel in the same direction, they meet in 7 hours, but if they travel towards each other, they meet in one hour. Find the speed of the two cars. **[CBSE 2002]**
2. A sailor goes 8 km downstream in 40 minutes and returns in 1 hour. Determine the speed of the sailor in still water and the speed of the current.
3. The boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km downstream. Determine the speed of stream and that of the boat in still water.

4. A boat goes 24 km upstream and 28 km downstream in 6 hrs. It goes 30 km upstream and 21 km downstream in $6\frac{1}{2}$ hrs. Find the speed of the boat in still water and also speed of the stream.

5. A man walks a certain distance with certain speed. If he walks 1/2 km an hour faster, he takes 1 hour less. But, if he walks 1 km an hour slower, he takes 3 more hours. Find the distance covered by the man and his original rate of walking.

6. A person rowing at the rate of 5 km/h in still water, takes thrice as much time in going 40 km upstream as in going 40 km downstream. Find the speed of the stream.

[NCERT EXEMPLAR]

7. Ramesh travels 760 km to his home partly by train and partly by car. He takes 8 hours if he travels 160 km. by train and the rest by car. He takes 12 minutes more if the travels 240 km by train and the rest by car. Find the speed of the train and car respectively.

8. A man travels 600 km partly by train and partly by car. If he covers 400 km by train and the rest by car, it takes him 6 hours and 30 minutes. But, if he travels 200 km by train and the rest by car, he takes half an hour longer. Find the speed of the train and that of the car.

[CBSE 2001]

9. Places A and B are 80 km apart from each other on a highway. A car starts from A and other from B at the same time. If they move in the same direction, they meet in 8 hours and if they move in opposite directions, they meet in 1 hour and 20 minutes. Find the speeds of the cars.

[CBSE 2002]

10. A boat goes 12 km upstream and 40 km downstream in 8 hours. It can go 16 km upstream and 32 km downstream in the same time. Find the speed of the boat in still water and the speed of the stream.

11. Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.

12. Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current. [NCERT]

13. A motor boat can travel 30 km upstream and 28 km downstream in 7 hours. It can travel 21 km upstream and return in 5 hours. Find the speed of the boat in still water and the speed of the stream.

[NCERT EXEMPLAR]

14. Abdul travelled 300 km by train and 200 km by taxi, it took him 5 hours 30 minutes. But if he travels 260 km by train and 240 km by taxi he takes 6 minutes longer. Find the speed of the train and that of the taxi.

[CBSE 2006C]

15. A train covered a certain distance at a uniform speed. If the train could have been 10 km/hr. faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/hr; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.

[NCERT]

16. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of two cars?

[NCERT, CBSE 2009]

LEVEL-2

17. While covering a distance of 30 km. Ajeet takes 2 hours more than Amit. If Ajeet doubles his speed, he would take 1 hour less than Amit. Find their speeds of walking.
18. A takes 3 hours more than B to walk a distance of 30 km. But, if A doubles his pace (speed) he is ahead of B by $1\frac{1}{2}$ hours. Find the speeds of A and B.

ANSWERS

1. Speed of car starting from point $A = 40$ km/hr.
   Speed of car starting from point $B = 30$ km/hr.
2. Speed of sailor = 10 km/hr, speed of current = 2 km/hr.
3. Speed of stream = 3 km/hr, Speed of boat = 8 km/hr.
4. Speed of stream = 4 km/hr, Speed of boat = 10 km/hr.
5. Distance = 36 km, Original speed = 4 km/hr.
6. 2.5 km/hr
7. Train; 80 km/hr, car: 100 km/hr.
8. 100 km/hr, 80 km/hr.
9. 35 km/hr, 25 km/hr.
10. 6 km/hr, 2 km/hr.
11. 60 km/hr, 80 km/hr.
12. 6 km/hr, 4 km/hr.
13. 10 km/hr, 4 km/hr.
14. 100 km/hr, 80 km/hr.
15. 600 km.
16. 60 km/hr, 40 km/hr
17. Ajit's speed = 5 km/hr, Amit's speed = 7.5 km/hr.
18. $\frac{10}{3}$ km/hr, 5 km/hr.

HINT TO SELECTED PROBLEM

5. Let the original speed be y km/hr and total time taken be x hrs. Then,
   Distance = (xy) km.
   Since distance covered in each case is same.
   $$\therefore \quad \left(y + \frac{1}{2}\right)(x - 1) = xy \text{ and, } (y - 1)(x + 3) = xy$$
   $$\Rightarrow \quad \frac{x}{2} - y - \frac{1}{2} = 0 \text{ and } - x + 3y - 3 = 0.$$

3.7.6 APPLICATIONS TO MISCELLANEOUS PROBLEMS

Following examples will illustrate the applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The taxi charges in a city comprise of a fixed charge together with the charge for the distance covered. For a journey of 10 km the charge paid is ₹75 and for a journey of 15 km the charge paid is ₹110. What will a person have to pay for travelling a distance of 25 km?*

[NCERT, CBSE 2000]

SOLUTION Let the fixed charges of taxi be ₹ x and the running charges be ₹ y km/hr.

According to the given condition, we have

$$x + 10y = 75 \quad \ldots (i)$$

$$x + 15y = 110 \quad \ldots (ii)$$

Subtracting equation (ii) from equation (i), we get

$$-5y = -35 \Rightarrow y = 7$$

Putting $y = 7$ in equation (i), we get $x = 5$.

$\therefore$ Total charges from travelling a distance of 25 km $= x + 25y$

$= ₹\,(5 + 25 \times 7) = ₹\,180.$

EXAMPLE 2 *The total expenditure per month of a household consists of a fixed rent of the house and mess charges depending upon the number of people sharing the house. The total monthly expenditure is ₹ 3900 for 2 people and ₹ 7500 for 5 people. Find the rent of the house and the mess charges per head per month.*

SOLUTION Let the monthly rent of the house be ₹ x and the mess charges per head per month be ₹ y.

According to the given conditions, we have,

$$x + 2y = 3900 \quad \ldots (i)$$

$$x + 5y = 7500 \quad \ldots (ii)$$

Subtracting equation (ii) from equation (i), we get

$$-3y = -3600 \Rightarrow y = 1200$$

Putting $y = 1200$ in equation (i), we get $x = 1500$

Hence, monthly rent = ₹ 1500 and mess charges per head per month = ₹ 1200.

EXAMPLE 3 *The area of a rectangle gets reduced by 9 square units if its length is reduced by 5 units and the breadth is increased by 3 units. If we increase the length by 3 units and breadth by 2 units, the area is increased by 67 square units. Find the length and breadth of the rectangle.* **[NCERT]**

SOLUTION Let the length and breadth of the rectangle be x and y units respectively. Then,

Area $= xy$ sq. units.

If length is reduced by 5 units and the breadth is increased by 3 units, then area is reduced by 9 square units.

$\therefore \quad xy - 9 = (x - 5)(y + 3)$

$\Rightarrow \quad xy - 9 = xy + 3x - 5y - 15$

$\Rightarrow \quad 3x - 5y - 6 = 0 \quad \ldots (i)$

When length is increased by 3 units and breadth by 2 units, the area is increased by 67 sq. units.

$\therefore \quad xy + 67 = (x + 3)(y + 2)$

$\Rightarrow \quad xy + 67 = xy + 2x + 3y + 6$

$\Rightarrow \quad 2x + 3y - 61 = 0 \quad \ldots (ii)$

Thus, we get the following system of linear equations:

$$3x - 5y - 6 = 0$$

$$2x + 3y - 61 = 0$$

By using cross-multiplication, we have

$$\frac{x}{305+18} = \frac{-y}{-183+12} = \frac{1}{9+10}$$

$$\Rightarrow \quad x = \frac{323}{19} = 17 \text{ and } y = \frac{171}{19} = 9$$

Hence, the length and breadth of the rectangle are 17 units and 9 units respectively.

EXAMPLE 4 *A man starts his job with a certain monthly salary and earns a fixed increment every year. If his salary was ₹1500 after 4 year of service and ₹1800 after 10 years of service, what was his starting salary and what is the annual increment?*

SOLUTION Let the starting salary of the man be ₹ x and the fixed annual increment be ₹ y. Then,

Salary after 4 years of service = ₹ $(x + 4y)$

Salary after 10 years of service = ₹ $(x + 10y)$

$$\therefore \quad x + 4y = 1500 \quad \ldots \text{(i)}$$

$$x + 10y = 1800 \quad \ldots \text{(ii)}$$

Subtracting equation (i) from equation (ii), we get

$$6y = 300 \Rightarrow y = 50$$

Putting $y = 50$ in equation (i), we get $x = 1300$.

Hence the starting salary was ₹ 1300 and annual increment is ₹ 50.

EXAMPLE 5 *A person invested some amount at the rate of 12% simple interest and some other amount at the rate of 10% simple interest. He received yearly interest of ₹ 130. But, if he had interchanged the amounts invested, he would have received ₹4 more as interest. How much amount did he invest at different rates?*

SOLUTION Suppose the person invested ₹ x at the rate of 12% simple interest and ₹ y at the rate of 10% simple interest. Then,

$$\text{Yearly interest} = \frac{12x}{100} + \frac{10y}{100}$$

$$\therefore \quad \frac{12x}{100} + \frac{10y}{100} = 130$$

$$\Rightarrow \quad 12x + 10y = 13000$$

$$\Rightarrow \quad 6x + 5y = 6500 \quad \ldots \text{(i)}$$

If the invested amounts are interchanged, then yearly interest increases by ₹ 4.

$$\therefore \quad \frac{10x}{100} + \frac{12y}{100} = 134$$

$$\Rightarrow \quad 10x + 12y = 13400$$

$$\Rightarrow \quad 5x + 6y = 6700 \quad \ldots \text{(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$x - y = -200 \quad \ldots \text{(iii)}$$

Adding equation (ii) and (i), we get

$$11x + 11y = 13200$$

$$\Rightarrow \quad x + y = 1200 \quad \ldots \text{(iv)}$$

Adding equations (iii) and (iv), we get

$$2x = 1000 \Rightarrow x = 500$$

Putting $x = 500$ in equation (iii), we get $y = 700$

Thus, the person invested ₹ 500 at the rate of 12% per year and ₹ 700 at the rate of 10% per year.

EXAMPLE 6 *The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them saves ₹200 per month, find their monthly incomes.* **[NCERT]**

SOLUTION Let the income of first person be ₹ $9x$ and the income of second person be ₹ $7x$. Further, let the expenditures of first and second person be $4y$ and $3y$ respectively. Then,

Saving of first person $= 9x - 4y$

Saving of second person $= 7x - 3y$

$\therefore \quad 9x - 4y = 200 \Rightarrow 9x - 4y - 200 = 0$ ... (i)

and, $\quad 7x - 3y = 200 \Rightarrow 7x - 3y - 200 = 0$ ... (ii)

Solving equation (i) and (ii) by cross-multiplication, we have

$$\frac{x}{800 - 600} = \frac{-y}{-1800 + 1400} = \frac{1}{-27 + 28}$$

$\Rightarrow \quad x = 200$ and $y = 400$.

Thus, monthly income of first person = ₹ $9x$ = ₹ (9×200) = ₹ 1800

and, monthly income of second person = ₹ $7x$ = ₹ (7×200) = ₹ 1400

EXAMPLE 7 *In a ΔABC, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.* **[NCERT]**

SOLUTION Let $\angle A = x°$, $\angle B = y°$. Then,

$$\angle C = 3\angle B \Rightarrow \angle C = 3y°$$

$\therefore \quad 3\angle B = 2(\angle A + \angle B)$

$\Rightarrow \quad 3y = 2(x + y) \Rightarrow y = 2x \Rightarrow 2x - y = 0$ ... (i)

Since $\angle A$, $\angle B$ and $\angle C$ are angles of a triangle.

$\therefore \quad \angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad x + y + 3y = 180 \Rightarrow x + 4y = 180$ ... (ii)

Putting $y = 2x$ in equation (ii), we get

$$x + 8x = 180 \Rightarrow 9x = 180 \Rightarrow x = 20°$$

Putting the value of x in equation (i), we get $y = 40°$

Hence, $\angle A = 20° \angle B = 40°$ and $\angle C = 3y° = (3 \times 40°) = 120°$

EXAMPLE 8 *Find the four angles of a cyclic quadrilateral ABCD in which $\angle A = (2x - 1)°$, $\angle B = (y + 5)°$ $\angle C = (2y + 15)°$ and $\angle D = (4x - 7)°$.*

SOLUTION We know that the sum of the opposite angles of a cyclic quadrilateral is 180°. In the cyclic quadrilateral $ABCD$, angles A and C and and angles B and D form pairs of opposite angles.

$\therefore \quad \angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

$\Rightarrow \quad 2x - 1 + 2y + 15 = 180$ and $y + 5 + 4x - 7 = 180$

$\Rightarrow \quad 2x + 2y = 166$ and $4x + y = 182$

$\Rightarrow \qquad x + y = 83 \qquad \ldots(i)$

and, $\qquad 4x + y = 182 \qquad \ldots(ii)$

Subtracting equation (i) from equation (ii), we get

$$3x = 99 \Rightarrow x = 33$$

Substituting $x = 33$ in equation (i), we get $y = 50$

Hence, $\angle A = (2 \times 33 - 1)^\circ = 65^\circ$, $\angle B = (y + 5)^\circ = (50 + 5)^\circ = 55^\circ$

$\angle C = (2y + 15)^\circ = (2 \times 50 + 15)^\circ = 115^\circ$ and $\angle D = (4 \times 33 - 7)^\circ = 125^\circ$

LEVEL-2

EXAMPLE 9 *A man sold a chair and a table together for ₹1520 thereby making a profit of 25% on the chair and 10% on table. By selling them together for ₹1535 he would have made a profit of 10% on the chair and 25% on the table. Find the cost price of each.*

SOLUTION Let the cost price of one chair be ₹ x and that of one table be ₹ y. Profit on a chair = 25%.

$\therefore$ Selling price of one chair $= x + \frac{25}{100}x = \frac{125}{100}x$

Profit on a table = 10%

$\therefore$ Selling price of one table $= y + \frac{10y}{100} = \frac{110}{100}y$

According to the given condition, we have

$$\frac{125}{100}x + \frac{110}{100}y = 1520 \Rightarrow 125x + 110y = 152000 \Rightarrow 25x + 22y = 30400 \qquad \ldots(i)$$

If profit on a chair is 10% and on a table is 25%, then total selling price is ₹ 1535.

$$\therefore \qquad \left(x + \frac{10}{100}x\right) + \left(y + \frac{25}{100}y\right) = 1535$$

$$\Rightarrow \qquad \frac{110}{100}x + \frac{125}{100}y = 1535$$

$$\Rightarrow \qquad 110x + 125y = 153500$$

$$\Rightarrow \qquad 22x + 25y = 30700 \qquad \ldots(ii)$$

Subtracting equation (ii) from equation (i), we get

$$3x - 3y = -300 \Rightarrow x - y = -100 \qquad \ldots(iii)$$

Adding equation (ii) and (i), we get

$$47x + 47y = 61100 \Rightarrow x + y = 1300 \qquad \ldots(iv)$$

Solving equations (iii) and (iv), we get

$$x = 600 \text{ and } y = 700$$

Hence, the cost price of a chair is ₹ 600 and that of a table is ₹ 700.

EXAMPLE 10 *Students of a class are made to stand in rows. If one student is extra in a row, there would be 2 rows less. If one student is less in a row there would be 3 rows more. Find the number of students in the class.*

SOLUTION Let the number of students be x and the number of rows be y. Then,

Number of students in each row $= x/y$

When one student is extra in each row, there are 2 rows less i.e., when each row has $\left(\frac{x}{y}+1\right)$ students, the number of rows is $(y-2)$.

$\therefore$ Total number of students = No. of rows × No. of students in each row

$$\Rightarrow \quad x=\left(\frac{x}{y}+1\right)(y-2)$$

$$\Rightarrow \quad x=x-\frac{2x}{y}+y-2 \Rightarrow -\frac{2x}{y}+y-2=0 \qquad \ldots\text{(i)}$$

If one student is less in each row, then there are 3 rows more i.e., when each row has $\left(\frac{x}{y}-1\right)$ students, the number of rows is $(y+3)$

$\therefore$ Total number of students = No. of rows × No. of students in each row

$$\Rightarrow \quad x=\left(\frac{x}{y}-1\right)(y+3) \Rightarrow x=x+\frac{3x}{y}-y-3 \Rightarrow \frac{3x}{y}-y-3=0 \qquad \ldots\text{(ii)}$$

Putting $\frac{x}{y}=u$ in (i) and (ii), we get

$$\Rightarrow \quad -2u+y-2=0 \qquad \ldots\text{(iii)}$$

$$\Rightarrow \quad 3u-y-3=0 \qquad \ldots\text{(iv)}$$

Adding (iii) and (iv), we get

$$\Rightarrow \quad u-5=0 \Rightarrow u=5$$

Putting $u=5$ in (iii), we get $y=12$

Now, $\quad u=5 \Rightarrow \frac{x}{y}=5 \Rightarrow \frac{x}{12}=5 \Rightarrow x=60$

Hence, the number of students in the class is 60.7

EXAMPLE 11 *8 men and 12 boys can finish a piece of work in 10 days while 6 men and 8 boys can finish it in 14 days. Find the time taken by one man alone and that by one boy alone to finish the work.*

SOLUTION Suppose that one man alone can finish the work in x days and one boy alone can finish it in y days. Then,

One man's one day's work $=\frac{1}{x}$

One boy's one day's work $=\frac{1}{y}$

$\therefore$ Eight men's one day's work $=\frac{8}{x}$

12 boy's one day's work $=\frac{12}{y}$

Since 8 men and 12 boys can finish the work in 10 days

$$10\left(\frac{8}{x}+\frac{12}{y}\right)=1 \Rightarrow \frac{80}{x}+\frac{120}{y}=1 \qquad \ldots\text{(i)}$$

Again, 6 men and 8 boys can finish the work in 14 days.

$$\therefore \qquad 14\left(\frac{6}{x}+\frac{8}{y}\right)=1 \Rightarrow \frac{84}{x}+\frac{112}{y}=1 \qquad \ldots \text{(ii)}$$

Putting $\frac{1}{x}=u$ and $\frac{1}{y}=v$ in equations (i) and (ii), we get

$$80u+120v-1=0$$
$$84u+112v-1=0$$

By using cross-multiplication, we have

$$\frac{u}{-120+112}=\frac{-v}{-80+84}=\frac{1}{80\times 112-120\times 84}$$

$$\Rightarrow \qquad \frac{u}{-8}=\frac{v}{-4}=\frac{1}{-1120}$$

$$\Rightarrow \qquad u=\frac{-8}{-1120}=\frac{1}{140} \text{ and } v=\frac{-4}{-1120}=\frac{1}{280}$$

Now, $\quad u=\frac{1}{140} \Rightarrow \frac{1}{x}=\frac{1}{140} \Rightarrow x=140$

and, $\quad v=\frac{1}{280} \Rightarrow \frac{1}{y}=\frac{1}{280} \Rightarrow y=280.$

Thus, one man alone can finish the work in 140 days and one boy alone can finish the work in 280 days.

EXAMPLE 12 *On selling a tea-set at 5% loss and a lemon-set at 15% gain, a crockery seller gains ₹ 7. If he sells the tea-set at 5% gain and the lemon-set at 10% gain, he gains ₹13. Find the actual price of the tea-set and the lemon-set.*

SOLUTION Let the cost price of the tea-set and the lemon-set be ₹ x and ₹ y respectively.

CASE I *When tea set is sold at 5% loss and lemon-set at 15% gain.*

Loss on tea-set $=₹\,\frac{5x}{100}=₹\,\frac{x}{20}$

Gain on lemon-set $=₹\,\frac{15y}{100}=₹\,\frac{3y}{20}$

$$\therefore \qquad \text{Net gain} = ₹\,\frac{3y}{20}-\frac{x}{20}$$

$$\Rightarrow \qquad \frac{3y}{20}-\frac{x}{20}=7 \Rightarrow 3y-x=140 \Rightarrow x-3y+140=0 \qquad \ldots \text{(i)}$$

CASE II *When tea-set is sold at 5% gain and the lemon-set at 10% gain.*

Gain on tea-set $=₹\,\frac{5x}{100}=₹\,\frac{x}{20}$

Gain on lemon-set $=₹\,\frac{10y}{100}=₹\,\frac{y}{10}$

$$\therefore \qquad \text{Total gain} = ₹\,\frac{x}{20}+\frac{y}{10}$$

$$\Rightarrow \qquad \frac{x}{20} + \frac{y}{10} = 13 \Rightarrow x + 2y = 260 \Rightarrow x + 2y - 260 = 0 \qquad \text{...(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$-5y + 400 = 0 \Rightarrow y = 80$$

Putting $y = 80$ in equation (i), we get

$$x - 240 + 140 = 0 \Rightarrow x = 100$$

Hence, cost prices of tea-set and lemon-set are ₹ 100 and ₹ 80 respectively.

EXAMPLE 13 *It can take 12 hours to fill a swimming pool using two pipes. If the pipe of larger diameter is used for four hours and the pipe of smaller diameter for 9 hours, only half of the pool can be filled. How long would it take for each pipe to fill the pool separately?* **[NCERT EXEMPLAR]**

SOLUTION Let the time taken by the pipes of larger and smaller diameters alone to fill the pool be x hours and y hours respectively. Let the total volume of the pool be V cubic units.

The pipe of larger diameter fills the pool in x hours. This means that in x hours the volume of water that comes out of the pipe of larger diameter is V cubic units.

$\therefore$ In 1 hour volume of the water that comes out of the pipe of larger diameter is $\frac{V}{x}$ cubic units.

So, in four hours, the volume of the water that comes out of the pipe of larger diameter is $\frac{4V}{x}$ cubic units.

Similarly, the volume of the water that comes out of the pipe of smaller diameter in 9 hours is $\frac{9V}{y}$.

It is given that if the pipe of larger diameter is used for 4 hours and that of smaller diameter for 9 hours, only half of the pool can be filled.

$$\therefore \qquad \frac{4V}{x} + \frac{9V}{y} = \frac{1}{2}V \Rightarrow \frac{4}{x} + \frac{9}{y} = \frac{1}{2} \qquad \text{...(i)}$$

If both the pipes are used for 12 hours, they completely fill the tank.

$$\therefore \qquad \frac{12V}{x} + \frac{12V}{y} = V \Rightarrow \frac{12}{x} + \frac{12}{y} = 1 \qquad \text{...(ii)}$$

Putting $\frac{1}{x} = u$ and $\frac{1}{y} = v$ in (i) and (ii), we obtain

$$4u + 9v = \frac{1}{2} \qquad \text{...(iii)}$$

$$12u + 12v = 1 \qquad \text{...(iv)}$$

Multiplying (iii) by 3 and subtracting from (iv), we get

$$-15v = -\frac{1}{2} \Rightarrow v = \frac{1}{30} \Rightarrow \frac{1}{y} = \frac{1}{30} \Rightarrow y = 30$$

Substituting $v = \frac{1}{30}$ in (iii), we get

$$4u + \frac{9}{30} = \frac{1}{2} \Rightarrow 4u = \frac{1}{2} - \frac{9}{30} = \frac{1}{5} \Rightarrow u = \frac{1}{20} \Rightarrow \frac{1}{x} = \frac{1}{20} \Rightarrow x = 20$$

Thus, the pipes of larger and smaller diameters fill the swimming pool alone in 20 hours and 30 hours respectively.

EXERCISE 3.11

LEVEL-1

1. If in a rectangle, the length is increased and breadth reduced each by 2 units, the area is reduced by 28 square units. If, however the length is reduced by 1 unit and the breadth increased by 2 units, the area increases by 33 square units. Find the area of the rectangle.
2. The area of a rectangle remains the same if the length is increased by 7 metres and the breadth is decreased by 3 metres. The area remains unaffected if the length is decreased by 7 metres and breadth is increased by 5 metres. Find the dimensions of the rectangle.
3. In a rectangle, if the length is increased by 3 metres and breadth is decreased by 4 metres, the area of the rectangle is reduced by 67 square metres. If length is reduced by 1 metre and breadth is increased by 4 metres, the area is increased by 89 sq. metres. Find the dimensions of the rectangle.
4. The incomes of X and Y are in the ratio of $8:7$ and their expenditures are in the ratio $19:16$. If each saves ₹ 1250, find their incomes.
5. A and B each has some money. If A gives ₹ 30 to B, then B will have twice the money left with A. But, if B gives ₹ 10 to A, then A will have thrice as much as is left with B. How much money does each have?
6. $ABCD$ is a cyclic quadrilateral such that $\angle A = (4y + 20)°$, $\angle B = (3y - 5)°$, $\angle C = (4x)°$ and $\angle D = (7x + 5)°$. Find the four angles. **[NCERT]**
7. 2 men and 7 boys can do a piece of work in 4 days. The same work is done in 3 days by 4 men and 4 boys. How long would it take one man and one boy to do it?
8. In a ΔABC, $\angle A = x°$, $\angle B = (3x - 2)°$, $\angle C = y°$. Also, $\angle C - \angle B = 9°$. Find the three angles.
9. In a cyclic quadrilateral $ABCD$, $\angle A = (2x + 4)°$, $\angle B = (y + 3)°$, $\angle C = (2y + 10)°$, $\angle D = (4x - 5)°$. Find the four angles.
10. Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test? **[NCERT]**
11. In a ΔABC, $\angle A = x°$, $\angle B = 3x°$ and $\angle C = y°$. If $3y - 5x = 30$, prove that the triangle is right angled.
12. The car hire charges in a city comprise of a fixed charges together with the charge for the distance covered. For a journey of 12 km, the charge paid is ₹ 89 and for a journey of 20 km, the charge paid is ₹ 145. What will a person have to pay for travelling a distance of 30 km? **[CBSE 2000]**

13. A part of monthly hostel charges in a college are fixed and the remaining depend on the number of days one has taken food in the mess. When a student A takes food for 20 days, he has to pay ₹ 1000 as hostel charges whereas a student B, who takes food for 26 days, pays ₹ 1180 as hostel charges. Find the fixed charge and the cost of food per day. **[NCERT, CBSE 2000]**

14. Half the perimeter of a garden, whose length is 4 more than its width is 36 m. Find the dimensions of the garden.

15. The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them. **[NCERT]**

16. 2 Women and 5 men can together finish a piece of embroidery in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the embroidery, and that taken by 1 man alone. **[NCERT]**

17. Meena went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes ₹ 50 and ₹ 100 she received. **[NCERT]**

LEVEL-2

18. There are two examination rooms A and B. If 10 candidates are sent from A to B, the number of students in each room is same. If 20 candidates are sent from B to A, the number of students in A is double the number of students in B. Find the number of students in each room. **[NCERT EXEMPLAR]**

19. A railway half ticket costs half the full fare and the reservation charge is the same on half ticket as on full ticket. One reserved first class ticket from Mumbai to Ahmedabad costs ₹ 216 and one full and one half reserved first class tickets cost ₹ 327. What is the basic first class full fare and what is the reservation charge? **[NCERT EXEMPLAR]**

20. A wizard having powers of mystic in candations and magical medicines seeing a cock, fight going on, spoke privately to both the owners of cocks. To one he said; if your bird wins, than you give me your stake-money, but if you do not win, I shall give you two third of that'. Going to the other, he promised in the same way to give three fourths. From both of them his gain would be only 12 gold coins. Find the stake of money each of the cock-owners have.

21. The students of a class are made to stand in rows. If 3 students are extra in a row, there would be 1 row less. If 3 students are less in a row there would be 2 rows more. Find the number of students in the class.

22. One says, "give me hundred, friend! I shall then become twice as rich as you" The other replies, "If you give me ten, I shall be six times as rich as you". Tell me what is the amount of their respective capital? **[NCERT]**

23. A shopkeeper sells a saree at 8% profit and a sweater at 10% discount, thereby getting a sum of ₹ 1008. If she had sold the saree at 10% profit and the sweater at 8% disciount, she would have got ₹ 1028. Find the cost price of the saree and the list price (price before discount) of the sweater. **[NCERT EXEMPLAR]**

24. In a competitive examination, one mark is awarded for each correct answer while 1/2 mark is deducted for every wrong answer. Jayanti answered 120 questions and got 90 marks. How many questions did she answer correctly. **[NCERT EXEMPLAR]**

25. A shopkeeper gives books on rent for reading. She takes a fixed charge for the first two days, and an additional charge for each day thereafter. Latika paid ₹ 22 for a book kept for 6 days, while Anand paid ₹ 16 for the book kept for four days. Find the fixed charges and charge for each extraday. [NCERT EXEMPLAR]

ANSWERS

1. 253 sq. units
2. Length = 28 m, Breadth = 15 m
3. Length = 28 m, Breadth = 19 m
4. X's income = ₹ 6000, Y's income = ₹ 5250
5. A : ₹ 62, B : ₹ 34
6. $\angle A = 120°, \angle B = 70°, \angle C = 60°, \angle D = 110°$
7. Man: 15 days Boy : 60 days
8. $A = 25°; B = 73°; C = 82°$
9. $A = 70°, B = 53°, C = 110°, D = 127°$
10. 20
12. ₹ 215
13. Fixed charge = ₹ 400; Cost of food per day = ₹ 30
14. Length = 20 m, Width = 16 m
15. 99°, 81°
16. 36 days, 18 days
17. 10, 15
18. 100, 80
19. Fare = ₹ 210, Reservation charge = ₹ 6
20. 42, 40 Gold Coins.
21. 36
22. ₹ 40, ₹ 170
23. ₹ 600, ₹ 400
24. 100
25. ₹ 10, ₹ 3

HINTS TO SELECTED PROBLEMS

10. Let x and y denote the number of right and wrong answers respectively, then

$$3x - y = 40,\ 2x - y = 25$$

19. Suppose basic first class full fare is ₹ x and reservation charge is ₹ y per ticket. Then, $x + y = 216$ and $x + y + (x/2) + y = 327$.

20. Let the stake money of first and second cock-owners be ₹ x and ₹ y respectively. Then, we have

$$x - \frac{3}{4}y = 12 \text{ and } y - \frac{2}{3}x = 12 \Rightarrow \quad 4x - 3y = 48 \text{ and } -2x + 3y = 36$$

22. Let the money with the first person be ₹ x and the money with the second person be ₹ y. Then,

$$x + 100 = 2(y - 100) \text{ and } y + 10 = 6(x - 10).$$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Write the value of k for which the system of equations $x + y - 4 = 0$ and $2x + ky - 3 = 0$ has no solution.

2. Write the value of k for which the system of equations

$$2x - y = 5$$
$$6x + ky = 15$$

has infinitely many solutions.

3. Write the value of k for which the system of equations $3x - 2y = 0$ and $kx + 5y = 0$ has infinitely many solutions.
4. Write the values of k for which the system of equations $x + ky = 0,\ 2x - y = 0$ has unique solution.
5. Write the set of values of a and b for which the following system of equations has infinitely many solutions.
$$2x + 3y = 7$$
$$2ax + (a + b)\, y = 28$$
6. For what value of k, the following pair of linear equations has infinitely many solutions?
$$10x + 5y - (k - 5) = 0$$
$$20x + 10y - k = 0$$
7. Write the number of solutions of the following pair of linear equations:
$$x + 2y - 8 = 0$$
$$2x + 4y = 16$$
[CBSE 2009]
8. Write the number of solutions of the following pair of linear equations:
$$x + 3y - 4 = 0$$
$$2x + 6y = 7$$

ANSWERS

1. $k = 2$ 2. $k = -3$ 3. $k = \frac{-15}{2}$ 4. $k \neq \frac{-1}{2}$ 5. $a = 4, b = 8$
6. $k = 10$ 7. Infinite 8. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of k for which the system of equations
$kx - y = 2$ and, $6x - 2y = 3$ has a unique solution, is
(a) $= 3$ (b) $\neq 3$ (c) $\neq 0$ (d) $= 0$
2. The value of k for which the system of equations $2x + 3y = 5$ and, $4x + ky = 10$ has infinite number of solutions, is
(a) 1 (b) 3 (c) 6 (d) 0
3. The value of k for which the system of equations $x + 2y - 3 = 0$ and $5x + ky + 7 = 0$ has no solution, is
(a) 10 (b) 6 (c) 3 (d) 1
4. The value of k for which the system of equations $3x + 5y = 0$ and $kx + 10y = 0$ has a non-zero solution, is
(a) 0 (b) 2 (c) 6 (d) 8
5. If the system of equations $2x + 3y = 7$ and, $(a + b)\, x + (2a - b)\, y = 21$ has infinitely many solutions, then
(a) $a = 1, b = 5$ (b) $a = 5, b = 1$ (c) $a = -1, b = 5$ (d) $a = 5, b = -1$
6. If the system of equations $3x + y = 1$ and, $(2k - 1)\, x + (k - 1)\, y = 2k + 1$ is inconsistent, then $k =$
(a) 1 (b) 0 (c) -1 (d) 2

7. If $am \neq bl$, then the system of equations $ax + by = c$ and, $lx + my = n$

(a) has a unique solution (b) has no solution
(c) has infinitely many solutions (d) may or may not have a solution.

8. If the system of equations

$$2x + 3y = 7$$
$$2ax + (a + b)y = 28$$

has infinitely many solutions, then

(a) $a = 2b$ (b) $b = 2a$ (c) $a + 2b = 0$ (d) $2a + b = 0$

9. The value of k for which the system of equations

$$x + 2y = 5$$
$$3x + ky + 15 = 0$$

has no solution is

(a) 6 (b) −6 (c) 3/2 (d) none of these

10. If $2x - 3y = 7$ and $(a + b)x - (a + b - 3)y = 4a + b$ represent coincident lines, then a and b satisfy the equation

(a) $a + 5b = 0$ (b) $5a + b = 0$ (c) $a - 5b = 0$ (d) $5a - b = 0$

11. If a pair of linear equations in two variables is consistent, then the lines represented by two equations are

(a) intersecting (b) parallel
(c) always coincident (d) intersecting or coincident

12. The area of the triangle formed by the line $\frac{x}{a} + \frac{y}{b} = 1$ with the coordinate axes is

(a) ab (b) $2ab$ (c) $\frac{1}{2}ab$ (d) $\frac{1}{4}ab$

13. The area of the triangle formed by the lines $y = x$, $x = 6$ and $y = 0$ is

(a) 36 sq. units (b) 18 sq. units (c) 9 sq. units (d) 72 sq. units

14. If the system of equations $2x + 3y = 5$, $4x + ky = 10$ has infinitely many solutions, then $k =$

(a) 1 (b) $\frac{1}{2}$ (c) 3 (d) 6

15. If the system of equations $kx - 5y = 2$, $6x + 2y = 7$ has no solution, then $k =$

(a) −10 (b) −5 (c) −6 (d) −15

16. The area of the triangle formed by the lines $x = 3$, $y = 4$ and $x = y$ is

(a) 1/2 sq. unit (b) 1 sq. unit (c) 2 sq. unit (d) None of these

17. The area of the triangle formed by the lines $2x + 3y = 12$, $x - y - 1 = 0$ and $x = 0$ (as shown in Fig. 3.24), is

(a) 7 sq. units (b) 7.5 sq. units (c) 6.5 sq. units (d) 6 sq. units

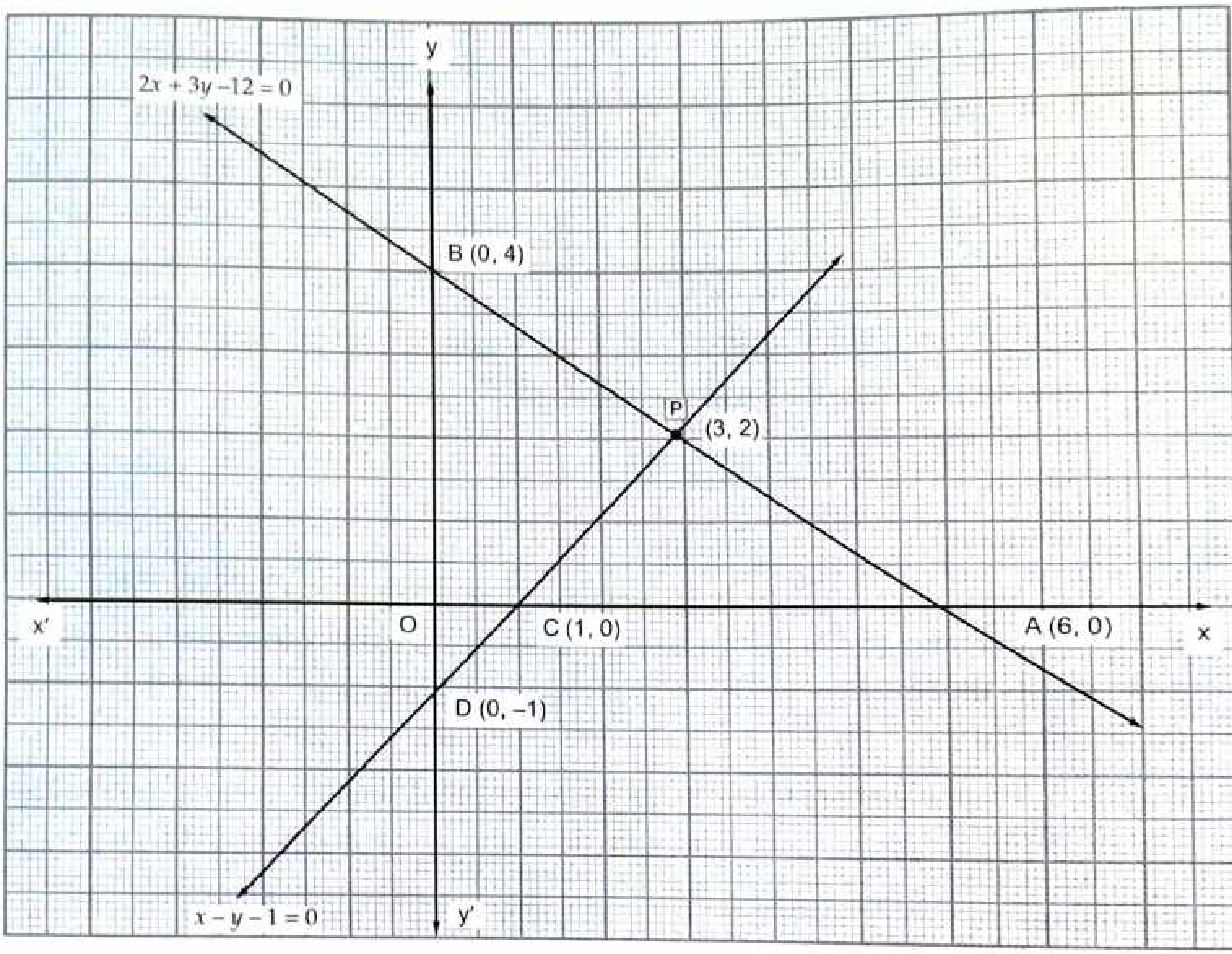


Fig. 3.24

18. The sum of the digits of a two digit number is 9. If 27 is added to it, the digits of the number get reversed. The number is

(a) 25 (b) 72 (c) 63 (d) 36

19. If $x = a, y = b$ is the solution of the systems of equations $x - y = 2$ and $x + y = 4$, then the values of a and b are, respectively

(a) 3 and 1 (b) 3 and 5 (c) 5 and 3 (d) – 1 and – 3

20. For what value k, do the equations $3x - y + 8 = 0$ and $6x - ky + 16 = 0$ represent coincident lines?

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 2 (d) – 2

21. Aruna has only ₹ 1 and ₹ 2 coins with her. If the total number of coins that she has is 50 and the amount of money with her is ₹ 75, then the number of ₹ 1 and ₹ 2 coins are, respectively

(a) 35 and 15 (b) 35 and 20 (c) 15 and 35 (d) 25 and 25

ANSWERS

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (d)	**7.** (a)
8. (b)	**9.** (a)	**10.** (c)	**11.** (d)	**12.** (c)	**13.** (b)	**14.** (d)
15. (d)	**16.** (a)	**17.** (b)	**18.** (d)	**19.** (a)	**20.** (c)	**21.** (b)

SUMMARY

1. A pair of linear equations in two variables x and y can be represented algebraically as follows:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0,$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers such that $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$.

2. Graphically or geometrically a pair of linear equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x = b_2y + c_2 = 0$$

in two variables represents a pair of straight lines which are

(i) intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (ii) parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

3. A pair of linear equations in two variables can be solved by the:
   (i) Graphical method (ii) Algebraic method.

4. To solve a pair of linear equations in two variables by Graphical method, we first draw the lines represented by them.
   (i) If the pair of lines intersect at a point, then we say that the pair is consistent and the coordinates of the point provide us the unique solution.
   (ii) If the pair of lines are parallel, then the pair has no solution and is called inconsistent pair of equations.
   (iii) If the pair of lines are coincident, then it has infinitely many solutions – each point on the line being of solution. In this case, we say that the pair of linear equations is consistent with infinitely many solutions.

5. To solve a pair of linear equations in two variables algebraically, we have following methods:
   (i) Substitution method. (ii) Elimination method.
   (iii) Cross-multiplication method.

6. If $a_1x + b_1y + c_1 = 0$

$$a_2x + b_2y + c_2 = 0$$

is a pair of linear equations in two variables x and y such that

(i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the pair of linear equations is consistent with a unique solution.

(ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of linear equations is inconsistent.

(iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the pair of linear equations is consistent with infinitely many solutions.

CHAPTER 4

QUADRATIC EQUATIONS

4.1 INTRODUCTION

In chapter 2, we have learnt about polynomials and their zeros. We have also learnt about graphical representation of linear, quadratic and cubic polynomials. When a polynomial $f(x)$ is equated to zero, we get an equation which is known as a polynomial equation. If $f(x)$ is a linear polynomial, then $f(x) = 0$ is called a linear equation. For example, $3x - 2 = 0$, $4t + \frac{3}{5} = 0$ etc. are linear equations. In earlier classes, we have learnt about the method of solving a linear equation. If $f(x)$ is a quadratic polynomial i.e., $f(x) = ax^2 + bx + c$, $a \neq 0$. Then, $f(x) = 0$ i.e., $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation. Such equations arise in many real life situations. In this chapter, we will learn about quadratic equations and various ways of finding their zeros or roots. In the end of the chapter, we will also discuss some applications of quadratic equations in daily life situations.

4.2 QUADRATIC EQUATION

QUADRATIC EQUATION *If $p(x)$ is a quadratic polynomial, then $p(x) = 0$ is called a quadratic equation.*

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

ROOTS OF A QUADRATIC EQUATION *Let $p(x) = 0$ be a quadratic equation, then the zeros of the polynomial $p(x)$ are called the roots of the equation $p(x) = 0$.*

Thus, $x = \alpha$ is a roots of $p(x) = 0$ if and only if $p(\alpha) = 0$.

As we have seen in section 2.2 that a quadratic polynomial may or may not have real zeros. In case a quadratic polynomial has real zeros, it can have at most two zeros. It follows from this that a quadratic equation can have at most two real roots.

Finding the roots of a quadratic equation is known as solving the quadratic equation.

Various concepts discussed so far are illustrated by the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON DETERMINING WHETHER A GIVEW EQUATION IS QUADRATIC OR NOT**

EXAMPLE 1 *Which of the following are quadratic equations?*

(i) $x^2 - 6x + 4 = 0$ (ii) $2x^2 - 7x = 0$

(iii) $x + \frac{3}{x} = x^2$ (iv) $x^2 + \frac{1}{x^2} = 2$

(v) $x^2 + 2\sqrt{x} - 3 = 0$ (vi) $3x^2 - 4x + 2 = 2x^2 - 2x + 4$

SOLUTION (i) Let $p(x) = x^2 - 6x + 4$

Clearly, $p(x) = x^2 - 6x + 4$ is a quadratic polynomial. Therefore, $x^2 - 6x + 4 = 0$ is a quadratic equation.

(ii) Clearly, $2x^2 - 7x$ is a quadratic polynomial. So, the given equation is a quadratic equation.

(iii) We have,

$$x + \frac{3}{x} = x^2 \Rightarrow \frac{x^2 + 3}{x} = x^2 \Rightarrow x^2 + 3 = x^3 \Rightarrow x^3 - x^2 - 3 = 0$$

Clearly, $x^3 - x^2 - 3$, being a polynomial of degree 3, is not a quadratic polynomial. So, the given equation is not a quadratic equation.

(iv) We have,

$$x^2 + \frac{1}{x^2} = 2 \Rightarrow \frac{x^4 + 1}{x^2} = 2 \Rightarrow x^4 - 2x^2 + 1 = 0$$

Clearly, $x^4 - 2x^2 + 1$ is not a quadratic polynomial. So, the given equation is not a quadratic equation.

(v) Clearly, $x^2 + 2\sqrt{x} - 3$ is not a quadratic polynomial because it contains a term involving $x^{1/2}$, where 1/2 is not an integer. So, the given equation is not a quadratic equation.

(vi) We have,

$$3x^2 - 4x + 2 = 2x^2 - 2x + 4 \quad \Rightarrow \quad x^2 - 2x - 2 = 0$$

Clearly, $x^2 - 2x - 2$ is a quadratic polynomial. So, the given equation is a quadratic equation.

Type II ON DETERMINING WHETHER THE GIVEN VALUES ARE SOLUTIONS OF THE GIVEN EQUATION OR NOT

EXAMPLE 2 *In each of the following determine whether the given values are solution of the given equation or not:*

(i) $3x^2 - 2x - 1 = 0, x = 1$ (ii) $6x^2 - x - 2 = 0, x = -1/2, x = 2/3$

(iii) $x^2 - x + 1 = 0, x = 1, x = -1$ (iv) $x^2 + \sqrt{2}x - 4 = 0, x = \sqrt{2}, x = -2\sqrt{2}$

SOLUTION (i) Substituting $x = 1$ on the LHS of the given equation, we get

$$\text{LHS} = 3 \times 1^2 - 2 \times 1 - 1 = 0 = \text{RHS}$$

So, $x = 1$ is a solution of the given equation.

(ii) Substituting $x = -\frac{1}{2}$ in the LHS of the given equation, we get

$$\text{LHS} = 6 \times \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 2 = \frac{6}{4} + \frac{1}{2} - 2 = 0 = \text{RHS}$$

So, $x = \frac{1}{2}$ is a solution of the given equation.

For $x = \frac{2}{3}$, we have

$$\text{LHS} = 6 \times \left(\frac{2}{3}\right)^2 - \frac{2}{3} - 2 = 0 = \text{RHS}$$

So, $x = \frac{2}{3}$ is also a solution of the given equation.

(iii) Substituting $x = 1$ on the LHS of the given equation, we get

$$\text{LHS} = 1^2 - 1 + 1 = 1 \neq \text{RHS}$$

So, $x = 1$ is not a solution of the given equation.
Similarly, $x = -1$ is not a solution of the given equation.

(iv) Substituting $x = \sqrt{2}$ on the LHS of the given equation, we get

$$\text{LHS} = (\sqrt{2})^2 + \sqrt{2} \times \sqrt{2} - 4 = 2 + 2 - 4 = 0 = \text{RHS}$$

So, $x = \sqrt{2}$ is solution of the given equation.

Substituting $x = -2\sqrt{2}$ on the LHS of the given equation, we get

$$\text{LHS} = (-2\sqrt{2})^2 + \sqrt{2} \times (-2\sqrt{2}) - 4 = 8 - 4 - 4 = 0 = \text{RHS}$$

So, $x = -2\sqrt{2}$ is also a solution of the given equation.

Type III **ON DETERMINING AN UNKNOWN INVOLVED IN A QUADRATIC EQUATION WHEN ITS ROOT(S) IS (ARE) GIVEN**

EXAMPLE 3 *In each of the following, determine the value of k for which the given value is a solution of the equation:*

(i) $kx^2 + 2x - 3 = 0, x = 2$ (ii) $3x^2 + 2kx - 3 = 0, x = -\frac{1}{2}$

(iii) $x^2 + 2ax - k = 0, x = -a$

SOLUTION (i) Since $x = 2$ is a root of the given equation. Therefore, it satisfies the equation

i.e. $k(2)^2 + 2 \times 2 - 3 = 0 \Rightarrow 4k + 4 - 3 = 0 \Rightarrow k = -1/4$

(ii) Since $x = -1/2$ is a root of the given equation. So, it satisfies the equation

i.e. $3\left(-\frac{1}{2}\right)^2 + 2k\left(-\frac{1}{2}\right) - 3 = 0 \Rightarrow \frac{3}{4} - k - 3 = 0 \Rightarrow k = -\frac{9}{4}$

(iii) Since $x = -a$ is a root of the equation $x^2 + 2ax - k = 0$

$\therefore$ $a^2 + 2a \times -a - k = 0 \Rightarrow k = -a^2$

EXAMPLE 4 *If one root of the quadratic equation* $2x^2 + kx - 6 = 0$ *is 2, find the value of k. Also, find the other root.* [CBSE 2002]

SOLUTION Since $x = 2$ is a root of the equation $2x^2 + kx - 6 = 0$.

$\therefore$ $2 \times 2^2 + 2k - 6 = 0 \Rightarrow 8 + 2k - 6 = 0 \Rightarrow 2k + 2 = 0 \Rightarrow k = -1$

Putting $k = -1$ in the equation $2x^2 + kx - 6 = 0$, we get

$$2x^2 - x - 6 = 0$$
$$\Rightarrow 2x^2 - 4x + 3x - 6 = 0$$
$$\Rightarrow 2x(x-2) + 3(x-2) = 0$$
$$\Rightarrow (x-2)(2x+3) = 0$$
$$\Rightarrow x - 2 = 0, 2x + 3 = 0 \Rightarrow x = 2, x = -3/2$$

Hence, the other root is $-3/2$.

LEVEL-2

EXAMPLE 5 *If $x = 2$ and $x = 3$ are roots of the equation $3x^2 - 2kx + 2m = 0$, find the value of k and m.*

SOLUTION It is given that $x = 2$ and $x = 3$ are roots of the equation $3x^2 - 2kx + 2m = 0$.

$\therefore$ $3 \times 2^2 - 2k \times 2 + 2m = 0$ and $3 \times 3^2 - 2k \times 3 + 2m = 0$

$\Rightarrow$ $12 - 4k + 2m = 0$ and $27 - 6k + 2m = 0$

$\Rightarrow$ $12 = 4k - 2m$ and $27 = 6k - 2m$

Solving these two equations, we get $k = \frac{15}{2}$ and $m = 9$

EXERCISE 4.1

LEVEL-1

1. Which of the following are quadratic equations?

(i) $x^2 + 6x - 4 = 0$
(ii) $\sqrt{3}x^2 - 2x + \frac{1}{2} = 0$
(iii) $x^2 + \frac{1}{x^2} = 5$
(iv) $x - \frac{3}{x} = x^2$
(v) $2x^2 - \sqrt{3x} + 9 = 0$
(vi) $x^2 - 2x - \sqrt{x} - 5 = 0$
(vii) $3x^2 - 5x + 9 = x^2 - 7x + 3$
(viii) $x + \frac{1}{x} = 1$
(ix) $x^2 - 3x = 0$
(x) $\left(x + \frac{1}{x}\right)^2 = 3\left(x + \frac{1}{x}\right) + 4$
(xi) $(2x + 1)(3x + 2) = 6(x - 1)(x - 2)$
(xii) $x + \frac{1}{x} = x^2, x \neq 0$
(xiii) $16x^2 - 3 = (2x + 5)(5x - 3)$
(xiv) $(x + 2)^3 = x^3 - 4$
(xv) $x(x + 1) + 8 = (x + 2)(x - 2)$

2. In each of the following, determine whether the given values are solutions of the given equation or not:

(i) $x^2 - 3x + 2 = 0, x = 2, x = -1$
(ii) $x^2 + x + 1 = 0, x = 0, x = 1$
(iii) $x^2 - 3\sqrt{3}x + 6 = 0, x = \sqrt{3}, x = -2\sqrt{3}$
(iv) $x + \frac{1}{x} = \frac{13}{6}, x = \frac{5}{6}, x = \frac{4}{3}$
(v) $2x^2 - x + 9 = x^2 + 4x + 3, x = 2, x = 3$
(vi) $x^2 - \sqrt{2}x - 4 = 0, x = -\sqrt{2}, x = -2\sqrt{2}$
(vii) $a^2x^2 - 3abx + 2b^2 = 0, x = a/b, x = b/a$

3. In each of the following, find the value of k for which the given value is a solution of the given equation:

(i) $7x^2 + kx - 3 = 0, x = 2/3$
(ii) $x^2 - x(a + b) + k = 0, x = a$
(iii) $kx^2 + \sqrt{2}x - 4 = 0, x = \sqrt{2}$
(iv) $x^2 + 3ax + k = 0, x = -a$

4. Determine, if 3 is a root of the equation given below:

$$\sqrt{x^2-4x+3}+\sqrt{x^2-9}=\sqrt{4x^2-14x+16}$$

LEVEL-2

5. If $x = 2/3$ and $x = -3$ are the roots of the equation $ax^2 + 7x + b = 0$, find the values of a and b. [CBSE 2016]

ANSWERS

1. (i), (ii), (vii), (viii), (ix), (xiii), (xiv)
2. (i) $x = 2$ is a solution but $x = -1$ is not a solution
   (ii) $x = 0$ and $x = 1$ are not solutions
   (iii) $x = \sqrt{3}$ is a solution but $x = -2\sqrt{3}$ is not a solution
   (iv) $x = \frac{5}{6}$ and $x = \frac{4}{3}$ are not solutions.
   (v) $x = 2$ and $x = 3$ are solutions
   (vi) $x = -\sqrt{2}$ is a solution but $x = -2\sqrt{2}$ is not a solution
   (vii) $x = \frac{b}{a}$ is a solution but $x = \frac{a}{b}$ is not a solution
3. (i) $k = -\frac{1}{6}$ (ii) $k = ab$ (iii) $k = 1$ (iv) $k = 2a^2$
4. $x = 3$ is not a root of the given equation
5. $a = 3, b = -6$

4.3 FORMULATION OF QUADRATIC EQUATIONS

Quadratic equations arise in several situations in the world around us and in different fields of mathematics. Following examples will illustrate the formulation of quadratic equations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The product of two consecutive positive integers is 240. Formulate the quadratic equation whose roots are these integers.*

SOLUTION Let two consecutive positive integers be x and $x + 1$. Then, their product is $x(x+1)$.

It is given that the product is 240.

$\therefore \quad x(x+1) = 240 \Rightarrow x^2 + x - 240 = 0$

This is the required quadratic equation.

EXAMPLE 2 *The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one more than twice its breadth. Formulate the quadratic equation to determine the length and breadth of the plot.* [NCERT]

SOLUTION Let the breadth of the plot be x metres.

It is given that the length of the plot is one more than twice its breadth.

$\therefore$ Length $= (2x + 1)$ metres

Now, Area of the plot $= 528\ m^2$

$\Rightarrow$ Length $\times$ Breadth $= 528\ m^2$

$\Rightarrow (2x+1) \times x = 528 \Rightarrow 2x^2 + x - 528 = 0$

This is the required quadratic equation.

EXAMPLE 3 *A two digit number is such that the product of the digits is 12. When 36 is added to the number the digits interchange their places. Formulate the quadratic equation whose root(s) is (are) digit(s) of the number.*

SOLUTION Let the ten's digit of the number be x.
It is given that the product of digits is 12.

$$\therefore \quad \text{Unit's digit} = \frac{12}{x}$$

$$\therefore \quad \text{Number} = 10x + \frac{12}{x}$$

If 36 is added to the number the digits interchange their places.

$$\therefore \quad 10x + \frac{12}{x} + 36 = 10 \times \frac{12}{x} + x$$

$$\Rightarrow \quad 10x + \frac{12}{x} + 36 = \frac{120}{x} + x$$

$$\Rightarrow \quad 9x - \frac{108}{x} + 36 = 0$$

$$\Rightarrow \quad 9x^2 - 108 + 36x = 0$$

$$\Rightarrow \quad x^2 + 4x - 12 = 0 \qquad \text{[Dividing throughout by 9]}$$

Hence, required quadratic equation is $x^2 + 4x - 12 = 0$.

EXAMPLE 4 *Rohan's mother is 26 years older than him. The product of their ages 3 years from now will be 360. Formulate the quadratic equation to find their ages.* [NCERT]

SOLUTION Let Rohan's present age be x years. Then, his mother's age is $(x + 26)$ years.
Rohan's age after 3 years = $(x + 3)$ years

After 3 years the age of Rohan's mother = $(x + 26 + 3)$ years = $(x + 29)$ years.

It is given that after 3 years from now, the product of Rohan's and his mother's ages will be 360 years.

$$\therefore \quad (x + 3)(x + 29) = 360 \Rightarrow x^2 + 32x - 273 = 0$$

This is the required equation.

LEVEL-2

EXAMPLE 5 *A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/hr less, then it would have taken 3 hours more to cover the same distance. Formulate the quadratic equation in terms of the speed of the train.* [NCERT]

SOLUTION Let the speed of the train be x km/hr. Then,

Time taken to travel a distance of 480 km $= \frac{480}{x}$ hr

Also,

Time taken by the train to travel a distance of 480 km with the speed $(x-8)$ km/hr $= \frac{480}{x-8}$ hr

It is given that if the speed had been 8 km/hr less, then the train would have taken 3 hours more to cover the same distance

$$\therefore \quad \frac{480}{x-8} = \frac{480}{x} + 3$$

$\Rightarrow \quad \frac{480}{x-8} - \frac{480}{x} = 3$

$\Rightarrow \quad \frac{480\,(x - x + 8)}{x\ (x-8)} = 3$

$\Rightarrow \quad \frac{480 \times 8}{x\,(x-8)} = 3$

$\Rightarrow \quad 3x\,(x-8) = 480 \times 8$

$\Rightarrow \quad x\,(x-8) = 160 \times 8 \ \Rightarrow\ x^2 - 8x - 1280 = 0,$ which is the required equation.

EXAMPLE 6 *Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, formulate the quadratic equation to find the sides of the two squares.*

SOLUTION Let the length of each side of a square be x metres. Then, its perimeter is $4x$.

It is given that the difference of the perimeters of two squares is 24 m.

$\therefore \quad$ Perimeter of second square $= 24 + 4x$ metres

$\Rightarrow \quad$ Length of each side of second square $= \frac{24 + 4x}{4}$ metres $= (6 + x)$ metres

It is given that the sum of the areas of two squares is 468 m^2.

$\therefore \quad x^2 + (6 + x)^2 = 468$

$\Rightarrow \quad x^2 + (36 + 12x + x^2) = 468$

$\Rightarrow \quad 2x^2 + 12x - 432 = 0$

$\Rightarrow \quad x^2 + 6x - 216 = 0$

This is the required equation.

EXAMPLE 7 *Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The larger takes 10 hours less than the smaller one to fill the tank separately. Formulate the quadratic equation to find the time in which each tap can separately fill the tank.* [CBSE 2016]

SOLUTION Suppose the smaller tap fills the tank in x hours. Then, the larger tap will fill the tank in $(x - 10)$ hours.

Since the smaller tap takes x hours to fill the tank.

$\therefore \quad$ Portion of the tank filled by the smaller tank in one hour $= \frac{1}{x}$

$\Rightarrow \quad$ Portion of the tank filled by the smaller tap in $9\frac{3}{8}$ hours i.e., in $\frac{75}{8}$ hours

$$= \frac{75}{8} \times \frac{1}{x} = \frac{75}{8x}$$

Similarly, we have,

Portion of the tank filled by the larger tap in $\frac{75}{8}$ hours $= \left(\frac{75}{8} \times \frac{1}{x-10}\right)$

$$= \frac{75}{8\,(x-10)}$$

It is given that the two taps fill the tank in $\frac{75}{8}$ hours.

$$\therefore \qquad \frac{75}{8x} + \frac{75}{8(x-10)} = 1$$

$$\Rightarrow \qquad \frac{1}{x} + \frac{1}{x-10} = \frac{8}{75}$$

$$\Rightarrow \qquad \frac{x-10+x}{x(x-10)} = \frac{8}{75}$$

$$\Rightarrow \qquad \frac{2x-10}{x^2-10x} = \frac{8}{75}$$

$$\Rightarrow \qquad (2x-10) \times 75 = 8(x^2 - 10x)$$

$$\Rightarrow \qquad 150x - 750 = 8x^2 - 80x \Rightarrow 8x^2 - 230x + 750 = 0 \Rightarrow 4x^2 - 115x + 375 = 0$$

This is the required quadratic equation.

EXERCISE 4.2

LEVEL-1

1. The product of two consecutive positive integers is 306. Form the quadratic equation to find the integers, if x denotes the smaller integer. [NCERT]
2. John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 128. Form the quadratic equation to find how many marbles they had to start with, if John had x marbles.
3. A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of articles produced in a day. On a particular day, the total cost of production was ₹ 750. If x denotes the number of toys produced that day, form the quadratic equation fo find x. [NCERT]
4. The height of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, form the quadratic equation to find the base of the triangle. [NCERT]

LEVEL-2

5. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore. If the average speed of the express train is 11 km/hr more than that of the passenger train, form the quadratic equation to find the average speed of express train.
6. A train travels 360 km at a uniform speed. If the speed had been 5 km/hr more, it would have taken 1 hour less for the same journey. Form the quadratic equation to find the speed of the train.

ANSWERS

1. $x^2 + x - 306 = 0$	2. $x^2 - 45x + 324 = 0$	3. $x^2 - 55x + 750 = 0$
4. $x^2 - 7x - 60 = 0$	5. $x^2 + 11x - 1452 = 0$	6. $x^2 + 5x - 1800 = 0$

4.4 SOLUTION OF A QUADRATIC EQUATION BY FACTORIZATION METHOD

In earlier class, we have learnt how to factorize quadratic and other simple polynomials. In this section, we will apply the method of factorization to solve simple quadratic equations.

Let the quadratic equation be $ax^2 + bx + c = 0, a \neq 0$.

Let the quadratic polynomial $ax^2 + bx + c$ be expressible as the product of two linear factors, say $(px + q)$ and $(rx + s)$, where p, q, r, s are real numbers such that $p \neq 0$ and $r \neq 0$. Then,

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad (px + q)(rx + s) = 0$$

$$\Rightarrow \quad px + q = 0 \text{ or, } rx + s = 0$$

Solving these linear equations, we get the possible roots of the given quadratic equation as

$$x = -\frac{p}{q} \text{ and } x = -\frac{s}{r}$$

Following examples will illustrate the above procedure for solving quadratic equations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following quadratic equations by factorization:*

(i) $x^2 + 6x + 5 = 0$ (ii) $8x^2 - 22x - 21 = 0$ (iii) $9x^2 - 3x - 2 = 0$

SOLUTION (i) We have,

$$x^2 + 6x + 5 = 0$$

$$\Rightarrow \quad x^2 + 5x + x + 5 = 0$$

$$\Rightarrow \quad x(x + 5) + (x + 5) = 0$$

$$\Rightarrow \quad (x + 5)(x + 1) = 0 \Rightarrow x + 5 = 0 \text{ or, } x + 1 = 0 \Rightarrow x = -5 \text{ or, } x = -1$$

Thus, $x = -5$ and $x = -1$ are two roots of the equation $x^2 + 6x + 5 = 0$

(ii) We have,

$$8x^2 - 22x - 21 = 0$$

$$\Rightarrow \quad 8x^2 - 28x + 6x - 21 = 0$$

$$\Rightarrow \quad 4x(2x - 7) + 3(2x - 7) = 0$$

$$\Rightarrow \quad (2x - 7)(4x + 3) = 0 \Rightarrow 2x - 7 = 0 \text{ or, } 4x + 3 = 0 \Rightarrow x = \frac{7}{2} \text{ or, } x = -\frac{3}{4}$$

Thus, $x = \frac{7}{2}$ and $x = -\frac{3}{4}$ are two roots of the equation $8x^2 - 22x - 21 = 0$

(iii) We have,

$$9x^2 - 3x - 2 = 0$$

$$\Rightarrow \quad 9x^2 - 6x + 3x - 2 = 0$$

$$\Rightarrow \quad 3x(3x - 2) + (3x - 2) = 0$$

$$\Rightarrow \quad (3x - 2)(3x + 1) = 0$$

$$\Rightarrow \quad 3x - 2 = 0 \text{ or, } 3x + 1 = 0 \Rightarrow x = \frac{2}{3} \text{ or, } x = -\frac{1}{3}$$

Thus, $x = \frac{2}{3}$ and $x = -\frac{1}{3}$ are two roots of the equation $9x^2 - 3x - 2 = 0$

EXAMPLE 2 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 9 = 0$ (ii) $x^2 - 8x + 16 = 0$

SOLUTION (i) We have,

$$x^2 - 9 = 0$$

$$\Rightarrow \quad (x-3)(x+3) = 0$$

$$\Rightarrow \quad x - 3 = 0 \text{ or, } x + 3 = 0 \Rightarrow x = 3 \text{ or, } x = -3 \Rightarrow x = \pm 3$$

Thus, $x = 3$ and $x = -3$ are roots of the given equation.

(ii) We have,

$$x^2 - 8x + 16 = 0 \Rightarrow (x-4)^2 = 0 \Rightarrow x = 4, x = 4$$

Thus, both the roots of the given equation are equal and are equal to 4.

EXAMPLE 3 *Solve the following quadratic equations by factorization method:*

(i) $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{34}{15}, x \neq 0, x \neq -1$ (ii) $\dfrac{x+3}{x-2} - \dfrac{1-x}{x} = \dfrac{17}{4}$

(iii) $\dfrac{1}{x-2} + \dfrac{2}{x-1} = \dfrac{6}{x}$ [CBSE 2013]

SOLUTION (i) We have,

$$\frac{x}{x+1} + \frac{x+1}{x} = \frac{34}{15}$$

$$\Rightarrow \quad \frac{x^2 + (x+1)^2}{x(x+1)} = \frac{34}{15}$$

$$\Rightarrow \quad \frac{x^2 + x^2 + 2x + 1}{x^2 + x} = \frac{34}{15}$$

$$\Rightarrow \quad 34x^2 + 34x = 15x^2 + 15x^2 + 30x + 15$$

$$\Rightarrow \quad 4x^2 + 4x - 15 = 0$$

$$\Rightarrow \quad 4x^2 + 10x - 6x - 15 = 0$$

$$\Rightarrow \quad 2x(2x+5) - 3(2x+5) = 0$$

$$\Rightarrow \quad (2x-3)(2x+5) = 0 \Rightarrow 2x - 3 = 0 \text{ or, } 2x + 5 = 0 \Rightarrow x = \frac{3}{2} \text{ or, } x = -\frac{5}{2}.$$

(ii) We have,

$$\frac{x+3}{x-2} - \frac{1-x}{x} = \frac{17}{4}$$

$$\Rightarrow \quad \frac{x(x+3) - (1-x)(x-2)}{x(x-2)} = \frac{17}{4}$$

$$\Rightarrow \quad \frac{x^2 + 3x - (x - 2 - x^2 + 2x)}{x^2 - 2x} = \frac{17}{4}$$

$$\frac{2x^2 + 2}{x^2 - 2x} = \frac{17}{4}$$

$$\Rightarrow \quad 8x^2 + 8 = 17x^2 - 34x$$

$\Rightarrow \quad 9x^2 - 34x - 8 = 0$

$\Rightarrow \quad 9x^2 - 36x + 2x - 8 = 0$

$\Rightarrow \quad 9x(x-4) + 2(x-4) = 0$

$\Rightarrow \quad (x-4)(9x+2) = 0 \Rightarrow x - 4 = 0 \text{ or, } 9x + 2 = 0 \Rightarrow x = 4 \text{ or, } x = -\frac{2}{9}$

(iii) We have,

$$\frac{1}{x-2} + \frac{2}{x-1} = \frac{6}{x}$$

$$\Rightarrow \quad \frac{(x-1)+2(x-2)}{(x-2)(x-1)} = \frac{6}{x}$$

$$\Rightarrow \quad \frac{3x-5}{x^2-3x+2} = \frac{6}{x}$$

$\Rightarrow \quad 3x^2 - 5x = 6x^2 - 18x + 12$

$\Rightarrow \quad 3x^2 - 13x + 12 = 0$

$\Rightarrow \quad 3x^2 - 9x - 4x + 12 = 0$

$\Rightarrow \quad 3x(x-3) - 4(x-3) = 0$

$\Rightarrow \quad (x-3)(3x-4) = 0 \Rightarrow x - 3 = 0 \text{ or, } 3x - 4 = 0 \Rightarrow x = 3 \text{ or, } x = \frac{4}{3}$

EXAMPLE 4 *Solve the following quadratic equations by factorization method:*

(i) $\frac{4}{x} - 3 = \frac{5}{2x+3}$ $x \neq 0, \frac{-3}{2}$ [CBSE 2006C, 2014]

(ii) $\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(x-3)(2x+3)} = 0, x \neq 3, \frac{-3}{2}$ [CBSE 2016]

SOLUTION (i) We have,

$$\frac{4}{x} - 3 = \frac{5}{2x+3}$$

$$\Rightarrow \quad \frac{4-3x}{x} = \frac{5}{2x+3}$$

$\Rightarrow \quad (4-3x)(2x+3) = 5x$

$\Rightarrow \quad 12 - x - 6x^2 = 5x$

$\Rightarrow \quad 6x^2 + 6x - 12 = 0$

$\Rightarrow \quad x^2 + x - 2 = 0$

$\Rightarrow \quad x^2 + 2x - x - 2 = 0$

$\Rightarrow \quad x(x+2) - (x+2) = 0$

$\Rightarrow \quad (x+2)(x-1) = 0 \Rightarrow x + 2 = 0 \text{ or, } x - 1 = 0 \Rightarrow x = -2 \text{ or, } x = 1$

(ii) Clearly, the given equation is valid if $x - 3 \neq 0$ and $2x + 3 \neq 0$ i.e. when $x \neq \frac{-3}{2}, 3$.

Now, $$\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(x-3)(2x+3)} = 0$$

$\Rightarrow \quad 2x(2x+3)+(x-3)+3x+9=0$ [Multiplying throughout by $(x-3)(2x+3)$]

$\Rightarrow \quad 4x^2+6x+x-3+3x+9=0$

$\Rightarrow \quad 4x^2+10x+6=0$

$\Rightarrow \quad 2x^2+5x+3=0$ $\left[\text{Multiplying through } \frac{1}{2}\right]$

$\Rightarrow \quad 2x^2+2x+3x+3=0$

$\Rightarrow \quad 2x(x+1)+3(x+1)=0$

$\Rightarrow \quad (2x+3)(x+1)=0$

$\Rightarrow \quad x+1=0$

$\Rightarrow \quad x=-1$ $[\because 2x+3 \neq 0]$

Hence, $x=-1$ is the only solution of the given equation.

ALITER $\quad \dfrac{2x}{x-3}+\dfrac{1}{2x+3}+\dfrac{2}{x-3}-\dfrac{1}{2x+3}=0$

$\Rightarrow \quad \dfrac{2(x+1)}{x-3}=0$

$\Rightarrow \quad x+1=0 \Rightarrow x=-1$

LEVEL-2

EXAMPLE 5 *Solve the following quadratic equations by factorization method:*

(i) $x^2+2\sqrt{2}x-6=0$ (ii) $\sqrt{3}x^2+10x+7\sqrt{3}=0$

SOLUTION (i) We have,

$x^2+2\sqrt{2}x-6=0$

$\Rightarrow \quad x^2+3\sqrt{2}x-\sqrt{2}x-6=0$

$\Rightarrow \quad x(x+3\sqrt{2})-\sqrt{2}(x+3\sqrt{2})=0$

$\Rightarrow \quad (x+3\sqrt{2})(x-\sqrt{2})=0 \Rightarrow x+3\sqrt{2}=0$ or, $x-\sqrt{2}=0 \Rightarrow x=-3\sqrt{2}$ or, $x=\sqrt{2}$

Thus, $x=-3\sqrt{2}$ and $x=\sqrt{2}$ are two roots of the given equation.

(ii) $\quad \sqrt{3}x^2+10x+7\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x^2+3x+7x+7\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x(x+\sqrt{3})+7(x+\sqrt{3})=0$

$\Rightarrow \quad (x+\sqrt{3})(\sqrt{3}x+7)=0 \Rightarrow x+\sqrt{3}=0$ or, $\sqrt{3}x+7=0$

$\Rightarrow \quad x=-\sqrt{3}$ or, $x=-7/\sqrt{3}$

Thus, $x=-\sqrt{3}$ and, $x=-\dfrac{7}{\sqrt{3}}$ are two roots of the given equation.

REMARK *In order to solve the quadratic equations in the following examples we may use the following algorithm:*

ALGORITHM

STEP I *Factorize the constant term of the given quadratic equation.*

STEP II Express the coefficient of middle term as the sum or difference of the factors obtained in step I. Clearly, the product of these two factors will be equal to the product of the coefficient of x^2 and constant term.

STEP III Split the middle term in two parts obtained in step II.

STEP IV Factorize the quadratic equation obtained in step III by grouping method.

EXAMPLE 6 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 2ax + a^2 - b^2 = 0$ (ii) $x^2 - 4ax + 4a^2 - b^2 = 0$

(iii) $4x^2 - 4ax + (a^2 - b^2) = 0$ [CBSE 2012] (iv) $4x^2 - 4a^2x + (a^4 - b^4) = 0$ [CBSE 2004, 2015]

SOLUTION (i) We have,

$$x^2 - 2ax + a^2 - b^2 = 0$$

Here, Factors of constant term $(a^2 - b^2)$ are $(a - b)$ and $(a + b)$.

Also, Coefficient of the middle term $= -2a = -\{(a - b) + (a + b)\}$

$\therefore$ $x^2 - 2ax + a^2 - b^2 = 0$

$\Rightarrow$ $x^2 - \{(a - b) + (a + b)\ x + (a - b)(a + b) = 0\}$

$\Rightarrow$ $x^2 - (a - b)\ x - (a + b)\ x + (a - b)(a + b) = 0$

$\Rightarrow$ $\{x^2 - (a - b)\ x\} - \{(a + b)\ x - (a - b)(a + b)\} = 0$

$\Rightarrow$ $x\{x - (a - b)\} - (a + b)\{x - (a - b)\} = 0$

$\Rightarrow$ $\{x - (a - b)\}\{x - (a + b)\} = 0$

$\Rightarrow$ $x - (a - b) = 0$ or, $x - (a + b) = 0 \Rightarrow x = a - b$ or, $x = a + b$

(ii) We have,

$$x^2 - 4ax + (4a^2 - b^2) = 0$$

Here, Constant term $= (4a^2 - b^2) = (2a - b)(2a + b)$

and, Coefficient of middle term $= -4a$

Also, Coefficient of middle term $= -\{(2a - b) + (2a + b)\}$

$\therefore$ $x^2 - 4ax + (4a^2 - b^2) = 0$

$\Rightarrow$ $x^2 - \{(2a - b) + (2a + b)\}\ x + (2a - b)(2a + b) = 0$

$\Rightarrow$ $x^2 - (2a - b)\ x - (2a + b)\ x + (2a - b)(2a + b) = 0$

$\Rightarrow$ $\{x^2 - (2a - b)\ x\} - \{(2a + b)\ x - (2a - b)(2a + b)\} = 0$

$\Rightarrow$ $x\{x - (2a - b)\} - (2a + b)\{x - (2a - b)\} = 0$

$\Rightarrow$ $\{x - (2a - b)\}\{x - (2a + b)\} = 0$

$\Rightarrow$ $x - (2a - b) = 0$ or, $x - (2a + b) = 0 \Rightarrow x = 2a - b$ or, $x = 2a + b$

(iii) We have,

$$4x^2 - 4ax + (a^2 - b^2) = 0$$

Here, Constant term $= (a^2 - b^2) = (a - b)(a + b)$

and, Coefficient of middle term $= -4a$

Also, Coefficient of the middle term $-4a = -\{2(a+b) + 2(a-b)\}$

$\therefore$ $4x^2 - 4ax + (a^2 - b^2) = 0$

$\Rightarrow$ $4x^2 - \{2(a+b) + 2(a-b)\}x + (a+b)(a-b) = 0$

$\Rightarrow$ $4x^2 - 2(a+b)x - 2(a-b)x + (a+b)(a-b) = 0$

$\Rightarrow$ $\{4x^2 - 2(a+b)x\} - \{2(a-b)x - (a+b)(a-b)\} = 0$

$\Rightarrow$ $2x\{2x - (a+b)\} - (a-b)\{2x - (a+b)\} = 0$

$\Rightarrow$ $\{2x - (a+b)\}\{2x - (a-b)\} = 0$

$\Rightarrow$ $\{2x - (a+b)\} = 0$ or, $\{2x - (a-b)\} = 0$

$\Rightarrow$ $2x = a + b$ or, $2x = a - b \Rightarrow x = \dfrac{a+b}{2}$ or, $x = \dfrac{a-b}{2}$

(iv) We have,

$$4x^2 - 4a^2x + (a^4 - b^4) = 0$$

Here, Constant term $= a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$

and, Coefficient of middle term $= -4a^2$

Also, Coefficient of the middle term $-4a^2 = -\{2(a^2 + b^2) + 2(a^2 - b^2)\}$

$\therefore$ $4x^2 - 4a^2x + (a^4 - b^4) = 0$

$\Rightarrow$ $4x^2 - \{2(a^2 + b^2) + 2(a^2 - b^2)\}x + (a^2 - b^2)(a^2 + b^2) = 0$

$\Rightarrow$ $4x^2 - 2(a^2 + b)x - 2(a^2 - b^2)x + (a^2 - b^2)(a^2 + b^2) = 0$

$\Rightarrow$ $\{4x^2 - 2(a^2 + b^2)x\} - \{2(a^2 - b^2)x - (a^2 - b^2)(a^2 + b^2)\} = 0$

$\Rightarrow$ $2x\{2x - (a^2 + b^2)\} - (a^2 - b^2)\{2x - (a^2 + b^2)\} = 0$

$\Rightarrow$ $\{2x - (a^2 + b^2)\}\{2x - (a^2 - b^2)\} = 0$

$\Rightarrow$ $2x - (a^2 + b^2) = 0$ or, $2x - (a^2 - b^2) = 0 \Rightarrow x = \dfrac{a^2 + b^2}{2}$ or, $x = \dfrac{a^2 - b^2}{2}$

EXAMPLE 7 *Solve the following quadratic equations by factorization method:*

(i) $4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$ [CBSE 2004]

(ii) $9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$ [CBSE 2004, 2009, 2016]

SOLUTION (i) We have,

$$4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$$

Here, Constant term $= a^2b^2 = a^2 \times b^2$

and, Coefficient of middle term $= -2(a^2 + b^2)$

$\therefore$ $4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$

$\Rightarrow$ $4x^2 - 2a^2x - 2b^2x + a^2b^2 = 0$

$\Rightarrow$ $(4x^2 - 2a^2x) - (2b^2x - a^2b^2) = 0$

$\Rightarrow \quad 2x(2x - a^2) - b^2(2x - a^2) = 0$

$\Rightarrow \quad (2x - a^2)(2x - b^2) = 0$

$\Rightarrow \quad (2x - a^2) = 0$ or, $(2x - b^2) = 0 \Rightarrow x = \frac{a^2}{2}$ or, $x = \frac{b^2}{2}$

(ii) We have,

$$9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$$

Here, Constant term $= 2a^2 + 5ab + 2b^2$

$= 2a^2 + 4ab + ab + 2b^2$

$= 2a(a + 2b) + b(a + 2b) = (2a + b)(a + 2b)$

and, Coefficient of middle term $= -9(a+b) = -3\{(2a+b) + (a+2b)$

$\therefore \quad 9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$

$\Rightarrow \quad 9x^2 - 3\{(2a+b) + (a+2b)\}x + (2a+b)(a+2b) = 0$

$\Rightarrow \quad 9x^2 - 3(2a+b)x - 3(a+2b)x + (2a+b)(a+2b) = 0$

$\Rightarrow \quad 3x\{3x - (2a+b)\} - (a+2b)\{3x - (2a+b)\} = 0$

$\Rightarrow \quad \{3x - (2a+b)\}\{3x - (a+2b)\} = 0$

$\Rightarrow \quad \{3x - (2a+b)\} = 0$ or, $\{3x - (a+2b)\} = 0 \Rightarrow x = \frac{2a+b}{3}$ or, $x = \frac{a+2b}{3}$

EXAMPLE 8 *Solve the following quadratic equations by factorization method:*

(i) $x^2 + \left(\frac{a}{a+b} + \frac{a+b}{a}\right)x + 1 = 0$ (ii) $x^2 + x - (a+1)(a+2) = 0$

(iii) $x^2 + 3x - (a^2 + a - 2) = 0$ (iv) $a^2b^2x^2 + b^2x - a^2x - 1 = 0$

[CBSE 2005]

SOLUTION (i) We have,

$$x^2 + \left(\frac{a}{a+b} + \frac{a+b}{a}\right)x + 1 = 0$$

$\Rightarrow \quad x^2 + \frac{ax}{a+b} + \frac{a+b}{a}x + \frac{a}{a+b} \times \frac{a+b}{a} = 0$

$\Rightarrow \quad x\left\{x + \frac{a}{a+b}\right\} + \frac{a+b}{a}\left\{x + \frac{a}{a+b}\right\} = 0$

$\Rightarrow \quad \left\{x + \frac{a}{a+b}\right\}\left\{x + \frac{a+b}{a}\right\} = 0$

$\Rightarrow \quad x + \frac{a}{a+b} = 0$ or, $x + \frac{a+b}{a} = 0 \Rightarrow x = -\frac{a}{a+b}$ or, $x = -\frac{a+b}{a}$

(ii) We have,

$$x^2 + x - (a+1)(a+2) = 0$$

$\Rightarrow$ $x^2 + x\{(a+2) - (a+1)\} - (a+1)(a+2) = 0$

$\Rightarrow$ $\{x^2 + x(a+2)\} - x(a+1) - (a+1)(a+2) = 0$

$\Rightarrow$ $x\{x + (a+2)\} - (a+1)\{x + (a+2)\} = 0$

$\Rightarrow$ $\{x + (a+2)\}\{x - (a+1)\} = 0$

$\Rightarrow$ $x + (a+2) = 0$ or, $x - (a+1) = 0 \Rightarrow x = -(a+2)$ or, $x = (a+1)$

(iii) We have,

$$x^2 + 3x - (a^2 + a - 2) = 0$$

$\Rightarrow$ $x^2 + 3x - (a+2)(a-1) = 0$

$\Rightarrow$ $x^2 + \{(a+2) - (a-1)\}x - (a+2)(a-1) = 0$

$\Rightarrow$ $\{x^2 + (a+2)x\} - (a-1)x - (a+2)(a-1) = 0$

$\Rightarrow$ $x\{x + (a+2)\} - (a-1)\{x + (a+2)\} = 0$

$\Rightarrow$ $\{x + (a+2)\}\{x - (a-1)\} = 0$

$\Rightarrow$ $x + (a+2) = 0$ or, $x - (a-1) = 0 \Rightarrow x = -(a+2)$ or, $x = a - 1$

(iv) We have,

$\Rightarrow$ $a^2b^2x^2 + b^2x - a^2x - 1 = 0$

$\Rightarrow$ $(a^2b^2x^2 + b^2x) - (a^2x + 1) = 0$

$\Rightarrow$ $(a^2x + 1)b^2x - (a^2x + 1) = 0$

$\Rightarrow$ $(a^2x + 1)(b^2x - 1) = 0$

$\Rightarrow$ $a^2x + 1 = 0, b^2x - 1 = 0$

$\Rightarrow$ $a^2x = -1, b^2x = 1 \Rightarrow x = -\dfrac{1}{a^2}, x = \dfrac{1}{b^2}$

EXAMPLE 9 *Solve the following quadratic equations by factorization method:*

$$\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}, a + b \neq 0$$

[CBSE 2005]

SOLUTION We have,

$$\frac{1}{a+b+x} - \frac{1}{x} = \frac{1}{a} + \frac{1}{b}$$

$\Rightarrow$ $\dfrac{x - (a+b+x)}{x(a+b+x)} = \dfrac{a+b}{ab}$

$\Rightarrow$ $\dfrac{-(a+b)}{x(a+b+x)} = \dfrac{a+b}{ab}$

$\Rightarrow$ $-ab(a+b) = (a+b)x(a+b+x)$

$\Rightarrow \quad (a+b)\{x(a+b+x)+ab\} = 0$

$\Rightarrow \quad x(a+b+x)+ab = 0 \qquad [\because\ a+b \neq 0]$

$\Rightarrow \quad x^2+ax+bx+ab = 0$

$\Rightarrow \quad x(x+a)+b(x+a) = 0$

$\Rightarrow \quad (x+a)(x+b) = 0 \Rightarrow x+a = 0 \text{ or, } x+b = 0 \Rightarrow x = -a \text{ or, } x = -b$

LEVEL-3

EXAMPLE 10 *Solve:*

$$x = \cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-x}}},\ x \neq 2$$

SOLUTION We have,

$$x = \cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-x}}}$$

$$\Rightarrow \quad x = \cfrac{1}{2-\cfrac{1}{\cfrac{2(2-x)-1}{2-x}}}$$

$$\Rightarrow \quad x = \cfrac{1}{2-\cfrac{2-x}{4-2x-1}}$$

$$\Rightarrow \quad x = \cfrac{1}{2-\cfrac{2-x}{3-2x}}$$

$$\Rightarrow \quad x = \frac{3-2x}{2(3-2x)-(2-x)}$$

$$\Rightarrow \quad x = \frac{3-2x}{4-3x}$$

$\Rightarrow \quad x(4-3x) = (3-2x)$

$\Rightarrow \quad 4x-3x^2 = 3-2x$

$\Rightarrow \quad 3x^2-6x+3 = 0 \Rightarrow x^2-2x+1 = 0 \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1, 1$

EXAMPLE 11 *Solve:*

(i) $x + \dfrac{1}{x} = 25\dfrac{1}{25}$ (ii) $(x-3)(x-4) = \dfrac{34}{(33)^2}$

SOLUTION (i) We have,

$$x + \frac{1}{x} = 25\frac{1}{25}$$

$$\Rightarrow \quad x + \frac{1}{x} = 25 + \frac{1}{25}$$

$$\Rightarrow \quad \frac{x^2 + 1}{x} = \left(25 + \frac{1}{25}\right)$$

$$\Rightarrow \quad x^2 + 1 = \left(25 + \frac{1}{25}\right)x$$

$$\Rightarrow \quad x^2 - \left(25 + \frac{1}{25}\right)x + 1 = 0$$

$$\Rightarrow \quad x^2 - \left(25 + \frac{1}{25}\right)x + 25 \times \frac{1}{25} = 0$$

$$\Rightarrow \quad x^2 - 25x - \frac{1}{25}x + 25 \times \frac{1}{25} = 0$$

$$\Rightarrow \quad (x^2 - 25x) - \left(\frac{x}{25} - 25 \times \frac{1}{25}\right) = 0$$

$$\Rightarrow \quad x(x - 25) - \frac{1}{25}(x - 25) = 0$$

$$\Rightarrow \quad (x - 25)\left(x - \frac{1}{25}\right) = 0$$

$$\Rightarrow \quad x - 25 = 0 \text{ or, } x - \frac{1}{25} = 0 \Rightarrow x = 25 \text{ or, } x = \frac{1}{25}$$

ALITER We have,

$$x + \frac{1}{x} = 25\frac{1}{25}$$

$$\Rightarrow \quad x + \frac{1}{x} = 25 + \frac{1}{25} \Rightarrow x = 25, \text{ or, } x = \frac{1}{25} \qquad \text{[By comparing two sides]}$$

(ii) We have,

$$(x - 3)(x - 4) = \frac{34}{33^2}$$

$$\Rightarrow \quad x^2 - 7x + 12 - \frac{34}{33^2} = 0$$

$$\Rightarrow \quad x^2 - 7x + \frac{13034}{33^2} = 0$$

$$\Rightarrow \quad x^2 - 7x + \frac{98}{33} \times \frac{133}{33} = 0$$

$$\Rightarrow \quad x^2 - \frac{231}{33}x + \frac{98}{33} \times \frac{133}{33} = 0$$

$$\Rightarrow \quad x^2 - \left(\frac{98}{33} + \frac{133}{33}\right)x + \frac{98}{33} \times \frac{133}{33} = 0$$

$$\Rightarrow \quad x^2 - \frac{98}{33}x - \frac{133}{33}x + \frac{98}{33} \times \frac{133}{33} = 0$$

$$\Rightarrow \quad \left(x^2 - \frac{98}{33}x\right) - \left(\frac{133}{33}x - \frac{98}{33} \times \frac{133}{33}\right) = 0$$

$$\Rightarrow \quad x\left(x - \frac{98}{33}\right) - \frac{133}{33}\left(x - \frac{98}{33}\right) = 0$$

$$\Rightarrow \quad \left(x - \frac{98}{33}\right)\left(x - \frac{133}{33}\right) = 0 \Rightarrow x = \frac{98}{33} \text{ or, } x = \frac{133}{33}$$

EXERCISE 4.3

LEVEL-1

Solve the following quadratic equations by factorization:

1. $(x-4)(x+2) = 0$
2. $(2x+3)(3x-7) = 0$
3. $3x^2 - 14x - 5 = 0$
4. $9x^2 - 3x - 2 = 0$
5. $\frac{1}{x-1} - \frac{1}{x+5} = \frac{6}{7}, x \neq 1, -5$ [CBSE 2010]
6. $6x^2 + 11x + 3 = 0$
7. $5x^2 - 3x - 2 = 0$
8. $48x^2 - 13x - 1 = 0$
9. $3x^2 = -11x - 10$
10. $25x(x+1) = -4$
11. $16x - \frac{10}{x} = 27$ [CBSE 2014]
12. $\frac{1}{x} - \frac{1}{x-2} = 3, \; x \neq 0, 2$ [NCERT, CBSE 2010]
13. $x - \frac{1}{x} = 3, \; x \neq 0$ [NCERT, CBSE 2010]
14. $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, \; x \neq 4, 7$ [NCERT]
15. $\frac{1}{x-3} + \frac{2}{x-2} = \frac{8}{x}, \; x \neq 0, 2, 3$ [CBSE 2013]
16. $a^2x^2 - 3abx + 2b^2 = 0$
17. $9x^2 - 6b^2x - (a^4 - b^4) = 0$ [CBSE 2015]
18. $4x^2 + 4bx - (a^2 - b^2) = 0$
19. $ax^2 + (4a^2 - 3b)x - 12ab = 0$
20. $2x^2 + ax - a^2 = 0$ [CBSE 2014]
21. $\frac{16}{x} - 1 = \frac{15}{x+1}, x \neq 0, -1$ [CBSE 2014]

22. $\dfrac{x+3}{x+2} = \dfrac{3x-7}{2x-3}$, $x \neq -2, \dfrac{3}{2}$ [CBSE 2017]

23. $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = \dfrac{25}{3}$, $x \neq 3, 4$

24. $\dfrac{x+3}{x-2} - \dfrac{1-x}{x} = \dfrac{17}{4}$, $x \neq 0, 2$ [CBSE 2017]

25. $\dfrac{x-3}{x+3} - \dfrac{x+3}{x-3} = \dfrac{48}{7}$, $x \neq 3, x \neq -3$ [CBSE 2015]

26. $\dfrac{1}{x-2} + \dfrac{2}{x-1} = \dfrac{6}{x}$, $x \neq 0$ [CBSE 2013]

27. $\dfrac{x+1}{x-1} - \dfrac{x-1}{x+1} = \dfrac{5}{6}$, $x \neq 1, -1$

28. $\dfrac{x-1}{2x+1} + \dfrac{2x+1}{x-1} = \dfrac{5}{2}$, $x \neq -\dfrac{1}{2}, 1$

29. $\dfrac{4}{x} - 3 = \dfrac{5}{2x+3}$, $x \neq 0, -\dfrac{3}{2}$ [CBSE 2013]

30. $\dfrac{x-4}{x-5} + \dfrac{x-6}{x-7} = \dfrac{10}{3}$; $x \neq 5, 7$ [CBSE 2014]

31. $\dfrac{x-2}{x-3} + \dfrac{x-4}{x-5} = \dfrac{10}{3}$; $x \neq 3, 5$ [CBSE 2014]

32. $\dfrac{5+x}{5-x} - \dfrac{5-x}{5+x} = 3\dfrac{3}{4}$; $x \neq 5, -5$ [CBSE 2014]

33. $\dfrac{3}{x+1} - \dfrac{1}{2} = \dfrac{2}{3x-1}$, $x \neq -1, \dfrac{1}{3}$ [CBSE 2014]

34. $\dfrac{3}{x+1} + \dfrac{4}{x-1} = \dfrac{29}{4x-1}$; $x \neq 1, -1, \dfrac{1}{4}$ [CBSE 2015]

35. $\dfrac{2}{x+1} + \dfrac{3}{2(x-2)} = \dfrac{23}{5x}$; $x \neq 0, -1, 2$ [CBSE 2015]

LEVEL-2

36. $x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$

37. $3\sqrt{5}x^2 + 25x - 10\sqrt{5} = 0$

38. $\sqrt{3}x^2 - 2\sqrt{2}x - 2\sqrt{3} = 0$ [CBSE 2015]

39. $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$ [CBSE 2014]

40. $\sqrt{2}x^2 - 3x - 2\sqrt{2} = 0$ [CBSE 2006C, 2013, 2016]

41. $x^2 - (\sqrt{2}+1)x + \sqrt{2} = 0$

42. $3x^2 - 2\sqrt{6}x + 2 = 0$

43. $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ [NCERT, CBSE 2010, 2012]

44. $\dfrac{m}{n}x^2 + \dfrac{n}{m} = 1 - 2x$ [CBSE 2017]

45. $\dfrac{x-a}{x-b} + \dfrac{x-b}{x-a} = \dfrac{a}{b} + \dfrac{b}{a}$

46. $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x-3)(x-4)}=\dfrac{1}{6}$

47. $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2,\ x\neq a, b$ [CBSE 2016]

48. $\dfrac{x+1}{x-1}+\dfrac{x-2}{x+2}=4-\dfrac{2x+3}{x-2};\ x\neq 1,-2,2$ [CBSE 2016]

49. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{2c}{x-c}$ [CBSE 2016]

50. $x^2+2ab=(2a+b)x$

51. $(a+b)^2x^2-4abx-(a-b)^2=0$

52. $a(x^2+1)-x(a^2+1)=0$

53. $x^2-x-a(a+1)=0$

54. $x^2+\left(a+\dfrac{1}{a}\right)x+1=0$

55. $abx^2+(b^2-ac)x-bc=0$ [CBSE 2005]

56. $a^2b^2x^2+b^2x-a^2x-1=0$ [CBSE 2005]

57. $\dfrac{x-1}{x-2}+\dfrac{x-3}{x-4}=3\dfrac{1}{3},\ x\neq 2,4$ [CBSE 2005]

58. $\dfrac{1}{2a+b+2x}=\dfrac{1}{2a}+\dfrac{1}{b}+\dfrac{1}{2x}$ [CBSE 2013]

59. $3\left(\dfrac{3x-1}{2x+3}\right)-2\left(\dfrac{2x+3}{3x-1}\right)=5;\ x\neq\dfrac{1}{3},-\dfrac{3}{2}$ [CBSE 2014]

60. $3\left(\dfrac{7x+1}{5x-3}\right)-4\left(\dfrac{5x-3}{7x+1}\right)=11;\ x\neq\dfrac{3}{5},-\dfrac{1}{7}$ [CBSE 2014]

LEVEL-3

61. $(x-5)(x-6)=\dfrac{25}{(24)^2}$

62. $7x+\dfrac{3}{x}=35\dfrac{3}{5}$

ANSWERS

1. $4,-2$
2. $-\dfrac{3}{2},\dfrac{7}{3}$
3. $5,-\dfrac{1}{3}$
4. $\dfrac{2}{3},\dfrac{-1}{3}$
5. $2,-6$
6. $-\dfrac{3}{2},\dfrac{-1}{3}$
7. $\dfrac{-2}{5},1$
8. $\dfrac{-1}{16},\dfrac{1}{3}$
9. $-\dfrac{5}{3},-2$
10. $-\dfrac{4}{5},-\dfrac{1}{5}$
11. $2,-\dfrac{5}{16}$
12. $\dfrac{3\pm\sqrt{3}}{2}$
13. $\dfrac{3\mp\sqrt{13}}{2}$
14. $1,2$
15. $4,\dfrac{12}{5}$
16. $\dfrac{2b}{a},\dfrac{b}{a}$
17. $\dfrac{a^2+b^2}{3},\dfrac{b^2-a^2}{3}$
18. $\dfrac{a-b}{2},-\dfrac{a+b}{2}$
19. $\dfrac{3b}{a},-4a$
20. $\dfrac{a}{2},-a$
21. ± 4
22. $-1,5$
23. $6,\dfrac{40}{13}$
24. $4,\dfrac{-2}{9}$

25. $-4, \frac{9}{4}$
26. $3, \frac{4}{3}$
27. $5, \frac{-1}{5}$
28. -1
29. $-2, 1$
30. $8, \frac{11}{2}$
31. $6, \frac{7}{2}$
32. $3, -\frac{25}{3}$
33. $3, 1$
34. $4, -7$
35. $4, -\frac{23}{11}$
36. $\sqrt{3}, 1$
37. $-2\sqrt{5}, \frac{\sqrt{5}}{3}$
38. $\sqrt{6}, -\sqrt{\frac{2}{3}}$
39. $\frac{\sqrt{3}}{4}, \frac{-2}{\sqrt{3}}$
40. $-\frac{1}{\sqrt{2}}, 2\sqrt{2}$
41. $\sqrt{2}, 1$
42. $\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$
43. $-\frac{5}{\sqrt{2}}, -\sqrt{2}$
44. $\frac{-n \pm \sqrt{mn}}{m}$
45. $0, a+b$
46. $-2, 7$
47. $\frac{a+b}{2}, a+b$
48. $-5, \frac{6}{5}$
49. $0, \frac{2ab-bc-ac}{a+b-2c}$
50. $2a, b$
51. $1, -\left(\frac{a-b}{a+b}\right)$
52. $a, \frac{1}{a}$
53. $-a, a+1$
54. $-a, -\frac{1}{a}$
55. $\frac{-b}{a}, \frac{c}{b}$
56. $-\frac{1}{a^2}, \frac{1}{b^2}$
57. $5, \frac{5}{2}$
58. $-a, \frac{-b}{2}$
59. $0, -7$
60. $0, 1$
61. $6\frac{1}{24}, 4\frac{23}{24}$
62. $5, \frac{3}{35}$

4.5 SOLUTION OF A QUADRATIC EQUATION BY COMPLETING THE SQUARE

In the previous section, we learnt about the factorization method to obtain the roots of a quadratic equation. In this section, we shall learn about the method of completing squares. We may use the following algorithm to obtain the roots of a quadratic equation by using the method of completing squares.

ALGORITHM

STEP I *Obtain the quadratic equation. Let the quadratic equation be* $ax^2 + bx + c = 0,\ a \neq 0$.

STEP II *Make the coefficient of* x^2 *unity by dividing throughout by it, if it is not unity. i.e., obtain*

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

STEP III *Shift the constant term* $\frac{c}{a}$ *on RHS to get* $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

STEP IV *Add square of half of the coefficient of x i.e.* $\left(\frac{b}{2a}\right)^2$ *on both sides to obtain*

$$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

STEP V *Write LHS as the perfect square of a binomial expression and simplify RHS to get*

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

STEP VI *Take square root of both sides to get* $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$

STEP VII *Obtain the values of x by shifting the constant term* $\frac{b}{2a}$ *on RHS.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the quadratic equation* $9x^2 - 15x + 6 = 0$ *by the method of completing the square.* [NCERT]

SOLUTION We have, $9x^2 - 15x + 6 = 0$

$\Rightarrow \quad x^2 - \frac{15}{9}x + \frac{6}{9} = 0$ [Dividing throughout by 9]

$\Rightarrow \quad x^2 - \frac{5}{3}x + \frac{2}{3} = 0$

$\Rightarrow \quad x^2 - \frac{5}{3}x = -\frac{2}{3}$ [Shifting the constant term on RHS]

$\Rightarrow \quad x^2 - 2\left(\frac{5}{6}\right)x + \left(\frac{5}{6}\right)^2 = \left(\frac{5}{6}\right)^2 - \frac{2}{3}$ [Adding square of half of coefficient of x on both sides]

$\Rightarrow \quad \left(x - \frac{5}{6}\right)^2 = \frac{25}{36} - \frac{2}{3}$

$\Rightarrow \quad \left(x - \frac{5}{6}\right)^2 = \frac{25 - 24}{36}$

$\Rightarrow \quad \left(x - \frac{5}{6}\right)^2 = \frac{1}{36}$

$\Rightarrow \quad x - \frac{5}{6} = \pm \frac{1}{6}$ [Taking square root of both sides]

$\Rightarrow \quad x = \frac{5}{6} \pm \frac{1}{6}$

$\Rightarrow \quad x = \frac{5}{6} + \frac{1}{6} = 1$ or, $x = \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \Rightarrow x = 1$ or, $x = 2/3$

Hence, the roots of the equation are 1 and 2/3.

EXAMPLE 2 *Solve the equation* $2x^2 - 5x + 3 = 0$ *by the method of completing square.* [NCERT]

SOLUTION We have, $2x^2 - 5x + 3 = 0$

$\Rightarrow \quad x^2 - \frac{5}{2}x + \frac{3}{2} = 0$ [Dividing throughout by 2]

$\Rightarrow \quad x^2 - \frac{5}{2}x = -\frac{3}{2}$ [Shifting the constant term on RHS]

$\Rightarrow \quad x^2 - 2\left(\frac{5}{4}\right)x + \left(\frac{5}{4}\right)^2 = \left(\frac{5}{4}\right)^2 - \frac{3}{2}$ $\left[\text{Adding} \left(\frac{1}{2}\text{Coeff. of } x\right)^2 \text{ on both sides}\right]$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{25}{16} - \frac{3}{2}$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{1}{16}$

$\Rightarrow \quad x - \frac{5}{4} = \pm \frac{1}{4} \Rightarrow x = \frac{5}{4} \pm \frac{1}{4}$

$\Rightarrow \quad x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4}$ or, $x = \frac{5}{4} - \frac{1}{4} = \frac{4}{4} \Rightarrow x = \frac{3}{2}$ or, $x = 1$

Hence, the roots of the equation $2x^2 - 5x + 3 = 0$ are $\frac{3}{2}$ and 1.

EXAMPLE 3 *Find the roots of the equation $5x^2 - 6x - 2 = 0$ by the method of completing the square.* [NCERT]

SOLUTION We have, $5x^2 - 6x - 2 = 0$

$$\Rightarrow \quad x^2 - \frac{6}{5}x - \frac{2}{5} = 0 \quad \text{[Dividing throughout by 5]}$$

$$\Rightarrow \quad x^2 - \frac{6}{5}x = \frac{2}{5}$$

$$\Rightarrow \quad x^2 - 2\left(\frac{3}{5}\right)x + \left(\frac{3}{5}\right)^2 = \frac{2}{5} + \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \quad \left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

$$\Rightarrow \quad x - \frac{3}{5} = \pm\frac{\sqrt{19}}{5}$$

$$\Rightarrow \quad x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$$

$$\Rightarrow \quad x = \frac{3 + \sqrt{19}}{5} \text{ or, } x = \frac{3 - \sqrt{19}}{5}$$

Hence, the roots of the given equation are $\frac{3 + \sqrt{19}}{5}$ and $\frac{3 - \sqrt{19}}{5}$.

EXAMPLE 4 *By using the method of completing the square, show that the equation $4x^2 + 3x + 5 = 0$ has no real roots.* [NCERT]

SOLUTION We have,

$$4x^2 + 3x + 5 = 0$$

$$\Rightarrow \quad x^2 + \frac{3}{4}x + \frac{5}{4} = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{8}x\right) = -\frac{5}{4}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{8}\right)x + \left(\frac{3}{8}\right)^2 = \left(\frac{3}{8}\right)^2 - \frac{5}{4}$$

$$\Rightarrow \quad \left(x + \frac{3}{8}\right)^2 = -\frac{71}{64}$$

Clearly, RHS is negative. But, $\left(x + \frac{3}{8}\right)^2$ cannot be negative for any real value of x.
Hence, the given equation has no real roots.

LEVEL-2

EXAMPLE 5 *Find the roots of the following equation $4x^2 + 4bx - (a^2 - b^2) = 0$ by the method of completing the square.*

SOLUTION We have,

$$4x^2 + 4bx - (a^2 - b^2) = 0$$

$$\Rightarrow \quad x^2 + bx - \left(\frac{a^2 - b^2}{4}\right) = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2}\right)x = \frac{a^2 - b^2}{4}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2 = \frac{a^2 - b^2}{4} + \left(\frac{b}{2}\right)^2$$

$$\Rightarrow \quad \left(x + \frac{b}{2}\right)^2 = \frac{a^2}{4}$$

$$\Rightarrow \quad x + \frac{b}{2} = \pm\frac{a}{2}$$

$$\Rightarrow \quad x = \frac{-b}{2} \pm \frac{a}{2} \Rightarrow x = \frac{-b - a}{2}, \frac{-b + a}{2}$$

Hence, the roots are $-\left(\frac{a+b}{2}\right)$ and $\left(\frac{a-b}{2}\right)$

EXAMPLE 6 *Find the roots of the equation* $a^2x^2 - 3abx + 2b^2 = 0$ *by the method of completing the square.*

SOLUTION We have,

$$a^2x^2 - 3abx + 2b^2 = 0$$

$$\Rightarrow \quad x^2 - \frac{3b}{a}x + 2\frac{b^2}{a^2} = 0$$

$$\Rightarrow \quad x^2 - \frac{3b}{a}x = -\frac{2b^2}{a^2}$$

$$\Rightarrow \quad x^2 - 2\left(\frac{3b}{2a}\right)x + \left(\frac{3b}{2a}\right)^2 = -\frac{2b^2}{a^2} + \left(\frac{3b}{2a}\right)^2$$

$$\Rightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = -\frac{2b^2}{a^2} + \frac{9b^2}{4a^2}$$

$$\Rightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x - \frac{3b}{2a}\right) = \pm\frac{b}{2a}$$

$$\Rightarrow \quad x = \frac{3b}{2a} \pm \frac{b}{2a} \Rightarrow x = \frac{3b}{2a} + \frac{b}{2a} = \frac{2b}{a} \text{ or, } x = \frac{3b}{2a} - \frac{b}{2a} = \frac{b}{a}$$

Hence, the roots are $\frac{2b}{a}$ and $\frac{b}{a}$.

EXAMPLE 7 *Solve the equation* $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$ *by the method of completing the square.*

SOLUTION We have,

$$x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$$

$$\Rightarrow \quad x^2 - (\sqrt{3} + 1)x = -\sqrt{3}$$

$$\Rightarrow \quad \left(x - \frac{\sqrt{3} + 1}{2}\right)^2 = \frac{-4\sqrt{3} + (\sqrt{3} + 1)^2}{4}$$

$$\Rightarrow \quad \left(x-\frac{\sqrt{3}+1}{2}\right)^2=\left(\frac{\sqrt{3}-1}{2}\right)^2$$

$$\Rightarrow \quad x-\frac{\sqrt{3}+1}{2}=\pm\frac{\sqrt{3}-1}{2} \Rightarrow x=\frac{\sqrt{3}+1}{2}\pm\frac{\sqrt{3}-1}{2} \Rightarrow x=\sqrt{3},1$$

Hence, the roots are $\sqrt{3}$ and 1.

EXERCISE 4.4

LEVEL-1

Find the roots of the following quadratic equations (if they exist) by the method of completing the square.

1. $x^2-4\sqrt{2x}+6=0$
2. $2x^2-7x+3=0$
3. $3x^2+11x+10=0$
4. $2x^2+x-4=0$
5. $2x^2+x+4=0$

LEVEL-2

6. $4x^2+4\sqrt{3}x+3=0$
7. $\sqrt{2}x^2-3x-2\sqrt{2}=0$
8. $\sqrt{3}x^2+10x+7\sqrt{3}=0$
9. $x^2-(\sqrt{2}+1)\,x+\sqrt{2}=0$
10. $x^2-4ax+4a^2-b^2=0$

ANSWERS

1. $\sqrt{2}, 3\sqrt{2}$
2. $3, \frac{1}{2}$
3. $-\frac{5}{3}, -2$
4. $\frac{\sqrt{33}-1}{4}, \frac{-\sqrt{33}-1}{4}$
5. No real roots
6. $-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$
7. $-\frac{1}{\sqrt{2}}, 2\sqrt{2}$
8. $-\sqrt{3}, -\frac{7}{\sqrt{3}}$
9. $\sqrt{2}, 1$
10. $2a-b, 2a+b$

4.6 SOLUTION OF A QUADRATIC EQUATION BY USING THE QUADRATIC FOR FORMULA (SHREEDHARACHARYA'S RULE)

In the previous section, we have learnt about factorization method of solving quadratic equations. In some cases, it is not convenient to solve quadratic equations by factorization method. For example, consider the equation $x^2+4x+2=0$. In order to solve this equation by factorization method we will have to split the coefficient of the middle term 4 into two integers whose sum is 4 and product is 2. Clearly, this not possible in integers. Therefore, this equation cannot be solved by using factorization method. In this section, we shall discuss a method to solve such quadratic equations. The method which we will discuss below is popularly known as Shreedharacharya's formula as it was first given by an ancient Indian mathematician Shreedharacharya around 1025 A.D.

Consider the quadratic equation

$$ax^2+bx+c=0, a\neq 0$$

$$\Rightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \text{[Dividing throught out by } a] \quad \ldots \text{(i)}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \left[\text{Adding } \left(\frac{b}{2a}\right)^2 \text{ i.e. } \left(\frac{1}{2}\text{Coeff. of } x\right)^2 \text{ on both sides}\right]$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \left[\text{Taking square root of both sides and assuming } b^2 - 4ac \geq 0\right]$$

$$\Rightarrow \quad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or, } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b + \sqrt{D}}{2a} \text{ or, } x = \frac{-b - \sqrt{D}}{2a}, \text{ where } D = b^2 - 4ac$$

Thus, if $D = b^2 - 4ac \geq 0$, then the quadratic equation $ax^2 + bx + c = 0$ has real roots α and β given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}$$

DISCRIMINANT *If $ax^2 + bx + c = 0, a \neq 0$ is a quadratic equation, then the expression $b^2 - 4ac$ is known as its discriminant and is generally denoted by D.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON DETERMINING THE DISCRIMINANT OF A QUADRATIC EQUATION**

EXAMPLE 1 *Write the discriminant of the following quadratic equations:*

(i) $x^2 - 4x + 2 = 0$

(ii) $3x^2 + 2x - 1 = 0$

(iii) $x^2 - 4x + a = 0$

(iv) $\sqrt{3}x^2 - 2\sqrt{2}x - 2\sqrt{3} = 0$

(v) $x^2 + x + 1 = 0$

(vi) $x^2 + px + 2q = 0$

SOLUTION (i) The given equation is $x^2 - 4x + 2 = 0$

Here, $a = 1, b = -4$ and, $c = 2$

$\therefore \quad D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 2 = 16 - 8 = 8$

(ii) The given equation is $3x^2 + 2x - 1 = 0$

Here, $a = 3, b = 2$ and, $c = -1$

$\therefore$ $D = b^2 - 4ac = 2^2 - 4 \times 3 \times -1 = 4 + 12 = 16$

(iii) The given equation is $x^2 - 4x + a = 0$

Here, $a = 1, b = -4$ and, $c = a$.

$\therefore$ $D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times a = 16 - 4a$

(iv) The given equation is $\sqrt{3}x^2 - 2\sqrt{2}x - 2\sqrt{3} = 0$

Here, $a = \sqrt{3}, b = -2\sqrt{2}$ and, $c = -2\sqrt{3}$

$\therefore$ $D = b^2 - 4ac = (-2\sqrt{2})^2 - 4 \times \sqrt{3} \times -2\sqrt{3} = 8 + 24 = 32$

(v) The given equation is $x^2 + x + 1 = 0$

Here, $a = 1, b = 1$ and, $c = 1$.

$\therefore$ $D = b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -3$

(vi) The given equation is $x^2 + px + 2q = 0$

Here, $a = 1, b = p$ and, $c = 2q$

$\therefore$ $D = b^2 - 4ac = p^2 - 4 \times 1 \times 2q = p^2 - 8q$

Type II ON SOLVING A QUADRATIC EQUATION HAVING REAL ROOTS BY USING QUADRATIC FORMULA

EXAMPLE 2 *In the following, determine whether the given quadratic equations have real roots and if so, find the roots*

(i) $9x^2 + 7x - 2 = 0$

(ii) $2x^2 + 5\sqrt{3}x + 6 = 0$

(iii) $3x^2 + 2\sqrt{5}x - 5 = 0$

(iv) $x^2 + 5x + 5 = 0$

(v) $6x^2 + x - 2 = 0$

(vi) $25x^2 + 20x + 7 = 0$

SOLUTION (i) The given equation is $9x^2 + 7x - 2 = 0$

Here, $a = 9, b = 7$ and $c = -2$

$\therefore$ $D = b^2 - 4ac = 7^2 - 4 \times 9 \times -2 = 49 + 72 = 121 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{-7 + \sqrt{121}}{2 \times 9} = \frac{-7 + 11}{18} = \frac{4}{18} = \frac{2}{9}$$

and, $$\beta = \frac{-b - \sqrt{D}}{2a} = \frac{-7 - \sqrt{121}}{2 \times 9} = \frac{-7 - 11}{18} = -1$$

(ii) The given equation is $2x^2 + 5\sqrt{3}x + 6 = 0$

Here, $a = 2, b = 5\sqrt{3}$ and $c = 6$

$\therefore$ $D = b^2 - 4ac = 75 - 4 \times 2 \times 6 = 27 > 0$

So, the given equation has real roots given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-5\sqrt{3}+\sqrt{27}}{2\times 2} = \frac{-5\sqrt{3}+3\sqrt{3}}{4} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$

and, $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-5\sqrt{3}-\sqrt{27}}{2\times 2} = \frac{-5\sqrt{3}-3\sqrt{3}}{4} = -2\sqrt{3}$$

(iii) The given equation is $3x^2 + 2\sqrt{5}x - 5 = 0$

Here, $a = 3, b = 2\sqrt{5}$ and, $c = -5$

$\therefore$ $D = b^2 - 4ac = (2\sqrt{5})^2 - 4\times 3\times -5 = 20 + 60 = 80 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-2\sqrt{5}+\sqrt{80}}{2\times 3} = \frac{-2\sqrt{5}+4\sqrt{5}}{6} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$$

and, $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-2\sqrt{5}-\sqrt{80}}{2\times 3} = \frac{-2\sqrt{5}-4\sqrt{5}}{6} = -\sqrt{5}$$

(iv) The given equation is $x^2 + 5x + 5 = 0$

Here, $a = 1, b = 5$ and, $c = 5$

$\therefore$ $D = b^2 - 4ac = 25 - 4\times 1\times 5 = 5 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-5+\sqrt{5}}{2} \text{ and } \beta = \frac{-b-\sqrt{D}}{2a} = \frac{-5-\sqrt{5}}{2}$$

(v) The given equation is $6x^2 + x - 2 = 0$

Here, $a = 6, b = 1$ and, $c = -2$

$\therefore$ $D = b^2 - 4ac = 1 - 4\times 6\times -2 = 49 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-1+\sqrt{49}}{2\times 6} = \frac{-1+7}{12} = \frac{6}{12} = \frac{1}{2}$$

and, $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-1-\sqrt{49}}{2\times 6} = \frac{-1-7}{12} = \frac{-8}{12} = \frac{-2}{3}$$

(vi) The given equation is $25x^2 + 20x + 7 = 0$

Here, $a = 25, b = 20$ and, $c = 7$

$\therefore$ $D = b^2 - 4ac = (20)^2 - 4\times 25\times 7 = 400 - 700 = -300 < 0$

So, the given equation has no real roots.

EXAMPLE 3 *Solve for x:* $\frac{x-1}{x+2} + \frac{x-3}{x-4} = \frac{10}{3}, x \neq -2, 4$

SOLUTION We have,

$$\frac{x-1}{x+2} + \frac{x-3}{x-4} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{(x^2-5x+4)+(x^2-x-6)}{(x+2)(x-4)} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{2x^2-6x-2}{x^2-2x-8} = \frac{10}{3}$$

$$\Rightarrow \quad 6x^2-18x-6 = 10x^2-20x-80$$

$$\Rightarrow \quad 4x^2-2x-74 = 0$$

$$\Rightarrow \quad 2x^2-x-37 = 0$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1+296}}{4} \Rightarrow x = \frac{1 \pm \sqrt{297}}{4}$$

EXAMPLE 4 *Solve for x:* $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}, x \neq 1, -2, -4.$

SOLUTION We have,

$$\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$$

$$\Rightarrow \quad \frac{1}{x+1} + \frac{2}{x+2} = \frac{1}{x+4} + \frac{3}{x+4}$$

$$\Rightarrow \quad \frac{1}{x+1} - \frac{1}{x+4} = \frac{3}{x+4} - \frac{2}{x+2}$$

$$\Rightarrow \quad \frac{x+4-x-1}{(x+1)(x+4)} = \frac{3x+6-2x-8}{(x+4)(x+2)}$$

$$\Rightarrow \quad \frac{3}{(x+1)(x+4)} = \frac{x-2}{(x+4)(x+2)}$$

$$\Rightarrow \quad \frac{3}{x+1} = \frac{x-2}{x+2}$$

[Multiplying both sides by $(x+4)$]

$$\Rightarrow \quad 3x+6 = x^2-x-2$$

$$\Rightarrow \quad x^2-4x-8 = 0 \Rightarrow x = \frac{4 \pm \sqrt{16+32}}{2} = \frac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}$$

LEVEL-2

EXAMPLE 5 *Using quadratic formula solve the following quadratic equations:*

(i) $p^2x^2 + (p^2-q^2)x - q^2 = 0, p \neq 0$ [CBSE 2004]

(ii) $9x^2 - 9(a+b)x + (2a^2+5ab+2b^2) = 0$ [CBSE 2004, 2009]

SOLUTION (i) We have, $p^2x^2 + (p^2-q^2)x - q^2 = 0$

Comparing this equation with $ax^2+bx+c=0$, we have

$$a = p^2, b = p^2-q^2 \text{ and } c = -q^2$$

$$D = b^2-4ac = (p^2-q^2)^2 - 4 \times p^2 \times -q^2 = (p^2-q^2)^2 + 4p^2q^2 = (p^2+q^2)^2 > 0$$

So, the given equation has real roots given by

$$\alpha = -\frac{b-\sqrt{D}}{2a} = -\frac{(p^2-q^2)+(p^2+q^2)}{2p^2} = \frac{q^2}{p^2}$$

and, $\beta = \frac{-b - \sqrt{D}}{2a} = \frac{-(p^2 - q^2) - (p^2 + q^2)}{2p^2} = -1$

ALITER We have,

$$p^2x^2 + (p^2 - q^2)x - q^2 = 0$$

$$\Rightarrow \quad p^2x^2 + p^2x - q^2x - q^2 = 0$$

$$\Rightarrow \quad (p^2x^2 + p^2x) - (q^2x + q^2) = 0$$

$$\Rightarrow \quad p^2x(x+1) - q^2(x+1) = 0$$

$$\Rightarrow \quad (x+1)(p^2x - q^2) = 0$$

$$\Rightarrow \quad x + 1 = 0 \text{ or, } p^2x - q^2 = 0 \Rightarrow x = -1 \text{ or, } x = \frac{q^2}{p^2}$$

(ii) We have,

$$9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$$

Comparing this equation with $Ax^2 + Bx + C = 0$, we have

$$A = 9, B = -9(a+b) \text{ and } C = 2a^2 + 5ab + 2b^2$$

$$\therefore \quad D = B^2 - 4AC$$

$$\Rightarrow \quad D = 81(a+b)^2 - 36(2a^2 + 5ab + 2b^2)$$

$$\Rightarrow \quad D = 9a^2 + 9b^2 - 18ab$$

$$\Rightarrow \quad D = 9(a-b)^2 \geq 0$$

So, the roots of the given equation are real and are given by

$$\alpha = \frac{-B + \sqrt{D}}{2A} = \frac{9(a+b) + 3(a-b)}{18} = \frac{12a + 6b}{18} = \frac{2a + b}{3}$$

and, $\beta = \frac{-B - \sqrt{D}}{2A} = \frac{9(a+b) - 3(a-b)}{18} = \frac{6a + 12b}{18} = \frac{a + 2b}{3}$

ALITER See Ex 7 (i) on page 4.14.

EXAMPLE 6 *Using quadratic formula, solve the following equation for x:*

$$abx^2 + (b^2 - ac)x - bc = 0$$ [CBSE 2005]

SOLUTION We have,

$$abx^2 + (b^2 - ac)x - bc = 0$$

$$\therefore \quad x = \frac{-(b^2 - ac) \pm \sqrt{(b^2 - ac)^2 - 4(ab)(-bc)}}{2ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) \pm \sqrt{(b^2 - ac)^2 - 4ab^2c}}{2ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) \pm \sqrt{b^4 - 2ab^2c + a^2c^2 + 4ab^2c}}{2ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) \pm \sqrt{(b^2 + ac)^2}}{2\,ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) \pm (b^2 + ac)}{2\,ab}$$

$$\Rightarrow \quad x = \frac{-(b^2 - ac) + (b^2 + ac)}{2\,ab}, x = \frac{-(b^2 - ac) - (b^2 + ac)}{2\,ab}$$

$$\Rightarrow \quad x = \frac{2\,ac}{2\,ab}, x = \frac{-2b^2}{2\,ab} \Rightarrow x = \frac{c}{b}, x = \frac{-b}{a}$$

EXERCISE 4.5

LEVEL-1

1. Write the discriminant of the following quadratic equations:
   (i) $2x^2 - 5x + 3 = 0$
   (ii) $x^2 + 2x + 4 = 0$
   (iii) $(x - 1)(2x - 1) = 0$
   (iv) $x^2 - 2x + k = 0, k \in R$
   (v) $\sqrt{3}x^2 + 2\sqrt{2}x - 2\sqrt{3} = 0$
   (vi) $x^2 - x + 1 = 0$
2. In the following, determine whether the given quadratic equations have real roots and if so, find the roots:
   (i) $16x^2 = 24x + 1$
   (ii) $x^2 + x + 2 = 0$
   (iii) $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$
   (iv) $3x^2 - 2x + 2 = 0$
   (v) $2x^2 - 2\sqrt{6}x + 3 = 0$
   (vi) $3a^2x^2 + 8abx + 4b^2 = 0, a \neq 0$
   (vii) $3x^2 + 2\sqrt{5}x - 5 = 0$
   (viii) $x^2 - 2x + 1 = 0$
   (ix) $2x^2 + 5\sqrt{3}x + 6 = 0$
   (x) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ [CBSE 2013, NCERT]
   (xi) $2x^2 - 2\sqrt{2}x + 1 = 0$ [NCERT]
   (xii) $3x^2 - 5x + 2 = 0$ [NCERT]
3. Solve for x:
   (i) $\dfrac{x-1}{x-2} + \dfrac{x-3}{x-4} = 3\dfrac{1}{3}; x \neq 2, 4$ [CBSE 2005]
   (ii) $\dfrac{1}{x} + \dfrac{2}{2x-3} = \dfrac{1}{x-2},\ x \neq 0, \dfrac{3}{2},\ 2$ [CBSE 2016]
   (iii) $x + \dfrac{1}{x} = 3,\ x \neq 0$ [NCERT]
   (iv) $\dfrac{16}{x} - 1 = \dfrac{15}{x+1}, x \neq 0, -1$ [CBSE 2014]
   (v) $\dfrac{1}{x-3} - \dfrac{1}{x+5} = \dfrac{1}{6},\ x \neq 3, -5$ [CBSE 2016]

ANSWERS

1. (i) 1 (ii) – 12 (iii) 1 (iv) $4 - 4k$
   (v) 32 (vi) – 3
2. (i) $\dfrac{3 \pm \sqrt{10}}{4}$ (ii) Not real (iii) $-4\sqrt{3}, \dfrac{2}{\sqrt{3}}$ (iv) Not real

(v) Real and equal, $\sqrt{\frac{3}{2}}$ (vi) $\frac{-2b}{a}, \frac{-2b}{3a}$ (vii) $\frac{\sqrt{5}}{3}, -\sqrt{5}$ (viii) 1

(ix) $-2\sqrt{3}, \frac{-\sqrt{3}}{2}$ (x) $-\sqrt{2}, -\frac{5}{\sqrt{2}}$ (xi) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (xii) $1, \frac{2}{3}$

3. (i) $5, \frac{5}{2}$ (ii) 1, 3 (iii) $\frac{3 \pm \sqrt{5}}{2}$ (iv) ± 4 (v) $-9, 7$

4.7 NATURE OF ROOTS

Let $ax^2 + bx + c = 0, a \neq 0$ be a quadratic equation. In the previous section, we have seen that the roots of this equation are given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}, \text{ provided that } D = b^2 - 4ac \geq 0.$$

If $D = b^2 - 4ac > 0$, then α and β are real.

Also,

$$\alpha - \beta = \left(\frac{-b + \sqrt{D}}{2a}\right) - \left(\frac{-b - \sqrt{D}}{2a}\right) = \frac{-b + \sqrt{D} + b + \sqrt{D}}{2a} = \frac{2\sqrt{D}}{2a} = \frac{\sqrt{D}}{a}$$

$$\Rightarrow \quad \alpha - \beta \neq 0 \Rightarrow \alpha \neq \beta$$

Thus, if $D = b^2 - 4ac > 0$ i.e. the discriminant of the equation is positive, then the equation has real and distinct roots α and β given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}$$

If $D = b^2 - 4ac = 0$, then α and β are real.

Also, $\alpha = -\frac{b}{2a} = \beta$ [Putting $D = 0$ in the expression for α and β]

Thus, if $D = b^2 - 4ac = 0$ i.e. the discriminant of the equation is zero, then the equation has real and equal roots both equal to $-\frac{b}{2a}$.

Now, a natural question arises: what is the nature of the roots of the equation $ax^2 + bx + c = 0$ when its discriminant D is negative? To answer this question, let us go back to the equation

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{[Dividing Throughout by } a\text{]}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \dots\text{(i)}$$

If $D = b^2 - 4ac < 0$, then $\frac{b^2 - 4ac}{4a^2} < 0$

Therefore, LHS of equation (i) is positive (being perfect square of a real number) and its RHS is negative. So, there is no real value of x satisfying equation (i). Hence, there is no real root of the given quadratic equation in this case.

Thus, if $D = b^2 - 4ac < 0$, i.e. the discriminant of the quadratic equation is negative, then the equation has not real roots.

Let us now discuss problems on determining the nature of the roots of quadratic equations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON DETERMINING THE NATURE OF ROOTS OF QUADRATIC EQUATION**

EXAMPLE 1 *Determine the nature of the roots of the following quadratic equations:*

(i) $2x^2 + x - 1 = 0$ (ii) $x^2 - 4x + 4 = 0$ (iii) $x^2 + x + 1 = 0$

(iv) $4x^2 - 4x + 1 = 0$ (v) $2x^2 + 5x + 5 = 0$

SOLUTION (i) The given quadratic equation is $2x^2 + x - 1 = 0$. Here, $a = 2, b = 1$ and $c = -1$.

$$\therefore \quad D = b^2 - 4ac = 1^2 - 4 \times 2 \times -1 = 9$$

We find that $D > 0$. So, roots of the given equation are real and distinct.

(ii) The given equation is $x^2 - 4x + 4 = 0$. Here, $a = 1, b = -4$ and, $c = 4$.

$$\therefore \quad D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 4 = 0$$

We find that $D = 0$. Therefore, roots of the given equation are real and equal.

(iii) The given equation is $x^2 + x + 1 = 0$. Here, $a = 1$, $b = 1$ and, $c = 1$

$$\therefore \quad D = b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -3$$

We find that $D < 0$. Therefore, roots of the given equation are not real.

(iv) The given equation is $4x^2 - 4x + 1 = 0$. Here, $a = 4, b = -4$ and, $c = 1$

$$\therefore \quad D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1 = 0$$

We find that $D = 0$. Therefore, roots of the given equation are real and equal.

(v) The given equation is $2x^2 + 5x + 5 = 0$. Here, $a = 2, b = 5$ and, $c = 5$

$$\therefore \quad D = b^2 - 4ac = 5^2 - 4 \times 2 \times 5 = 25 - 40 = -15$$

We find that $D < 0$. Therefore, roots of the given equation are not real.

Type II **ON DETERMINING THE VALUES OF AN UNKNOWN INVOLVED IN THE QUADRATIC EQUATION WHEN THE NATURE OF ITS ROOTS IS GIVEN**

Results to Remember:

(i) $ax - b > 0 \Rightarrow x > \frac{b}{a}$ if, $a > 0$ and $x < \frac{b}{a}$ if, $a < 0$

(ii) $x^2 - a^2 > 0 \Rightarrow x < -a$ or, $x > a$ (iii) $x^2 - a^2 \geq 0 \Rightarrow x \leq -a$ or, $x \geq a$

(iv) $x^2 - a^2 < 0 \Rightarrow -a < x < a$ (v) $(x - a)(x - b) > 0, a < b \Rightarrow x < a$ or, $x > b$

(vi) $(x - a)(x - b) < 0, a < b \Rightarrow a < x < b$

EXAMPLE 2 *Find the values of k for which the given equation has real and equal roots:*

(i) $2x^2 - 10x + k = 0$ (ii) $9x^2 + 3kx + 4 = 0$

(iii) $12x^2 + 4kx + 3 = 0$ (iv) $2x^2 + 3x + k = 0$

(v) $2x^2 - kx + 1 = 0$ (vi) $kx^2 - 5x + k = 0$

(vii) $x^2 + k(4x + k - 1) + 2 = 0$ (viii) $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$ [CBSE 2002 C]

(ix) $(k + 1)x^2 - 2(k - 1)x + 1 = 0$ [CBSE 2015]

SOLUTION (i) The given equation is $2x^2 - 10x + k = 0$. Here, $a = 2, b = -10$ and $c = k$

$\therefore$ $D = b^2 - 4ac = (-10)^2 - 4 \times 2 \times k = 100 - 8k$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 100 - 8k = 0 \Rightarrow k = \frac{100}{8} = \frac{25}{2}$$

(ii) The given equation is $9x^2 + 3kx + 4 = 0$. Here, $a = 9, b = 3k$ and $c = 4$.

$\therefore$ $D = b^2 - 4ac = (3k)^2 - 4 \times 9 \times 4 = 9k^2 - 144$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 9k^2 - 144 = 0 \Rightarrow k^2 = \frac{144}{9} \Rightarrow k^2 = 16 \Rightarrow k = \pm 4$$

(iii) The given equation is $12x^2 + 4kx + 3 = 0$. Here, $a = 12, b = 4k$ and, $c = 3$

$\therefore$ $D = b^2 - 4ac = (4k)^2 - 4 \times 12 \times 3 = 16k^2 - 144$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 16k^2 - 144 = 0 \Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$$

(iv) The given equation is $2x^2 + 3x + k = 0$. Here, $a = 2, b = 3$ and, $c = k$

$\therefore$ $D = b^2 - 4ac = 9 - 4 \times 2 \times k = 9 - 8k$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 9 - 8k = 0 \Rightarrow k = \frac{9}{8}$$

(v) The given equation is $2x^2 - kx + 1 = 0$. Here, $a = 2, b = -k$ and, $c = 1$

$\therefore$ $D = b^2 - 4ac = (-k)^2 - 4 \times 2 \times 1 = k^2 - 8$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow k^2 - 8 = 0 \Rightarrow k^2 = 8 \Rightarrow k = \pm 2\sqrt{2}$$

(vi) The given equation is $kx^2 - 5x + k = 0$. Here, $a = k, b = -5$ and, $c = k$

$\therefore$ $D = b^2 - 4ac = (-5)^2 - 4 \times k \times (k) = 25 - 4k^2$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 25 - 4k^2 = 0 \Rightarrow 25 = 4k^2 \Rightarrow k^2 = \frac{25}{4} \Rightarrow k = \pm \frac{5}{2}$$

(vii) The given equation is $x^2 + k(4x + k - 1) + 2 = 0$ or, $x^2 + 4kx + k(k - 1) + 2 = 0$.

Here, $a = 1, b = 4k, c = k(k-1)+2 = 0$

$\therefore \quad D = (4k)^2 - 4 \times 1 \times \{k(k-1)+2\}$

$\Rightarrow \quad D = 16k^2 - 4k(k-1) - 8$

$\Rightarrow \quad D = 16k^2 - 4k^2 + 4k - 8$

$\Rightarrow \quad D = 12k^2 + 4k - 8$

$\Rightarrow \quad D = 4(3k^2 + k - 2)$

$\Rightarrow \quad D = 4(3k^2 + 3k - 2k - 2) = 4[3k(k+1) - 2(k+1)] = 4(3k-2)(k+1)$

The given equation will have equal roots, if

$$D = 0 \Rightarrow 4(3k-2)(k+1) = 0 \Rightarrow 3k - 2 = 0 \text{ or, } k+1 = 0 \Rightarrow k = \frac{2}{3} \text{ or, } k = -1$$

(viii) The given equation is $x^2 - 2x(1+3k) + 7(3+2k) = 0$. Here, $a = 1, b = -2(1+3k)$ and $c = 7(3+2k)$.

$\therefore \quad D = 4(1+3k)^2 - 4 \times \{7(3+2k)\}$

$\Rightarrow \quad D = 4(3k+1)^2 - 4 \times 1 \times 7(3+2k) = 4(9k^2 + 6k + 1 - 21 - 14k) = 4(9k^2 - 8k - 20)$

The given equation will have equal roots, if

$D = 0$

$\Rightarrow \quad 4(9k^2 - 8k - 20) = 0$

$\Rightarrow \quad 9k^2 - 8k - 20 = 0$

$\Rightarrow \quad 9k^2 - 18k + 10k - 20 = 0$

$\Rightarrow \quad (k-2)(9k+10) = 0 \Rightarrow k - 2 = 0 \text{ or, } 9k + 10 = 0 \Rightarrow k = 2 \text{ or, } k = -\frac{10}{9}$

(ix) The given equation is $(k+1)x^2 - 2(k-1)x + 1 = 0$. Here, $a = k+1, b = -2(k-1), c = 1$. Let D be the discriminant of the given equation. Then,

$$D = b^2 - 4ac = 4(k-1)^2 - 4(k+1) = 4(k^2 - 3k)$$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 4(k^2 - 3k) = 0 \Rightarrow k^2 - 3k = 0 \Rightarrow k(k-3) = 0 \Rightarrow k = 0, 3$$

EXAMPLE 3 *Find the values of k for which the following equation has equal roots:*

$$(k-12)x^2 + 2(k-12)x + 2 = 0$$ [CBSE 2013, 2017]

SOLUTION The quadratic equation is $(k-12)x^2 + 2(k-12)x + 2 = 0$. Here, $a = k - 12, b = 2(k-12)$ and $c = 2$.

$\therefore \quad D = b^2 - 4ac = 4(k-12)^2 - 4(k-12) \times 2$

$\Rightarrow \quad D = 4(k-12)\{(k-12) - 2\} = 4(k-12)(k-14)$

The given equation will have equal roots, if

$$D = 0 \Rightarrow 4(k-12)(k-14) = 0 \Rightarrow k - 12 = 0 \text{ or, } k - 14 = 0 \Rightarrow k = 12 \text{ or, } k = 14$$

EXAMPLE 4 *If -4 is a root of the quadratic equation $x^2 + px - 4 = 0$ and the quadratic equation $x^2 + px + k = 0$ has equal roots, find the value of k*

SOLUTION It is given that -4 is a root of the equation $x^2 + px - 4 = 0$.

$\therefore$ $(-4)^2 + p \times (-4) - 4 = 0$ [$\because$ A root always satisfies the equation]

$\Rightarrow$ $16 - 4p - 4 = 0 \Rightarrow 4p = 12 \Rightarrow p = 3$

The equation $x^2 + px + k = 0$ has equal roots. Therefore, its discriminant is zero.

i.e. $p^2 - 4k = 0$ [$\because a = 1, b = p$ and $c = k$]

$\Rightarrow$ $9 - 4k = 0$ [$\because p = 3$]

$\Rightarrow$ $k = 9/4$

EXAMPLE 5 *Show that the equation $x^2 + ax - 4 = 0$ has real and distinct roots for all real values of a.*

SOLUTION The given equation is $x^2 + ax - 4 = 0$. Let D be its discriminant. Then,

$$D = a^2 - 4 \times -4 = a^2 + 16$$

Clearly, $D = a^2 + 16 > 0$ for all $a \in R$.

Hence, the given equation has real and distinct roots.

EXAMPLE 6 *Find the value of k for which the given equation has equal roots. Also, find the roots.*

(i) $9x^2 - 24x + k = 0$ (ii) $2kx^2 - 40x + 25 = 0$

SOLUTION (i) The given equation is $9x^2 - 24x + k = 0$. Here, $a = 9, b = -24$ and $c = k$.

$\therefore$ $D = b^2 - 4ac = (-24)^2 - 4 \times 9 \times k = 576 - 36k$

The given equation will have real and equal roots, if

$$D = 0 \Rightarrow 576 - 36k = 0 \Rightarrow k = 16$$

Putting $k = 16$ in the given equation, we get

$$9x^2 - 24x + 16 = 0 \Rightarrow (3x - 4)^2 = 0 \Rightarrow 3x - 4 = 0 \Rightarrow x = 4/3.$$

Hence, both the roots of the given equation are equal to $4/3$.

(ii) The given equation is $2kx^2 - 40x + 25 = 0$. Here, $a = 2k$, $b = -40$ and $c = 25$

$\therefore$ $D = b^2 - 4ac = (-40)^2 - 4 \times 2k \times 25 = 1600 - 200k$

The equation will have equal roots, if

$$D = 0 \Rightarrow 1600 - 200\,k = 0 \Rightarrow k = 8$$

Substituting $k = 8$ in the given equation, we get

$$16x^2 - 40x + 25 = 0 \Rightarrow (4x - 5)^2 = 0 \Rightarrow x = \frac{5}{4}$$

Hence, the roots of the given equation are each equal to $5/4$.

EXAMPLE 7 *Find the value of k for which the quadratic equation $(k + 4)\,x^2 + (k + 1)\,x + 1 = 0$ has equal roots.* **[CBSE 2000 C, 2013]**

SOLUTION Here, $a = k + 4, b = k + 1$ and $c = 1$. Let D be the discriminatn of this equation. Then,

$$D = b^2 - 4ac = (k+1)^2 - 4(k+4) = k^2 - 2k - 15 = (k-5)(k+3)$$

If the roots of the given equation are equal, then

$$D = 0 \Rightarrow (k-5)(k+3) = 0 \Rightarrow k = 5, -3.$$

EXAMPLE 8 *If* -5 *is a root of the quadratic equation* $2x^2 + px - 15 = 0$ *and the quadratic equation* $p(x^2 + x) + k = 0$ *has equal roots, find the value of k.* [CBSE 2002, 2009]

SOLUTION It is given -5 is a root of the equation $2x^2 + px - 15 = 0$. Therefore, $x = -5$ satisfies it.

i.e. $$2(-5)^2 - 5p - 15 = 0 \Rightarrow 50 - 5p - 15 = 0 \Rightarrow 5p = 35 \Rightarrow p = 7$$

Putting $p = 7$ in $p(x^2 + x) + k = 0$, we get $7x^2 + 7x + k = 0$.

This equation will have equal roots, if its disriminant is zero.

i.e. $$49 - 4 \times 7 \times k = 0 \Rightarrow k = \frac{49}{28} \Rightarrow k = \frac{7}{4}$$

LEVEL-2

EXAMPLE 9 *Find the values of k for which the equation* $x^2 - 4x + k = 0$ *has distinct real roots.*

SOLUTION The given equation is $x^2 - 4x + k = 0$. Here, $a = 1, b = -4$ and $c = k$.

$\therefore$ $$D = (-4)^2 - 4 \times 1 \times k = 16 - 4k$$

The given equation will have real and distinct roots, if

$$D > 0 \Rightarrow 16 - 4k > 0 \Rightarrow 16 > 4k \Rightarrow 4k < 16 \Rightarrow k < \frac{16}{4} = 4$$

Hence, the given equation will have distinct roots, if $k < 4$.

EXAMPLE 10 *Determine the positive values of 'k' for which the equation* $x^2 + kx + 64 = 0$ *and* $x^2 - 8x + k = 0$ *will both have real roots.* [CBSE 2016]

SOLUTION Given equations are

$$x^2 + kx + 64 = 0 \quad \ldots(i)$$

and, $$x^2 - 8x + k = 0 \quad \ldots(ii)$$

Let D_1 and D_2 be the discriminants of equations (i) and (ii) respectively. Then,

$$D_1 = k^2 - 4 \times 64 = k^2 - 256 \text{ and, } D_2 = (-8)^2 - 4k = 64 - 4k$$

Both the equations will have real roots, if

$$D_1 \geq 0 \text{ and } D_2 \geq 0$$

$\Rightarrow$ $k^2 - 256 \geq 0$ and $64 - 4k \geq 0$

$\Rightarrow$ $k^2 \geq 256$ and $64 \geq 4k$

$\Rightarrow$ $k \geq 16\ (\because k > 0)$ and $k \leq 16$

$\Rightarrow$ $k = 16$

Hence, both the equations will have real roots, when $k = 16$.

EXAMPLE 11 *Find the values of k for which the given equation has real roots:*

(i) $kx^2 - 6x - 2 = 0$ (ii) $9x^2 + 3kx + 4 = 0$ (iii) $5x^2 - kx + 1 = 0$

SOLUTION (i) We have, $kx^2 - 6x - 2 = 0$. Here, $a = k, b = -6$ and $c = -2$

$\therefore \quad D = b^2 - 4ac = (-6)^2 - 4 \times k \times -2 = 36 + 8k$

The given equation will have real roots, if

$$D \geq 0 \Rightarrow 36 + 8k \geq 0 \Rightarrow 8k \geq -36 \Rightarrow k \geq \frac{36}{8} \Rightarrow k \geq -\frac{9}{2}$$

(ii) The given equation is $9x^2 + 3kx + 4 = 0$. Here, $a = 9, b = 3k$ and $c = 4$

$\therefore \quad D = b^2 - 4ac = 9k^2 - 4 \times 9 \times 4 = 9k^2 - 144$

The given equation will have real roots, if

$D \geq 0$

$\Rightarrow \quad 9k^2 - 144 \geq 0$

$\Rightarrow \quad 9(k^2 - 16) \geq 0$

$\Rightarrow \quad k^2 - 16 \geq 0 \qquad [\because ab > 0 \text{ and } a > 0 \Rightarrow b > 0]$

$\Rightarrow \quad k \leq -4 \text{ or } k \geq 4 \qquad [\because x^2 - a^2 \geq 0 \Rightarrow x \leq -a \text{ or, } x \geq a]$

(iii) The given equation is $5x^2 - kx + 1 = 0$. Here, $a = 5, b = -k$ and $c = 1$.

$\therefore \quad D = b^2 - 4ac = (-k)^2 - 4 \times 5 \times 1 = k^2 - 20$

The given equation will have real roots, if

$D \geq 0$

$\Rightarrow \quad k^2 - 20 \geq 0$

$\Rightarrow \quad k \leq -\sqrt{20} \text{ or, } k \geq \sqrt{20} \qquad [\because x^2 - a^2 \geq 0 \Rightarrow x \leq -a \text{ or, } x \geq a]$

EXAMPLE 12 *If p, q, r are real and $p \neq q$, then show that the roots of the equation $(p-q)x^2 + 5(p+q)x - 2(p-q) = 0$ are real and unequal.*

SOLUTION The given equation is $(p-q)x^2 + 5(p+q)x - 2(p-q) = 0$.

Here, $\quad a = p - q, b = 5(p+q)$ and $c = -2(p-q)$

Let D be the discriminant of the given equation. Then,

$$D = b^2 - 4ac = 25(p+q)^2 - 4(p-q) \times -2(p-q) = 25(p+q)^2 + 8(p-q)^2$$

We find that: $25(p+q)^2 > 0$ and $8(p-q)^2 > 0 \qquad [\because p \neq q]$

$\therefore \quad D = 25(p+q)^2 + 8(p-q)^2 > 0$

Hence, roots of the given equation are real and unequal.

EXAMPLE 13 *Find the values of k for which the equation $x^2 + 5kx + 16 = 0$ has no real roots.*

SOLUTION The given equation is $x^2 + 5kx + 16 = 0$. Here, $a = 1, b = 5k$ and $c = 16$.

The discriminant D of this equation is given by

$$D = b^2 - 4ac = (5k)^2 - 4 \times 1 \times 16 = 25k^2 - 16$$

The given equation will have no real roots, if

$D < 0$

$\Rightarrow \quad 25k^2 - 64 < 0$

$\Rightarrow \quad 25\left(k^2-\frac{64}{25}\right)<0$

$\Rightarrow \quad k^2-\frac{64}{25}<0 \qquad [\because ab<0 \text{ and } a>0 \Rightarrow b<0]$

$\Rightarrow \quad -\frac{8}{5}<k<\frac{8}{5} \qquad \left[\because x^2-a^2<0 \Rightarrow -a<x<a\right]$

Type III **ON DETERMINING OR PROVING THE NATURE OF THE ROOTS**

EXAMPLE 14 *If p, q, r and s are real numbers such that $pr = 2(q+s)$, then show that at least one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.*

SOLUTION We have,

$$x^2+px+q=0 \qquad \text{...(i)}$$

and, $$x^2+rx+s=0 \qquad \text{...(ii)}$$

Let D_1 and D_2 be the discriminants of equations (i) and (ii) respectively. Then,

$$D_1=p^2-4q \text{ and } D_2=r^2-4s$$

$\Rightarrow \quad D_1+D_2=p^2-4q+r^2-4s=(p^2+r^2)-4(q+s)$

$\Rightarrow \quad D_1+D_2=p^2+r^2-4\left(\frac{pr}{2}\right) \qquad \left[\because pr=2(q+s) \therefore q+s=\frac{pr}{2}\right]$

$\Rightarrow \quad D_1+D_2=p^2+r^2-2pr=(p-r)^2 \geq 0 \qquad \left[\because (p-r)^2 \geq 0 \text{ for all real } p, r\right]$

$\Rightarrow$ At least one of D_1 and D_2 is greater than or equal to zero

$\Rightarrow$ At least one of the two equations has real roots.

EXAMPLE 15 *If the roots of the equation $x^2+2cx+ab=0$ are real unequal, prove that the equation $x^2-2(a+b)x+a^2+b^2+2c^2=0$ has no real roots.* **[CBSE 2016]**

SOLUTION The two equations are

$$x^2+2cx+ab=0 \qquad \text{...(i)}$$

and, $$x^2-2(a+b)x+a^2+b^2+2c^2=0 \qquad \text{... (ii)}$$

Let D_1 and D_2 be the discriminants of equations (i) and (ii) respectively. Then,

$$D_1=(2c)^2-4\times1\times ab=4c^2-4ab=4(c^2-ab)$$

and, $$D_2=[-2(a+b)]^2-4\times1\times(a^2+b^2+2c^2)$$

$\Rightarrow \quad D_2=4(a+b)^2-4(a^2+b^2+2c^2)$

$\Rightarrow \quad D_2=4\{a^2+b^2+2ab-a^2-b^2-2c^2\}$

$\Rightarrow \quad D_2=4(2ab-2c^2)=-8(c^2-ab)$

It is given that the roots of equation (i) are real and unequal. Therefore,

$$D_1>0$$

$\Rightarrow \quad 4(c^2 - ab) > 0$

$\Rightarrow \quad c^2 - ab > 0$

$\Rightarrow \quad -8(c^2 - ab) < 0$

$\Rightarrow \quad D_2 < 0$

$\Rightarrow$ Roots of equations (ii) are not real.

EXAMPLE 16 *Prove that the equation* $x^2(a^2 + b^2) + 2x(ac + bd) + (c^2 + d^2) = 0$ *has no real root, if* $ad \neq bc$.

SOLUTION Let D be the discriminant of the equation $(a^2+b^2)x^2+2x(ac+bd)+(c^2+d^2)=0$. Then,

$$D = 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2)$$

$$\Rightarrow \quad D = 4[(ac + bd)^2 - (a^2 + b^2)(c^2 + d^2)]$$

$$\Rightarrow \quad D = 4[a^2c^2 + b^2d^2 + 2ac.bd - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2]$$

$$\Rightarrow \quad D = 4[2ac.bd - a^2d^2 - b^2c^2] = -4[a^2d^2 + b^2c^2 - 2ad.bc] = -4(ad - bc)^2$$

It is given that $ad \neq bc$.

$$\therefore \quad ad - bc \neq 0 \Rightarrow (ad - bc)^2 > 0 \Rightarrow -4(ad - bc)^2 < 0 \Rightarrow D < 0$$

Hence, the given equation has no real roots.

EXERCISE 4.6

LEVEL-1

1. Determine the nature of the roots of the following quadratic equations:
   (i) $2x^2 - 3x + 5 = 0$ [NCERT]
   (ii) $2x^2 - 6x + 3 = 0$ [NCERT]
   (iii) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$
   (iv) $3x^2 - 4\sqrt{3}x + 4 = 0$ [NCERT]
   (v) $3x^2 - 2\sqrt{6}x + 2 = 0$
2. Find the values of k for which the roots are real and equal in each of the following equations:
   (i) $kx^2 + 4x + 1 = 0$
   (ii) $kx^2 - 2\sqrt{5}x + 4 = 0$
   (iii) $3x^2 - 5x + 2k = 0$
   (iv) $4x^2 + kx + 9 = 0$
   (v) $2kx^2 - 40x + 25 = 0$
   (vi) $9x^2 - 24x + k = 0$
   (vii) $4x^2 - 3kx + 1 = 0$
   (viii) $x^2 - 2(5 + 2k)x + 3(7 + 10k) = 0$
   (ix) $(3k + 1)x^2 + 2(k + 1)x + k = 0$
   (x) $kx^2 + kx + 1 = -4x^2 - x$
   (xi) $(k + 1)x^2 + 2(k + 3)x + (k + 8) = 0$
   (xii) $x^2 - 2kx + 7k - 12 = 0$
   (xiii) $(k + 1)x^2 - 2(3k + 1)x + 8k + 1 = 0$
   (xiv) $5x^2 - 4x + 2 + k(4x^2 - 2x - 1) = 0$
   (xv) $(4 - k)x^2 + (2k + 4)x + (8k + 1) = 0$
   (xvi) $(2k+1)x^2 + 2(k+3)x + (k+5) = 0$
   (xvii) $4x^2 - 2(k + 1)x + (k + 4) = 0$
   (xviii) $4x^2 - 2(k + 1)x + (k + 1) = 0$ [CBSE 2017]

3. In the following, determine the set of values of k for which the given quadratic equation has real roots:

(i) $2x^2 + 3x + k = 0$ (ii) $2x^2 + x + k = 0$ (iii) $2x^2 - 5x - k = 0$

(iv) $kx^2 + 6x + 1 = 0$ (v) $3x^2 + 2x + k = 0$

4. Find the values of k *for which the following equations have real and equal roots:*

(i) $x^2 - 2(k+1)x + k^2 = 0$ **[CBSE 2001 C, 2013]**

(ii) $k^2x^2 - 2(2k-1)x + 4 = 0$ **[CBSE 2001 C]**

(iii) $(k+1)x^2 - 2(k-1)x + 1 = 0$ **[CBSE 2002 C]**

(iv) $x^2 + k(2x + k - 1) + 2 = 0$ **[CBSE 2017]**

5. Find the values of k for which the following equations have real roots

(i) $2x^2 + kx + 3 = 0$ **[NCERT]** (ii) $kx(x-2) + 6 = 0$ **[NCERT, 2013]**

(iii) $x^2 - 4kx + k = 0$ **[CBSE 2012]** (iv) $kx(x - 2\sqrt{5}) + 10 = 0$ **[CBSE 2013]**

(v) $kx(x-3) + 9 = 0$ **[CBSE 2014]** (vi) $4x^2 + kx + 3 = 0$ **[CBSE 2014]**

6. Find the values of k for which the given quadratic equation has real and distinct roots:

(i) $kx^2 + 2x + 1 = 0$ (ii) $kx^2 + 6x + 1 = 0$

7. For what value of k, $(4-k)x^2 + (2k+4)x + (8k+1) = 0$, is a perfect square.

8. Find the least positive value of k for which the equation $x^2 + kx + 4 = 0$ has real roots.

9. Find the values of k for which the quadratic equation $(3k+1)x^2 + 2(k+1)x + 1 = 0$ has equal roots. Also, find the roots. **[CBSE 2014]**

10. Find the values of p for which the equadratic equation $(2p+1)x^2 - (7p+2)x + (7p-3) = 0$ has equal roots. Also, find these roots. **[CBSE 2014]**

11. If -5 is a root of the quadratic equation $2x^2 + px - 15 = 0$ and the quadratic equation $p(x^2 + x) + k = 0$ has equal roots, find the value of k. **[CBSE 2014]**

12. If 2 is a root of the quadratic equation $3x^2 + px - 8 = 0$ and the quadratic equation $4x^2 - 2px + k = 0$ has equal roots, find the value of k. **[CBSE 2014]**

13. If 1 is a root of the quadratic equation $3x^2 + ax - 2 = 0$ and the quadratic equation $a(x^2 + 6x) - b = 0$ has equal roots, find the value of b.

14. Find the value of p for which the quadratic equation $(p+1)x^2 - 6(p+1)x + 3(p+q) = 0$, $p \neq -1$ has equal roots. Hence, find the roots of the equation. **[CBSE 2015]**

15. Determine the nature of the roots of the following quadratic equations:

(i) $(x - 2a)(x - 2b) = 4ab$ (ii) $9a^2b^2x^2 - 24abcdx + 16c^2d^2 = 0, a \neq 0, b \neq 0$

(iii) $2(a^2 + b^2)x^2 + 2(a+b)x + 1 = 0$ (iv) $(b+c)x^2 - (a+b+c)x + a = 0$

16. Determine the set of values of k for which the following quadratic equations have real roots:

(i) $x^2 - kx + 9 = 0$ (ii) $2x^2 + kx + 2 = 0$

(iii) $4x^2 - 3kx + 1 = 0$ (iv) $2x^2 + kx - 4 = 0$

LEVEL-2

If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, then prove that $2b = a + c$. **[CBSE 2002 C]**

18. If the roots of the equation $(a^2+b^2)x^2-2(ac+bd)x+(c^2+d^2)=0$ are equal, prove that $\frac{a}{b}=\frac{c}{d}$. [CBSE 2017]

19. If the roots of the equations $ax^2+2bx+c=0$ and $bx^2-2\sqrt{ac}\,x+b=0$ are simultaneously real, then prove that $b^2=ac$.

20. If p, q are real and $p \neq q$, then show that the roots of the equation $(p-q)x^2+5\ (p+q)x-2(p-q)=0$ are real and unequal.

21. If the roots of the equation $(c^2-ab)x^2-2(a^2-bc)x+b^2-ac=0$ are equal, prove that either $a=0$ or $a^3+b^3+c^3=3abc$.

22. Show that the equation $2(a^2+b^2)x^2+2(a+b)x+1=0$ has no real roots, when $a \neq b$.

23. Prove that both the roots of the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are real but they are equal only when $a=b=c$.

24. If a, b, c are real numbers such that $ac \neq 0$, then show that at least one of the equations $ax^2+bx+c=0$ and $-ax^2+bx+c=0$ has real roots.

25. If the equation $(1+m^2)x^2+2mcx+(c^2-a^2)=0$ has equal roots, prove that $c^2=a^2(1+m^2)$. [CBSE 2017]

ANSWERS

1. (i) Not real (ii) Real and distinct (iii) Not real
(iv) Real and equal (v) Real and equal

2. (i) $k=4$ (ii) $k=\frac{5}{4}$ (iii) $k=\frac{25}{24}$
(iv) $k=\pm 12$ (v) $k=8$ (vi) $k=16$
(vii) $k=\pm\frac{4}{3}$ (viii) $k=2, \frac{1}{2}$ (ix) $k=\frac{-1}{2}, 1$
(x) $k=5, -3$ (xi) $k=\frac{1}{3}$ (xii) $k=4, 3$
(xiii) $k=0, 3$ (xiv) $k=-\frac{6}{5}, 1$ (xv) $k=0, 3$
(xvi) $k=\frac{-5\pm\sqrt{41}}{2}$ (xvii) $k=-3, -7$ (xviii) $k=-1, 3$

3. (i) $k \leq \frac{9}{8}$ (ii) $k \leq \frac{1}{8}$ (iii) $k \geq -\frac{25}{8}$ (iv) $k \leq 9$

4. (i) $k=\frac{-1}{2}$ (ii) $k=\frac{1}{4}$ (iii) $k=0, 3$ (iv) $k=2$

5. (i) $k=\pm 2\sqrt{6}$ (ii) $k=6$ (iii) $k=0, \frac{1}{4}$
(iv) $k=2$ (v) $k=4$ (vi) $k=\pm 4\sqrt{3}$

6. (i) $k<1$ (ii) $k<9$ 7. $k=0, 3$ 8. $k=4$

9. $k=0, 1; x=-1, -\frac{1}{2}$ 10. $p=4, -\frac{4}{7}; x=\frac{5}{3}, 7$ 11. $k=2$ 12. 1 13. -9

14. $p=3; x=3$ 15. (i) Real and distinct (ii) Real and equal
(iii) Not real (iv) Real and unequal

16. (i) $k \leq -6$ or $k \geq 6$ (ii) $k \leq -4$ or, $k \geq 4$ (iii) $k \leq -\frac{4}{3}$ or, $k \geq \frac{4}{3}$ (iv) $k \in R$

HINTS TO SELECTED PROBLEMS

17. $D = 0 \Rightarrow (c - a)^2 - 4(b - c)(a - b) = 0$

$$\Rightarrow \quad c^2 + a^2 + 4b^2 - 2ac - 4ab + 4ac - 4bc = 0 \Rightarrow (c + a - 2b)^2 = 0 \Rightarrow c + a - 2b = 0$$

ALITER Clearly, $x = 1$ satisfies the given equation. Since it has equal roots. So, both roots are equal to 1.

$$\therefore \qquad \text{Product of the roots} = 1 \Rightarrow \frac{a-b}{b-c} = 1 \Rightarrow a - b = b - c \Rightarrow 2b = a + c$$

19. Let D_1 and D_2 be the discriminants of the two equations. Then,

$$D_1 \geq 0 \text{ and } D_2 \geq b^2 \Rightarrow 4b^2 - 4ac \geq 0 \text{ and } 4ac - 4b^2 \geq 0 \Rightarrow b^2 \geq ac \text{ and } ac \geq b^2 \Rightarrow b^2 = ac.$$

21. We have, $D = 4a(a^3 + b^3 + c^3 - 3abc)$. For the roots to be equal, we must have

$$D = 0 \Rightarrow 4a(a^3 + b^3 + c^3 - 3abc) = 0 \Rightarrow a = 0 \text{ or, } a^3 + b^3 + c^3 = 3abc$$

23. The given equation is $3x^2 - 2x(a + b + c) + (ab + bc + ca) = 0$. Let D be its discriminant. Then,

$$D = 4(a + b + c)^2 - 12(ab + bc + ca)$$

$$\Rightarrow \quad D = 4[(a + b + c)^2 - 3(ab + bc + ca)]$$

$$\Rightarrow \quad D = 4[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$\Rightarrow \quad D = 2[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 2[(a - b)^2 + (b - c)^2 + (c - a)^2$$

Clearly $D \geq 0$. If $D = 0$, then,

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0 \Rightarrow a - b = 0, b - c = 0, c - a = 0.$$

4.8 SOLUTIONS OF PROBLEMS INVOLVING QUADRATIC EQUATIONS

In this section, we will discuss some simple problems on practical applications of quadratic equation. In this type of problems we first formulate a quadratic equation whose solution is a solution of the given problem. Sometimes it may happen that, out of the roots of the quadratic equation only one has a meaning for the problem. Any root of the quadratic equation, which does not satisfy the condition of the problem will be rejected.

In order to solve this type of problems, we may use the following algorithm.

ALGORITHM

STEP I *Translate the word problem into symbolic language and formulate the quadratic equation.*

STEP II *Solve the quadratic equation formed in Step I.*

STEP III *Translate the solution into verbal language and reject the solution which does not have a meaning for the problem.*

4.8.1 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON NUMBERS

In this section, we will discuss problems on numbers. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The sum of the squares of two consecutive natural numbers is 313. Find the numbers.*

SOLUTION Let two consecutive natural numbers be x and $x+1$. Then,

$$x^2+(x+1)^2=313 \quad \text{[Given]}$$

$\Rightarrow \quad 2x^2+2x+1=313$

$\Rightarrow \quad 2x^2+2x-312=0$

$\Rightarrow \quad x^2+x-156=0$

$\Rightarrow \quad x^2+13x-12x-156=0$

$\Rightarrow \quad x(x+13)-12(x+13)=0$

$\Rightarrow \quad (x+13)(x-12)=0$

$\Rightarrow \quad x+13=0$ or, $x-12=0 \Rightarrow x=12$ or, $x=-13$

Since x, being a natural number, cannot be negative. Therefore, $x=12$.
Hence, the two consecutive natural numbers are 12 and 13.

EXAMPLE 2 *The sum of two numbers is 15. If the sum of their reciprocals is $\frac{3}{10}$, find the numbers.* [CBSE 2000, 2005]

SOLUTION Let the required numbers be x and $15-x$. Then,

$$\frac{1}{x}+\frac{1}{15-x}=\frac{3}{10}$$

$\Rightarrow \quad \dfrac{15-x+x}{x(15-x)}=\dfrac{3}{10}$

$\Rightarrow \quad \dfrac{15}{x(15-x)}=\dfrac{3}{10}$

$\Rightarrow \quad 150=3x(15-x)$

$\Rightarrow \quad 150=45x-3x^2$

$\Rightarrow \quad x^2-15x+50=0$

$\Rightarrow \quad x^2-10x-5x+50=0$

$\Rightarrow \quad x(x-10)-5(x-10)=0$

$\Rightarrow \quad (x-10)(x-5)=0$

$\Rightarrow \quad x-10=0$ or, $x-5=0 \Rightarrow x=10$ or, $x=5$

Hence, the two numbers are 10 and 5.

EXAMPLE 3 *The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.*

SOLUTION Let the required number be x. Then,

$$x+\frac{1}{x}=2\frac{1}{30}$$

$\Rightarrow \quad \dfrac{x^2+1}{x}=\dfrac{61}{30}$

$\Rightarrow \quad 30x^2 + 30 = 61x$

$\Rightarrow \quad 30x^2 - 61x + 30 = 0$

$\Rightarrow \quad 30x^2 - 36x - 25x + 30 = 0$

$\Rightarrow \quad 6x(5x - 6) - 5(5x - 6) = 0$

$\Rightarrow \quad (6x - 5)(5x - 6) = 0$

$\Rightarrow \quad 6x - 5 = 0$ or, $5x - 6 = 0 \Rightarrow x = \frac{5}{6}$ or, $x = \frac{6}{5}$

Hence, the required number is 5/6 or, 6/5

EXAMPLE 4 *Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.*

SOLUTION Let the larger part be x. Then, the smaller part $= 16 - x$.
By hypothesis, we have

$2x^2 = (16 - x)^2 + 164$

$\Rightarrow \quad 2x^2 - (16 - x)^2 - 164 = 0$

$\Rightarrow \quad x^2 + 32x - 420 = 0$

$\Rightarrow \quad (x + 42)(x - 10) = 0$

$\Rightarrow \quad x = -42$ or, $x = 10$

$\Rightarrow \quad x = 10$ $\quad [\because x > 0 \therefore x = -42$ is not possible$]$

Hence, the required parts are 10 and 6.

EXAMPLE 5 *The sum of the squares of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers.*

SOLUTION Let the smaller number be x. Then,

Square of larger number $= 18x$

Also, Square of the smaller number $= x^2$

It is given that the sum of the square of the integers is 208.

$\therefore \quad x^2 + 18x = 208$

$\Rightarrow \quad x^2 + 18x - 208 = 0$

$\Rightarrow \quad x^2 + 26x - 8x - 208 = 0$

$\Rightarrow \quad (x + 26)(x - 8) = 0 \Rightarrow x = 8, x = -26$

But, the numbers are positive. Therefore, $x = 8$

$\therefore$ Square of the larger number $= 18x = 18 \times 8 = 144$

$\Rightarrow$ Larger number $= \sqrt{144} = 12$

Hence, the numbers are 8 and 12.

EXAMPLE 6 *The difference of the squares of two numbers is 45. The square of the smaller number is 4 times the larger number. Determine the numbers.*

SOLUTION Let the larger number be x. Then,

Square of the smaller number $= 4x$

Also, Square of the larger number $= x^2$

It is given that the difference of the squares of the numbers is 45.

$\therefore \quad x^2 - 4x = 45$

$\Rightarrow \quad x^2 - 4x - 45 = 0$

$\Rightarrow \quad x^2 - 9x + 5x - 45 = 0$

$\Rightarrow \quad x(x-9) + 5(x-9) = 0$

$\Rightarrow \quad (x-9)(x+5) = 0$

$\Rightarrow \quad x - 9 = 0 \text{ or, } x + 5 = 0 \Rightarrow x = 9, -5$

CASE I *When* $x = 9$: In this case, we have

Square of the smaller number $= 4x = 36$

$\therefore$ Smaller number $= \pm 6$.

Thus, the numbers are 9, 6 or 9, − 6

CASE II *When* $x = -5$: In this case, we have

Square of the smaller number $= 4x = -20$. But, square of a number is always positive. Therefore, $x = -5$ is not possible.

Hence, the numbers are 9, 6 or 9, − 6.

EXAMPLE 7 *A two digit number is such that the product of its digits is 18. When 63 is subtracted from the number, the digits interchange their places. Find the number.* [CBSE 2006C]

SOLUTION Let the tens digit be x. Then, the units digits $= \frac{18}{x}$.

$\therefore \quad \text{Number} = 10x + \frac{18}{x}$

and, Number obtained by interchanging the digits $= 10 \times \frac{18}{x} + x$

$\therefore \quad \left(10x + \frac{18}{x}\right) - \left(10 \times \frac{18}{x} + x\right) = 63$

$\Rightarrow \quad 10x + \frac{18}{x} - \frac{180}{x} - x = 63$

$\Rightarrow \quad 9x - \frac{162}{x} - 63 = 0$

$\Rightarrow \quad 9x^2 - 63x - 162 = 0$

$\Rightarrow \quad x^2 - 7x - 18 = 0$

$\Rightarrow \quad (x-9)(x+2) = 0 \Rightarrow x = 9 \text{ or, } x = -2$

But, a digit can never be negative. So, $x = 9$. Hence, the required number $= 10 \times 9 + \frac{18}{9} = 92$.

EXAMPLE 8 *A two digit number is such that the product of the digits is 14. When 45 is added to the number, then the digits are reversed. Find the number.*

SOLUTION Let the tens digit be x. Then, units digit $= \frac{14}{x}$.

$\therefore \quad \text{Number} = 10x + \frac{14}{x}$

and, Number formed by reversing the digits $= 10 \times \frac{14}{x} + x$

$$\therefore \quad 10x + \frac{14}{x} + 45 = 10 \times \frac{14}{x} + x \qquad \text{[Given]}$$

$$\Rightarrow \quad 10x + \frac{14}{x} + 45 = \frac{140}{x} + x$$

$$\Rightarrow \quad 9x - \frac{126}{x} + 45 = 0$$

$$\Rightarrow \quad 9x^2 + 45x - 126 = 0$$

$$\Rightarrow \quad x^2 + 5x - 14 = 0$$

$$\Rightarrow \quad x^2 + 7x - 2x - 14 = 0$$

$$\Rightarrow \quad (x+7)(x-2) = 0 \Rightarrow x = -7 \text{ or, } x = 2 \Rightarrow x = 2$$

Hence, the required number $= 10 \times 2 + \frac{14}{2} = 27$.

EXAMPLE 9 *Find two consecutive odd positive integers, sum of whose squares is 290.* [NCERT, CBSE 2014]

SOLUTION Let x be an odd positive integer. Then, an odd positive integer just greater than x is $x + 2$. It is given that

$$x^2 + (x+2)^2 = 290$$

$$\Rightarrow \quad 2x^2 + 4x + 4 = 290$$

$$\Rightarrow \quad 2x^2 + 4x - 286 = 0$$

$$\Rightarrow \quad x^2 + 2x - 143 = 0$$

$$\Rightarrow \quad x^2 + 13x - 11x - 143 = 0$$

$$\Rightarrow \quad x(x+13) - 11(x+13) = 0$$

$$\Rightarrow \quad (x+13)(x-11) = 0$$

$$\Rightarrow \quad x - 11 = 0 \qquad [\because x > 0 \therefore x + 13 \neq 0]$$

$$\Rightarrow \quad x = 11$$

Hence, required integers are 11 and 13.

LEVEL-2

EXAMPLE 10 *If the sum of first n even natural numbers is 420, find the value of n.*

SOLUTION We have,

$$2 + 4 + 6 + 8 + \dots \text{ to } n \text{ terms} = 420$$

$$\Rightarrow \quad \frac{n}{2}[2 \times 2 + (n-1) \times 2] = 420$$

$$\Rightarrow \quad n(2 + n - 1) = 420$$

$$\Rightarrow \quad n(n+1) = 420$$

$$\Rightarrow \quad n^2 + n - 420 = 0$$

$$\Rightarrow \quad n^2 + 21n - 20n - 420 = 0$$

$$\Rightarrow \quad n(n+21) - 20(n+21) = 0$$

$$\Rightarrow \quad (n+21)(n-20) = 0$$

$$\Rightarrow \quad n = 20, -21 \Rightarrow n = 20 \qquad [\because n \text{ is a natural number} \therefore n > 0]$$

EXAMPLE 11 *The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.* [CBSE 2016]

SOLUTION Let the numerator of the fraction be x. Then,

Denominator $= 2x + 1$ [Given]

$\therefore$ Fraction $= \frac{x}{2x+1} \Rightarrow$ Reciprocal of the fraction $= \frac{2x+1}{x}$

It is given that the sum of the fraction and its reciprocal is $2\frac{16}{21}$.

$$\therefore \quad \frac{x}{2x+1} + \frac{2x+1}{x} = 2\frac{16}{21}$$

$$\Rightarrow \quad \frac{x^2 + (2x+1)^2}{x(2x+1)} = \frac{58}{21}$$

$$\Rightarrow \quad \frac{5x^2 + 4x + 1}{2x^2 + x} = \frac{58}{21}$$

$$\Rightarrow \quad 21(5x^2 + 4x + 1) = 58(2x^2 + x)$$

$$\Rightarrow \quad 105x^2 + 84x + 21 = 116x^2 + 58x$$

$$\Rightarrow \quad 11x^2 - 26x - 21 = 0$$

$$\Rightarrow \quad 11x^2 - 33x + 7x - 21 = 0$$

$$\Rightarrow \quad 11x(x-3) + 7(x-3) = 0$$

$$\Rightarrow \quad (11x+7)(x-3) = 0$$

$$\Rightarrow \quad x = 3, -\frac{7}{11} \Rightarrow x = 3 \qquad [\because x \text{ is a natural number } \therefore x > 0]$$

Hence, fraction $= \frac{x}{2x+1} = \frac{3}{7}$

EXAMPLE 12 *A two-digit number is four times the sum and three times the product of its digits. Find the number.* [CBSE 2016]

SOLUTION Let the digits at tens and units place of the number be x and y respectively. Then,

Number $= 10x + y$

It is given that

Number $= 4 \times$ Sum of the digits. Also, Number $= 3 \times$ Product of digits

$\Rightarrow \quad 10x + y = 4(x + y)$ and $10x + y = 3xy$

$\Rightarrow \quad 6x - 3y = 0$ and $10x + y = 3xy$

$\Rightarrow \quad y = 2x$ and $10x + y = 3xy$

$\Rightarrow \quad 10x + 2x = 3x \times 2x$ [On eliminating y]

$\Rightarrow \quad 6x^2 - 12x = 0$

$\Rightarrow \quad 6x(x-2) = 0 \Rightarrow x = 0$ or, $x = 2$

Since the given number is a two-digit number. So, its tens digit cannot be zero.

$\therefore \quad x = 2 \Rightarrow y = 2 \times 2 = 4$ [$\because y = 2x$]

Hence, required number $= 10x + y = 10 \times 2 + 4 = 24$.

EXAMPLE 13 *If the sum of n successive odd natural numbers starting from 3 is 48, find the value of n.*

SOLUTION We have,

$$3+5+7+9+\dots \text{ to } n \text{ terms} = 48$$

$$\Rightarrow \quad \frac{n}{2}\{2\times 3+(n-1)\times 2\} = 48 \qquad \left[\text{Using}: S_n = \frac{n}{2}\{2a+(n-1)d\} \text{ where } a=3 \text{ and } d=2\right]$$

$$\Rightarrow \quad n(3+n-1) = 48$$

$$\Rightarrow \quad n^2+2n-48 = 0$$

$$\Rightarrow \quad n^2+8n-6n-48 = 0$$

$$\Rightarrow \quad n(n+8)-6(n+8) = 0$$

$$\Rightarrow \quad (n+8)(n-6) = 0$$

$$\Rightarrow \quad n = -8 \text{ or, } n = 6 \Rightarrow n = 6 \qquad [\because n > 0]$$

LEVEL-3

EXAMPLE 14 *One-fourth of a herd of camels was seen in the forest. Twice the square root of the herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels.*

SOLUTION Let the total number of camels be x. Then,

Number of camels seen in the forest $= \frac{x}{4}$

Number of camels gone to mountains $= 2\sqrt{x}$

Number of camels on the bank of river = 15

Total number of camels $= \frac{x}{4}+2\sqrt{x}+15$

By hypothesis, we have

$$\frac{x}{4}+2\sqrt{x}+15 = x$$

$$\Rightarrow \quad 3x-8\sqrt{x}-60 = 0$$

$$\Rightarrow \quad 3y^2-8y-60 = 0, \text{ where } x = y^2$$

$$\Rightarrow \quad 3y^2-18y+10y-60 = 0$$

$$\Rightarrow \quad 3y(y-6)+10(y-6) = 0$$

$$\Rightarrow \quad (3y+10)(y-6) = 0$$

$$\Rightarrow \quad y = 6 \text{ or, } y = -\frac{10}{3}$$

Now, $y = -\frac{10}{3} \Rightarrow x = \left(-\frac{10}{3}\right)^2 = \frac{100}{9}$ $\qquad [\because x = y^2]$

But, the number of camels cannot be a fraction.

$\therefore \quad y = 6 \Rightarrow x = 6^2 = 36$ $\qquad [\because x = y^2]$

Hence, the number of camels = 36.

EXAMPLE 15 *O Girl! Out of a group of swans, $\frac{7}{2}$ times the square root of the number are playing on the shore of a tank. The two remaining ones are playing, with amorous fight, in the water. What is the total number of swans?*

SOLUTION Let the total number of swans be x. Then,

$$\text{Number of swans playing on the shore of the tank} = \frac{7}{2}\sqrt{x}$$

It is given that there are two remaining swans.

$$\therefore \quad x = \frac{7}{2}\sqrt{x} + 2$$

$$\Rightarrow \quad x - \frac{7}{2}\sqrt{x} - 2 = 0$$

$$\Rightarrow \quad y^2 - \frac{7}{2}y - 2 = 0, \text{ where } y^2 = x$$

$$\Rightarrow \quad 2y^2 - 7y - 4 = 0$$

$$\Rightarrow \quad 2y^2 - 8y + y - 4 = 0$$

$$\Rightarrow \quad 2y(y-4) + (y-4) = 0$$

$$\Rightarrow \quad (y-4)(2y+1) = 0$$

$$\Rightarrow \quad y = 4 \text{ or, } y = -\frac{1}{2}$$

$$\Rightarrow \quad y = 4 \qquad \left[\because y = -\frac{1}{2} \text{ is not possible}\right]$$

$$\Rightarrow \quad x = y^2 = 4^2 = 16$$

Hence, the total number of swans is 16.

EXERCISE 4.7

LEVEL-1

1. Find two consecutive numbers whose squares have the sum 85. [CBSE 2000]
2. Divide 29 into two parts so that the sum of the squares of the parts is 425.
3. Two squares have sides x cm and $(x+4)$ cm. The sum of their areas is 656 cm^2. Find the sides of the squares.
4. The sum of two numbers is 48 and their product is 432. Find the numbers.
5. If an integer is added to its square, the sum is 90. Find the integer with the help of quadratic equation.
6. Find the whole number which when decreased by 20 is equal to 69 times the reciprocal of the number.
7. Find two consecutive natural numbers whose product is 20.
8. The sum of the squares of two consecutive odd positive integers is 394. Find them. [CBSE 2009, 2017]
9. The sum of two numbers is 8 and 15 times the sum of their reciprocals is also 8. Find the numbers.
10. The sum of a number and its positive square root is 6/25. Find the number.
11. The sum of a number and its square is 63/4, find the numbers.

12. There are three consecutive integers such that the square of the first increased by the product of the other two gives 154. What are the integers?
13. The product of two successive integral multiples of 5 is 300. Determine the multiples.
14. The sum of the squares of two numbers is 233 and one of the numbers is 3 less than twice the other number. Find the numbers.
15. Find the consecutive even integers whose squares have the sum 340.
16. The difference of two numbers is 4. If the difference of their reciprocals is $\frac{4}{21}$, find the numbers. [CBSE 2008]
17. Find two natural numbers which differ by 3 and whose squares have the sum 117.
18. The sum of the squares of three consecutive natural numbers is 149. Find the numbers.
19. The sum of two numbers is 16. The sum of their reciprocals is 1/3. Find the numbers. [CBSE 2005]
20. Determine two consecutive multiples of 3 whose product is 270.
21. The sum of a number and its reciprocal is 17/4. Find the number.
22. A two-digit number is such that the product of its digits is 8. When 18 is subtracted from the number, the digits interchange their places. Find the number.
23. A two-digit number is such that the product of the digits is 12. When 36 is added to the number the digits interchange their places. Determine the number.
24. A two-digit number is such that the product of the digits is 16. When 54 is subtracted from the number, the digits are interchanged. Find the number.
25. Two numbers differ by 3 and their product is 504. Find the numbers. [CBSE 2002 C]
26. Two numbers differ by 4 and their product is 192. Find the numbers. [CBSE 2000 C]
27. A two digit number is 4 times the sum of its digits and twice the product of its digits. Find the number.
28. The difference of the squares of two positive integers is 180. The square of the smaller number is 8 times the larger, find the numbers. [CBSE 2014]
29. The sum of two numbers is 18. The sum of their reciprocals is 1/4. Find the numbers. [CBSE 2005]
30. The sum of two numbers a and b is 15, and the sum of their reciprocals $\frac{1}{a}$ and $\frac{1}{b}$ is 3/10. Find the numbers a and b. [CBSE 2005]
31. The sum of two numbers is 9. The sum of their reciprocals is 1/2. Find the numbers. [CBSE 2012]
32. Three consecutive positive integers are such that the sum of the square of the first and the product of other two is 46, find the integers. [CBSE 2010]
33. The difference of squares of two numbers is 88. If the larger number is 5 less than twice the smaller number, then find the two numbers. [CBSE 2010]
34. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find two numbers. [NCERT]
35. Find two consecutive odd positive integers, sum of whose squares is 970. [CBSE 2014]
36. The difference of two natural numbers is 3 and the difference of their reciprocals is $\frac{3}{28}$. Find the numbers. [CBSE 2014]
37. The sum of the squares of two consecutive odd numbers is 394. Find the numbers. [CBSE 2014]

38. The sum of the squares of two consecutive multiples of 7 is 637. Find the multiples. [CBSE 2014]

39. The sum of the squares of two consecutive even numbers is 340. Find the numbers. [CBSE 2014]

40. The numerator of a fraction is 3 less than the denominator. If 2 is added to both the numerator and the denominator, then the sum of the new fraction and the original fraction is $\frac{29}{20}$. Find the original fraction. [CBSE 2015]

41. Find a natural number whose square diminished by 84 is equal to thrice of 8 more than the given number. . [NCERT EXEMPLAR]

42. A natural number when increased by 84 equals 160 times its reciprocal. Find the number. [NCERT EXEMPLAR]

ANSWERS

1. 6, 7 or $-6, -7$	2. 13, 16	3. 16 cm, 20 cm	4. 36, 12
5. $-10, 9$	6. 23	7. 4, 5	8. 13, 15
9. 3, 5	10. $\frac{1}{25}$	11. $\frac{7}{2}$ or $-\frac{9}{2}$	12. 8, 9, 10
13. 15, 20 or $-20, -15$		14. 8, 13	15. 12, 14
16. 7, 3 or $-3, -7$	17. 6, 9	18. 6, 7, 8	19. 4, 12
20. 15, 18	21. 4 or $\frac{1}{4}$	22. 42	23. 26
24. 82	25. 21, 24, or $-24, -21$	26. 12, 16, or $-16, -12$	
27. 36	28. 8, 12	29. 12, 6	
30. $a = 5, b = 10$ or $a = 10, b = 5$		31. 3, 6	32. 4, 5, 6
33. 13, 9	34. 18, 12; 18, -12	35. 21, 23	36. 7, 4
37. 13, 15	38. 14, 21	39. 12, 14	40. $\frac{7}{10}$
41. 12	42. 8		

HINTS TO SELECTED PROBLEMS

1. Let the natural numbers be x and $x + 1$. Then, by hypothesis, we have
$$x^2 + (x+1)^2 = 85.$$

2. Let the two parts be x and $29 - x$. Then, by hypothesis, we have
$$x^2 + (29 - x)^2 = 425.$$

4. Let the numbers be x and $48 - x$. Then, by using the given condition, we have
$$x(48 - x) = 432.$$

5. We have, $x + x^2 = 90 \Rightarrow x^2 + x - 90 = 0 \Rightarrow (x + 10)(x - 9) = 0$

6. Let the whole number be x. It is given that
$$(x - 20) = 60\left(\frac{1}{x}\right) \Rightarrow x^2 - 20x - 69 = 0 \Rightarrow (x - 23)(x + 3) = 0 \Rightarrow x = 23, -3.$$

7. Let the numbers be x and $x + 1$. It is given that
$$x(x + 1) = 20 \Rightarrow x^2 + x - 20 = 0$$

8. Let the consecutive odd positive integers be $2x - 1$ and $2x + 1$. Then,
$$(2x - 1)^2 + (2x + 1)^2 = 394 \Rightarrow 8x^2 + 2 = 394 \Rightarrow 4x^2 = 392 \Rightarrow x = 6$$

9. Let the numbers be x and $8 - x$. It is given that

$$15\left(\frac{1}{x}+\frac{1}{8-x}\right)=8 \Rightarrow 15=x(8-x) \Rightarrow x^2-8x+15=0$$

10. Let the number be x. It is given that

$$x+\sqrt{x}=\frac{6}{25} \Rightarrow y^2+y=6/25, \text{ where } x=y^2 \Rightarrow 25y^2+25y-6=0.$$

11. Let the number be x. It is given that: $x+x^2=\frac{63}{4}$.
12. Let the integers be x, $x+1$ and $x+2$. Then, $x^2+(x+1)(x+2)=154$.
13. Let the successive multiples of 5 be $5x$ and $5(x+1)$. Then, $5x.5(x+1)=300 \Rightarrow x^2+x=12 \Rightarrow x^2+x-12=0$
14. Let one number be x. Then, other number $=2x-3$. It is given that: $x^2+(2x-3)^2=233$.
15. Let the consecutive even integers be $2x$ and $2x+2$. Then, by hypothesis $(2x)^2+(2x+2)^2=340 \Rightarrow 8x^2+8x-336=0 \Rightarrow x^2+x-42=0$.
17. Let the numbers be x and $x-3$. Then, $x^2+(x-3)^2=117$.
18. Let the numbers be x, $x+1$ and $x+2$. Then, $x^2+(x+1)^2+(x+2)^2=149$.
19. Let the two parts be x and $57-x$. Then, $x(57-x)=782$.
20. Let the required numbers be $3x$ and $3x+3$. Then, $(3x)(3x+3)=270$

4.8.2 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON TIME AND DISTANCE

For solving problems on time and distance, we use the following formulae:

$$\text{Speed}=\frac{\text{Distance}}{\text{Time}}, \text{Time}=\frac{\text{Distance}}{\text{Speed}}$$

Following example will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A train travels a distance of 300 km at constant speed. If the speed of the train is increased by 5 km an hour, the journey would have taken 2 hours less. Find the original speed of the train.*

SOLUTION Let x km/hr be the constant speed of the train. Then,

Time taken to cover 300 km $=\frac{300}{x}$ hrs.

Time taken to cover 300 km when the speed is increased by 5 km/hr $=\frac{300}{x+5}$ hours.

It is given that the time to cover 300 km is reduced by 2 hours.

$$\therefore \quad \frac{300}{x}-\frac{300}{x+5}=2$$

$$\Rightarrow \quad \frac{300(x+5)-300x}{x(x+5)}=2$$

$$\Rightarrow \quad \frac{300x+1500-300x}{x^2+5x}=2$$

$$\Rightarrow \quad 2x^2+10x=1500$$

$$\Rightarrow \quad x^2+5x-750=0$$

$$\Rightarrow \quad x^2+30x-25x-750=0$$

$$\Rightarrow \quad x(x+30)-25(x+30)=0$$

$\Rightarrow \quad (x+30)(x-25)=0 \Rightarrow x=25$ or, $x=-30$.

But, x cannot be negative. Therefore, $x = 25$.

Hence, the original speed of the train is 25 km/hr.

EXAMPLE 2 *The speed of a boat in still water is 15 km/hr. It can go 30 km upstream and return downstream to the original point in 4 hours 30 minutes. Find the speed of the stream.* [CBSE 2017]

SOLUTION Let the speed of the stream be x km/hr. Then,

Speed downstream = $(15 + x)$ km/hr.

$\therefore$ Speed upstream = $(15 - x)$ km/hr.

Time taken by the boat to go 30 km upstream $= \dfrac{30}{15-x}$ hours.

Time taken by the boat to return 30 km downstream $= \dfrac{30}{15+x}$ hours.

It is given that the boat returns to the same point in 4 hours 30 minutes

$$\therefore \quad \frac{30}{15-x}+\frac{30}{15+x}=\frac{9}{2}$$

$$\Rightarrow \quad \frac{30(15+x)+30(15-x)}{(15+x)(15-x)}=\frac{9}{2}$$

$$\Rightarrow \quad \frac{450+30x+450-30x}{225-x^2}=\frac{9}{2}$$

$$\Rightarrow \quad \frac{900}{225-x^2}=\frac{9}{2}$$

$$\Rightarrow \quad 9(225-x^2)=1800$$

$$\Rightarrow \quad 225-x^2=200 \Rightarrow x^2=25 \Rightarrow x=\pm 5$$

But, the speed of the stream can never be negative.

Hence, the speed of the stream is 5 km/hr.

EXAMPLE 3 *A fast train takes 3 hours less than a slow train for a journey of 600 km. If the speed of the slow train is 10 km/hr less than that of the fast train, find the speeds of the two trains.*

SOLUTION Let the speed of the slow train be x km/hr. Then, speed of the fast train is $(x + 10)$ km/hr.

Time taken by the slow train to cover 600 km $= \dfrac{600}{x}$ hrs

Time taken by the fast train to cover 600 km $= \dfrac{600}{x+10}$ hrs

$$\therefore \quad \frac{600}{x}-\frac{600}{x+10}=3$$

$$\Rightarrow \quad \frac{600(x+10)-600x}{x(x+10)}=3$$

$$\Rightarrow \quad \frac{6000}{x^2+10x}=3$$

$$\Rightarrow \quad 3(x^2+10x)=6000$$

$$\Rightarrow \quad x^2+10x-2000=0$$

$$\Rightarrow \quad x^2+50x-40x-2000=0$$

$$\Rightarrow \quad x(x+50)-40(x+50)=0$$

$\Rightarrow \quad (x+50)(x-40)=0$

$\Rightarrow \quad x=-50$ or $x=40 \Rightarrow x=40$ [$\because$ x cannot be negative]

Hence, the speeds of two trains are 40 km/hr and 50 km/hr.

EXAMPLE 4 *A plane left 30 minutes later than the schedule time and in order to reach its destination 1500 km away in time it has to increase its speed by 250 km/hr from its usual speed. Find its usual speed.* [CBSE 2016]

SOLUTION Let the usual speed of the plane be x km/hr. Then,

Time taken to cover 1500 km with the usual speed $= \frac{1500}{x}$ hrs

Time taken to cover 1500 km with the speed of $(x+250)$ km/hr $= \frac{1500}{x+250}$

$$\therefore \quad \frac{1500}{x} = \frac{1500}{x+250} + \frac{1}{2}$$

$$\Rightarrow \quad \frac{1500}{x} - \frac{1500}{x+250} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1500x + 1500 \times 250 - 1500x}{x(x+250)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1500 \times 250}{x^2 + 250x} = \frac{1}{2}$$

$$\Rightarrow \quad 750000 = x^2 + 250x$$

$$\Rightarrow \quad x^2 + 250\,x - 750000 = 0$$

$$\Rightarrow \quad x^2 + 1000x - 750x - 750000 = 0$$

$$\Rightarrow \quad x(x+1000) - 750(x+1000) = 0$$

$$\Rightarrow \quad (x+1000)(x-750) = 0$$

$\Rightarrow \quad x = -1000$ or, $x = 750 \Rightarrow x = 750$ [$\because$ speed cannot be negative]

Hence, the usual speed of the plane is 750 km/hr.

EXAMPLE 5 *In a flight of 600 km, a aircraft was slowed down due to bad weather. Its average speed for the trip was reduced by 200 km/hr and the time of flight increased by 30 minutes. Find the duration of flight.*

SOLUTION Let the original speed of the aircraft be x km/hr. Then,

New speed $= (x-200)$ km/hr.

Duration of flight at original speed $= \left(\frac{600}{x}\right)$ hr

Duration of flight at reduced speed $= \left(\frac{600}{x-200}\right)$ hr

$$\therefore \quad \frac{600}{x-200} - \frac{600}{x} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{600x - 600(x-200)}{x(x-200)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{120000}{x^2 - 200x} = \frac{1}{2}$$

$$\Rightarrow \quad x^2 - 200x - 240000 = 0$$

$\Rightarrow \quad x^2 - 600x + 400x - 240000 = 0$

$\Rightarrow \quad x(x - 600) + 400(x - 600) = 0$

$\Rightarrow \quad (x - 600)(x + 400) = 0$

$\Rightarrow \quad x - 600 = 0$ or, $x + 400 = 0$

$\Rightarrow \quad x = 600$ or, $x = -400 \Rightarrow x = 600$ $\quad$ [$\because x$ cannot be negative]

So, the original speed of the aircraft was 600 km/hr.

Hence, $\quad$ Duration of flight $= \left(\frac{600}{x}\right)$hr $= \left(\frac{600}{600}\right)$hr $= 1$ hr

LEVEL-2

EXAMPLE 6 *Swati can row her boat at a speed of 5 km/hr in still water. If it takes her 1 hour more to row the boat 5.25 km upstream than to return downstream, find the speed of the stream.*

SOLUTION Let the speed of the stream be x km/hr.

$\therefore$ Speed of the boat upstream = $(5 - x)$ km/hr.

Speed of the boat downstream = $(5 + x)$ km/hr.

Time taken for going 5.25 km upstream $= \frac{5.25}{5 - x}$ hours.

Time taken for going 5.25 km downstream $= \frac{5.25}{5 + x}$ hours.

Obviously, time taken for going 5.25 km upstream is more than the time taken for going 5.25 km. downstream.

It is given that the time taken for going 5.25 km. upstream is 1 hour more than the time taken for going 5.25 downstream.

$$\therefore \quad \frac{5.25}{5 - x} - \frac{5.25}{5 + x} = 1$$

$$\Rightarrow \quad 5.25\left\{\frac{1}{5 - x} - \frac{1}{5 + x}\right\} = 1$$

$$\Rightarrow \quad \frac{21}{4}\left\{\frac{5 + x - 5 + x}{(5 - x)(5 + x)}\right\} = 1$$

$$\Rightarrow \quad \frac{21}{4} \times \frac{2x}{25 - x^2} = 1$$

$$\Rightarrow \quad \frac{21}{2} \times \frac{x}{25 - x^2} = 1$$

$$\Rightarrow \quad 21x = 50 - 2x^2$$

$$\Rightarrow \quad 2x^2 + 21x - 50 = 0$$

$$\Rightarrow \quad 2x^2 + 25x - 4x - 50 = 0$$

$$\Rightarrow \quad x(2x + 25) - 2(2x + 25) = 0$$

$$\Rightarrow \quad (2x + 25)(x - 2) = 0$$

$$\Rightarrow \quad x - 2 = 0, 2x + 25 = 0 \Rightarrow x = 2 \qquad \left[\because x \neq -\frac{25}{2} \text{ as } x > 0\right]$$

Hence, the speed of the stream is 2 km/hr.

EXAMPLE 7 *Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels 5 km/hr faster than the second train. If after two hours, they are 50 km apart, find the average speed of each train.*

SOLUTION Let the speed of the second train be x km/hr. Then, the speed of the first train is $(x+5)$ km/hr.

Let O be the position of the railway station from which the two trains leave.

Distance travelled by the first train in 2 hours $= OA = \text{Speed} \times \text{Time}$
$= 2(x+5)$ km

Distance travelled by the second train in 2 hours $= OB = \text{Speed} \times \text{Time} = 2x$ km

By using pythagoras theorem, we have,

$$AB^2 = OA^2 + OB^2$$
$$\Rightarrow \quad 50^2 = \{2(x+5)\}^2 + \{2x\}^2$$
$$\Rightarrow \quad 2500 = 4(x+5)^2 + 4x^2$$
$$\Rightarrow \quad 8x^2 + 40x - 2400 = 0$$
$$\Rightarrow \quad x^2 + 5x - 300 = 0$$
$$\Rightarrow \quad x^2 + 20x - 15x - 300 = 0$$
$$\Rightarrow \quad x(x+20) - 15(x+20) = 0$$
$$\Rightarrow \quad (x+20)(x-15) = 0$$
$$\Rightarrow \quad x = -20 \text{ or, } x = 15$$
$$\Rightarrow \quad x = 15 \qquad [\because x \text{ cannot be negative}]$$

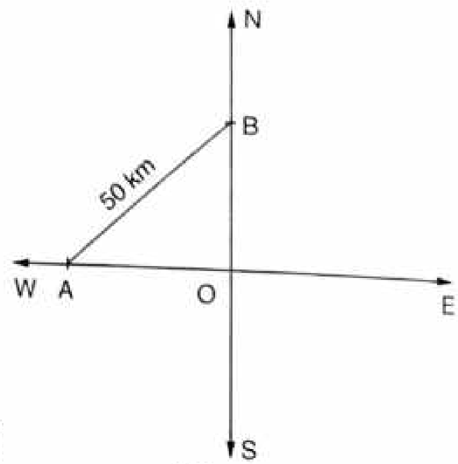


Fig. 4.1

Hence, the speed of the second train is 15 km/hr and, the speed of the first train is 20 km/hr.

EXERCISE 4.8

LEVEL-1

1. The speed of a boat in still water is 8 km/hr It can go 15 km upstream and 22 km downstream in 5 hours. Find the speed of the stream.
2. A train, travelling at a uniform speed for 360 km, would have taken 48 minutes less to travel the same distance if its speed were 5 km/hr more. Find the original speed of the train. **[NCERT EXEMPLAR]**
3. A fast train takes one hour less than a slow train for a journey of 200 km. If the speed of the slow train is 10 km/hr less than that of the fast train, find the speed of the two trains.
4. A passenger train takes one hour less for a journey of 150 km if its speed is increased by 5 km/hr from its usual speed. Find the usual speed of the train.
5. The time taken by a person to cover 150 km was 2.5 hrs more than the time taken in the return journey. If he returned at a speed of 10 km/hr more than the speed of going, what was the speed per hour in each direction?
6. A plane left 40 minutes late due to bad weather and in order to reach its destination, 1600 km away in time, it had to increase its speed by 400 km/hr from its usual speed. Find the usual speed of the plane. **[CBSE 2018]**
7. An aeroplane takes 1 hour less for a journey of 1200 km if its speed is increased by 100 km/hr from its usual speed. Find its usual speed.
8. A train travels at a certain average speed for a distance 63 km and then travels a distance of 72 km at an average speed of 6 km/hr more than the original speed. If it takes 3 hours to complete total journey, what is its original asverage speed? **[NCERT EXEMPLAR, CBSE 2018]**

9. A train covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hour more, it would have taken 30 minutes less for the journey. Find the original speed of the train. [CBSE 2006C]

10. A train travels 360 km at a uniform speed. If the speed had been 5 km/hr more, it would have taken 1 hour less for the same journey. Find the speed of the train. [NCERT]

11. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11 km/hr more than that of the passenger train, find the average speeds of the two trains. [NCERT]

12. An aeroplane left 50 minutes later than its scheduled time, and in order to reach the destination, 1250 km away, in time, it had to increase its speed by 250 km/hr from its usual speed. Find its usual speed. [CBSE 2010]

13. While boarding an aeroplane, a passenger got hurt. The pilot showing promptness and concern, made arrangements to hospitalise the injured and so the plane started late by 30 minutes to reach the destination, 1500 km away in time, the pilot increased the speed by 100 km/hr. Find the original speed/hour of the plane. [CBSE 2013]

14. A motor boat whose speed in still water is 18 km/hr takes 1 hour more to go 24 km up stream that to return down stream to the same spot. Find the speed of the stream. [CBSE 2014, 2018]

15. A car moves a distance of 2592 km with uniform speed. The number of hours taken for the journey is one-half the number representing the speed, in km/hour. Find the time taken to cover the distance. [CBSE 2017]

ANSWERS

1. 3 km/hr 2. 45 km/hr 3. 50 km/hr, 40 km/hr
4. 25 km/hr 5. 20 km/hr, 30 km/hr 6. 800 km/hr
7. 300 km/hr 8. 42 km/hr 9. 45 km/hr 10. 40 km/hr
11. Speed of the passenger train = 33 km/hr, Speed of the express train = 44 km/hr
12. 500 km/hr 13. 500 km/hr 14. 6 km/hr 15. 36 hours

HINTS TO SELECTED PROBLEMS

2. Let the usual speed of the train be x km/hr. Then,

$$\frac{360}{x} - \frac{360}{x+10} = 3 \Rightarrow 1200 = x(x+10) \Rightarrow x^2 + 10x - 1200 = 0$$

3. Let the speed of the fast train be x km/hr. Then, speed of slow train = $(x-10)$ km/hr. Since, fast train takes one hour less than a slow train to cover 200 km.

$$\therefore \quad \frac{200}{x-10} - \frac{200}{x} = 1$$

4. Let the usual speed be x km./hr. Then, $\frac{150}{x} - \frac{150}{x+5} = 1$.

10. Let the speed in the upward journey be x km/hr. Then, the speed in the return journey = $(x+5)$ km/hr.

$$\therefore \quad \frac{360}{x} - \frac{360}{x+5} = 1$$

4.8.3 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON AGES

The following illustrations will illustrate the problems on ages.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their present ages.*

SOLUTION Suppose, one year ago, son's age be x years.

Then, man's age one year ago = $8x$ years.

$\therefore$ Present age of son = $(x + 1)$ years and, Present age of man = $(8x + 1)$ years.

$\therefore$ $8x + 1 = (x + 1)^2$ [Given]

$\Rightarrow$ $x^2 - 6x = 0$

$\Rightarrow$ $x(x - 6) = 0$

$\Rightarrow$ $x = 0$ or, $x = 6$

$\Rightarrow$ $x = 6$ [$\because$ Son's age cannot be 0]

So, Present age of son = $(x + 1)$ years = 7 years.

and, Present age of man = $(8x + 1)$ years = 49 years.

EXAMPLE 2 *The product of Ramu's age (in years) five years ago with his age (in years) 9 years later is 15. Find Ramu's present age.*

SOLUTION Let Ramu's present age be x years. Then,

His age 5 years ago = $(x - 5)$ years.

His age 9 years later = $(x + 9)$ years.

It is given that the product of these ages is 15.

$\therefore$ $(x - 5)(x + 9) = 15$

$\Rightarrow$ $x^2 + 4x - 60 = 0$

$\Rightarrow$ $x^2 + 10x - 6x - 60 = 0$

$\Rightarrow$ $(x + 10)(x - 6) = 0$

$\Rightarrow$ $x = 6$ or, $x = -10$

But, $x \neq -10$. So, $x = 6$.

Hence, Ramu's present age is 6 years.

EXAMPLE 3 *The sum of ages of a father and his son is 45 years. Five years ago, the product of their ages (in years) was 124. Determine their present ages.*

SOLUTION Let the present age of father be x years. Then,

Son's present age = $(45 - x)$ yeras.

Five years ago:

Father's age = $(x - 5)$ years

Son's age = $(45 - x - 5)$ years = $(40 - x)$ years.

It is given that five years ago, the product of their ages was 124.

$\therefore$ $(x - 5)(40 - x) = 124$

$\Rightarrow$ $40x - x^2 - 200 + 5x = 124$

$\Rightarrow$ $x^2 - 45x + 324 = 0$

$\Rightarrow$ $x^2 - 36x - 9x + 324 = 0$

$\therefore$ $x(x - 36) - 9(x - 36) = 0$

$(x - 9)(x - 36) = 0 \Rightarrow x = 9$, or, $x = 36$

When $x = 36$, we have

Father's present age = 36 years

Son's present age = 9 years

When $x = 9$, we have

Father's present age = 9 years

Son's present age = 36 years

Clearly, this is not possible.

Hence, Father's present age = 36 years and Son's present age = 9 years.

LEVEL-2

EXAMPLE 4 *Seven years ago Varun's age was five times the square of Swati's age. Three years hence Swati's age will be two fifth of Varun's age. Find their present ages.* [CBSE 2006C]

SOLUTION Seven years ago, let Swati's age be x years. Then, seven years ago Varun's age was $5x^2$ years.

$\therefore$ Swati's present age $= (x + 7)$ years, Varun's present age $= (5x^2 + 7)$ years

Three years hence, we have

Swati's age $= (x + 7 + 3)$ years $= (x + 10)$ years

Varun's age $= (5x^2 + 7 + 3)$ years $= (5x^2 + 10)$ years

It is given that three years hence Swati's age will be $\frac{2}{5}$ of Varun's age.

$$\therefore \quad x + 10 = \frac{2}{5}(5x^2 + 10)$$

$$\Rightarrow \quad x + 10 = 2x^2 + 4$$

$$\Rightarrow \quad 2x^2 - x - 6 = 0$$

$$\Rightarrow \quad 2x^2 - 4x + 3x - 6 = 0$$

$$\Rightarrow \quad 2x(x - 2) + 3(x - 2) = 0$$

$$\Rightarrow \quad (2x + 3)(x - 2) = 0$$

$$\Rightarrow \quad x - 2 = 0 \qquad [\because 2x + 3 \neq 0 \text{ as } x > 0]$$

$$\Rightarrow \quad x = 2$$

Hence, Swati's present age $= (2 + 7)$ years $= 9$ years

Varun's present age $= (5 \times 2^2 + 7)$ years $= 27$ years

EXERCISE 4.9

LEVEL-1

1. Ashu is x years old while his mother Mrs Veena is x^2 years old. Five years hence Mrs Veena will be three times old as Ashu. Find their present ages.
2. The sum of the ages of a man and his son is 45 years. Five years ago, the product of their ages was four times the man's age at the time. Find their present ages.
3. The product of Shikha's age five years ago and her age 8 years later is 30, her age at both times being given in years. Find her present age.
4. The product of Ramu's age (in years) five years ago and his age (in years) nine years later is 15. Determine Ramu's present age.
5. Is the following situation possible? If so, determine their present ages. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48. [NCERT]
6. A girl is twice as old as her sister. Four years hence, the product of their ages (in years) will be 160. Find their present ages. [CBSE 2010]

7. The sum of the reciprocals of Rehman's ages (in years) 3 years ago and 5 years from now is 1/3. Find his present age. [NCERT]

LEVEL-2

8. If Zeba were younger by 5 years than what she really is, then the square of her age (in years) would have been 11 more than 5 times her actual age. What is her age now? [NCERT EXEMPLAR]

9. At present Asha's age (in years) is 2 more than the square of her daughter Nisha's age. When Nisha grows to her mother's present age, Asha's age would be one year less than 10 times the present age of Nisha. Find the present ages of both Asha and Nisha. [NCERT EXEMPLAR]

ANSWERS

1. 5 years, 25 years
2. 36 years, 9 years
3. 7 years
4. 6 years
5. No.
6. 6 years, 12 years
7. 7 years
8. 14 years
9. Asha: 27 years, Nisha: 5 years

HINTS TO SELECTED PROBLEMS

2. Let the present age of the man be x years. Then, present age of his son is $(45-x)$ years. Five years ago, man's age $= (x-5)$ years. Son's age $= (45-x-5)$ years.
   $\therefore \quad (x-5)(45-x-5) = 4(x-5)$
3. Let the present age be x years. Then, $(x-5)(x+8) = 30$.
5. Let the present ages of two friends be x years and $(20-x)$ years respectively. According to the given condition, we have
   $$(x-4)(20-x-4) = 48 \Rightarrow (x-4)(16-x) = 48 \Rightarrow x^2 - 20x + 112 = 0$$
   Let D be the discriminant of this quadratic. Then,
   $$D = 400 - 448 = -48 < 0$$
   So, above equation does not have real roots. Hence, the given situation is not possible.
7. Let his present age be x years. Then,
   $$\frac{1}{x-3} + \frac{1}{x+5} = \frac{1}{3} \Rightarrow x^2 - 4x - 21 = 0 \Rightarrow x = 7$$

4.8.4 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS IN GEOMETRY

The following examples will illustrate the above application.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The hypotenuse of right-angled triangle is 6 metres more than twice the shortest side. If the third side is 2 metres less than the hypotenuse, find the sides of the triangle.*

SOLUTION Let the length of the shortest side be x metres. Then,

Hypotenuse $= (2x+6)$ metres

And, The third side $= (2x+6-2)$ metres $= (2x+4)$ metres

By, Pythagoras theorem, we have

$$(2x+6)^2 = x^2 + (2x+4)^2$$

$$\Rightarrow \quad x^2 - 8x - 20 = 0$$

$$\Rightarrow \quad x^2 - 10x - 2x - 20 = 0$$

$\Rightarrow \quad (x-10)(x+2)=0$

$\Rightarrow \quad x=10$ or, $x=-2 \Rightarrow x=10$ [Since side of a triangle is never negative]

$\therefore$ Length of the shortest side = 10 metres.

Length of the hypotenuse = $(2x+6)$ metres = 26 metres

Length of the third side = $(2x+4)$ metres = 24 metres

Hence, the sides of the triangle are 10 m, 26 m and 24 m.

EXAMPLE 2 *The hypotenuse of a grassy land in the shape of a right triangle is 1 metre more than twice the shortest side. If the third side is 7 metres more than the shortest side, find the sides of the grassy land.*

SOLUTION Let the length of the shortest side be x metres. Then, by hypothesis

Hypotenuse = $(2x+1)$ metres, Third side = $(x+7)$ metres.

By Pythagoras theorem, we have

$(\text{Hypotenuse})^2$ = Sum of the square of the remaining two sides

$$\Rightarrow \quad (2x+1)^2 = x^2 + (x+7)^2$$

$$\Rightarrow \quad 4x^2 + 4x + 1 = 2x^2 + 14x + 49$$

$$\Rightarrow \quad 2x^2 - 10x - 48 = 0$$

$$\Rightarrow \quad x^2 - 5x - 24 = 0$$

$$\Rightarrow \quad x^2 - 8x + 3x - 24 = 0$$

$$\Rightarrow \quad x(x-8) + 3(x-8) = 0$$

$$\Rightarrow \quad (x-8)(x+3) = 0$$

$$\Rightarrow \quad x = 8, -3$$

$$\Rightarrow \quad x = 8 \qquad [\because x=-3 \text{ is not possible}]$$

Hence, the lengths of the sides of the grassy land are 8 metres, 17 metres and 15 metres.

LEVEL-2

EXAMPLE 3 *The hypotenuse of a right triangle is $3\sqrt{5}$ cm. If the smaller side is tripled and the larger side is doubled, the new hypotenuse will be 15 cm. Find the length of each side.*

SOLUTION Let the smaller side of the right triangle be x cm and the larger side by y cm. Then,

$$x^2 + y^2 = (3\sqrt{5})^2 \qquad \text{[Using Pythagoras Theorem]}$$

$$\Rightarrow \quad x^2 + y^2 = 45 \qquad \text{...(i)}$$

If the smaller side is tripled and the larger side be doubled, the new hypotenuse is 15 cm.

$$\therefore \quad (3x)^2 + (2y)^2 = 15^2 \Rightarrow 9x^2 + 4y^2 = 225 \qquad \text{...(ii)}$$

From equation (i), we get $y^2 = 45 - x^2$

Putting $y^2 = 45 - x^2$ in equation (ii), we get

$$9x^2 + 4(45 - x^2) = 225$$

$\Rightarrow \qquad 5x^2 + 180 = 225 \Rightarrow 5x^2 = 45 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

But, length of a side cannot be negative. Therefore, $x = 3$.

Putting $x = 3$ in (i), we get

$$9 + y^2 = 45 \Rightarrow y^2 = 36 \Rightarrow y = 6$$

Hence, the length of the smaller side is 3 cm and the length of the larger side is 6 cm.

EXAMPLE 4 *Vikram wishes to fit three rods together in the shape of a right triangle. The hypotenuse is to be 2 cm longer than the base and 4 cm longer than the altitude. What should be the lengths of the rods?*

SOLUTION Let the length of the hypotenuse be x cm. Then,
Base $= (x - 2)$ cm and, Altitude $= (x - 4)$ cm.
By Pythagoras theorem, we have

$$(\text{Base})^2 + (\text{Altitude})^2 = (\text{Hypotenuse})^2$$

$\Rightarrow \qquad (x-2)^2 + (x-4)^2 = x^2$

$\Rightarrow \qquad x^2 - 4x + 4 + x^2 - 8x + 16 = x^2$

$\Rightarrow \qquad x^2 - 12x + 20 = 0$

$\Rightarrow \qquad x^2 - 10x - 2x + 20 = 0$

$\Rightarrow \qquad x(x - 10) - 2(x - 10) = 0$

$\Rightarrow \qquad (x - 10)(x - 2) = 0$

$\Rightarrow \qquad x = 2, \text{ or, } x = 10 \Rightarrow x = 10$ [For $x = 2$, Base = 0 cm which is not possible]

Hence, the length of the rods are 8 cm 6 cm and 10 cm.

EXERCISE 4.10

LEVEL-1

1. The hypotenuse of a right triangle is 25 cm. The difference between the lengths of the other two sides of the triangle is 5 cm. Find the lengths of these sides.
2. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

LEVEL-2

3. The hypotenuse of a right triangle is $3\sqrt{10}$ cm. If the smaller leg is tripled and the longer leg doubled, new hypotenuse will be $9\sqrt{5}$ cm. How long are the legs of the triangle?
4. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it the possible to do so? If yes, at what distances from the two gates should the pole be erected? [NCERT]

ANSWERS

1. 15 cm, 20 cm 2. 120 m, 90 m 3. 3 cm, 9 cm
4. At a distance of 5 metres from the gate B

HINTS TO SELECTED PROBLEMS

4. Let P be the required location of the pole such that its distance from gate B is x metres. i.e. $BP = x$ metres. Therefore, $AP = x + 7$. Applying Pythagoras theorem in right triangle APB, we obtain

$$AP^2 + PB^2 = AB^2$$

$$\Rightarrow \quad (x+7)^2 + x^2 = 13^2$$

$$\Rightarrow \quad 2x^2 + 14x - 120 = 0 \Rightarrow x^2 + 7x - 60 = 0 \Rightarrow (x+12)(x-5) = 0 \Rightarrow x = 5$$

4.8.5 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON MENSURATION

Following examples will illustrate the above applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The area of a right angled triangle is 600 cm^2. If the base of the triangle exceeds the altitude by 10 cm, find the dimensions of the triangle.*

SOLUTION Let the altitude BC of right-angled triangle ABC be x cm. Then,

$$\text{Base} = BC = (x+10) \text{ cm}.$$

$$\therefore \quad \text{Area} = \frac{1}{2}(\text{Base} \times \text{Height})$$

$$\Rightarrow \quad \text{Area} = \frac{1}{2}(x+10)\,x \text{ cm}^2$$

$$\Rightarrow \quad \frac{1}{2}x(x+10) = 600 \qquad [\because \text{Area} = 600 \text{ cm}^2]$$

$$\Rightarrow \quad x^2 + 10x = 1200$$

$$\Rightarrow \quad x^2 + 10x - 1200 = 0$$

$$\Rightarrow \quad x^2 + 40x - 30x - 1200 = 0$$

$$\Rightarrow \quad x(x+40) - 30(x+40) = 0$$

$$\Rightarrow \quad (x+40)(x-30) = 0$$

$$\Rightarrow \quad x = 30, -40 \Rightarrow x = 30 \qquad [\because x > 0]$$

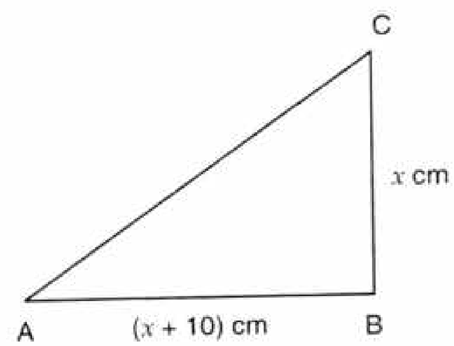


Fig. 4.2

Hence, Base = (30 + 10) cm = 40 cm and, Altitude = 30 cm.

EXAMPLE 2 *The perimeter of a rectangular field is 82 cm and its area is 400 m^2. Find the breadth of the rectangle.*

SOLUTION Let the breadth of the rectangle be x metres. Then,

$$\text{Perimeter} = 82 \text{ m}$$

$$\Rightarrow \quad 2(\text{Length} + \text{Breadth}) = 82$$

$$\Rightarrow \quad \text{Length} + x = 41$$

$$\Rightarrow \quad \text{Length} = 41 - x \text{ metres}$$

Now, $\quad \text{Area} = 400 \text{ m}^2$

$\Rightarrow \quad \text{Length} \times \text{Breadth} = 400$

$\Rightarrow \quad (41 - x)\, x = 400$

$\Rightarrow \quad 41x - x^2 = 400$

$\Rightarrow \quad x^2 - 41x + 400 = 0$

$\Rightarrow \quad x^2 - 25x - 16x + 400 = 0$

$\Rightarrow \quad x(x - 25) - 16(x - 25) = 0$

$\Rightarrow \quad (x - 25)(x - 16) = 0 \Rightarrow x = 25 \text{ or, } x = 16$

Hence, breadth = 25 m or, 16 m.

EXAMPLE 3 *The length of the sides forming right angle of a right angled triangle are $5x$ cm and $(3x - 1)$ cm. If the area of the triangle is 60 cm², find its hypotenuse.*

SOLUTION Let ABC be a right angled triangle with right angle at B.

Let $AB = 5x$ and $BC = 3x - 1$. Then,

$$\text{Area} = \Delta ABC = \frac{1}{2}(\text{Base} \times \text{Height})$$

$\Rightarrow \quad 60 = \frac{1}{2}(AB \times BC)$

$\Rightarrow \quad 60 = \frac{1}{2} \times 5x(3x - 1)$

$\Rightarrow \quad 120 = 5x(3x - 1)$

$\Rightarrow \quad 24 = x(3x - 1)$

$\Rightarrow \quad 3x^2 - x - 24 = 0$

$\Rightarrow \quad 3x^2 - 9x + 8x - 24 = 0$

$\Rightarrow \quad 3x(x - 3) + 8(x - 3) = 0$

$\Rightarrow \quad (x - 3)(3x + 8) = 0$

$\Rightarrow \quad x - 3 = 0 \text{ or } 3x + 8 = 0$

$\Rightarrow \quad x = 3 \text{ or, } x = -\frac{8}{3} \Rightarrow x = 3 \qquad [\because x \neq -8/3]$

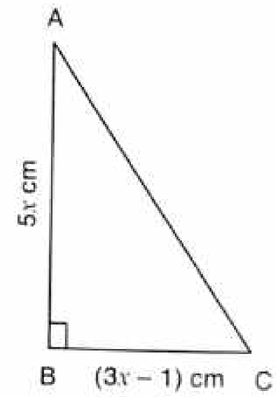


Fig. 4.3

$\therefore \quad AB = 5x = 5 \times 3 = 15 \text{ cm and } BC = (3x - 1) = (3 \times 3 - 1) = 8 \text{ cm}.$

Now, $AC^2 = AB^2 + BC^2 \Rightarrow AC^2 = (15)^2 + (8)^2 \Rightarrow AC^2 = 289 \Rightarrow AC = 17 \text{ cm}.$

Hence, Hypotenues = 17 cm.

EXAMPLE 4 *The length of a rectangle exceeds its width by 8 cm and the area of the rectangle is 240 sq. cm. Find the dimensions of the rectangle.*

SOLUTION Let the breadth of the given rectangle be x cm. Then, length = $(x + 8)$ cm.

Now, $\text{Area} = 240 \text{ cm}^2$

$\Rightarrow \quad \text{length} \times \text{breadth} = 240$

$\Rightarrow \quad (x + 8)\, x = 240$

$\Rightarrow \quad x^2 + 8x - 240 = 0$

$\Rightarrow \quad x^2 + 20x - 12x - 240 = 0$

$$\Rightarrow \quad x(x+20) - 12(x+20) = 0$$

$$\Rightarrow \quad (x+20)(x-12) = 0$$

$$\Rightarrow \quad x = 12 \text{ or, } x = -20$$

But, x cannot be negative. So, $x = 12$.
Hence, length $= x + 8 = 12 + 8 = 20$ cm and breadth $= 12$ cm.

EXAMPLE 5 *The side of a square exceeds the side of the another square by 4 cm and the sum of the areas of the two squares is 400 sq. cm. Find the dimensions of the squares.*

SOLUTION Let S_1 and S_2 be two squares. Let the side of the square S_2 be x cm in length. Then, the side of square S_1 is $(x+4)$ cm.

$$\therefore \quad \text{Area of square } S_1 = (x+4)^2 \qquad [\because \text{Area} = (\text{side})^2]$$

$$\text{and,} \quad \text{Area of square } S_2 = x^2$$

It is given that

$$\text{Area of square } S_1 + \text{Area of square } S_2 = 400 \text{ cm}^2$$

$$\Rightarrow \quad (x+4)^2 + x^2 = 400$$

$$\Rightarrow \quad (x^2 + 8x + 16) + x = 400$$

$$\Rightarrow \quad 2x^2 + 8x - 384 = 0$$

$$\Rightarrow \quad x^2 + 4x - 192 = 0$$

$$\Rightarrow \quad x^2 + 16x - 12x - 192 = 0$$

$$\Rightarrow \quad x(x+16) - 12(x+16) = 0$$

$$\Rightarrow \quad (x+16)(x-12) = 0$$

$$\Rightarrow \quad x = 12 \text{ or, } x = -16$$

As the length of the side of a square cannot be negative. Therefore, $x = 12$.

$$\therefore \quad \text{Side of square } S_1 = x + 4 = 12 + 4 = 16 \text{ cm and, Side of square } S_2 = 12 \text{ cm.}$$

LEVEL-2

EXAMPLE 6 *If twice the area of a smaller square is subtracted from the area of a larger square, the result is 14 cm². However, if twice the area of the larger square is added to three times the area of the smaller square, the result is 203 cm². Determine the sides of the square.*

SOLUTION Let the lengths of each side of the smaller square be x cm and that of the larger square be y cm. Then,

Area of the smaller square $= x^2$ cm², Area of the larger square $= y^2$ cm²

It is given that

$$y^2 - 2x^2 = 14 \qquad \ldots \text{(i)}$$

$$\text{and,} \quad 2y^2 + 3x^2 = 203 \qquad \ldots \text{(ii)}$$

From (i), we have

$$y^2 = 14 + 2x^2$$

Substituting this value of y^2 in (ii), we get

$$2(14 + 2x^2) + 3x^2 = 203$$

$$\Rightarrow \quad 28 + 4x^2 + 3x^2 = 203$$

$$\Rightarrow \quad 7x^2 = 203 - 28$$

$$\Rightarrow \quad 7x^2 = 175 \Rightarrow x^2 = 25 \Rightarrow x = 5 \text{ cm.}$$

Putting $x = 5$ in (i), we get

$$y^2 - 2 \times 5^2 = 14 \Rightarrow y^2 = 64 \Rightarrow y = 8$$

Hence, the lengths of the sides of the square are 5 cm and 8 cm respectively.

EXAMPLE 7 *A farmer wishes to grow a 100 m^2 rectangular vegetable garden. Since he has with the only 30 m barbed wire, he fences three sides of the rectangular garden letting compound wall of his house act as the fourth side-fence. Find the dimensions of his garden.*

SOLUTION Let the length of one side be x metres and other side by y metres. Then,

$$x + y + x = 30 \Rightarrow y = 30 - 2x \quad \ldots(i)$$

$\therefore$ Area of the vegetable garden = 100 m^2

$\Rightarrow$ $xy = 100$

$\Rightarrow$ $x(30 - 2x) = 100$

$\Rightarrow$ $30x - 2x^2 = 100$

$\Rightarrow$ $15x - x^2 = 50$

$\Rightarrow$ $x^2 - 15x + 50 = 0$

$\Rightarrow$ $x^2 - 10x - 5x + 50 = 0$

$\Rightarrow$ $(x - 10)(x - 5) = 0$

$\Rightarrow$ $x = 5, 10$

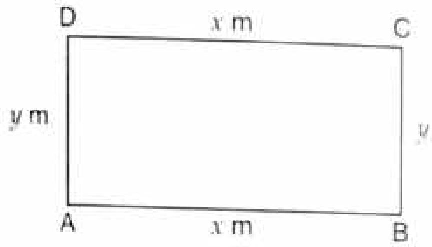


Fig. 4.4

When $x = 5$, we have

$$y = 30 - 2 \times 5 = 20 \quad \text{[Using (i)]}$$

When $x = 10$, we have

$$y = 30 - 2 \times 10 = 10 \quad \text{[Using (i)]}$$

Hence, the dimensions of the vegetable garden are: $5\text{m} \times 20\text{m}$ or, $10\text{m} \times 10\text{m}$

EXAMPLE 8 *The area of an isosceles triangle is 60 cm^2 and the length of each one of its equal sides is 13 cm. Find its base.* [CBSE 2015]

SOLUTION Let ABC be the given isosceles triangle in which $AB = AC = 13$ cm. Draw AD perpendicular from A on BC. Let $BC = 2x$ cm. Then, $BD = DC = x$ cm.

In ΔABD, we have

$$AB^2 = AD^2 + BD^2 \quad \text{[By Pythagoras Theorem]}$$

$\Rightarrow$ $13^2 = AD^2 + x^2$

$\Rightarrow$ $AD = \sqrt{13^2 - x^2} = \sqrt{169 - x^2}$

Now, Area = 60 cm^2

$\Rightarrow$ $\frac{1}{2}(BC \times AD) = 60$

$\Rightarrow$ $\frac{1}{2}\left\{(2x \times \sqrt{169 - x^2}\right\} = 60$

$\Rightarrow$ $x\sqrt{169 - x^2} = 60$

$\Rightarrow$ $x^2(169 - x^2) = 3600$

$\Rightarrow$ $x^4 - 169x^2 + 3600 = 0$

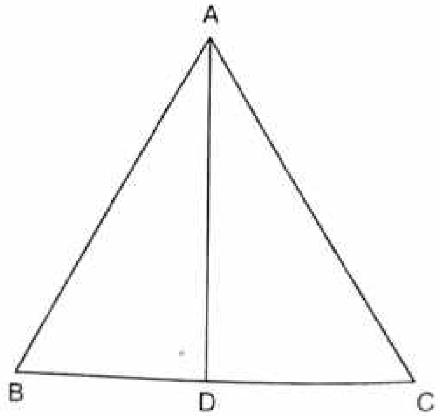


Fig. 4.5

$\Rightarrow \quad (x^2 - 144)(x^2 - 25) = 0$

$\Rightarrow \quad x^2 = 144$ or, $x^2 = 25 \Rightarrow x = 12$ or, $x = 5$

Hence, Base $= 2x = 24$ cm or, 10 cm.

EXAMPLE 9 *The perimeter of a right triangle is 60 cm. Its hypotenues is 25 cm. Find the area of the triangle.* [CBSE 2016]

SOLUTION Let ABC be the given right angled triangle such that base $= BC = x$ cm and hypotenuse $AC = 25$ cm.

Now, Perimeter $= 60$ cm

$\Rightarrow \quad AB + BC + AC = 60$

$\Rightarrow \quad AB + x + 25 = 60$

$\Rightarrow \quad AB = 35 - x$

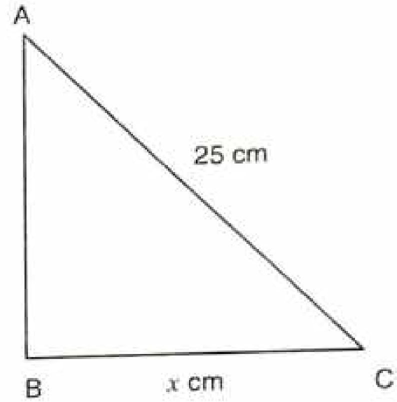


Fig. 4.6

By Pythagoras theorem, we have

$$AB^2 + BC^2 = AC^2$$

$\Rightarrow \quad (35 - x)^2 + x^2 = 25^2$

$\Rightarrow \quad 2x^2 - 70x + 600 = 0$

$\Rightarrow \quad x^2 - 35x + 300 = 0$

$\Rightarrow \quad x^2 - 20x - 15x + 300 = 0$

$\Rightarrow \quad (x - 20)(x - 15) = 0 \Rightarrow x = 20$ or, $x = 15$

If $x = 20$, then $AB = 35 - x = 15$ and $BC = x = 20$.

$\therefore \quad \text{Area} = \frac{1}{2}(BC \times AB) = \frac{1}{2}(20 \times 15) = 150 \text{ cm}^2$

If $x = 15$, then $AB = 35 - x = 20$ and $BC = x = 15$

$\therefore \quad \text{Area} = \frac{1}{2}(BC \times AB) = \frac{1}{2}(15 \times 20) = 150 \text{ cm}^2$

Hence, Area $= 150$ cm^2.

EXAMPLE 10 *There is a square field whose side is 44 m. A square flower bed is prepared in its centre leaving a gravel path all round the flower bed. The total cost of laying the flower bed and gravelling the path at ₹ 2.75 and ₹ 1.50 per square metre, respectively, is ₹ 4904. Find the width of the gravel path.*

SOLUTION Let the width of the gravel path be x metres. Then,
Each side of the square flower bed is $(44 - 2x)$ metres.

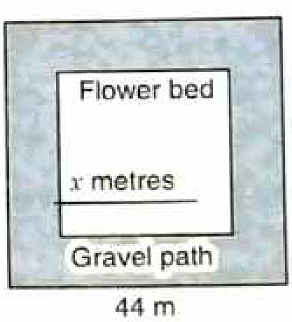


Fig. 4.7

Now, Area of the square field $= 44 \times 44 = 1936 \text{ m}^2$

Area of the flower bed $= (44 - 2x)^2 \text{ m}^2$

$\therefore$ Area of the gravel path
$= \text{Area of the field} - \text{Area of the flower bed}$
$= 1936 - (44 - 2x)^2$
$= 1936 - (1936 - 176x + 4x^2)$
$= (176x - 4x^2) \text{ m}^2$

Cost of laying the flower bed = (Area of the flower bed) (Rate per sq. m)

$$= (44-2x)^2 \times \frac{275}{100} = \frac{11}{4}(44-2x)^2 = 11(22-x)^2$$

Cost of gravelling the path = (Area of the path) × (Rate per sq. m)

$$= (176x - 4x^2)\frac{150}{100} = 6(44x - x^2)$$

It is given that the total cost of laying the flower bed and gravelling the path is ₹ 4904.

$\therefore \quad 11(22-x)^2 + 6(44x - x^2) = 4904$

$\Rightarrow \quad 11(484 - 44x + x^2) + (264x - 6x^2) = 4904$

$\Rightarrow \quad 5x^2 - 220x + 5324 = 4908$

$\Rightarrow \quad 5x^2 - 220x + 420 = 0$

$\Rightarrow \quad x^2 - 44x + 84 = 0$

$\Rightarrow \quad x^2 - 42x - 2x + 84 = 0$

$\Rightarrow \quad x(x-42) - 2(x-42) = 0$

$\Rightarrow \quad (x-2)(x-42) = 0$

$\Rightarrow \quad x = 2$ or, $x = 42$

But, $x \neq 42$, as the side of the square is 44 m. Therefore, $x = 2$.

Hence, the width of the gravel path is 2 metres.

EXAMPLE 11 *A chess board contains 64 equal squares and the area of each square is 6.25 cm². A border round the board is 2 cm wide. Find the length of the side of the chess board.*

SOLUTION Let the length of the side of the chess board be x cm. Then,

Area of 64 squares $= (x-4)^2$

$\therefore \quad (x-4)^2 = 64 \times 6.25$

$\Rightarrow \quad x^2 - 8x + 16 = 400$

$\Rightarrow \quad x^2 - 8x - 384 = 0$

$\Rightarrow \quad x^2 - 24x + 16x - 384 = 0$

$\Rightarrow \quad (x-24)(x+16) = 0$

$\Rightarrow \quad x = 24$ cm.

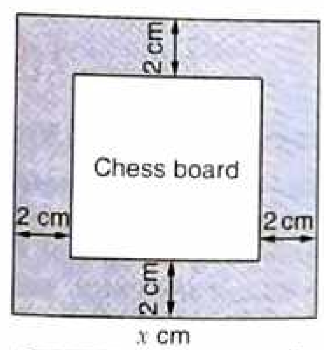


Fig. 4.8

EXERCISE 4.11

LEVEL-1

1. The perimeter of a rectangular field is 82 m and its area is 400 m². Find the breadth of the rectangle.
2. The length of a hall is 5 m more than its breadth. If the area of the floor of the hall is 84 m², what are the length and breadth of the hall?
3. Two squares have sides x cm and $(x+4)$ cm. The sum of their areas is 656 cm². Find the sides of the squares.

4. The area of a right angled triangle is 165 m^2. Determine its base and altitude if the latter exceeds the former by 7m.
5. Is it possible to design a rectangular mango grove whose length is twice its breadth and the area is 800 m^2? If so, find its length and breadth. [NCERT]
6. Is it possible to design a rectangular park of perimeter 80 m and area 400 m^2? If so, find its length and breadth. [NCERT]
7. Sum of the areas of two squares is 640 m^2. If the difference of their perimeters is 64 m, find the sides of the two squares. [NCERT, CBSE 2008]
8. Sum of the areas of two squares is 400 cm^2. If the difference of their perimeters is 16 cm, find the sides of two squares. [CBSE 2013]
9. The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one metre more then twice its breadth. Find the length and the breadth of the plot. [CBSE 2014]
10. In the centre of a rectangular lawn of dimensions 50 m × 40 m, a rectangular pond has to be constructed so that the area of the grass surrounding the pond would be 1184 m^2. Find the length and breadth of the pond. [NCERT EXEMPLAR]

ANSWERS

1. 16 m
2. Breadth = 7 m, Length = 12 m
3. 16 cm, 20 cm
4. Base = 15 m, Altitude = 22 m
5. Yes, 40 m, 20 m
6. Yes. 20m, 20 m
7. 24 m, 8 m
8. 16 cm, 12 cm
9. 33 m, 16 m
10. Length: 34 m, Breadth: 24 m.

HINTS TO SELECTED PROBLEMS

1. Let the breadth be x metres. Then,

$$2\text{ (length + breadth)} = 82 \Rightarrow \text{ length} = 41 - x \text{ metres.}$$

$$\therefore \quad \text{Area} = 400\text{ m}^2 \Rightarrow x(41 - x) = 400 \Rightarrow x^2 - 41x + 400 = 0$$

4.8.6 APPLICATIONS OF QUADRATIC EQUATIONS FOR SOLVING PROBLEMS ON TIME AND WORK

Following examples will illustrate the above applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A takes 6 days less than the time taken by B to finish a piece of work. If both A and B together can finish it in 4 days, find the time taken by B to finish the work.* [CBSE 2017]

SOLUTION Suppose B alone takes x days to finish the work. Then, A alone can finish it in $(x-6)$ days.

Now, $\quad (A\text{'s one day's work}) + (B\text{'s one day's work}) = \dfrac{1}{x} + \dfrac{1}{x-6}$

and, $\quad (A+B)\text{'s one day's work} = \dfrac{1}{4}$

$$\therefore \quad \frac{1}{x} + \frac{1}{x-6} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{x-6+x}{x(x-6)} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{2x-6}{x^2-6x} = \frac{1}{4}$$

$$\Rightarrow \quad 8x - 24 = x^2 - 6x$$

$\Rightarrow \quad x^2 - 14x + 24 = 0$

$\Rightarrow \quad x^2 - 12x - 2x + 24 = 0 \Rightarrow (x-12)(x-2) = 0 \Rightarrow x = 12$ or, $x = 2$

But, x cannot be less than 6. So, $x = 12$. Hence, B alone can finish the work in 12 days.

LEVEL-2

EXAMPLE 2 *A swimming pool is filled with three pipes with uniform flow. The first two pipes operating simultaneously, fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool five hours faster than the first pipe and four hours slower than the third pipe. Find the time required by each pipe to fill the pool separately.*

SOLUTION Let V be the volume of the pool and x the number of hours required by the second pipe alone to fill the pool. Then, the first pipe takes $(x+5)$ hours, while the third pipe takes $(x-4)$ hours to fill the pool. So, the parts of the pool filled by the first, second and third pipes in one hour are respectively

$$\frac{V}{x+5}, \frac{V}{x} \text{ and } \frac{V}{x-4}$$

Let the time taken by the first and second pipes to fill the pool simultaneously be t hours. Then, the third pipe also takes the same time to fill the pool.

$$\therefore \quad \left(\frac{V}{x+5} + \frac{V}{x}\right)t = \text{Volume of the pool.}$$

Also, $$\frac{V}{x-4}t = \text{Volume of the pool.}$$

$$\Rightarrow \quad \left(\frac{V}{x+5} + \frac{V}{x}\right)t = \frac{V}{x-4}t$$

$$\Rightarrow \quad \frac{1}{x+5} + \frac{1}{x} = \frac{1}{x-4}$$

$$\Rightarrow \quad (2x+5)(x-4) = x^2 + 5x$$

$$\Rightarrow \quad x^2 - 8x - 20 = 0$$

$$\Rightarrow \quad x^2 - 10x + 2x - 20 = 0 \Rightarrow (x-10)(x+2) = 0 \Rightarrow x = 10 \text{ or, } x = -2$$

But, x cannot be negative. So, $x = 10$.

Hence, the timings required by first, second and third pipes to fill the pool individually are 15 hours, 10 hours and 6 hours respectively.

EXAMPLE 3 *Two pipes running together can fill a cistern in $3\frac{1}{13}$ minutes. If one pipe takes 3 minutes more than the other to fill it, find the time in which each pipe would fill the cistern.*

SOLUTION Suppose the faster pipe takes x minutes to fill the cistern. Therefore, the slower pipe will take $(x+3)$ minutes to fill the cistern.

Since the faster pipe takes x minutes to fill the cistern.

$\therefore$ Portion of the cistern filled by the faster pipe in one minute $= \dfrac{1}{x}$

$\Rightarrow$ Portion of the cistern filled by the faster pipe in $\frac{40}{13}$ minutes $= \frac{1}{x} \times \frac{40}{13} = \frac{40}{13x}$

Similarly,

Portion of the cistern filled by the slower pipe in $\frac{40}{13}$ minutes

$$= \frac{1}{x+3} \times \frac{40}{13} = \frac{40}{13(x+3)}$$

It is given that the cistern is filled in $\frac{40}{13}$ minutes.

$$\therefore \quad \frac{40}{13x} + \frac{40}{13(x+3)} = 1$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{x+3} = \frac{13}{40}$$

$$\Rightarrow \quad \frac{x+3+x}{x(x+3)} = \frac{13}{40}$$

$$\Rightarrow \quad 40(2x+3) = 13x(x+3)$$

$$\Rightarrow \quad 80x + 120 = 13x^2 + 39x$$

$$\Rightarrow \quad 13x^2 - 41x - 120 = 0$$

$$\Rightarrow \quad 13x^2 - 65x + 24x - 120 = 0$$

$$\Rightarrow \quad 13x(x-5) + 24(x-5) = 0$$

$$\Rightarrow \quad (x-5)(13x+24) = 0$$

$$\Rightarrow \quad x - 5 = 0 \text{ or, } 13x + 24 = 0$$

$$\Rightarrow \quad x = 5 \text{ or, } x = \frac{-24}{13} \Rightarrow x = 5 \qquad [\because x > 0]$$

Hence, the faster pipe fills the cistern in 5 minutes and the slower pipe takes 8 minutes to fill the cistern.

EXERCISE 4.12

LEVEL-1

1. A takes 10 days less than the time taken by B to finish a piece of work. If both A and B together can finish the work in 12 days, find the time taken by B to finish the work.
2. If two pipes function simultaneously, a reservoir will be filled in 12 hours. One pipe fills the reservoir 10 hours faster than the other. How many hours will the second pipe take to fill the reservoir?
3. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank. [NCERT]
4. Two pipes running together can fill a tank in $11\frac{1}{9}$ minutes. If one pipe takes 5 minutes more than the other to fill the tank separately, find the time in which each pipe would fill the tank separately. [CBSE 2010]

5. To fill a swimming pool two pipes are used. If the pipe of larger diameter used for 4 hours and the pipe of smaller diameter for 9 hours, only half of the pool can be filled. Find, how long it would take for each pipe to fill the pool separately, if the pipe of smaller diameter takes 10 hours more than the pipe of larger diameter to fill the pool? [CBSE 2015]

ANSWERS

1. 30 days 2. 30 hours 3. 15 hours, 25 hours
5. Larger diameter pipe fills in 20 hours, Smaller diameter pipe fills in 30 hours.

HINT TO SELECTED PROBLEM

1. Suppose B alone takes x days to finish the work. Then, A alone can finish it in $(x-10)$ days.

   Now,

   A's one day's work + B's one day's work = $(A+B)$'s one day's work

   $$\Rightarrow \quad \frac{1}{x}+\frac{1}{x-10}=\frac{1}{12}$$

4.8.7 MISCELLANEOUS APPLICATIONS OF QUADRATIC EQUATIONS

Following examples will illustrate the above applications.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A person on tour has ₹360 for his expenses. If he extends his tour for 4 days, he has to cut down his daily expenses by ₹3. Find the original duration of the tour.*

SOLUTION Let the original duration of the tour be x days.

Total expenditure on tour = ₹ 360

$\therefore$ Expenditure per day = ₹ $\frac{360}{x}$

Duration of the extended tour = $(x+4)$ days

$\therefore$ Expenditure per day according to new schedule = ₹ $\frac{360}{x+4}$

It is given that the daily expenses are cut down by ₹ 3.

$$\therefore \quad \frac{360}{x}-\frac{360}{x+4}=3$$

$$\Rightarrow \quad \frac{360(x+4)-360x}{x(x+4)}=3$$

$$\Rightarrow \quad \frac{360x+1440-360x}{x(x+4)}=3$$

$$\Rightarrow \quad \frac{1440}{x^2+4x}=3$$

$$\Rightarrow \quad x^2+4x=480$$

$$\Rightarrow \quad x^2+4x-480=0$$

$$\Rightarrow \quad x^2+24x-20x-480=0$$

$$\Rightarrow \quad x(x+24)-20(x+24)=0$$

$$\Rightarrow \quad (x-20)(x+24)=0 \Rightarrow x-20=0 \text{ or, } x+24=0 \Rightarrow x=20 \text{ or, } x=-24$$

But, the number of days cannot be negative. So, $x = 20$.
Hence, the original duration of the tour was of 20 days.

EXAMPLE 2 *A piece of cloth costs ₹200. If the piece was 5 m longer and each metre of cloth costs ₹2 less the cost of the piece would have remained unchanged. How long is the piece and what is the original rate per metre?*

SOLUTION Let the length of the piece be x metres. Then, rate per metre = ₹ $\frac{200}{x}$

New length $= (x+5)$ metres.

Since the cost remains same.

$\therefore$ New rate per metre = ₹ $\frac{200}{x+5}$

It is given that

$$\frac{200}{x} - \frac{200}{x+5} = 2$$

$$\Rightarrow \quad \frac{200(x+5) - 200x}{x(x+5)} = 2$$

$$\Rightarrow \quad \frac{1000}{x(x+5)} = 2$$

$$\Rightarrow \quad x^2 + 5x = 500$$

$$\Rightarrow \quad x^2 + 5x - 500 = 0$$

$$\Rightarrow \quad x^2 + 25x - 20x - 500 = 0$$

$$\Rightarrow \quad (x+25)(x-20) = 0$$

$$\Rightarrow \quad x + 25 = 0 \text{ or, } x - 20 = 0$$

$$\Rightarrow \quad x + 25 = 0 \text{ or, } x - 20 = 0 \Rightarrow x = 20 \text{ or, } x = -25$$

But, x cannot be negative. So, $x = 20$.

Rate per metre = ₹ $\frac{200}{x}$ = ₹ $\frac{200}{20}$ = ₹ 10

Hence, the length of the piece of cloth is 20 metres and rate = ₹ 10 per metre.

EXAMPLE 3 *₹6500 were divided equally among a certain number of persons. Had there been 15 more persons, each would have got ₹30 less. Find the original number of persons.*

SOLUTION Let the original number of persons be x. Then,

Share of each person = ₹ $\frac{6500}{x}$

If the number of persons is increased by 15. Then,

New share of each person = ₹ $\frac{6500}{x+15}$

Since each person gets ₹ 30 less, if number of persons is increased by 15.

$$\therefore \quad \frac{6500}{x} - \frac{6500}{x+15} = 30$$

$$\Rightarrow \quad \frac{6500\,(x+15) - 6500x}{x\,(x+15)} = 30$$

$$\Rightarrow \quad \frac{6500 \times 15}{x\,(x+15)} = 30$$

$$\Rightarrow \quad \frac{3250}{x\,(x+15)} = 1$$

$$\Rightarrow \quad x^2 + 15x - 3250 = 0$$

$$\Rightarrow \quad x^2 + 65x - 50x - 3250 = 0$$

$$\Rightarrow \quad x\,(x+65) - 50\,(x+65) = 0$$

$$\Rightarrow \quad (x+65)\,(x-50) = 0$$

$$\Rightarrow \quad x - 50 = 0 \text{ or, } x + 65 = 0 \;\Rightarrow\; x = 50 \text{ or, } x = -65$$

Since the number of persons cannot be negative. Therefore, $x = 50$.
Hence, the original number of persons is 50.

EXAMPLE 4 *A shopkeeper buys a number of books for ₹80. If he had bought 4 more books for the same amount, each book would have cost ₹1 less. How many books did he buy?* [CBSE 2012]

SOLUTION Let the number of books bought be x. Then,

$$\text{Cost of } x \text{ books} = ₹\,80 \;\Rightarrow\; \text{Cost of one book} = ₹\,\frac{80}{x}$$

If the number of books bought is $x + 4$, then

$$\text{Cost of one book} = ₹\,\frac{80}{x+4}$$

It is given that the cost of one book is reduced by one rupee.

$$\therefore \quad \frac{80}{x} - \frac{80}{x+4} = 1$$

$$\Rightarrow \quad 80\left(\frac{1}{x} - \frac{1}{x+4}\right) = 1$$

$$\Rightarrow \quad 80\left\{\frac{x+4-x}{x\,(x+4)}\right\} = 1$$

$$\Rightarrow \quad \frac{320}{x^2 + 4x} = 1$$

$$\Rightarrow \quad x^2 + 4x = 320$$

$$\Rightarrow \quad x^2 + 4x - 320 = 0$$

$$\Rightarrow \quad x^2 + 20x - 16x - 320 = 0$$

$$\Rightarrow \quad x\,(x+20) - 16\,(x+20) = 0$$

$$\Rightarrow \quad (x+20)\,(x-16) = 0$$

$$\Rightarrow \quad x = -20 \text{ or, } x = 16 \;\Rightarrow\; x = 16 \qquad [\because x \text{ cannot be negative}]$$

Hence, the number of books is 16.

EXAMPLE 5 *If the price of a book is reduced by ₹5, a person can buy 5 more books for ₹300. Find the original list price of the book.*

SOLUTION Let the original list price of the book be ₹ x.

$\therefore$ Number of books bought for ₹ 300 $= \dfrac{300}{x}$

Reduced list price of the book = ₹ $(x-5)$

$\therefore$ Number of books bought for ₹ 300 $= \dfrac{300}{x-5}$

It is given that

$$\frac{300}{x-5} - \frac{300}{x} = 5$$

$$\Rightarrow \quad \frac{300x - 300x + 1500}{x(x-5)} = 5$$

$$\Rightarrow \quad \frac{1500}{x^2 - 5x} = 5$$

$$\Rightarrow \quad x^2 - 5x = 300$$

$$\Rightarrow \quad x^2 - 5x - 300 = 0$$

$$\Rightarrow \quad x^2 - 20x + 15x - 300 = 0$$

$$\Rightarrow \quad (x-20)(x+15) = 0$$

$$\Rightarrow \quad x - 20 = 0 \text{ or, } x + 15 = 0$$

$$\Rightarrow \quad x = 20, x = -15 \Rightarrow x = 20 \qquad [\because x = -15 \text{ is not possible}]$$

Hence, the list price of the book = ₹ 20

LEVEL-2

EXAMPLE 6 *A factory kept increasing its output by the same percentage every year. Find the percentage if it is known that the output is doubled in the last two years.*

SOLUTION Let P be the initial production (2 years ago), and let the increase in product every year be $x\%$. Then,

Product at the end of first year $= P + \dfrac{Px}{100} = P\left(1 + \dfrac{x}{100}\right)$

Product at the end of the second year

$$= P\left(1+\frac{x}{100}\right) + \frac{x}{100}\left\{P\left(1+\frac{x}{100}\right)\right\}$$

$$= P\left(1+\frac{x}{100}\right)\left(1+\frac{x}{100}\right) = P\left(1+\frac{x}{100}\right)^2$$

Since product is doubled in last two years

$$\therefore \quad P\left(1+\frac{x}{100}\right)^2 = 2P$$

$$\Rightarrow \quad \left(1+\frac{x}{100}\right)^2 = 2$$

$$\Rightarrow \quad (100+x)^2 = 2 \times 100^2$$

$$\Rightarrow \quad x^2 + 200x - 10000 = 0$$

$$\Rightarrow \quad x = \frac{-200 \pm \sqrt{(200)^2 + 40000}}{2} = -100 \pm 100\sqrt{2} = 100(-1 \pm \sqrt{2})$$

$$\Rightarrow \quad x = 100(-1+\sqrt{2})$$

[$\because x$ cannot be negative]

EXAMPLE 7 *A dealer sells a toy for ₹24 and gains as much per cent as the cost price of the toy. Find the cost price of the toy.*

SOLUTION Let the cost price of the toy be ₹ x. Then,

$$\text{Gain} = x\% \Rightarrow \text{Gain} = ₹\left(x \times \frac{x}{100}\right) = ₹\,\frac{x^2}{100}$$

$$\therefore \quad \text{S.P.} = \text{C.P.} + \text{Gain} = x + \frac{x^2}{100}$$

But, S.P. = ₹ 24.

$$\therefore \quad x + \frac{x^2}{100} = 24$$

[Given]

$$\Rightarrow \quad 100x + x^2 = 2400$$

$$\Rightarrow \quad x^2 + 100x - 2400 = 0$$

$$\Rightarrow \quad x^2 + 120x - 20x - 2400 = 0$$

$$\Rightarrow \quad x(x+120) - 20(x+120) = 0$$

$$\Rightarrow \quad (x+120)(x-20) = 0$$

$$\Rightarrow \quad x = 20, -120 \Rightarrow x = 20$$

Hence, the cost price of the toy is ₹ 20. [$\because x > 0$]

EXAMPLE 8 *A peacock is sitting on the top of a pillar, which is 9 m high. From a point 27 m away from the bottom of the pillar, a snake is coming to its hole at the base of the pillar. Seeing the snake the peacock pounces on it. If their speeds are equal, at what distance from the whole is the snake caught?* **[CBSE 2008]**

SOLUTION Let PQ be the pole and the peacock is sitting at the top P of the pole. Let the hole be at Q. Initially, the snake is at S when the peacock notices the snake such that $QS = 27$ m.

Suppose v m/sec is the common speed of both the snake and the peacock and the peacock catches the snake after t seconds at point T. Clearly, distance travelled by the snake in t seconds is same as the distance flown by peacock.

$$\therefore \quad PT = ST = x \quad \text{(say)}$$

Thus, in right triangle PQT, we have

$$QT = 27 - x, \; PT = x \text{ and } PQ = 9$$

Using Pythagoras theorem, we have

$$PT^2 = PQ^2 + QT^2$$

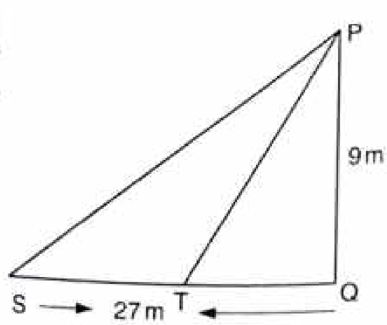


Fig. 4.9

$\Rightarrow \quad x^2 = 9^2 + (27 - x)^2$

$\Rightarrow \quad x^2 = 81 + 729 - 54x + x^2$

$\Rightarrow \quad 0 = 810 - 54x$

$\Rightarrow \quad 54x = 810$

$\Rightarrow \quad x = 15$

$\therefore \quad QT = SQ - ST = (27 - 15) \text{ m} = 12 \text{ m}$

Hence, the snake is caught at a distance of 12 m from the hole.

EXAMPLE 9 *The angry Arjun carried some arrows for fighting with Bheeshm. With half the arrows, he cut down the arrows thrown by Bheeshm on him and with six other arrows he killed the rath driver of Bheeshm. With one arrow each he knocked down respectively the rath, flag and the bow of Bheeshm. Finally, with one more than four times the square root of arrrows he laid Bheeshm unconscious on an arrow bed. Find the total number of arrows Arjun had.*

SOLUTION Suppose Arjun had x arrows.

Number of arrows used to cut arrows of Bheeshm = $x/2$

Number of arrows used to kill the rath driver = 6

Number of other arrows used = 3

Remaining arrows $= 4\sqrt{x} + 1$

By hypothesis, we have

$\therefore \quad \frac{x}{2} + 6 + 3 + 4\sqrt{x} + 1 = x$

$\Rightarrow \quad x + 20 + 8\sqrt{x} = 2x$

$\Rightarrow \quad x = 20 + 8\sqrt{x}$

Putting $x = y^2$, the above equation becomes

$y^2 = 20 + 8y$

$\Rightarrow \quad y^2 - 8y - 20 = 0$

$\Rightarrow \quad y^2 - 10y + 2y - 20 = 0$

$\Rightarrow \quad (y - 10)(y + 2) = 0$

$\Rightarrow \quad y = 10 \text{ or, } y = -2$

$\Rightarrow \quad y = 10$ [$\because$ y cannot be negative]

$\Rightarrow \quad x = y^2 \Rightarrow x = 100$

Hence, the number of arrows which Arjun had = 100.

EXAMPLE 10 *One-fourth of a herd of camels was seen in the forest. Twice the square root of the herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels.*

SOLUTION Let the total number of camels be x. Then,

Number of camels seen in the forest $= \frac{x}{4}$

Number of camels gone to mountains $= 2\sqrt{x}$

Number of camels on the bank of river = 15

Total number of camels $= \frac{x}{4} + 2\sqrt{x} + 15$

By hypothesis, we have

$\therefore \quad \frac{x}{4} + 2\sqrt{x} + 15 = x$

$\Rightarrow \quad 3x - 8\sqrt{x} - 60 = 0$

$\Rightarrow \quad 3y^2 - 8y - 60 = 0$, where $x = y^2$

$\Rightarrow \quad 3y^2 - 18y + 10y - 60 = 0$

$\Rightarrow \quad 3y(y-6) + 10(y-6) = 0$

$\Rightarrow \quad (3y+10)(y-6) = 9$

$\Rightarrow \quad y = 6$ or, $y = -\frac{10}{3}$

Now, $\quad y = -\frac{10}{3} \Rightarrow x = \left(-\frac{10}{3}\right)^2 = \frac{100}{9}$ $\quad [\because x = y^2]$

But, the number of camels cannot be a fraction.

$\therefore \quad y = 6 \Rightarrow x = 6^2 = 36$ $\quad [\because x = y^2]$

Hence, the number of camels = 36.

EXERCISE 4.13

LEVEL-1

1. A piece of cloth costs ₹ 35. If the piece were 4 m longer and each metre costs ₹ one less, the cost would remain unchanged. How long is the piece?
2. Some students planned a picnic. The budget for food was ₹ 480. But eight of these failed to go and thus the cost of food for each member increased by ₹ 10. How many students attended the picnic?
3. A dealer sells an article for ₹ 24 and gains as much percent as the cost price of the article. Find the cost price of the article.
4. Out of a group of swans, 7/2 times the square root of the total number are playing on the share of a pond. The two remaining ones are swinging in water. Find the total number of swans.
5. If the list price of a toy is reduced by ₹ 2, a person can buy 2 toys more for ₹ 360. Find the original price of the toy. [CBSE 2002 C]
6. ₹ 9000 were divided equally among a certain number of persons. Had there been 20 more persons, each would have got ₹ 160 less. Find the original number of persons.
7. Some students planned a picnic. The budget for food was ₹ 500. But, 5 of them failed to go and thus the cost of food for each member increased by ₹ 5. How many students attended the picnic?
8. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it the possible to do so? If yes, at what distances from the two gates should the pole be erected? [CBSE 2016]
9. In a class test, the sum of the marks obtained by P in Mathematics and science is 28. Had he got 3 marks more in Mathematics and 4 marks less in Science. The product of his marks, would have been 180. Find his marks in the two subjects. [CBSE 2008]

10. In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of her marks would have been 210. Find her marks in two subjects. [CBSE 2014, NCERT]

LEVEL-2

11. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹ 90, find the number of articles produced and the cost of each article. [NCERT]

12. At t minutes past 2 pm the time needed by the minutes hand and a clock to show 3 pm was found to be 3 minutes less than $\frac{t^2}{4}$ minutes. Find t. [NCERT EXEMPLAR]

ANSWERS

1. 10 m 2. 16 3. ₹ 20 4. 16 5. ₹ 20 6. 25 7. 20
8. At a distance of 5 metres from the gate B
9. Marks in Mathematics = 12, Marks in Science = 16

or

Marks in Mathematics = 9, Marks in Science = 19

10. Marks in Mathematics = 12, Marks in English = 18

or

Marks in Mathematics = 13, Marks in English = 17

11. Number of articles = 6, Cost of each article = 15 12. 14

HINTS TO SELECTED PROBLEMS

2. Suppose x students planned a picnic. Then, share of each student ₹ $\frac{480}{x}$

Share of each student when 8 students failed to go is ₹ $\frac{480}{x-8}$

$$\therefore \quad \frac{480}{x-8} - \frac{480}{x} = 10.$$

3. Let the cost price be ₹ x. Then, gain percent = x

$$\therefore \quad \text{Selling Price} = ₹\left(x + \frac{x}{100} \times x\right) = ₹\left(\frac{x^2 + 100x}{100}\right)$$

$$\Rightarrow \quad \frac{x^2 + 100x}{100} = 24 \Rightarrow x^2 + 100x - 2400 = 0.$$

4. Let the number of swans be x. Then,

$$\frac{7}{2}\sqrt{x} + 2 = x \Rightarrow 2x - 7\sqrt{x} - 4 = 0 \Rightarrow 2y^2 - 7y - 4 = 0, \text{ where } x = y^2.$$

8. Let P be the required location of the pole such that its distance from gate B is x metres. i.e. $BP = x$ metres.

$$\therefore \quad AP = x + 7$$

In right triangle APB, we have

$$AP^2 + PB^2 = AB^2$$

$$\Rightarrow \quad (x+7)^2 + x^2 = 13^2$$

$$\Rightarrow \quad 2x^2 + 14x - 120 = 0 \Rightarrow x^2 + 7x - 60 = 0 \Rightarrow (x+12)(x-5) = 0 \Rightarrow x = 5$$

12. Time needed by the minutes hand to show 3 pm = $(60 - t)$ minutes.

$$\therefore \quad 60 - t = \frac{t^2}{4} - 3 \Rightarrow t^2 + 4t - 252 = 0$$

VERY SHORT ANSWER TYPE QUESTION (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the question:

1. Write the value of k for which the quadratic equation $x^2 - kx + 4 = 0$ has equal roots.
2. What is the nature of roots of the quadratic equation $4x^2 - 12x - 9 = 0$?
3. If $1 + \sqrt{2}$ is a root of a quadratic equation with rational coefficients, write its other root.
4. Write the number of real roots of the equation $x^2 + 3|x| + 2 = 0$.
5. Write the sum of real roots of the equation $x^2 + |x| - 6 = 0$.
6. Write the set of values of 'a' for which the equation $x^2 + ax - 1 = 0$ has real roots.
7. Is there any real value of 'a' for which the equation $x^2 + 2x + (a^2 + 1) = 0$ has real roots?
8. Write the value of λ for which $x^2 + 4x + \lambda$ is a perfect square.
9. Write the condition to be satisfied for which equations $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}\,x + b = 0$ have equal roots.
10. Write the set of values of k for which the quadratic equation has $2x^2 + kx - 8 = 0$ has real roots.
11. Write a quadratic polynomial, sum of whose zeros is $2\sqrt{3}$ and their product is 2.
12. Show that $x = -3$ is a solution of $x^2 + 6x + 9 = 0$. [CBSE 2008]
13. Show that $x = -2$ is a solution of $3x^2 + 13x + 14 = 0$. [CBSE 2008]
14. Find the discriminant of the quadratic equation $3\sqrt{3}\,x^2 + 10x + \sqrt{3} = 0$ [CBSE 2009]
15. If $x = \frac{-1}{2}$, is a solution of the quadratic equation $3x^2 + 2kx - 3 = 0$, find the value of k. [CBSE 2015]
16. If $x = 3$ is one root of the quadratic equation $x^2 - 2kx - 6 = 0$, then find the value of k. [CBSE 2018]

ANSWERS

1. $k = \pm 4$ 2. Real and distinct 3. $1 - \sqrt{2}$ 4. No real root
5. 0 6. All real values 7. No 8. $\lambda = 4$
9. $b^2 = ac$ 10. All real values

11. $f(x) = k(x^2 - 2\sqrt{3}x + 2)$, where k is any real number 14. 64 15. $\frac{9}{4}$ 16. $k = \frac{1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If the equation $x^2 + 4x + k = 0$ has real and distinct roots, then

(a) $k < 4$ (b) $k > 4$ (c) $k \geq 4$ (d) $k \leq 4$

2. If the equation $x^2 - ax + 1 = 0$ has two distinct roots, then
(a) $|a| = 2$ (b) $|a| < 2$ (c) $|a| > 2$ (d) None of these

3. If the equation $9x^2 + 6kx + 4 = 0$ has equal roots, then the roots are both equal to
(a) $\pm\frac{2}{3}$ (b) $\pm\frac{3}{2}$ (c) 0 (d) ± 3

4. If $ax^2 + bx + c = 0$ has equal roots, then $c =$
(a) $\frac{-b}{2a}$ (b) $\frac{b}{2a}$ (c) $\frac{-b^2}{4a}$ (d) $\frac{b^2}{4a}$

5. If the equation $ax^2 + 2x + a = 0$ has two distinct roots, if
(a) $a = \pm 1$ (b) $a = 0$ (c) $a = 0, 1$ (d) $a = -1, 0$

6. The positive value of k for which the equation $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ will both have real roots, is
(a) 4 (b) 8 (c) 12 (d) 16

7. The value of $\sqrt{6 + \sqrt{6 + \sqrt{6 +}}} \ldots$ is
(a) 4 (b) 3 (c) -2 (d) 3.5

8. If 2 is a root of the equation $x^2 + bx + 12 = 0$ and the equation $x^2 + bx + q = 0$ has equal roots, then $q =$
(a) 8 (b) -8 (c) 16 (d) -16

9. If the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + c^2 + d^2 = 0$ has equal roots, then
(a) $ab = cd$ (b) $ad = bc$ (c) $ad = \sqrt{bc}$ (d) $ab = \sqrt{cd}$

10. If the roots of the equation $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are equal, then
(a) $2b = a + c$ (b) $b^2 = ac$ (c) $b = \frac{2ac}{a + c}$ (d) $b = ac$

11. If the equation $x^2 - bx + 1 = 0$ does not possess real roots, then
(a) $-3 < b < 3$ (b) $-2 < b < 2$ (c) $b > 2$ (d) $b < -2$

12. If $x = 1$ is a common root of the equations $ax^2 + ax + 3 = 0$ and $x^2 + x + b = 0$, then $ab =$
(a) 3 (b) 3.5 (c) 6 (d) -3

13. If p and q are the roots of the equation $x^2 - px + q = 0$, then
(a) $p = 1, q = -2$ (b) $b = 0, q = 1$ (c) $p = -2, q = 0$ (d) $p = -2, q = 1$

14. If a and b can take values 1, 2, 3, 4. Then the number of the equations of the form $ax^2 + bx + 1 = 0$ having real roots is
(a) 10 (b) 7 (c) 6 (d) 12

15. The number of quadratic equations having real roots and which do not change by squaring their roots is
(a) 4 (b) 3 (c) 2 (d) 1

16. If $(a^2+b^2)x^2+2(ab+bd)x+c^2+d^2=0$ has no real roots, then

(a) $ad=bc$ (b) $ab=cd$ (c) $ac=bd$ (d) $ad \neq bc$

17. If the sum of the roots of the equation $x^2-x=\lambda(2x-1)$ is zero, then $\lambda=$

(a) -2 (b) 2 (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

18. If $x=1$ is a common root of $ax^2+ax+2=0$ and $x^2+x+b=0$ then, $ab=$

(a) 1 (b) 2 (c) 4 (d) 3

19. The value of c for which the equation $ax^2+2bx+c=0$ has equal roots is

(a) $\frac{b^2}{a}$ (b) $\frac{b^2}{4a}$ (c) $\frac{a^2}{b}$ (d) $\frac{a^2}{4b}$

20. If $x^2+k(4x+k-1)+2=0$ has equal roots, then $k=$

(a) $-\frac{2}{3}, 1$ (b) $\frac{2}{3}, -1$ (c) $\frac{3}{2}, \frac{1}{3}$ (d) $-\frac{3}{2}, -\frac{1}{3}$

21. If the sum and product of the roots of the equation $kx^2+6x+4k=0$ are equal, then $k=$

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

22. If $\sin\alpha$ and $\cos\alpha$ are the roots of the equation $ax^2+bx+c=0$, then $b^2=$

(a) a^2-2ac (b) a^2+2ac (c) a^2-ac (d) a^2+ac

23. If 2 is a root of the equation $x^2+ax+12=0$ and the quadratic equation $x^2+ax+q=0$ has equal roots, then $q=$

(a) 12 (b) 8 (c) 20 (d) 16

24. If the sum of the roots of the equation $x^2-(k+6)x+2(2k-1)=0$ is equal to half of their product, then $k=$

(a) 6 (b) 7 (c) 1 (d) 5

25. If a and b are roots of the equation $x^2+ax+b=0$, then $a+b=$

(a) 1 (b) 2 (c) -2 (d) -1

26. A quadratic equation whose one root is 2 and the sum of whose roots is zero, is

(a) $x^2+4=0$ (b) $x^2-4=0$ (c) $4x^2-1=0$ (d) $x^2-2=0$

27. If one root of the equation $ax^2+bx+c=0$ is three times the other, then $b^2:ac=$

(a) $3:1$ (b) $3:16$ (c) $16:3$ (d) $16:1$

28. If one root of the equation $2x^2+kx+4=0$ is 2, then the other root is

(a) 6 (b) -6 (c) -1 (d) 1

29. If one root of the equation $x^2+ax+3=0$ is 1, then its other root is

(a) 3 (b) -3 (c) 2 (d) -2

[redacted]

30. If one root of the equation $4x^2 - 2x + (\lambda - 4) = 0$ be the reciprocal of the other, then $\lambda =$

(a) 8 (b) −8 (c) 4 (d) −4

31. If $y = 1$ is a common root of the equations $ay^2 + ay + 3 = 0$ and $y^2 + y + b = 0$, then ab equals

(a) 3 (b) −7/2 (c) 6 (d) − 3

[CBSE 2012]

32. The values of k for which the quadratic equation $16x^2 + 4kx + 9 = 0$ has real and equal roots are

(a) $6, -\frac{1}{6}$ (b) 36, − 36 (c) 6, − 6 (d) $\frac{3}{4}, -\frac{3}{4}$

[CBSE 2014]

ANSWERS

1. (a)	2. (c)	3. (a)	4. (d)	5. (a)	6. (d)
7. (b)	8. (c)	9. (d)	10. (b)	11. (b)	12. (a)
13. (a)	14. (a)	15. (c)	16. (d)	17. (c)	18. (b)
19. (a)	20. (b)	21. (a)	22. (b)	23. (d)	24. (b)
25. (d)	26. (b)	27. (c)	28. (d)	29. (a)	30. (a)
31. (a)	32. (c)				

SUMMARY

1. A polynomial of degree 2 is called a quadratic polynomial. The general form of a quadratic polynomial is $ax^2 + bx + c$, where a, b, c are real numbers such that $a \neq 0$ and x is a real variable.
2. If $p(x) = ax^2 + bx = + c, a \neq 0$ is a quadratic polynomial and α is a real number, then $p(\alpha) = a\alpha^2 + b\alpha + c$ is known as the value of the quadratic polynomial $p(x)$.
3. A real number α is said to be a zero of the quadratic polynomial $p(x) = ax^2 + bx + c$, if $p(\alpha) = 0.0$
4. If $p(x) = ax^2 + bx + c$ is a quadratic polynomial, then $p(x) = 0$ i.e., $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation.
5. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$.

   In other words, α is a root of $ax^2 + bx + c = 0$ if and only if α is a zero of the polynomial $p(x) = ax^2 + bx + c$.
6. If $ax^2 + bx + c, a \neq 0$ is factorizable into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.
7. The roots of a quadratic equation can also be found by using the method of completing the square.

8. The roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ can be found by using the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided that $b^2 - 4ac \geq 0$.

9. Nature of the roots of quadratic equation $ax^2 + bx + c = 0, a \neq 0$ depends upon the value of $D = b^2 - 4ac$, which is known as the discriminate of the quadratic equation.

10. The quadratic equation $ax^2 + bx + c = 0, a \neq 0$ has:
    (i) two distinct real roots, if $D = b^2 - 4ac > 0$
    (ii) two equal roots i.e. coincident real roots if $D = b^2 - 4ac = 0$
    (iii) no real roots, if $D = b^2 - 4ac < 0$.

CHAPTER 5

ARITHMETIC PROGRESSIONS

5.1 INTRODUCTION

In earlier classes, you might have come across various patterns of numbers like

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad \ldots\ldots\ldots$$

$$0, \quad -2, \quad -4, \quad -6, \quad -8, \quad \ldots\ldots\ldots$$

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad \ldots\ldots\ldots$$

etc.

These patterns are generally known as sequences. In this chapter, we intend to study a particular type of sequences which are known as arithmetic progressions.

5.2 SEQUENCES

As mentioned above that an arrangement of numbers of which one number is designated as the first, another as the second, another as the third and so on is known as a sequence.

Consider the following arrangement of numbers:

$$1, \quad 8, \quad 27, \quad 64, \quad 125, \quad \ldots\ldots\ldots$$

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \ldots\ldots\ldots$$

$$2, \quad 4, \quad 6, \quad 8, \quad 10, \quad \ldots\ldots\ldots$$

In each of the above arrangements numbers are arranged in a definite order according to some rule. In the first arrangement the numbers are cubes of natural numbers and in the second arrangement the numbers are reciprocals of natural numbers whereas in the third arrangement the numbers are even natural number. Each of the above arrangements is a sequence. Thus, we may define a sequence formally as follows:

DEFINITION *A sequence is an arrangement of numbers in a definite order according to some rule.*

The various numbers occurring in a sequence are called its terms. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots$ etc. or $x_1, x_2, x_3, \ldots$ etc. Here, the subscripts denote the positions of the terms. First number or the number at first place is called its first term of the sequence and is denoted by a_1. The number at the second place is called the second term and is denoted by a_2 and so on. In general, the number at the nth place is called the nth term of the sequence and is denoted by a_n. The nth term is also called the *general term* of the sequence.

For example, 2, 4, 6, 8, 10, ... is a sequence whose

first term is 2 i.e., $a_1 = 2$; second term is 4 i.e., $a_2 = 4$

third term is 6 i.e., $a_3 = 6$; fourth term is 8 i.e., $a_4 = 8$

and so on.

Similarly, 1, 4, 9, 16, 25, ... is a sequence such that

$$a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_5 = 25 \text{ and so on.}$$

Often, it is possible to express the rule which generates the various terms of a sequence in terms of an algebraic formula. For example, consider the sequence of even natural numbers i.e., 2, 4, 6, 8, 10, ...

We have,

$$a_1 = \text{First term} = 2 = 2 \times 1, \quad a_2 = \text{Second term} = 4 = 2 \times 2$$

$$a_3 = \text{Third term} = 6 = 2 \times 3, \ a_4 = \text{Fourth term} = 8 = 2 \times 4$$

$$a_5 = \text{Fifth term} = 10 = 2 \times 5, \ a_6 = \text{sixth term} = 12 = 2 \times 6$$

and so on.

It is evident from this that

$$a_n = n\text{th term} = 2 \times n = 2n$$

Let us now consider the sequence of squares of natural numbers i.e., 1, 4, 9, 16, 25,....

Here, we have

$$a_1 = 1 = 1^2, \quad a_2 = 4 = 2^2, a_3 = 9 = 3^2, a_4 = 16 = 4^2 \ a_5 = 25 = 5^2, \ a_6 = 36 = 6^2$$

and so on.

It follows from this that

$$a_n = n\text{th term} = n^2$$

Similarly, consider the sequence of odd natural numbers i.e., 1, 3, 5, 7, 9, 11, ...

We find that

$$a_1 = 1 = 2 \times 1 - 1, \ a_2 = 3 = 2 \times 2 - 1, a_3 = 5 = 2 \times 3 - 1, a_4 = 7 = 2 \times 4 - 1$$

and so on.

In general, $a_n = 2 \times n - 1 = 2n - 1$.

It follows from the above discussion that a sequence can be described either by listing its first few terms till the rule for writing down the other terms becomes clear or, by writing the algebraic formula for the nth term of the sequence.

For example, the sequence of even natural numbers i.e., 2, 4, 6, 8, 10,... can be described as

$$a_n = 2n, \text{ where } n = 1, 2, 3, ...$$

Similarly, the sequence of odd natural numbers i.e., 1, 3, 5, 7, 9, ... can be described as

$$a_n = 2n - 1, \text{ where } n = 1, 2, 3, 4, ...$$

The sequence, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ can be described as

$$a_n = \frac{1}{n}, \text{ where } n = 1, 2, 3, \cdots$$

The sequence 1, 4, 9, 16,... can be described as

$$a_n = n^2, \text{ where } n = 1, 2, 3, 4, ...$$

In the above discussion, we have seen that a sequence can be described by listing its first few terms till the rule for writing down the other terms becomes clear. We can also describe a sequence by writing the algebraic formula for its nth term or general term. In some cases, the terms of the sequence do not follow some fixed pattern but they are generated by some recursive relation.

Consider for instance, the sequence, 1, 1, 2, 3, 5, 8, ...

Here, we have

$$a_1 = 1, a_2 = 1$$
$$a_3 = 2 = 1 + 1 = a_1 + a_2$$
$$a_4 = 3 = 1 + 2 = a_2 + a_3$$
$$a_5 = 5 = 2 + 3 = a_3 + a_4$$
$$a_6 = 8 = 3 + 5 = a_4 + a_5 \text{ and so on}$$

In general

$$a_n = a_{n-1} + a_{n-2} \text{ for } n > 2.$$

Thus, the above sequence is described as

$$a_1 = 1, a_2 = 1 \text{ and } a_n = a_{n-1} + a_{n-2} \text{ for all } n > 2.$$

Let us now discuss some examples to illustrate the applications of what we have discussed so far.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write the first three terms in each of the sequence defined by the following:*

(i) $a_n = 3n + 2$ (ii) $a_n = n^2 + 1$

SOLUTION (i) We have, $a_n = 3n + 2$

Putting $n = 1, 2$, and 3, we get

$$a_1 = 3 \times 1 + 2 = 3 + 2 = 5,$$
$$a_2 = 3 \times 2 + 2 = 6 + 2 = 8,$$
$$a_3 = 3 \times 3 + 2 = 9 + 2 = 11$$

Thus, the required first three terms of the sequence defined by $a_n = 3n + 2$ are 5, 8, and 11.

(ii) We have, $a_n = n^2 + 1$

Putting $n = 1, 2$, and 3, we get

$$a_1 = 1^2 + 1 = 1 + 1 = 2,$$
$$a_2 = 2^2 + 1 = 4 + 1 = 5,$$

and, $a_3 = 3^2 + 1 = 9 + 1 = 10.$

Thus, the first three terms of the sequence defined by $a_n = n^2 + 1$ are 2, 5, and 10.

EXAMPLE 2 *Write the first five terms of the sequence defined by* $a_n = (-1)^{n-1} \cdot 2^n$

SOLUTION We have, $a_n = (-1)^{n-1} \cdot 2^n$

Putting $n = 1, 2, 3, 4$, and 5, we get

$$a_1 = (-1)^{1-1} \times 2^1 = (-1)^0 \times 2 = 2$$
$$a_2 = (-1)^{2-1} \times 2^2 = (-1)^1 \times 4 = -4$$
$$a_3 = (-1)^{3-1} \times 2^3 = (-1)^2 \times 8 = 8$$
$$a_4 = (-1)^{4-1} \times 2^4 = (-1)^3 \times 16 = -16$$

and, $a_5 = (-1)^{5-1} \times 2^5 = (-1)^4 \times 32 = 32.$

EXAMPLE 3 *What is 18th term of the sequence defined by* $a_n = \dfrac{n(n-3)}{n+4}$

SOLUTION We have, $a_n = \dfrac{n(n-3)}{n+4}$

Putting $n = 18$, we get

$$a_{18} = \frac{18 \times (18-3)}{18+4} = \frac{18 \times 15}{22} = \frac{135}{11}$$

EXAMPLE 4 *A sequence is defined by* $a_n = n^3 - 6n^2 + 11n - 6$. *Show that the first three terms of the sequence are zero and all other terms are positive.*

SOLUTION We have, $a_n = n^3 - 6n^2 + 11n - 6$

Putting $n = 1, 2, 3$, we get

$$a_1 = (1)^3 - 6 \times 1^2 + 11 \times 1 - 6 = 1 - 6 + 11 - 6 = 12 - 12 = 0$$

$$a_2 = 2^3 - 6 \times 2^2 + 11 \times 2 - 6 = 8 - 24 + 22 - 6 = 30 - 30 = 0$$

and, $$a_3 = 3^3 - 6 \times 3^2 + 11 \times 3 - 6 = 27 - 54 + 33 - 6 = 60 - 60 = 0$$

Thus, we have

$$a_1 = a_2 = a_3 = 0$$

We observe that $a_n = n^3 - 6n^2 + 11n - 6$ is a cubic polynomial in n and it vanishes for $n = 1$, 2, and 3. Therefore, by factor theorem $(n-1)$, $(n-2)$ and $(n-3)$ are factors of a_n. Thus, we have

$$a_n = (n-1)(n-2)(n-3)$$

In this expression, if we substitute any value of n which is greater than 3, then each factor on the RHS is positive. Therefore,

$$a_n > 0 \quad \text{for all } n > 3.$$

Hence, first three terms of the sequence are zero and all other terms are positive.

LEVEL-2

EXAMPLE 5 *Let a sequence be defined by* $a_1 = 3$, $a_n = 3a_{n-1} + 1$ *for all* $n > 1$. *Find the first four terms of the sequence.*

SOLUTION We have, $a_1 = 3$

and, $a_n = 3a_{n-1} + 1$ for all $n > 1$.

Putting $n = 2, 3$, and 4, we get

$$a_2 = 3a_1 + 1 = 3 \times 3 + 1 = 10$$

$$a_3 = 3a_2 + 1 = 3 \times 10 + 1 = 31$$

and, $$a_4 = 3a_3 + 1 = 3 \times 31 + 1 = 94.$$

Hence, the first four terms of the sequence are 3, 10, 31 and 94.

EXAMPLE 6 *Let a sequence be defined by* $a_1 = 1$, $a_2 = 1$ *and,* $a_n = a_{n-1} + a_{n-2}$ *for all* $n > 2$. *Find* $\dfrac{a_{n+1}}{a_n}$ *for* $n = 1, 2, 3, 4$.

SOLUTION We have, $a_1 = 1, a_2 = 1$

and, $a_n = a_{n-1} + a_{n-2}$ for all $n > 2$

Putting $n = 3, 4$, and 5, we get

$$a_3 = a_2 + a_1 = 1 + 1 = 2$$

$$a_4 = a_3 + a_2 = 2 + 1 = 3$$

and, $a_5 = a_4 + a_3 = 3 + 2 = 5$

Thus, we have

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3 \text{ and } a_5 = 5$$

Now, putting $n = 1, 2, 3$ and 4 in $\frac{a_{n+1}}{a_n}$, we get

$$\frac{a_2}{a_1} = \frac{1}{1} = 1 \qquad [\because a_1 = a_2 = 1]$$

$$\frac{a_3}{a_2} = \frac{2}{1} = 2 \qquad [\because a_2 = 1 \text{ and } a_3 = 2]$$

$$\frac{a_4}{a_3} = \frac{3}{2} \qquad [\because a_3 = 2 \text{ and } a_4 = 3]$$

$$\frac{a_5}{a_4} = \frac{5}{3} \qquad [\because a_4 = 3 \text{ and } a_5 = 5]$$

EXERCISE 5.1

LEVEL-1

1. Write the first five terms of each of the following sequences whose nth terms are:
   (i) $a_n = 3n + 2$ (ii) $a_n = \frac{n-2}{3}$ (iii) $a_n = 3^n$
   (iv) $a_n = \frac{3n-2}{5}$ (v) $a_n = (-1)^n \cdot 2^n$ (vi) $a_n = \frac{n(n-2)}{2}$
   (vii) $a_n = n^2 - n + 1$ (viii) $a_n = 2n^2 - 3n + 1$ (ix) $a_n = \frac{2n-3}{6}$
2. Find the indicated terms in each of the following sequences whose nth terms are:
   (i) $a_n = 5n - 4;\ a_{12}$ and a_{15} (ii) $a_n = \frac{3n-2}{4n+5};\ a_7$ and a_8
   (iii) $a_n = n(n-1)(n-2);\ a_5$ and a_8 (iv) $a_n = (n-1)(2-n)(3+n);\ a_1, a_2, a_3$
   (v) $a_n = (-1)^n n;\ a_3, a_5, a_8$
3. Find the next five terms of each of the following sequences given by:
   (i) $a_1 = 1, a_n = a_{n-1} + 2, n \geq 2$ (ii) $a_1 = a_2 = 2, a_n = a_{n-1} - 3,\ n > 2$
   (iii) $a_1 = -1, a_n = \frac{a_{n-1}}{n}, n \geq 2$ (iv) $a_1 = 4, a_n = 4a_{n-1} + 3, n > 1$.

ANSWERS

1. (i) $a_1 = 5, a_2 = 8, a_3 = 11, a_4 = 14, a_5 = 17$ (ii) $a_1 = -\frac{1}{3}, a_2 = 0, a_3 = \frac{1}{3}, a_4 = \frac{2}{3}, a_5 = 1$

   (iii) $a_1 = 3, a_2 = 9, a_3 = 27, a_4 = 81, a_5 = 243$ (iv) $a_1 = \frac{1}{5}, a_2 = \frac{4}{5}, a_3 = \frac{7}{5}, a_4 = 2, a_5 = \frac{13}{5}$

   (v) $a_1 = -2, a_2 = 4, a_3 = -8, a_4 = 16, a_5 = -32$ (vi) $a_1 = \frac{-1}{2}, a_2 = 0, a_3 = \frac{3}{2}, a_4 = 4, a_5 = \frac{15}{2}$

   (vii) $a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 13, a_5 = 21$ (viii) $a_1 = 0, a_2 = 3, a_3 = 10, a_4 = 21, a_5 = 36$

   (ix) $a_1 = -\frac{1}{6}, a_2 = \frac{1}{6}, a_3 = \frac{1}{2}, a_4 = \frac{5}{6}, a_5 = \frac{7}{6}$

2. (i) $a_{12} = 56, a_{15} = 71$ (ii) $a_7 = \frac{19}{33}, a_8 = \frac{22}{37}$ (iii) $a_5 = 60, a_8 = 336$

   (iv) $a_1 = 0, a_2 = 0, a_3 = -12$ (v) $a_3 = -3, a_5 = -5, a_8 = 8$

3. (i) $a_2 = 3, a_3 = 5, a_4 = 7, a_5 = 9, a_6 = 11$

   (ii) $a_3 = -1, a_4 = -4, a_5 = -7, a_6 = -10, a_7 = -13$

   (iii) $a_2 = -\frac{1}{2}, a_3 = -\frac{1}{6}, a_4 = -\frac{1}{24}, a_5 = -\frac{1}{120}, a_6 = -\frac{1}{720}$

   (iv) $a_2 = 19, a_3 = 79, a_4 = 319, a_5 = 1279, a_6 = 5119$

5.3 ARITHMETIC PROGRESSION (A.P.)

In this section, we shall discuss a particular type of sequences in which each term, except the first, progresses in a definite manner. Consider for instance, the following sequences

(i) 1, 4, 7, 10, 13, ... (ii) 12, 7, 2, – 3, – 8, ...

(iii) – 9, – 7, – 5, – 3, – 2, 1, 3, ...

In each of these sequences every term except the first is obtained by adding a fixed number (positive or negative) to the preceding term. For example, in the sequence given in (i), each term is obtained by adding 3 to the preceding term. In the sequence given in (ii) each term is – 5 more than the preceding term and in the sequence given in (iii) each term is obtained by adding 2 to the preceding term.

All these sequences are called arithmetic sequences or arithmetic progressions abbreviated as A.P. Thus, we may define an arithmetic sequence as follows:

ARITHMETIC PROGRESSION (A.P.) *A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called an arithmetic progression, if there exists a constant number d such that*

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d \\ a_4 &= a_3 + d \\ &\vdots \\ a_n &= a_{n-1} + d \text{ and so on.} \end{aligned}$$

The constant 'd' is called the common difference of the A.P.

Thus, if the first term is a and the common difference is d, then

$$a, a+d, a+2d, a+3d, a+4d, \ldots\ldots\ldots$$

is an arithmetic progression.

In other words, a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called an arithmetic progression if the difference of a term and the preceding term is always constant. This constant is called the common difference of the A.P.

Thus, if $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an A.P. with common difference 'd', then,

$$\begin{aligned} a_2 - a_1 &= d \\ a_3 - a_2 &= d \\ a_4 - a_3 &= d \\ \vdots \quad & \quad \vdots \\ a_n - a_{n-1} &= d \text{ and so on.} \end{aligned}$$

ILLUSTRATION 1 *The sequence 1, 4, 7, 10, 13, ... is an A.P. whose first term is 1 and the common difference is equal to 3.*

ILLUSTRATION 2 *The sequence 11, 7, 3, – 1, ... is an A.P. whose first term is 11 and the common difference is equal to – 4.*

It follows from the above discussion that the sequence $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$ is an A.P. with common difference 'd' if and only if

$$a_{n+1} - a_n = d \text{ for } n = 1, 2, 3, 4, \ldots$$

This suggests us the following algorithm to determine whether a sequence is an A.P. or not when we are given an algebraic formula for the general term of the sequence.

ALGORITHM

STEP I *Obtain a_n*

STEP II *Replace n by $(n+1)$ in a_n to get a_{n+1}*

STEP III *Calculate $a_{n+1} - a_n$*

STEP IV *Check the value of $a_{n+1} - a_n$. If $a_{n+1} - a_n$ is independent of n, then the given sequence is an A.P. Otherwise it is not an A.P.*

ILLUSTRATION 3 *Show that the sequence defined by $a_n = 4n + 5$ is an A.P. Also, find its common difference.*

SOLUTION We have, $a_n = 4n + 5$

Replacing n by $(n+1)$, we get

$$a_{n+1} = 4(n+1) + 5 = 4n + 9$$

Now, $a_{n+1} - a_n = (4n + 9) - (4n + 5) = 4$

Clearly, $a_{n+1} - a_n$ is independent of n and is equal to 4.

So, the given sequence is an A.P. with common difference 4.

ILLUSTRATIVE EXAMPLES

LEVEL-1

ILLUSTRATION 1 *Show that the sequence defined by $a_n = 2n^2 + 1$ is not an A.P.*

SOLUTION We have, $a_n = 2n^2 + 1$

Replacing n by $(n+1)$ in a_n, we obtain

$$a_{n+1} = 2(n+1)^2 + 1 = 2n^2 + 4n + 3$$

Now, $$a_{n+1} - a_n = (2n^2 + 4n + 3) - (2n^2 + 1) = 4n + 2$$

Clearly, $a_{n+1} - a_n$ is not independent of n and is therefore not constant. So, the given sequence is not an A.P.

ILLUSTRATION 2 *Show that a sequence is an A.P. if its nth term is a linear expression in n and in such a case the common difference is equal to the coefficient of n.*

SOLUTION Let there be a sequence whose nth term is a linear expression in n

i.e. $a_n = An + B$, where A, B are constants.

$\Rightarrow$ $a_{n+1} = A(n+1) + B$

$\therefore$ $a_{n+1} - a_n = \{A(n+1) + B\} - \{An + B\} = A$

Clearly, $a_{n+1} - a_n$ is independent of n and is therefore a constant. So, the sequence is an A.P. with common difference A.

NOTE *Readers may use the above statement as a standard result.*

ILLUSTRATION 3 *The nth term of a sequence is $3n - 2$. Is the sequence an A.P.? If so, find its 10th term.*

SOLUTION We have, $a_n = 3n - 2$.

Clearly, a_n is a linear expression in n. So, the given sequence is an A.P. with common difference 3.

Putting $n = 10$, we get

$$a_{10} = 3 \times 10 - 2 = 28.$$

REMARK *It is evident from the above examples that a sequence is not an A.P. if its nth term is not a linear expression in n.*

EXERCISE 5.2

LEVEL-1

1. Show that the sequence defined by $a_n = 5n - 7$ is an A.P., find its common difference.
2. Show that the sequence defined by $a_n = 3n^2 - 5$ is not an A.P.
3. The general term of a sequence is given by $a_n = -4n + 15$. Is the sequence an A.P.? If so, find its 15th term and the common difference.
4. Write the sequence with nth term:

   (i) $a_n = 3 + 4n$ (ii) $a_n = 5 + 2n$ (iii) $a_n = 6 - n$ (iv) $a_n = 9 - 5n$

   Show that all of the above sequences form A.P.
5. The n^{th} term of an A.P. is $6n + 2$. Find the common difference. [CBSE 2008]

6. Justify whether it is true to say that the sequence having following n^{th} term is an A.P.

(i) $a_n = 2n - 1$ (ii) $a_n = 3n^2 + 5$ (iii) $a_n = 1 + n + n^2$

ANSWERS

1. 5
3. $a_{15} = -45$, Common difference $= -4$
4. (i) $7, 11, 15, 19, \ldots$ (ii) $7, 9, 11, 13, \ldots$ (iii) $5, 4, 3, 2, 1, 0, -1, \ldots$ (iv) $4, -1, -6, -11, \ldots$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write an A.P. whose first term is 10 and common difference is 3.*

SOLUTION We know that if a is the first term and d is the common difference, then the arithmetic progression is

$$a, a + d, a + 2d, a + 3d, \ldots$$

Here, $a = 10$ and $d = 3$.

So, the arithmetic progression is $10, 13, 16, 19, 22, \ldots$

EXAMPLE 2 *Write an A.P. having 4 as the first term and -3 as the common difference.*

SOLUTION The arithmetic progression with first term a and common difference d is given by

$$a, a + d, a + 2d, a + 3d, \ldots$$

i.e., each term is obtained by adding 'd' to the preceding term.

Here, $a = 4$ and $d = -3$.

So, the arithmetic progression is

$$4, 4 + (-3), 4 + 2 \times (-3), 4 + 3(-3), 4 + 4(-3), \ldots$$

or, $$4, 1, -2, -5, -8, \ldots$$

EXAMPLE 3 *Write an A.P. whose first term and common difference are -1.25 and -0.25 respectively.*

SOLUTION Here, $a = -1.25$ and $d = -0.25$

So, the arithmetic progression is

$$-1.25, -1.25 + (-0.25), -1.25 + 2(-0.25), -1.25 + 3(-0.25), \ldots$$

or, $$-1.25, -1.50, -1.75, -2, \ldots$$

EXAMPLE 4 *For the following arithmetic progressions write the first term and common difference*

(i) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, \ldots$ (ii) $0.6, 1.7, 2.8, 3.9, \ldots$

SOLUTION (i) We have,

$$\frac{5}{3} - \frac{1}{3} = \frac{4}{3}, \frac{9}{3} - \frac{5}{3} = \frac{4}{3}, \frac{13}{3} - \frac{9}{3} = \frac{4}{3}$$

Clearly, the difference between a term and the preceding term is same and is equal to $\frac{4}{3}$.

So, the given sequence is an A.P. with first term $\frac{1}{3}$ and common difference $\frac{4}{3}$.

(ii) We have,

$$1.7 - 0.6 = 1.1, 2.8 - 1.7 = 1.1, 3.9 - 2.8 = 1.1$$

So, the given sequence is an A.P. with first term 0.6 and common difference 1.1.

EXAMPLE 5 *In which of the following situations, the sequence formed will form an A.P.?*

(i) *Number of students left in the school auditorium from the total strength of 1000 students when they leave the auditorium in batches of 25.*

(ii) *The amount of money in the account every year when ₹100 are deposited annually to accumulate at compound interest at 4% per annum.*

SOLUTION (i) We have,

Total strength of students in the auditorium = 1000

Number of students left in the auditorium when first batch of 25 students leaves the auditorium = 1000 − 25 = 975

Number of students left in the auditorium when second batch of 25 students leaves the auditorium = 975 − 25 = 950

Number of students left in the auditorium when third batch of 25 students leaves the audiotirum = 950 − 25 = 925 and so on.

Thus, the number of students left in the auditorium at different stages are

1000, 975, 950, 925, ...

Clearly, it is an A.P. with first term 1000 and common difference −25.

(ii) We know that if P is the principal and r% per annum is the rate of interest, compound annually, then the amount A_n at the end of n years is given by

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Here, $P = ₹\,100$ and $r = 4$.

$$\therefore \quad A_n = 100\left(1 + \frac{4}{100}\right)^n = 100 \times \left(\frac{26}{25}\right)^n = 100 \times (1.04)^n$$

Thus, the amount of money in the account at the end of different years is given by

$₹\,100 \times 1.04,\ ₹\,100 \times (1.04)^2,\ ₹\,100 \times (1.04)^3, \ldots$

or, ₹ 104, ₹ 108.16, ₹ 112.48, ...

Clearly, it is not forming an A.P.

EXAMPLE 6 *Find the common difference and write the next three terms of the A.P.* 3, −2, −7, −12, ...

SOLUTION We have,

Second term − First term = − 2 − (3) = − 5

Third term − Second term = − 7 − (− 2) = − 5

So, the given sequence is an A.P. with common difference − 5.

Since, each term of an A.P. is obtained by adding common difference to the preceding term.

$$\therefore \quad a_5 = a_4 + (-5) = -12 + (-5) = -17$$

$$a_6 = a_5 + (-5) = -17 + (-5) = -22$$

and, $$a_7 = a_6 + (-5) = -22 + (-5) = -27$$

EXERCISE 5.3

LEVEL-1

1. For the following arithmetic progressions write the first term a and the common difference d:

   (i) $-5, -1, 3, 7, \ldots$ [NCERT] (ii) $\frac{1}{5}, \frac{3}{5}, \frac{5}{5}, \frac{7}{5}, \cdots$

   (iii) $0.3, 0.55, 0.80, 1.05, \ldots$ (iv) $-1.1, -3.1, -5.1, -7.1, \ldots$

2. Write the arithmetic progression when first term a and common difference d are as follows:

   (i) $a = 4, \quad d = -3$ [NCERT] (ii) $a = -1, \quad d = \frac{1}{2}$ [NCERT]

   (iii) $a = -1.5, \quad d = -0.5$

3. In which of the following situations, the sequence of numbers formed will form an A.P.?

   (i) The cost of digging a well for the first metre is ₹ 150 and rises by ₹ 20 for each succeeding metre. [NCERT]

   (ii) The amount of air present in the cylinder when a vacuum pump removes each time $\frac{1}{4}$ of their remaining in the cylinder. [NCERT]

   (iii) Divya deposited ₹ 1000 at compound interest at the rate of 10% per annum. The amount at the end of first year, second year, third year, ..., and so on. [NCERT EXEMPLAR]

4. Find the common difference and write the next four terms of each of the following arithmetic progressions :

   (i) $1, -2, -5, -8, \ldots$ (ii) $0, -3, -6, -9, \ldots$

   (iii) $-1, \frac{1}{4}, \frac{3}{2}, \cdots$ (iv) $-1, -\frac{5}{6}, -\frac{2}{3}, \cdots$

5. Prove that no matter what the real numbers a and b are, the sequence with nth term $a + nb$ is always an A.P. What is the common difference?

6. Find out which of the following sequences are arithmetic progressions. For those which are arithmetic progressions, find out the common difference.

   (i) $3, 6, 12, 24, \ldots$ (ii) $0, -4, -8, -12, \ldots$

   (iii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots$ (iv) $12, 2, -8, -18, \ldots$

   (v) $3, 3, 3, 3, \ldots$ (vi) $p, p + 90, p + 180, p + 270, \ldots$ where $p = (999)^{999}$

   (vii) $1.0, 1.7, 2.4, 3.1, \ldots$ (viii) $-225, -425, -625, -825, \ldots$

   (ix) $10, 10 + 2^5, 10 + 2^6, 10 + 2^7, \ldots$

   (x) $a + b, (a + 1) + b, (a + 1) + (b + 1), (a + 2) + (b + 1), (a + 2) + (b + 2), \ldots$

   (xi) $1^2, 3^2, 5^2, 7^2, \ldots$ (xii) $1^2, 5^2, 7^2, 73, \ldots$

7. Find the common difference of the A.P. and write the next two terms:

   (i) $51, 59, 67, 75, \ldots$ (ii) $75, 67, 59, 51, \ldots$

   (iii) $1.8, 2.0, 2.2, 2.4, \ldots$ (iv) $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \cdots$

   (v) $119, 136, 153, 170, \ldots$

ANSWERS

1. (i) $a=-5, d=4$ (ii) $a=\frac{1}{5}, d=\frac{2}{5}$ (iii) $a=0.3, d=0.25$ (iv) $a=-1.1, d=-2$
2. (i) $4, 1, -2, -5, -8, \ldots$ (ii) $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$ (iii) $-1.5, -2, -2.5, -3, \ldots$
3. (i) A.P. (ii) Does not form an A.P. (iii) A.P.
4. (i) $-3; a_5=-11, a_6=-14, a_7=-17, a_8=-20$

   (ii) $-3; a_5=-12, a_6=-15, a_7=-18, a_8=-21$

   (iii) $\frac{5}{4}; a_4=\frac{11}{4}, a_5=\frac{16}{4}, a_6=\frac{21}{4}, a_7=\frac{26}{4}$ (iv) $\frac{1}{6}; a_4=-\frac{1}{2}, a_5=-\frac{1}{3}, a_6=-\frac{1}{6}, a_7=0$
5. (i) b
6. (i) No (ii) Yes, $d=-4$ (iii) No (iv) Yes, $d=-104$ (v) Yes, $d=0$ (vi) Yes, $d=90$ (vii) Yes, $d=0.7$ (viii) Yes, $d=-200$ (ix) No (x) Yes, $d=1$ (xi) No (xii) Yes, $d=24$
7. (i) $d=8, a_5=83, a_6=91$ (ii) $d=-8, a_5=43, a_6=35$

   (iii) $d=0.2, a_5=2.6, a_6=2.8$ (iv) $d=\frac{1}{4}; a_5=1, a_6=\frac{5}{4}$

   (v) $d=17; a_5=187, a_6=204$

5.4 GENERAL TERM OF AN A.P.

In this section, we shall find the formula for the n th term or general term of an A.P. in terms of its first term and the common difference. The same will be used to solve some problems on A.P.

THEOREM *Let a be the first term and d be the common difference of an A.P. Then, its nth term or general term is given by*

$$a_n = a + (n-1)\,d.$$

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the given A.P. Then,

$$a_1 = a$$

$$\Rightarrow \quad a_1 = a + (1-1)\,d \qquad \text{...(i)}$$

Since each term of an A.P. is obtained by adding common difference to the preceding term. Therefore,

$$a_2 = a + d$$

$$\Rightarrow \quad a_2 = a + (2-1)\,d \qquad \text{...(ii)}$$

Similarly, we have

$$a_3 = a_2 + d$$

$$\Rightarrow \quad a_3 = (a+d) + d$$

$$\Rightarrow \quad a_3 = a + 2d$$

$$\Rightarrow \quad a_3 = a + (3-1)\,d \qquad \text{...(iii)}$$

and, $a_4 = a_3 + d$

$\Rightarrow$ $a_4 = (a + 2d) + d$

$\Rightarrow$ $a_4 = a + 3d$

$\Rightarrow$ $a_4 = a + (4 - 1)\,d$ ...(iv)

Observing the pattern in equation (i), (ii), (iii) and (iv), we find that

$$a_n = a + (n - 1)\,d$$

Q.E.D.

REMARK *It is evident from the above theorem that*
General term of an A.P. = First term + (Term number − 1) × (Common difference)

5.4.1 n^{th} TERM OF AN A.P. FROM THE END

Let there be an A.P. with first term a and common difference d. If there are m terms in the A.P., then

nth term from the end $= (m - n + 1)$th term the beginning

$\Rightarrow$ nth term from the end $= a_{m-n+1}$

$\Rightarrow$ nth term from the end $= a + (m - n + 1 - 1)\,d$

$\Rightarrow$ nth term from the end $= a + (m - n)\,d$

Also, if l is the last term of the A.P., then nth term from the end is the nth term of an A.P. whose first term is l and common difference is $-d$.

$\therefore$ nth term from the end = Last term $+ (n - 1)(-d)$

$\Rightarrow$ nth term from the end $= l - (n - 1)\,d$

ILLUSTRATION *Find the 6^{th} term from the end of the A.P. 17, 14, 11, ..., −40* [CBSE 2005]

SOLUTION We have,

l = Last term = −40 and, d = Common difference = −3

$\therefore$ 6^{th} term from the end $= l - (6 - 1)\,d = -40 - 5 \times -3 = -25$

5.4.2 MIDDLE TERM(S) OF A FINITE A.P.

Let there be a finite A.P. with first term a, common difference d and number of terms n.

If n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ term is the middle term and is given by $a + \left(\frac{n+1}{2} - 1\right)d$.

If n is even, then $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2} + 1\right)^{th}$ are middle terms given by $a + \left(\frac{n}{2} - 1\right)d$ and $a + \left(\frac{n}{2} + 1 - 1\right)d = a + \frac{n}{2}d$ respectively.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the 12th, 24th and nth term of the A.P. given by 9, 13, 17, 21, 25,...*

SOLUTION We have, a = First term = 9

and, d = Common difference = 4 [$\because$ 13 − 9 = 4, 17 − 13 = 4, 21 − 17 = 4 etc.]

We know that the nth term of an A.P. with first term a and common difference d is given by

$$a_n = a + (n - 1)\,d$$

$\therefore$ $a_{12} = a + (12 - 1)\,d = a + 11d = 9 + 11 \times 4 = 53$

$$a_{24} = a + (24-1)d \quad = a + 23d = 9 + 23 \times 4 = 101$$

and, $a_n = a + (n-1)d = 9 + (n-1) \times 4 = 4n + 5$

Thus, we have

$$a_{12} = 53, a_{24} = 101 \text{ and } a_n = 4n + 5$$

EXAMPLE 2 *Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its 16th term and the general term.*

SOLUTION We have,

$$(12-9) = (15-12) = (18-15) = 3$$

Therefore, the given sequence is an A.P. with common difference 3.

$$a = \text{First term} = 9$$

$\therefore$ 16th term $= a_{16} = a + (16-1)d = a + 15d$ $\qquad [\because a_n = a + (n-1)d]$

$\Rightarrow$ $a_{16} = 9 + 15 \times 3 = 54$

$\because$ General term $=$ nth term $= a + (n-1)d$

$\therefore$ $a_n = 9 + (n-1) \times 3 = 3n + 6$

EXAMPLE 3 *The first term of an A.P. is –7 and the common difference 5. Find its 18th term and the general term.*

SOLUTION We have,

$$a = \text{First term} = -7 \text{ and, } d = \text{Common difference} = 5$$

$\therefore$ $a_{18} = a + (18-1)d$ $\qquad [\because a_n = a + (n-1)d]$

$\Rightarrow$ $a_{18} = a + 17d = -7 + 17 \times 5 = 78$

and, $a_n = a + (n-1) \times 5 = -7 + (n-1) \times 5$

$\Rightarrow$ $a_n = -7 + 5n - 5 = 5n - 12$

EXAMPLE 4 *Determine the 10th term from the end of the A.P. 4, 9, 14, ..., 254.*

SOLUTION We have,

$$l = \text{Last term} = 254 \text{ and, } d = \text{Common difference} = 5.$$

$\therefore$ 10th term from the end $= l - (10-1)d = l - 9d = 254 - 9 \times 5 = 209$

EXAMPLE 5 *Which term of the sequence –1, 3, 7, 11, ... is 95 ?*

SOLUTION Clearly, the given sequence is an A.P such that

$$a = \text{First term} = -1 \text{ and, } d = \text{Common difference} = 4$$

Let 95 be the nth term of the given A.P. Then,

$$a_n = 95$$

$\Rightarrow$ $a + (n-1)d = 95$

$\Rightarrow$ $-1 + (n-1) \times 4 = 95$

$\Rightarrow$ $-1 + 4n - 4 = 95$

$\Rightarrow$ $4n - 5 = 95 \Rightarrow 4n = 100 \Rightarrow n = 25$

Thus, 95 is 25th term of the given sequence.

EXAMPLE 6 *Which term of the sequence 4, 9, 14, 19,... is 124 ?*

SOLUTION Clearly, the given sequence is an A.P. with first term a (= 4) and common difference d (= 5)

Let 124 be the nth term of the given sequence. Then,

$$a_n = 124 \Rightarrow a+(n-1)\,d = 124 \Rightarrow 4+(n-1)\times 5 = 124 \Rightarrow 5n-1 = 124 \Rightarrow 5n = 125 \Rightarrow n = 25$$

Hence, 25th term of the given sequence is 124.

EXAMPLE 7 *How many terms are there in the sequence 3, 6, 9, 12, ..., 111?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 3$. Let there be n terms in the given sequence. Then,

$$n\text{th term} = 111$$

$$\Rightarrow \quad a + (n-1)\,d = 111$$

$$\Rightarrow \quad 3 + (n-1)\times 3 = 111 \Rightarrow n = 37$$

Thus, the given sequence contains 37 terms.

EXAMPLE 8 *Find the middle term of the A.P. 6, 13, 20, ..., 216.* [CBSE 2015]

SOLUTION Clearly, 6, 13, 20, ..., 216 is an A.P. with first term $a = 6$ and common difference $d = 7$. Let there be n terms in the given A.P. Then,

$$a_n = 216$$

$$\Rightarrow \quad a + (n-1)\,d = 216$$

$$\Rightarrow \quad 6 + 7\,(n-1) = 216 \qquad [\because a = 6 \text{ and } d = 7]$$

$$\Rightarrow \quad 7n = 217 \Rightarrow n = 31$$

Here, n is odd so $\left(\frac{n+1}{2}\right)^{\text{th}}$ *i.e.* $\left(\frac{31+1}{2}\right)^{\text{th}} = 16^{\text{th}}$ term is the middle term and is given by

$$a_{16} = a + (16-1)\,d = a + 15d = 6 + 15\times 7 = 111$$

EXAMPLE 9 *Find the middle term(s) of the A.P. 7, 13, 19, ..., 241.*

SOLUTION Clearly, 7, 13, 19, ..., 241 is an A.P. with first term $a = 7$ and common difference $d = 6$. Let there be n terms in the A.P. Then,

$$a_n = 241$$

$$\Rightarrow \quad a + (n-1)\,d = 241$$

$$\Rightarrow \quad 7 + 6\,(n-1) = 241$$

$$\Rightarrow \quad 6n = 240 \Rightarrow n = 40$$

Clearly, n is even. So, $\left(\frac{n}{2}\right)^{\text{th}} = 20^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}} = 21^{\text{th}}$ are middle terms and are given by

$$a_{20} = a + (20-1)\,d = a + 19d = 7 + 19\times 6 = 121$$

and, $$a_{21} = a + (21-1)\,d = a + 20d = 7 + 20\times 6 = 127$$

EXAMPLE 10 *Consider the A.P. 2, 5, 8, 11, ..., 302. Show that twice of the middle term of the above A.P. is equal to the sum of its first and last term.*

SOLUTION Clearly, 2, 5, 8, 11, ..., 302 is an A.P. with first term $a = 2$ and common difference $d = 3$. Let there be n terms in the given A.P.. Then,

$$n^{\text{th}} \text{ term} = 302$$

$$\Rightarrow \quad a + (n-1)\,d = 302$$

$$\Rightarrow \quad 2 + 3\,(n-1) = 302$$

$$\Rightarrow \quad 3n = 303$$

$\Rightarrow \quad n = 101$

Clearly, n is odd. Therefore, $\left(\frac{n+1}{2}\right)^{th}$ *i.e.* 51^{st} term is the middle term.

Now,

$$\text{Middle term} = a_{51} = a + 50d = 2 + 50 \times 3 = 152$$
$$\text{First term} + \text{Last term} = 2 + 302 = 304$$

Clearly, twice the middle term is equal to the sum of the first and last term.

EXAMPLE 11 *In the A.P. 1, 7, 13, 19, ..., 415, prove that the sum of the middle terms is equal to the sum of first and last terms.*

SOLUTION We observe that 1, 7, 13, 19, ..., 415 is an A.P. with first term $a = 1$ and common difference $d = 6$. Let there be n terms in the given A.P. Then,

$$n^{th} \text{ term} = 415$$
$$\Rightarrow \quad a + (n-1)d = 415$$
$$\Rightarrow \quad 1 + 6(n-1) = 415$$
$$\Rightarrow \quad 6n = 420$$
$$\Rightarrow \quad n = 70$$

So, there are 70 terms in the given A.P. Therefore, $\left(\frac{70}{2}\right)^{th} = 35^{th}$ and $\left(\frac{70}{2}+1\right)^{th} = 36^{th}$ are the middle terms.

Now,

$$a_{35} = a + 34d = 1 + 34 \times 6 = 205 \text{ and, } a_{36} = a + 35d = 1 + 35 \times 6 = 211$$
$$\therefore \quad a_{35} + a_{36} = 205 + 211 = 416$$

Also, $\quad a_1 + a_{70} = 1 + 415 = 416$

Clearly, $a_{35} + a_{36} = a_1 + a_{70}$.

EXAMPLE 12 *For what value of n are the n^{th} terms of the following two A.P's the same?*
(i) 1, 7, 13, 19, ... *(ii) 69, 68, 67, ...* **[CBSE 2006C]**

SOLUTION Clearly, 1, 7, 13, 19, ... forms an A.P. with first term 1 and common difference 6. Therefore, its n^{th} term is given by

$$a_n = 1 + (n-1) \times 6 = 6n - 5$$

Also, 69, 68, 67, 66, ... forms an A.P. with first term 69 and common difference -1.
So, its n^{th} term is given by

$$a_n' = 69 + (n-1) \times (-1) = -n + 70$$

The two A.Ps will have identical nth terms, if

$$a_n = a_n'$$
$$\Rightarrow \quad 6n - 5 = -n + 70$$
$$\Rightarrow \quad 7n = 75$$
$$\Rightarrow \quad n = \frac{75}{7}, \text{ which is not a natural number.}$$

Hence, there is no value of n for which the two A.Ps will have identical terms.

EXAMPLE 13 *If the 8^{th} term of an A.P. is 31 and the 15^{th} term is 16 more than the 11^{th} term, find the A.P.* **[CBSE 2006C]**

SOLUTION Let a be the first term and d be the common difference of the A.P.

We have,

$$a_8 = 31 \text{ and } a_{15} = 16 + a_{11}$$

$\Rightarrow \quad a + 7d = 31 \text{ and } a + 14d = 16 + a + 10d$

$\Rightarrow \quad a + 7d = 31 \text{ and } 4d = 16$

$\Rightarrow \quad a + 7d = 31 \text{ and } d = 4$

$\Rightarrow \quad a + 7 \times 4 = 31 \Rightarrow a + 28 = 31 \Rightarrow a = 3$

Hence, the A.P. is $a, a+d, a+2d, a+3d, \ldots$ i.e., $3, 7, 11, 15, 19, \ldots$

EXAMPLE 14 *Which term of the arithmetic progression 5, 15, 25, ... will be 130 more than its 31st term?* **[CBSE 2006C, 2017]**

SOLUTION We have, $a = 5$ and $d = 10$

$\therefore \quad a_{31} = a + 30d = 5 + 30 \times 10 = 305$

Let n^{th} term of the given A.P. be 130 more than its 31^{st} term. Then,

$$a_n = 130 + a_{31}$$

$\therefore \quad a + (n-1)d = 130 + 305$

$\Rightarrow \quad 5 + 10(n-1) = 435$

$\Rightarrow \quad 10(n-1) = 430$

$\Rightarrow \quad n - 1 = 43$

$\Rightarrow \quad n = 44$

Hence, 44^{th} term of the given A.P. is 130 more than its 31^{st} term.

EXAMPLE 15 *If the 10^{th} term of an A.P. is 52 and 17^{th} term is 20 more than the 13^{th} term, find the A.P.* **[CBSE 2006C]**

SOLUTION Let a be the first term and d be the common difference of the A.P.

We have,

$$a_{10} = 52 \text{ and } a_{17} = a_{13} + 20$$

$\Rightarrow \quad a + 9d = 52 \text{ and } a + 16d = a + 12d + 20$

$\Rightarrow \quad a + 9d = 52 \text{ and } 4d = 20$

$\Rightarrow \quad a + 9d = 52 \text{ and } d = 5$

$\Rightarrow \quad a + 45 = 52 \text{ and } d = 5$

$\Rightarrow \quad a = 7 \text{ and } d = 5$

Hence, the A.P. is $a, a+d, a+2d, a+3d, \ldots$ i.e., $7, 12, 17, 22, \ldots$

EXAMPLE 16 *Is 184 a term of the sequence 3, 7, 11, ...?*

SOLUTION Clearly, the given sequence is an A.P. with first term a (= 3) and common difference d (= 4).

Let the nth term of the given sequence be 184. Then,

$$a_n = 184$$

$\Rightarrow \quad a + (n-1)d = 184$

$\Rightarrow \quad 3 + (n-1) \times 4 = 184 \Rightarrow 4n = 185 \Rightarrow n = 46\frac{1}{4}$

Since n is not a natural number. So, 184 is not a term of the given sequence.

EXAMPLE 17 *The 10^{th} term of an A.P. is 52 and 16^{th} term is 82. Find the 32nd term and the general term.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Let the A.P. be $a_1, a_2, a_3, \ldots, a_n, \ldots$
It is given that

$$a_{10} = 52 \text{ and, } a_{16} = 82$$

$$\Rightarrow \quad a + (10-1)d = 52 \text{ and, } a + (16-1)d = 82$$

$$\Rightarrow \quad a + 9d = 52 \quad \ldots(i)$$

$$\text{and,} \quad a + 15d = 82 \quad \ldots(ii)$$

Subtracting equation (ii) from equation (i), we get

$$-6d = -30 \Rightarrow d = 5$$

Putting $d = 5$ in equation (i), we get

$$a + 45 = 52 \Rightarrow a = 7$$

$$\therefore \quad a_{32} = a + (32-1)d = 7 + 31 \times 5 = 162$$

$$\text{and,} \quad a_n = a + (n-1)d = 7 + (n-1) \times 5 = 5n + 2$$

Hence, $a_{32} = 162$ and $a_n = 5n + 2$

EXAMPLE 18 *The sum of 5th and 9th terms of an A.P. is 72 and the sum of 7th and 12th terms is 97. Find the A.P.* [CBSE 2009]

SOLUTION Let a be the first term and 'd' be the common difference of the A.P. It is given that

$$a_5 + a_9 = 72 \text{ and, } a_7 + a_{12} = 97$$

$$\Rightarrow \quad (a + 4d) + (a + 8d) = 72 \text{ and, } (a + 6d) + (a + 11d) = 97$$

Thus, we have

$$\Rightarrow \quad 2a + 12d = 72 \quad \ldots(i)$$

$$\Rightarrow \quad 2a + 17d = 97 \quad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$5d = 25 \Rightarrow d = 5$$

Putting $d = 5$ in (i), we get

$$2a + 60 = 72 \Rightarrow 2a = 12 \Rightarrow a = 6$$

$$\therefore \quad a = 6 \text{ and } d = 5$$

Hence, the A.P. is 6, 11, 16, 21, 26, ...

EXAMPLE 19 *Determine the general term of an A.P. whose 7th term is –1 and 16th term 17.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Let the A.P. be $a_1, a_2, a_3, \ldots, a_n, \ldots$
It is given that

$$a_7 = -1 \text{ and } a_{16} = 17$$

$$\Rightarrow \quad a + (7-1)d = -1 \text{ and, } a + (16-1)d = 17$$

$$\Rightarrow \quad a + 6d = -1 \quad \ldots(i)$$

$$\text{and,} \quad a + 15d = 17 \quad \ldots(ii)$$

Subtracting equation (i) from equation (ii), we get

$$9d = 18 \Rightarrow d = 2$$

Putting $d = 2$ in equation (i), we get

$$a + 12 = -1 \Rightarrow a = -13$$

Hence, General term $= a_n = a + (n-1)d = -13 + (n-1) \times 2 = 2n - 15$

EXAMPLE 20 *If five times the fifth term of an A.P. is equal to 8 times its eighth term, show that its 13^{th} term is zero.*

SOLUTION Let $a_1, a_2, a_3, ..., a_n, ...$ be the A.P. with its first term a and common difference d. It is given that

$$5a_5 = 8a_8$$
$$\Rightarrow \quad 5(a + 4d) = 8(a + 7d)$$
$$\Rightarrow \quad 5a + 20d = 8a + 56d$$
$$\Rightarrow \quad 3a + 36d = 0$$
$$\Rightarrow \quad 3(a + 12d) = 0$$
$$\Rightarrow \quad a + 12d = 0 \Rightarrow a + (13 - 1)d = 0 \Rightarrow a_{13} = 0$$

Hence, 13th term is zero.

EXAMPLE 21 *How many numbers of two digits are divisible by 7?*

SOLUTION We observe that 14 is the first two digit number divisible by 7 and 98 is the last two digit number divisible by 7. Thus, we have to determine the number of terms in the sequence

$$14, 21, 28, ..., 98$$

Clearly, it is an A.P. with first term = 14 and common difference = 7 i.e. $a = 14$ and $d = 7$.

Let there be n terms in this A.P. Then,

$$n\text{th term} = 98$$
$$\Rightarrow \quad 14 + (n - 1) \times 7 = 98$$
$$\Rightarrow \quad 14 + 7n - 7 = 98$$
$$\Rightarrow \quad 7n = 91 \Rightarrow n = 13$$

Hence, there are 13 numbers of two digits which are divisible by 7.

EXAMPLE 22 *Find the number of integers between 50 and 500 which are divisible by 7.*

SOLUTION We observe that 56 is the first integer between 50 and 500 which is divisible by 7. Also, when we divide 500 by 7 the remainder is 3. Therefore, $500 - 3 = 497$ is the largest integer divisible by 7 and lying between 50 and 500. Thus, we have to find the number of terms in an A.P. with first term = 56, last term = 497 and common difference = 7 (as the numbers are divisible by 7).

Let there be n terms in the A.P. Then,

$$a_n = 497$$
$$\Rightarrow \quad a + (n - 1)d = 497$$
$$\Rightarrow \quad 56 + (n - 1) \times 7 = 497 \qquad [\because a = 56 \text{ and } d = 7]$$
$$\Rightarrow \quad 7n + 49 = 497$$
$$\Rightarrow \quad 7n = 448 \Rightarrow n = 64$$

Thus, there are 64 integers between 50 and 500 which are divisible by 7.

EXAMPLE 23 *Which term of the A.P. 3, 15, 27, 39,... will be 132 more than its 54th term?* [NCERT]

SOLUTION Given A.P. is 3, 15, 27, 39,...

Clearly, its

First term = 3 and, Common difference = 12.

Let n^{th} term of the A.P. be 132 more than its 54th term

i.e., $$a_n = 132 + a_{54}$$
$$\Rightarrow \quad a + (n - 1)d = 132 + (a + 53d)$$

$\Rightarrow \quad 3 + 12(n-1) = 132 + (3 + 53 \times 12)$

$\Rightarrow \quad 12n - 9 = 771$

$\Rightarrow \quad 12n = 780 \Rightarrow n = 65$

Hence, 65th term of the given A.P. is 132 more than its 54th term.

EXAMPLE 24 *Two A.P's have the same common difference. The first term of one of these is 3, and that of the other is 8. What is the difference between their*

(i) *2nd terms ?* (ii) *4th terms ?* (iii) *10th terms ?* (iv) *30th terms ?*

SOLUTION Let the common difference of the two A.P's be d. Then, their n^{th} terms are

$$a_n = 3 + (n-1)d \text{ and } b_n = 8 + (n-1)d$$

$\Rightarrow \quad a_n - b_n = [3 + (n-1)d] - [8 + (n-1)d]$

$\Rightarrow \quad a_n - b_n = -5$ for all $n \in N$.

Hence, $a_2 - b_2 = -5, a_4 - b_4 = -5, a_{10} - b_{10} = -5$ and $a_{30} - b_{30} = -5$.

EXAMPLE 25 *A sum of ₹1000 is invested at 8% simple interest per annum. Calculate the interest at the end of 1, 2, 3,... years. Is the sequence of interests an A.P.? Find the interest at the end of 30 years.* [NCERT]

SOLUTION Let P be the principle, R rate of interest and I_n be the interest at the end of n years. We know that

$$I_n = \frac{PRn}{100} \qquad \left[\text{Using : Interest} = \frac{PRT}{100}\right]$$

Here, we have

$$P = ₹\,1000, \text{ and } R = 8\% \text{ per annum}$$

$\therefore \quad I_n = ₹\left(\frac{1000 \times 8 \times n}{100}\right) = ₹\,80\,n$

Putting $n = 1, 2, 3, \ldots$, we have

$$I_1 = ₹\,80, I_2 = ₹\,160, I_3 = ₹\,240 \text{ and so on.}$$

Since, I_n is a linear expression in n. Therefore, the sequence of interest forms an A.P. with common difference 80.

Also, Interest at the end of 30 years $= I_{30} = ₹\,(80 \times 30) = ₹\,2400$

EXAMPLE 26 *In a flower bed there are 23 rose plants in the first row, twenty one in the second row, nineteen in the third row and so on. There are five plants in the last row. How many rows are there in the flower bed?* [NCERT]

SOLUTION The number of rose plants in first, second third, ..., and last row are respectively 23, 21, 19, ..., 5.

Let the number of rows of rose plants be n.

The sequence 23, 21, 19, ..., 5 is an A.P. with first term a (= 23), common difference d (= −2) and nth term (= 5).

$\therefore \quad a_n = a + (n-1)d$

$\Rightarrow \quad 5 = 23 + (n-1) \times -2$

$\Rightarrow \quad 5 = 23 - 2n + 2 \Rightarrow 20 = 2n \Rightarrow n = 10$

Hence, there are 10 rows of rose plants.

EXAMPLE 27 *Suba Rao started work in 1995 at an annual salary of ₹ 5000 and received a ₹200 raise each year. In what year did his annual salary will reach ₹7000?* [NCERT]

SOLUTION Annual salary received by Suba Rao in 1995, 1996, 1997, ... is

₹ 5000, ₹ 5200, ₹ 5400, ...

Clearly, it is an arithmetic progression with first term $a = 5000$ and common difference $d = 200$.

Suppose Suba Rao's annual salary reaches to ₹ 7000 in nth years. Then,

nth term of the above A.P. = ₹ 7000

$$\Rightarrow \quad a + (n-1)d = 7000$$

$$\Rightarrow \quad 5000 + (n-1) \times 200 = 7000$$

$$\Rightarrow \quad (n-1) \times 200 = 2000$$

$$\Rightarrow \quad n - 1 = \frac{2000}{200} \Rightarrow n - 1 = 10 \Rightarrow n = 11$$

Thus, 11th annual salary received by Suba Rao will be ₹ 7000. This means that after 10 years i.e., in the year 2005 his annual salary will reach to ₹ 7000.

EXAMPLE 28 *Jasleen saved ₹5 in the first week of the year and then increased her weekly savings by ₹1.75 each week. In what week will her weekly savings be ₹20.75?* [NCERT]

SOLUTION Suppose Jasleen's weekly savings will be ₹ 20.75 in the n^{th} week.

Clearly, Jasleen's weekly savings form an A.P. with first term $a = 5$ and common difference $d = 1.75$.

$$\therefore \quad n\text{th term} = 20.75$$

$$\Rightarrow \quad a + (n-1)d = 20.75$$

$$\Rightarrow \quad 5 + (n-1) \times 1.75 = 20.75$$

$$\Rightarrow \quad (n-1) \times 1.75 = 15.75$$

$$\Rightarrow \quad n - 1 = \frac{15.75}{1.75} \Rightarrow n - 1 = 9 \Rightarrow n = 10$$

Hence, Jasleen's weekly savings will be ₹ 20.75 in 10th week.

LEVEL-2

EXAMPLE 29 *If the m^{th} term of an A.P. be $1/n$ and n^{th} term be $1/m$, then show that its $(mn)^{th}$ term is 1.* [CBSE 2017]

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Then,

$$\frac{1}{n} = m\text{th term} \Rightarrow \frac{1}{n} = a + (m-1)d \qquad \text{...(i)}$$

$$\frac{1}{m} = n\text{th term} \Rightarrow \frac{1}{m} = a + (n-1)d \qquad \text{...(ii)}$$

On subtracting equation (ii) from equation (i), we get

$$\frac{1}{n} - \frac{1}{m} = (m-n)d \Rightarrow \frac{m-n}{mn} = (m-n)d \Rightarrow d = \frac{1}{mn}$$

Putting $d = \dfrac{1}{mn}$ in equation (i), we get

$$\frac{1}{n} = a + \frac{(m-1)}{mn} \Rightarrow \frac{1}{n} = a + \frac{1}{n} - \frac{1}{mn} \Rightarrow a = \frac{1}{mn}$$

$\therefore$ (mn) th term $= a + (mn-1)\,d = \frac{1}{mn} + (mn-1)\frac{1}{mn} = 1$ $\quad \left[\because\ a = \frac{1}{mn} = d\right]$

EXAMPLE 30 *If the p^{th} term of an A.P. is q and the q^{th} term is p, prove that its n^{th} term is $(p+q-n)$.* [CBSE 2008, 2017]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

p th term $= q \Rightarrow a + (p-1)\,d = q$ ...(i)

q th term $= p \Rightarrow a + (q-1)\,d = p$ ...(ii)

Subtracting equation (ii) from equation (i), we get

$$(p-q)\,d = (q-p) \Rightarrow d = -1$$

Putting $d = -1$ in equation (i), we get

$$a + (p-1) \times (-1) = q \Rightarrow a = (p+q-1)$$

$\therefore$ n th term $= a + (n-1)d = (p+q-1) + (n-1) \times (-1) = (p+q-n)$

EXAMPLE 31 *If m times the m^{th} term of an A.P. is equal to n times its n^{th} term, show that the $(m+n)^{th}$ term of the A.P. is zero.* [CBSE 2008]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

(m times m^{th} term) = (n times n^{th} term)

$\Rightarrow$ $m\,a_m = n\,a_n$

$\Rightarrow$ $m\{a + (m-1)\,d\} = n\{a + (n-1)\,d\}$

$\Rightarrow$ $m\{a + (m-1)\,d\} - n\{a + (n-1)\,d\} = 0$

$\Rightarrow$ $a(m-n) + \{m(m-1) - n(n-1)\}\,d = 0$

$\Rightarrow$ $a(m-n) + \{(m^2 - n^2) - (m-n)\}\,d = 0$

$\Rightarrow$ $a(m-n) + (m-n)(m+n-1)\,d = 0$

$\Rightarrow$ $(m-n)\{a + (m+n-1)\,d\} = 0$

$\Rightarrow$ $a + (m+n-1)\,d = 0$ $\quad [\because\ m \neq n]$

$\Rightarrow$ $a_{m+n} = 0$

Hence, $(m+n)^{\text{th}}$ term of the given A.P. is zero.

EXAMPLE 32 *If p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively, then show that*

(i) $a(q-r) + b(r-p) + c(p-q) = 0$ [CBSE 2016]

(ii) $(a-b)\,r + (b-c)\,p + (c-a)\,q = 0$

SOLUTION (i) Let A be the first term and D be the common difference of the given A.P. Then,

$a = p$ th term $\Rightarrow a = A + (p-1)\,D$ ... (i)

$b = q$ th term $\Rightarrow b = A + (q-1)\,D$ ... (ii)

$c = r$ th term $\Rightarrow c = A + (r-1)\,D$ ... (iii)

We have,

$$a(q-r) + b(r-p) + c(p-q)$$

$$= \{A + (p-1)\,D\}(q-r) + \{A + (q-1)\,D\}(r-p) + \{A + (r-1)\,D\}(p-q)$$

[Using (i), (ii) and (iii)]

$$= A\{(q-r)+(r-p)+(p-q)\} + D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$$
$$= A \times 0 + D\{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)\}$$
$$= A \times 0 + D \times 0 = 0$$

(ii) On subtracting equation (ii) from equation (i), equation (iii) from equation (ii) and equation (i) from equation (iii), we get

$$a-b=(p-q)D, (b-c)=(q-r)D \text{ and } c-a=(r-p)D$$

$\therefore$ $(a-b)r+(b-c)p+(c-a)q$
$$=(p-q)Dr+(q-r)Dp+(r-p)Dq$$
$$=D\{(p-q)r+(q-r)p+(r-p)q\} = D \times 0 = 0$$

EXAMPLE 33 *Which term of the sequence* $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ *is the first negative term?*

SOLUTION The given sequence is an A.P. in which first term a (= 20) and common difference d (= $-3/4$). Let the n^{th} term of the given A.P. be the first negative term. Then,

$$a_n < 0$$
$$\Rightarrow \quad a+(n-1)d < 0$$
$$\Rightarrow \quad 20+(n-1)\times(-3/4) < 0$$
$$\Rightarrow \quad \frac{83}{4}-\frac{3n}{4} < 0$$
$$\Rightarrow \quad 83-3n < 0$$
$$\Rightarrow \quad 3n > 83 \Rightarrow n > 27\frac{2}{3} \Rightarrow n \geq 28 \qquad [\because n \text{ is a natural number}]$$

Thus, 28^{th} term of the given sequence is the first negative term.

EXAMPLE 34 *Two A.P's have the same common difference. The difference between their* 100^{th} *terms is 111 222 333. What is the difference between their Millionth terms?*

SOLUTION Let the two A.P's be $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$

Also, let d be the common difference of two A.P's. Then,

$$a_n = a_1 + (n-1)d \text{ and } b_n = b_1 + (n-1)d$$
$$\Rightarrow \quad a_n - b_n = \{a_1 + (n-1)d\} - \{b_1 + (n-1)d\}$$
$$\Rightarrow \quad a_n - b_n = a_1 - b_1$$

Clearly, $a_n - b_n$ is independent of n and is equal to $a_1 - b_1$. In other words

$$a_n - b_n = a_1 - b_1 \text{ for all } n \in N.$$
$$\Rightarrow \quad a_{100} - b_{100} = a_1 - b_1$$

and, $\quad a_k - b_k = a_1 - b_1$, where $k = 10,00,000$.

But, $\quad a_{100} - b_{100} = 111\,222\,333$

$\therefore \quad a_1 - b_1 = 111\,222\,333$

$\Rightarrow \quad a_k - b_k = a_1 - b_1 = 111\,222\,333$, where $k = 10,00,000$.

Hence, the difference between millionth terms is same as the difference between 100th terms i.e., 111 222 333.

EXAMPLE 35 *Find a, b and c if it is given that the numbers a, 7, b, 23, c. are in A.P:*

[NCERT EXEMPLAR]

SOLUTION Let d be the common difference of the A.P formed by the numbers $a, 7, b, 23, c$. Then,

$$7 = a + d, b = a + 2d, 23 = a + 3d \text{ and } c = a + 4d$$

$\Rightarrow$ $d = 7 - a, b = a + 2d, 23 = a + 3d$ and $c = a + 4d$

Putting $d = 7 - a$ in $23 = a + 3d$, we obtain

$$23 = a + 3(7 - a) \Rightarrow 2a = -2 \Rightarrow a = -1$$

$\therefore$ $d = 7 - a \Rightarrow d = 7 + 1 = 8$

Thus, $b = a + 2d = -1 + 16 = 15$ and $c = a + 4d = -1 + 4 \times 8 = 31$

Hence, $a = -1, b = 15$ and $c = 31$.

EXERCISE 5.4

LEVEL-1

1. Find:
   (i) 10^{th} term of the A.P. 1, 4, 7, 10,...
   (ii) 18^{th} term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2},...$
   (iii) n^{th} term of the A.P. 13, 8, 3, −2,...
   (iv) 10^{th} term of the A.P. −40, −15, 10, 35, ...
   (v) 8^{th} term of the A.P. 117, 104, 91, 78,...
   (vi) 11^{th} term of the A.P. 10.0, 10.5, 11.0, 11.5, ...
   (vii) 9^{th} term of the A.P. $\frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4},...$
2. (i) Which term of the A.P. 3, 8, 13, ... is 248?
   (ii) Which term of the A.P. 84, 80, 76,... is 0?
   (iii) Which term of the A.P. 4, 9, 14, ... is 254?
   (iv) Which term of the A.P. 21, 42, 63, 84, ... is 420?
   (v) Which term of the A.P. 121, 117, 113, ... is its first negative term?
3. (i) Is 68 a term of the A.P. 7, 10, 13,...?
   (ii) Is 302 a term of the A.P. 3, 8, 13,...?
   (iii) Is −150 a term of the A.P. 11, 8, 5, 2, ...? [CBSE 2017]
4. How many terms are there in the A.P.?
   (i) 7, 10, 13, ... 43.
   (ii) $-1, -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, ..., \frac{10}{3}$.
   (iii) 7, 13, 19, ..., 205.
   (iv) $18, 15\frac{1}{2}, 13, ..., -47$.
5. The first term of an A.P. is 5, the common difference is 3 and the last term is 80; find the number of terms.
6. The 6^{th} and 17^{th} terms of an A.P. are 19 and 41 respectively, find the 40^{th} term.
7. If 9^{th} term of an A.P. is zero, prove that its 29^{th} term is double the 19^{th} term.
8. If 10 times the 10^{th} term of an A.P. is equal to 15 times the 15^{th} term, show that 25^{th} term of the A.P. is zero.
9. The 10^{th} and 18^{th} terms of an A.P. are 41 and 73 respectively. Find 26^{th} term.
10. In a certain A.P. the 24^{th} term is twice the 10^{th} term. Prove that the 72nd term is twice the 34^{th} term.

11. The 26^{th}, 11^{th} and last term of an A.P. are 0, 3 and $-\frac{1}{5}$, respectively. Find the common difference and the number of terms. [NCERT EXEMPLAR]
12. If the n^{th} term of the A.P. 9, 7, 5, ... is same as the n^{th} term of the A.P. 15, 12, 9, ... find n.
13. Find the 12^{th} term from the end of the following arithmetic progressions:
    (i) 3, 5, 7, 9, ... 201 (ii) 3, 8, 13, ... , 253 [NCERT] (iii) 1, 4, 7, 10, ..., 88
14. The 4^{th} term of an A.P. is three times the first and the 7^{th} term exceeds twice the third term by 1. Find the first term and the common difference.
15. Find the second term and n^{th} term of an A.P. whose 6^{th} term is 12 and the 8^{th} term is 22.
16. How many numbers of two digit are divisible by 3 ?
17. An A.P. consists of 60 terms. If the first and the last terms be 7 and 125 respectively, find 32nd term.
18. The sum of 4^{th} and 8^{th} terms of an A.P. is 24 and the sum of the 6^{th} and 10^{th} terms is 34. Find the first term and the common difference of the A.P. [NCERT]
19. The first term of an A.P. is 5 and its 100^{th} term is -292. Find the 50^{th} term of this A.P.
20. Find $a_{30} - a_{20}$ for the A.P.
    (i) $-9, -14, -19, -24, \ldots$ (ii) $a, a+d, a+2d, a+3d, \ldots$
21. Write the expression $a_n - a_k$ for the A.P. $a, a+d, a+2d, \ldots$
    Hence, find the common difference of the A.P. for which
    (i) 11^{th} term is 5 and 13^{th} term is 79.
    (ii) $a_{10} - a_5 = 200$
    (iii) 20^{th} term is 10 more than the 18^{th} term.
22. Find n if the given value of x is the n^{th} term of the given A.P.
    (i) $25, 50, 75, 100, \ldots; x = 1000$ (ii) $-1, -3, -5, -7, \ldots; x = -151$
    (iii) $5\frac{1}{2}, 11, 16\frac{1}{2}, 22, \ldots; x = 550$ (iv) $1, \frac{21}{11}, \frac{31}{11}, \frac{41}{11}, \ldots, x = \frac{171}{11}$
23. The eighth term of an A.P. is half of its second term and the eleventh term exceeds one third of its fourth term by 1. Find the 15^{th} term.
24. Find the arithmetic progression whose third term is 16 and seventh term exceeds its fifth term by 12. [NCERT]
25. The 7^{th} term of an A.P. is 32 and its 13^{th} term is 62. Find the A.P. [CBSE 2004]
26. Which term of the A.P. 3, 10, 17, ... will be 84 more than its 13^{th} term? [CBSE 2004]
27. Two arithmetic progressions have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms? [NCERT]
28. For what value of n, the nth terms of the arithmetic progressions 63, 65, 67, ... and 3, 10, 17, ... are equal? [CBSE 2008]
29. How many multiples of 4 lie between 10 and 250? [NCERT]
30. How many three digit numbers are divisible by 7? [CBSE 2013, NCERT]
31. Which term of the arithmetic progression 8, 14, 20, 26, ... will be 72 more than its 41^{st} term? [CBSE 2006C]

32. Find the term of the arithmetic progression 9, 12, 15, 18, ... which is 39 more than its 36^{th} term. [CBSE 2006C]
33. Find the 8^{th} term from the end of the A.P. 7, 10, 13, ..., 184 [CBSE 2005]
34. Find the 10^{th} term from the end of the A.P. 8, 10, 12,..., 126. [CBSE 2006]
35. The sum of 4^{th} and 8^{th} terms of an A.P. is 24 and the sum of 6^{th} and 10^{th} terms is 44. Find the A.P. [CBSE 2009]
36. Which term of the A.P. 3, 15, 27, 39, ... will be 120 more than its 21^{st} term? [CBSE 2009]
37. The 17^{th} term of an A.P. is 5 more than twice its 8^{th} term. If the 11^{th} term of the A.P. is 43, find the n^{th} term. [CBSE 2012]
38. Find the number of all three digit natural numbers which are divisible by 9. [CBSE 2013]
39. The 19^{th} term of an A.P. is equal to three times its sixth term. If its 9th term is 19, find the A.P. [CBSE 2013]
40. The 9^{th} term of an A.P. is equal to 6 times its second term. If its 5^{th} term is 22, find the A.P. [CBSE 2013]
41. The 24^{th} term of an A.P. is twice its 10^{th} term. Show that its 72^{nd} term is 4 times its 15^{th} term. [CBSE 2013]
42. Find the number of natural numbers between 101 and 999 which are divisible by both 2 and 5. [CBSE 2014]
43. If the seventh term of an A.P. is 1/9 and its ninth term is 1/7, find its $(63)^{rd}$ term. [CBSE 2014]
44. The sum of 5^{th} and 9^{th} terms of an A.P. is 30. If its 25^{th} term is three times its 8^{th} term, find the AP. [CBSE 2014]
45. Find whether 0 (zero) is a term of the A.P. 40, 37, 34, 31,.... [CBSE 2014]
46. Find the middle term of the A.P. 213, 205, 197, ..., 37. [CBSE 2015]
47. If the 5^{th} term of an A.P. is 31 and 25^{th} term is 140 more than the 5^{th} term, find the A.P. [CBSE 2015]

LEVEL-2

48. Find the sum of two middle terms of the A.P.: $-\frac{4}{3}, -1, \frac{-2}{3}, -\frac{1}{3}, ..., 4\frac{1}{3}$. [NCERT EXEMPLAR]
49. If $(m+1)^{th}$ term of an A.P. is twice the $(n+1)^{th}$ term, prove that $(3m+1)^{th}$ term is twice the $(m+n+1)^{th}$ term.
50. If an A.P. consists of n terms with first term a and n^{th} term l show that the sum of the m^{th} term from the beginning and the m^{th} term from the end is $(a+l)$.
51. How many numbers lie between 10 and 300, which when divided by 4 leave a remainder 3? [NCERT EXEMPLAR]
52. Find the 12^{th} term from the end of the A.P. $-2, -4, -6, ..., -100$. [NCERT EXEMPLAR]
53. For the A.P.: $-3, -7, -11, ...$, can we find $a_{30} - a_{20}$ without actually finding a_{30} and a_{20}? Give reasons for your answer. [NCERT EXEMPLAR]
54. Two A.P.s have the same common difference. The first term of one A.P. is 2 and that of the other is 7. The difference between their 10^{th} terms is the same as the difference between their 21^{st} terms, which is the same as the difference between any two corresponding terms. Why? [NCERT EXEMPLAR]

ANSWERS

1. (i) 28	(ii) $35\sqrt{2}$	(iii) $-5n+18$	(iv) 185
(v) 26	(vi) 15	(vii) $\frac{19}{4}$	
2. (i) 50	(ii) 22	(iii) 51	(iv) 20^{th} term
(v) 32^{nd}	3. (i) No	(ii) No	(iii) No
4. (i) 13	(ii) 27	(iii) 34	(iv) 27
5. (i) 26	6. 87	9. 105	11. $d=-\frac{1}{5}, n=27$
12. 7	13. (i) 179	(ii) 198	(iii) 55

14. First term = 3, Common difference = 2

15. $a_2=-8, a_n=5n-18$		16. 30	17. 69	
18. $-\frac{1}{2}, \frac{5}{2}$	19. -142	20. (i) -50	(ii) $10d$	
21. (i) $(n-k)d, 37$		(ii) 40	(iii) 5	
22. (i) 40	(ii) 76	(iii) 100	(iv) 17	23. 3
24. 4, 10, 16, 22, ...	25. 2, 7, 12, 17,....	26. 25^{th}	27. 100	
28. 13	29. 60	30. 128	31. 53^{rd}	
32. 49^{th}	33. 163	34. 108	35. $-13, -8, -3, 2, 7, \ldots$	
36. 31^{st}	37. $4n-1$	38. 100	39. 3, 5, 7, 9, ...	
40. 2, 7, 12, 17, ...	42. 89	43. 1	44. 3, 5, 7, 9, 11,...	
45. No	46. 125	47. 3, 10, 17, 24, ...		48. 3
51. 73	52. -78	53. Yes, $a_{30}-a_{20}=(30-20)d=-40$		

54. The difference between any two corresponding terms of such A.P's is the same as the difference between their first term.

5.5 SELECTION OF TERMS IN AN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

Number of terms	*Terms*	*Common difference*
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

It should be noted that in case of an odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are $a-d, a+d$ and the common differences is $2d$.

REMARK 1 *If three numbers a, b, c in order are in A.P. Then,*

$$b-a = \text{Common difference} = c-b$$

$$\Rightarrow \quad b-a=c-b$$

$$\Rightarrow \quad 2b=a+c$$

Thus, a, b, c are in A.P. if and only if $2b=a+c$.

REMARK 2 *If a, b, c are in A.P., then b is known as the arithmetic mean (AM) between a and c.*

REMARK 3 *If a, c, b are in A.P. Then,*

$$2c=a+b \Rightarrow c=\frac{a+b}{2}$$

Thus, A.M. *between a and b is* $\frac{a+b}{2}$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If $2x, x + 10, 3x + 2$ are in A.P., find the value of x.*

SOLUTION Since, $2x, x + 10, 3x + 2$ are in A.P.

$\therefore \quad 2(x + 10) = 2x + (3x + 2)$

$\Rightarrow \quad 2x + 20 = 5x + 2$

$\Rightarrow \quad 3x = 18 \Rightarrow x = 6$

EXAMPLE 2 *The sum of three numbers in A.P. is -3, and their product is 8. Find the numbers.*

SOLUTION Let the numbers be $(a - d), a, (a + d)$. It is given that the sum of the numbers is -3.

$\therefore \quad (a - d) + a + (a + d) = -3 \Rightarrow 3a = -3 \Rightarrow a = -1$

It is also given that the product of the product of the numbers is 8.

$\therefore \quad (a - d)(a)(a + d) = 8$

$\Rightarrow \quad a(a^2 - d^2) = 8$

$\Rightarrow \quad (-1)(1 - d^2) = 8 \qquad [\because a = -1]$

$\Rightarrow \quad d^2 = 9 \Rightarrow d = \pm 3$

If $d = 3$, the numbers are $-4, -1, 2$. If $d = -3$, the numbers are $2, -1, -4$.
Thus, the numbers are $-4, -1, 2$, or $2, -1, -4$.

EXAMPLE 3 *Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.*

SOLUTION Let the numbers be $(a - 3d), (a - d), (a + d), (a + 3d)$. Then,

Sum of numbers = 20

$\Rightarrow \quad (a - 3d) + (a - d) + (a + d) + (a + 3d) = 20 \Rightarrow 4a = 20 \Rightarrow a = 5$

It is given that, sum of the squares = 120

$\Rightarrow \quad (a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$

$\Rightarrow \quad 4a^2 + 20d^2 = 120$

$\Rightarrow \quad a^2 + 5d^2 = 30$

$\Rightarrow \quad 25 + 5d^2 = 30 \qquad [\because a = 5]$

$\Rightarrow \quad 5d^2 = 5 \Rightarrow d = \pm 1$

If $d = 1$, then the numbers are 2, 4, 6, 8. If $d = -1$, then the numbers are 8, 6, 4, 2.
Thus, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

EXAMPLE 4 *Divide 32 into four parts which are in A.P. such that the product of extremes is to the product of means is 7:15.*

SOLUTION Let the four parts be $(a - 3d), (a - d), (a + d)$ and $(a + 3d)$. Then,

Sum of the numbers = 32

$\Rightarrow \quad (a - 3d) + (a - d) + (a + d) + (a + 3d) = 32 \Rightarrow 4a = 32 \Rightarrow a = 8$

It is given that

$$\frac{(a - 3d)(a + 3d)}{(a - d)(a + d)} = \frac{7}{15}$$

$$\Rightarrow \quad \frac{a^2 - 9d^2}{a^2 - d^2} = \frac{7}{15}$$

$$\Rightarrow \quad \frac{64-9d^2}{64-d^2}=\frac{7}{15} \Rightarrow 128d^2 = 512 \Rightarrow d^2 = 4 \Rightarrow d = \pm 2$$

Thus, the four parts are $a-3d, a-d, a+d$ and $a+3d$ i.e., 2, 6, 10, 14.

LEVEL-2

EXAMPLE 5 *If the numbers a, b, c, d, e form an A.P., then find the value of* $a-4b+6c-4d+e$.

SOLUTION Let D be the common difference of the given A.P. Then,

$$b = a+D, c = a+2D, d = a+3D \text{ and } e = a+4D$$

$$\therefore \quad a-4b+6c-4d+e = a-4(a+D)+6(a+2D)-4(a+3D)+(a+4D)$$

$$\Rightarrow \quad a-4b+6c-4d+e = a-4a-4D+6a+12D-4a-12D+a+4D$$

$$\Rightarrow \quad a-4b+6c-4d+e = a-4a+6a-4a+a-4D+12D-12D+4D$$

$$\Rightarrow \quad a-4b+6c-4d+e = 0$$

EXAMPLE 6 *The sum of the first three terms of an A.P. is 33. If the product of first and the third term exceeds the second term by 29, find the AP.* [NCERT EXEMPLAR]

SOLUTION Let the first three terms in A.P. be $a-d, a, a+d$. It is given that the sum of these terms is 33.

$$\therefore \quad a-d+a+a+d = 33 \Rightarrow 3a = 33 \Rightarrow a = 11$$

It is given that

$$(a-d)(a+d) = a+29$$

$$\Rightarrow \quad a^2-d^2 = a+29$$

$$\Rightarrow \quad 121-d^2 = 11+29$$

$$\Rightarrow \quad d^2 = 81$$

$$\Rightarrow \quad d = \pm 9$$

Thus, we have $a = 11, d = 9$ or $a = 11$ and $d = -9$.
Hence, the two AP's are 2, 11, 20, 29, ... and 20, 11, 2, ...

EXAMPLE 7 *Determine k so that* $k^2+4k+8, 2k^2+3k+6, 3k^2+4k+4$ *are three conse- cutive terms of an A.P.* [NCERT EXEMPLAR]

SOLUTION We know that if a, b, c are three consecutive terms of an A.P., then

$$b-a = c-b \text{ i.e. } 2b = a+c$$

Thus, if $k^2+4k+8, 2k^2+3k+6$ and $3k^2+4k+4$ are three consecutive terms of an A.P., then

$$2(2k^2+3k+6) = (k^2+4k+8)+(3k^2+4k+4)$$

$$\Rightarrow \quad 4k^2+6k+12 = 4k^2+8k+12$$

$$\Rightarrow \quad 2k = 0 \Rightarrow k = 0.$$

EXAMPLE 8 *If* $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ *is the A.M. between a and b. Then, find the value of n.*

SOLUTION We know that the A.M. between a and b is $\dfrac{a+b}{2}$.

It is given that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the A.M. between a and b.

$$\therefore \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{a+b}{2}$$

$$\Rightarrow \quad 2(a^{n+1}+b^{n+1})=(a^n+b^n)(a+b)$$

$$\Rightarrow \quad 2a^{n+1}+2b^{n+1}=a^{n+1}+ab^n+a^nb+b^{n+1}$$

$$\Rightarrow \quad a^{n+1}+b^{n+1}=ab^n+a^nb$$

$$\Rightarrow \quad a^{n+1}-a^nb=ab^n-b^{n+1}$$

$$\Rightarrow \quad a^n(a-b)=b^n(a-b)$$

$$\Rightarrow \quad a^n=b^n \Rightarrow \frac{a^n}{b^n}=1 \Rightarrow \left(\frac{a}{b}\right)^n=\left(\frac{a}{b}\right)^0 \Rightarrow n=0$$

EXERCISE 5.5

LEVEL-1

1. Find the value of x for which $(8x+4)$, $(6x-2)$ and $(2x+7)$ are in A.P.
2. If $x+1$, $3x$ and $4x+2$ are in A.P., find the value of x.
3. Show that $(a-b)^2, (a^2+b^2)$ and $(a+b)^2$ are in A.P.
4. The sum of three terms of an A.P. is 21 and the product of the first and the third terms exceeds the second term by 6, find three terms.
5. Three numbers are in A.P. If the sum of these numbers be 27 and the product 648, find the numbers.
6. Find the four numbers in A.P., whose sum is 50 and in which the greatest number is 4 times the least.
7. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers. [CBSE 2016]
8. Divide 56 in four parts in A.P. such that the ratio of the product of their extremes to the product of their means is $5:6$. [CBSE 2016]

LEVEL-2

9. The angles of a quadrilateral are in A.P. whose common difference is $10°$. Find the angles.
10. Split 207 into three parts such that these are in A.P. and the product of the two smaller parts is 4623. [NCERT EXEMPLAR]
11. The angles of a triangle are in A.P. The greatest angle is twice the least. Find all the angles. [NCERT EXEMPLAR]
12. The sum of four consecutive numbers in A.P. is 32 and the ratio of the product of the first and last terms to the product of two middle terms is $7:15$. Find the number. [NCERT EXEMPLAR, CBSE 2018]

ANSWERS

1. $15/2$ 2. 3 4. 1, 7, 13 5. 6, 9, 12
6. 5, 10, 15, 20 7. 2, 4, 6 or 6, 4, 2 8. 8, 12, 16, 20
9. $75°, 85°, 95°, 105°$ 10. 67, 69, 71 11. $40°, 60°, 80°$ 12. 2, 6, 10, 14

5.6 SUM TO n TERMS OF AN A.P.

THEOREM *The sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is*

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

or, $$S_n = \frac{n}{2}\{a + l\}, \text{ where } l = \text{last term} = a + (n-1)d$$

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with first term a and common difference d. Then,

$$a_1 = a, a_2 = a + d, a_3 = a + 2d, a_4 = a + 3d, \ldots, a_n = a + (n-1)d$$

Now, $S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$

$\Rightarrow$ $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-2)d) + \{a + (n-1)d\}$ ...(i)

Writing the above series in a reverse order, we get

$S_n = \{a + (n-1)d\} + \{a + (n-2)d\} + \ldots + (a + d) + a$ ...(ii)

Adding the corresponding terms of equations (i) and (ii), we get

$$2S_n = \{2a + (n-1)d\} + \{2a + (n-1)d\} + \ldots + \{2a + (n-1)d\}$$

$\Rightarrow$ $2S_n = n\{2a + (n-1)d\}$ [$\because 2a + (n-1)d$ repeats n times]

$\Rightarrow$ $$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

Now, l = last term = n^{th} term = $a + (n-1)d$

$\therefore$ $$S_n = \frac{n}{2}\{2a + (n-1)d\} = \frac{n}{2}\left[a + \{a + (n-1)d\}\right] = \frac{n}{2}\{a + l\}$$

Q.E.D.

NOTE 1 *In the formula $S_n = \frac{n}{2}[2a + (n-1)d]$, there are four quantities viz. S_n, a, n and d. If any three of these are known, the fourth can be determined. Sometimes two of these quantities are given, in such cases remaining two quantities are provided by some other relation.*

NOTE 2 *In the sum S_n of n terms of a sequence is given, then n^{th} term a_n of the sequence can be determined by using the following formula*

$$a_n = S_n - S_{n-1}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SUM OF AN A.P.**

EXAMPLE 1 *Find the sum of 20 terms of the A.P. 1, 4, 7, 10 ...*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then, we have

$$a = 1 \text{ and } d = 3.$$

We have to find the sum of 20 terms of the given A.P.

Putting $a = 1, d = 3, n = 20$ in $S_n = \frac{n}{2}\{2a + (n-1)d\}$, we get

$$S_{20} = \frac{20}{2}\{2 \times 1 + (20-1) \times 3\} = 10 \times 59 = 590$$

EXAMPLE 2 *If the n^{th} term of an A.P. is $(2n + 1)$, find the sum of first n terms of the A.P.* [CBSE 2005]

SOLUTION We have,

$$a_n = (2n+1) \Rightarrow a_1 = 2 \times 1 + 1 = 3$$

So, the given sequence is an A.P. with first term $a = a_1 = 3$ and last term $l = a_n = 2n+1$. Therefore, the sum of n terms is given by

$$S_n = \frac{n}{2}(a+l) = \frac{n}{2}\{3 + (2n+1)\} = \frac{n}{2}(2n+4) = n(n+2)$$

EXAMPLE 3 *Find the sum of first 30 terms of an A.P. whose second term is 2 and seventh term is 22.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$a_2 = 2 \text{ and } a_7 = 22$$

$$\Rightarrow \quad a + d = 2 \text{ and } a + 6d = 22$$

Solving these two equations, we get $a = -2$ and $d = 4$. Putting $n = 30, a = -2$ and $d = 4$ in $S_n = \frac{n}{2}\{2a + (n-1)d\}$, we obtain

$$\therefore \quad S_{30} = \frac{30}{2}\{2 \times (-2) + (30-1) \times 4\}$$

$$\Rightarrow \quad S_{30} = 15(-4 + 116) = 15 \times 112 = 1680$$

Hence, the sum of first 30 terms is 1680.

EXAMPLE 4 *Find the sum of first 20 terms of an A.P., in which 3rd term is 7 and 7^{th} term is two more than thrice of its 3rd term.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$a_3 = 7 \text{ and } a_7 = 3a_3 + 2 \qquad \text{[Given]}$$

$$\Rightarrow \quad a + 2d = 7 \text{ and } a + 6d = 3(a + 2d) + 2$$

$$\Rightarrow \quad a + 2d = 7 \text{ and } a + 6d = 3a + 6d + 2$$

$$\Rightarrow \quad a + 2d = 7 \text{ and } a = -1$$

$$\Rightarrow \quad a = -1, d = 4$$

Putting $n = 20, a = -1$ and $d = 4$ in $S_n = \frac{n}{2}\{2a + (n-1)d\}$, we get

$$S_{20} = \frac{20}{2}\{2 \times -1 + (20-1) \times 4\} = \frac{20}{2}(-2 + 76) = 740$$

EXAMPLE 5 *Find the sum of first n natural numbers.*

SOLUTION First n natural numbers are $1, 2, 3, 4, \ldots, (n-1), n$.

Clearly, it is an A.P. with first term 1 and common difference 1.

Let S_n denote the required. Then,

$$S_n = \frac{n}{2}(1+n) \qquad \left[\text{Using} : S_n = \frac{n}{2}(a+l)\right]$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}$$

EXAMPLE 6 *The sum of first six terms of an arithmetic progression is 42. The ratio of its 10th term to its 30th term is 1 : 3. Calculate the first and the thirteenth term of the A.P.* **[CBSE 2009]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_6 = 42 \Rightarrow \frac{6}{2}\{2a + (6-1)d\} = 42 \Rightarrow 2a + 5d = 14 \quad \ldots(i)$$

It is given that

$$a_{10} : a_{30} = 1 : 3$$

$$\Rightarrow \quad \frac{a + 9d}{a + 29d} = \frac{1}{3}$$

$$\Rightarrow \quad 3a + 27d = a + 29d$$

$$\Rightarrow \quad 2a - 2d = 0$$

$$\Rightarrow \quad a = d \quad \ldots(ii)$$

Solving (i) and (ii), we get $a = d = 2$

$\therefore \quad a_{13} = a + 12d = 2 + 2 \times 12 = 26$

Hence, first term = 2 and thirteenth term = 26.

EXAMPLE 7 *Find the sum of all three digit natural numbers, which are divisible by 7.* [CBSE 2006C]

SOLUTION The smallest and the largest numbers of three digits, which are divisible by 7 are 105 and 994 respectively. So, the sequence of three digit numbers which are divisible by 7 are 105, 112, 119, ..., 994. Clearly, it is an A.P. with first term $a = 105$ and common difference $d = 7$. Let there be n terms in this sequence. Then,

$$a_n = 994 \Rightarrow a + (n-1)d = 994 \Rightarrow 105 + (n-1) \times 7 = 994 \Rightarrow n = 128$$

Now,

$$\text{Required sum} = \frac{n}{2}\{2a + (n-1)d\}$$

$$\Rightarrow \quad \text{Required sum} = \frac{128}{2}\{2 \times 105 + (128-1) \times 7\} = 70336$$

EXAMPLE 8 *Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.*

SOLUTION Clearly, the numbers between 250 and 1000 which are divisible by 3 are 252, 255, 258, ..., 999.

This is an A.P. with first term $a = 252$, Common difference = 3 and last term = 999.

Let there be n terms in this A.P. Then,

$$a_n = 999$$

$$\Rightarrow \quad a + (n-1)d = 999$$

$$\Rightarrow \quad 252 + (n-1) \times 3 = 999$$

$$\Rightarrow \quad n = 250$$

$$\therefore \quad \text{Required sum} = S_n = \frac{n}{2}(a + l) = \frac{250}{2}(252 + 999) = 156375$$

EXAMPLE 9 *Find the sum of all odd integers between 2 and 100 divisible by 3.*

SOLUTION The odd integers between 2 and 100 which are divisible by 3 are 3, 9, 15, 21, ..., 99.

Clearly, it is an A.P. with first term $a = 3$ and, common difference $d = 6$.

Let there be n terms in this sequence. Then,

$$a_n = 99$$

$$\Rightarrow \quad a + (n-1)d = 99$$

$$\Rightarrow \quad 3 + (n-1) \times 6 = 99 \Rightarrow n = 17$$

$$\therefore \quad \text{Required sum} = S_n = \frac{n}{2}(a + l) = \frac{17}{2}(3 + 99) = 867$$

EXAMPLE 10 *If the ratio of the* 11^{th} *term of an A.P. to its* 18^{th} *term is 2 : 3, find the ratio of the sum of first five terms to the the sum of its first 10 terms.* [CBSE 2017]

SOLUTION Let 'a' be the first term and 'd' be the common difference of the given A.P. It is given that

$$\frac{a_{11}}{a_{18}} = \frac{2}{3} \Rightarrow \frac{a+10}{a+17d} = \frac{2}{3} \Rightarrow 3a + 30d = 2a + 34d \Rightarrow a = 4d$$

$$\therefore \quad \frac{S_5}{S_{10}} = \frac{\frac{5}{2}\{2a+(5-1)d\}}{\frac{10}{2}\{2a+(10-1)d\}} = \frac{1}{2}\left\{\frac{2a+4d}{2a+9d}\right\} = \frac{1}{2}\left\{\frac{2\times 4d+4d}{2\times 4d+9d}\right\} = \frac{1}{2}\left(\frac{12d}{17d}\right) = \frac{6}{17}$$

Hence, $S_5 : S_{10} = 6 : 17$

Type II ON FIDING THE NUMBER OF TERMS IN AN A.P. WHEN THEIR SUM IS GIVEN

EXAMPLE 11 *How many terms of the series 54, 51, 48,... be taken so that their sum is 513? Explain the double answer.* [CBSE 2005]

SOLUTION Clearly, the given sequence is an A.P. with first term a (= 54) and common differenced d (= – 3). Let the sum of n terms be 513. Then,

$$S_n = 513$$

$$\Rightarrow \quad \frac{n}{2}\{2a + (n-1)d\} = 513$$

$$\Rightarrow \quad \frac{n}{2}[108 + (n-1) \times -3] = 513$$

$$\Rightarrow \quad n^2 - 37n + 342 = 0$$

$$\Rightarrow \quad (n-18)(n-19) = 0 \Rightarrow n = 18 \text{ or, } 19$$

Here, the common difference is negative. So, 19th term is given by

$$a_{19} = 54 + (19-1) \times -3 = 0$$

Thus, the sum of 18 terms as well as that of 19 terms is 513.

EXAMPLE 12 *Find the number of terms in the series* $20 + 19\frac{1}{3} + 18\frac{2}{3} + \ldots$ *of which the sum is 300, explain the double answer.*

SOLUTION The given series is an arithmetic series with first term a (= 20) and the common difference $d\left(= -\frac{2}{3}\right)$. Let the sum of n terms be 300. Then,

$$S_n = 300$$

$$\Rightarrow \quad \frac{n}{2}\{2a + (n-1)d\} = 300$$

$$\Rightarrow \quad \frac{n}{2}\{2 \times 20 + (n-1)(-2/3)\} = 300$$

$$\Rightarrow \quad n^2 - 61n + 900 = 0 \Rightarrow (n-25)(n-36) = 0 \Rightarrow n = 25 \text{ or}, 36$$

So, sum of 25 terms = sum of 36 terms = 300.

Here, the common difference is negative therefore terms go on diminishing and 31st term becomes zero. All terms after 31st term are negative. These negative terms when added to positive terms from 26th term to 30th term, they cancel out each other and the sum remains same.

Hence, the sum of 25 terms as well as that of 36 terms is 300.

Type III **ON FIDING THE DESIRED TERM WHEN THE SUM OF n TERMS OF AN A.P. IS GIVEN**

EXAMPLE 13 *If S_n, the sum of first n terms of an A.P., is given by $S_n = 5n^2 + 3n$, then find its n^{th} term.* **[CBSE 2009]**

SOLUTION Let a_n be the n^{th} term of the A.P. Then,

$$a_n = S_n - S_{n-1}$$

$$\Rightarrow \quad a_n = (5n^2 + 3n) - \{5(n-1)^2 + 3(n-1)\} \quad \left[\begin{array}{l}\text{Replacing } n \text{ by } (n-1) \text{ in } S_n \\ \text{to get } S_{n-1} = 5(n-1)^2 + 3(n-1)\end{array}\right]$$

$$\Rightarrow \quad a_n = (5n^2 + 3n) - (5n^2 - 7n + 2)$$

$$\Rightarrow \quad a_n = 10n - 2$$

EXAMPLE 14 *In an A.P., the sum of first n terms is $\frac{3n^2}{2} + \frac{5n}{2}$. Find its 25th term.* **[CBSE 2006C]**

SOLUTION Let S_n denote the sum of n terms of an A.P. whose nth term is a_n.
We have,

$$S_n = \frac{3n^2}{2} + \frac{5n}{2}$$

$$\Rightarrow \quad S_{n-1} = \frac{3}{2}(n-1)^2 + \frac{5}{2}(n-1) \quad [\text{Replacing } n \text{ by } (n-1)]$$

$$\therefore \quad S_n - S_{n-1} = \left\{\frac{3n^2}{2} + \frac{5n}{2}\right\} - \left\{\frac{3}{2}(n-1)^2 + \frac{5}{2}(n-1)\right\}$$

$$\Rightarrow \quad a_n = \frac{3}{2}\{n^2 - (n-1)^2\} + \frac{5}{2}\{n - (n-1)\} \quad [\because a_n = S_n - S_{n-1}]$$

$$\Rightarrow \quad a_n = \frac{3}{2}(2n-1) + \frac{5}{2}$$

$$\Rightarrow \quad a_{25} = \frac{3}{2}(2 \times 25 - 1) + \frac{5}{2} = \frac{3}{2} \times 49 + \frac{5}{2} = 76 \quad [\text{Replacing } n \text{ by } 25]$$

EXAMPLE 15 *A manufacturer of TV sets produced 600 units in the third year and 700 units in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find the production in (i) the first year (ii) the 10th year (iii) 7 years.* **[CBSE 2015, NCERT]**

SOLUTION Since the production increases uniformly by a fixed number every year. Therefore, the sequence formed by the production in different years is an A.P.

Let a be the first term and d be the common difference of the A.P. formed i.e., 'a' denotes the production in the first year and d denotes the number of units by which the production increases every year.

We have,

$$a_3 = 600 \text{ and } a_7 = 700 \Rightarrow a + 2d = 600 \text{ and } a + 6d = 700$$

Solving these equations, we get: $a = 550$ and $d = 25$

(i) We have, $a = 550$

$\therefore$ Production in the first year is of 550 TV sets.

(ii) The production in the 10th term is given by a_{10}.

$\therefore$ Production in the 10th year $= a_{10} = a + 9d = 550 + 9 \times 25 = 775$

So, production in 10th year is of 775 TV sets.

(iii) Total production in 7 years

= Sum of 7 terms of the A.P. with first term a (= 550) and common difference d (= 25).

$$= \frac{7}{2}\{2 \times 550 + (7-1) \times 25\} = \frac{7}{2}(1100 + 150) = 4375$$

Thus, the total production in 7 years is of 4375 TV sets.

EXAMPLE 16 *A sum of ₹280 is to be used to award four prizes. If each prize after the first is ₹20 less than its preceding prize, find the value of each of the prizes.* [CBSE 2014, NCERT]

SOLUTION The values of four prizes form an A.P. with common difference $d = -20$ the sum of whose terms is 280. Let the value of first prize be ₹ a. Then,

$$\text{Sum} = 280$$

$$\Rightarrow \quad \frac{4}{2}\{2a + (4-1) \times -20\} = 280$$

$$\Rightarrow \quad 2(2a - 60) = 280 \Rightarrow a - 30 = 70 \Rightarrow a = 100$$

Hence, the values of 4 prizes are ₹ 100, ₹ 80, ₹ 60 and ₹ 40.

EXAMPLE 17 *In a school, students thought of planting trees in and around the school to reduce noise polution and air polution. It was decided that the number of trees that each section of each class will plant be the same as the class in which they are studying e.g. – a section of I class will plant 1 tree, a section of II class will plant 2 trees and so on a section of class XII will plant 12 trees. There are three sections of each class. How many trees will be planted by the students?*

[CBSE 2014, NCERT]

SOLUTION Since each section of each class plants the same number of trees as the class number and there are three sections of each class.

$\therefore$ Total number of trees planted by the students

$$= 3[1 + 2 + 3 + \dots + 12]$$

$$= 3\left[\frac{12}{2}\{2 \times 1 + (12-1) \times 1\}\right] = 3\{6(2+11)\} = 18 \times 13 = 234$$

LEVEL-2

EXAMPLE 18 *An A.P. consists of 37 terms. The sum of the three middle most terms is 225 and the sum of the last three terms is 429. Find the A.P.* [NCERT EXEMPLAR]

SOLUTION In 38 terms of the A.P., we find that 19th term is the middle term and 18th 19th and 20th terms are three middle most terms.

Let a be the first term and d be the common difference of the A.P. Then,

$$a_{18} + a_{19} + a_{20} = 225 \text{ and, } a_{35} + a_{36} + a_{37} = 429 \quad \text{[Given]}$$

$$\Rightarrow \quad (a + 17d) + (a + 18d) + (a + 19d) = 225 \text{ and, } (a + 34d) + (a + 35d) + (a + 36d) = 429$$

$$\Rightarrow \quad 3a + 54d = 225 \text{ and } 3a + 105d = 429$$

$\Rightarrow \quad a + 18d = 75$ and $a + 35d = 143$

$\Rightarrow \quad a = 3$ and $d = 4$

Hence, the A.P. is 3, 7, 11, 15, ...

EXAMPLE 19 *Find the sum of all 11 terms of an A.P. whose middle most term is 30.* [NCERT EXEMPLAR]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Clearly, in an A.P. consisting of 11 terms, $\left(\frac{11+1}{2}\right)^{th}$ *i.e.* 6th term is the middle most term. It is given that the middle most term is 30.

$$\therefore \quad a + 5d = 30 \qquad \text{...(i)}$$

$$\therefore \quad S_{11} = \frac{11}{2}\{2a + (11-1)d\} = 11(a+5d) = 11 \times 30 = 330 \qquad \text{[Using (i)]}$$

EXAMPLE 20 *If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}(mn+1)$.* [CBSE 2015, 2017]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$a_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n} \qquad \text{...(i)}$$

and, $$a_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m} \qquad \text{...(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$(m-n)d = \frac{1}{n} - \frac{1}{m} \Rightarrow (m-n)d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in equation (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n} \Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$$

$$\therefore \quad S_{mn} = \frac{mn}{2}\{2a + (mn-1)d\}$$

$$\Rightarrow \quad S_{mn} = \frac{mn}{2}\left\{\frac{2}{mn} + (mn-1) \times \frac{1}{mn}\right\} = \frac{1}{2}(mn+1)$$

EXAMPLE 21 *The sum of n, $2n$, $3n$ terms of an A.P. are S_1, S_2, S_3 respectively. Prove that*

$$S_3 = 3(S_2 - S_1)$$

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_1 = \text{Sum of } n \text{ terms} \quad \Rightarrow S_1 = \frac{n}{2}\{2a + (n-1)d\} \qquad \text{...(i)}$$

$$S_2 = \text{Sum of } 2n \text{ terms} \quad \Rightarrow S_2 = \frac{2n}{2}\{2a + (2n-1)d\} \qquad \text{...(ii)}$$

and, $$S_3 = \text{Sum of } 3n \text{ terms} \quad \Rightarrow S_3 = \frac{3n}{2}\{2a + (3n-1)d\} \qquad \text{...(iii)}$$

Now,

$$S_2 - S_1 = \frac{2n}{2}\{2a + (2n-1)d\} - \frac{n}{2}\{2a + (n-1)d\}$$

$\Rightarrow$ $$S_2 - S_1 = \frac{n}{2}\left[2\{2a + (2n-1)d\} - \{2a + (n-1)d\}\right] = \frac{n}{2}\{2a + (3n-1)d\}$$

$\therefore$ $$3(S_2 - S_1) = \frac{3n}{2}\{2a + (3n-1)d\} = S_3 \qquad \text{[Using (iii)]}$$

Hence, $S_3 = 3(S_2 - S_1)$.

EXAMPLE 22 *The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that* $S_1 + S_3 = 2S_2$.

SOLUTION We have,

S_1 = Sum of n terms of an A.P. with first term 1 and common difference 1

$\Rightarrow$ $$S_1 = \frac{n}{2}\{2 \times 1 + (n-1) \times 1\} = \frac{n}{2}(n+1)$$

S_2 = Sum of n terms of an A.P. with first term 1 and common difference 2

$\Rightarrow$ $$S_2 = \frac{n}{2}\{2 \times 1 + (n-1) \times 2\} = n^2$$

S_3 = Sum of n terms of an A.P. with first term 1 and common difference 3

$\Rightarrow$ $$S_3 = \frac{n}{2}\{2 \times 1 + (n-1) \times 3\} = \frac{n}{2}(3n-1)$$

Now, $S_1 + S_3 = \frac{n}{2}(n+1) + \frac{n}{2}(3n-1) = 2n^2$ and $S_2 = n^2$

Hence, $S_1 + S_3 = 2S_2$

EXAMPLE 23 *The sum of the third and seventh terms of an A.P. is 6 and their product is 8. Find the sum of first sixteen terms of the A.P.*

SOLUTION Let a be the first term and d be the common difference of the A.P.

We have,

$a_3 + a_7 = 6$ and $a_3 a_7 = 8$

$\Rightarrow$ $(a + 2d) + (a + 6d) = 6$ and $(a + 2d)(a + 6d) = 8$

$\Rightarrow$ $2a + 8d = 6$ and $(a + 2d)(a + 6d) = 8$

$\Rightarrow$ $a + 4d = 3$ and $(a + 2d)(a + 6d) = 8$

$\Rightarrow$ $a = 3 - 4d$ and $(a + 2d)(a + 6d) = 8$

$\Rightarrow$ $(3 - 4d + 2d)(3 - 4d + 6d) = 8$ [Putting $a = 3 - 4d$ in the second equation]

$\Rightarrow$ $(3 - 2d)(3 + 2d) = 8$

$\Rightarrow$ $9 - 4d^2 = 8 \Rightarrow 4d^2 = 1 \Rightarrow d^2 = \frac{1}{4} \Rightarrow d = \pm\frac{1}{2}$

CASE I *When* $d = \frac{1}{2}$: Putting $d = \frac{1}{2}$ in $a = 3 - 4d$, we get

$$a = 3 - 4 \times \frac{1}{2} = 3 - 2 = 1$$

$\therefore$ $$S_{16} = \frac{16}{2}\{2a + (16-1)d\} = 8\left\{2 \times 1 + 15 \times \frac{1}{2}\right\} = 8 \times \frac{19}{2} = 76$$

CASE II When $d = -\frac{1}{2}$: Putting $d = -\frac{1}{2}$ in $a = 3 - 4d$, we get $a = 3 + 2 = 5$

$$\therefore \quad S_{16} = \frac{16}{2}\{2a + (16 - 1)d\} = 8\left\{10 + 15 \times -\frac{1}{2}\right\} = 8 \times \frac{5}{2} = 20$$

EXAMPLE 24 *If in an A.P. the sum of m terms is equal to n and the sum of n terms is equal to m, then prove that the sum of (m + n) terms is – (m + n).*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = n$$

$$\Rightarrow \quad \frac{m}{2}\{2a + (m - 1)d\} = n$$

$$\Rightarrow \quad 2am + m(m - 1)d = 2n \qquad \text{...(i)}$$

and, $$S_n = m$$

$$\Rightarrow \quad \frac{n}{2}\{2a + (n - 1)d\} = m$$

$$\Rightarrow \quad 2an + n(n - 1)d = 2m \qquad \text{...(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$2a(m - n) + \{m(m - 1) - n(n - 1)\}d = 2n - 2m$$

$$\Rightarrow \quad 2a(m - n) + \{(m^2 - n^2) - (m - n)\}d = -2(m - n)$$

$$\Rightarrow \quad 2a + (m + n - 1)d = -2 \qquad \text{[On dividing both sides by } (m - n)] \quad \text{...(iii)}$$

Now,

$$S_{m+n} = \frac{m + n}{2}\{2a + (m + n - 1)d\}$$

$$\Rightarrow \quad S_{m+n} = \frac{(m + n)}{2}(-2) \qquad \text{[Using (iii)]}$$

$$\Rightarrow \quad S_{m+n} = -(m + n)$$

EXAMPLE 25 *If the sum of m terms of an A.P. is the same as the sum of its n terms, show that the sum of its (m + n) terms is zero.* [CBSE 2017]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = S_n$$

$$\Rightarrow \quad \frac{m}{2}\{2a + (m - 1)d\} = \frac{n}{2}\{2a + (n - 1)d\}$$

$$\Rightarrow \quad 2a(m - n) + \{m(m - 1) - n(n - 1)\}d = 0$$

$$\Rightarrow \quad 2a(m - n) + \{(m^2 - n^2) - (m - n)\}d = 0$$

$$\Rightarrow \quad (m - n)\{2a + (m + n - 1)d\} = 0$$

$$\Rightarrow \quad 2a + (m + n - 1)d = 0 \qquad [\because m - n \neq 0] \quad \text{...(i)}$$

Now,

$$S_{m+n} = \frac{m + n}{2}\{2a + (m + n - 1)d\} = \frac{m + n}{2} \times 0 = 0 \qquad \text{[Using (i)]}$$

EXAMPLE 26 *The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that*

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

a = Sum of p terms

$$\Rightarrow \quad a=\frac{p}{2}\{2A+(p-1)D\}$$

$$\Rightarrow \quad \frac{2a}{p}=\{2A+(p-1)D\} \qquad ...(i)$$

b = Sum of q terms

$$\Rightarrow \quad b=\frac{q}{2}\{2A+(q-1)D\}$$

$$\Rightarrow \quad \frac{2b}{q}=\{2A+(q-1)D\} \qquad ...(ii)$$

and, c = Sum of r terms

$$\Rightarrow \quad c=\frac{r}{2}\{2A+(r-1)D\}$$

$$\Rightarrow \quad \frac{2c}{r}=\{2A+(r-1)D\} \qquad ...(iii)$$

Multiplying equation (i), (ii) and (iii) by $(q-r),(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{2a}{p}(q-r)+\frac{2b}{q}(r-p)+\frac{2c}{r}(p-q)$$

$$=\{2A+(p-1)D\}(q-r)+\{2A+(q-1)D\}(r-p)+\{2A+(r-1)D\}(p-q)$$

$$=2A(q-r+r-p+p-q)+D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$$

$$=2A\times 0+D\times 0=0$$

EXAMPLE 27 *The ratio of the sum of n terms of two A.P's is (7n + 1) : (4n + 27). Find the ratio of their m^{th} terms.*

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P's. Then, the sums of their n terms are given by

$$S_n=\frac{n}{2}\{2a_1+(n-1)d_1\} \text{ and, } S_n'=\frac{n}{2}\{2a_2+(n-1)d_2\}$$

$$\therefore \quad \frac{S_n}{S_n'}=\frac{\frac{n}{2}\{2a_1+(n-1)d_1\}}{\frac{n}{2}\{2a_2+(n-1)d_2\}}=\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}$$

It is given that

$$\frac{S_n}{S_n'}=\frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27} \qquad \text{...(i)}$$

To find the ratio of the m^{th} terms of the two given A.P's, we replace n by $(2m-1)$ in equation (i).

Replacing n by $(2m-1)$ in equation (i), we get

$$\therefore \quad \frac{2a_1 + (2m-2)d_1}{2a_2 + (2m-2)d_2} = \frac{7(2m-1)+1}{4(2m-1)+27}$$

$$\Rightarrow \quad \frac{a_1 + (m-1)d_1}{a_2 + (m-1)d_2} = \frac{14m-6}{8m+23}$$

Hence, the ratio of the m^{th} terms of the two A.P's is $(14m-6):(8m+23)$.

EXAMPLE 28 *The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of the m^{th} and n^{th} terms is $(2m-1):(2n-1)$.* [CBSE 2016, 2017]

SOLUTION Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}\{2a + (m-1)d\} \text{ and, } S_n = \frac{n}{2}\{2a + (n-1)d\} \text{ respectively.}$$

Then,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{\frac{m}{2}\{2a + (m-1)d\}}{\frac{n}{2}\{2a + (n-1)d\}} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}$$

$$\Rightarrow \quad \{2a + (m-1)d\}\, n = \{2a + (n-1)d\}\, m$$

$$\Rightarrow \quad 2a(n-m) = d\{(n-1)m - (m-1)n\}$$

$$\Rightarrow \quad 2a(n-m) = d(n-m)$$

$$\Rightarrow \quad d = 2a$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a + (m-1)d}{a + (n-1)d} = \frac{a + (m-1)2a}{a + (n-1)2a} = \frac{2m-1}{2n-1}$$

EXAMPLE 29 *If there are $(2n+1)$ terms in A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n+1):n$.*

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Let a_k denote the k^{th} terms of the given A.P. Then,

$$a_k = a + (k-1)d$$

Now, $\quad S_1 = \text{Sum of odd terms}$

$$\Rightarrow \quad S_1 = a_1 + a_3 + a_5 + \dots + a_{2n+1}$$

$$\Rightarrow \quad S_1 = \frac{n+1}{2}\{a_1 + a_{2n+1}\}$$

$$\Rightarrow \quad S_1 = \frac{n+1}{2}\{a + a + (2n+1-1)d\} \qquad [\because a_{2n+1} = a + (2n+1-1)d]$$

$$\Rightarrow \quad S_1 = (n+1)(a+nd)$$

and, $\quad S_2$ = Sum of even terms

$$\Rightarrow \quad S_2 = a_2 + a_4 + a_6 + \ldots + a_{2n}$$

$$\Rightarrow \quad S_2 = \frac{n}{2}[a_2 + a_{2n}]$$

$$\Rightarrow \quad S_2 = \frac{n}{2}[(a+d) + \{a + (2n-1)d\}] \qquad [\because a_{2n} = a + (2n-1)d]$$

$$\Rightarrow \quad S_2 = n(a+nd)$$

$$\therefore \quad S_1 : S_2 = (n+1)(a+nd) : n(a+nd) = (n+1) : n$$

EXAMPLE 30 *Show that the sum of an A.P. whose first term is a, the second term is b and the last term is c, is equal to* $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$. [NCERT EXEMPLAR]

SOLUTION Let there be n terms in the given A.P.

We have,

First term $= a$, second term $= b$

$\therefore$ Common difference $d = b - a$

It is given that the last term is c i.e. nth term $= c$.

$$\therefore \quad c = a + (n-1)d$$

$$\Rightarrow \quad c = a + (n-1)(b-a)$$

$$\Rightarrow \quad n - 1 = \frac{c-a}{b-a} \Rightarrow n = \frac{b+c-2a}{b-a} \qquad \ldots\text{(i)}$$

Let S_n be the sum of n terms of the A.P. Then,

$$S_n = \frac{n}{2}(a+c) = \frac{(b+c-2a)(a+c)}{2(b-a)} \qquad \text{[Using (i)]}$$

EXAMPLE 31 *Solve the equation:* $1 + 4 + 7 + 10 + \cdots + x = 287$. [NCERT EXEMPLAR]

SOLUTION Here, $1, 4, 7, 10, \ldots, x$, is an A.P. with first term $a = 1$ and common difference $d = 3$. Let there be n terms in the A.P. Then,

$$x = n\text{th term} \Rightarrow x = 1 + (n-1) \times 3 = 3n - 2 \qquad \ldots\text{(i)}$$

Now,

$$1 + 4 + 7 + 10 + \cdots + x = 287$$

$$\Rightarrow \quad \frac{n}{2}(1 + x) = 287 \qquad \left[\text{Using } S_n = \frac{n}{2}(a + l)\right]$$

$$\Rightarrow \quad \frac{n}{2}(1 + 3n - 2) = 287 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 3n^2 - n = 574 \Rightarrow 3n^2 - n - 574 = 0 \Rightarrow 3n^2 - 42n + 41n - 574 = 0$$

$$\Rightarrow \quad 3n(n-14) + 41(n-14) = 0$$

$\Rightarrow \quad (n-14)(3n+41)=0 \Rightarrow n-14=0 \qquad [\because 3n+4 \neq 0]$

$\Rightarrow \quad n=14$

Putting $n = 14$ in (i), we get $x = 3 \times 14 - 2 = 40$.

EXAMPLE 32 *A contract on construction job specifies a penalty for delay of completion beyond a certain date as follows: ₹200 for the first day, ₹250 for the second day, ₹300 for the third day, etc; the penalty for each succeeding day being ₹50 more than for the preceding day. How much does a delay of 30 days cost the contractor?* [NCERT]

SOLUTION Since the penalty for each succeeding day is ₹ 50 more than for the preceding day. Therefore, amount of penalty for different days forms an A.P. with first term a (= 200) and common difference d (= 50). We have to find how much does a delay of 30 days cost the contractor? In other words, we have to find the sum of 30 terms of the A.P.

$\therefore \quad \text{Required sum} = \frac{30}{2}\{2 \times 200 + (30-1) \times 50\} \qquad \left[\because S_n = \frac{n}{2}[2a+(n-1)d]\right]$

$\Rightarrow \quad \text{Required sum} = 15(400 + 29 \times 50)$

$\Rightarrow \quad \text{Required sum} = 15(400 + 1450)$

$\Rightarrow \quad \text{Required sum} = 15 \times 1850 = 27750$

Thus, a delay of 30 days will cost the contractor of ₹ 27750.

EXAMPLE 33 *The digits of a positive integer, having three digits are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.*

SOLUTION Let the digits at ones, tens and hundreds place be $(a-d)$, a and $(a+d)$ respectively. Then, the number is

$$(a+d) \times 100 + a \times 10 + (a-d) = 111a + 99d$$

The number obtained by reversing the digits is

$$(a-d) \times 100 + a \times 10 + (a+d) = 111a - 99d$$

It is given that the sum of the digits is 15.

$$(a-d) + a + (a+d) = 15 \qquad \text{...(i)}$$

Also, it is given that the number obtained by reversing the digits is 594 less than the original number.

$\therefore \quad 111a - 99d = 111a + 99d - 594 \qquad \text{...(ii)}$

$\Rightarrow \quad 3a = 15 \text{ and } 198d = 594$

$\Rightarrow \quad a = 5 \text{ and } d = 3$

So, the number is $111 \times 5 + 99 \times 3 = 852$.

EXAMPLE 34 *A man repays a loan of ₹3250 by paying ₹20 in the first month and then increases the payment by ₹15 every month. How long will it take him to clear the loan?*

SOLUTION Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

$\therefore \quad \text{Sum of the amounts} = 3250$

$\Rightarrow \quad \frac{n}{2}\{2 \times 20 + (n-1) \times 15\} = 3250$

$\Rightarrow \quad \frac{n}{2}(40 + 15n - 15) = 3250$

$\Rightarrow \quad n(15n + 25) = 6500$

$\Rightarrow \quad 15n^2 + 25n - 6500 = 0$

$\Rightarrow \quad 3n^2 + 5n - 1300 = 0$

$\Rightarrow \quad (n - 20)(3n + 65) = 0$

$\Rightarrow \quad n = 20 \text{ or, } n = -\frac{65}{3} \Rightarrow n = 20 \qquad \left[\because n \neq -\frac{65}{3}\right]$

Thus, the loan is cleared in 20 months.

EXAMPLE 35 *A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete. Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m (See Fig. 5.1). Calculate the total volume of concrete required to build the terrace.* [NCERT]

SOLUTION We observe that the length and width of each step are 50 m and $\frac{1}{2}$ m respectively. Also, we have

Height of first step $= \frac{1}{4}$ m.

Height of second step $= \left(\frac{1}{4} + \frac{1}{4}\right)$ m $= \left(2 \times \frac{1}{4}\right)$ m

Height of third step $= \frac{3}{4}$ m and so on.

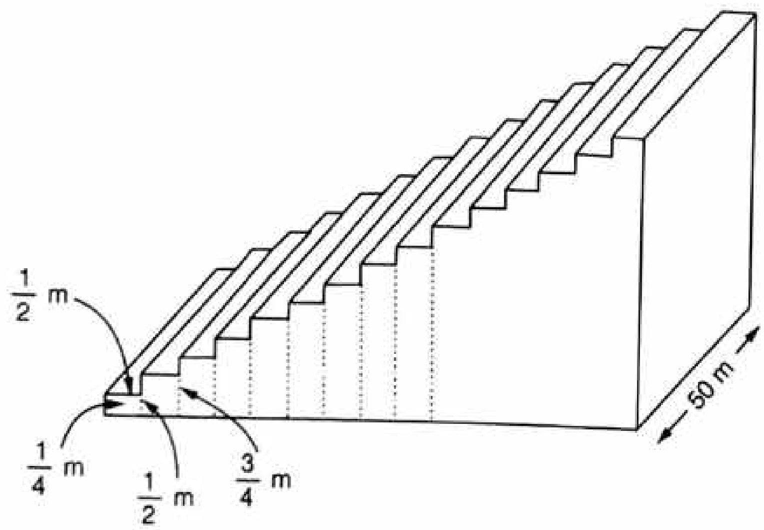


Fig. 5.1

Let $V_1, V_2, V_3, \ldots, V_{15}$ denote respectively the volumes of the concrete required to build the first, second, third, ..., fifteenth step. Then,

$$V_1 = \left(50 \times \frac{1}{2} \times \frac{1}{4}\right) \text{m}^3,\ V_2 = \left\{50 \times \frac{1}{2} \times \left(2 \times \frac{1}{4}\right)\right\} \text{m}^3,\ V_3 = \left\{50 \times \frac{1}{2} \times \left(3 \times \frac{1}{4}\right)\right\} \text{m}^3,$$

$$V_4 = \left(50 \times \frac{1}{2} \times 1\right) \text{m}^3 \text{ and so on.}$$

$\therefore$ Total volume of the concrete $= V_1 + V_2 + V_3 + \cdots + V_{15}$

$$= \left\{50 \times \frac{1}{2} \times \frac{1}{4}\right\} + \left\{50 \times \frac{1}{2} \times \left(2 \times \frac{1}{4}\right)\right\} + \left\{50 \times \frac{1}{2}\left(3 \times \frac{1}{4}\right)\right\} + \cdots + \left\{50 \times \frac{1}{2} \times \left(15 \times \frac{1}{4}\right)\right\} \text{m}^3$$

$$= \left(50 \times \frac{1}{2}\right)\left\{\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \cdots + \frac{15}{4}\right\} \text{m}^3$$

$$= 25\left\{\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \cdots + \frac{15}{4}\right\} \text{m}^3$$

$$= \frac{25}{4}\{1 + 2 + 3 + \cdots + 15\} \text{m}^3 = \frac{25}{4} \times \frac{15}{2}(1 + 15) \text{m}^3 = \frac{25}{4} \times \frac{15}{2} \times 16 \text{m}^3 = 750 \text{m}^3$$

EXAMPLE 36 *200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on (see Fig. 5.2). In how many rows 200 logs are placed and how many logs are in the top row?* [NCERT]

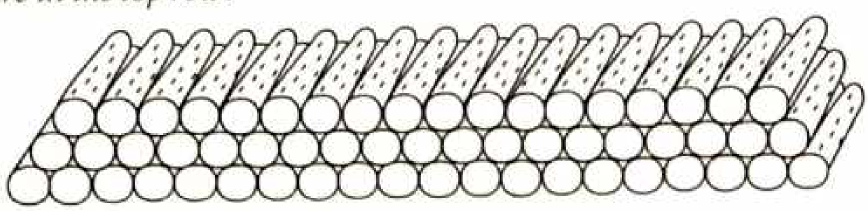


Fig. 5.2

SOLUTION Suppose 200 logs are stacked in n rows.

There are 20 logs in the first row and the number of logs in a row is one less than the number logs in the preceding row. So, number of logs in various rows form an A.P. with first term a (= 20) and common difference d (= – 1). As there are 200 logs in all rows.

$\therefore$ (Sum of n terms of an A.P. with $a = 20$ and $d = -1$) = 200

$\Rightarrow \quad \frac{n}{2}\{2a + (n-1)d\} = 200$

$\Rightarrow \quad \frac{n}{2}\{2 \times 20 + (n-1) \times -1\} = 200$

$\Rightarrow \quad \frac{n}{2}(40 - n + 1) = 200$

$\Rightarrow \quad n(41 - n) = 400$

$\Rightarrow \quad n^2 - 41n + 400 = 0$

$\Rightarrow \quad (n - 25)(n - 16) = 0 \Rightarrow n = 16 \text{ or, } n = 25$

Now,

If $n = 25$, then number of logs in 25th row is equal to 25th terms of an A.P. with first term 20 and common difference – 1.

$\therefore$ Number of logs in 25th row $= a + 24d = 20 - 24 = -4$

Clearly, this is not meaningful.

$\therefore \quad n = 16$

Thus, logs are placed in 16 rows.

Number of logs in top row

= Number of logs in 16th row

$= 16^{th}$ term of an A.P. with $a = 20$ and $d = -1$

$= a + 15d$

$= 20 + 15 \times -1 = 5$

Hence, there are 5 logs in the top row.

EXAMPLE 37 *Raghav buys a shop for ₹1, 20,000. He pays half of the amount in cash and agrees to pay the balance in 12 annual instalments of ₹5000 each. If the rate of interest is 12% and he pays with the instalment the interest due on the unpaid amount, find the total cost of the shop.*

SOLUTION Raghav pays half of ₹ 1,20,000 i.e. ₹ 60,000 in cash and the balance ₹ 60,000 in 12 annual instalments of ₹ 5000 each. With each instalment he pays interest on the unpaid amount at the rate of 12% per annum.

$\therefore$ Amount of first instalment = ₹ 5000 + Interest on unpaid amount of ₹ 60000

$$= ₹\,5000 + ₹\left(\frac{12}{100} \times 60000\right) = ₹\,5000 + ₹\,7200 = ₹\,12200$$

$\therefore$ Amount of second instalment = ₹ 5000 + Interest on unpaid amount of ₹ 55000

$$= ₹\,5000 + ₹\left(\frac{12}{100} \times 55000\right) = ₹\,5000 + ₹\,6600 = ₹\,11600$$

$\therefore$ Amount of third instalment = ₹ 5000 + Interest on unpaid amount of ₹ 50000

$$= ₹\,5000 + ₹\left(\frac{12}{100} \times 50000\right) = ₹\,5000 + ₹\,6000 = ₹\,11000$$

Clearly, amount of various instalments form an A.P. with first term ₹ 12200 and common difference – 600

$\therefore$ Total cost of the shop = ₹ [60,000 + Sum of 12 instalments]

$$= ₹\left[60{,}000 + \frac{12}{2}\{2 \times 12200 + (12-1) \times (-600)\}\right]$$

$$= ₹\,[60{,}000 + 6\,(24{,}400 - 6{,}600)]$$

$$= ₹\,[60{,}000 + 1{,}06{,}800] = ₹\,1{,}66{,}800$$

EXAMPLE 38 *Two cars start together in the same direction from the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by 1/2 km in each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?*

SOLUTION Suppose the second car overtakes the first car after t hours. Then, the two cars travel the same distance in t hours.

Distance travelled by the first car in t hours = $10\,t$ km.

Distance travelled by the second car in t hours.

= Sum of t terms of an A.P. with first term 8 and common difference 1/2.

$$= \frac{t}{2}\left\{2 \times 8 + (t-1) \times \frac{1}{2}\right\} = \frac{t\,(t+31)}{4}$$

When the second car overtakes the first car, we have

$$10t = \frac{t\,(t+31)}{4} \Rightarrow 40t = t^2 + 31t \Rightarrow t^2 - 9t = 0 \Rightarrow t\,(t-9) = 0 \Rightarrow t = 9 \qquad [\because t \neq 0]$$

Thus, the second car will overtake the first car in 9 hours.

EXAMPLE 39 *150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.*

SOLUTION Suppose the work is completed in n days when the workers started dropping. Since 4 workers are dropped on every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference -4

i.e., $$\frac{n}{2}\{2 \times 150 + (n-1) \times -4\} = n(152 - 2n)$$

Had the workers not dropped then the work would have finished it in $(n-8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150(n-8)$.

$\therefore \quad n(152 - 2n) = 150(n-8)$

$\Rightarrow \quad 152n - 2n^2 = 150n - 1200$

$\Rightarrow \quad 2n^2 - 2n - 1200 = 0$

$\Rightarrow \quad n^2 - n - 600 = 0$

$\Rightarrow \quad (n-25)(n+24) = 0 \qquad [\because n > 0]$

$\Rightarrow \quad n = 25$

Hence, the work is completed in 25 days.

EXAMPLE 40 *Along a road lie an odd number of stones placed at intervals of 10 metres. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones he covered a distance of 3 km. Find the number of stones.*

SOLUTION Let there be $(2n+1)$ stones. Clearly, one stone lies in the middle and n stones on each side of it in a row. Let P be the mid-stone and let A and B be the end stones on the left and right of P respectively.

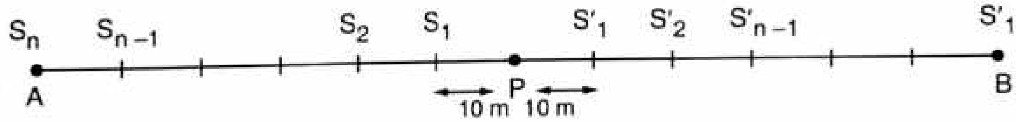


Fig. 5.3

Clearly, there are n intervals each of length 10 metres on both the sides of P. Now, suppose the man starts from A. He picks up the end stone on the left of mid-stone and goes to the mid-stone, drops it and goes to $(n-1)$ th stone on left, picks it up, goes to the mid-stone and drops it. This process is repeated till he collects all stones on the left of the mid-stone at the mid-stone. So, distance covered in collecting stones on the left of the mid-stones is

$$10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1].$$

After collecting all stones on left of the mid-stone the man goes to the stone B on the right side of the mid-stone, picks it up, goes to the mid-stone and drops it. Then, he goes to $(n-1)^{\text{th}}$ stone on the right and the process is repeated till he collects all stones at the mid-stone.

Distance covered in collecting the stones on the right side of the mid-stone.

$$= 2\left[10 \times n + 10 \times (n-1) + 10 \times (n-2) + \ \ldots \ + 10 \times 2 + 10 \times 1\right]$$

$\therefore$ Total distance covered

$$= 10 \times n + 2\left[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1\right] + 2\left[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1\right]$$

$$= 4\left[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1\right] - 10 \times n$$

$$= 40\{1+2+3+\ldots+n\} - 10n = 40\left\{\frac{n}{2}(1+n)\right\} - 10n = 20n(n+1) - 10n = 20n^2 + 10n$$

But, the total distance covered is 3 km i.e. 3000 m

$\therefore \quad 20n^2 + 10n = 3000$

$\Rightarrow \quad 2n^2 + n - 300 = 0$

$\Rightarrow \quad (n-12)(2n+25) = 0 \qquad [\because 2n + 25 \neq 0]$

$\Rightarrow \quad n = 12$

Hence, the number of stones $= 2n + 1 = 25$

EXAMPLE 41 *The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. Find this value of x.* [NCERT]

SOLUTION Let there be a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it i.e.,

House:	H_1	H_2	H_3	$\cdots$	H_{x-1}	H_x	H_{x+1}	$\cdots$	H_{49}
House No.	1	2	3		$(x-1)$	x	$(x+1)$	$\cdots$	49

$1 + 2 + 3 + \cdots + (x-1) = (x+1) + (x+2) + \cdots + 49$

$\Rightarrow \quad 1+2+3+\cdots+(x-1) = \{1+2+3+\cdots+x+(x+1)+\cdots+49\} - (1+2+3+\cdots+x)$

$\Rightarrow \quad \frac{x-1}{2}\{1+(x-1)\} = \frac{49}{2}(1+49) - \frac{x}{2}(1+x) \qquad \left[\text{Using: } S_n = \frac{n}{2}(a+l)\right]$

$\Rightarrow \quad \frac{x(x-1)}{2} = \frac{49 \times 50}{2} - \frac{x(x+1)}{2}$

$\Rightarrow \quad x(x-1) = 49 \times 50 - x(x+1) \qquad$ [Multiplying both sides by 2]

$\Rightarrow \quad (x^2 - x) + (x^2 + x) = 49 \times 50$

$\Rightarrow \quad 2x^2 = 49 \times 50$

$\Rightarrow \quad x^2 = 49 \times 25$

$\Rightarrow \quad x^2 = 7^2 \times 5^2$

$\Rightarrow \quad x = 7 \times 5 = 35$

Since, x is not a fraction. Hence, the value of x satisfying the given condition exists and is equal to 35.

EXAMPLE 42 *A ladder has rungs 25 cm apart (See Fig 5.4). The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and bottom rungs are 2.5 metre apart, what is the length of the wood required for the rungs?* [NCERT]

SOLUTION It is given that the gap between two consecutive rungs is 25 cm and the top and bottom rungs are 2.5 metre i.e., 250 cm apart.

$\therefore$ Number of rungs $= \frac{250}{25} + 1 = 11.$

It is given that the rungs are decreasing uniformly in length from 45 cm at the bottom to 25 cm at the top. Therefore, lengths of the rungs form an A.P. with first term $a = 45$ cm and 11th term $= 25$ cm.

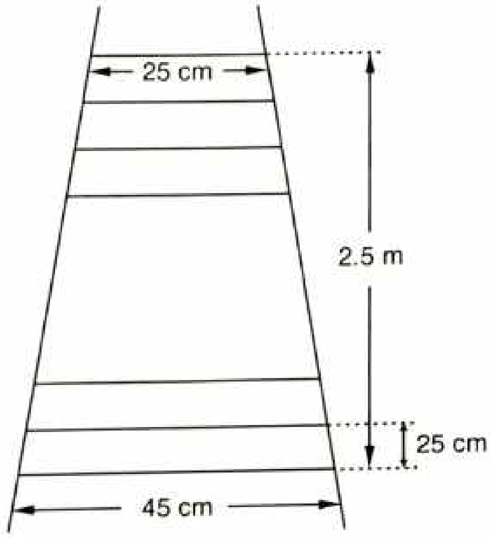


Fig. 5.4

$\therefore$ Length of the wood required for rungs

= Sum of 11 terms of an A.P. with first term 45 cm and last term = 25 cm

$= \frac{11}{2}(45 + 25)$ cm $\left[\because S_n = \frac{n}{2}(a + l)\right]$

$= 385$ cm $= 3.85$ metres

EXAMPLE 43 *A spiral is made up of successive semi-circles, with centres alternately at A and B, starting with centre at A, of radii 0.5 cm, 1.0 cm, 1.5 cm, 2.0 cm, ... as shown in Fig. 5.5 what is the total length of such a spiral made up of thirteen consecutive semi-circles?* $\left(\text{Take } \pi = \frac{22}{7}\right)$. **[NCERT]**

SOLUTION Let $l_1, l_2, l_3, l_4, \ldots, l_{13}$ be the lengths (circumferences) of semi-circles of radii $r_1 = 0.5$ cm, $r_2 = 1.0$ cm, $r_3 = 1.5$ cm, $r_4 = 2.0$ cm, $r_5 = 2.5$ cm, ... respectively. Then,

$$l_1 = \pi r_1 = \pi \times 0.5 \text{ cm} = \frac{\pi}{2} \text{ cm}$$

$$l_2 = \pi r_2 = \pi \times 1 \text{ cm} = 2\left(\frac{\pi}{2}\right) \text{ cm}$$

$$l_3 = \pi r_3 = \pi \times \frac{3}{2} \text{ cm} = 3\left(\frac{\pi}{2}\right) \text{ cm}$$

$$l_4 = \pi r_4 = \pi \times 2 \text{ cm} = 4\left(\frac{\pi}{2}\right) \text{ cm}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$l_{13} = \pi r_{13} = \pi \times \frac{13}{2} \text{ cm} = 13\left(\frac{\pi}{2}\right) \text{ cm}$$

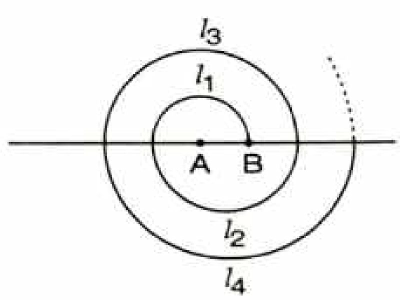


Fig. 5.5

$\therefore$ Total length of the spiral $= l_1 + l_2 + l_3 + \cdots + l_{13}$

$$= \left\{\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right) + 3\left(\frac{\pi}{2}\right) + \cdots + 13\left(\frac{\pi}{2}\right)\right\} \text{ cm}$$

$$= \frac{\pi}{2}(1 + 2 + 3 + \cdots + 13) \text{ cm}$$

$$= \frac{\pi}{2} \times \frac{13}{2}(1 + 13) \text{ cm} \quad \left[\text{Using } S_n = \frac{n}{2}(a + l)\right]$$

$$= \frac{\pi}{2} \times \frac{13}{2} \times 14 \text{ cm} = \frac{1}{2} \times \frac{22}{7} \times 13 \times 7 \text{ cm} = 143 \text{ cm}$$

EXAMPLE 44 *In a potato race, a bucket is placed at the starting point, which is 5 m from the first potato, and the other potatoes are placed 3 m apart in a straight line. There are n potatoes in the line (See Fig. 5.6). Each competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in the bucket, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run?* [NCERT

Fig. 5.6

SOLUTION We have,

d_1 = Distance run by the competitor to pick up first potato $= 2 \times 5$ m

d_2 = Distance run by the competitor to pick up second potato $= 2\,(5 + 3)$ m

d_3 = Distance run by the competitor to pick up third potato $= 2\,(5 + 2 \times 3)$ m

d_4 = Distance run by the competitor to pick up fourth potato $= 2\,(5 + 3 \times 3)$ m

$\vdots \qquad\qquad \vdots$

d_n = Distance run by the competitor to pick up n^{th} potato $= 2\,\{5 + (n - 1) \times 3\}$ m

$\therefore$ Total distance run by the competitor to pick up n potatoes

$$= d_1 + d_2 + d_3 + \cdots + d_n$$

$$= 2 \times 5 + 2(5 + 3) + 2(5 + 2 \times 3) + 2(5 + 3 \times 3) + \cdots + 2\{5 + (n - 1) \times 3\} \text{ metres}$$

$$= 2\left[5 + \{5 + 3\} + \{5 + (2 \times 3)\} + \{5 + (3 \times 3\} + \cdots + \{5 + (n - 1) \times 3\}\right]$$

$$= 2\left[\underbrace{(5 + 5 + \cdots + 5)}_{n\text{-times}} + \{3 + (2 \times 3) + (3 \times 3) + \cdots + (n - 1) \times 3\}\right]$$

$$= 2\left[5n + 3\{1 + 2 + 3 + \cdots + (n - 1)\}\right]$$

$$= 2\left[5n + 3\left(\frac{n - 1}{2}\right)\{1 + (n - 1)\}\right] \qquad \left[\text{Using: } S_n = \frac{n}{2}(a + l)\right]$$

$$= 2\left\{5n + \frac{3n(n - 1)}{2}\right\} = [10n + 3n(n - 1)] = 3n^2 + 7n = n(3n + 7) \text{ metres}$$

EXERCISE 5.6

LEVEL-1

1. Find the sum of the following arithmetic progressions:

 (i) 50, 46, 42, ... to 10 terms (ii) 1, 3, 5, 7, ... to 12 terms

(iii) $3, 9/2, 6, 15/2, \ldots$ to 25 terms (iv) $41, 36, 31, \ldots$ to 12 terms

(v) $a+b, a-b, a-3b, \ldots$ to 22 terms (vi) $(x-y)^2, (x^2+y^2), (x+y)^2 \ldots$, to n terms

(vii) $\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y}, \ldots$ to n terms

(viii) $-26, -24, -22, \ldots$ to 36 terms.

2. Find the sum to n term of the A.P. $5, 2, -1, -4, -7, \ldots,$
3. Find the sum of n terms of an A.P. whose n^{th} terms is given by $a_n = 5 - 6n$.
4. Find the sum of last ten terms of the A.P.: 8, 10, 12, 14, .., 126. [NCERT EXEMPLAR]
5. Find the sum of the first 15 terms of each of the following sequences having n^{th} term as
   (i) $a_n = 3 + 4n$ (ii) $b_n = 5 + 2n$ (iii) $x_n = 6 - n$ (iv) $y_n = 9 - 5n$ [NCERT]
6. Find the sum of first 20 terms of the sequence whose n^{th} term is $a_n = An + B$.
7. Find the sum of the first 25 terms of an A.P. whose n^{th} term is given by $a_n = 2 - 3n$. [CBSE 2004]
8. Find the sum of the first 25 terms of an A.P. whose n^{th} term is given by $a_n = 7 - 3n$. [CBSE 2004]
9. If the sum of a certain number of terms starting from first term of an A.P. is $25, 22, 19, \ldots,$ is 116. Find the last term.
10. (i) How many terms of the sequence 18, 16, 14, ... should be taken so that their sum is zero?
    (ii) How many terms are there in the A.P. whose first and fifth terms are -14 and 2 respectively and the sum of the terms is 40 ?
    (iii) How many terms of the A.P. 9, 17, 25, ... must be taken so that their sum is 636? [NCERT]
    (iv) How many terms of the A.P. 63, 60, 57, ... must be taken so that their sum is 693? [CBSE 2005]
    (v) How many terms of the A.P. 27, 24, 21... should be taken so that their sum is zero? [CBSE 2016]
11. Find the sum of the first
    (i) 11 terms of the A.P : 2, 6, 10, 14, ...
    (ii) 13 terms of the A.P : $-6, 0, 6, 12, \ldots$.
    (iii) 51 terms of the A.P : whose second term is 2 and fourth term is 8.
12. Find the sum of
    (i) the first 15 multiples of 8 [NCERT, CBSE 2017]
    (ii) the first 40 positive integers divisible by (a) 3 (b) 5 (c) 6. [NCERT]
    (iii) all 3 – digit natural numbers which are divisible by 13. [CBSE 2006C]
    (iv) all 3-digit natural numbers, which are multiples of 11. [CBSE 2012]
    (v) all 2-digit natural numbers divisible by 4. [CBSE 2017]
    (vi) first 8 multiples of 3. [CBSE 2018]
13. Find the sum:
    (i) $2 + 4 + 6 + \ldots + 200$ (ii) $3 + 11 + 19 + \ldots + 803$
    (iii) $(-5) + (-8) + (-11) + \ldots + (-230)$ (iv) $1 + 3 + 5 + 7 + \ldots + 199$
    (v) $7 + 10\frac{1}{2} + 14 + \ldots + 84$ (vi) $34 + 32 + 30 + \ldots + 10$
    (vii) $25 + 28 + 31 + \ldots + 100$ [CBSE 2006C]
    (viii) $18 + 15\frac{1}{2} + 13 + \cdots + \left(-49\frac{1}{2}\right)$ [CBSE 2013]

14. The first and the last terms of an A.P. are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

15. The third term of an A.P. is 7 and the seventh term exceeds three times the third term by 2. Find the first term, the common difference and the sum of first 20 terms.

16. The first term of an A.P. is 2 and the last term is 50. The sum of all these terms is 442. Find the common difference.

17. If 12^{th} term of an A.P. is -13 and the sum of the first four terms is 24, what is the sum of first 10 terms ?

18. Find the sum of n terms of the series $\left(4-\frac{1}{n}\right)+\left(4-\frac{2}{n}\right)+\left(4-\frac{3}{n}\right)+\ldots$ [CBSE 2017]

19. In an A.P., if the first term is 22, the common difference is -4 and the sum to n terms is 64, find n.

20. In an A.P., if the 5^{th} and 12^{th} terms are 30 and 65 respectively, what is the sum of first 20 terms?

21. Find the sum of first 51 terms of an A.P. whose second and third terms are 14 and 18 respectively. [NCERT]

22. If the sum of 7 terms of an A.P. is 49 and that of 17 terms is 289, find the sum of n terms. [CBSE 2013, 2016, NCERT]

23. The first term of an A.P. is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference. [NCERT]

24. In an A.P. the first term is 8, nth term is 33 and the sum to first n terms is 123. Find n and d, the common differences. [CBSE 2008]

25. In an A.P., the first term is 22, nth term is -11 and the sum to first n terms is 66. Find n and d, the common difference. [CBSE 2008]

26. The first and the last terms of an A.P. are 7 and 49 respectively. If sum of all its terms is 420, find its common difference. [CBSE 2014]

27. The first and the last terms of an A.P. are 5 and 45 respectively. If the sum of all its terms is 400, find its common difference. [CBSE 2014]

28. The sum of first q terms of an A.P. is 162. The ratio of its 6^{th} term to its 13^{th} term is $1:2$. Find the first and 15^{th} term of the A.P. [CBSE 2015]

29. If the 10th term of an A.P. is 21 and the sum of its first ten terms is 120, find its n^{th} term. [CBSE 2014]

30. The sum of the first 7 terms of an A.P. is 63 and the sum of its next 7 terms is 161. Find the 28^{th} term of this A.P. [CBSE 2014]

31. The sum of first seven terms of an A.P. is 182. If its 4th and the 17th terms are in the ratio $1:5$, find the A.P. [CBSE 2014]

32. The nth term of an A.P. is given by $(-4n+15)$. Find the sum of first 20 terms of this A.P. [CBSE 2013]

33. In an A.P., the sum of first ten terms is -150 and the sum of its next ten terms is -550. Find the A.P. [CBSE 2010]

34. Sum of the first 14 terms of an A.P. is 1505 and its first term is 10. Find its 25^{th} term. [CBSE 2012]

35. In an A.P., the first term is 2, the last term is 29 and the sum of the terms is 155. Find the common difference of the A.P. [CBSE 2010]

36. The first and the last term of an A.P. are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum? [NCERT]

37. Find the number of terms of the A.P. $-12, -9, -6, \ldots, 21$. If 1 is added to each term of this A.P., then find the sum of all terms of the A.P. thus obtained. [CBSE 2013]

38. The sum of the first n terms of an A.P. is $3n^2 + 6n$. Find the nth term of this A.P. [CBSE 2014]

39. The sum of first n terms of an A.P. is $5n - n^2$. Find the n^{th} term of this A.P. [CBSE 2014]

40. The sum of the first n terms of an A.P. is $4n^2 + 2n$. Find the n^{th} term of this A.P. [CBSE 2014]

41. The sum of first n terms of an A.P. is $3n^2 + 4n$. Find the 25th term of this A.P. [CBSE 2013]

42. The sum of first n terms of an A.P. is $5n^2 + 3n$. If its mth term is 168, find the value of m. Also, find the 20th term of this A.P. [CBSE 2013]

43. The sum of first q terms of an A.P. is $63q - 3q^2$. If its pth term is -60, find the value of p. Also, find the 11th term of this A.P. [CBSE 2013]

44. The sum of first m terms of an A.P. is $4m^2 - m$. If its nth term is 107, find the value of n. Also, find the 21st term of this A.P. [CBSE 2013]

45. If the sum of the first n terms of an A.P. is $4n - n^2$, what is the first term? What is the sum of first two terms? What is the second term? Similarly, find the third, the tenth and the nth terms. [NCERT]

46. If the sum of first n terms of an A.P. is $\frac{1}{2}(3n^2 + 7n)$, then find its n^{th} term. Hence write its 20^{th} term. [CBSE 2015]

47. In an A.P., the sum of first n terms is $\frac{3n^2}{2} + \frac{13}{2}n$. Find its 25^{th} term. [CBSE 2006C]

48. Find the sum of all natural numbers between 1 and 100, which are divisible by 3.

49. Find the sum of first n odd natural numbers.

50. Find the sum of all odd numbers between
   (i) 0 and 50 [CBSE 2017]
   (ii) 100 and 200.

51. Show that the sum of all odd integers between 1 and 1000 which are divisible by 3 is 83667.

52. Find the sum of all integers between 84 and 719, which are multiples of 5.

53. Find the sum of all integers between 50 and 500, which are divisible by 7.

54. Find the sum of all even integers between 101 and 999.

55. (i) Find the sum of all integers between 100 and 550, which are divisible by 9.
   (ii) all integers between 100 and 550 which are not divisible by 9.
   (iii) all integers between 1 and 500 which are multiplies of 2 as well as of 5.
   (iv) all integers from 1 to 500 which are multiplies 2 as well as of 5.
   (v) all integers from 1 to 500 which are multiples of 2 or 5.

56. Let there be an A.P. with first term 'a', common difference 'd'. If a_n denotes its n^{th} term and S_n the sum of first n terms, find.
   (i) n and S_n, if $a = 5, d = 3$ and $a_n = 50$.
   (ii) n and a, if $a_n = 4, d = 2$ and $S_n = -14$.
   (iii) d, if $a = 3, n = 8$ and $S_n = 192$.
   (iv) a, if $a_n = 28, S_n = 144$ and $n = 9$.
   (v) n and d, if $a = 8, a_n = 62$ and $S_n = 210$.

(vi) n and a_n, if $a = 2, d = 8$ and $S_n = 90$. [NCERT]

(vii) k, if $S_n = 3n^2 + 5n$ and $a_k = 164$. (viii) S_{22}, if $d = 22$ and $a_{22} = 149$

57. If S_n denotes the sum of first n terms of an A.P., prove that $S_{12} = 3(S_8 - S_4)$. [NCERT EXEMPLAR, CBSE 2015]

58. A thief, after committing a theft runs at a uniform speed of 50 m/minute. After 2 minutes, a policeman runs to catch him. He goes 60 m in first minute and increases his speed by 5 m/minute every suceeding minute. After how many minutes, the policeman will catch the thief? [CBSE 2016]

59. The sums of first n terms of three A.P.s are S_1, S_2 and S_3. The first term of each is 5 and their common differences are 2, 4 and 6 respectively. Prove that $S_1 + S_3 = 2S_2$. [CBSE 2016]

60. Resham wanted to save at least ₹ 6500 for sending her daughter to school next year (after 12 months). She saved ₹ 450 in the first month and raised her savings by ₹ 20 every next month. How much will she be able to save in next 12 months ? Will she be able to send her daughter to the school next year? [CBSE 2016]

61. In a school, students decided to plant trees in and around the school to reduce air polution. It was decided that the number of trees, that each section of each class will plant, will be double of the class in which they are studying. If there are 1 to 12 classes in the school and each class has two sections, find how many trees were planted by the studetns. [CBSE 2014]

62. Ramkali would need ₹ 1800 for admission fee and books etc., for her daughter to start going to school from next year. She saved ₹ 50 in the first month of this year and increased her monthly saving by ₹ 20. After a year, how much money will she save? Will she be able to fulfil her dream of sending her daughter to school?

[CBSE 2005, 2014]

63. A man saved ₹ 16500 in ten years. In each year after the first he saved ₹ 100 more than he did in the preceding year. How much did he save in the first year?

64. A man saved ₹ 32 during the first year, ₹ 36 in the second year and in this way he increases his savings by ₹ 4 every year. Find in what time his saving will be ₹ 200.

65. A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which form an arithmetic series. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid, find the value of the first instalment.

66. There are 25 trees at equal distances of 5 metres in a line with a well, the distance of the well from the nearest tree being 10 metres. A gardener waters all the trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.

67. A man is employed to count ₹ 10710. He counts at the rate of ₹180 per minute for half an hour. After this he counts at the rate of ₹ 3 less every minute than the preceding minute. Find the time taken by him to count the entire amount.

68. A piece of equipment cost a certain factory ₹ 600,000. If it depreciates in value, 15% the first, 13.5% the next year, 12% the third year, and so on. What will be its value at the end of 10 years, all percentages applying to the original cost?

69. A sum of ₹ 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹ 20 less than its preceding prize, find the value of each prize.

70. If S_n denotes the sum of the first n terms of an A.P., prove that $S_{30} = 3(S_{20} - S_{10})$. [CBSE 2014]

71. Solve the question: $(-4) + (-1) + 2 + 5 + \cdots + x = 437$. [NCERT EXEMPLAR]

72. Which term of the A.P. $-2, -7, -12, \ldots$ will be -77? Find the sum of this A.P. upto the term -77.

73. The sum of first n terms of an A.P. whose first term is 8 and the common difference is 20 is equal to the sum of first $2n$ terms of another A.P. whose first term is -30 and common difference is 8. Find n. [NCERT EXEMPLAR]

74. The students of a school decided to beautify the school on the annual day by fixing colourful on the straight passage of the school. They have 27 flags to be fixed at intervals of every 2 metre. The flags are stored at the position of the middle most flag. Ruchi was given the responsibility of placing the flags. Ruchi kept her books where the flags were stored. She could carry only one flag at a time. How much distance did she cover in completing this job and returning back to collect her books? What is the maximum distance she travelled carrying a flag? [NCERT EXEMPLAR]

ANSWERS

1. (i) 320 (ii) 144 (iii) 525 (iv) 162 (v) $22a - 440b$

(vi) $n\left\{(x-y)^2 + (n-1)xy\right\}$ (vii) $\frac{n}{2(x+y)}\left\{n(2x-y) - y\right\}$ (viii) 324

2. $\frac{n}{2}(13 - 3n)$ 3. $n(2 - 3n)$ 4. 1170 5. (i) 525 (ii) 315 (iii) -30 (iv) -465

6. $210A + 20B$ 7. -925 8. -800 9. 4

10. (i) 19 (ii) 10 (iii) 12 (iv) 21, 22 (v) 19

11. (i) 242 (ii) 390 (iii) 3774

12. (i) 960 (ii) (a) 2460 (b) 4100 (c) 4920 (iii) 37674 (iv) 44550 (v) 1188 (vi) 108

13. (i) 10100 (ii) 40703 (iii) -8930 (iv) 10000 (v) $\frac{2093}{2}$ (vi) 286

(vii) 1625 (viii) -441 14. 38, 6973 15. $-1, 4, 740$

16. 3 17. 0 18. $\frac{1}{2}(7n - 1)$ 19. 4 or 8 20. 1150 21. 5610

22. n^2 23. $n = 16, d = 8/3$ 24. $n = 6, d = 5$ 25. $n = 12, d = -3$

26. 3 27. $d = \frac{8}{3}$ 28. 6, 48 29. $2n + 1$ 30. 57

31. 2, 10, 18, 26, … 32. 760 33. $a = 3, d = -4$ 34. 370

35. 3 36. $n = 38, S = 6973$ 37. 12, 66 38. $6n + 3$ 39. $6 - 2n$

40. $4n - 2$ 41. 151 42. $m = 17, a_{20} = 198$ 43. $p = 21, a_{11} = 0$

44. $n = 14, a_{21} = 163$ 45. $S_1 = 1, S_2 = 4, a_2 = 1, S_3 = 3, a_3 = -1, a_{10} = -15$

46. $a_n = 3n + 2, a_{20} = 62$ 47. 80 48. 1683 49. n^2

50. (i) 625 (ii) 7500 52. 50800 53. 17696 54. 246950

55. (i) 16425 (ii) 129500 (iii) 12250 (iv) 12750 (v) 75250
56. (i) $n = 16, S_n = 440$ (ii) $n = 7, a = -8$ (iii) $d = 6$ (iv) $a = 4$,
(v) $n = 6,\ d = \dfrac{54}{5}$ (vi) $n = 5, a_n = 34$ (vii) 27 (viii) – 1804

58. 5 60. ₹ 6720, Yes 61. 312 62. 1920, Yes 63. ₹ 1200
64. 5 years 65. ₹ 51 66. 3500 m 67. 89 minutes
68. ₹ 15000 69. Values of the prizes (in ₹) are: 160, 140, 120, 100, 80, 60, 40
71. 50 72. 16th term, –632 73. 11 74. 728 m, 26 m

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Define an arithmetic progression.
2. Write the common difference of an A.P. whose nth term is $a_n = 3n + 7$.
3. Which term of the sequence 114, 109, 104, ... is the first negative term?
4. Write the value of $a_{30} - a_{10}$ for the A.P. 4, 9, 14, 19,......
5. Write 5th term from the end of the A.P. 3, 5, 7, 9, ..., 201.
6. Write the value of x for which $2x$, $x + 10$ and $3x + 2$ are in A.P.
7. Write the nth term of an A.P. the sum of whose n terms is S_n.
8. Write the sum of first n odd natural numbers.
9. Write the sum of first n even natural numbers.
10. If the sum of n terms of an A.P. is $S_n = 3n^2 + 5n$. Write its common difference.
11. Write the expression for the common difference of an A.P. whose first term is a and nth term is b.
12. The first term of an A.P. is p and its common difference is q. Find its 10th term. [CBSE 2008]
13. For what value of p are $2p + 1, 13, 5p - 3$ are three consecutive terms of an A.P.? [CBSE 2009]
14. If $\frac{4}{5}$, a, 2 are three consecutive terms of an A.P., then find the value of a. [CBSE 2009]
15. If the sum of first p term of an A.P. is $ap^2 + bp$, find its common difference. [CBSE 2010]
16. Find the 9th term from the end of the A.P. 5, 9, 13,..., 185. [CBSE 2016]
17. For what value of k will the consecutive terms $2k + 1, 3k + 3$ and $5k - 1$ form on A.P.? [CBSE 2016]
18. Write the nth term of the A.P. $\frac{1}{m}, \frac{1+m}{m}, \frac{1+2m}{m}, ...$ [CBSE 2017]
19. In an A.P., if the common difference $d = -4$, and the seventh term a_7 is 4, then find the first term. [CBSE 2018]

ANSWERS

2. 3 3. 24th 4. 100 5. 193 6. 6 7. $a_n = S_n - S_{n-1}$ 8. n^2

9. $n(n+1)$ 10. 6 11. $\frac{b-a}{n-1}$ 12. $p+9q$ 13. 4 14. $\frac{7}{5}$

15. $2a$ 16. 153 17. 6 18. $\frac{m(n-1)+1}{m}$ 19. 28

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If 7th and 13th terms of an A.P. be 34 and 64 respectively, then its 18th term is
   (a) 87 (b) 88 (c) 89 (d) 90
2. If the sum of P terms of an A.P. is q and the sum of q terms is p, then the sum of $p+q$ terms will be
   (a) 0 (b) $p-q$ (c) $p+q$ (d) $-(p+q)$
3. If the sum of n terms of an A.P. be $3n^2+n$ and its common difference is 6, then its first term is
   (a) 2 (b) 3 (c) 1 (d) 4
4. The first and last terms of an A.P. are 1 and 11. If the sum of its terms is 36, then the number of terms will be
   (a) 5 (b) 6 (c) 7 (d) 8
5. If the sum of n tems of an A.P. is $3n^2+5n$ then which of its terms is 164?
   (a) 26th (b) 27th (c) 28th (d) none of these.
6. If the sum of n terms of an A.P. is $2n^2+5n$, then its nth term is
   (a) $4n-3$ (b) $3n-4$ (c) $4n+3$ (d) $3n+4$
7. If the sum of three consecutive terms of an increasing A.P. is 51 and the product of the first and third of these terms is 273, then the third term is
   (a) 13 (b) 9 (c) 21 (d) 17
8. If four numbers in A.P. are such that their sum is 50 and the greatest number is 4 times the least, then the numbers are
   (a) 5, 10, 15, 20 (b) 4, 10, 16, 22 (c) 3, 7, 11, 15 (d) none of these
9. Let S_n denote the sum of n terms of an A.P. whose first term is a. If the common difference d is given by $d = S_n - kS_{n-1} + S_{n-2}$, then $k=$
   (a) 1 (b) 2 (c) 3 (d) none of these.
10. The first and last term of an A.P. are a and l respectively. If S is the sum of all the terms of the A.P. and the common difference is given by $\frac{l^2-a^2}{k-(l+a)}$, then $k=$
    (a) S (b) $2S$ (c) $3S$ (d) none of these
11. If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then $k=$
    (a) $\frac{1}{n}$ (b) $\frac{n-1}{n}$ (c) $\frac{n+1}{2n}$ (d) $\frac{n+1}{n}$
12. If the first, second and last term of an A.P. are a, b and $2a$ respectively, its sum is
    (a) $\frac{ab}{2(b-a)}$ (b) $\frac{ab}{b-a}$ (c) $\frac{3ab}{2(b-a)}$ (d) none of these

13. If S_1 is the sum of an arithmetic progression of 'n' odd number of terms and S_2 the sum of the terms of the series in odd places, then $\frac{S_1}{S_2} =$

(a) $\frac{2n}{n+1}$ (b) $\frac{n}{n+1}$ (c) $\frac{n+1}{2n}$ (d) $\frac{n+1}{n}$

14. If in an A.P., $S_n = n^2 p$ and $S_m = m^2 p$, where S_r denotes the sum of r terms of the A.P., then S_p is equal to

(a) $\frac{1}{2}p^3$ (b) mnp (c) p^3 (d) $(m+n)p^2$

15. If S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3S_n$, then $S_{3n} : S_n$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10

16. In an AP, $S_p = q, S_q = p$ and S_r denotes the sum of first r terms. Then, S_{p+q} is equal to

(a) 0 (b) $-(p+q)$ (c) $p+q$ (d) pq

17. If S_r denotes the sum of the first r terms of an A.P. Then, $S_{3n} : (S_{2n} - S_n)$ is

(a) n (b) $3n$ (c) 3 (d) none of these

18. If the first term of an A.P. is 2 and common difference is 4, then the sum of its 40 terms is

(a) 3200 (b) 1600 (c) 200 (d) 2800

19. The number of terms of the A.P. 3, 7, 11, 15, ... to be taken so that the sum is 406 is

(a) 5 (b) 10 (c) 12 (d) 14

20. Sum of n terms of the series $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \cdots$ is

(a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$ (c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1

21. The 9th term of an A.P. is 449 and 449th term is 9. The term which is equal to zero is

(a) 501^{th} (b) 502^{th} (c) 508^{th} (d) none of these

22. If $\frac{1}{x+2}, \frac{1}{x+3}, \frac{1}{x+5}$ are in A.P. Then, $x =$

(a) 5 (b) 3 (c) 1 (d) 2

23. The n^{th} term of an A.P., the sum of whose n terms is S_n, is

(a) $S_n + S_{n-1}$ (b) $S_n - S_{n-1}$ (c) $S_n + S_{n+1}$ (d) $S_n - S_{n+1}$

24. The common difference of an A.P., the sum of whose n terms is S_n, is

(a) $S_n - 2S_{n-1} + S_{n-2}$ (b) $S_n - 2S_{n-1} - S_{n-2}$
(c) $S_n - S_{n-2}$ (d) $S_n - S_{n-1}$

25. If the sums of n terms of two arithmetic progressions are in the ratio $\frac{3n+5}{5n+7}$, then their n^{th} terms are in the ratio

(a) $\frac{3n-1}{5n-1}$ (b) $\frac{3n+1}{5n+1}$ (c) $\frac{5n+1}{3n+1}$ (d) $\frac{5n-1}{3n-1}$

26. If S_n denote the sum of n terms of an A.P. with first term a and common difference d such that $\frac{S_x}{S_{kx}}$ is independent of x, then
(a) $d = a$ (b) $d = 2a$ (c) $a = 2d$ (d) $d = -a$

27. If the first term of an A.P. is a and n^{th} term is b, then its common difference is
(a) $\frac{b-a}{n+1}$ (b) $\frac{b-a}{n-1}$ (c) $\frac{b-a}{n}$ (d) $\frac{b+a}{n-1}$

28. The sum of first n odd natural numbers is
(a) $2n-1$ (b) $2n+1$ (c) n^2 (d) n^2-1

29. Two A.P.'s have the same common difference. The first term of one of these is 8 and that of the other is 3. The difference between their 30^{th} terms is
(a) 11 (b) 3 (c) 8 (d) 5

30. If $18, a, b, -3$ are in A.P., the $a + b =$
(a) 19 (b) 7 (c) 11 (d) 15

31. The sum of n terms of two A.P.'s are in the ratio $5n+9 : 9n+6$. Then, the ratio of their 18^{th} term is
(a) $\frac{179}{321}$ (b) $\frac{178}{321}$ (c) $\frac{175}{321}$ (d) $\frac{176}{321}$

32. If $\frac{5+9+13+\cdots \text{ to } n \text{ terms}}{7+9+11+\cdots \text{ to } (n+1) \text{ terms}} = \frac{17}{16}$, then $n =$
(a) 8 (b) 7 (c) 10 (d) 11

33. The sum of n terms of an A.P. is $3n^2 + 5n$, then 164 is its
(a) 24^{th} term (b) 27^{th} term (c) 26^{th} term (d) 25^{th} term

34. If the n^{th} term of an A.P. is $2n+1$, then the sum of first n terms of the A.P. is
(a) $n(n-2)$ (b) $n(n+2)$ (c) $n(n+1)$ (d) $n(n-1)$

35. If 18^{th} and 11^{th} term of an A.P. are in the ratio $3:2$, then its 21^{st} and 5^{th} terms are in the ratio
(a) $3:2$ (b) $3:1$ (c) $1:3$ (d) $2:3$

36. The sum of first 20 odd natural numbers is
(a) 100 (b) 210 (c) 400 (d) 420 **[CBSE 2012]**

37. The common difference of the A.P. is $\frac{1}{2q}, \frac{1-2q}{2q}, \frac{1-4q}{2q}, \cdots$ is
(a) -1 (b) 1 (c) q (d) $2q$ **[CBSE 2013]**

38. The common difference of the A.P. $\frac{1}{3}, \frac{1-3b}{3}, \frac{1-6b}{3}, \cdots$ is
(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $-b$ (d) b **[CBSE 2013]**

39. The common difference of the A.P. $\frac{1}{2b}, \frac{1-6b}{2b}, \frac{1-12b}{2b}, \cdots$ is
(a) $2b$ (b) $-2b$ (c) 3 (d) -3 **[CBSE 2013]**

40. If $k, 2k-1$ and $2k+1$ are three consecutive terms of an AP, the value of k is
(a) -2 (b) 3 (c) -3 (d) 6 [CBSE 2014]

41. The next term of the A.P. $\sqrt{7}, \sqrt{28}, \sqrt{63}, \ldots$
(a) $\sqrt{70}$ (b) $\sqrt{84}$ (c) $\sqrt{97}$ (d) $\sqrt{112}$ [CBSE 2014]

42. The first three terms of an A.P. respectively are $3y-1, 3y+5$ and $5y+1$. Then, y equals
(a) -3 (b) 4 (c) 5 (d) 2 [CBSE 2014]

ANSWERS

1. (c)	2. (d)	3. (a)	4. (b)	5. (b)	6. (c)
7. (c)	8. (a)	9. (b)	10. (b)	11. (d)	12. (c)
13. (a)	14. (c)	15. (b)	16. (b)	17. (c)	18. (a)
19. (d)	20. (c)	21. (c)	22. (c)	23. (b)	24. (a)
25. (b)	26. (b)	27. (b)	28. (c)	29. (d)	30. (d)
31. (a)	32. (b)	33. (b)	34. (b)	35. (b)	36. (c)
37. (a)	38. (c)	39. (d)	40. (b)	41. (d)	42. (c)

SUMMARY

1. A sequence is an arrangement of numbers or objects in a definite order.
2. A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called an arithmetic progression, if there exists a constant d such that

   $a_2 - a_1 = d, a_3 - a_2 = d, a_4 - a_3 = d, \ldots, a_{n+1} - a_n = d$ and so on.

   The constant 'd' is called the common difference.
3. If 'a' is the first term and 'd' the common difference of an AP, then the A.P. is
$$a, a+d, a+2d, a+3d, a+4d, \ldots$$
4. A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an AP, if $a_{n+1} - a_n$ is independent of n.
5. A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an AP, if and only if its n^{th} term a_n is a linear expression in n and in such a case the coefficient of n is the common difference.
6. The n^{th} term a_n of an A.P. with first term 'a' and common difference 'd' is given by
$$a_n = a + (n-1)d.$$
7. Let there be an A.P. with first term 'a' and common difference d. If there are m terms in the AP, then
$$n^{\text{th}} \text{ term from the end} = (m-n+1)^{\text{th}} \text{ term from the beginning}$$
$$= a + (m-n)d$$
   Also,
$$n^{\text{th}} \text{ term from the end} = \text{Last term} + (n-1)(-d)$$
$$= l - (n-1)d, \text{ where } l \text{ denotes the last term.}$$
8. Various terms is an A.P. can be chosen in the following manner.

Number of terms	*Terms*	*Common difference*
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a+d, a+3d, a+5d$	$2d$

9. The sum to n terms of an A.P. with first term 'a' and common difference 'd' is given by

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

Also, $S_n = \frac{n}{2}\{a + l\}$, where l = last term $= a + (n-1)d$

10. If the ratio of the sums of n terms of two AP's is given, then to find the ratio of their n^{th} terms, we replace n by $(2n-1)$ in the ratio of the sums of n terms.

11. A sequence is an A.P. if and only if the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants. In such a case the common difference is $2A$.

CHAPTER 6

CO-ORDINATE GEOMETRY

6.1 INTRODUCTION

In class IX, we have seen that to locate the position of a point on a plane, we require a pair of mutually perpendicular lines which are known as the coordinate axes. The horizontal line is known as the x-axis and the vertical line is known as the y-axis. The intersection point of the coordinate axes is known as the origin. The distance of a point from the y-axis is called its x-coordinate, or abscissa and the distance from the x-axis is called its y-coordinate, or ordinate. We have seen that the coordinates of a point on the x-axis are of the form $(x, 0)$, and that of a point on y-axis are of the form $(0, y)$. We have also learnt about plotting of points in a plane when their coordinates are given. Also, we have seen that a linear equation in two variables, when represented graphically, gives a straight line.

In this chapter, we will see how we can find the distance between two points whose coordinates are given. We will also find the coordinates of the point which divides the line segment joining two given points in a given ratio. Finally, we will learn about the method of finding the area of a triangle in terms of the coordinates of its vertices.

6.2 RECAPITULATION

RECTANGULAR COORDINATE AXES Let $X'OX$ and $Y'OY$ be two mutually perpendicular lines through any point O in the plane of the paper. We call the point O, the origin. Now, choose a convenient unit of length and starting from the origin as zero, mark-off a number scale on the horizontal line $X'OX$, positive to the right of the origin O and negative to the left of origin O. Also, mark-off the same scale on the vertical line $Y'OY$, positive upwards and negative downwards of the origin O.

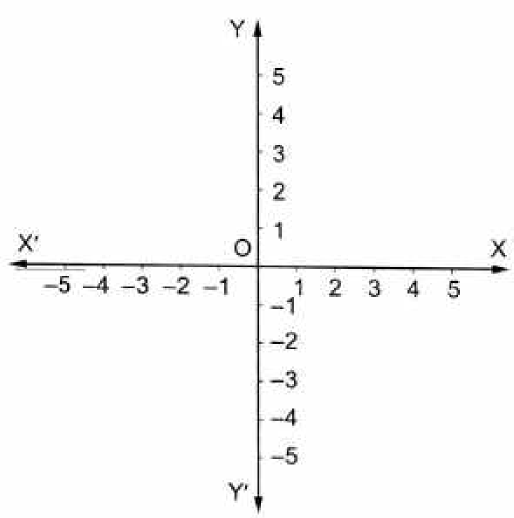


Fig. 6.1

The line $X'OX$ is called the x-axis or axis of x, the line $Y'OY$ is known as the y-axis or axis of y, and the two lines taken together are called the coordinate axes or the axes of coordinates.

CARTESIAN COORDINATES OF A POINT Let $X'OX$ and $Y'OY$ be the coordinate axes, and let P be any point in the plane. Draw perpendiculars PM and PN from P on x and y-axis respectively. The length of the directed line segment OM in the units of scale chosen is called the *x-coordinate* or *abscissa* of point P. Similarly, the length of the directed line segment ON on the same scale is called the *y-coordinate* or *ordinate* of point P. Let $OM = x$ and $ON = y$. Then the position of the point P in the plane with respect to the coordinate axes is represented by the ordered (x, y). The ordered pair (x, y) is called the coordinates of point P.

Thus, for a given point, the abscissa and ordinate are the distances of the given point from y-axis and x-axis respectively.

The above system of coordinating an ordered pair (x, y) with every point in a plane is called the *Rectangular Cartesian coordinate system.*

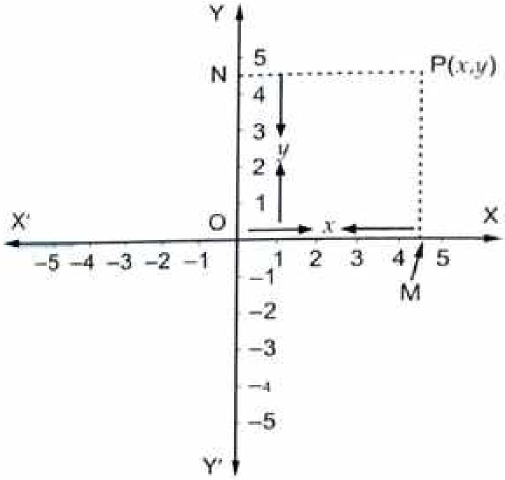


Fig. 6.2

It follows from the above discussion that corresponding to every point P in the Euclidean plane there is a unique ordered pair (x, y) of real numbers called its cartesian coordinates. Conversely, when we are given an ordered pair (x, y) and a Cartesian coordinate system, we can determine a point in the Euclidean plane having its coordinates (x, y). For this we mark-off a directed line segment $OM = x$ on the x-axis and another directed line segment $ON = y$ on y-*axis*. Now, draw perpendiculars at M and N to X and Y axes respectively. The point of intersection of these two perpendiculars determines point P in the Euclidean space having coordinates (x, y).

Thus, there is one-to-one correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called the Cartesian plane and is denoted by R^2.

QUADRANTS Let $X'OX$ and $Y'OY$ be the coordinate axes. We observe that the two axes divide the Euclidean plane into four regions, called the quadrants. The regions XOY, $X'OY$, $X'OY'$ and $Y'OX$ are known as the first, the second, the third and the fourth quadrants respectively. The ray OX is taken as positive x-axis, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. In view of the above sign convention the four quadrants are characterised by the following signs of *abscissa* and *ordinate*.

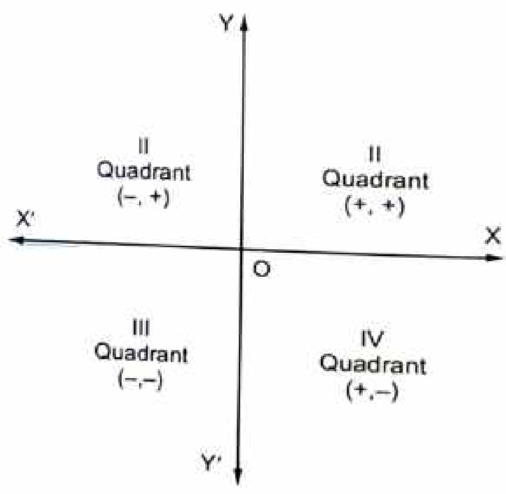


Fig. 6.3

I quadrant: $x > 0, y > 0$

II quadrant: $x < 0, y > 0$

III quadrant: $x < 0, y < 0$

IV quadrant: $x > 0, y < 0$

The coordinates of the origin are taken as (0, 0). The coordinates of any point on x-axis are of the form (x, 0) and the coordinates of any point on y-axis are of the form (0, y). Thus, if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis.

It follows from the above discussion that by simply looking at the coordinates of a point we can tell in which quadrant it would lie as discussed in the following illustration.

REMARK 1 *If the coordinates of a point P are (x, y), we shall frequently refer to it as P(x, y).*

REMARK 2 *It is evident from the above discussion that:*

(i) *The abscissa of a point is its perpendicular distance from y-axis.*

(ii) *The ordinate of a point is its perpendicular distance from x-axis.*
(iii) *The abscissa of every point situated on the right side of y-axis is positive and the abscissa of every point situated on the left side of y-axis is negative.*
(iv) *The ordinate of every point situated above x-axis is positive and that of every point below x-axis is negative.*
(v) *The abscissa of every point on y-axis is zero.*
(vi) *The ordinate of every point on x-axis is zero.*
(vii) *Coordinates of the origin are O (0, 0).*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The base AB of two equilateral triangles ABC and ABC′ with side 2a lies along the X-axis such that the mid-point of AB is at the origin as shown in Fig. 6.4. Find the coordinates of the vertices C and C′ of the triangles.*

SOLUTION Since the mid-point of AB is at the origin O and $AB = 2a$.

$\therefore \quad OA = OB = a.$

Thus, the coordinates of A and B are $(a, 0)$ and $(-a, 0)$ respectively.

Since triangles ABC and ABC' are equilateral. Therefore, their third vertices C and C′ lie on the perpendicular bisector of base AB. Clearly, $Y'OY$ is the perpendicular bisector of AB. Thus, C and C′ lie on Y-axis. Consequently, their x-coordinates are equal to zero.

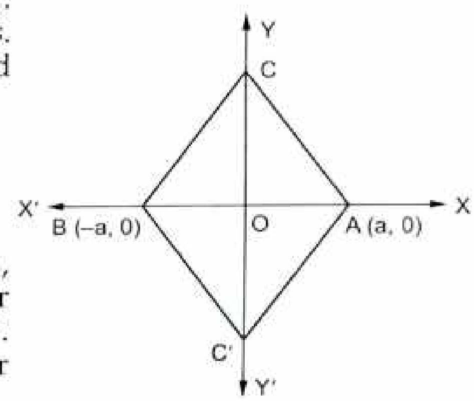


Fig. 6.4

In $\Delta\, AOC$, we have

$OA^2 + OC^2 = AC^2$ [Using Pythagoras theorem]

$\Rightarrow \quad a^2 + OC^2 = (2a)^2$ $[\because AB = AC = BC$ and $AB = 2a \therefore AC = 2a]$

$\Rightarrow \quad OC^2 = 4a^2 - a^2$

$\Rightarrow \quad OC^2 = 3a^2$

$\Rightarrow \quad OC = \sqrt{3}a$

Similarly, by applying Pythagoras theorem in $\Delta\, AOC'$, we have

$$OC' = \sqrt{3}\, a$$

Thus, the coordinates of C and C′ are $(0, \sqrt{3}\, a)$ and $(0, -\sqrt{3}\, a)$ respectively.

LEVEL-2

EXAMPLE 2 *Find the coordinates of the vertices of an equilateral triangle of side 2a as shown in Fig. 6.5.*

SOLUTION Since OAB is an equilateral triangle of side $2a$. Therefore,

$$OA = AB = OB = 2a$$

Let BL perpendicular from B on OA. Then,

$$OL = LA = a$$

In $\Delta\, OLB$, we have

$OB^2 = OL^2 + LB^2$

$\Rightarrow \quad (2a)^2 = a^2 + LB^2$

$\Rightarrow \quad LB^2 = 3a^2$

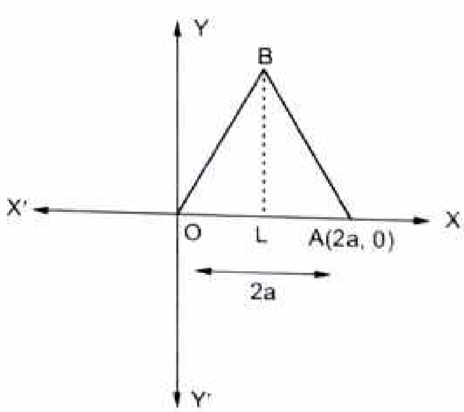


Fig. 6.5

$\Rightarrow \quad LB = \sqrt{3}\, a$

Clearly, coordinates of O are $(0, 0)$ and that of A are $(2a, 0)$. Since $OL = a$ and $LB = \sqrt{3}\, a$. So, the coordinates of B are $(a, \sqrt{3}\, a)$.

EXERCISE 6.1

LEVEL-1

1. On which axis do the following points lie?
   (i) $P(5, 0)$ (ii) $Q(0 -2)$ (iii) $R(-4, 0)$ (iv) $S(0, 5)$
2. Let $ABCD$ be a square of side $2a$. Find the coordinates of the vertices of this square when
   (i) A coincides with the origin and AB and AD are along OX and OY respectively.
   (ii) The centre of the square is at the origin and coordinate axes are parallel to the sides AB and AD respectively.

LEVEL-2

3. The base PQ of two equilateral triangles PQR and PQR' with side $2a$ lies along y-axis such that the mid-point of PQ is at the origin. Find the coordinates of the vertices R and R' of the triangles.

ANSWERS

1. P on x-axis, Q on y-axis, R on x-axis, S on y-axis
2. (i) $A(0,0)$, $B(2a,0)$, $C(2a,2a)$, $D(0,2a)$ (ii) $A(a,a)$, $B(-a,a)$, $C(-a,-a)$, $D(a,-a)$
3. $R(\sqrt{3}a, 0)$, $R'(-\sqrt{3}\, a, 0)$

6.3 DISTANCE BETWEEN TWO POINTS

The distance between any two points in the plane is the length of the line segment joining them.

THEOREM *The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., $$PQ = \sqrt{(\text{Difference of abscissae})^2 + (\text{Difference of ordinates})^2}$$

PROOF Let $X'OX$ and $Y'OY$ be the coordinate axes. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points in the plane. Draw PL and QM perpendicular from P and Q on x-axis. From P draw PN perpendicular to QM. Then,

$$OL = x_1, OM = x_2, PL = y_1 \text{ and } QM = y_2$$

$\therefore \quad PN = LM = OM - OL = x_2 - x_1$

and, $\quad QN = QM - NM = QM - PL = y_2 - y_1$

Clearly, $\Delta\, PNQ$ is a right triangle right angled at N. Therefore, by Pythagoras theorem, we have

$$PQ^2 = PN^2 + QN^2$$

$\Rightarrow \quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\Rightarrow \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

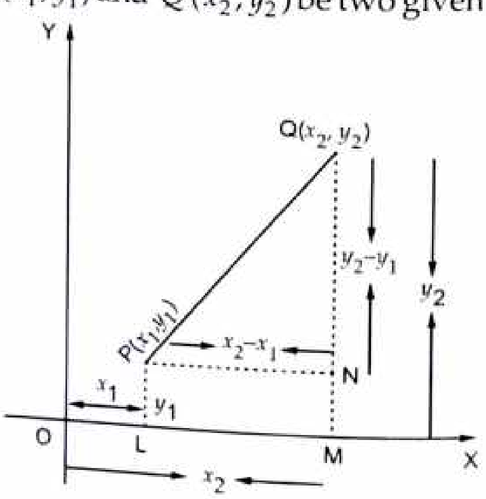


Fig. 6.6

Hence, distance btween any two points is given by

$$\sqrt{(\text{Diff. of abscissae})^2 + (\text{Diff. of ordinates})^2}$$

Q.E.D

NOTE *If O is the origin and P (x, y) is any point, then from the above formula, we have*

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

SOME USEFUL POINTS

(I) *In order to prove that a given figure is a*

(i) *square, prove that the four sides are equal and the diagonals are also equal.*

(ii) *rhombus, prove that the four sides are equal.*

(iii) *rectangle, prove that opposite sides are equal and the diagonals are also equal.*

(iv) *parallelogram, prove that the opposite sides are equal.*

(v) *parallelogram but not a rectangle, prove that its opposite sides are equal but the diagonals are not equal.*

(vi) *rhombus but not a square, prove that its all sides are equal but the diagonals are not equal.*

(II) *For three points to be collinear, prove that the sum of the distances between two pairs of points is equal to the third pair of points.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the points*

(i) $P(-6, 7)$ *and* $Q(-1, -5)$ (ii) $R(a+b, a-b)$ *and* $S(a-b, -a-b)$

(iii) $A(at_1^2, 2at_1)$ *and* $B(at_2^2, 2at_2)$

SOLUTION (i) Here, $x_1 = -6,\ y_1 = 7$ and $x_2 = -1,\ y_2 = -5$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow \quad PQ = \sqrt{(-1+6)^2 + (-5-7)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

(ii) Using distance formula, we obtain

$$RS = \sqrt{(a-b-a-b)^2 + (-a-b-a+b)^2} = \sqrt{4b^2 + 4a^2} = 2\sqrt{a^2 + b^2}$$

(iii) Using distance formula, we obtain

$$AB = \sqrt{\left(at_2^2 - at_1^2\right)^2 + \left(2at_2 - 2at_1\right)^2}$$

$$\Rightarrow \quad AB = \sqrt{a^2(t_2 - t_1)^2(t_2 + t_1)^2 + 4a^2(t_2 - t_1)^2}$$

$$\Rightarrow \quad AB = a(t_2 - t_1)\sqrt{(t_2 + t_1)^2 + 4}$$

EXAMPLE 2 *If the point* (x, y) *is equidistant from the points* $(a+b, b-a)$ *and* $(a-b, a+b)$, *prove that* $bx = ay$.

SOLUTION Let $P(x, y)$, $Q(a+b, b-a)$ and $R(a-b, a+b)$ be the given points. Then,

$$PQ = PR \quad \text{[Given]}$$

$$\Rightarrow \quad \sqrt{\{x-(a+b)\}^2 + \{y-(b-a)\}^2} = \sqrt{\{x-(a-b)\}^2 + \{y-(a+b)\}^2}$$

$$\Rightarrow \quad \{x-(a+b)\}^2 + \{y-(b-a)\}^2 = \{x-(a-b)\}^2 + \{y-(a+b)\}^2$$

$$\Rightarrow \quad x^2 - 2x(a+b) + (a+b)^2 + y^2 - 2y(b-a) + (b-a)^2$$
$$= x^2 + (a-b)^2 - 2x(a-b) + y^2 - 2y(a+b) + (a+b)^2$$
$$\Rightarrow \quad -2x(a+b) - 2y(b-a) = -2x(a-b) - 2y(a+b)$$
$$\Rightarrow \quad ax + bx + by - ay = ax - bx + ay + by$$
$$\Rightarrow \quad 2bx = 2ay \Rightarrow bx = ay$$

REMARK *We know that a point which is equidistant from points P and Q lies on the perpendicular bisector of PQ. Therefore, $bx = ay$ is the equation of the perpendicular bisector of PQ.*

EXAMPLE 3 *Find the equation of the perpendicular bisector of AB, where A and B are the points (3, 6) and (–3, 4) respectively. Also, find its point of intersecction with (i) x-axis (ii) y-axis.*

SOLUTION Let $P(x, y)$ be any point on the perpendicular bisector of AB. Then,

$$PA = PB$$
$$\Rightarrow \quad \sqrt{(x-3)^2 + (y-6)^2} = \sqrt{(x+3)^2 + (y-4)^2}$$
$$\Rightarrow \quad (x-3)^2 + (y-6)^2 = (x+3)^2 + (y-4)^2$$
$$\Rightarrow \quad x^2 - 6x + 9 + y^2 - 12y + 36 = x^2 + 6x + 9 + y^2 - 8y + 16$$
$$\Rightarrow \quad 12x + 4y - 20 = 0$$
$$\Rightarrow \quad 3x + y - 5 = 0 \qquad \ldots(i)$$

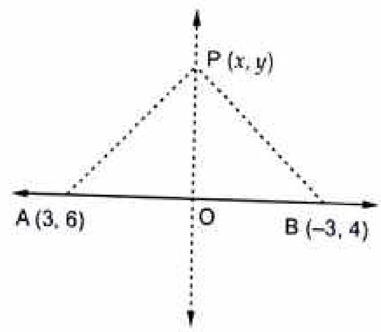


Fig. 6.7

Hence, the equation of the perpendicular bisector of AB is $3x + y - 5 = 0$.

(i) We know that the coordinates of any point on x-axis are of the form $(x, 0)$. In other words, y-coordinate of every point on x-axis is zero. So, putting $y = 0$ in (i), we get

$$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$$

Thus, the perpendicular bisector of AB cuts x-axis at $(5/3, 0)$.

(ii) The coordinates of any point on y-axis are of the form $(0, y)$. Putting $x = 0$ in (i), we get

$$y - 5 = 0 \Rightarrow y = 5$$

Thus, the perpendicular bisector of AB intersects y-axis at $(0, 5)$.

EXAMPLE 4 *Find the value of x, if the distance between the points (x, –1) and (3, 2) is 5.*

SOLUTION Let $P(x, -1)$ and $Q(3, 2)$ be the given points. Then,

$$PQ = 5$$
$$\Rightarrow \quad \sqrt{(x-3)^2 + (-1-2)^2} = 5 \qquad \text{[Given]}$$
$$\Rightarrow \quad (x-3)^2 + 9 = 5^2$$

$\Rightarrow \quad x^2 - 6x + 18 = 25$

$\Rightarrow \quad x^2 - 6x - 7 = 0 \Rightarrow (x-7)(x+1) = 0 \Rightarrow x = 7$ or, $x = -1$

EXAMPLE 5 *If the points $A(4, 3)$ and $B(x, 5)$ are on the circle with centre $O(2, 3)$, find the value of x.* **[CBSE 2009]**

SOLUTION Since A and B lie on the circle having centre O.

$\therefore \quad OA = OB$ [Each equal to radius]

$\Rightarrow \quad \sqrt{(4-2)^2 + (3-3)^2} = \sqrt{(x-2)^2 + (5-3)^2}$

$\Rightarrow \quad 2 = \sqrt{(x-2)^2 + 4}$

$\Rightarrow \quad 4 = (x-2)^2 + 4 \Rightarrow (x-2)^2 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2.$

EXAMPLE 6 *Find a point on x-axis which is equidistant from $A(2, -5)$ and $B(-2, 9)$.* **[NCERT, CBSE 2009, 2017]**

SOLUTION We know that a point on x-axis is of the form $(x, 0)$. So, let $P(x, 0)$ be the point equidistant from $A(2, -5)$ and $B(-2, 9)$. Then,

$PA = PB$

$\Rightarrow \quad \sqrt{(x-2)^2 + (0+5)^2} = \sqrt{(x+2)^2 + (0-9)^2}$

$\Rightarrow \quad (x-2)^2 + 25 = (x+2)^2 + 81$

$\Rightarrow \quad x^2 - 4x + 4 + 25 = x^2 + 4x + 4 + 81 \Rightarrow -8x = 56 \Rightarrow x = -7$

Hence, the required point is $(-7, 0)$.

EXAMPLE 7 *Find a point on the y-axis which is equidistant from the point $A(6, 5)$ and $B(-4, 3)$.* **[NCERT, CBSE 2017]**

SOLUTION We know that a point on y-axis is of the form $(0, y)$. So, let the required point be $P(0, y)$. Then,

$PA = PB$

$\Rightarrow \quad \sqrt{(0-6)^2 + (y-5)^2} = \sqrt{(0+4)^2 + (y-3)^2}$

$\Rightarrow \quad 36 + (y-5)^2 = 16 + (y-3)^2$

$\Rightarrow \quad 36 + y^2 - 10y + 25 = 16 + y^2 - 6y + 9 \Rightarrow 4y = 36 \Rightarrow y = 9$

So, the required point is $(0, 9)$.

EXAMPLE 8 *The x-coordinate of a point P is twice its y-coordinate. If P is equidistant from $Q(2, -5)$ and $R(-3, 6)$, then find the coordinates of P.* **[CBSE 2010, 2016]**

SOLUTION Let the coordinates of P be (x, y). It is given that $x = 2y$. It is also given that

$PQ = PR$

$\Rightarrow \quad \sqrt{(x-2)^2 + (y+5)^2} = \sqrt{(x+3)^2 + (y-6)^2}$

$\Rightarrow \quad \sqrt{(2y-2)^2 + (y+5)^2} = \sqrt{(2y+3)^2 + (y-6)^2}$

$\Rightarrow \quad \sqrt{5y^2 + 2y + 29} = \sqrt{5y^2 + 45}$

$\Rightarrow \quad 5y^2 + 2y + 29 = 5y^2 + 45 \Rightarrow 2y = 16 \Rightarrow y = 8$

Hence, the coordinates of P are $(16, 8)$.

EXAMPLE 9 *Do the points A (3, 2), B (–2, –3) and C (2, 3) form a triangle? If so, name the type of triangle formed.* **[NCERT]**

SOLUTION Using distance formula, we obtain

$$AB = \sqrt{(-2-3)^2 + (-3-2)^2} = \sqrt{25+25} = \sqrt{50}$$

$$BC = \sqrt{(2+2)^2 + (3+3)^2} = \sqrt{16+36} = \sqrt{52}$$

and, $$AC = \sqrt{(2-3)^2 + (3-2)^2} = \sqrt{1+1} = \sqrt{2}$$

Clearly, $AB + BC > AC$, $AC + BC > AB$ and $AB + AC > BC$. Therefore, points A, B and C form a triangle. We also observe that $BC^2 = AB^2 + AC^2$. Therefore, ΔABC is a right triangle, right angled at A.

EXAMPLE 10 *Show that the points* $(a, a), (-a, -a)$ *and* $(-\sqrt{3}a, \sqrt{3}a)$ *are the vertices of an equilateral triangle. Also, find its area.* **[CBSE 2015]**

SOLUTION Let $A(a, a)$, $B(-a, -a)$ and $C(-\sqrt{3}a, \sqrt{3}a)$ be the given points. Then,

$$AB = \sqrt{(-a-a)^2 + (-a-a)^2} = \sqrt{4a^2 + 4a^2} = 2\sqrt{2}\,a$$

$$BC = \sqrt{(-\sqrt{3}\,a + a)^2 + (\sqrt{3}\,a + a)^2}$$

$$\Rightarrow \quad BC = \sqrt{a^2(1-\sqrt{3})^2 + a^2(\sqrt{3}+1)^2}$$

$$\Rightarrow \quad BC = a\sqrt{(1-\sqrt{3})^2 + (1+\sqrt{3})^2}$$

$$\Rightarrow \quad BC = a\sqrt{1+3-2\sqrt{3}+1+3+2\sqrt{3}} = a\sqrt{8} = 2\sqrt{2}\,a$$

and, $$AC = \sqrt{(-\sqrt{3}\,a - a)^2 + (\sqrt{3}\,a - a)^2}$$

$$\Rightarrow \quad AC = \sqrt{a^2(\sqrt{3}+1)^2 + a^2(\sqrt{3}-1)^2}$$

$$\Rightarrow \quad AC = a\sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$$

$$\Rightarrow \quad AC = a\sqrt{3+1+2\sqrt{3}+3+1-2\sqrt{3}} = a\sqrt{8} = 2\sqrt{2}\,a$$

Clearly, we have

$$AB = BC = AC$$

Hence, the triangle ABC formed by the given points is an equilateral triangle.

Now,

$$\text{Area of } \Delta\, ABC = \frac{\sqrt{3}}{4}(\text{Side})^2$$

$$= \frac{\sqrt{3}}{4} \times AB^2 = \frac{\sqrt{3}}{4} \times (2\sqrt{2}a)^2 \text{ sq. units} = 2\sqrt{3}a^2 \text{ sq. units}$$

EXAMPLE 11 *Show that the points* $(1, -1), (5, 2)$ *and* $(9, 5)$ *are collinear.* **[CBSE 2006 C]**

SOLUTION Let $A(1, -1)$, $B(5, 2)$ and $(9, 5)$ be the given points. Then, we have

$$AB = \sqrt{(5-1)^2 + (2+1)^2} = \sqrt{16+9} = 5$$

$$BC = \sqrt{(5-9)^2 + (2-5)^2} = \sqrt{16+9} = 5$$

and, $$AC = \sqrt{(1-9)^2 + (-1-5)^2} = \sqrt{64+36} = 10$$

Clearly, $AC = AB + BC$. Hence, A, B, C are collinear points.

EXAMPLE 12 *Show that four points (0, –1), (6, 7), (–2, 3) and (8, 3) are the vertices of a rectangle. Also, find its area.* **[CBSE 2013]**

SOLUTION Let $A(0-1)$, $B(6,7)$, $C(-2,3)$ and $D(8,3)$ be the given points. Then,

$$AD = \sqrt{(8-0)^2 + (3+1)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$BC = \sqrt{(6+2)^2 + (7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2 + (3+1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

and, $BD = \sqrt{(8-6)^2 + (3-7)^2} = \sqrt{4+16} = 2\sqrt{5}$

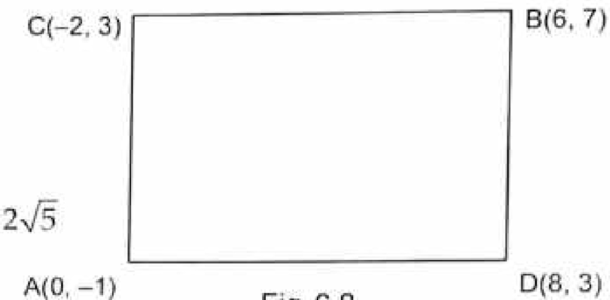


Fig. 6.8

$\therefore$ $AD = BC$ and $AC = BD$

So, $ADBC$ is a parallelogram.

Now, $AB = \sqrt{(6-0)^2 + (7+1)^2} = \sqrt{36+64} = 10$ and, $CD = \sqrt{(8+2)^2 + (3-3)^2} = 10$

Clearly, $AB^2 = AD^2 + DB^2$ and $CD^2 = CB^2 + BD^2$.

Hence, $ADBC$ is a rectangle.

Area of rectangle $ADBC = AD \times DB = (4\sqrt{5} \times 2\sqrt{5})$ sq. units $= 40$ sq. units

EXAMPLE 13 *Show that $A(6,4)$, $B(5,-2)$ and $C(7,-2)$ are the vertices of an isosceles triangle. Also, find the length of the median through A.* **[CBSE 2010]**

SOLUTION We have

$$AB = \sqrt{(6-5)^2 + (4+2)^2} = \sqrt{37}, \quad AC = \sqrt{(6-7)^2 + (4+2)^2} = \sqrt{37}$$

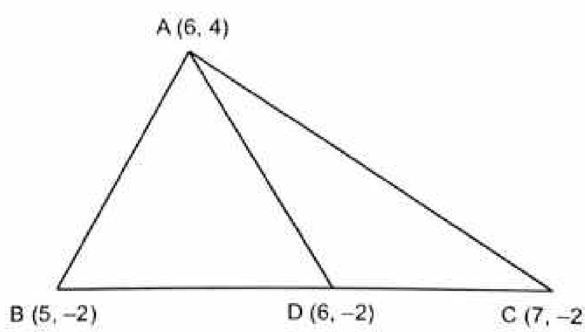


Fig. 6.9

$\therefore$ $AB = AC$

So, ΔABC is isosceles.

Let D be the mid-point of BC. Then, coordinates of D are $\left(\frac{5+7}{2}, \frac{-2-2}{2}\right)$ i.e. $(6,-2)$.

$\therefore$ $AD = \sqrt{(6-6)^2 + (4+2)^2} = \sqrt{36} = 6$

EXAMPLE 14 *If $P(2,-1)$, $Q(3,4)$, $R(-2,3)$ and $S(-3,-2)$ be four points in a plane, show that PQRS is a rhombus but not a square. Find the area of the rhombus.* **[CBSE 2013]**

SOLUTION The given points are $P(2,-1)$, $Q(3,4)$, $R(-2,3)$ and $S(-3,-2)$.

We have,

$$PQ = \sqrt{(3-2)^2 + (4+1)^2} = \sqrt{1^2 + 5^2} = \sqrt{26} \text{ units}$$

$$QR = \sqrt{(-2-3)^2 + (3-4)^2} = \sqrt{25+1} = \sqrt{26} \text{ units}$$

$$RS = \sqrt{(-3+2)^2 + (-2-3)^2} = \sqrt{1+25} = \sqrt{26} \text{ units}$$

$$SP = \sqrt{(-3-2)^2 + (-2+1)^2} = \sqrt{26} \text{ units}$$

$$PR = \sqrt{(-2-2)^2 + (3+1)^2} = \sqrt{16+16} = 4\sqrt{2} \text{ units}$$

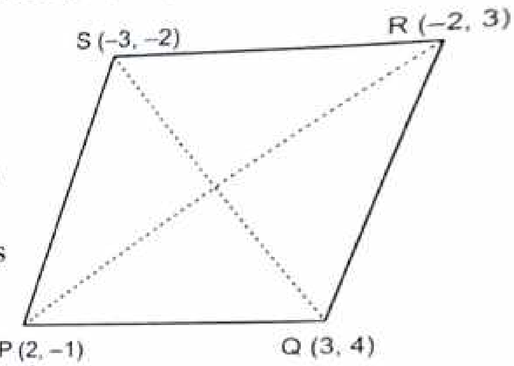


Fig. 6.10

and, $$QS = \sqrt{(-3-3)^2 + (-2-4)^2} = \sqrt{36+36} = 6\sqrt{2} \text{ units}$$

$\therefore$ $$PQ = QR = RS = SP = \sqrt{26} \text{ units}$$

and, $PR \neq QS$

This means that $PQRS$ is a quadrilateral whose sides are equal but diagonals are not equal. Thus, $PQRS$ is a rhombus but not a square.

Now, Area of rhombus $PQRS = \frac{1}{2} \times$ (Product of lengths of diagonals)

$$= \frac{1}{2} \times (PR \times QS) = \left(\frac{1}{2} \times 4\sqrt{2} \times 6\sqrt{2}\right) \text{sq. units} = 24 \text{ sq. units}$$

EXAMPLE 15 *Find the coordinates of the centre of the circle passing through the points $(0, 0)$, $(-2, 1)$ and $(-3, 2)$. Also, find its radius.*

SOLUTION Let $P(x, y)$ be the centre of the circle passing through the points $O(0, 0)$, $A(-2, 1)$ and $B(-3, 2)$. Then,

$$OP = AP = BP$$

Now, $OP = AP$

$\Rightarrow$ $OP^2 = AP^2$

$\Rightarrow$ $x^2 + y^2 = (x+2)^2 + (y-1)^2$

$\Rightarrow$ $x^2 + y^2 = x^2 + y^2 + 4x - 2y + 5$

$\Rightarrow$ $4x - 2y + 5 = 0$ ...(i)

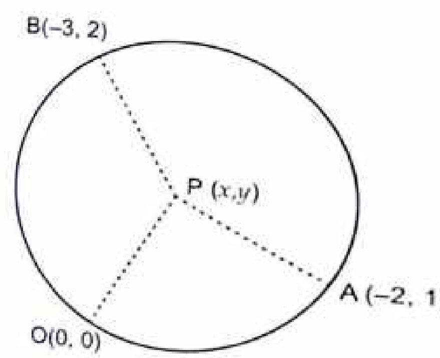


Fig. 6.11

and, $OP = BP$

$\Rightarrow$ $OP^2 = BP^2$

$\Rightarrow$ $x^2 + y^2 = (x+3)^2 + (y-2)^2$

$\Rightarrow$ $x^2 + y^2 = x^2 + y^2 + 6x - 4y + 13$

$\Rightarrow$ $6x - 4y + 13 = 0$ ...(ii)

On solving equations (i) and (ii), we get: $x = \frac{3}{2}$ and $y = \frac{11}{2}$.

Thus, the coordinates of the centre are $\left(\frac{3}{2}, \frac{11}{2}\right)$

$\therefore$ Radius $= OP = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{4} + \frac{121}{4}} = \frac{1}{2}\sqrt{130}$ units.

EXAMPLE 16 *If $(-4, 0)$ and $(4,0)$ are two vertices of an equilateral triangle, find the coordinates of its third vertex.* **[CBSE 2014]**

SOLUTION Let $C(x, y)$ be the third vertex of triangle ABC having two vertices at $A(-4,0)$ and $B(4, 0)$. Since ΔABC is equilateral. Therefore,

$$AC = BC = AB$$

Now, $AC = BC$

$\Rightarrow \sqrt{(x+4)^2 + (y-0)^2} = \sqrt{(x-4)^2 + (y-0)^2}$

$\Rightarrow (x+4)^2 + y^2 = (x-4)^2 + y^2$

$\Rightarrow 16x = 0$

$\Rightarrow x = 0$

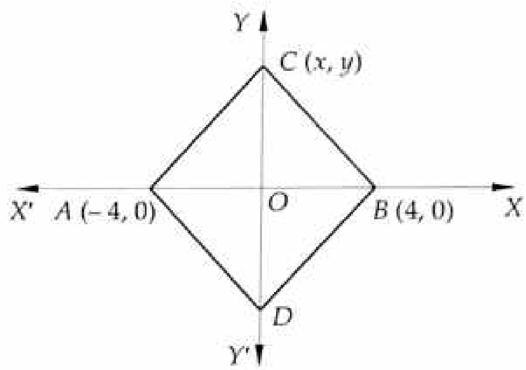


Fig. 6.12

Again,

$AC = BC = AB$

$\Rightarrow AC = AB$

$\Rightarrow \sqrt{(x+4)^2 + (y-0)^2} = \sqrt{(4+4)^2 + 0^2}$

$\Rightarrow (x+4)^2 + y^2 = 64$

$\Rightarrow (0+4)^2 + y^2 = 64$ $[\because x = 0]$

$\Rightarrow y^2 = 48$

$\Rightarrow y = \pm 4\sqrt{3}$

Hence, the coordinates of the third vertex are $C(0, 4\sqrt{3})$ and $D(0, -4\sqrt{3})$.

EXAMPLE 17 *Points $A(-1, y)$ and $B(5, 7)$ lie on a circle with centre $O(2, -3y)$. Find the values of y. Hence, find the radius of the circle.* **[CBSE 2014]**

SOLUTION Since O is the centre of the circle and A, B are points on its circumference.

$\therefore OA = OB =$ Radius

$\Rightarrow OA = OB$

$\Rightarrow \sqrt{(2+1)^2 + (-3y-y)^2} = \sqrt{(2-5)^2 + (-3y-7)^2}$

$\Rightarrow 9 + 16y^2 = 9 + (3y+7)^2$

$\Rightarrow 16y^2 = 9y^2 + 42y + 49$

$\Rightarrow 7y^2 - 42y - 49 = 0$

$\Rightarrow y^2 - 6y - 7 = 0$

$\Rightarrow (y-7)(y+1) = 0 \Rightarrow y = -1, 7$

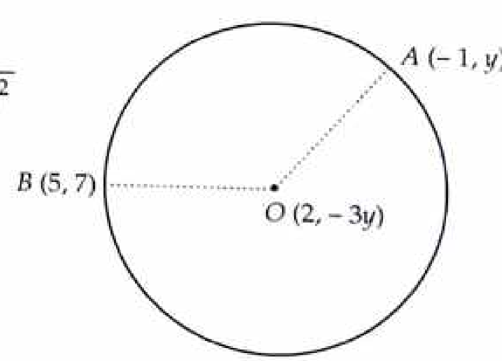


Fig. 6.13

CASE I *When* $y = -1$: In this case

The coordinates of O, A and B are $O(2, 3)$, $A(-1, -1)$ and $B(5, 7)$ respectively.

$\therefore$ Radius $= OA = \sqrt{(2+1)^2 + (3+1)^2} = 5$

CASE II *When* $y = 7$ *:* In this case

The coordinates O, A and B are $O(2, -21)$, $A(-1, 7)$ and $B(5, 7)$ respectively.

$\therefore$ $$\text{Radius} = OA = \sqrt{(2+1)^2 + (-21-7)^2} = \sqrt{9 + 784} = \sqrt{793}$$

EXAMPLE 18 *If* $A(5, 2)$, $B(2, -2)$ *and* $C(-2, t)$ *are the vertices of right angled triangle w* $\angle B = 90°$ *, then find the value of* t. [CBSE 201

SOLUTION Using Pythagoras theorem in right triangle ABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $$(5+2)^2 + (2-t)^2 = \{(5-2)^2 + (2+2)^2\} + \{(2+2)^2 + (-2-t)^2\}$$

$\Rightarrow$ $$49 + (4 - 4t + t^2) = (9 + 16) + (16 + 4 + 4t + t^2)$$

$\Rightarrow$ $$t^2 - 4t + 53 = t^2 + 4t \times 45$$

$\Rightarrow$ $$-8t = -8$$

$\Rightarrow$ $$t = 1$$

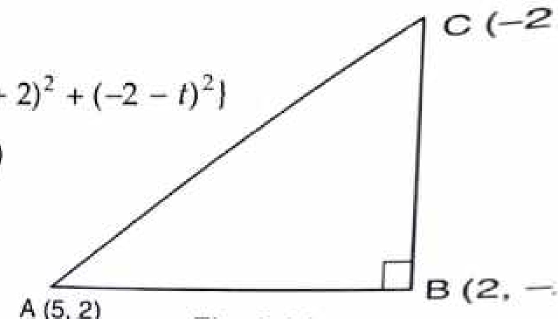


Fig. 6.14

LEVEL-2

EXAMPLE 19 *If P and Q are two points whose coordinates are* $(at^2, 2at)$ *and* $\left(\frac{a}{t^2}, \frac{2a}{t}\right)$ *respectively and S is the point* $(a, 0)$*. Show that* $\frac{1}{SP} + \frac{1}{SQ}$ *is independent of* t.

SOLUTION Using distance formula, we obtain

$$SP = \sqrt{(at^2 - a)^2 + (2at - 0)^2} = a\sqrt{(t^2 - 1)^2 + 4t^2} = a(t^2 + 1)$$

and, $$SQ = \sqrt{\left(\frac{a}{t^2} - a\right)^2 + \left(\frac{2a}{t} - 0\right)^2}$$

$\Rightarrow$ $$SQ = \sqrt{\frac{a^2(1-t^2)^2}{t^4} + \frac{4a^2}{t^2}} = \frac{a}{t^2}\sqrt{(1-t^2)^2 + 4t^2} = \frac{a}{t^2}\sqrt{(1+t^2)^2} = \frac{a}{t^2}(1 + t^2)$$

$\therefore$ $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t^2+1)} + \frac{t^2}{a(t^2+1)}$$

$\Rightarrow$ $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1 + t^2}{a(t^2+1)} = \frac{1}{a}$$, which is independent of t.

EXAMPLE 20 *If two vertices of an equilateral triangle be* $(0, 0)$, $(3, \sqrt{3})$*, find the third vertex.*

SOLUTION $O(0, 0)$ and $A(3, \sqrt{3})$ be the given points and let $B(x, y)$ be the third vertex of equilateral ΔOAB. Then,

$$OA = OB = AB$$

$\Rightarrow$ $$OA^2 = OB^2 = AB^2$$

We have, $$OA^2 = (3-0)^2 + (\sqrt{3} - 0)^2 = 12,$$

$$OB^2 = x^2 + y^2$$

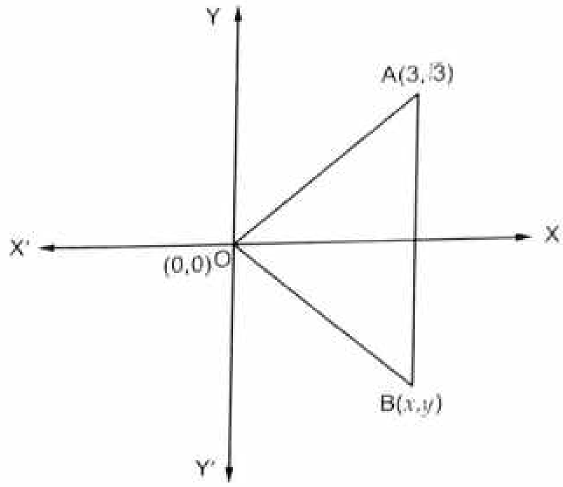


Fig. 6.15

and, $AB^2 = (x-3)^2 + (y-\sqrt{3})^2$

$\Rightarrow$ $AB^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

$\therefore$ $OA^2 = OB^2 = AB^2$

$\Rightarrow$ $OA^2 = OB^2$ and $OB^2 = AB^2$

$\Rightarrow$ $x^2 + y^2 = 12$

and, $x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $6x + 2\sqrt{3}\,y = 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $3x + \sqrt{3}\,y = 6$

$\Rightarrow$ $x^2 + \left(\dfrac{6-3x}{\sqrt{3}}\right)^2 = 12$ $\left[\because\ 3x + \sqrt{3}\,y = 6 \therefore\ y = \dfrac{6-3x}{\sqrt{3}}\right]$

$\Rightarrow$ $3x^2 + (6-3x)^2 = 36$

$\Rightarrow$ $12x^2 - 36x = 0$

$\Rightarrow$ $x = 0, 3$

$\therefore$ $x = 0 \Rightarrow \sqrt{3}\,y = 6 \Rightarrow y = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$ $\left[\text{Putting } x = 0 \text{ in } 3x + \sqrt{3}y = 6\right]$

and, $x = 3 \Rightarrow 9 + \sqrt{3}\,y = 6 \Rightarrow y = \dfrac{6-9}{\sqrt{3}} = -\sqrt{3}$ $\left[\text{Putting } x = 3 \text{ in } 3x + \sqrt{3}y = 6\right]$

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3})$ or, $(3, -\sqrt{3})$.

EXAMPLE 21 *Find the coordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2). Also, find its circum-radius.*

SOLUTION Recall that the circumcentre of a triangle is equidistant from the vertices of a triangle. Let $A(8, 6)$, $B(8, -2)$ and $C(2, -2)$ be the vertices of the given triangle and let $P(x, y)$ be the circumcentre of this triangle. Then,

$PA = PB = PC$

$\Rightarrow$ $PA^2 = PB^2 = PC^2$

Now, $PA^2 = PB^2$

$\Rightarrow$ $(x-8)^2 + (y-6)^2 = (x-8)^2 + (y+2)^2$

$\Rightarrow$ $x^2 + y^2 - 16x - 12y + 100 = x^2 + y^2 - 16x + 4y + 68$

$\Rightarrow$ $16y = 32$

$\Rightarrow$ $y = 2$

and, $PB^2 = PC^2$

$\Rightarrow$ $(x-8)^2 + (y+2)^2 = (x-2)^2 + (y+2)^2$

$\Rightarrow$ $x^2 + y^2 - 16x + 4y + 68 = x^2 + y^2 - 4x + 4y + 8$

$\Rightarrow$ $12x = 60$

$\Rightarrow$ $x = 5$

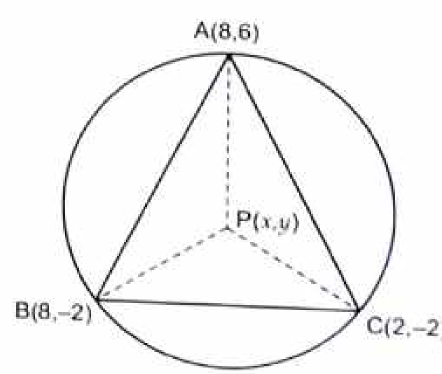


Fig. 6.16

So, the coordinates of the circumcentre P are (5, 2).

Also, Circum-radius $= PA = PB = PC = \sqrt{(5-8)^2 + (2-6)^2} = 5$

EXAMPLE 22 *Let the opposite angular points of a square be (3, 4) and (1, –1). Find the coordinates of the remaining angular points.*

SOLUTION Let $ABCD$ be a square and let $A(3, 4)$ and $C(1, -1)$ be the given angular points. Let $B(x, y)$ be the unknown vertex.

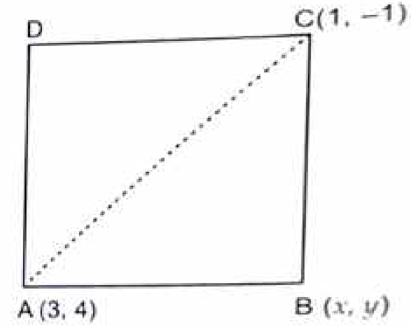


Fig. 6.17

Then, $AB = BC$

$\Rightarrow \quad AB^2 = BC^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$

$\Rightarrow \quad 4x + 10y - 23 = 0$

$\Rightarrow \quad x = \dfrac{23 - 10y}{4}$ ...(i)

In right-angled triangle ABC, we have

$$AB^2 + BC^2 = AC^2$$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2 = (3-1)^2 + (4+1)^2$ ...(ii)

$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0$

Substituting the value of x from (i) into (ii), we get

$$\left(\frac{23-10y}{4}\right)^2 + y^2 - (23 - 10y) - 3y - 1 = 0$$

$\Rightarrow \quad 4y^2 - 12y + 5 = 0 \Rightarrow (2y-1)(2y-5) = 0 \Rightarrow y = \dfrac{1}{2}$ or, $\dfrac{5}{2}$

Putting $y = \dfrac{1}{2}$ and $y = \dfrac{5}{2}$ respectively in (i), we get $x = \dfrac{9}{2}$ and $x = \dfrac{-1}{2}$ respectively.

Hence, the required vertices of the square are $(9/2, 1/2)$ and $(-1/2, 5/2)$.

EXAMPLE 23 *Prove that the points (–3, 0), (1, –3) and (4, 1) are the vertices of an isosceles right-angled triangle. Find the area of this triangle.*

SOLUTION Let $A(-3, 0)$, $B(1, -3)$ and $C(4, 1)$ be the given points. Then,

$$AB = \sqrt{\{1-(-3)\}^2 + (-3-0)^2} = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = 5 \text{ units}$$

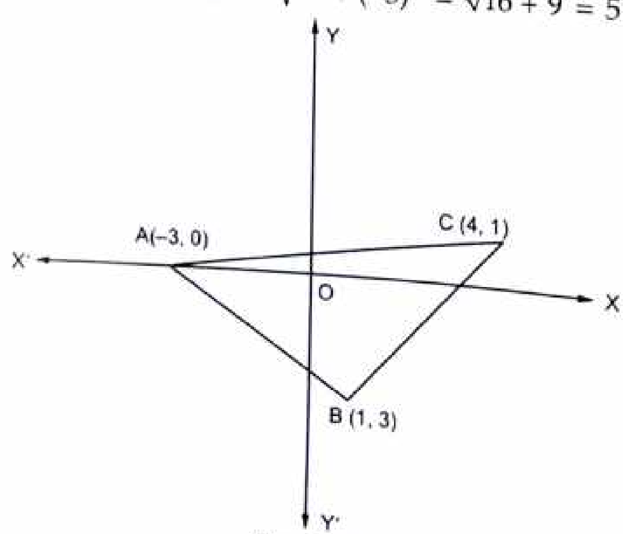


Fig. 6.18

$$BC = \sqrt{(4-1)^2 + (1+3)^2} = \sqrt{9+16} = 5 \text{ units}$$

and, $\quad CA = \sqrt{(4+3)^2 + (1-0)^2} = \sqrt{49+1} = 5\sqrt{2}$ units.

Clearly, $\quad AB = BC$. Therefore, ΔABC is isosceles.

Also, $\quad AB^2 + BC^2 = 25 + 25 = (5\sqrt{2})^2 = CA^2$

$\therefore \quad \Delta ABC$ is right-angled at B.

Thus, ΔABC is a right-angled isosceles triangle.

Now, Area of $\Delta ABC = \frac{1}{2}(\text{Base} \times \text{Height}) = \frac{1}{2}(AB \times BC) = \left(\frac{1}{2} \times 5 \times 5\right)$ sq. units $= \frac{25}{2}$ sq. units

EXERCISE 6.2

LEVEL-1

1. Find the distance between the following pair of points :
   (i) $(-6,7)$ and $(-1,-5)$
   (ii) $(a+b, b+c)$ and $(a-b, c-b)$
   (iii) $(a \sin\alpha, -b\cos\alpha)$ and $(-a\cos\alpha, b\sin\alpha)$
   (iv) $(a, 0)$ and $(0, b)$
2. Find the value of a when the distance between the points $(3, a)$ and $(4, 1)$ is $\sqrt{10}$.
3. If the points $(2, 1)$ and $(1,-2)$ are equidistant from the point (x, y), show that $x + 3y = 0$.
4. Find the values of x, y if the distances of the point (x, y) from $(-3, 0)$ as well as from $(3, 0)$ are 4.
5. The length of a line segment is of 10 units and the coordinates of one end-point are $(2,-3)$. If the abscissa of the other end is 10, find the ordinate of the other end.
6. Show that the points $(-4,-1)$, $(-2,-4)$, $(4, 0)$ and $(2, 3)$ are the vertices points of a rectangle. [CBSE 2006 C]
7. Show that the points $A(1,-2)$, $B(3, 6)$, $C(5, 10)$ and $D(3, 2)$ are the vertices of a parallelogram.
8. Prove that the points $A(1,7)$, $B(4,2)$, $C(-1,-1)$ and $D(-4,4)$ are the vertices of a square. [NCERT]
9. Prove that the points $(3, 0)$, $(6, 4)$ and $(-1, 3)$ are vertices of a right-angled isosceles triangle. [CBSE 2006 C]
10. Prove that $(2,-2)$, $(-2,1)$ and $(5,2)$ are the vertices of a right angled triangle. Find the area of the triangle and the length of the hypotenuse. [CBSE 2016]
11. Prove that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle.
12. Prove that the points $(2, 3)$, $(-4, -6)$ and $(1, 3/2)$ do not form a triangle.
13. The points $A(2, 9)$, $B(a, 5)$ and $C(5, 5)$ are the vertices of a triangle ABC right angled at B. Find the values of a and hence the area of ΔABC. [NCERT EXEMPLAR]
14. Show that the quadrilateral whose vertices are $(2, -1)$, $(3, 4)$, $(-2, 3)$ and $(-3, -2)$ is a rhombus.
15. Two vertices of an isosceles triangle are $(2, 0)$ and $(2, 5)$. Find the third vertex if the length of the equal sides is 3.

16. Which point on x-axis is equidistant from (5, 9) and (– 4, 6)?
17. Prove that the points (– 2, 5), (0, 1) and (2, – 3) are collinear.
18. The coordinates of the point P are (–3, 2). Find the coordinates of the point Q which lies on the line joining P and origin such that $OP = OQ$.
19. Which point on y-axis is equidistant from (2, 3) and (– 4, 1)?
20. The three vertices of a parallelogram are (3, 4), (3, 8) and (9, 8). Find the fourth vertex.
21. Find a point which is equidistant from the points A (– 5, 4) and B (–1, 6). How many such points are there? **[NCERT EXEMPLAR]**
22. The centre of a circle is $(2a, a-7)$. Find the values of a if the circle passes through the point (11, –9) and has diameter $10\sqrt{2}$ units. **[NCERT EXEMPLAR]**
23. Ayush starts walking from his house to office. Instead of going to the office directly, he goes to a bank first, from there to his daughter's school and then reaches the office. What is the extra distance travelled by Ayush in reaching the office? (Assume that all distances covered are in straight lines). If the house is situated at (2, 4), bank at (5, 8), school at (13, 14) and office at (13, 26) and coordinates are in kilometers. **[NCERT EXEMPLAR]**
24. Find the value of k, if the point P (0, 2) is equidistant from (3, k) and (k, 5).
25. If (–4, 3) and (4, 3) are two vertices of an equilateral triangle, find the coordinates of the third vertex, given that the origin lies in the (i) interior, (ii) exterior of the triangle. **[NCERT EXEMPLAR]**
26. Show that the points (–3, 2), (–5, –5), (2, –3) and (4, 4) are the vertices of a rhombus. Find the area of this rhombus.
27. Find the coordinates of the circumcentre of the triangle whose vertices are (3, 0), (–1, –6) and (4, –1). Also, find its circumradius.
28. Find a point on the x-axis which is equidistant from the points (7, 6) and (–3, 4). **[CBSE 2005]**
29. (i) Show that the points A(5, 6), B (1, 5), C(2, 1) and D(6, 2) are the vertices of a square. **[CBSE 2004]**
    (ii) Prove that the points A (2, 3), B (–2, 2), C (–1, –2), and D (3, –1) are the vertices of a square $ABCD$. **[CBSE 2013]**
    (iii) Name the type of triangle PQR formed by the points $P(\sqrt{2}, \sqrt{2})$, $Q(-\sqrt{2}, -\sqrt{2})$ and $R(-\sqrt{6}, \sqrt{6})$ **[NCERT EXEMPLAR]**
30. Find the point on x-axis which is equidistant from the points (–2, 5) and (2, –3). **[CBSE 2004]**
31. Find the value of x such that $PQ = QR$ where the coordinates of P, Q and R are (6, –1), (1, 3) and (x, 8) respectively. **[CBSE 2005]**
32. Prove that the points (0, 0), (5, 5) and (–5, 5) are the vertices of a right isosceles triangle. **[CBSE 2005]**
33. If the point $P(x, y)$ is equidistant from the points A(5, 1) and B (1, 5), prove that $x = y$. **[CBSE 2005]**
34. If Q (0, 1) is equaidistant from P (5, –3) and R (x, 6), find the values of x. Also, find the distances QR and PR. **[NCERT]**
35. Find the values of y for which the distance between the points P (2, –3) and Q (10, y) is 10 units. **[NCERT]**
36. If the point $P(k-1, 2)$ is equidistant from the points $A(3, k)$ and $B(k, 5)$, find the values of k. **[CBSE 2014]**

37. If the point $A(0, 2)$ is equidistant from the points $B(3, p)$ and $C(p, 5)$, find p. Also, find the length of AB. [CBSE 2014]

38. Name the quadrilateral formed, if any, by the following points, and give reasons for your answers:

(i) $A(-1, -2), B(1, 0), C(-1, 2), D(-3, 0)$

(ii) $A(-3, 5), B(3, 1), C(0, 3), D(-1, -4)$

(iii) $A(4, 5), B(7, 6), C(4, 3), D(1, 2)$ [NCERT]

39. Find the equation of the perpendicular bisector of the line segment joining points $(7, 1)$ and $(3, 5)$. [NCERT]

40. Prove that the points $(3, 0)$, $(4, 5)$, $(-1, 4)$ and $(-2, -1)$, taken in order, form a rhombus. Also, find its area. [NCERT]

41. In the seating arrangement of desks in a classroom three students Rohini, Sandhya and Bina are seated at $A(3, 1)$, $B(6, 4)$ and $C(8, 6)$. Do you think they are seated in a line?

42. Find a point on y-axis which is equidistant from the points $(5, -2)$ and $(-3, 2)$. [CBSE 2009]

43. Find a relation between x and y such that the point (x, y) is equidistant from the points $(3, 6)$ and $(-3, 4)$. [NCERT]

44. If a point $A(0, 2)$ is equidistant from the points $B(3, p)$ and $C(p, 5)$, then find the value of p. [CBSE 2012, 2013]

45. Prove that the points $(7, 10)$, $(-2, 5)$ and $(3, -4)$ are the vertices of an isosceles right triangle. [CBSE 2013]

46. If the point $P(x, 3)$ is equidistant from the points $A(7, -1)$ and $B(6, 8)$, find the value of x and find the distance AP. [CBSE 2014]

47. If $A(3, y)$ is equidistant from points $P(8, -3)$ and $Q(7, 6)$, find the vaue of y and find the distance AQ. [CBSE 2014]

48. If $(0, -3)$ and $(0, 3)$ are the two vertices of an equilateral triangle, find the coordinates of its third vertex. [CBSE 2014]

49. If the point $P(2, 2)$ is equidistant from the points $A(-2, k)$ and $B(-2k, -3)$, find k. Also, find the length of AP. [CBSE 2014]

50. Show that ΔABC, where $A(-2, 0)$, $B(2, 0)$, $C(0, 2)$ and ΔPQR, where $P(-4, 0)$, $Q(4, 0)$, $R(0, 4)$ are similar. [CBSE 2017]

LEVEL-2

51. An equilateral triangle has two vertices at the points $(3, 4)$ and $(-2, 3)$, find the coordinates of the third vertex.

52. Find the circumcentre of the triangle whose vertices are $(-2, -3)$, $(-1, 0)$, $(7, -6)$.

53. Find the angle subtended at the origin by the line segment whose end points are $(0, 100)$ and $(10, 0)$.

54. Find the centre of the circle passing through $(5, -8)$, $(2, -9)$ and $(2, 1)$.

55. If two opposite vertices of a square are $(5, 4)$ and $(1, -6)$, find the coordinates of its remaining two vertices.

56. Find the centre of the circle passing through $(6, -6)$, $(3, -7)$ and $(3, 3)$.

57. Two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of other two vertices.

ANSWERS

1. (i) 13 (ii) $2\sqrt{2}b$ (iii) $\sqrt{a^2 + b^2}\,(\sin\alpha + \cos\alpha)$ (iv) $\sqrt{a^2 + b^2}$

2. 4, –2

4. $x = 0, y = \pm\sqrt{7}$ 5. 3, –9 10. $\frac{25}{2}$ sq. units, $5\sqrt{2}$

13. $a = 2$, Area = 6 sq. units. 15. $\left(2-\frac{\sqrt{11}}{2}, \frac{5}{2}\right), \left(2+\frac{\sqrt{11}}{2}, \frac{5}{2}\right)$ 16. (3, 0) 18. (3, –2)

19. (0, – 1) 20. (9, 4)

21. (–3, 5). Infinite number of points. Infact all the points which are solutions of the equation $2x + y + 1 = 0$.

22. $a = 5, 3$ 23. 2.4 km 24. 1

25. (i) $(0, 3 - 4\sqrt{3})$ (ii) $(0, 3 + 4\sqrt{3})$ 26. 45 sq. units

27. (1, –3), $\sqrt{13}$ units 28. (3, 0) 29. (iii) Equilateral

30. (–2, 0) 31. 5, – 3

34. $x = -4, 4$; $QR = \sqrt{41}$ units ; $PR = \sqrt{82}, 9\sqrt{2}$ units 35. $y = 3, -9$

37. $p = 1, \sqrt{10}$ 36. $k = 1, 5$

38. (i) Square (ii) Not a quadrilateral (iii) parallelogram

39. $x - y = 2$ 40. 24 sq. units 41. Yes. 42. (0, –2)

43. $3x + y = 5$ 44. $p = 1$ 46. $x = 2, \sqrt{41}$ units

47. $y = 1, AQ = \sqrt{41}$ units 48. $(3\sqrt{3}, 0), (-3\sqrt{3}, 0)$ 49. $k = -1, -3$; $AP = 5$

51. $\left(\frac{1+\sqrt{3}}{2}, \frac{7-5\sqrt{3}}{2}\right), \left(\frac{1-\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2}\right)$ 52. (3, –3) 53. 90°

54. (2, –4) 55. (8, – 3), and (–2, 1) 56. (3, –2) 57. (1, 0) and (1, 4)

6.4 SECTION FORMULAE

Let A and B be two points in the plane of the paper as shown in Fig. 6.19 and P be a point on the segment joining A and B such that $AP : BP = m : n$. Then, we say that the point P divides segment AB internally in the ratio $m : n$.

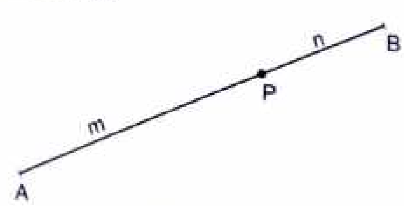


Fig. 6.19

If P is a point on AB produced such that $AP : BP = m : n$, then point P is said to divide AB externally in the ratio $m : n$

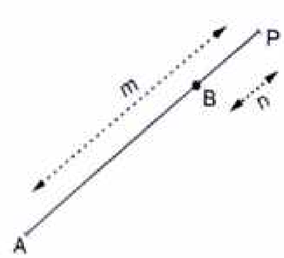


Fig. 6.20

In this section, we shall develop a formula, generally known as section formula, for finding the coordinates of P when we are given the coordinates of A and B and the ratio in which P divides AB internally.

THEOREM *Prove that the coordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ are given by*

$$\left(x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}\right)$$

PROOF Let O be the origin and let OX and OY be the x-axis and y-axis respectively. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points. Let (x, y) be the coordinates of the point P which divides AB internally in the ratio $m : n$. Draw $AL \perp OX$, $BM \perp OX$, $PN \perp OX$. Also, draw AH and PK perpendiculars from A and P on PN and BM respectively. Then,

$$OL = x_1, ON = x, OM = x_2, AL = y_1, PN = y \text{ and } BM = y_2$$

$\therefore$ $AH = LN = ON - OL = x - x_1,\ PH = PN - HN = PN - AL = y - y_1,$

$PK = NM = OM - ON = x_2 - x$

and, $BK = BM - MK = BM - PN = y_2 - y$

Clearly, triangle AHP and PKB are similar.

$$\therefore \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Now, $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x}$

$\Rightarrow$ $mx_2 - mx = nx - nx_1$

$\Rightarrow$ $mx + nx = mx_2 + nx_1$

$\Rightarrow$ $x = \dfrac{mx_2 + nx_1}{m+n}$

and, $\dfrac{m}{n} = \dfrac{y - y_1}{y_2 - y}$

$\Rightarrow$ $my_2 - my = ny - ny_1$

$\Rightarrow$ $my + ny = my_2 + ny_1$

$\Rightarrow$ $y = \dfrac{my_2 + ny_1}{m+n}$

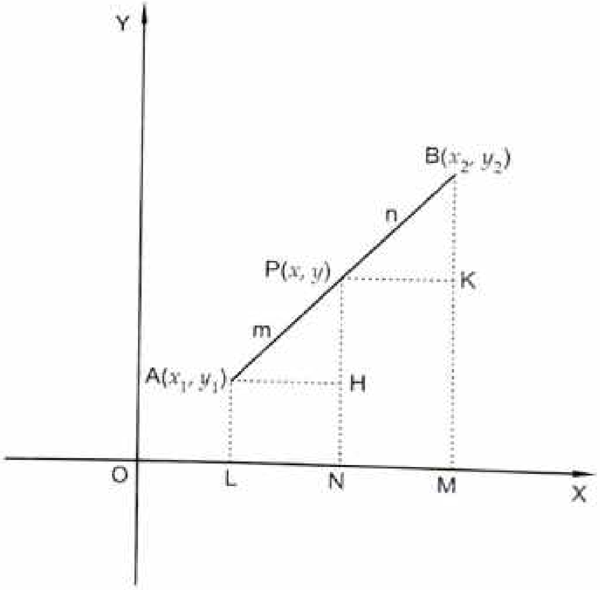


Fig. 6.21

Thus, the coordinates of P are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

NOTE 1 *If P is the mid-point of AB, then it divides AB in the ratio $1 : 1$, so its coordinates are*

$$\left(\frac{1 \cdot x_1 + 1 \cdot x_2}{1+1}, \frac{1 \cdot y_1 + 1 \cdot y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

NOTE 2 *Fig. 6.22 will help to remember the section formula.*

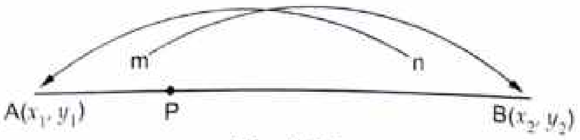


Fig. 6.22

NOTE 3 *The ratio m : n can also be written as* $\frac{m}{n}:1$, or $\lambda : 1$, *where* $\lambda = \frac{m}{n}$.

So, the coordinates of point P dividing the line segment joining the points $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *are*

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) = \left(\frac{\frac{m}{n}x_2 + x_1}{\frac{m}{n}+1}, \frac{\frac{m}{n}y_2 + y_1}{\frac{m}{n}+1}\right) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SECTION POINT WHEN THE SECTION RATIO IS GIVEN**

EXAMPLE 1 *Find the coordinates of the point which divides the line segment joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 internally.*

SOLUTION Let $P(x, y)$ be the required point. Then,

$$x = \frac{3 \times -4 + 2 \times 6}{3+2} \text{ and } y = \frac{3 \times 5 + 2 \times 3}{3+2}$$

$$\Rightarrow \quad x = 0 \text{ and } y = \frac{21}{5}$$

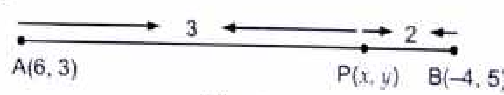


Fig. 6.23

So, the coordinates of P are (0, 21/5).

EXAMPLE 2 *Find the coordinates of points which trisect the line segment joining (1, –2) and (–3, 4).* **[CBSE 2017]**

SOLUTION Let $A(1, -2)$ and $B(-3, 4)$ be the given points. Let the points of trisection be P and Q. Then, $AP = PQ = QB = \lambda$ (say).

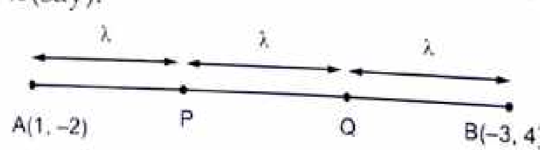


Fig. 6.24

$$\therefore \quad PB = PQ + QB = 2\lambda \text{ and } AQ = AP + PQ = 2\lambda$$

$$\Rightarrow \quad AP : PB = \lambda : 2\lambda = 1 : 2 \text{ and } AQ : QB = 2\lambda : \lambda = 2 : 1$$

So, P divides AB internally in the ratio 1 : 2 while Q divides internally in the ratio 2 : 1. Thus, the coordinates of P and Q are

$$P\left(\frac{1 \times -3 + 2 \times 1}{1+2}, \frac{1 \times 4 + 2 \times -2}{1+2}\right) = P\left(\frac{-1}{3}, 0\right)$$

$$Q\left(\frac{2 \times -3 + 1 \times 1}{2+1}, \frac{2 \times 4 + 1 \times (-2)}{2+1}\right) = Q\left(\frac{-5}{3}, 2\right) \text{ respectively}$$

Hence, the two points of trisection are (–1/3, 0) and (–5/3, 2).

REMARK *As Q is the mid-point of BP. So, the coordinates of Q can also be obtained by using mid-point formula.*

Type II ON FINDING THE SECTION RATIO OR AN END POINT OF THE SEGMENT WHEN THE SECTION POINT IS GIVEN

EXAMPLE 3 *In what ratio does the x-axis divide the line segment joining the points (2, –3) and (5, 6)? Also, find the coordinates of the point of intersection.*

SOLUTION Let the required ratio be $\lambda : 1$. Then, the coordinates of the point of division are,

$$R\left(\frac{5\lambda+2}{\lambda+1}, \frac{6\lambda-3}{\lambda+1}\right)$$

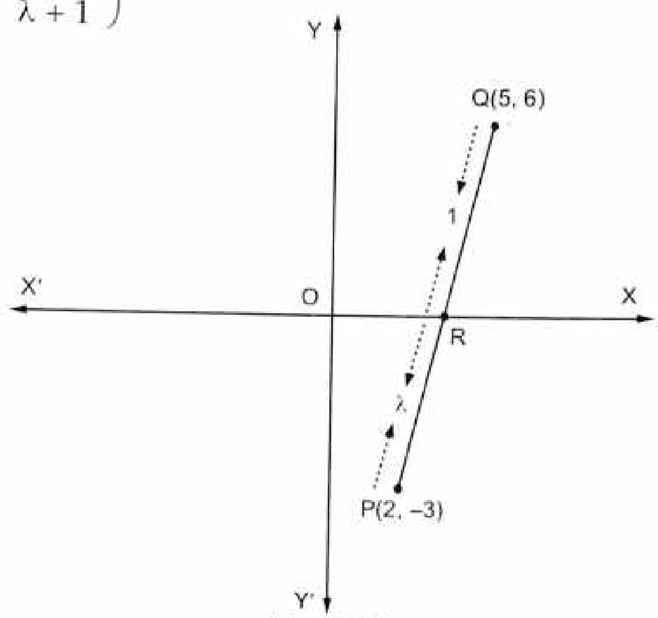


Fig. 6.25

But, it is a point on x-axis on which y-coordinates of every point is zero.

$$\therefore \qquad \frac{6\lambda-3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{1}{2}$$

Thus, the required ratio is $\frac{1}{2} : 1$ or, $1:2$.

Putting $\lambda = 1/2$ in the coordinates of R, we find that its coordinates are (3, 0).

EXAMPLE 4 *In what ratio does the y-axis divide the line segment joining the point P (– 4, 5) and Q (3, –7)? Also, find the coordinates of the point of intersection.*

SOLUTION Suppose x-axis divides PQ in the ratio $\lambda : 1$. Then, the coordinates of the point of division are

$$R\left(\frac{3\lambda-4}{\lambda+1}, \frac{-7\lambda+5}{\lambda+1}\right)$$

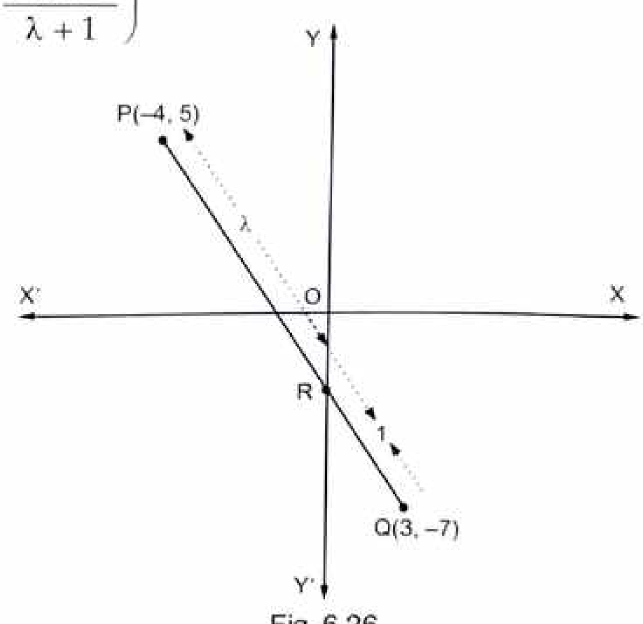


Fig. 6.26

Since R lies on y-axis and x-coordinate of every point on y-axis is zero.

$$\therefore \quad \frac{3\lambda - 4}{\lambda + 1} = 0 \Rightarrow 3\lambda - 4 = 0 \Rightarrow \lambda = \frac{4}{3}$$

Hence, the required ratio is $\frac{4}{3} : 1$ i.e., $4 : 3$

Putting $\lambda = 4/3$ in the coordinates of R, we find that its coordinates are $\left(0, \frac{-13}{7}\right)$.

EXAMPLE 5 *In what ratio does the point C (3/5, 11/5) divide the line segment joining the points A (3, 5) and B (−3, −2)?*

SOLUTION Let the point C divide AB in the ratio $\lambda : 1$. Then, the coordinates of C are

$$\left(\frac{-3\lambda + 3}{\lambda + 1}, \frac{-2\lambda + 5}{\lambda + 1}\right)$$

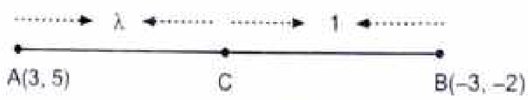


Fig. 6.27

But, the coordinates of C are given as $(3/5, 11/5)$.

$$\therefore \quad \frac{-3\lambda + 3}{\lambda + 1} = \frac{3}{5} \text{ and } \frac{-2\lambda + 5}{\lambda + 1} = \frac{11}{5}$$

$$\Rightarrow \quad -15\lambda + 15 = 3\lambda + 3 \text{ and } -10\lambda + 25 = 11\lambda + 11$$

$$\Rightarrow \quad 18\lambda = 12 \text{ and } 21\lambda = 14$$

$$\Rightarrow \quad \lambda = \frac{2}{3}$$

Hence, the point C divides AB in the ratio $2 : 3$.

EXAMPLE 6 *If the point C (−1, 2) divides internally the line segment joining A (2, 5) and B in ratio 3 : 4, find the coordinates of B.*

SOLUTION Let the coordinates of B be (α, β). It is given that $AC : BC = 3 : 4$. So, the co-ordinates of C are

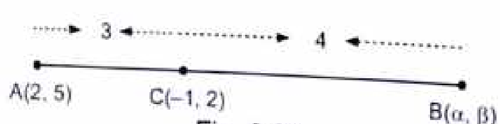


Fig. 6.28

$$\left(\frac{3\alpha + 4 \times 2}{3 + 4}, \frac{3\beta + 4 \times 5}{3 + 4}\right) = \left(\frac{3\alpha + 8}{7}, \frac{3\beta + 20}{7}\right)$$

But, the coordinates of C are $(-1, 2)$.

$$\therefore \quad \frac{3\alpha + 8}{7} = -1 \text{ and } \frac{3\beta + 20}{7} = 2$$

$$\Rightarrow \quad \alpha = -5 \text{ and } \beta = -2$$

Thus, the coordinates of B are $(-5, -2)$.

EXAMPLE 7 *Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points (1, 3) and (2, 7).* **[CBSE 2008]**

SOLUTION Suppose the line $3x + y - 9 = 0$ divides the line segment joining A (1, 3) and B (2, 7) in the ratio $k : 1$ at point C. Then, the coordinates of C are

$$\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1}\right)$$

But, C lies on $3x + y - 9 = 0$. Therefore,

$$3\left(\frac{2k+1}{k+1}\right) + \frac{7k+3}{k+1} - 9 = 0 \Rightarrow 6k + 3 + 7k + 3 - 9k - 9 = 0 \Rightarrow k = \frac{3}{4}$$

So, the required ratio is 3 : 4 internally.

EXAMPLE 8 *Find the ratio in which the point $(-3, p)$ divides the line segment joining the points $(-5, -4)$ and $(-2, 3)$. Hence, find the value of p.* **[CBSE 2016]**

SOLUTION Suppose the point $P(-3, p)$ divides the line segment joining points $A(-5, -4)$ and $B(-2, 3)$ in the ratio $k : 1$.

Then, the coordinates of P are $\left(\frac{-2k-5}{k+1}, \frac{3k-4}{k+1}\right)$

But, the coordinates of P are given as $(-3, p)$.

$$\therefore \quad \frac{-2k-5}{k+1} = -3 \text{ and } \frac{3k-4}{k+1} = p$$

$$\Rightarrow \quad -2k - 5 = -3k - 3 \text{ and } \frac{3k-4}{k+1} = p$$

$$\Rightarrow \quad k = 2 \text{ and } p = \frac{3k-4}{k+1}$$

$$\Rightarrow \quad k = 2 \text{ and } p = 2/3$$

Hence, the ratio is 2 : 1 and $p = 2/3$.

Type III ON DETERMINATION OF THE TYPE OF A GIVEN QUADRILATERAL

EXAMPLE 9 *Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$ and $(1, 2)$ are the vertices of a parallelogram. Is it a rectangle?*

SOLUTION Let the given point be A, B, C and D respectively. Then,

$$\text{Coordinates of the mid-point of } AC \text{ are } \left(\frac{-2+4}{2}, \frac{-1+3}{2}\right) = (1, 1)$$

$$\text{Coordinates of the mid-point of } BD \text{ are } \left(\frac{1+1}{2}, \frac{0+2}{2}\right) = (1, 1)$$

Thus, AC and BD have the same mid-point. Hence, $ABCD$ is a parallelogram.
Now, we shall see whether $ABCD$ is a rectangle or not.
We have,

$$AC = \sqrt{(4-(-2))^2 + (3-(-1))^2} = 2\sqrt{13}$$

and, $$BD = \sqrt{(1-1)^2 + (0-2)^2} = 2$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a rectangle.

EXAMPLE 10 *Prove that $(4, -1)$, $(6, 0)$, $(7, 2)$ and $(5, 1)$ are the vertices of a rhombus. Is it a square?*

SOLUTION Let the given points be A, B, C and D respectively. Then,

$$\text{Coordinates of the mid-point of } AC \text{ are } \left(\frac{4+7}{2}, \frac{-1+2}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$$

Coordinates of the mid-point of BD are $\left(\frac{6+5}{2}, \frac{0+1}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$

Thus, AC and BD have the same mid-point.
Hence, $ABCD$ is a parallelogram.
Now,

$$AB = \sqrt{(6-4)^2 + (0+1)^2} = \sqrt{5}, \ BC = \sqrt{(7-6)^2 + (2-0)^2} = \sqrt{5}$$

$\therefore \quad AB = BC$

So, $ABCD$ is a parallelogram whose adjacent sides are equal.
Hence, $ABCD$ is a rhombus.
We have,

$$AC = \sqrt{(7-4)^2 + (2+1)^2} = 3\sqrt{2}, \text{ and, } BD = \sqrt{(6-5)^2 + (0-1)^2} = \sqrt{2}$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a square.

Type IV ON FINDING THE UNKNOWN VERTEX FROM GIVEN POINTS

EXAMPLE 11 *The three vertices of a parallelogram taken in order are (–1, 0), (3, 1) an (2, 2) respectively. Find the coordinates of the fourth vertex.*

SOLUTION Let $A(-1, 0)$, $B(3, 1)$, $C(2, 2)$ and $D(x, y)$ be the vertices of a parallelogram ABC taken in order. Since, the diagonals of a parallelogram bisect each other.

$\therefore$ Coordinates of the mid-point of AC = Coordinates of the mid-point of BD

$$\Rightarrow \quad \left(\frac{-1+2}{2}, \frac{0+2}{2}\right) = \left(\frac{3+x}{2}, \frac{1+y}{2}\right)$$

$$\Rightarrow \quad \left(\frac{1}{2}, 1\right) = \left(\frac{3+x}{2}, \frac{y+1}{2}\right)$$

$$\Rightarrow \quad \frac{3+x}{2} = \frac{1}{2} \text{ and } \frac{y+1}{2} = 1$$

$$\Rightarrow \quad x = -2 \text{ and } y = 1$$

Hence, the fourth vertex of the parallelogram is $(-2, 1)$.

EXAMPLE 12 *If the points $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(p, 3)$ are the vertices of a parallelogram taken in order, find the value of p.*

SOLUTION We know that the diagonals of a parallelogram bisect each other. So, coordinates of the mid-point of diagonal AC are same as the coordinates of the mid-point of diagonal BD.

$$\therefore \quad \left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

$$\Rightarrow \quad \left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\Rightarrow \quad \frac{15}{2} = \frac{8+p}{2} \Rightarrow 15 = 8 + p \Rightarrow p = 7$$

EXAMPLE 13 *If $A(-2, -1)$, $B(a, 0)$, $C(4, b)$ and $D(1, 2)$ are the vertices of a parallelogram, find the values of a and b.* [CBSE 2006 C, 2018]

SOLUTION We know that the diagonals of a parallelogram bisect each other. Therefore, the coordinates of the mid-point of AC are same as the coordinates of the mid-point of BD i.e.,

$$\left(\frac{-2+4}{2}, \frac{-1+b}{2}\right) = \left(\frac{a+1}{2}, \frac{0+2}{2}\right)$$

$$\Rightarrow \quad \left(1, \frac{b-1}{2}\right) = \left(\frac{a+1}{2}, 1\right)$$

$$\Rightarrow \quad \frac{a+1}{2} = 1 \text{ and } \frac{b-1}{2} = 1$$

$$\Rightarrow \quad a+1 = 2 \text{ and } b-1 = 2$$

$$\Rightarrow \quad a = 1 \text{ and } b = 3$$

Hence, $a = 1$ and $b = 3$

EXAMPLE 14 *If the coordinates of the mid-points of the sides of a triangle are (1, 2) (0, –1) and (2, –1). Find the coordinates of its vertices.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC. Let $D(1, 2)$, $E(0, -1)$, and $F(2, -1)$ be the mid-points of sides BC, CA and AB respectively.

Since D is the mid-point of BC.

$$\therefore \quad \frac{x_2+x_3}{2} = 1 \text{ and } \frac{y_2+y_3}{2} = 2$$

$$\Rightarrow \quad x_2 + x_3 = 2 \text{ and } y_2 + y_3 = 4 \quad \ldots \text{(i)}$$

Similarly, E and F are the mid-points of CA and AB respectively.

$$\therefore \quad \frac{x_1+x_3}{2} = 0 \text{ and } \frac{y_1+y_3}{2} = -1$$

$$\Rightarrow \quad x_1 + x_3 = 0 \text{ and } y_1 + y_3 = -2 \quad \ldots \text{(ii)}$$

$$\text{and,} \quad \frac{x_1+x_2}{2} = 2 \text{ and } \frac{y_1+y_2}{2} = -1$$

$$\Rightarrow \quad x_1 + x_2 = 4 \text{ and } y_1 + y_2 = -2 \quad \ldots \text{(iii)}$$

From (i), (ii) and (iii), we get

$$(x_2+x_3)+(x_1+x_3)+(x_1+x_2) = 2+0+4 \text{ and, } (y_2+y_3)+(y_1+y_3)+(y_1+y_2) = 4-2-2$$

$$\Rightarrow \quad 2(x_1+x_2+x_3) = 6 \text{ and } 2(y_1+y_2+y_3) = 0 \quad \ldots \text{(iv)}$$

$$\Rightarrow \quad x_1+x_2+x_3 = 3 \text{ and } y_1+y_2+y_3 = 0$$

From (i) and (iv), we get

$$x_1 + 2 = 3 \text{ and } y_1 + 4 = 0$$

$$\Rightarrow \quad x_1 = 1 \text{ and } y_1 = -4$$

So, the coordinates of A are (1, –4)

From (ii) and (iv), we get

$$x_2 + 0 = 3 \text{ and } y_2 - 2 = 0$$

$$\Rightarrow \quad x_2 = 3 \text{ and } y_2 = 2$$

So, coordinates of B are (3, 2)

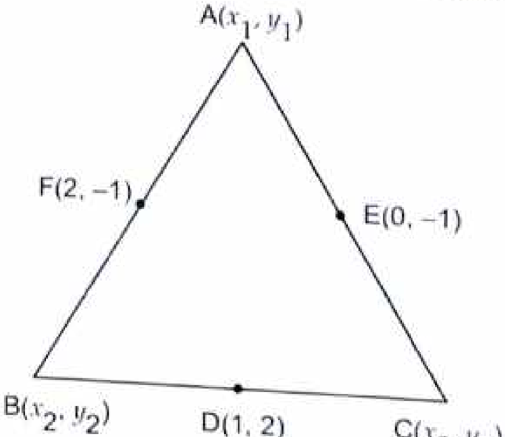


Fig. 6.29

From (iii) and (iv), we get

$$x_3 + 4 = 3 \text{ and } y_3 - 2 = 0$$

$$\Rightarrow \quad x_3 = -1 \text{ and } y_3 = 2$$

So, coordinates of C are $(-1, 2)$

Hence, the vertices of the triangle ABC are $A(1, -4)$, $B(3, 2)$ and $C(-1, 2)$.

EXAMPLE 15 *The coordinates of one end point of a diameter of a circle are $(4, -1)$ ... coordinates of the centre of the circle are $(1, -3)$. Find the coordinates of the other end of the di...*

SOLUTION Let AB be a diameter of the circle having its centre at $C(1, -3)$ such th... coordinates of one end A are $(4, -1)$.

Let the coordinates of B be (x, y).

Since C is the mid-point of AB. Therefore, the coordinates of C are $\left(\frac{x+4}{2}, \frac{y-1}{2}\right)$.

But, the coordiantes of C are given to be $(1, -3)$.

$$\therefore \quad \frac{x+4}{2} = 1 \text{ and } \frac{y-1}{2} = -3$$

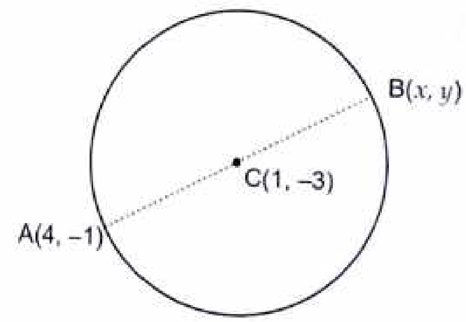


Fig. 6.30

$$\Rightarrow \quad x + 4 = 2 \text{ and } y - 1 = -6$$

$$\Rightarrow \quad x = -2 \text{ and } y = -5$$

Hence, the coordinates of B are $(-2, -5)$.

EXAMPLE 16 *Find the lengths of the medians of a ΔABC whose vertices are $A(7, -3)$, $B(5, 3)$ ... $C(3, -1)$.*

SOLUTION Let D, E, F be the mid-points of the sides BC, CA and AB respectively. Then coordinates of D, E and F are

$$D\left(\frac{5+3}{2}, \frac{3-1}{2}\right) = D(4, 1), \quad E\left(\frac{3+7}{2}, \frac{-1-3}{2}\right) = E(5, -2)$$

and, $$F\left(\frac{7+5}{2}, \frac{-3+3}{2}\right) = F(6, 0)$$

$$\therefore \quad AD = \sqrt{(7-4)^2 + (-3-1)^2} = \sqrt{9+16} = 5 \text{ units}$$

$$BE = \sqrt{(5-5)^2 + (-2-3)^2} = \sqrt{0+25} = 5 \text{ units}$$

and, $$CF = \sqrt{(6-3)^2 + (0+1)^2} = \sqrt{9+1} = \sqrt{10} \text{ units.}$$

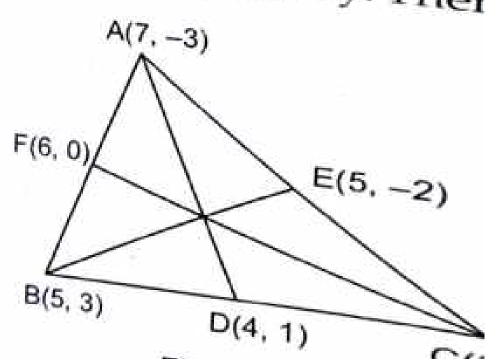


Fig. 6.31

EXAMPLE 17 *If $A(5, -1)$, $B(-3, -2)$ and $C(-1, 8)$ are the vertices of triangle ABC, fina... length of median through A and the coordinates of the centroid.*

[CBSE 200...

SOLUTION Let AD be the median through the vertex A of ΔABC. Then, D is the mid-poi... BC. So, the coordinates of D are $\left(\frac{-3-1}{2}, \frac{-2+8}{2}\right)$ i.e., $(-2, 3)$.

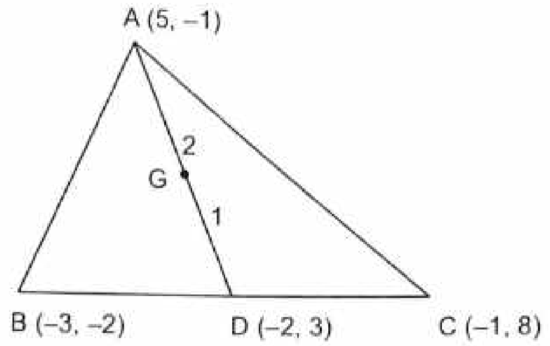


Fig. 6.32

$\therefore \quad AD = \sqrt{(5+2)^2 + (-1-3)^2} = \sqrt{49+16} = \sqrt{65}$ units

Let G be the centroid of $\Delta\ ABC$. Then, G lies on median AD and divides it in the ratio 2:1. So, coordinates of G are

$$\left(\frac{2\times -2 + 1\times 5}{2+1}, \frac{2\times 3 + 1\times -1}{2+1}\right) = \left(\frac{-4+5}{3}, \frac{6-1}{3}\right) = \left(\frac{1}{3}, \frac{5}{3}\right)$$

LEVEL-2

EXAMPLE 18 *Point P divides the line segment joining the points A (–1, 3) and B (9, 8) such that $\frac{AP}{BP} = \frac{k}{1}$. If P lies on the line $x - y + 2 = 0$, find the value of k.* **[CBSE 2010]**

SOLUTION It is given that P divides the line segment joining A (–1, 3) and B (9, 8) in the ratio $k:1$. So, coordinates of P are $\left(\frac{9k-1}{k+1}, \frac{8k+3}{k+1}\right)$.

$P\left(\frac{9k-1}{k+1}, \frac{8k+3}{k+1}\right)$ lines on the line $x - y + 2 = 0$

$\therefore \quad \frac{9k-1}{k+1} - \frac{8k+3}{k+1} + 2 = 0$

Fig. 6.33

$\Rightarrow \quad 9k - 1 - 8k - 3 + 2k + 2 = 0$

$\Rightarrow \quad 3k - 2 = 0$

$\Rightarrow \quad k = 2/3$

EXAMPLE 19 *Point P divides the line segment joining the points A (2, 1) and B (5, –8) such that $\frac{AP}{AB} = \frac{1}{3}$. If P lies on the line $2x - y + k = 0$, find the value of k.* **[CBSE 2010]**

SOLUTION We have,

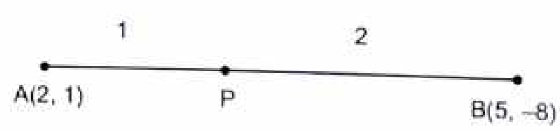


Fig. 6.34

$\frac{AP}{AB} = \frac{1}{3}$

$\Rightarrow \quad \frac{AP}{AP + PB} = \frac{1}{3}$

$\Rightarrow \quad 3AP = AP + BP$

$\Rightarrow \quad 2AP = BP$

$\Rightarrow \quad \frac{AP}{BP} = \frac{1}{2}$

So, P divides AB in the ratio $1:2$.

$\therefore$ Coordinates of P are $\left(\frac{1\times 5+2\times 2}{1+2}, \frac{1\times -8+2\times 1}{1+2}\right)=(3, 2)$

Since, $P(3, 2)$ lies on the line $2x-y+k=0$.

$\therefore$ $2\times 3-2+k=0 \Rightarrow k=-4$

EXAMPLE 20 *The vertices of a ΔABC are $A(5,5)$, $B(1,5)$ and $C(9,1)$. A line is drawn to intersect sides AB and AC at P and Q respectively, such that $\frac{AP}{AB}=\frac{AQ}{AC}=\frac{3}{4}$. Find the length of the line segment PQ.* **[CBSE 2014]**

SOLUTION We have,

$$\frac{AP}{AB}=\frac{AQ}{AC}=\frac{3}{4}$$

$$\Rightarrow \quad \frac{AP}{AP+PB}=\frac{AQ}{AQ+QC}=\frac{3}{4}$$

$$\Rightarrow \quad \frac{AP}{AP+PB}=\frac{3}{4}, \frac{AQ}{AQ+QC}=\frac{3}{4}$$

$$\Rightarrow \quad 4\,AP=3\,AP+3\,PB \text{ and } 4\,AQ=3\,AQ+3\,QC$$

$$\Rightarrow \quad AP=3\,PB \text{ and } AQ=3\,QC$$

$$\Rightarrow \quad \frac{AP}{PB}=\frac{3}{1} \text{ and } \frac{AQ}{QC}=\frac{3}{1}$$

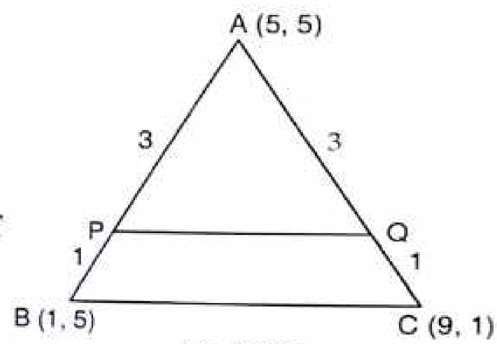


Fig. 6.35

$\Rightarrow$ P and Q divide AB and AC respectively in the same ratio $3:1$

Thus, the coordinates of P and Q are

$$\left(\frac{3\times 1+1\times 5}{3+1}, \frac{3\times 5+1\times 5}{3+1}\right)=(2, 5) \text{ and } \left(\frac{3\times 9+1\times 5}{3+1}, \frac{3\times 1+1\times 5}{3+1}\right)=(8, 2)$$

$\therefore$ $PQ=\sqrt{(2-8)^2+(5-2)^2}=\sqrt{45}=3\sqrt{5}$ units

EXERCISE 6.3

LEVEL-1

1. Find the coordinates of the point which divides the line segment joining $(-1, 3)$ and $(4, -7)$ internally in the ratio $3:4$.
2. Find the points of trisection of the line segment joining the points:
   (i) $(5, -6)$ and $(-7, 5)$, (ii) $(3, -2)$ and $(-3, -4)$, (iii) $(2, -2)$ and $(-7, 4)$. **[NCERT]**
3. Find the coordinates of the point where the diagonals of the parallelogram formed by joining the points $(-2, -1)$, $(1, 0)$, $(4, 3)$ and $(1, 2)$ meet.
4. Prove that the points $(3, -2)$, $(4, 0)$, $(6, -3)$ and $(5, -5)$ are the vertices of a parallelogram.
5. If $P(9a-2, -b)$ divides the line segment joining $A(3a+1, -3)$ and $B(8a, 5)$ in the ratio $3:1$, find the values of a and b. **[NCERT EXEMPLAR]**
6. If (a, b) is the mid-point of the line segment joining the points $A(10, -6)$, $B(k, 4)$ and $a-2b=18$, find the value of k and the distance AB. **[NCERT EXEMPLAR]**

7. Find the ratio in which the point $(2, y)$ divides the line segment joining the points $A(-2, 2)$ and $B(3, 7)$. Also, find the value of y. [CBSE 2009]
8. If $A(-1, 3)$, $B(1, -1)$ and $C(5, 1)$ are the vertices of a triangle ABC, find the length of the median through A.
9. If the points P, $Q(x, 7)$, R, $S(6, y)$ in this order divide the line segment joining $A(2, p)$ and $B(7, 10)$ in 5 equal parts, find x, y and p. [CBSE 2015]
10. If a vertex of a triangle be $(1, 1)$ and the middle points of the sides through it be $(-2, 3)$ and $(5, 2)$, find the other vertices.
11. (i) In what ratio is the line segment joining the points $(-2, -3)$ and $(3, 7)$ divided by the y-axis? Also, find the coordinates of the point of division. [CBSE 2006C]
    (ii) In what ratio is the line segment joining $(-3, -1)$ and $(-8, -9)$ divided at the point $(-5, -21/5)$?
12. If the mid-point of the line joining $(3, 4)$ and $(k, 7)$ is (x, y) and $2x + 2y + 1 = 0$ find the value of k. [NCERT EXEMPLAR]
13. Find the ratio in which the points $P(3/4, 5/12)$ divides the line segments joining the points $A(1/2, 3/2)$ and $B(2, -5)$. [CBSE 2015]
14. Find the ratio in which the line segment joining $(-2, -3)$ and $(5, 6)$ is divided by (i) x-axis (ii) y-axis. Also, find the coordinates of the point of division in each case. [CBSE 2013]
15. Prove that the points $(4, 5)$, $(7, 6)$, $(6, 3)$, $(3, 2)$ are the vertices of a parallelogram. Is it a rectangle.
16. Prove that $(4, 3)$, $(6, 4)$, $(5, 6)$ and $(3, 5)$ are the angular points of a square.
17. Prove that the points $(-4, -1)$, $(-2, -4)$, $(4, 0)$ and $(2, 3)$ are the vertices of a rectangle.
18. Find the lengths of the medians of a triangle whose vertices are $A(-1,3)$, $B(1,-1)$ and $C(5, 1)$.
19. Find the ratio in which the line segment joining the points $A(3,-3)$ and $B(-2,7)$ is divided by x-axis. Also, find the coordinates of the point of division. [CBSE 2014]
20. Find the ratio in which the point $P(x, 2)$ divides the line segment joining the points $A(12,5)$ and $B(4,-3)$. Also, find the value of x. [CBSE 2014]
21. Find the ratio in which the point $P(-1, y)$ lying on the line segment joining $A(-3, 10)$ and $B(6, -8)$ divides it. Also find the value of y. [CBSE 2013]
22. Find the coordinates of a point A, where AB is a diameter of the circle whose centre is $(2, -3)$ and B is $(1, 4)$. [NCERT]
23. If the points $(-2, -1)$, $(1, 0)$, $(x, 3)$ and $(1, y)$ form a parallelogram, find the values of x and y.
24. The points $A(2,0)$, $B(9,1)$, $C(11,6)$ and $D(4,4)$ are the vertices of a quadrilateral $ABCD$. Determine whether $ABCD$ is a rhombus or not.
25. In what ratio does the point $(-4, 6)$ divide the line segment joining the points $A(-6, 10)$ and $B(3, -8)$? [NCERT, CBSE 2017]
26. Find the ratio in which the y-axis divides the line segment joining the points $(5, -6)$ and $(-1, -4)$. Also, find the coordinates of the point of division. [CBSE 2010, 2016]
27. Show that $A(-3, 2)$, $B(-5, -5)$, $C(2, -3)$ and $D(4, 4)$ are the vertices of a rhombus.
28. Find the lengths of the medians of a ΔABC having vertices at $A(0,-1)$, $B(2, 1)$ and $C(0, 3)$.
29. Find the ratio in which $P(4, m)$ divides the line segment joining the points $A(2, 3)$ and $B(6, -3)$. Hence, find m. [CBSE 2018]

30. Find the coordinates of the points which divide the line segment joining the points $(-4, 0)$ and $(0, 6)$ in four equal parts.
31. Show that the mid-point of the line segment joining the points $(5, 7)$ and $(3, 9)$ is also the mid-point of the line segment joining the points $(8, 6)$ and $(0, 10)$.
32. Find the distance of the point $(1, 2)$ from the mid-point of the line segment joining the points $(6, 8)$ and $(2, 4)$.
33. If A and B are $(1, 4)$ and $(5, 2)$ respectively, find the coordinates of P when $AP/BP = 3/4$.
34. Show that the points $A\ (1, 0)$, $B\ (5, 3)$, $C\ (2, 7)$ and $D\ (-2, 4)$ are the vertices of a parallelogram.
35. Determine the ratio in which the point $P\ (m, 6)$ divides the join of $A(-4, 3)$ and $B(2, 8)$. Also, find the value of m. **[CBSE 2004]**
36. Determine the ratio in which the point $(-6, a)$ divides the join of $A(-3, 1)$ and $B(-8, 9)$. Also find the value of a. **[CBSE 2004]**
37. $ABCD$ is a rectangle formed by joining the points $A\ (-1, -1)$, $B\ (-1, 4)$, $C\ (5, 4)$ and $D\ (5, -1)$. P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer. **[NCERT]**
38. Points P, Q, R and S divide the line segment joining the points $A\ (1, 2)$ and $B\ (6, 7)$ in 5 equal parts. Find the coordinates of the points P, Q and R. **[CBSE 2014]**
39. If A and B are two points having coordinates $(-2, -2)$ and $(2, -4)$ respectively, find the coordinates of P such that $AP = \frac{3}{7} AB$. **[NCERT, CBSE 2008, 2009]**
40. Find the coordinates of the points which divide the line segment joining $A\ (-2, 2)$ and $B\ (2, 8)$ into four equal parts. **[NCERT]**

LEVEL-2

41. Three consecutive vertices of a parallelogram are $(-2, -1)$, $(1, 0)$ and $(4, 3)$. Find the fourth vertex.
42. The points $(3, -4)$ and $(-6, 2)$ are the extremities of a diagonal of a parallelogram. If the third vertex is $(-1, -3)$. Find the coordinates of the fourth vertex.
43. If the coordinates of the mid-points of the sides of a triangle are $(1, 1)$, $(2, -3)$ and $(3, 4)$, find the vertices of the triangle.
44. Determine the ratio in which the straight line $x - y - 2 = 0$ divides the line segment joining $(3, -1)$ and $(8, 9)$.
45. Three vertices of a parallelogram are $(a + b, a - b)$, $(2a + b, 2a - b)$, $(a - b, a + b)$. Find the fourth vertex.
46. If two vertices of a parallelogram are $(3, 2)$, $(-1, 0)$ and the diagonals cut at $(2, -5)$, find the other vertices of the parallelogram.
47. If the coordinates of the mid-points of the sides of a triangle are $(3, 4)$, $(4, 6)$ and $(5, 7)$, find its vertices. **[CBSE 2008]**
48. The line segment joining the points $P\ (3, 3)$ and $Q\ (6, -6)$ is trisected at the points A and B such that A is nearer to P. If A also lies on the line given by $2x + y + k = 0$, find the value of k. **[CBSE 2009]**
49. If three consecutive vertices of a parallelogram are $(1, -2)$, $(3, 6)$ and $(5, 10)$, find its fourth vertex.
50. If the points $A\ (a, -11)$, $B\ (5, b)$, $C\ (2, 15)$ and $D\ (1, 1)$ are the vertices of a parallelogram $ABCD$, find the values of a and b.

51. If the coordinates of the mid-points of the sides of a triangle be (3, –2), (–3, 1) and (4, –3), then find the coordinates of its vertices.

52. The line segment joining the points (3, –4) and (1, 2) is trisected at the points P and Q. If the coordinates of P and Q are $(p, -2)$ and $(5/3, q)$ respectively. Find the values of p and q. **[CBSE 2005]**

53. The line joining the points (2, 1) and (5,–8) is trisected at the points P and Q. If point P lies on the line $2x - y + k = 0$. Find the value of k. **[CBSE 2005]**

54. $A(4, 2)$, $B(6, 5)$ and $C(1, 4)$ are the vertices of ΔABC.
   (i) The median from A meets BC in D. Find the coordinates of the point D.
   (ii) Find the coordinates of point P on AD such that $AP : PD = 2 : 1$.
   (iii) Find the coordinates of the points Q and R on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
   (iv) What do you observe? **[NCERT, CBSE, 2009, 10]**

55. If the points $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(k, p)$ are the vertices of a parallelogram taken in order, then find the values of k and p.

56. A point P divides the line segment joining the points $A(3, -5)$ and $B(-4, 8)$ such that $\frac{AP}{PB} = \frac{k}{1}$. If P lies on the line $x + y = 0$, then find the value of k. **[CBSE 2012]**

57. The mid-point P of the line segment joining the points $A(-10, 4)$ and $B(-2, 0)$ lies on the line segment joining the points $C(-9, -4)$ and $D(-4, y)$. Find the ratio in which P divides CD. Also, find the value of y. **[CBSE 2014]**

58. If the point $C(-1, 2)$ divides internally the line segment joining the points $A(2, 5)$ and $B(x, y)$ in the ratio 3 : 4, find the value of $x^2 + y^2$. **[CBSE 2016]**

59. $ABCD$ is a parallelogram with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Find the coordinates of the fourth vertex D in terms of x_1, x_2, x_3, y_1, y_2 and y_3.

[NCERT EXEMPLAR]

60. The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of ΔABC.
   (i) The median from A meets BC at D. Find the coordinates of the point D.
   (ii) Find the coordinates of the point P on AD such that $AP : PD = 2 : 1$.
   (iii) Find the points of coordinates Q and R on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
   (iv) What are the coordinates of the centroid of the triangle ABC?

[NCERT EXEMPLAR]

ANSWERS

1. (8/7, –9/7) 2. (i) (1, –7/3), (–3, 4/3) (ii) (1, –8/3), (–1, –10/3) (iii) (–1, 0), (–4, 2)

3. (1, 1) 5. $a = 1, b = -3$ 6. $k = 22$, $AB = 2\sqrt{61}$

7. $4 : 1, y = 6$ 8. 5 9. $x = 4, y = 9, p = 5$

10. (–5, 5), (9, 3) 11. (i) 2 : 3 internally; (0, 1) (ii) 2 : 3 internally

12. $k = -15$ 13. 1 : 5 14. (i) $2 : 5; \left(0, \frac{-3}{7}\right)$ (ii) $2 : 5; \left(0, \frac{-3}{7}\right)$

15. No 18. $AD = 5, BE = \sqrt{10}, CF = 5$ 19. $3 : 7, (3/2, 0)$

20. $3:5; x = 9$
21. $2:7, 6$
22. $(3, -10)$
23. $x = 4, y = 2$
24. No
25. $2:7$
28. $AD = \sqrt{10}$ units, $BE = 2$ units, $CF = \sqrt{10}$ units
29. $1:1, m = 0$
30. $(-3, 1.5), (-2, 3), (-1, 4.5)$
32. 5 units
33. $\left(\frac{19}{7}, \frac{22}{7}\right)$
35. $3:2, m = \frac{-2}{5}$
36. $3:2, a = 5$
38. $P(2, 3), Q(3, 4), R(4, 5)$
39. $\left(\frac{-2}{7}, \frac{-20}{7}\right)$
40. $\left(-1, \frac{7}{2}\right), (0, 5), \left(1, \frac{13}{2}\right)$
41. $(1, 2)$
42. $(-2, 1)$
43. $(4, 0), (2, 8), (0, -6)$
44. $2:3$ internally
45. $(-b, b)$
46. $(1, -12), (5, -10)$
47. $(6, 9), (4, 5), (2, 3)$
48. -8
49. $(3, 2)$
50. $a = 4, b = 3$51. $A(-2, 0), B(10, -6), C(-4, 2)$
52. $p = \frac{7}{3}, q = 0$
53. $k = -8$
55. $k = 7, p = 3$
56. $1/2$
57. $3:2, y = 6$
58. 29
59. $(x_1 + x_3 - x_2, y_1 + y_3 - y_2)$
60. (i) $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$ (ii) $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{4}\right)$

(iii) $Q\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right), R\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

HINT TO THE SELECTED PROBLEM

39. We have, $AP = \frac{3}{7}AB$. Also, $AP + BP = AB$

$\therefore \quad \frac{3}{7}AB + BP = AB \Rightarrow BP = \frac{4}{7}AB$

Hence, $AP : BP = 3 : 4$

6.5 SOME APPLICATIONS OF SECTION FORMULA

In this section, we shall discuss an application of the section formula learnt in the previous section to find the coordinates of the centroid of a triangle in terms of the coordinates of its vertices.

THEOREM *Prove that the coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) are*

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Also, deduce that the medians of a triangle are concurrent.

[NCERT, CBSE 2004]

PROOF Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC whose medians are AD, BE and CF respectively. So D, E and F are respectively the mid-points of BC, CA and AB.

Coordinates of D are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$

Coordinates of a point dividing AD in the ratio 2 : 1 are

$$\left(\frac{1.x_1 + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{1.y_1 + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

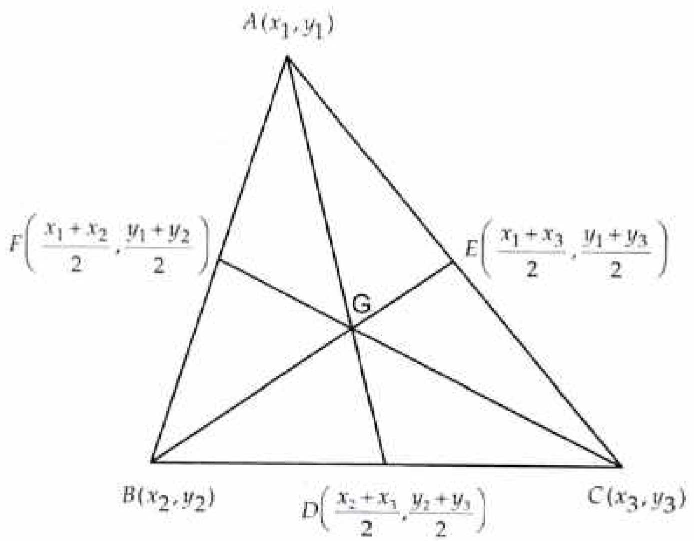


Fig. 6.36

The coordinates of E are $\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$

The coordinates of a point dividing BE in the ratio 2 : 1 are

$$\left(\frac{1.x_2 + \frac{2(x_1 + x_3)}{2}}{1+2}, \frac{1.y_2 + \frac{2(y_1 + y_3)}{2}}{1+2}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly the coordinates of a point dividing CF in the ratio 2 : 1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Thus, the point having coordinates $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ is common to AD, BE and CF and divides them in the ratio 1 : 2.

Hence, medians of a triangle are concurrent and the coordinates of the centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the coordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0).*

SOLUTION We know that the coordinates of the centroid of a triangle whose angular points are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

So, the coordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0) are

$$\left(\frac{0+8+8}{3}, \frac{6+12+0}{3}\right) \text{ or, } \left(\frac{16}{3}, 6\right)$$

EXAMPLE 2 *If $x - 2y + k = 0$ is a median of the triangle whose vertices are at points $A(-1, 3)$, $B(0, 4)$ and $C(-5, 2)$ find the value of k.*

SOLUTION The coordinates of the centroid G of ΔABC are

$$\left(\frac{-1+0-5}{3}, \frac{3+4+2}{3}\right) \text{ i.e. } (-2, 3)$$

Since G lies on the median $x - 2y + k = 0$. So, coordinates of G satisfy its equation.

$$\therefore \quad -2 - 6 + k = 0 \Rightarrow k = 8.$$

EXAMPLE 3 *If the coordinates of the mid-points of the sides of a triangle are (1, 1), (2, –3) and (3, 4). Find its centroid.*

SOLUTION Let $P(1, 1)$, $Q(2, -3)$, $R(3, 4)$ be the mid-points of sides AB, BC and CA respectively of triangle ABC. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle. ABC. Then,

P is the mid-point of BC

$$\Rightarrow \quad \frac{x_1 + x_2}{2} = 1, \frac{y_1 + y_2}{2} = 1$$

$$\Rightarrow \quad x_1 + x_2 = 2 \text{ and } y_1 + y_2 = 2 \quad \ldots(i)$$

Q is the mid-point of BC

$$\Rightarrow \quad \frac{x_2 + x_3}{2} = 2, \frac{y_2 + y_3}{2} = -3$$

$$\Rightarrow \quad x_2 + x_3 = 4 \text{ and } y_2 + y_3 = -6 \quad \ldots(ii)$$

R is the mid-point of AC

$$\Rightarrow \quad \frac{x_1 + x_3}{2} = 3 \text{ and } \frac{y_1 + y_3}{2} = 4$$

$$\Rightarrow \quad x_1 + x_3 = 6 \text{ and } y_1 + y_3 = 8 \quad \ldots(iii)$$

From (i), (ii) and (iii), we get

$$x_1 + x_2 + x_2 + x_3 + x_1 + x_3 = 2 + 4 + 6$$

$$\text{and,} \quad y_1 + y_2 + y_2 + y_3 + y_1 + y_3 = 2 - 6 + 8$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 6 \text{ and } y_1 + y_2 + y_3 = 2 \quad \ldots(iv)$$

The coordinates of the centroid of ΔABC are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)=\left(\frac{6}{3}, \frac{2}{3}\right)=\left(2, \frac{2}{3}\right) \qquad \text{[Using (iv)]}$$

EXAMPLE 4 *Two vertices of a triangle are (3, –5) and (–7, 4). If its centroid is (2, –1), find the third vertex.*

SOLUTION Let the coordinates of the third vertex be (x, y). Then,

$$\frac{x+3-7}{3}=2 \text{ and } \frac{y-5+4}{3}=-1$$

$$\Rightarrow \quad x-4=6 \text{ and } y-1=-3$$

$$\Rightarrow \quad x=10 \text{ and } y=-2$$

Thus, the coordinates of the third vertex are (10, –2).

LEVEL-2

EXAMPLE 5 *Use analytical geometry to prove that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.*

SOLUTION Let AOB be a right-angled triangle with base OA taken along x-axis and the perpendicular OB taken along y-axis. Let $OA=a$ and $OB=b$.

Let D be the mid-point of the hypotenuse AB. Then, the coordinates of A, B and D are respectively $(a, 0)$, $(0, b)$ and $(a/2, b/2)$.

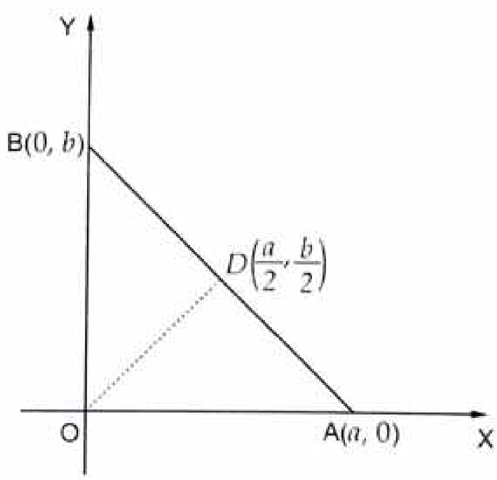


Fig. 6.37

Now, $$DO=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-0\right)^2}=\frac{1}{2}\sqrt{a^2+b^2},$$

$$DA=\sqrt{\left(\frac{a}{2}-a\right)^2+\left(\frac{b}{2}-0\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}$$

and, $$DB=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-b\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}$$

Hence, $DA=DB=DC$ i.e., D is equidistant from the vertices of triangle ABC.

EXAMPLE 6 *Using analytical geometry, prove that the diagonals of a rhombus are perpendicular to each other.*

SOLUTION Let $OABC$ be a rhombus such that OA is along x-axis. Let BL and CM be perpendiculars from B and C respectively on x-axis.

Clearly, triangles ABL and OCM are congruent.

$\therefore \quad OM = AL$ and $CM = BL$

Let the coordinates of A and C be $(x_1, 0)$ and (x_2, y_2) respectively. Then, $OM = x_2$ an[d] $OA = x_1$.

$\therefore \quad OL = OA + AL = OA + OM = x_1 + x_2$ and $BL = CM = y_2$

So, the coordinates of B are $(x_1 + x_2, y_2)$.

Now, $\quad OA = OC \Rightarrow OA^2 = OC^2 \Rightarrow x_1^2 = x_2^2 + y_2^2 \quad$ ...(i)

In order to prove that the diagonals OB and AC are mutually perpendicular, it is sufficient to show that $\angle ODA = \pi/2$.

Since the diagonals of a rhombus bisect each other. Therefore, coordinates of D are $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$.

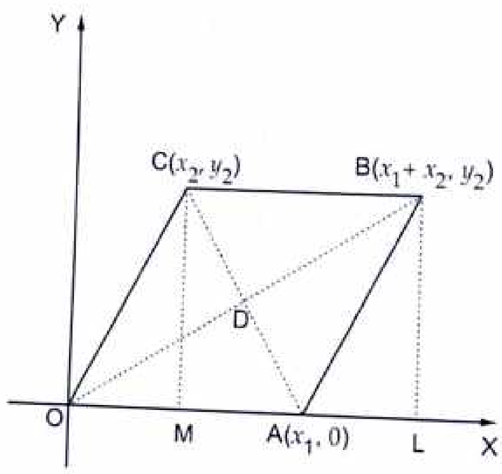


Fig. 6.38

Now,

$$OD^2 = \left(\frac{x_1 + x_2}{2} - 0\right)^2 + \left(\frac{y_2}{2} - 0\right)^2 = \left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2$$

$$AD^2 = \left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_2}{2} - 0\right)^2 \Rightarrow AD^2 = \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2$$

and, $\quad OA^2 = (x_1 - 0)^2 + (0 - 0)^2 = x_1^2$

$$\therefore \quad OD^2 + AD^2 = \left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2 + \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2}{y}\right)^2$$

$$\Rightarrow \quad OD^2 + AD^2 = \frac{1}{4}\left\{2x_1^2 + 2x_2^2 + 2y_2^2\right\}$$

$$\Rightarrow \quad OD^2 + AD^2 = \frac{1}{2}(x_1^2 + x_2^2 + y_2^2) = \frac{1}{2}(x_1^2 + x_1^2)$$

$$\Rightarrow \quad OD^2 + AD^2 = x_1^2 = OA^2 \qquad \text{[Using (i)]}$$

$\therefore \quad \Delta ODA$, is a right-angled triangle such that $\angle ODA = \pi/2$.

Hence, the diagonals of a rhombus are at right angles.

EXAMPLE 7 *Prove that the diagonals of a rectangle bisect each other and are equal.*

SOLUTION Let $OACB$ be a rectangle such that OA is a along x-axis and OB is along y-axis. Let $OA = a$ and $OB = b$.

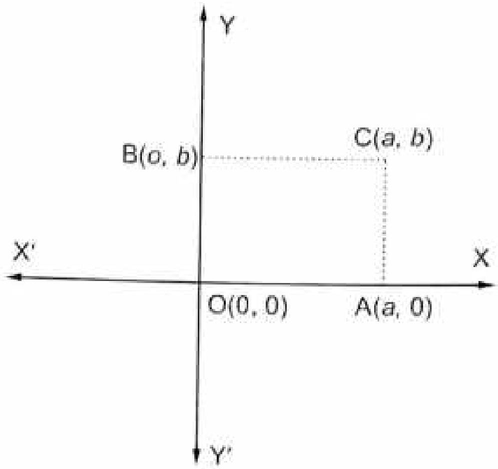


Fig. 6.39

Then, the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.
Since, $OACB$ is a rectangle. Therefore,

$$AC = Ob \Rightarrow AC = b$$

Thus, we have

$$OA = a \text{ and } AC = b$$

So, the coordinates of C are (a, b).

The coordinates of the mid-point of OC are $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Also, the coordinates of the mid-points of AB are $\left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Clearly, coordinates of the mid-point of OC and AB are same.
Hence, OC and AB bisect each other.

Also, $\quad OC = \sqrt{a^2 + b^2}$ and $AB = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}$

$\therefore \quad OC = AB$

EXERCISE 6.4

LEVEL-1

1. Find the centroid of the triangle whose vertices are:
   (i) (1, 4), (–1, –1), (3, –2) (ii) (– 2, 3), (2, –1), (4, 0)
2. Two vertices of a triangle are (1, 2), (3, 5) and its centroid is at the origin. Find the coordinates of the third vertex.
3. Find the third vertex of a triangle, if two of its vertices are at (–3, 1) and (0, –2) and the centroid is at the origin.
4. A (3, 2) and B (–2, 1) are two vertices of a triangle ABC whose centroid G has the coordinates $(5/3, -1/3)$. Find the coordinates of the third vertex C of the triangle. [CBSE 2004]
5. If (–2, 3), (4, –3) and (4, 5) are the mid-points of the sides of a triangle, find the coordinates of its centroid.

LEVEL-2

6. Prove analytically that the line segment joining the middle points of two sides of a triangle is equal to half of the third side.

7. Prove that the lines joining the middle points of the opposite sides of a quadrilateral and the join of the middle points of its diagonals meet in a point and bisect one another.

8. If G be the centroid of a triangle ABC and P be any other point in the plane, prove that $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$.

9. If G be the centroid of a triangle ABC, prove that:
$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$

10. In Fig. 6.40, a right triangle BOA is given. C is the mid-point of the hypotenuse AB. Show that it is equidistant from the vertices O, A and B.

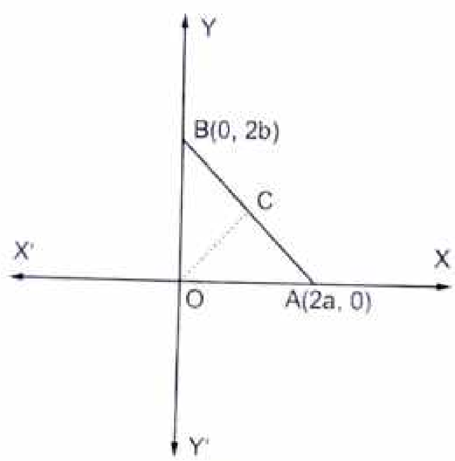


Fig. 6.40

ANSWERS

1.(i) $\left(1, \frac{1}{3}\right)$ (ii) $\left(\frac{4}{3}, \frac{2}{3}\right)$ 2. $(-4, -7)$ 3. $(3, 1)$

4. $(4, -4)$ 5. $\left(2, \frac{5}{3}\right)$

6.6 AREA OF A TRIANGLE

In earlier classes, we have computed the area of a triangle by using the formula

$$\text{Area of a triangle} = \frac{1}{2} \times \text{Base} \times \text{Altitude}$$

In class IX, we have used Heron's formula to find the area of a triangle when the lengths of its sides are given. In this section, we will find the area of a triangle in terms of the coordinates of its vertices. We can find the lengths of three sides of triangle by using distance formula and then, we can use the Heron's formula. But, this becomes tedious, particularly when the lengths of the sides are irrational numbers. That is why, we prefer to compute the area in terms of the coordinates of the vertices of the triangle. In the following theorem, we state and prove the same.

THEOREM *The area of a triangle, the coordinates of whose vertices are* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *is*

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

PROOF Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw AL, BM and CN perpendiculars from A, B, C on the x-axis.

Clearly, $ABML$, $ALNC$ and $BMNC$ are all trapeziums.

We know that

$$\text{Area of trapezium} = \frac{1}{2}\ \text{(Sum of parallel sides) (Distance between them)}$$

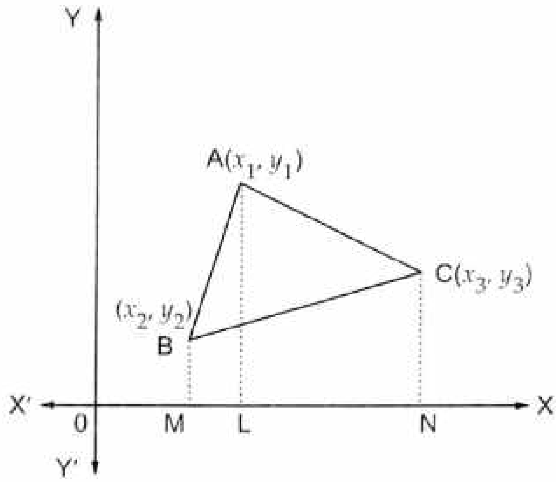


Fig. 6.41

We have,

$$\text{Area of } \Delta ABC = \text{Area of trapezium } ABML + \text{Area of trapezium } ALNC - \text{Area of trapezium } BMNC$$

Let Δ denote the area of ΔABC. Then,

$$\Delta = \frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)$$

$$\Rightarrow \quad \Delta = \left|\frac{1}{2}(y_2 - y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)\right|$$

$$\Rightarrow \quad \Delta = \left|\frac{1}{2}\left\{x_1 (y_2 + y_1 - y_1 - y_3) + x_2 (-y_2 - y_1 + y_2 + y_3) + x_3 (y_1 + y_3 - y_2 - y_3\right\}\right|$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\left|x_1 (y_2 - y_3) + x_2(y_3 - y_1) + x_3 (y_1 - y_2)\right|$$

REMARK 1 *To find the area of a polygon we divide it in triangles and take numerical value of the area of each of the triangles.*

REMARK 2 *The area of ΔABC can also be computed by using the following steps:*

STEP I *Write the coordinates of the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ in three columns as shown below and augment the coordinates of $A(x_1, y_1)$ as fourth column.*

STEP II *Draw broken parallel lines pointing down wards from left to right and right to left.*

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{array}$$

STEP III *Compute the sum of the products of numbers at the ends of the lines pointing downwards from left to right and subtract from this sum the sum of the products of numbers at the ends of the lines pointing downward from right to left i.e., compute*

$$(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)$$

STEP IV *Find the absolute of the number obtained in step III and take its half to obtain the area.*

REMARK 3 *Three points* $A(x_1, y_1), B(x_2, y_2)$ *and* $C(x_3, y_3)$ *are collinear iff*

Area of $\Delta ABC = 0$ i.e., $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE AREA OF A TRIANGLE WHEN COORDINATES OF ITS VERTICES AR GIVEN**

EXAMPLE 1 *Find the area of a triangle whose vertices are A (3, 2), B (11, 8) and C (8, 12).*

SOLUTION Let $A = (x_1, y_1) = (3, 2)$, $B = (x_2, y_2) = (11, 8)$ and $C = (x_3, y_3) = (8, 12)$ be th given points. Then,

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}\left|\{3(8 - 12) + 11(12 - 2) + 8(2 - 8)\}\right|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(-12 + 110 - 48)| = 25 \text{ sq. units}$$

ALITER We have,

$$\begin{array}{cccc} 3 & 11 & 8 & 3 \\ 2 & 8 & 12 & 2 \end{array}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}|(3 \times 8 + 11 \times 12 + 8 \times 2) - (11 \times 2 + 8 \times 8 + 3 \times 12)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(24 + 132 + 16) - (22 + 64 + 36|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|172 - 122| = 25 \text{ sq. units}$$

EXAMPLE 2 *Find the area of the triangle formed by the points A (5, 2), B (4, 7) and C (7, –4).*

SOLUTION Here, $x_1 = 5, y_1 = 2, x_2 = 4, y_2 = 7, x_3 = 7$ and $y_3 = -4$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|5(7 + 4) + 4(-4 - 2) + 7(2 - 7)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(5 \times 11 + 4 \times -6 + 7 \times -5)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(55 - 24 - 35)| = \frac{1}{2}|-4| = 2 \text{ sq. units}$$

ALITER We have,

$$\begin{array}{cccc} 5 & 4 & 7 & 5 \\ 2 & 7 & -4 & 2 \end{array}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}|(5 \times 7 + 4 \times -4 + 7 \times 2) - (4 \times 2 + 7 \times 7 + 5 \times -4)|$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|(35 - 16 + 14) - (8 + 49 - 20)|$$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|33-(37)| = \frac{1}{2}|-4| = 2$ sq. units

EXAMPLE 3 *Prove that the area of triangle whose vertices are $(t,\ t-2)$, $(t+2,\ t+2)$ and $(t+3,\ t)$ is independent of t.* **[CBSE 2016]**

SOLUTION Let $A=(x_1, y_1)=(t, t-2)$, $B=(x_2, y_2)=(t+2, t+2)$ and $C=(x_3, y_3)=(t+3, t)$ be the vertices of the given triangle. Then,

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|\{t(t+2-t)+(t+2)(t-t+2)+(t+3)(t-2-t-2)\}|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|\{(2t+2t+4-4t-12)\}| = |-4| = 4$ sq. units

Clearly, area of $\Delta\, ABC$ is independent of t.

ALITER We have,

$$\begin{array}{cccc} t & t+2 & t+3 & t \\ t-2 & t+2 & t & t-2 \end{array}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}\left|\{t(t+2)+(t+2)t+(t+3)(t-2)\} - \{(t+2)(t-2) + (t+3)(t+2)+t\times t\}\right|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|(t^2+2t+t^2+2t+t^2+t-6)-(t^2-4+t^2+5t+6+t^2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|(3t^2+5t-6)-(3t^2+5t+2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|-6-2|$

$\Rightarrow$ Area of $\Delta\, ABC = 4$ sq. units

Hence, Area of $\Delta\, ABC$ is independent of t.

EXAMPLE 4 *Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $(0, -1)$, $(2, 1)$ and $(0, 3)$. Find the ratio of area of the triangle formed to the area of the given triangle.* **[NCERT]**

SOLUTION Let $A(0, -1)$, $B(2, 1)$ and $C(0, 3)$ be the vertices of $\Delta\, ABC$. Let D, E, F be the mid-points of sides BC, CA and AB respectively. Then, the coordinates of D, E and F are $(1, 2)$, $(0, 1)$ and $(1, 0)$ respectively.

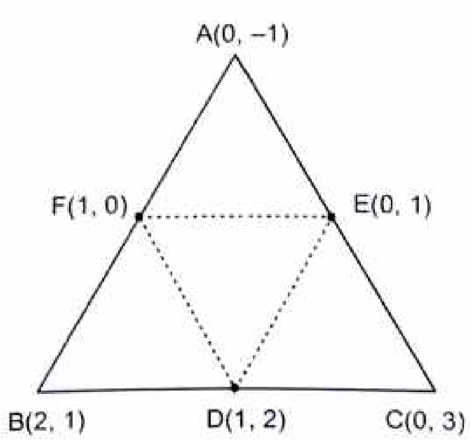


Fig. 6.42

Now,

$$\text{Area of } \Delta ABC = \frac{1}{2}\left| x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta ABC = \frac{1}{2}\left| 0 (1 - 3) + 2 (3 - (-1) + 0 (0 - 1) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta ABC = \frac{1}{2}\left| 0 + 8 + 0 \right| = 4 \text{ sq. units}$$

$$\text{Area of } \Delta DEF = \frac{1}{2}\left| x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta DEF = \frac{1}{2}\left| 1 (1 - 0) + 0 (0 - 2) + 1 (2 - 1) \right|$$

$\Rightarrow$ $$\text{Area of } \Delta DEF = \frac{1}{2}\left| 1 + 1 \right| = 1 \text{ sq. units}$$

$\therefore$ $$\text{Area of } \Delta DEF : \text{Area of } \Delta ABC = 1 : 4$$

EXAMPLE 5 *If $P(1, 2)$, $Q(1, 0)$ and $R(0, 1)$ are the mid-points of the sides AB, BC ar respectively of ΔABC, find the coordinates of the vertices A, B and C, and hence find th of ΔABC.*

[CBSE

SOLUTION Let the coordinates of the vertices A, B and C of ΔABC be (x_1, y_1), $(x_2, y_2$ (x_3, y_3) respectively. It is given that $P(1, 2)$ is the mid-point of AB.

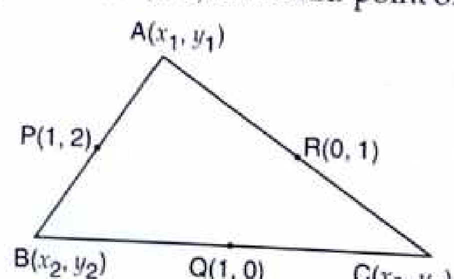


Fig. 6.43

$\therefore$ $$\frac{x_1 + x_2}{2} = 1 \text{ and } \frac{y_1 + y_2}{2} = 2$$

$\Rightarrow$ $x_1 + x_2 = 2$ ...(i) and $y_1 + y_2 = 4$

$Q(1, 0)$ is the mid-point of BC.

$\therefore$ $$\frac{x_2 + x_3}{2} = 1 \text{ and } \frac{y_2 + y_3}{2} = 0$$

$\Rightarrow$ $x_2 + x_3 = 2$ ...(iii) and $y_2 + y_3 = 0$

Point $R(0, 1)$ is the mid-point of AC.

$\therefore$ $$\frac{x_1 + x_3}{2} = 0 \text{ and } \frac{y_1 + y_3}{2} = 1$$

$\Rightarrow$ $x_1 + x_3 = 0$ ...(v) and $y_1 + y_3 = 2$

Adding (i), (iii) and (v), we obtain

$$2(x_1 + x_2 + x_3) = 2 + 2 + 0$$

$\Rightarrow$ $$x_1 + x_2 + x_3 = 2$$

Adding (ii), (iv) and (vi), we obtain

$$2(y_1 + y_2 + y_3) = 4 + 0 + 2 \Rightarrow y_1 + y_2 + y_3 = 3 \qquad \text{...(viii)}$$

Subtracting (i), (iii) and (v) respectively from (vii), we obtain

$$x_3 = 0, x_1 = 0, x_2 = 2$$

Subtracting (ii), (iv) and (vi) respectively from (viii), we obtain

$$y_3 = -1, y_1 = 3, y_2 = 1$$

Hence, the coordinates of the vertices of ΔABC are $A\,(0, 3)$, $B\,(2, 1)$ and $C\,(0, -1)$.

$$\therefore \quad \text{Area of } \Delta\, ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\left|0\,(1 - (-1)) + 2\,(-1 - 3) + 0\,(3 - 1)\right| = 4 \text{ sq. units}$$

EXAMPLE 6 *The vertices of ΔABC are $A\,(4, 6)$, $B\,(1, 5)$ and $C\,(7, 2)$. A line is drawn to intersect sides AB and AC at D and E respectively such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of ΔADE and compare it with the area of ΔABC.* **[NCERT]**

SOLUTION We have,

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE} = 4$$

$$\Rightarrow \quad \frac{AD + DB}{AD} = \frac{AE + EC}{AE} = 4$$

$$\Rightarrow \quad 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE} = 4$$

$$\Rightarrow \quad \frac{DB}{AD} = \frac{EC}{AE} = 3$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC} = \frac{1}{3}$$

$$\Rightarrow \quad AD : DB = AE : EC = 1 : 3$$

$\Rightarrow$ D and E divide AB and AC respectively in the ratio 1 : 3.

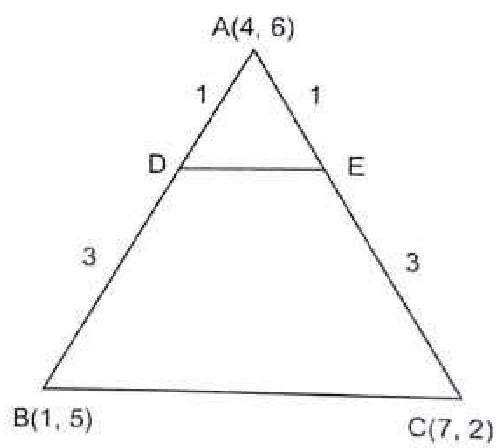


Fig. 6.44

So, the coordinates of D and E are

$$\left(\frac{1+12}{1+3}, \frac{5+18}{1+3}\right) = \left(\frac{13}{4}, \frac{23}{4}\right) \text{ and } \left(\frac{7+12}{1+3}, \frac{2+18}{1+3}\right) = \left(\frac{19}{4}, 5\right) \text{ respectively.}$$

We have,

$$\begin{array}{cccc} 4 & \frac{13}{4} & \frac{19}{4} & 4 \\ 6 & \frac{23}{4} & 5 & 6 \end{array}$$

$$\therefore \quad \text{Area of } \Delta\, ADE = \frac{1}{2}\left|\left(4 \times \frac{23}{4} + \frac{13}{4} \times 5 + \frac{19}{4} \times 6\right) - \left(\frac{13}{4} \times 6 + \frac{19}{4} \times \frac{23}{4} + 4 \times 5\right)\right|$$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}\left|\left(\frac{92}{4}+\frac{65}{4}+\frac{114}{4}\right)-\left(\frac{78}{4}+\frac{437}{16}+20\right)\right|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}\left|\frac{271}{4}-\frac{1069}{16}\right| = \frac{1}{2}\times\frac{15}{16} = \frac{15}{32}$ sq. units

Also, we have

$$\begin{array}{cccc} 4 & 1 & 7 & 4 \\ 6 & 5 & 2 & 6 \end{array}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|(4\times 5 + 1\times 2 + 7\times 6) - (1\times 6 + 7\times 5 + 4\times 2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|(20+2+42)-(6+35+8)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|64-49| = \frac{15}{2}$ sq. units

$\therefore$ $\frac{\text{Area of } \Delta\, ADE}{\text{Area of } \Delta\, ABC} = \frac{15/32}{15/2} = \frac{1}{16}$

Hence, Area of $\Delta\, ADE$: Area of $\Delta\, ABC = 1:16$

EXAMPLE 7 *If $A(4, -6)$, $B(3, -2)$ and $C(5, 2)$ are the vertices of $\Delta\, ABC$, then verify the fact a median of a triangle ABC divides it into two triangles of equal areas.* **[CBSE 2013, 2(**

SOLUTION Let D be the mid-point of BC. Then, the coordinates of D are $(4, 0)$.
We have,

$$\begin{array}{cccc} 4 & 3 & 5 & 4 \\ -6 & -2 & 2 & -6 \end{array}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|(4\times -2 + 3\times 2 + 5\times -6) - (3\times -6 + 5\times -2 + 4\times 2)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|(-8+6-30)-(-18-10+8)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|-32+20| = 6$ sq. units

Also, we have

$$\begin{array}{cccc} 4 & 3 & 4 & 4 \\ -6 & -2 & 0 & -6 \end{array}$$

$\therefore$ Area of $\Delta\, ABD = \frac{1}{2}\left|\begin{array}{l}\{(4\times(-2)+3\times 0+4\times(-6)\} \\ -\{3\times(-6)+4\times(-2)+4\times 0\}\end{array}\right|$

$\Rightarrow$ Area of $\Delta\, ABD = \frac{1}{2}|(-8+0-24)-(-18-8+0)|$

$\Rightarrow$ Area of $\Delta\, ABD = \frac{1}{2}|(-32+26| = 3$ sq. units

$\therefore$ $\frac{\text{Area of } \Delta\, ABC}{\text{Area of } \Delta\, ABD} = \frac{6}{3} = \frac{2}{1}$

$\Rightarrow$ Area of $\Delta\, ABC = 2$ (Area of $\Delta\, ABD$)

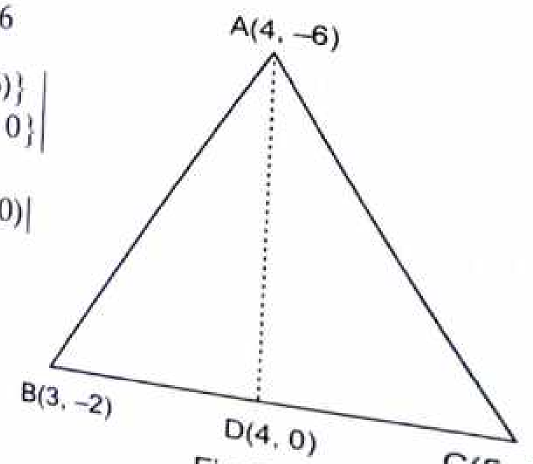


Fig. 6.45

EXAMPLE 8 *Find the area of the triangle ABC with A* $(1, -4)$ *and mid-points of sides through A being* $(2, -1)$ *and* $(0, -1)$. **[NCERT EXEMPLAR, CBSE 2015]**

SOLUTION Let the coordinates of B and C be (x_1, y_1) and (x_2, y_2) respectively. It is given that the points E and F are the mid-points of AB and AC respectively.

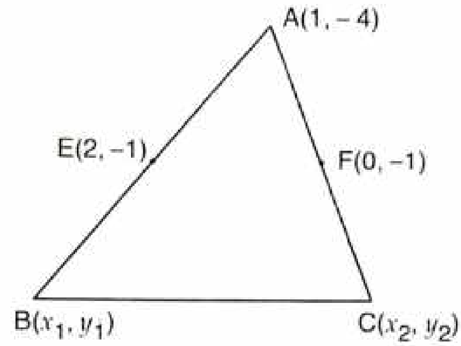


Fig. 6.46

$$\frac{x_1+1}{2}=2, \frac{y_1-4}{2}=-1 \text{ and } \frac{x_2+1}{2}=0, \frac{y_2-4}{2}=-1$$

$\Rightarrow$ $\quad x_1 = 3, y_1 = 2$ and $x_2 = -1, y_2 = 2$

Thus, the coordinates of B and C are $(3, 2)$ and $(-1, 2)$ respectively.

$\therefore$ $\quad$ Area of $\Delta ABC = \frac{1}{2}|(2+6+4)-(-12-2+2)|$

$$\begin{array}{cccc} 1 & 3 & -1 & 1 \\ -4 & 2 & 2 & -4 \end{array}$$

$$= \frac{1}{2}|12-(-12)| = 12 \text{ sq. units}$$

Type II **ON FINDING THE AREA OF A QUADRILATERAL WHEN COORDINATES OF ITS VERTICES ARE GIVEN**

EXAMPLE 9 *Find the area of the quadrilateral ABCD whose vertices are respectively* $A(1, 1), B(7, -3)$ $C(12, 2)$ *and* $D(7, 21)$. **[CBSE 2017]**

SOLUTION We have,

Area of quadrilateral $ABCD = |\text{Area of } \Delta ABC| + |\text{Area of } \Delta ACD|$

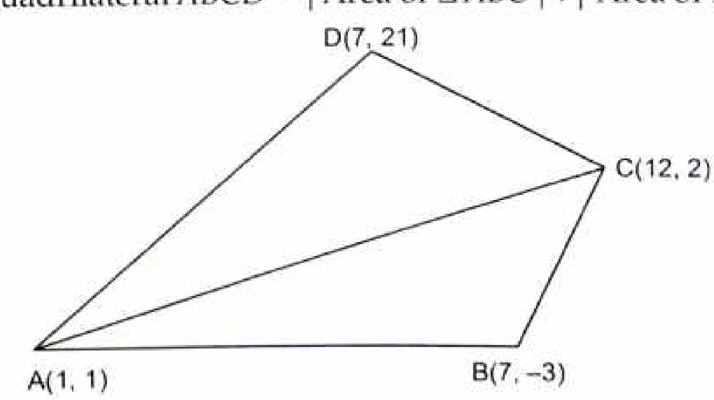


Fig. 6.47

We have,

$$\begin{array}{cccc} 1 & 7 & 12 & 1 \\ 1 & -3 & 2 & 1 \end{array}$$

$\therefore$ Area of $\Delta\, ABC = \dfrac{1}{2}\left|(1 \times -3 + 7 \times 2 + 12 \times 1) - (7 \times 1 + 12 \times (-3) + 1 \times 2)\right|$

$\Rightarrow$ Area of $\Delta\, ABC = \dfrac{1}{2}\left|(-3 + 14 + 12) - (7 - 36 + 2)\right|$

$\Rightarrow$ Area of $\Delta\, ABC = \dfrac{1}{2}\left|23 + 27\right| = 25$ sq. units

Also, we have

$$\begin{array}{cccc} 1 & 12 & 7 & 1 \\ 1 & 2 & 21 & 1 \end{array}$$

$\therefore$ Area of $\Delta\, ACD = \dfrac{1}{2}\left|(1 \times 2 + 12 \times 21 + 7 \times 1) - (12 \times 1 + 7 \times 2 + 1 \times 21)\right|$

$\Rightarrow$ Area of $\Delta\, ACD = \dfrac{1}{2}\left|(2 + 252 + 7) - (12 + 14 + 21)\right|$

$\Rightarrow$ Area of $\Delta\, ACD = \dfrac{1}{2}\left|261 - 47\right| = 107$ sq. units

$\therefore$ Area of quadrilateral $ABCD = 25 + 107 = 132$ sq. units

Type III ON COLLINEARITY OF THREE POINTS

FORMULA *Three points* $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) *are collinear iff*

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

or,

$$(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_1 y_3 + x_3 y_2 + x_2 y_1) = 0$$

EXAMPLE 10 *Prove that the points* $(2, -2), (-3, 8)$ *and* $(-1, 4)$ *are collinear.*

SOLUTION Let Δ be the area of the triangle formed by the given points.

We have,

$$\begin{array}{cccc} 2 & -3 & -1 & 2 \\ -2 & 8 & 4 & -2 \end{array}$$

$\therefore$ $\Delta = \dfrac{1}{2}\left|\{2 \times 8 + (-3) \times 4 + (-1) \times (-2)\} - \{(-3) \times (-2) + (-1) \times 8 + 2 \times 4\}\right|$

$\Rightarrow$ $\Delta = \dfrac{1}{2}\left|(16 - 12 + 2) - (6 - 8 + 8)\right|$

$\Rightarrow$ $\Delta = \dfrac{1}{2}\left|6 - 6\right| = 0$

Hence, given points are collinear.

EXAMPLE 11 *Prove that the points* $(a, b + c), (b, c + a)$ *and* $(c, a + b)$ *are collinear.* [CBSE

SOLUTION Let Δ be the area of the triangle formed by the points $(a, b + c), (b, c + a)$ $(c, a + b)$.

We have,

$$\begin{array}{cccc} a & b & c & a \\ b+c & c+a & a+b & b+c \end{array}$$

$\therefore$ $\Delta = \dfrac{1}{2}\left|\{a(c + a) + b(a + b) + c(b + c)\} - \{b(b + c) + c(c + a) + a(a + b)\}\right|$

$$\Rightarrow \qquad \Delta = \frac{1}{2}\left|(ac + a^2 + ab + b^2 + bc + c^2) - (b^2 + bc + c^2 + ca + a^2 + ab)\right|$$

$$\Rightarrow \qquad \Delta = 0$$

Hence, the given points are collinear.

Type IV ON FINDING THE DESIRED RESULT OR UNKNOWN WHEN THREE POINTS ARE COLLINEAR

EXAMPLE 12 *For what value of k are the points (k, 2 – 2k) (–k + 1, 2k) and (–4 – k, 6 – 2k) are collinear?*

SOLUTION Given points will be collinear, if area of the triangle formed by them is zero. We have,

$$\begin{array}{cccc} k & -k+1 & -4-k & k \\ 2-2k & 2k & 6-2k & 2-2k \end{array}$$

i.e.,

$$\left|\left\{2k^2 + (-k+1)(6-2k) + (-4-k)(2-2k)\right\} - \left\{(-k+1)(2-2k) + (-4-k)(2k) + k(6-2k)\right\}\right| = 0$$

$$\Rightarrow \qquad \left|(2k^2 + 6 - 8k + 2k^2 + 2k^2 + 6k - 8) - (2 - 4k + 2k^2 - 8k - 2k^2 + 6k - 2k^2)\right| = 0$$

$$\Rightarrow \qquad (6k^2 - 2k - 2) - (-2k^2 - 6k + 2) = 0$$

$$\Rightarrow \qquad 8k^2 + 4k - 4 = 0$$

$$\Rightarrow \qquad 2k^2 + k - 1 = 0 \Rightarrow (2k - 1)(k + 1) = 0 \Rightarrow k = 1/2 \text{ or, } k = -1$$

Hence, the given points are collinear for $k = 1/2$ or, $k = -1$.

EXAMPLE 13 *For what value of x will the points (x, –1), (2, 1) and (4, 5) lie on a line?* **[CBSE 2013]**

SOLUTION Given points will be collinear if the area of the triangle formed by them is zero. We have,

$$\begin{array}{cccc} x & 2 & 4 & x \\ -1 & 1 & 5 & -1 \end{array}$$

$\therefore$ Area of the triangle = 0

$$\Rightarrow \qquad \left|\{x \times 1 + 2 \times 5 + 4 \times (-1)\} - \{(2 \times -1 + 4 \times 1 + x \times 5\}\right| = 0$$

$$\Rightarrow \qquad (x + 10 - 4) - (-2 + 4 + 5x) = 0$$

$$\Rightarrow \qquad (x + 6) - (5x + 2) = 0$$

$$\Rightarrow \qquad -4x + 4 = 0$$

$$\Rightarrow \qquad x = 1$$

Hence, the given points lie on a line, if $x = 1$.

EXAMPLE 14 *Find the condition that the point (x, y) may lie on the line joining (3, 4) and (–5, –6).*

SOLUTION Since the point $P(x, y)$ lies on the line joining A (3, 4) and B (–5, –6). Therefore, P, A and B are collinear points.

$$\begin{array}{cccc} x & 3 & -5 & x \\ y & 4 & -6 & y \end{array}$$

$$\therefore \qquad \{4x + 3 \times -6 + (-5) \times y\} - \{3y + (-5) \times 4 + x \times (-6)\} = 0$$

$\Rightarrow \qquad (4x - 18 - 5y) - (3y - 6x - 20) = 0$

$\Rightarrow \qquad 10x - 8y + 2 = 0 \Rightarrow 5x - 4y + 1 = 0$

Hence, the point (x, y) lies on the line joining $(3, 4)$ and $(-5, -6)$, if $5x - 4y + 1 = 0$.

EXAMPLE 15 *If $P(x, y)$ is any point on the line joining the points $A(a, 0)$ and $B(0, b)$, then show that $\frac{x}{a} + \frac{y}{b} = 1$.* **[CBSE 2009]**

SOLUTION It is given that the point $P(x, y)$ lies on the line segment joining points $A(a, 0)$ and $B(0, b)$. Therefore, points $P(x, y)$, $A(a, 0)$ and $B(0, b)$ are collinear points.

$$\begin{array}{cccc} x & a & 0 & x \\ y & 0 & b & y \end{array}$$

$\therefore \qquad (x \times 0 + a \times b + 0 \times y) - (a \times y + 0 \times 0 + x \times b) = 0$

$\Rightarrow \qquad ab - (ay + bx) = 0$

$\Rightarrow \qquad ab = ay + bx$

$\Rightarrow \qquad \frac{ab}{ab} = \frac{ay}{ab} + \frac{bx}{ab}$ [Dividing throughout by ab]

$\Rightarrow \qquad 1 = \frac{y}{b} + \frac{x}{a}$ or $\frac{x}{a} + \frac{y}{b} = 1.$

EXAMPLE 16 *If the points (p, q), (m, n) and $(p - m, q - n)$ are collinear, show that $pn = qm$.* **[CBSE 2010]**

SOLUTION Given points are collinear. Therefore

$$\begin{array}{cccc} p & m & p-m & p \\ q & n & q-n & q \end{array}$$

$\{p \times n + m(q - n) + (p - m) q\} - \{m \times q + (p - m) n + p(q - n)\} = 0$

$\Rightarrow \qquad (pn + qm - mn + pq - mq) - (mq + pn - mn + pq - pn) = 0$

$\Rightarrow \qquad (pn + pq - mn) - (mq - mn + pq) = 0$

$\Rightarrow \qquad pn - mq = 0$

$\Rightarrow \qquad pn = qm$

EXAMPLE 17 *Find k so that the point $P(-4, 6)$ lies on the line segment joining $A(k, 10)$ and $B(3, -8)$. Also, find the ratio in which P divides AB.* **[CBSE 2010]**

SOLUTION If $P(-4, 6)$ lies on the line segment joining $A(k, 10)$ and $B(3, -8)$, then P, A and B are collinear.

$$\begin{array}{cccc} -4 & k & 3 & -4 \\ 6 & 10 & -8 & 6 \end{array}$$

$\therefore \qquad (-4 \times 10 + k \times -8 + 3 \times 6) - (6k + 30 + -4 \times -8) = 0$

$\Rightarrow \qquad (-40 - 8k + 18) - (6k + 30 + 32) = 0$

$\Rightarrow \qquad (-22 - 8k) - (6k + 62) = 0$

$\Rightarrow \qquad -14k - 84 = 0$

$\Rightarrow \qquad k = -6$

Suppose P divides AB in the ratio $\lambda : 1$. Then, the coordinates of P are $\left(\frac{3\lambda - 6}{\lambda + 1}, \frac{-8\lambda + 10}{\lambda + 1}\right)$. But, the coordinates of P are $(-4, 6)$.

Fig. 6.48

$$\therefore \quad \frac{3\lambda - 6}{\lambda + 1} = -4 \text{ and } \frac{-8\lambda + 10}{\lambda + 1} = 6$$

$$\Rightarrow \quad \lambda = \frac{2}{7}$$

Hence, P divides AB in the ratio $\frac{2}{7} : 1$ or $2 : 7$.

EXAMPLE 18 *If the points $A(1, -2)$, $B(2, 3)$, $C(-3, 2)$ and $D(-4, -3)$ are the vertices of parallelogram $ABCD$, then taking AB as the base, find the height of the parallelogram.*

[CBSE 2013]

SOLUTION Let $DM = h$ be the height of parallelogram $ABCD$ when AB is taken as the base. From Fig. 6.49,

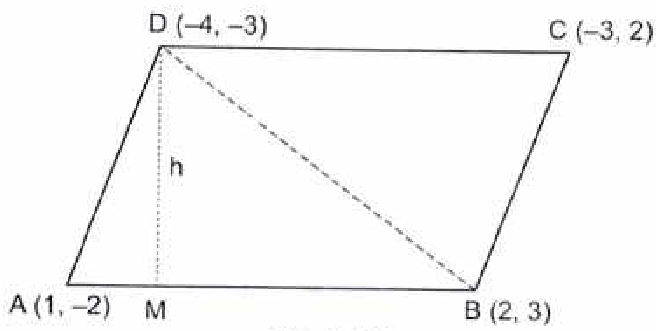


Fig. 6.49

$$\text{Area of } \Delta ABD = \frac{1}{2}(AB \times DM)$$

$$\Rightarrow \quad \text{Area of } \Delta ABD = \frac{1}{2} AB \times h$$

$$\Rightarrow \quad h = \frac{2\,(\text{Area of } \Delta ABD)}{AB} \qquad \ldots\text{(i)}$$

Using distance formula, we obtain

$$AB = \sqrt{(2-1)^2 + (3+2)^2} = \sqrt{26}$$

The coordinates of vertices of ΔABD are $A(1, -2)$, $B(2, 3)$ and $D(-4, -3)$.

$$\begin{array}{cccc} 1 & 2 & -4 & 1 \\ -2 & 3 & -3 & -2 \end{array}$$

$$\therefore \quad \text{Area of } \Delta ABD = \frac{1}{2}\left|\{1 \times 3 + 2(-3) + (-4)(-2)\} - \{1 \times (-3) + (-4) \times 3 + 2 \times (-2)\}\right|$$

$$= \frac{1}{2}\left|(3 - 6 + 8) - (-3 - 12 - 4)\right| = \frac{1}{2}(5 + 19) = 12 \text{ sq. units}$$

Substituting the values of AB and area of ΔABD in (i), we obtain

$$h = \frac{2 \times 12}{\sqrt{26}} = \frac{24}{\sqrt{26}} \text{ Units}$$

EXAMPLE 19 *Three vertices of a parallelogram ABCD are A(3, −4), B(−1, −3) and C(−6, 2). Find the coordinates of vertex D and find the area of parallellogram ABCD.* **[CBSE 2013]**

SOLUTION Let (x, y) be the coordinates of vertex D. We know that the diagonals of a parallelogram bisect each other. Therefore, mid-points of diagonals AC and BD are same. Consequently, the coordinates of their mid-points are same and hence,

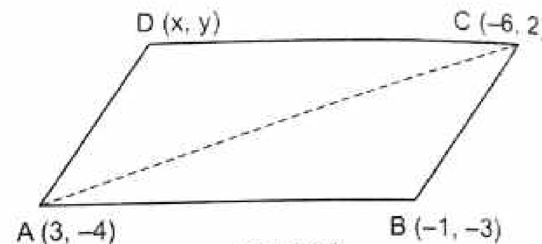


Fig. 6.50

$$\left(\frac{x-1}{2}, \frac{y-3}{2}\right) = \left(\frac{3-6}{2}, \frac{-4+2}{2}\right)$$

$$\Rightarrow \quad \left(\frac{x-1}{2}, \frac{y-3}{2}\right) = \left(-\frac{3}{2}, -1\right)$$

$$\Rightarrow \quad \frac{x-1}{2} = -\frac{3}{2} \text{ and } \frac{y-3}{2} = -1$$

$$\Rightarrow \quad x - 1 = -3 \text{ and } y - 3 = -2$$

$$\Rightarrow \quad x = -2 \text{ and } y = 1$$

Hence, the coordinates of D are $(-2, 1)$.

We know that each diagonal of a parallelogram divides it in two triangles of equal area.

$$\therefore \quad \text{Area of parallelogram } ABCD = 2\,(\text{Area of } \Delta ABC) \qquad \ldots(i)$$

$$\begin{array}{cccc} 3 & -1 & -6 & 3 \\ -4 & -3 & 2 & -4 \end{array}$$

Now, Area of $\Delta ABC = \frac{1}{2}\left|\{3 \times (-3) + (-1) \times 2 + (-6)(-4)\} - \{3 \times 2 + (-6)(-3) + (-1)(-4)\}\right|$

$$= \frac{1}{2}\left|(-9 - 2 + 24) - (6 + 18 + 4)\right| = \frac{1}{2}(13 - 28) = \frac{15}{2} \text{ square units.}$$

Substituting this in (i), we obtain

$$\text{Area of parallelogram } ABCD = 2 \times \frac{15}{2} = 15 \text{ square units.}$$

EXAMPLE 20 *If the area of ΔABC formed by $A(x, y)$, $B(1, 2)$ and $C(2, 1)$ is 6 square units, then prove that $x + y = 15$ or, $x + y + 9 = 0$.* **[CBSE 2013]**

SOLUTION We have,

$$\begin{array}{cccc} x & 1 & 2 & x \\ y & 2 & 1 & y \end{array}$$

$$\text{Area of } \Delta ABC = 6$$

$$\Rightarrow \quad \frac{1}{2}\left|(2x + 1 + 2y) - (x + 4 + y)\right| = 6$$

$$\Rightarrow \quad |x + y - 3| = 12$$

$$\Rightarrow \quad x + y - 3 = \pm 12$$

$$\Rightarrow \quad x + y - 15 = 0 \text{ or, } x + y + 9 = 0$$

$$\Rightarrow \quad x + y = 15 \text{ or, } x + y + 9 = 0$$

EXAMPLE 21 *If the points $P(-3, 9)$, $Q(a, b)$ and $R(4, -5)$ are collinear and $a + b = 1$, find the values of a and b.* **[CBSE 2014]**

SOLUTION It is given that the points $P(-3, 9)$, $Q(a, b)$ and $R(4, -5)$ are collinear.

$$\begin{array}{cccc} -3 & a & 4 & -3 \\ 9 & b & -5 & 9 \end{array}$$

$$\therefore \quad \text{Area of } \Delta PQR = 0$$

$$\Rightarrow \quad \left|\{-3b - 5a + 36\} - \{15 + 4b + 9a\}\right| = 0$$

$\Rightarrow \quad |(-14a - 7b + 21)| = 0$

$\Rightarrow \quad 14a + 7b - 21 = 0$

$\Rightarrow \quad 2a + b - 3 = 0$ ...(i)

It is given that $a + b = 1$ ...(ii)

Solving (i) and (ii), we obtain $a = 2$ and $b = -1$.

LEVEL-2

Type I ON FINDING THE AREA OF A TRIANGLE

EXAMPLE 22 *If D, E and F are the mid-points of sides BC, CA and AB respectively of a ΔABC, then using coordinate geometry prove that*

$$\text{Area of } \Delta DEF = \frac{1}{4}(\text{Area of } \Delta ABC)$$

SOLUTION Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of ΔABC. Then, the coordinates of D, E and F are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ and $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ respectively.

We have,

$$\Delta_1 = \text{Area of } \Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

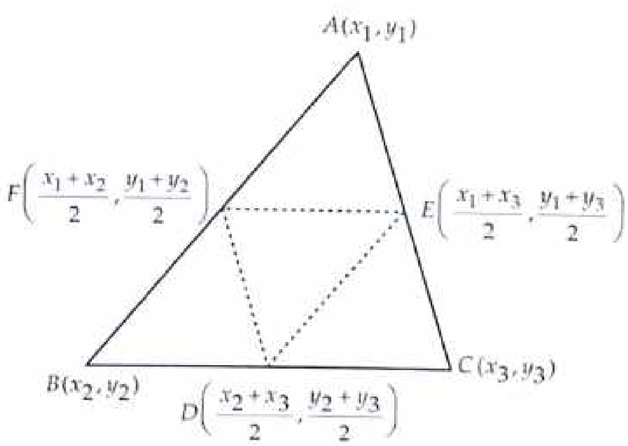


Fig. 6.51

$$\Delta_2 = \text{Area of } \Delta DEF = \frac{1}{2}\left|\left(\frac{x_2+x_3}{2}\right)\left(\frac{y_1+y_3}{2} - \frac{y_1+y_2}{2}\right) + \left(\frac{x_1+x_3}{2}\right)\left(\frac{y_1+y_2}{2} - \frac{y_2+y_3}{2}\right) + \left(\frac{x_1+x_2}{2}\right)\left(\frac{y_2+y_3}{2} - \frac{y_1+y_3}{2}\right)\right|$$

$$\Rightarrow \quad \Delta_2 = \frac{1}{8}|(x_2 + x_3)(y_3 - y_2) + (x_1 + x_3)(y_1 - y_3) + (x_1 + x_2)(y_2 - y_1)|$$

$$\Rightarrow \quad \Delta_2 = \frac{1}{8}|x_1(y_1 - y_3 + y_2 - y_1) + x_2(y_3 - y_2 + y_2 - y_1) + x_3(y_3 - y_2 + y_1 - y_3)|$$

$$\Rightarrow \quad \Delta_2 = \frac{1}{8}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$\Rightarrow \qquad \Delta_2 = \frac{1}{4}\,(\text{Area of } \Delta\, ABC) = \frac{1}{4}\,\Delta_1$$

Hence, Area of $\Delta DEF = \frac{1}{4}$ (Area of $\Delta\, ABC$)

Type II **MIXED PROBLEMS BASED UPON THE CONCEPT OF AREA OF A TRIANGLE**

EXAMPLE 23 *If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC, where $x_i, y_i, i = 1, 2, 3$ are integers. Then, the area of $\Delta\, ABC$ is given by

$$\Delta = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$\Rightarrow \qquad \Delta = \text{A rational number} \qquad [\because x_i, y_i \text{ are integers}]$$

If possible, let the triangle ABC be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{Side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \qquad [\because AB = BC = CA]$$

$$\Rightarrow \qquad \Delta = \frac{\sqrt{3}}{4}(\text{A positive integer}) \qquad \left[\begin{array}{l}\because \text{ vertices are integers} \\ \therefore AB^2 \text{ is a positive integer}\end{array}\right]$$

$$\Rightarrow \qquad \Delta = \text{An irrational number}$$

This is a contradiction to the fact that the area is a rational number.

Hence, the triangle cannot be equilateral.

EXAMPLE 24 *If the coordinates of two points A and B are (3, 4) and (5, −2) respectively. Find the coordinates of any point P, if PA = PB and Area of $\Delta PAB = 10$.*

SOLUTION Let the coordinates of P be (x, y). Then,

$$PA = PB$$

$$\Rightarrow \qquad PA^2 = PB^2$$

$$\Rightarrow \qquad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$$

$$\Rightarrow \qquad x - 3y - 1 = 0 \qquad \ldots(\text{i})$$

Now, Area of $\Delta PAB = 10$

$$\begin{array}{cccc} x & 3 & -5 & x \\ y & 4 & -2 & y \end{array}$$

$$\Rightarrow \qquad \frac{1}{2}\left|(4x + 3 \times -2 + 5y) - (3y + 20 - 2x)\right| = 10$$

$$\Rightarrow \qquad \left|(4x + 5y - 6) - (-2x + 3y + 20)\right| = 20$$

$$\Rightarrow \qquad \left|6x + 2y - 26\right| = 20$$

$$\Rightarrow \qquad 6x + 2y - 26 = \pm 20$$

$$\Rightarrow \qquad 6x + 2y - 46 = 0 \text{ or, } 6x + 2y - 6 = 0$$

$$\Rightarrow \qquad 3x + y - 23 = 0 \text{ or, } 3x + y - 3 = 0 \qquad \ldots(\text{ii})$$

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$ we get $x = 7, y = 2$.

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get $x = 1, y = -0$.

Thus, the coordinates of P are $(7, 2)$ or $(1, 0)$.

EXAMPLE 25 *The coordinates of A, B, C are (6, 3), (–3, 5) and (4, –2) respectively and P is any point (x, y). Show that the ratio of the areas of triangles PBC and ABC is* $\left|\frac{x + y - 2}{7}\right|$.

SOLUTION We have,

$$\begin{array}{cccc} x & -3 & 4 & x \\ y & 5 & -2 & y \end{array}$$

$\therefore$ Area of $\Delta PBC = \frac{1}{2}|(5x + 6 + 4y) - (-3y + 20 - 2x)|$

$\Rightarrow$ Area of $\Delta PBC = \frac{1}{2}|5x + 6 + 4y + 3y - 20 + 2x|$

$\Rightarrow$ Area of $\Delta PBC = \frac{1}{2}|7x + 7y - 14|$

$\Rightarrow$ Area of $\Delta PBC = \frac{7}{2}|x + y - 2|$

$\Rightarrow$ Area of $\Delta ABC = \frac{7}{2}|6 + 3 - 2|$ $\left[\begin{array}{l}\text{Replacing } x \text{ by 6 and } y = 3 \\ \text{in Area of } \Delta PBC\end{array}\right]$

$\Rightarrow$ Area of $\Delta ABC = \frac{49}{2}$

$$\therefore \quad \frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{7}{2}|x + y - 2|}{\frac{49}{2}} = \frac{|x + y - 2|}{7} = \left|\frac{x + y - 2}{7}\right|$$

EXERCISE 6.5

LEVEL-1

1. Find the area of a triangle whose vertices are
   (i) $(6, 3), (-3, 5)$ and $(4, -2)$ (ii) $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$
   (iii) $(a, c + a), (a, c)$ and $(-a, c - a)$
2. Find the area of the quadrilaterals, the coordinates of whose vertices are
   (i) $(-3, 2), (5, 4), (7, -6)$ and $(-5, -4)$ (ii) $(1, 2), (6, 2), (5, 3)$ and $(3, 4)$
   (iii) $(-4, -2), (-3, -5), (3, -2), (2, 3)$ [NCERT] [CBSE 2009]
3. The four vertices of a quadrilateral are $(1, 2), (-5, 6), (7, -4)$ and $(k, -2)$ taken in order. If the area of the quadrilateral is zero, find the value of k.
4. The vertices of ΔABC are $(-2, 1), (5, 4)$ and $(2, -3)$ respectively. Find the area of the triangle and the length of the altitude through A.
5. Show that the following sets of points are collinear.
   (a) $(2, 5), (4, 6)$ and $(8, 8)$ (b) $(1, -1), (2, 1)$ and $(4, 5)$.
6. Find the area of a quadrilateral $ABCD$, the coordinates of whose varities are $A(-3, 2)$, $B(5, 4)$, $C(7, -6)$ and $(-5, -4)$. [CBSE 2016]

7. In ΔABC, the coordinates of vertex A are $(0, -1)$ and $D(1, 0)$ and $E(0, 1)$ respectively the mid-points of the sides AB and AC. If F is the mid-point of side BC, find the area of ΔDEF. **[CBSE 2016]**
8. Find the area of the triangle PQR with $Q(3, 2)$ and the mid-points of the sides through Q being $(2, -1)$ and $(1, 2)$. **[CBSE 2015]**
9. If $P(-5, -3), Q(-4, -6), R(2, -3)$ and $S(1, 2)$ are the vertices of a quadrilateral $PQRS$, find its area. **[CBSE 2015]**
10. If $A(-3, 5), B(-2, -7), C(1, -8)$ and $D(6, 3)$ are the vertices of a quadrilateral $ABCD$, find its area. **[CBSE 2014, 2018]**
11. For what value of a the point $(a, 1), (1, -1)$ and $(11, 4)$ are collinear? **[CBSE 2017]**
12. Prove that the points $(a, b), (a_1, b_1)$ and $(a - a_1, b - b_1)$ are collinear if $ab_1 = a_1 b$.
13. If the vertices of a triangle are $(1, -3), (4, p)$ and $(-9, 7)$ and its area is 15 sq. units, findthe value(s) of p. **[CBSE 2012]**
14. If (x, y) be on the line joining the two points $(1, -3)$ and $(-4, 2)$, prove that $x + y + 2 = 0$.
15. Find the value of k if points $(k, 3), (6, -2)$ and $(-3, 4)$ are collinear. **[CBSE 2008]**
16. Find the value of k, if the points $A(7, -2), B(5, 1)$ and $C(3, 2k)$ are collinear. **[CBSE 2010]**
17. If the point $P(m, 3)$ lies on the line segment joining the points $A\left(-\frac{2}{5}, 6\right)$ and $B(2, 8)$, find the value of m. **[CBSE 2010]**
18. If $R(x, y)$ is a point on the line segment joining the points $P(a, b)$ and $Q(b, a)$, then prove that $x + y = a + b$. **[CBSE 2010]**
19. Find the value of k, if the points $A(8, 1), B(3, -4)$ and $C(2, k)$ are collinear. **[CBSE 2010]**
20. Find the value of a for which the area of the triangle formed by the points $A(a, 2a), B(-2, 6)$ and $C(3, 1)$ is 10 square units.
21. If $a \neq b \neq 0$, prove that the points $(a, a^2), (b, b^2), (0, 0)$ are never collinear. **[CBSE 2017]**
22. The area of a triangle is 5 sq. units. Two of its vertices are at $(2, 1)$ and $(3, -2)$. If the third vertex is $(7/2, y)$, find y. **[CBSE 2017]**

LEVEL-2

23. Prove that the points $(a, 0), (0, b)$ and $(1, 1)$ are collinear if, $\frac{1}{a} + \frac{1}{b} = 1$.
24. The point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio $k : 1$. Find the two values of k for which the area of ΔABC where B is $(1, 5)$ and $C(7, -2)$ is equal to 2 units.
25. The area of a triangle is 5. Two of its vertices are $(2, 1)$ and $(3, -2)$. The third vertex lies on $y = x + 3$. Find the third vertex.
26. If $a \neq b \neq c$, prove that the points $(a, a^2), (b, b^2), (c, c^2)$ can never be collinear.
27. Four points $A(6, 3), B(-3, 5), C(4, -2)$ and $D(x, 3x)$ are given in such a way that $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.

28. If three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on the same line, prove that

$$\frac{y_2 - y_3}{x_2 x_3} + \frac{y_3 - y_1}{x_3 x_1} + \frac{y_1 - y_2}{x_1 x_2} = 0.$$

29. Find the area of a parallelogram $ABCD$ if three of its vertices are $A(2, 4)$, $B(2 + \sqrt{3}, 5)$ and $C(2, 6)$. **[CBSE 2013]**

30. Find the value (s) of k for which the points $(3k - 1, k - 2), (k, k - 7)$ and $(k - 1, -k - 2)$ are collinear. **[CBSE 2014]**

31. If the points $A(-1, -4)$, $B(b, c)$ and $C(5, -1)$ are collinear and $2b + c = 4$, find the values of b and c. **[CBSE 2014]**

32. If the points $A(-2, 1)$, $B(a, b)$ and $C(4, -1)$ are collinear and $a - b = 1$, find the values of a and b. **[CBSE 2014]**

33. If the points $A(1, -2)$, $B(2, 3)$, $C(a, 2)$ and $D(-4, -3)$ form a parallelogram, find the value of a and height of the parallelogram taking AB as base. **[NCERT EXEMPLAR]**

34. $A(6, 1)$, $B(8, 2)$ and $C(9, 4)$ are three vertices of a parallelogram $ABCD$. If E is the mid-point of DC, find the area of ΔADE. **[NCERT EXEMPLAR]**

35. If $D(-1/5, 5/2)$, $E(7, 3)$ and $F(7/2, 7/2)$ are the mid-points of sides of ΔABC, find the area of ΔABC. **[NCERT EXEMPLAR]**

ANSWERS

1. (i) $\frac{49}{2}$ sq. units (ii) $a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ (iii) a^2
2. (i) 80 sq. units (ii) $\frac{11}{2}$ sq. units (iii) 28 sq. units
3. $k = 3$
4. 20 sq. units. $\frac{40}{\sqrt{58}}$
6. 85 sq. units
7. 4 sq. units, 1 sq. unit
8. 12 sq. units
9. 13 sq. units
10. 72 sq. units
11. $a = 5$
13. $p = -3, -9$
15. $k = \frac{-3}{2}$
16. $k = 2$
17. $m = -4$
19. $k = -5$
20. $a = 0, \frac{8}{3}$
22. $y = \frac{13}{2}, -\frac{27}{2}$
24. $k = 7$ or $\frac{31}{9}$
25. $(7/2, 13/2)$ or $(-3/2, 3/2)$
27. $\frac{11}{8}, \frac{-3}{8}$
29. $2\sqrt{3}$ sq. units.
30. $k = 0, 3$
31. $b = 3, c = -2$
32. $a = 1, b = 0$
33. $a = -3, h = \frac{12\sqrt{2}}{\sqrt{13}}$
34. $\frac{3}{4}$ sq. units
35. 11 sq. units

HINTS TO SELECTED PROBLEMS

1. (ii) Let Δ be the area of the triangle formed by the points $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$. Then,

We have,

$$\begin{array}{cccc} at_1^2 & at_2^2 & at_3^2 & at_1^2 \\ 2at_1 & 2at_2 & 2at_3 & 2at_1 \end{array}$$

$\therefore \quad \Delta = \frac{1}{2}\left|(at_1^2 \times 2at_2 + at_2^2 \times 2at_3 + at_3^2 \times 2at_1) - (at_2^2 \times 2at_1 + at_3^2 \times 2at_2 + at_1^2 \times 2at_3\right.$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(2a^2 t_1^2 t_2 + 2a^2 t_2^2 t_3 + 2a^2 t_3^2 t_1) - (2a^2 t_1 t_2^2 + 2a^2 t_2 t_3^2 + 2a^2 t_1\right.$

$\Rightarrow \quad \Delta = a^2 \left|(t_1^2 t_2 + t_2^2 t_3 + t_3^2 t_1) - (t_1 t_2^2 + t_2 t_3^2 + t_3 t_1^2)\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_1^2 t_2 - t_1^2 t_3) + (t_2^2 t_3 - t_2 t_3^2) + (t_3^2 t_1 - t_1 t_2^2)\right|$

$\Rightarrow \quad \Delta = a^2 \left|t_1^2 (t_2 - t_3) + t_2 t_3 (t_2 - t_3) - t_1(t_2^2 - t_3^3)\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_2 - t_3)\{t_1^2 + t_2 t_3 - t_1(t_2 + t_3)\}\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_2 - t_3)\{t_1^2 + t_2 t_3 - t_1 t_2 - t_1 t_3\}\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_2 - t_3)\{(t_1^2 - t_1 t_2) - (t_1 t_3 - t_2 t_3)\}\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_2 - t_3)\{t_1 (t_1 - t_2) - t_3 (t_1 - t_2)\}\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_2 - t_3)(t_1 - t_2)(t_1 - t_3)\right|$

$\Rightarrow \quad \Delta = a^2 \left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$

(iii) Let Δ be the area of the triangle formed by the points $(a, c + a)$, (a, c) and $(-a$
We have,

$$\begin{array}{cccc} a & a & -a & a \\ c + a & c & c - a & c + a \end{array}$$

$\therefore \quad \Delta = \frac{1}{2}\left|\{ac + a(c - a) - a(c + a)\} - \{a(c + a) - ac + a(c - a)\}\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|\{ac + ac - a^2 - ac - a^2\} - \{ac + a^2 - ac + ac - a^2\}\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|(ac - 2a^2) - (ac)\right|$

$\Rightarrow \quad \Delta = \frac{1}{2}\left|-2a^2\right| = a^2$ sq. units

2. (i) Let $A(-3,2)$, $B(5,4)$, $C(7,-6)$ and $D(-5,-4)$ be the vertices of the quadrila
We have,

$$\begin{array}{cccc} -3 & 5 & 7 & -3 \\ 2 & 4 & -6 & 2 \end{array}$$

$\therefore \quad$ Area of $\Delta\, ABC = \frac{1}{2}\left|(-12 - 30 + 14) - (10 + 28 + 18)\right|$

$\Rightarrow \quad$ Area of $\Delta\, ABC = \frac{1}{2}\left|-28 - 56\right| = 42$ sq. units

We have,

$$\begin{array}{cccc} -3 & 7 & -5 & -3 \\ 2 & -6 & -4 & 2 \end{array}$$

$\therefore$ Area of $\Delta\, ACD = \frac{1}{2}|(18 - 28 - 10) - (14 + 30 + 12)|$

$\Rightarrow$ Area of $\Delta\, ACD = \frac{1}{2}|-20 - 56| = 38$ sq. units

$\therefore$ Area of quadrilateral $ABCD = (42 + 38) = 80$ sq. units

(ii) Let A (1, 2), B (6, 2), C (5, 3) and D (3, 4) be the vertices of the given quadrilateral. We have,

$$\begin{array}{cccc} 1 & 6 & 5 & 1 \\ 2 & 2 & 3 & 2 \end{array}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|(2 + 18 + 10) - (12 + 10 + 3)|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}|30 - 25| = \frac{5}{2}$ sq. units

Also, we have

$$\begin{array}{cccc} 1 & 5 & 3 & 1 \\ 2 & 3 & 4 & 2 \end{array}$$

$\therefore$ Area of $\Delta\, ACD = \frac{1}{2}|(3 + 20 + 6) - (10 + 9 + 4)| = 3$ sq. units

$\therefore$ Area of quadrilateral $ABCD = \left(\frac{5}{2} + 3\right)$ sq. units $= \frac{11}{2}$ sq. units

3. Let $A\,(1, 2)$, $B\,(-5, 6)$, $C\,(7, -4)$ and $D\,(k, -2)$ be the vertices of the quadrilateral. We have,

$$\begin{array}{cccc} 1 & -5 & 7 & 1 \\ 2 & 6 & -4 & 2 \end{array}$$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}|\{(6 + 20 + 14) - (-10 + 42 - 4)\}|$

$\Rightarrow$ Area of $\Delta\, ABC = \frac{1}{2}(40 - 28) = 6$ sq. units

Also, we have

$$\begin{array}{cccc} 1 & 7 & k & 1 \\ 2 & -4 & -2 & 2 \end{array}$$

Area of $\Delta\, ACD = \frac{1}{2}\{(-4 - 14 + 2k) - (14 - 4k - 2)\}$

$\Rightarrow$ Area of $\Delta\, ACD = \frac{1}{2}\{(2k - 18) - (12 - 4k)\}$

$\Rightarrow$ Area of $\Delta\, ACD = \frac{1}{2}(6k - 30) = (3k - 15)$

$\therefore$ Area of quadrilateral $ABCD = 6 + 3k - 15 = 3k - 9$

It is given that the area of quadrilateral is zero.

$\therefore$ $3k - 9 = 0 \Rightarrow k = 3$

4. We have,

$$\begin{array}{cccc} -2 & 5 & 2 & -2 \\ 1 & 4 & -3 & 1 \end{array}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|(-8-15+2)-(5+8+6)| = 20$ sq. units

We have, $BC = \sqrt{(5-2)^2+(4+3)^2} = \sqrt{58}$

Now, Area of $\Delta ABC = \frac{1}{2} BC \times$ (Length of the altitude through A)

$\Rightarrow$ $20 = \frac{1}{2} \times \sqrt{58} \times$ Length of the altitude through A

$\therefore$ Length of the altitude through $= \frac{40}{\sqrt{58}}$

11. Points $(a, 1), (1, -1)$ and $(11, 4)$ will be collinear, if

$$\begin{array}{cccc} a & 1 & 11 & a \\ 1 & -1 & 4 & 1 \end{array}$$

$$(a \times -1 + 1 \times 4 + 11 \times 1) - (1 \times 1 + 11 \times -1 + a \times 4) = 0$$

$\Rightarrow$ $(-a+15)-(1-11+4a) = 0$

$\Rightarrow$ $-a+15+10-4a = 0$

$\Rightarrow$ $-5a+25 = 0$

$\Rightarrow$ $a = 5$

12. Points $(a, b), (a_1, b_1)$ and $(a-a_1, b-b_1)$ will be collinear, if

$$\begin{array}{cccc} a & a_1 & a-a_1 & a \\ b & b_1 & b-b_1 & b \end{array}$$

$$\{ab_1 + a_1(b-b_1) + (a-a_1)b\} - \{a_1b + (a-a_1)b_1 + a(b-b_1)\} = 0$$

$\Rightarrow$ $(ab_1 + a_1b - a_1b_1 + ab - a_1b) - (a_1b + ab_1 - a_1b_1 + ab - ab_1) = 0$

$\Rightarrow$ $(ab_1 - a_1b_1 + ab) - (a_1b - a_1b_1 + ab) = 0$

$\Rightarrow$ $ab_1 - a_1b = 0$

$\Rightarrow$ $ab_1 = a_1b$

23. Points $(a, 0), (0, b)$ and $(1, 1)$ are collinear.

$$\begin{array}{cccc} a & 0 & 1 & a \\ 0 & b & 1 & 0 \end{array}$$

$\therefore$ $(ab + 0 \times 1 + 1 \times 0) - (0 \times 0 + 1 \times b + a \times 1) = 0$

$\Rightarrow$ $ab - a - b = 0 \Rightarrow ab = a + b \Rightarrow 1 = \frac{a}{ab} + \frac{b}{ab} \Rightarrow \frac{1}{a} + \frac{1}{b} = 1$

24. It is given that the point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in $k : 1$. So, the coordinates of A are

$$\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$$

We have,

$$\begin{array}{cccc} \dfrac{3k-5}{k+1} & 1 & 7 & \dfrac{3k-5}{k+1} \\ \dfrac{5k+1}{k+1} & 5 & -2 & \dfrac{5k+1}{k+1} \end{array}$$

$\therefore$ Area of $\Delta\, ABC = 2$ sq. units

$\Rightarrow \quad \dfrac{1}{2}\left|\left\{\dfrac{3k-5}{k+1}\times 5 - 2 + 7\times\dfrac{5k+1}{k+1}\right\} - \left\{\dfrac{5k+1}{k+1}\times 1 + 35 - 2\times\dfrac{3k-5}{k+1}\right\}\right| = 2$

$\Rightarrow \quad \dfrac{1}{2}\left|\left(\dfrac{15k-25}{k+1} - 2 + \dfrac{35k+7}{k+1}\right) - \left(\dfrac{5k+1}{k+1} + 35 - \dfrac{6k-10}{k+1}\right)\right| = 2$

$\Rightarrow \quad \dfrac{1}{2}\left|\dfrac{(15k-25-2k-2+35k+7)-(5k+1+35k+35-6k+10)}{k+1}\right| = 2$

$\Rightarrow \quad \dfrac{1}{2}\left|\dfrac{(48k-20)-(34k+46)}{k+1}\right| = 2$

$\Rightarrow \quad \dfrac{1}{2}\left|\dfrac{14k-66}{k+1}\right| = 2$

$\Rightarrow \quad \left|\dfrac{7k-33}{k+1}\right| = 2$

$\Rightarrow \quad \dfrac{7k-33}{k+1} = \pm 2$

$\Rightarrow \quad 7k - 33 = \pm 2\,(k+1)$

$\Rightarrow \quad 7k - 33 = 2k + 2,\ 7k - 33 = -2k - 2$

$\Rightarrow \quad 5k = 35,\ \ 9k = 31$

$\Rightarrow \quad k = 7,\ k = \dfrac{31}{9}$

25. Let the third vertex be $A\ (x,\ y)$. Other two vertices of the triangle are $B\ (2,\ 1)$ and $C\ (3, -2)$.

We have,

$$\begin{array}{cccc} x & 2 & 3 & x \\ y & 1 & -2 & y \end{array}$$

$\therefore$ Area of $\Delta ABC = 5$ sq. units

$\Rightarrow \quad \dfrac{1}{2}|(x - 4 + 3y) - (2y + 3 - 2x)| = 5$

$\Rightarrow \quad \dfrac{1}{2}|x - 4 + 3y - 2y - 3 + 2x)| = 5$

$\Rightarrow \quad \dfrac{1}{2}|3x + y - 7| = 5$

$\Rightarrow \quad 3x + y - 7 = \pm 10$

$\Rightarrow \quad 3x + y - 17 = 0$ or, $3x + y + 3 = 0$

It is given that the vertex $A(x, y)$ lies on $y = x + 3$.

Solving $3x + y - 17 = 0$ and $y = x + 3$, we get $x = \dfrac{7}{2}$ and $y = \dfrac{13}{2}$

Solving $3x + y + 3 = 0$ and $y = x + 3$, we get $x = \dfrac{-3}{2}$ and $y = \dfrac{3}{2}$

Hence, the coordinates of the third vertex are $\left(\dfrac{7}{2}, \dfrac{13}{2}\right)$ or, $\left(\dfrac{-3}{2}, \dfrac{3}{2}\right)$.

26. Let Δ be the area of the triangle formed by the points $(a, a^2), (b, b^2)$ and (c, c^2)

$$\begin{array}{cccc} a & b & c & a \\ a^2 & b^2 & c^2 & a^2 \end{array}$$

$\therefore \quad \Delta = \dfrac{1}{2}\left|(ab^2 + bc^2 + ca^2) - (a^2b + b^2c + c^2a)\right|$

$\Rightarrow \quad \Delta = \dfrac{1}{2}\left|(a^2c - a^2b) + (ab^2 - ac^2) + (bc^2 - b^2c)\right|$

$\Rightarrow \quad \Delta = \dfrac{1}{2}\left|-a^2(b - c) + a(b^2 - c^2) - bc(b - c)\right|$

$\Rightarrow \quad \Delta = \dfrac{1}{2}\left|(b - c)\left\{-a^2 + a(b + c) - bc\right\}\right|$

$\Rightarrow \quad \Delta = \dfrac{1}{2}\left|(b - c)\left(-a^2 + ab + ac - bc\right)\right|$

$\Rightarrow \quad \Delta = \dfrac{1}{2}\left|(b - c)\left\{-a(a - b) + c(a - b)\right\}\right|$

$\Rightarrow \quad \Delta = \dfrac{1}{2}\left|(b - c)(a - b)(c - a)\right|$

It is given that $a \neq b \neq c$.

$\therefore \quad \Delta \neq 0$

Hence, given points are never collinear.

27. We have,

$$\begin{array}{cccc} x & -3 & 4 & x \\ 3x & 5 & -2 & 3x \end{array}$$

$\therefore \quad$ Area of $\Delta DBC = \dfrac{1}{2}\left|(5x + 6 + 12x) - (-9x + 20 - 2x)\right|$

$\Rightarrow \quad$ Area of $\Delta DBC = \dfrac{1}{2}\left|(28x - 14)\right| = |14x - 7| = 7|2x - 1|$

Also, we have

$$\begin{array}{cccc} 6 & -3 & 4 & 6 \\ 3 & 5 & -2 & 3 \end{array}$$

$\therefore \quad$ Area of $\Delta ABC = \dfrac{1}{2}\left|(30 + 6 + 12) - (-9 + 20 - 12)\right|$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{1}{2}|48+1| = \frac{49}{2}$$

Now,

$$\frac{\text{Area of } \Delta DBC}{\text{Area of } \Delta ABC} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{7\,|\,2x-1\,|}{\frac{49}{2}} = \frac{1}{2}$$

$$\Rightarrow \quad |\,2x-1\,| = \frac{7}{4}$$

$$\Rightarrow \quad 2x-1 = \pm\frac{7}{4}$$

$$\Rightarrow \quad 2x = \frac{11}{4} \text{ or, } 2x = -\frac{3}{4}$$

$$\Rightarrow \quad x = \frac{11}{8} \text{ or } x = -\frac{3}{8}$$

28. Points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y & y_2 & y_3 & y_1 \end{array}$$

$$\therefore \quad (x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3) = 0$$

$$\Rightarrow \quad (x_1 y_2 - x_1 y_3) + (x_2 y_3 - x_2 y_1) + (x_3 y_1 - x_3 y_2) = 0$$

$$\Rightarrow \quad x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0$$

$$\Rightarrow \quad \frac{x_1 (y_2 - y_3)}{x_1 x_2 x_3} + \frac{x_2 (y_3 - y_1)}{x_1 x_2 x_3} + \frac{x_3 (y_1 - y_2)}{x_1 x_2 x_3} = 0$$

$$\Rightarrow \quad \frac{y_2 - y_3}{x_2 x_3} + \frac{y_3 - y_1}{x_1 x_3} + \frac{y_1 - y_2}{x_1 x_2} = 0$$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Write the distance between the points $A\,(10 \cos\theta, 0)$ and $B\,(0, 10 \sin\theta)$.
2. Write the perimeter of the triangle formed by the points $O\,(0, 0)$, $A\,(a, 0)$ and $B\,(0, b)$.
3. Write the ratio in which the line segment joining points (2, 3) and (3, –2) is divided by X axis.
4. What is the distance between the points (5 sin 60°, 0) and (0, 5 sin 30°)?
5. If $A\,(-1, 3)$, $B\,(1, -1)$ and $C\,(5, 1)$ are the vertices of a triangle ABC, what is the length of the median through vertex A?
6. If the distance between points $(x, 0)$ and (0, 3) is 5, what are the values of x?
7. What is the area of the triangle formed by the points $O\,(0, 0)$, $A\,(6, 0)$ and $B\,(0, 4)$?
8. Write the coordinates of the point dividing line segment joining points (2, 3) and (3, 4) internally in the ratio 1 : 5.

9. If the centroid of the triangle formed by points $P(a, b)$, $Q(b, c)$ and $R(c, a)$ is at the origin, what is the value of $a + b + c$?

10. In Q. No. 9, what is the value of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$?

11. Write the coordinates of a point on X-axis which is equidistant from the points $(-3, 4)$ and $(2, 5)$.

12. If the mid-point of the segment joining $A(x, y+1)$ and $B(x+1, y+2)$ is $C(3/2, 5/2)$, find x, y.

13. Two vertices of a triangle have coordinates $(-8, 7)$ and $(9, 4)$. If the centroid of the triangle is at the origin, what are the coordinates of the third vertex?

14. Write the coordinates the reflections of point $(3, 5)$ in X and Y-axes.

15. If points Q and R reflections of point $P(-3, 4)$ in X and Y axes respectively, what is QR?

16. Write the formula for the area of the triangle having its vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3).

17. Write the condition of collinearity of points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

18. Find the values of x for which the distance between the point $P(2, -3)$ and $Q(x, 5)$ is 10.

19. Write the ratio in which the line segment doining the points $A(3, -6)$ and $B(5, 3)$ is divided by X-axis.

20. Find the distance between the points $(-8/2, 2)$ and $(2/5, 2)$.

21. Find the value of a so that the point $(3, a)$ lies on the line represented by $2x - 3y + 5 = 0$. **[CBSE 2009]**

22. What is the distance between the points $A(c, 0)$ and $B(0, -c)$? **[CBSE 2009]**

23. If $P(2, 6)$ is the mid-point of the line segment joining $A(6, 5)$ and $B(4, y)$, find y. **[CBSE 2010]**

24. If the distance between the points $(3, 0)$ and $(0, y)$ is 5 units and y is positive, then what is the value of y? **[CBSE 2010]**

25. If $P(x, 6)$ is the mid-point of the line segment joining $A(6, 5)$ and $B(4, y)$, find y. **[CBSE 2010]**

26. If $P(2, p)$ is the mid-point of the line segment joining the points $A(6, -5)$ and $B(-2, 11)$, find the value of p. **[CBSE 2010]**

27. If $A(1, 2)$, $B(4, 3)$ and $C(6, 6)$ are the three vertices of a parallelogram $ABCD$, find the coordinates of fourth vertex D. **[CBSE 2010]**

28. What is the distance between the points $A(\sin\theta - \cos\theta, 0)$ and $B(0, \sin\theta + \cos\theta)$? **[CBSE 2010]**

29. What are the coordinates of the point where the perpendicular bisector of the line segment joining the points $A(1, 5)$, and $B(4, 6)$ cuts the y-axis? **[CBSE 2015]**

30. Find the area of the triangle with vertices $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$.

31. If the points $A(1, 2)$, $O(0, 0)$, and $C(a, b)$ are collinear, then find $a : b$.

32. Find the coordinates of the point which is equidistant from the three vertices $A(2x, 0)$, $O(0, 0)$ and $B(0, 2y)$ of ΔAOB.

33. If the distance between the points $(4, k)$, and $(1, 0)$ is 5, then what can be the possible value of k? **[CBSE 2017]**

34. Find the distance of a point $P(x, y)$ from the origin. **[CBSE 2018]**

ANSWERS

1. 10 2. $\frac{1}{2}ab$ 3. 3 : 2 4. 5 5. 5 6. ± 4

7. 12 sq. units 8. (13/6, 19/6) 9. 0 10. 3

11. (2/5, 0) 12. (1, 1) 13. (−1, −11) 14. (3, −5), (−3, 5) 15. 10

16. $\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$

17. $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$ 18. 8, −4 19. 2 : 1

20. 2 21. $\frac{1}{3}$ 22. $\sqrt{2}c$ 23. 7 24. 4 25. 7

26. 3 27. (3, 6) 28. $\sqrt{2}$ 29. (0, 13) 30. 0 31. 1 : 2

32. (x, y) 33. ± 4 34. $\sqrt{x^2+y^2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The distance between the points $(\cos\theta, \sin\theta)$ and $(\sin\theta - \cos\theta)$ is
   (a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 2 (d) 1
2. The distance between the points $(a\cos 25°, 0)$ and $(0, a\cos 65°)$ is
   (a) a (b) $2a$ (c) $3a$ (d) None of these
3. If x is a positive integer such that the distance between points $P(x, 2)$ and $Q(3, -6)$ is 10 units, then $x =$
   (a) 3 (b) −3 (c) 9 (d) −9
4. The distance between the points $(a\cos\theta + b\sin\theta, 0)$ and $(0, a\sin\theta - b\cos\theta)$ is
   (a) a^2+b^2 (b) $a+b$ (c) a^2-b^2 (d) $\sqrt{a^2+b^2}$
5. If the distance between the points $(4, p)$ and $(1, 0)$ is 5, then $p =$
   (a) ± 4 (b) 4 (c) −4 (d) 0
6. A line segement is of length 10 units. If the coordinates of its one end are (2, −3) and the abscissa of the other end is 10, then its ordinate is
   (a) 9, 6 (b) 3, −9 (c) −3, 9 (d) 9, −6
7. The perimeter of the triangle formed by the points (0, 0), (1, 0) and (0, 1) is
   (a) $1 \pm \sqrt{2}$ (b) $\sqrt{2}+1$ (c) 3 (d) $2+\sqrt{2}$
8. If $A(2, 2)$, $B(-4, -4)$ and $C(5, -8)$ are the vertices of a triangle, then the length of the median through vertex C is
   (a) $\sqrt{65}$ (b) $\sqrt{117}$ (c) $\sqrt{85}$ (d) $\sqrt{113}$
9. If three points $(0, 0)$, $(3, \sqrt{3})$ and $(3, \lambda)$ form an equilateral triangle, then $\lambda =$
   (a) 2 (b) −3 (c) −4 (d) None of these
10. If the points $(k, 2k)$, $(3k, 3k)$ and $(3, 1)$ are collinear, then k
    (a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

11. The coordinates of the point on X-axis which are equidistant from the poin $(-3, 4)$ and $(2, 5)$ are
(a) $(20, 0)$ (b) $(-23, 0)$ (c) $(4/5, 0)$ (d) None of these

12. If $(-1, 2)$, $(2, -1)$ and $(3, 1)$ are any three vertices of a parallelogram, then
(a) $a = 2, b = 0$ (b) $a = -2, b = 0$
(c) $a = -2, b = 6$ (d) $a = 6, b = 2$

13. If $A(5, 3)$, $B(11, -5)$ and $P(12, y)$ are the vertices of a right triangle right angled at P, the $y =$
(a) $-2, 4$ (b) $-2, 4$ (c) $2, -4$ (d) $2, 4$

14. The area of the triangle formed by $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ is
(a) $a + b + c$ (b) abc (c) $(a + b + c)^2$ (d) 0

15. If $(x, 2)$, $(-3, -4)$ and $(7, -5)$ are collinear, then $x =$
(a) 60 (b) 63 (c) -63 (d) -60

16. If points $(t, 2t)$, $(-2, 6)$ and $(3, 1)$ are collinear, then $t =$
(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{5}{3}$ (d) $\frac{3}{5}$

17. If the area of the triangle formed by the points $(x, 2x)$, $(-2, 6)$ and $(3, 1)$ is 5 square units, then $x =$
(a) $2/3$ (b) $3/5$ (c) 3 (d) 5

18. If points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear, then $\frac{1}{a} + \frac{1}{b} =$
(a) 1 (b) 2 (c) 0 (d) -1

19. If the centroid of a triangle is $(1, 4)$ and two of its vertices are $(4, -3)$ and $(-9, 7)$, then the area of the triangle is
(a) 183 sq. units (b) $\frac{183}{2}$ sq. units (c) 366 sq. units (d) $\frac{183}{4}$ sq. units

20. The line segment joining points $(-3, -4)$, and $(1, -2)$ is divided by y-axis in the ratio
(a) $1:3$ (b) $2:3$ (c) $3:1$ (d) $2:3$

21. The ratio in which $(4, 5)$ divides the join of $(2, 3)$ and $(7, 8)$ is
(a) $-2:3$ (b) $-3:2$ (c) $3:2$ (d) $2:3$

22. The ratio in which the x-axis divides the segment joining $(3, 6)$ and $(12, -3)$ is
(a) $2:1$ (b) $1:2$ (c) $-2:1$ (d) $1:-2$

23. If the centroid of the triangle formed by the points (a, b), (b, c) and (c, a) is at the origin, then $a^3 + b^3 + c^3 =$
(a) abc (b) 0 (c) $a + b + c$ (d) $3abc$

24. If points $(1, 2)$, $(-5, 6)$ and $(a, -2)$ are collinear, then $a =$
(a) -3 (b) 7 (c) 2 (d) -2

25. If the centroid of the triangle formed by $(7, x)$, $(y, -6)$ and $(9, 10)$ is at $(6, 3)$, then $(x, y) =$
(a) $(4, 5)$ (b) $(5, 4)$ (c) $(-5, -2)$ (d) $(5, 2)$

26. The distance of the point (4, 7) from the x-axis is

(a) 4 (b) 7 (c) 11 (d) $\sqrt{65}$

27. The distance of the point (4, 7) from the y-axis is

(a) 4 (b) 7 (c) 11 (d) $\sqrt{65}$

28. If P is a point on x-axis such that its distance from the origin is 3 units, then the coordinates of a point Q on OY such that $OP = OQ$, are

(a) (0, 3) (b) (3, 0) (c) (0, 0) (d) (0, – 3)

29. If the point (x, 4) lies on a circle whose centre is at the origin and radius is 5, then $x =$

(a) ± 5 (b) ± 3 (c) 0 (d) ± 4

30. If the point $P(x, y)$ is equidistant from $A(5, 1)$ and $B(-1, 5)$, then

(a) $5x = y$ (b) $x = 5y$ (c) $3x = 2y$ (d) $2x = 3y$

31. If points $A(5, p)$, $B(1, 5)$, $C(2, 1)$ and $D(6, 2)$ form a square $ABCD$, then $p =$

(a) 7 (b) 3 (c) 6 (d) 8

32. The coordinates of the circumcentre of the triangle formed by the points O (0, 0), $A(a, 0)$ and $B(0, b)$ are

(a) (a, b) (b) $(a/2, b/2)$ (c) $(b/2, a/2)$ (d) (b, a)

33. The coordinates of a point on x-axis which lies on the perpendicular bisector of the line segment joining the points (7, 6) and (–3, 4) are

(a) (0, 2) (b) (3, 0) (c) (0, 3) (d) (2, 0)

34. If the centroid of the triangle formed by the points (3, –5), (–7, 4), (10, $-k$) is at the point (k, –1), then $k =$

(a) 3 (b) 1 (c) 2 (d) 4

35. If (–2, 1) is the centroid of the triangle having its vertices at (x, 2), (10, – 2), (–8, y), then x, y satisfy the relation

(a) $3x + 8y = 0$ (b) $3x - 8y = 0$ (c) $8x + 3y = 0$ (d) $8x = 3y$

36. The coordinates of the fourth vertex of the rectangle formed by the points (0, 0), (2, 0), (0, 3) are

(a) (3, 0) (b) (0, 2) (c) (2, 3) (d) (3, 2)

37. The length of a line segment joining $A(2, -3)$ and B is 10 units. If the abscissa of B is 10 units, then its ordinates can be

(a) 3 or –9 (b) –3 or 9 (c) 6 or 27 (d) –6 or –27

38. The ratio in which the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is divided by x-axis is

(a) $y_1 : y_2$ (b) $-y_1 : y_2$ (c) $x_1 : x_2$ (d) $-x_1 : x_2$

39. The ratio in which the line segment joining points $A(a_1, b_1)$ and $B(a_2, b_2)$ is divided by y-axis is

(a) $-a_1 : a_2$ (b) $a_1 : a_2$ (c) $b_1 : b_2$ (d) $-b_1 : b_2$

40. If the line segment joining the points (3, –4), and (1, 2) is trisected at points $P(a, -2)$ ar $Q\left(\frac{5}{3}, b\right)$. Then,

(a) $a = \frac{8}{3}, b = \frac{2}{3}$ (b) $a = \frac{7}{3}, b = 0$ (c) $a = \frac{1}{3}, b = 1$ (d) $a = \frac{2}{3}, b = \frac{1}{3}$

41. If the coordinates of one end of a diameter of a circle are (2, 3) and the coordinates of i centre are (– 2, 5), then the coordinates of the other end of the diameter are

(a) (– 6, 7) (b) (6, – 7) (c) (6, 7) (d) (– 6, – 7) **[CBSE 2012**

42. The coordinates of the point P dividing the line segment joining the points $A(1, 3)$ an $B(4, 6)$ in the ratio 2 : 1 are

(a) (2, 4) (b) (3, 5) (c) (4, 2) (d) (5, 3) **[CBSE 2012**

43. In Fig. 6.52, the area of ΔABC (in square units) is

(a) 15 (b) 10 (c) 7.5 (d) 2.5 **[CBSE 2013**

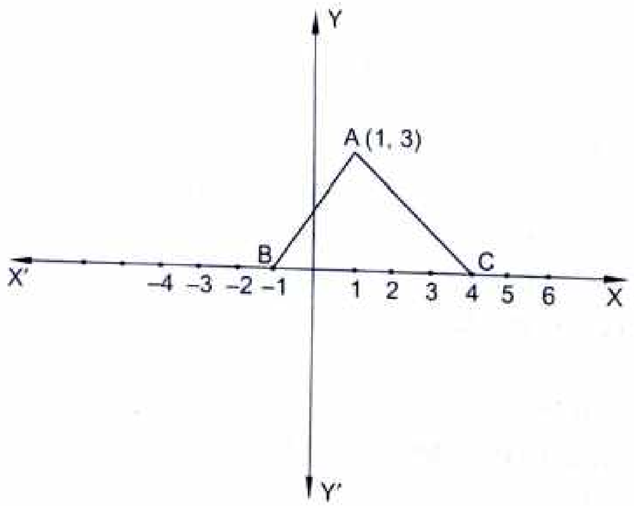


Fig. 6.52

44. The point on the x-axis which is equidistant from points (–1, 0) and (5, 0) is

(a) (0, 2) (b) (2, 0) (c) (3, 0) (d) (0, 3) **[CBSE 2013]**

45. If $A(4, 9)$, $B(2, 3)$ and $C(6, 5)$ are the vertices of ΔABC, then the length of median through C is

(a) 5 units (b) $\sqrt{10}$ units (c) 25 units (d) 10 units **[CBSE 2014]**

46. If $P(2, 4)$, $Q(0, 3)$, $R(3, 6)$ and $S(5, y)$ are the vertices of a parallelogram $PQRS$, then the value of y is

(a) 7 (b) 5 (c) – 7 (d) – 8 **[CBSE 2014]**

47. If $A(x, 2)$, $B(-3, -4)$ and $C(7, -5)$ are collinear, then the value of x is

(a) – 63 (b) 63 (c) 60 (d) – 60 **[CBSE 2014]**

48. The perimeter of a triangle with vertices (0, 4) and (0, 0) and (3, 0) is

(a) $7 + \sqrt{5}$ (b) 5 (c) 10 (d) 12 **[CBSE 2014]**

49. If the point $P(2, 1)$ lies on the line segment joining points $A(4, 2)$ and $B(8, 4)$, then

(a) $AP = \frac{1}{3}AB$ (b) $AP = BP$ (c) $PB = \frac{1}{3}AB$ (d) $AP = \frac{1}{2}AB$

50. A line intersects the y-axis and x-axis at P and Q, respectively. If $(2, -5)$ is the mid-point of PQ, then the coordinates of P and Q are, respectively

(a) $(0, -5)$ and $(2, 0)$ (b) $(0, 10)$ and $(-4, 0)$

(c) $(0, 4)$ and $(-10, 0)$ (d) $(0, -10)$ and $(4, 0)$

ANSWERS

1. (b)	2. (a)	3. (c)	4. (d)	5. (a)	6. (b)
7. (d)	8. (c)	9. (d)	10. (b)	11. (d)	12. (c)
13. (c)	14. (d)	15. (c)	16. (b)	17. (a)	18. (a)
19. (b)	20. (c)	21. (d)	22. (a)	23. (d)	24. (b)
25. (d)	26. (b)	27. (a)	28. (a)	29. (b)	30. (d)
31. (c)	32. (b)	33. (b)	34. (c)	35. (a)	36. (c)
37. (a)	38. (b)	39. (a)	40. (b)	41. (a)	42. (b)
43. (c)	44. (b)	45. (b)	46. (a)	47. (a)	48. (d)
49. (d)	50. (d)				

SUMMARY

1. The abscissa and ordinate of a given point are the distances of the point from y-axis and x-axis respectively.
2. The coordinates of any point on x-axis are of the form $(x, 0)$.
3. The coordinates of any point on y-axis are of the form $(0, y)$.
4. The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

5. Distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$OP = \sqrt{x^2 + y^2}$$

6. The coordinates of the point which divides the join of points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right).$$

7. The coordinates of the mid-point of the line segment joining the points $P(x_1, y_1)$ and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
8. The coordinates of the centroid of triangle formed by the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

9. The area of the triangle formed by the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3,$

$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

or, $\frac{1}{2}\left|(x_1 y_2 + x_2 y_3 x_3 y_1) - (x_1 y_3 + x_2 y_1 + x_3 y_2)\right|$

10. If points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then

$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

CHAPTER 7

TRIANGLES

7.1 CONCEPT OF SIMILARITY

In earlier classes, we have learnt about congruent figures. Two geometric figures having the same shape and size are known as congruent figures. Note that congruent figures are alike in every respect. In this chapter, we shall study about similarity of geometric figures. Geometric figures having the same shape but different sizes are known as similar figures. Two congruent figures are always similar but similar figures need not be congruent as discussed in the following illustrations.

ILLUSTRATION 1 *Any two line segments are always similar but they need not be congruent. They are congruent, if their lengths are equal.*

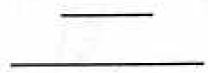


Fig. 7.1

ILLUSTRATION 2 *Any two circles are similar but not necessarily congruent. They are congruent if their radii are equal.*

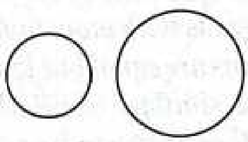


Fig. 7.2

ILLUSTRATION 3 (i) *Any two squares are similar. (See Fig. 7.3(i)).*

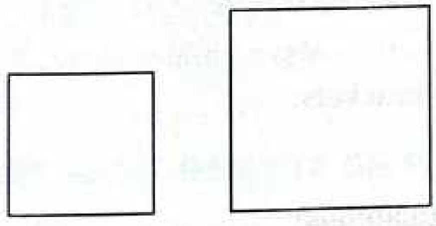


Fig. 7.3(i)

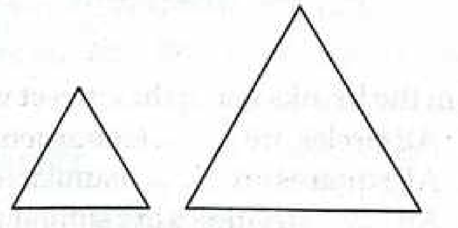


Fig. 7.3(ii)

(ii) *Any two equilateral triangles are similar. (see Fig. 7.3(ii)).*

If two figures are similar one can be obtained from the other either by shrinking or by stretching, without changing its shape. There is one-to-one correspondence between the parts of two similar figures [Fig. 7.3 (ii)].

7.2 SIMILAR POLYGONS

DEFINITION *Two polygons are said to be similar to each other, if*

(i) *their corresponding angles are equal, and*

(ii) *the lengths of their corresponding sides are proportional.*

If two polygons $ABCDE$ and $PQRST$ are similar, then from the above definition it follows that:

Angle at A = Angle at P, Angle at B = Angle at Q, Angle at C = Angle at R,

Angle at D = Angle at S, Angle at E = Angle at T

and, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$

If two polygons $ABCDE$ and $PQRST$, are similar, we write $ABCDE \sim PQRST$. Here, the symbol '$\sim$' stands for 'is similar to'.

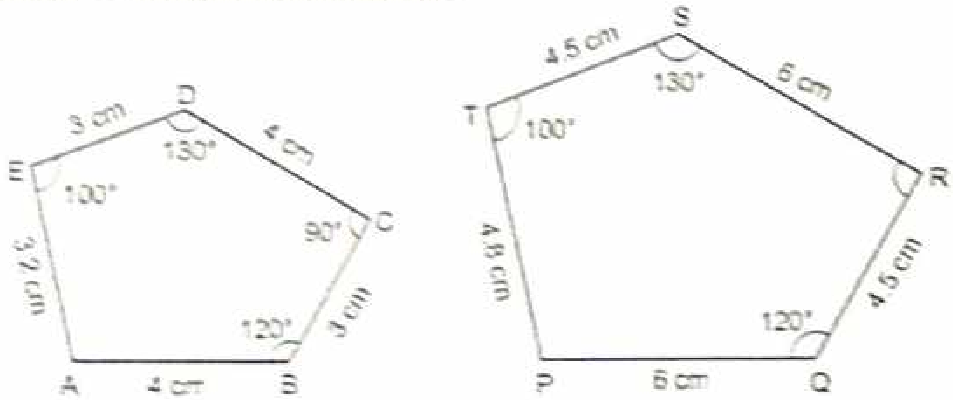


Fig. 7.4

NOTE 1 *It should be noted that for the similarity of polygons with more than three sides, the two conditions given in the definition are independent of each other i. e., either of the two conditions without the other is not sufficient for polygons with more than three sides to be similar. In other words, if the corresponding angles of two polygons are equal but lengths of their corresponding sides are not proportional, the polygons need not be similar. Similarly, if the corresponding angles of two polygons are not equal but length of the the corresponding sides are proportional, then the polygons need not be similar.*

NOTE 2 *Triangles are special type of polygons. In case of triangles, if either of the two conditions given in the above definition holds, then the other holds automatically.*

EXERCISE 7.1

LEVEL-1

1. Fill in the blanks using the correct word given in brackets:
   (i) All circles are ______ (congruent, similar).
   (ii) All squares are ______ (similar, congruent).
   (iii) All ______ triangles are similar (isosceles, equilaterals):
   (iv) Two triangles are similar, if their corresponding angles are ______ (proportional, equal)
   (v) Two triangles are similar, if their corresponding sides are ______ (proportional, equal)
   (vi) Two polygons of the same number of sides are similar, if (a) their corresponding angles are and (b) their corresponding sides are ______ (equal, proportional). [NCERT]

2. Write the truth value (T/F) of each of the following statements:
   (i) Any two similar figures are congruent.
   (ii) Any two congruent figures are similar.
   (iii) Two polygons are similar, if their corresponding sides are proportional.
   (iv) Two polygons are similar if their corresponding angles are proportional.
   (v) Two triangles are similar if their corresponding sides are proportional.
   (vi) Two triangles are similar if their corresponding angles are equal.

ANSWERS

1. (i) similar	(ii) similar	(iii) equilateral
(iv) equal	(v) proportional	(vi) equal, proportional
2. (i) False	(ii) True	(iii) False
(iv) False	(v) True	(vi) True

7.3 SIMILAR TRIANGLES AND THEIR PROPERTIES

DEFINITION *Two triangles are said to be similar, if their*

(i) *corresponding angles are equal and,*
(ii) *corresponding sides are proportional.*

It follows from this definition that two trianges ABC and DEF are similar, if

(i) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and, (ii) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

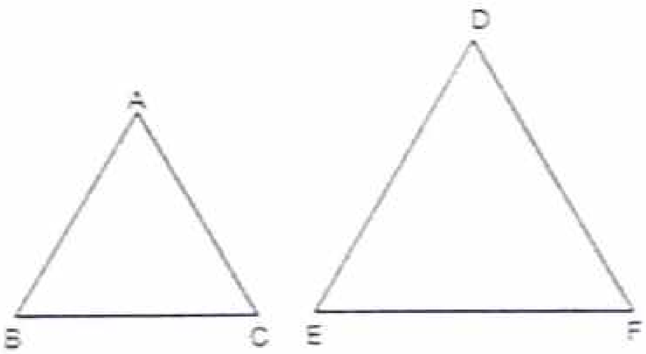


Fig. 7.5

NOTE 1 *In the later part of this chapter we shall show that the two conditions given in the above definition are not independent. In fact, if either of the two conditions holds, then the other holds automatically. So any one of the two conditions can be used to define similar triangles.*

NOTE 2 *If corresponding angles of two triangles are equal, then they are known as equiangular triangles.*

7.4 SOME BASIC RESULTS ON PROPORTIONALITY

In this section, we shall discuss some basic results on proportionality.

Let us first do the following activity.

ACTIVITY Draw any angle $\angle XAY$ and mark points P_1, P_2, D, P_3 and B on its arm AX such that $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$ unit.

Through point B, draw any line intersecting arm AY at point C. Also, through point D, draw a line parallel to BC to intersect AC at E.

We have,

$$AD = AP_1 + P_1P_2 + P_2D = 3 \text{ units}$$

and, $$DB = DP_3 + P_3B = 2 \text{ units.}$$

$$\therefore \quad \frac{AD}{DB} = \frac{3}{2}$$

Now, measure AE and EC and find $\frac{AE}{EC}$.

You will find that

$$\frac{AE}{EC} = \frac{3}{2}$$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$$

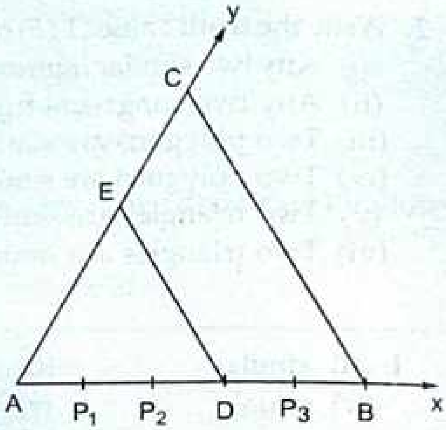


Fig. 7.6

Thus, we observe that in ΔABC if $DE \parallel BC$, then

$$\frac{AD}{DB} = \frac{AE}{EC}$$

We prove this result as a theorem known as basic proportionality theorem or Thale's Theorem as given below.

THEOREM 1 *(Basic proportionality Theorem or Thales Theorem) If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.*

[NCERT, CBSE 2002 C, 2005, 2006 C, 2007, 2008, 2009, 2010]

GIVEN A triangle ABC in which $DE \parallel BC$, and intersects AB in D and AC in E.

TO PROVE $\frac{AD}{DB} = \frac{AE}{EC}$

CONSTRUCTION Join BE, CD and draw $EF \perp BA$ and $DG \perp CA$.

PROOF Since EF is perpendicular to AB. Therefore, EF is the height of triangles ADE and DBE.

Now, $$\text{Area}(\Delta ADE) = \frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(AD \, . \, EF)$$

and, $$\text{Area}(\Delta DBE) = \frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(DB \, . \, EF)$$

$$\therefore \quad \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DBE)} = \frac{\frac{1}{2}(AD \, . \, EF)}{\frac{1}{2}(DB \, . \, EF)} = \frac{AD}{DB} \quad \text{...(i)}$$

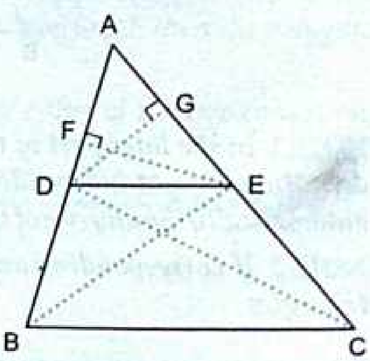


Fig. 7.7

Similarly, we have

$$\frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DEC)} = \frac{\frac{1}{2}(AE \, . \, DG)}{\frac{1}{2}(EC \, . \, DG)} = \frac{AE}{EC} \quad \text{...(ii)}$$

But, $\Delta\, DBE$ and $\Delta\, DEC$ are on the same base DE and between the same parallels DE and BC.

$$\therefore \quad \text{Area}(\Delta DBE) = \text{Area}(\Delta DEC)$$

$$\Rightarrow \quad \frac{1}{\text{Area}(\Delta DBE)} = \frac{1}{\text{Area}(\Delta DEC)}$$ [Taking reciprocals of both sides]

$$\Rightarrow \quad \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DBE)} = \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta DEC)} \qquad \text{[Multiplying both sides by Area } (\Delta ADE)\text{]}$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC} \qquad \text{[Using (i) and (ii)]}$$

Q.E.D.

COROLLARY *If in a ΔABC, a line $DE \parallel BC$, intersects AB in D and AC in E, then:*

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ **[NCERT]** (ii) $\frac{AB}{DB} = \frac{AC}{EC}$

PROOF (i) From the basic proportionality theorem, we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\Rightarrow \quad \frac{DB}{AD} = \frac{EC}{AE} \qquad \text{[Taking reciprocals of both sides]}$$

$$\Rightarrow \quad 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE} \qquad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE}$$

(ii) From the basic proportionality theorem, we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\Rightarrow \quad \frac{AD}{DB} + 1 = \frac{AE}{EC} + 1 \qquad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{AD + DB}{DB} = \frac{AE + EC}{EC}$$

$$\Rightarrow \quad \frac{AB}{DB} = \frac{AC}{EC}$$

Q.E.D.

The above results can be summarised as follows:

SUMMARY If in a ΔABC, $DE \parallel BC$, and intersects AB in D and AC in E, then we have

(i) $\frac{AD}{DB} = \frac{AE}{EC}$ (ii) $\frac{DB}{AD} = \frac{EC}{AE}$ (iii) $\frac{AB}{AD} = \frac{AC}{AE}$

(iv) $\frac{AD}{AB} = \frac{AE}{AC}$ (v) $\frac{AB}{DB} = \frac{AC}{EC}$ (vi) $\frac{DB}{AB} = \frac{EC}{AC}$

Let us now perform the following activity.

ACTIVITY Draw any angle XAY and mark points B_1, B_2, B_3, B_4 and B on its arm AX such that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B = 1$ unit. Also, mark points C_1, C_2, C_3, C_4 and C on arm AY such that $AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C = 1$ unit. Join $B_1C_1, B_2C_2, B_3C_3, B_4C_4$ and BC as shown in Fig. 7.8.

We observe that

$$AB_1 = 1 \text{ unit}, AC_1 = 1 \text{ unit},$$
$$B_1B = 4 \text{ units}, C_1C = 4 \text{ units},$$
$$\therefore \quad \frac{AB_1}{B_1B} = \frac{AC_1}{C_1C}\left(= \frac{1}{4}\right)$$

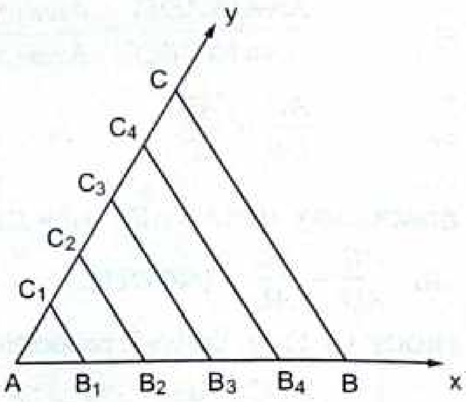


Fig. 7.8

You can also see that B_1C_1 and BC are parallel to each other.

Similarly, we observe that

$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C}\left(= \frac{2}{3}\right) \text{ and } B_2C_2 \parallel BC$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C}\left(= \frac{3}{2}\right) \text{ and } B_3C_3 \parallel BC$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C}\left(= \frac{4}{1}\right) \text{ and } B_4C_4 \parallel BC.$$

It follows from the above activity that if a line divides two sides of a triangle in the same ratio, then it is parallel to the third side of the triangle.

This fact is stated and proved as a theorem given below and it is the converse of the basic proportionality theorem.

THEOREM 2 *(Converse of Basic Proportionality Theorem) If a line divides any two sides of a triangle in the same ratio, then the line must be paralled to the third side.* **[NCERT]**

GIVEN A ΔABC and a line l intersecting AB in D and AC in E, such that $\frac{AD}{DB} = \frac{AE}{EC}$

TO PROVE $l : BC$ i.e. $DE \parallel BC$

PROOF If possible, let DE be not parallel to BC. Then, there must be another line parallel to BC. Let $DF \parallel BC$.

Since $DF \parallel BC$. Therefore, from Basic Proportionality Theorem, we get

$$\frac{AD}{DB} = \frac{AF}{FC} \qquad \text{...(i)}$$

But, $$\frac{AD}{DB} = \frac{AE}{EC} \qquad \text{(Given)} \qquad \text{...(ii)}$$

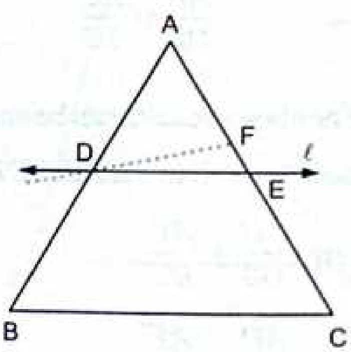


Fig. 7.9

From (i) and (ii), we get

$$\frac{AF}{FC} = \frac{AE}{EC}$$

$$\Rightarrow \quad \frac{AF}{FC} + 1 = \frac{AE}{EC} + 1 \qquad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{AF + FC}{FC} = \frac{AE + EC}{EC}$$

$$\Rightarrow \quad \frac{AC}{FC} = \frac{AC}{EC}$$

$$\Rightarrow \quad FC = EC$$

This is possible only when F and E coincide i.e. DF is the line l itself. But, $DF \parallel BC$.

Hence, $\quad l \parallel BC.$

Q.E.D.

We shall now discuss some examples which will illustrate the applications of the results discussed so far.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **BASED ON THE RESULT THAT THE LINE DRAWN PARALLEL TO ONE SIDE OF A TRIANGLE INTERSECTING THE OTHER TWO SIDES DIVIDES THEM IN THE SAME RATIO.**

EXAMPLE 1 *In Fig 7.10, PQ is parallel to MN. If* $\frac{KP}{PM} = \frac{4}{13}$ *and* $KN = 20.4$ *cm. Find KQ.*

SOLUTION In $\Delta\ KMN$, we have

$PQ \parallel MN$

$\therefore \quad \frac{KP}{PM} = \frac{KQ}{QN}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{KP}{PM} = \frac{KQ}{KN - KQ}$

$\Rightarrow \quad \frac{4}{13} = \frac{KQ}{20.4 - KQ}$

$\Rightarrow \quad 4(20.4 - KQ) = 13\ KQ$

$\Rightarrow \quad 81.6 - 4\ KQ = 13\ KQ$

$\Rightarrow \quad 17\ KQ = 81.6$

$\Rightarrow \quad KQ = \frac{81.6}{17} = 4.8$ cm

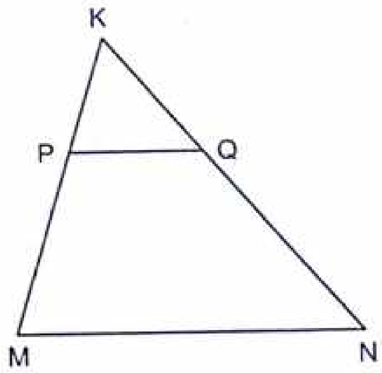


Fig. 7.10

EXAMPLE 2 *In a given* $\Delta\ ABC$, $DE \parallel BC$ *and* $\frac{AD}{DB} = \frac{3}{5}$. *If* $AC = 5.6$, *find AE.*

SOLUTION In $\Delta\ ABC$, we have

$DE \parallel BC$

$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{AC - AE}$

$\Rightarrow \quad \frac{3}{5} = \frac{AE}{5.6 - AE}$ [$\because AC = 5.6$]

$\Rightarrow \quad 3(5.6 - AE) = 5AE$

$\Rightarrow \quad 16.8 - 3AE = 5AE$

$\Rightarrow \quad 8AE = 16.8$

$\Rightarrow \quad AE = \frac{16.8}{8}$ cm $= 2.1$ cm.

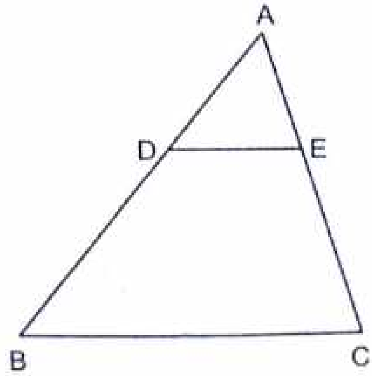


Fig. 7.11

EXAMPLE 3 *In Fig. 7.12, DE || BC. If AD = x, DB = x - 2, AE = x + 2 and EC = x - 1, find the value of x.*

SOLUTION In ΔABC, we have

$DE \parallel BC$

$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{x}{x-2} = \frac{x+2}{x-1}$

$\Rightarrow \quad x(x-1) = (x-2)(x+2)$

$\Rightarrow \quad x^2 - x = x^2 - 4 \Rightarrow x = 4$

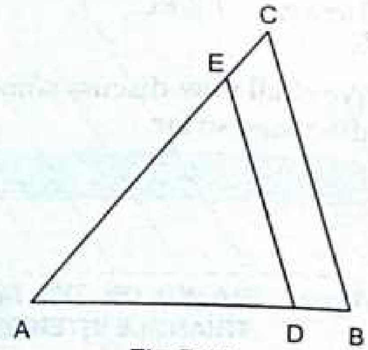


Fig. 7.12

EXAMPLE 4 *In Fig. 7.13, LM || AB. If AL = x – 3, AC = 2x, BM = x – 2 and BC = 2x + 3, find the value of x.*

SOLUTION In ΔABC, we have

$LM \parallel AB$

$\therefore \quad \frac{AL}{LC} = \frac{BM}{MC}$ [By Thaley's Theorem]

$\Rightarrow \quad \frac{AL}{AC - AL} = \frac{BM}{BC - BM}$

$\Rightarrow \quad \frac{x-3}{2x-(x-3)} = \frac{x-2}{(2x+3)-(x-2)}$

$\Rightarrow \quad \frac{x-3}{x+3} = \frac{x-2}{x+5}$

$\Rightarrow \quad (x-3)(x+5) = (x-2)(x+3)$

$\Rightarrow \quad x^2 + 2x - 15 = x^2 + x - 6$

$\Rightarrow \quad x = 9$

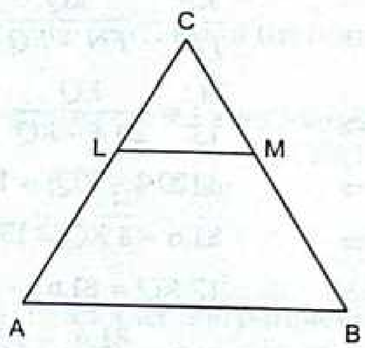


Fig. 7.13

EXAMPLE 5 *In Fig. 7.14, if ST || QR. Find PS.*

SOLUTION In ΔPRQ, we have

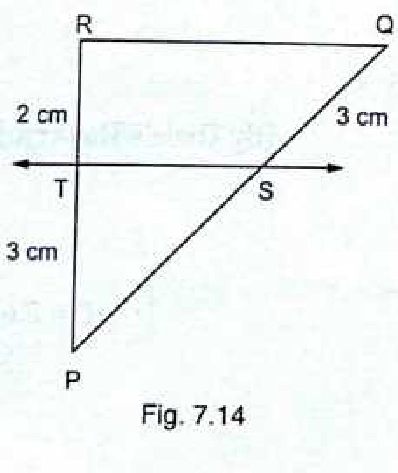


Fig. 7.14

$ST \parallel QR$

$\Rightarrow \quad \frac{PS}{QS} = \frac{PT}{RT}$ [By Thale's Theorem]

$$\Rightarrow \qquad \frac{PS}{3} = \frac{3}{2}$$

$$\Rightarrow \qquad PS = \frac{9}{2} \text{ cm}$$

EXAMPLE 6 *In Fig. 7.15 (i) and (ii), PQ || BC. Find QC in (i) and AQ in (ii).*

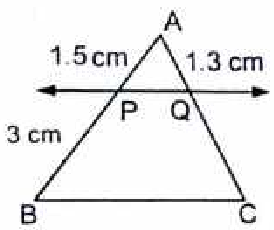


Fig. 7.15 (i)

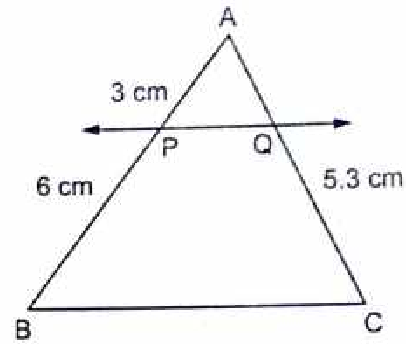


Fig. 7.15 (ii)

SOLUTION In Fig. 7.15 (i), we have

$PQ \parallel BC$

Therefore, by basic proportionality theorem, we have

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

$$\Rightarrow \qquad \frac{1.5}{3} = \frac{1.3}{QC}$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{1.3}{QC}$$

$$\Rightarrow \qquad QC = 2.6 \text{ cm}$$

In Fig. 7.15 (ii), it is given that $PQ \parallel BC$.

Therefore, by basic proportionality theorem, we have

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

$$\Rightarrow \qquad \frac{3}{6} = \frac{AQ}{5.3}$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{AQ}{5.3}$$

$$\Rightarrow \qquad AQ = \frac{5.3}{2} = 2.65 \text{ cm}$$

EXAMPLE 7 *In Fig. 7.16, if PQ || BC and PR || CD. Prove that (i)* $\frac{AR}{AD} = \frac{AQ}{AB}$ *(ii)* $\frac{QB}{AQ} = \frac{DR}{AR}$ **[CBSE 2010]**

SOLUTION In ΔABC, we have

$PQ \parallel BC$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{AQ}{AB} = \frac{AP}{AC} \qquad \dots \text{(i)}$$

In $\Delta\, ACD$, we have

$$PR \parallel CD$$

Therefore, by basic proportionality theorem, we have

$$\frac{AP}{AC} = \frac{AR}{AD} \qquad \ldots(ii)$$

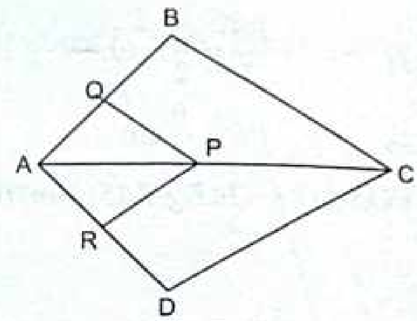


Fig. 7.16

From (i) and (ii), we obtain that

$$\frac{AQ}{AB} = \frac{AR}{AD} \text{ or, } \frac{AR}{AD} = \frac{AQ}{AB}$$

(ii) From (i) we have

$$\frac{AQ}{AB} = \frac{AR}{AD}$$

$$\Rightarrow \quad \frac{AB}{AQ} = \frac{AD}{AR}$$

$$\Rightarrow \quad \frac{AQ + QB}{AQ} = \frac{AR + RD}{AR} = 1 + \frac{QB}{AQ}$$

$$= 1 + \frac{RD}{AR} \Rightarrow \frac{QB}{AQ} = \frac{DR}{AR}$$

EXAMPLE 8 *In Fig. 7.17, DE $\parallel$ AC and DC $\parallel$ AP. Prove that* $\frac{BE}{EC} = \frac{BC}{CP}$ **[CBSE 2005]**

SOLUTION In $\Delta\, BPA$, we have

$$DC \parallel AP \qquad \text{[Given]}$$

Therefore, by basic proportionality theorem, we have

$$\frac{BC}{CP} = \frac{BD}{DA} \qquad \ldots(i)$$

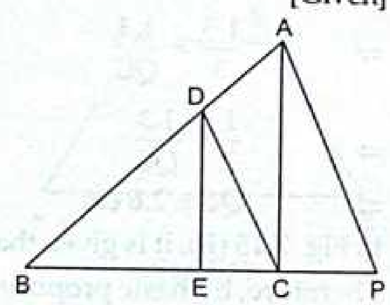


Fig. 7.17

In $\Delta\, BCA$, we have

$$DE \parallel AC \qquad \text{[Given]}$$

Therefore, by basic proportionality theorem, we have

$$\frac{BE}{EC} = \frac{BD}{DA} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{BC}{CP} = \frac{BE}{EC} \text{ or, } \frac{BE}{EC} = \frac{BC}{CP}$$

Type II PROBLEMS BASED UPON THE CONVERSE OF PROPORTIONALITY THEOREM

EXAMPLE 9 *D and E are respectively the points on the sides AB and AC of a $\Delta\, ABC$ such that $AB = 5.6\ cm$, $AD = 1.4\ cm$, $AC = 7.2\ cm$ and $AE = 1.8\ cm$, show that DE $\parallel$ BC.*

SOLUTION We have,

$$AB = 5.6 \text{ cm}, AD = 1.4 \text{ cm}, AC = 7.2 \text{ cm and } AE = 1.8 \text{ cm}.$$

$$\therefore \quad BD = AB - AD = (5.6 - 1.4) \text{ cm} = 4.2 \text{ cm}$$

and,

$$EC = AC - AE = (7.2 - 1.8) \text{ cm} = 5.4 \text{ cm}$$

Now, $$\frac{AD}{DB} = \frac{1.4}{4.2} = \frac{1}{3} \text{ and } \frac{AE}{EC} = \frac{1.8}{5.4} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

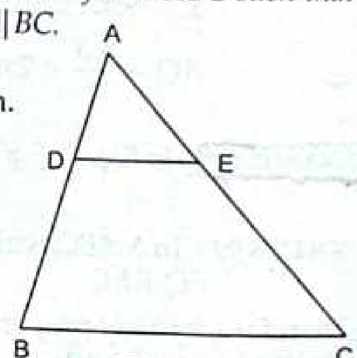


Fig. 7.18

Thus, DE divides sides AB and AC of $\Delta\ ABC$ in the same ratio. Therefore, by the converse of Basic Pro-portionality Theorem, we have

$$DE \parallel BC$$

EXAMPLE 10 *Any point X inside $\Delta\ DEF$ is joined to its vertices. From a point P in DX, PQ is drawn parallel to DE meeting XE at Q and QR is drawn parallel to EF meeting XF in R. Prove that $PR \parallel DF$.* **[NCERT, CBSE 2002]**

GIVEN A $\Delta\ DEF$ and a point X inside it. Point X is joined to the vertices D, E and F. P is any point on DX. $PQ \parallel DE$ and $QR \parallel EF$.

TO PROVE $PR \parallel DF$

CONSTRUCTION Join PR.

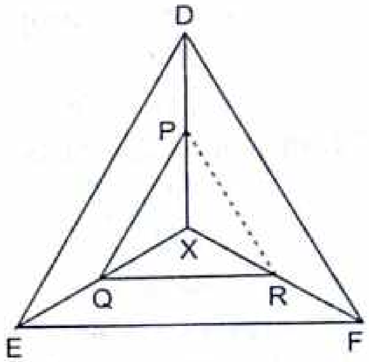


Fig. 7.19

PROOF In $\Delta\ XED$, we have

$$PQ \parallel DE$$

$$\therefore \quad \frac{XP}{PD} = \frac{XQ}{QE} \qquad \ldots(i) \quad \text{[By Thale's Theorem]}$$

In $\Delta\ XEF$, we have

$$QR \parallel EF$$

$$\therefore \quad \frac{XQ}{QE} = \frac{XR}{RF} \qquad \ldots(ii) \quad \text{[By Thale's Theorem]}$$

From (i) and (ii), we have

$$\frac{XP}{PD} = \frac{XR}{RF}$$

Thus, in $\Delta\ XFD$, points R and P are dividing sides XF and XD in the same ratio. Therefore, by the converse of Basic Proportionality Theorem, we have

$$PR \parallel DF$$

EXAMPLE 11 *In a $\Delta\ ABC$, D and E are points on sides AB and AC respectively such that $BD = CE$. If $\angle B = \angle C$, show that $DE \parallel BC$.*

SOLUTION In $\Delta\ ABC$, we have

$$\angle B = \angle C$$

$$\Rightarrow \quad AC = AB \qquad \text{[Sides opposite to equal angles are equal]}$$

$$\Rightarrow \quad AE + EC = AD + DB$$

$$\Rightarrow \quad AE + CE = AD + BD$$

$$\Rightarrow \quad AE + CE = AD + CE \qquad [\because BD = CE]$$

$$\Rightarrow \quad AE = AD$$

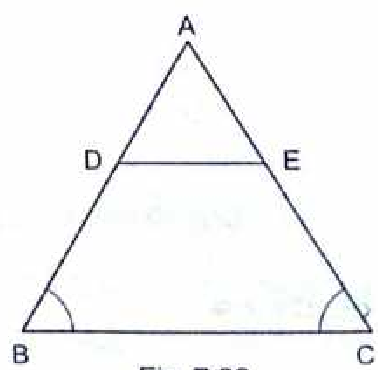


Fig. 7.20

Thus, we have

$$AD = AE \text{ and } BD = CE$$

$$\therefore \quad \frac{AD}{BD} = \frac{AE}{CE}$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

$$\Rightarrow \quad DE \parallel BC \qquad \text{[By the converse of Thale's Theorem]}$$

EXAMPLE 12 *In Fig. 7.21, if $DE \parallel AQ$ and $DF \parallel AR$. Prove that $EF \parallel QR$.* **[NCERT, CBSE 2008]**

SOLUTION In $\Delta\, PQA$, we have

$DE \parallel AQ$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{PE}{EQ} = \frac{PD}{DA} \quad \ldots(i)$$

In $\Delta\, PAR$, we have

$DF \parallel AD$ [Given]

Therefore, by basic proportionality theorem, we have

$$\frac{PD}{DA} = \frac{PF}{FR} \quad \ldots(ii)$$

From (i) and (ii), we have

$$\frac{PE}{EQ} = \frac{PF}{FR}$$

$\Rightarrow$ $EF \parallel QR$ [By the converse of Basic Proportionality Theorem]

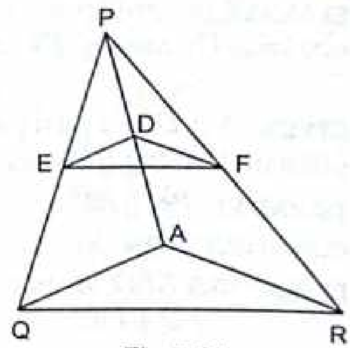


Fig. 7.21

EXAMPLE 13 *In Fig. 7.22, A, B and C are points on OP, OQ and OR respectively such that $AB \parallel PQ$ and $BC \parallel QR$. Show that $AC \parallel PR$.* **[NCERT]**

SOLUTION In $\Delta\, OPQ$, we have

$AB \parallel PQ$

$$\Rightarrow \frac{OA}{AP} = \frac{OB}{BQ} \quad \ldots(i)$$

In $\Delta\, OQR$, we have

$BC \parallel QR$

$$\Rightarrow \frac{OB}{BQ} = \frac{OC}{CR} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{OA}{AP} = \frac{OC}{CR}$$

Thus, A and C are points on sides OP and OR respectively of ΔOPR, such that

$$\frac{OA}{AP} = \frac{OC}{CR}$$

$\Rightarrow$ $AC \parallel PR$ [Using the converse of BPT]

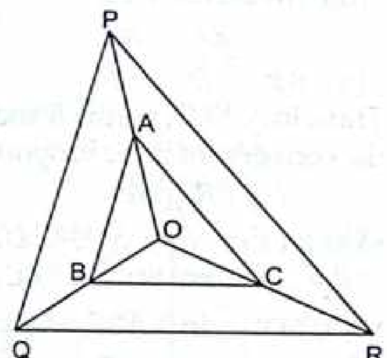


Fig. 7.22

LEVEL-2

Type I BASED UPON BASIC PROPORTIONALITY THEOREM

EXAMPLE 14 *In Fig. 7.23, if $EF \parallel DC \parallel AB$. prove that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$.*

GIVEN $EF \parallel DC \parallel AB$ in the given figure.

TO PROVE $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

CONSTRUCTION Produce DA and CB to meet at P (say).

PROOF In $\Delta\, PEF$, we have

$AB \parallel EF$

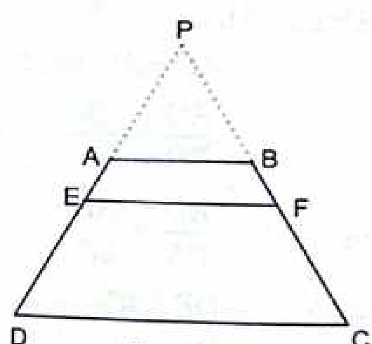


Fig. 7.23

$$\therefore \quad \frac{PA}{AE} = \frac{PB}{BF} \quad \text{[By Thale's Theorem]}$$

$$\Rightarrow \quad \frac{PA}{AE} + 1 = \frac{PB}{BF} + 1 \quad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{PA + AE}{AE} = \frac{PB + BF}{BF}$$

$$\Rightarrow \quad \frac{PE}{AE} = \frac{PF}{BF} \quad \ldots(i)$$

In ΔPDC, we have

$$EF \parallel DC$$

$$\therefore \quad \frac{PE}{ED} = \frac{PF}{FC} \quad \text{[By Basic Proportionality Theorem]} \quad \ldots(ii)$$

On dividing equation (i) by equation (ii), we get

$$\frac{\frac{PE}{AE}}{\frac{PE}{ED}} = \frac{\frac{PF}{BF}}{\frac{PF}{FC}}$$

$$\Rightarrow \quad \frac{ED}{AE} = \frac{FC}{BF}$$

$$\Rightarrow \quad \frac{AE}{ED} = \frac{BF}{FC}$$

EXAMPLE 15 *Let X be any point on the side BC of a triangle ABC. If XM, XN are drawn parallel to BA and CA meeting CA, BA in M, N respectively; MN meets BC produced in T, prove that* $TX^2 = TB \times TC$.

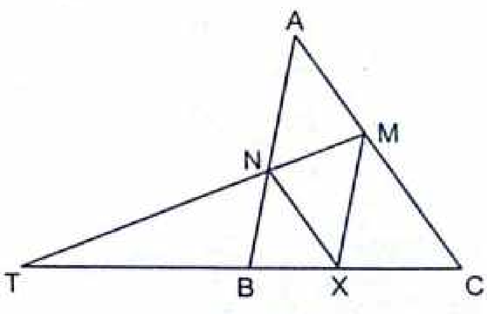


Fig. 7.24

SOLUTION In $\Delta\, TXM$, we have

$$XM \parallel BN$$

$$\therefore \quad \frac{TB}{TX} = \frac{TN}{TM} \quad \ldots(i)$$

In $\Delta\, TMC$, we have

$$XN \parallel CM$$

$$\therefore \quad \frac{TX}{TC} = \frac{TN}{TM} \quad \ldots(ii)$$

From equations (i) and (ii), we get

$$\frac{TB}{TX} = \frac{TX}{TC}$$

$$\Rightarrow \quad TX^2 = TB \times TC$$

EXAMPLE 16 *ABCD is a parallelogram, P is a point on side BC and DP when produced meets AB produced at L. Prove that*

(i) $\frac{DP}{PL} = \frac{DC}{BL}$ (ii) $\frac{DL}{DP} = \frac{AL}{DC}$

GIVEN A parallelogram $ABCD$ in which P is a point on side BC such that DP produced meets AB produced at L.

TO PROVE

(i) $\frac{DP}{PL} = \frac{DC}{BL}$ (ii) $\frac{DL}{DP} = \frac{AL}{DC}$

PROOF (i) In $\Delta\, ALD$, we have

$BP \parallel AD$

$\therefore \quad \frac{LB}{BA} = \frac{LP}{PD}$

$\Rightarrow \quad \frac{BL}{AB} = \frac{PL}{DP}$

$\Rightarrow \quad \frac{BL}{DC} = \frac{PL}{DP}$ $[\because AB = DC]$

$\Rightarrow \quad \frac{DP}{PL} = \frac{DC}{BL}$ [Taking reciprocals of both sides]

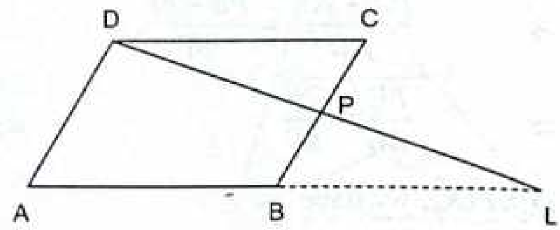


Fig. 7.25

(ii) From (i), we have

$\frac{DP}{PL} = \frac{DC}{BL}$

$\Rightarrow \quad \frac{PL}{DP} = \frac{BL}{DC}$ [Taking reciprocals of both sides]

$\Rightarrow \quad \frac{PL}{DP} = \frac{BL}{AB}$ $[\because DC = AB]$

$\Rightarrow \quad \frac{PL}{DP} + 1 = \frac{BL}{AB} + 1$ [Adding 1 on both sides]

$\Rightarrow \quad \frac{DP + PL}{DP} = \frac{BL + AB}{AB}$

$\Rightarrow \quad \frac{DL}{DP} = \frac{AL}{AB}$

$\Rightarrow \quad \frac{DL}{DP} = \frac{AL}{DC}$ $[\because AB = DC]$

EXAMPLE 17 *In Fig. 7.26, $EF \parallel AB \parallel DC$. Prove that* $\frac{AE}{ED} = \frac{BF}{FC}$ **[NCERT]**

SOLUTION We have,

$EF \parallel AB \parallel DC$

$\Rightarrow \quad EP \parallel DC$

Thus, in $\Delta\, ADC$, we have

$EP \parallel DC$

Therefore, by basic proportionality theorem, we have

$\frac{AE}{ED} = \frac{AP}{PC}$ ...(i)

Again, $EF \parallel AB \parallel DC$

$\Rightarrow \quad FP \parallel AB$

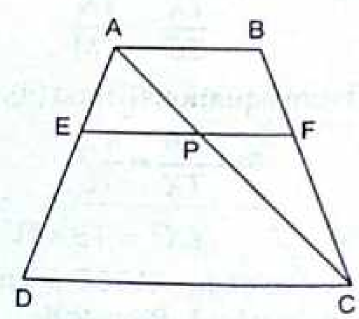


Fig. 7.26

Thus, in $\Delta\, CAB$, we have

$$FP \parallel BA$$

Therefore, by basic proportionality theorem, we have

$$\frac{BF}{FC} = \frac{AP}{PC} \qquad \text{... (ii)}$$

From (i) and (ii), we obtain $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

EXAMPLE 18 *In Fig. 7.27, $DE \parallel BC$ and $CD \parallel EF$. Prove that $AD^2 = AB \times AF$.* **[CBSE 2007]**

SOLUTION In $\Delta\, ABC$, we have

$$DE \parallel BC$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE} \qquad \text{...(i)}$$

In $\Delta\, ADC$, we have

$$FE \parallel DC$$

$$\Rightarrow \quad \frac{AD}{AF} = \frac{AC}{AE} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{AB}{AD} = \frac{AD}{AF}$$

$$\Rightarrow \quad AD^2 = AB \times AF$$

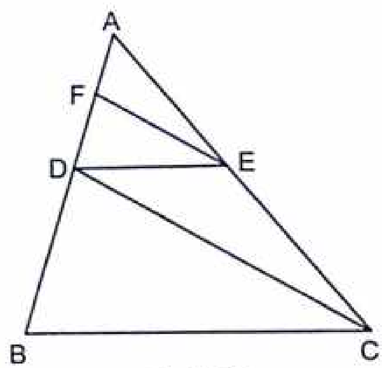


Fig. 7.27

Type II ON THALE'S THEOREM AND ITS CONVERSE

EXAMPLE 19 *Two triangles ABC and DBC lie on the same side of the base BC. From a point P on BC, $PQ \parallel AB$ and $PR \parallel BD$ are drawn. They meet AC in Q and DC in R respectively. Prove that $QR \parallel AD$*

GIVEN Two triangles ABC and DBC lie on the same side of the base BC. Points P, Q and R are points on BC, AC and CD respectively such that $PR \parallel BD$ and $PQ \parallel AB$.

TO PROVE $QR \parallel AD$

PROOF In $\Delta\, ABC$, we have

$$PQ \parallel AB$$

$$\therefore \quad \frac{CP}{PB} = \frac{CQ}{QA} \qquad \text{...(i) [By Basic Proportionality Theorem]}$$

In $\Delta\, BCD$, we have

$$PR \parallel BD$$

$$\therefore \quad \frac{CP}{PB} = \frac{CR}{RD} \qquad \text{...(ii) [By Thale's Theorem]}$$

From (i) and (ii), we have

$$\frac{CQ}{QA} = \frac{CR}{RD}$$

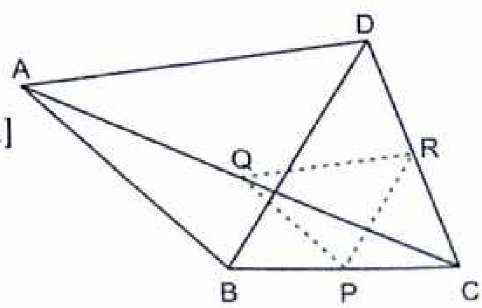


Fig. 7.28

Thus, in $\Delta\, ACD$, Q and R are points on AC and CD respectively such that

$$\frac{CQ}{QA} = \frac{CR}{RD}$$

$$\Rightarrow \quad QR \parallel AD \qquad \text{[By the converse of Basic Proportionality Theorem]}$$

EXAMPLE 20 *ABCD is a quadrilateral; P, Q, R and S are the points of trisection of sides AB, BC, CD and DA respectively and are adjacent to A and C; prove that PQRS is a parallelogram.*

GIVEN A quadrilateral $ABCD$ in which P, Q, R and S are the points of trisection of sides AB, BC, CD and DA respectively and are adjacent to A and C.

TO PROVE $PQRS$ is a parallelogram i. e., $PQ \parallel SR$ and $QR \parallel PS$.

CONSTRUCTION Join AC.

PROOF Since P, Q, R and S are the points of trisection of AB, BC, CD and DA respectively.

$\therefore \quad BP = 2\,PA, BQ = 2\,QC, DR = 2\,RC$ and, $DS = 2\,SA$

In $\Delta\, ADC$, we have

$$\frac{DS}{SA} = \frac{2SA}{SA} = 2 \text{ and, } \frac{DR}{RC} = \frac{2RC}{RC} = 2$$

$$\Rightarrow \quad \frac{DS}{SA} = \frac{DR}{RC}$$

$\Rightarrow$ S and R divide the sides DA and DC respectively in the same ratio.

$\Rightarrow$ $SR \parallel AC$ [By the converse of Thale's Theorem] ...(i)

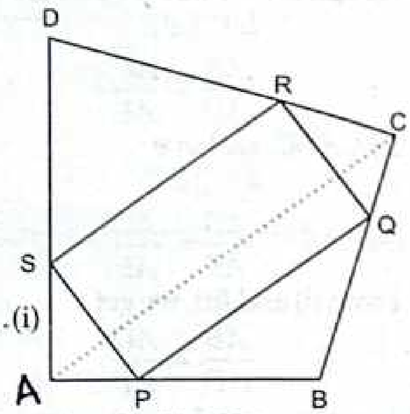


Fig. 7.29

In $\Delta\, ABC$, we have

$$\frac{BP}{PA} = \frac{2\,PA}{PA} = 2 \text{ and } \frac{BQ}{QC} = \frac{2\,QC}{QC} = 2$$

$$\Rightarrow \quad \frac{BP}{PA} = \frac{BQ}{QC}$$

$\Rightarrow$ P and Q divide the sides BA and BC respectively in the same ratio.

$\Rightarrow$ $PQ \parallel AC$ [By the converse of Thale's Theorem] ...(ii)

From (i) and (ii), we obtain

$$SR \parallel AC \text{ and } PQ \parallel AC \Rightarrow SR \parallel PQ$$

Similarly, by joining BD, we can prove that $QR \parallel PS$. Hence, $PQRS$ is a parallelogram.

EXAMPLE 21 *Let ABC be a triangle and D and E be two points on side AB such that AD = BE. If DP $\parallel$ BC and EQ $\parallel$ AC, then prove that PQ $\parallel$ AB.*

SOLUTION In $\Delta\, ABC$, we have

$DP \parallel BC$ and $EQ \parallel AC$

$$\therefore \quad \frac{AD}{DB} = \frac{AP}{PC} \text{ and } \frac{BE}{EA} = \frac{BQ}{QC}$$

$$\Rightarrow \frac{AD}{DB} = \frac{AP}{PC} \text{ and } \frac{AD}{DB} = \frac{BQ}{QC} \left[\begin{array}{l} EA = ED + DA = ED + BE = BD \\ (\because AD = BE) \end{array}\right]$$

$$\Rightarrow \frac{AP}{PC} = \frac{BQ}{QC}$$

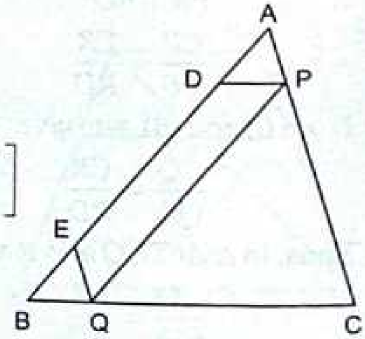


Fig. 7.30

$\Rightarrow$ In a $\Delta\, ABC$, P and Q divide sides CA and CB respectively in the same ratio.

$\Rightarrow$ $PQ \parallel AB$.

EXAMPLE 22 *In Fig. 7.31, ABC is a triangle in which AB = AC. Points D and E are points on the sides AB and AC respectively such that AD = AE. Show that the points B, C, E and D are concyclic.*

SOLUTION In order to prove that the points B, C, E and D are concyclic, it is sufficient to show that $\angle ABC + \angle CED = 180°$ and $\angle ACB + \angle BDE = 180°$.

In $\Delta\, ABC$, we have

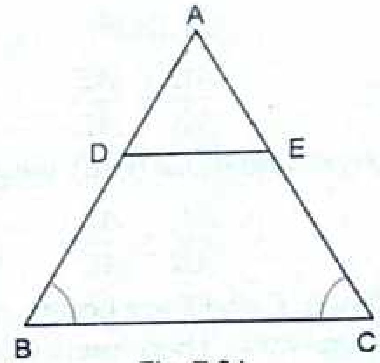


Fig. 7.31

$$AB = AC \text{ and } AD = AE$$

$\Rightarrow$ $AB - AD = AC - AE$

$\Rightarrow$ $DB = EC$

Thus, we have

$$AD = AE \text{ and } DB = EC$$

$\Rightarrow$ $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

$\Rightarrow$ $DE \parallel BC$ [By the converse of Thale's Theorem]

$\Rightarrow$ $\angle ABC = \angle ADE$ [Corresponding angles]

$\Rightarrow$ $\angle ABC + \angle BDE = \angle ADE + \angle BDE$ [Adding $\angle BDE$ on both sides]

$\Rightarrow$ $\angle ABC + \angle BDE = 180°$

$\Rightarrow$ $\angle ACB + \angle BDE = 180°$ [$\because AB = AC \therefore \angle ABC = \angle ACB$]

Again, $DE \parallel BC$

$\Rightarrow$ $\angle ACB = \angle AED$

$\Rightarrow$ $\angle ACB + \angle CED = \angle AED + \angle CED$ [Adding $\angle CED$ on both sides]

$\Rightarrow$ $\angle ACB + \angle CED = 180°$

$\Rightarrow$ $\angle ABC + \angle CED = 180°$ [$\because \angle ABC = \angle ACB$]

Thus, $BDEC$ is quadrilateral such that $\angle ACB + \angle BDE = 180°$

and $\angle ABC + \angle CED = 180°$

Therefore, $BDEC$ is a cyclic quadrilateral. Hence, B, C, E and D are concyclic points.

EXAMPLE 23 *The side BC of a triangle ABC is bisected at D; O is any point in AD. BO and CO produced meet AC and AB in E and F respectively and AD is produced to X so that D is the mid-point of OX. Prove that AO : AX = AF : AB and show that FE $\parallel$ BC.*

SOLUTION Join BX and CX.

We have,

$$BD = CD \text{ and } OD = DX.$$

Thus, BC and OX bisect each other.

$\Rightarrow$ $OBXC$ is a parallelogram.

$\Rightarrow$ $BX \parallel CO$ and $CX \parallel BO$

$\Rightarrow \quad BX \parallel CF$ and $CX \parallel BE$

$\Rightarrow \quad BX \parallel OF$ and $CX \parallel OE$

In ΔABX, we have

$BX \parallel OF$

$\Rightarrow \quad \dfrac{AO}{AX} = \dfrac{AF}{AB}$ ...(i)

In ΔACX, we have

$CX \parallel OE$

$\Rightarrow \quad \dfrac{AO}{AX} = \dfrac{AE}{AC}$ ...(ii)

From equations (i) (ii), we get

$$\frac{AF}{AB} = \frac{AE}{AC}$$

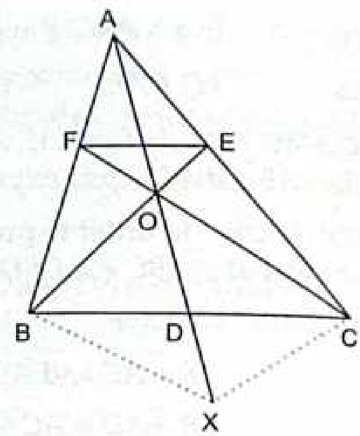


Fig. 7.32

Thus, E and F are points on AB and AC such that they divide AB and AC respectively in the same ratio. Therefore, by the converse of Thale's Theorem $FE \parallel BC$.

EXAMPLE 24 *In Fig. 7.33, if $\dfrac{AD}{DC} = \dfrac{BE}{EC}$ and $\angle CDE = \angle CED$, prove that ΔCAB is isosceles.*

SOLUTION In ΔABC, we have

$\dfrac{AD}{DC} = \dfrac{BE}{EC}$ [Given]

Therefore, by the converse of basic proportionality theorem, we have,

$DE \parallel AB$

$\Rightarrow \quad \angle CDE = \angle CAB$ and $\angle CED = \angle CBA$

[Corresponding angles]

But, $\quad \angle CDE = \angle CED$ [Given]

$\therefore \quad \angle CAB = \angle CBA$

$\Rightarrow \quad \angle A = \angle B$

$\Rightarrow \quad BC = AC$ [$\because$ Sides opposite to equal angles are equal]

$\Rightarrow \quad \Delta CAB$ is isosceles.

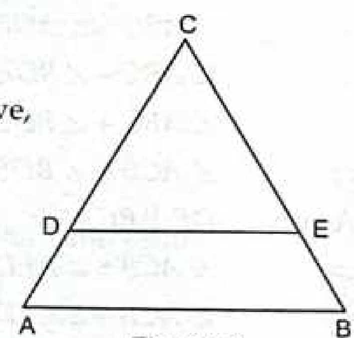


Fig. 7.33

EXAMPLE 25 *In Fig. 7.34, $\dfrac{PS}{SQ} = \dfrac{PT}{TR}$ and $\angle PST = \angle PRQ$. Prove that ΔPQR is an isosceles triangle.* **[NCERT]**

SOLUTION We have,

$\dfrac{PS}{SQ} = \dfrac{PT}{TR}$

$\Rightarrow \quad ST \parallel QR$ [By using the converse of Basic Proportionality Theorem]

$\Rightarrow \quad \angle PST = \angle PQR$ [Corresponding angles]

$\Rightarrow \quad \angle PRQ = \angle PQR$ [$\because \angle PST = \angle PRQ$ (Given)]

$\Rightarrow \quad PQ = PR$ [$\because$ Sides opposite to equal angles are equal]

$\Rightarrow \quad \Delta PQR$ is isosceles.

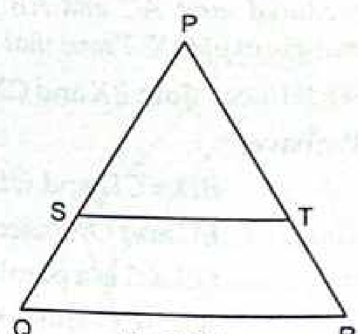


Fig. 7.34

EXERCISE 7.2

LEVEL-1

1. In a ΔABC, D and E are points on the sides AB and AC respectively such that $DE \parallel BC$.
   (i) If $AD = 6$ cm, $DB = 9$ cm and $AE = 8$ cm, find AC.
   (ii) If $\frac{AD}{DB} = \frac{3}{4}$ and $AC = 15$ cm, find AE.
   (iii) If $\frac{AD}{DB} = \frac{2}{3}$ and $AC = 18$ cm, find AE.
   (iv) If $AD = 4$, $AE = 8$, $DB = x - 4$, and $EC = 3x - 19$, find x
   (v) If $AD = 8$ cm, $AB = 12$ cm and $AE = 12$ cm, find CE.
   (vi) If $AD = 4$ cm, $DB = 4.5$ cm and $AE = 8$ cm, find AC.
   (vii) If $AD = 2$ cm, $AB = 6$ cm and $AC = 9$ cm, find AE.
   (viii) If $\frac{AD}{BD} = \frac{4}{5}$ and $EC = 2.5$ cm, find AE.
   (ix) If $AD = x$, $DB = x - 2$, $AE = x + 2$ and $EC = x - 1$, find the value of x.
   (x) If $AD = 8x - 7$, $DB = 5x - 3$, $AE = 4x - 3$ and $EC = (3x - 1)$, find the value of x.
   (xi) If $AD = 4x - 3$, $AE = 8x - 7$, $BD = 3x - 1$ and $CE = 5x - 3$, find the volume x. [CBSE 2002]
   (xii) If $AD = 2.5$ cm, $BD = 3.0$ cm and $AE = 3.75$ cm, find the length of AC. [CBSE 2006C]

2. In a ΔABC, D and E are points on the sides AB and AC respectively. For each of the following cases show that $DE \parallel BC$:
   (i) $AB = 12$ cm, $AD = 8$ cm, $AE = 12$ cm and $AC = 18$ cm.
   (ii) $AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm and $AE = 1.8$ cm.
   (iii) $AB = 10.8$ cm, $BD = 4.5$ cm, $AC = 4.8$ cm and $AE = 2.8$ cm.
   (iv) $AD = 5.7$ cm, $BD = 9.5$ cm, $AE = 3.3$ cm and $EC = 5.5$ cm.

3. In a ΔABC, P and Q are points on sides AB and AC respectively, such that $PQ \parallel BC$. If $AP = 2.4$ cm, $AQ = 2$ cm, $QC = 3$ cm and $BC = 6$ cm, find AB and PQ.

4. In a ΔABC, D and E are points on AB and AC respectively such that $DE \parallel BC$. If $AD = 2.4$ cm, $AE = 3.2$ cm, $DE = 2$ cm and $BC = 5$ cm, find BD and CE. [CBSE 2001C]

5. In Fig. 7.35, state if $PQ \parallel EF$.

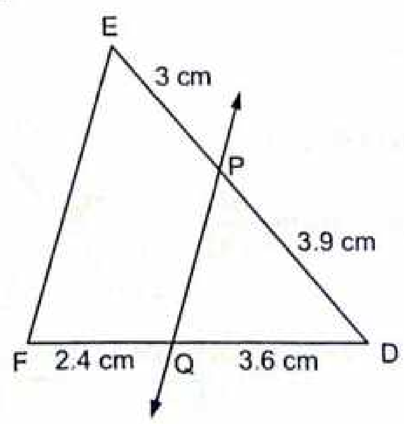


Fig. 7.35

6. M and N are points on the sides PQ and PR respectively of a $\Delta\, PQR$. For each of the following cases, state whether $MN \parallel QR$:
   (i) $PM = 4$ cm, $QM = 4.5$ cm, $PN = 4$ cm, $NR = 4.5$ cm
   (ii) $PQ = 1.28$ cm, $PR = 2.56$ cm, $PM = 0.16$ cm, $PN = 0.32$ cm

LEVEL-2

7. In three line segments OA, OB, and OC, points L, M, N respectively are so chosen that $LM \parallel AB$ and $MN \parallel BC$ but neither of L, M, N nor of A, B, C are collinear. Show that $LN \parallel AC$.
8. If D and E are points on sides AB and AC respectively of a $\Delta\, ABC$ such that $DE \parallel BC$ *and* $BD = CE$. Prove that $\Delta\, ABC$ is isosceles. **[CBSE 2007, 2009]**

ANSWERS

1. (i) 20 cm (ii) 6.43 cm (iii) 7.2 cm (iv) 11 cm (v) 6 cm (vi) 17 cm (vii) 3 cm (viii) 2 cm (xi) $x = 4$ (x) $x = 1$ (xi) 1 (xii) 8.25 cm
3. $AB = 6$ cm, $PQ = 2.4$ cm
4. $DB = 3.6$ cm, $CE = 4.8$ cm
5. No
6. (i) Yes (ii) Yes

HINT TO SELECTED PROBLEM

8. By Thale's Theorem, we obtain $\dfrac{AD}{BD} = \dfrac{AE}{EC} \Rightarrow AD = AE$

But, $BD = CE$ and $AD = AE$

$\therefore \quad AD + BD = AE + CE \Rightarrow AB = AC$

7.5 INTERNAL AND EXTERNAL BISECTORS OF AN ANGLE OF A TRIANGLE

In this section, we will derive some properties of internal and external bisectors of an angle of a triangle. These, properties will be stated and proved as theorems.

Let us first perform the following activity.

ACTIVITY Draw any angle $\angle XAY$ and mark points P_1, P_2, P_3, P_4, P_5 and B on its arm AX such that $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5B = 1$ unit. Also, mark points Q_1, Q_2, Q_3 and C on arm AY such that $AQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3C = 1$ unit. Join BC.

We have,

$$\frac{AB}{AC} = \frac{6}{4} = \frac{3}{2}$$

Drow bisector of $\angle XAY$ to intersect BC at D.

Measure lengths BD and DC and compute $\dfrac{BD}{DC}$.

You will find that

$$\frac{BD}{DC} = \frac{3}{2}$$

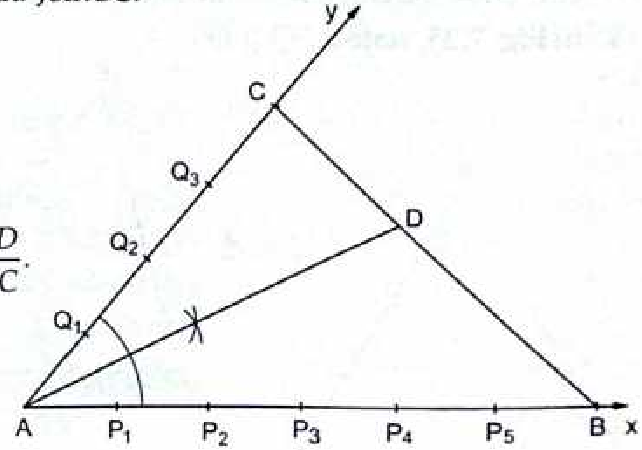


Fig. 7.36

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

This means that the bisector of $\angle A$ of ΔABC divides opposite side BC in the ratio $AB : AC$.

This fact is stated and proved as a theorem given below.

THEOREM 1 *The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.*

GIVEN A ΔABC in which AD is the internal bisector of $\angle A$ and meets BC in D.

TO PROVE $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

CONSTRUCTION Draw $CE \parallel DA$ to meet BA produced in E.

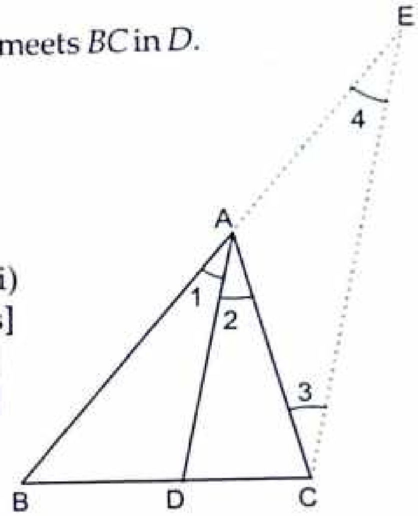


Fig. 7.37

PROOF Since $CE \parallel DA$ and AC cuts them.

$\therefore \quad \angle 2 = \angle 3$ ...(i) [Alternate angles]

and, $\quad \angle 1 = \angle 4$ ...(ii) [Corresponding angles]

But, $\quad \angle 1 = \angle 2$ [$\because AD$ is the bisector of $\angle A$]

From (i) and (ii), we get

$\angle 3 = \angle 4$

Thus, in ΔACE, we have

$\angle 3 = \angle 4$

$\Rightarrow \quad AE = AC$ ...(iii) [Sides opposite to equal angles are equal]

Now, in ΔBCE, we have

$DA \parallel CE$

$\Rightarrow \quad \dfrac{BD}{DC} = \dfrac{BA}{AE}$ [Using Basic Proportionality Theorem]

$\Rightarrow \quad \dfrac{BD}{DC} = \dfrac{AB}{AC}$ [$\because BA = AB$ and $AE = AC$ (From (iii)]

Hence, $\quad \dfrac{BD}{DC} = \dfrac{AB}{AC}$

In order to see whether the converse of the above theorem is true on not. Let us perform the following activity.

ACTIVITY Draw any angle $\angle XAY$ and mark points P_1, P_2, P_3, P_4, P_5 and B on its arm AX such that $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5B = 1$ unit. Also, mark points Q_1, Q_2 and C on arm AY such that $AQ_1 = Q_1Q_2 = Q_2C = 1$ unit. Join BC.

Compute $AB : AC$.

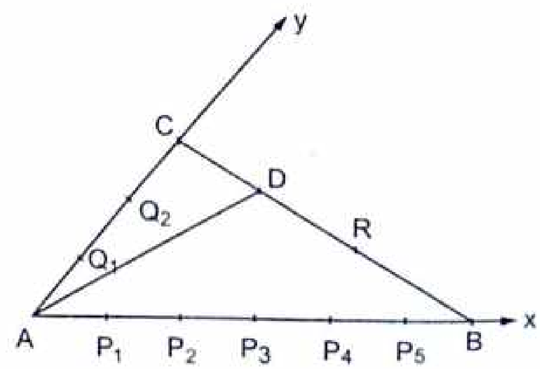


Fig. 7.38

We have,

$AB = 6$ units and $AC = 3$ units.

$$\therefore \quad \frac{AB}{AC} = \frac{6}{3} = \frac{2}{1}$$

Divide BC into 3 (= 2 + 1) equal parts and mark the points of division as R and D.

We have,

$BD = BR + RD = 2$ units and $CD = 1$ unit.

$\therefore \quad \frac{BD}{CD} = \frac{2}{1}$

That is D divides BC in the ratio 2 : 1.

Join AD and measure $\angle XAD$ and $\angle YAD$.

You will find that $\angle XAD$ and $\angle YAD$. That is, AD is the bisector of $\angle BAC$ of ΔABC.

This means that if D is a point on side BC of ΔABC such that it divides BC in the ratio $AB : AC$. Then, AD is the bisector of $\angle A$ of ΔABC.

We state and prove this fact as a theorem given below.

THEOREM 2 *In a triangle ABC, if D is a point on BC such that* $\frac{BD}{DC} = \frac{AB}{AC}$, *prove that AD is the bisector of* $\angle A$.

OR

In a triangle ABC, if D is a point on BC such that D divides BC in the ratio AB : AC, then AD is the bisector of $\angle A$.

OR

If a line through one vertex of a triangle divides the opposite sides in the ratio of other two sides, then the line bisects the angle at the vertex.

GIVEN A ΔABC, in which D is a point on BC such that $\frac{BD}{DC} = \frac{AB}{AC}$

TO PROVE AD is the bisector of $\angle A$.

CONSTRUCTION Produce BA to E such that $AE = AC$. Join EC.

PROOF In ΔACE, we have

$AE = AC$ [By construction]

$\Rightarrow \quad \angle 3 = \angle 4$ ...(i)

Now, $\quad \frac{BD}{DC} = \frac{AB}{AC}$

$\Rightarrow \quad \frac{BD}{DC} = \frac{AB}{AE}$ [$\because AC = AE$]

Thus, in ΔBCE, we have

$\frac{BD}{DC} = \frac{BA}{AE}$

Therefore, by the converse of Basic Proportionality Theorem, we have

$DA \parallel CE$

$\Rightarrow \quad \angle 1 = \angle 4$ ...(ii) [Corresponding angles]

and, $\quad \angle 2 = \angle 3$ ...(iii) [Alternate angles]

But, $\quad \angle 3 = \angle 4$ [From (i)]

$\therefore \quad \angle 1 = \angle 2$ [From (ii) and (iii)]

Hence, AD is the bisector of $\angle A$.

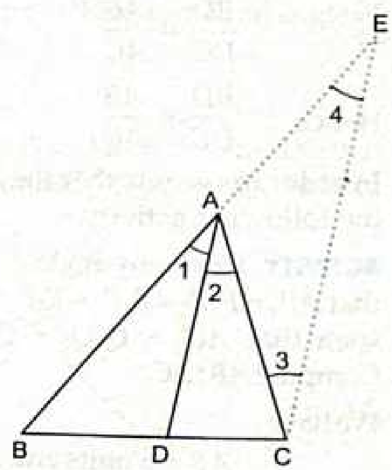


Fig. 7.39

Q.E.D.

REMARK *In the previous two theorems we have seen that the internal bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle and vice-versa. In the following theorem, we shall prove that the bisector of the exterior of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.*

THEOREM 3 *The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.*

GIVEN A ΔABC, in which AD is the bisector of the exterior of angle $\angle A$ and intersects BC produced in D.

TO PROVE $\dfrac{BD}{CD} = \dfrac{AB}{AC}$

CONSTRUCTION Draw $CE \parallel DA$ meeting AB in E.

PROOF Since $CE \parallel DA$ and AC intersects them.

$\therefore \quad \angle 1 = \angle 3 \quad$ ...(i)

Also, $CE \parallel DA$ and BK intersects them.

$\therefore \quad \angle 2 = \angle 4 \quad$ ...(ii)

But, $\quad \angle 1 = \angle 2 \quad \left[\because AD \text{ is the bisector of } \angle CAK \therefore \angle 1 = \angle 2\right]$

$\therefore \quad \angle 3 = \angle 4 \quad$ [From (i) and (ii)]

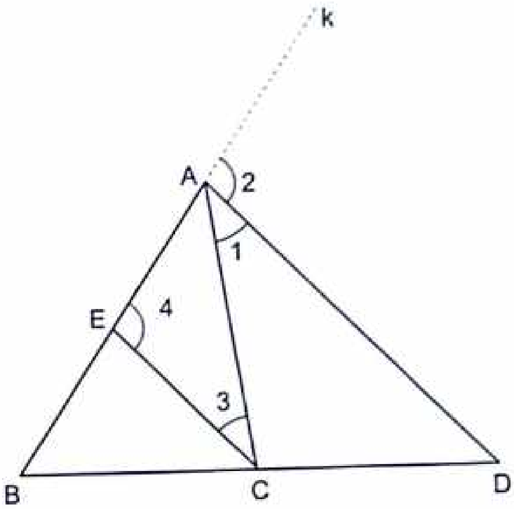


Fig. 7.40

Thus, in ΔACE, we have

$\angle 3 = \angle 4$

$\Rightarrow \quad AE = AC \quad$ [$\because$ Sides opposite to equal angles in a Δ are equal] ...(iii)

Now, in ΔBAD, we have

$EC \parallel AD$

$\therefore \quad \dfrac{BD}{CD} = \dfrac{BA}{EA} \quad$ [Using corollary of Basic Proportionality Theorem]

$\Rightarrow \quad \dfrac{BD}{CD} = \dfrac{AB}{AE} \quad$ [$\because BA = AB$ and $EA = AE$]

$\Rightarrow \quad \dfrac{BD}{CD} = \dfrac{AB}{AC} \quad$ [$\because AE = AC$, From (iii)]

Q.E.D.

The following examples will illustrate the applications of the above results.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In Fig 7.41, AD is the bisector of $\angle A$. If $BD = 4$ cm, $DC = 3$ cm and $AB = 6$ cm, determine AC.*

SOLUTION In ΔABC, AD is the bisector of $\angle A$.

$\therefore \quad \dfrac{BD}{DC} = \dfrac{AB}{AC}$

$\Rightarrow \quad \dfrac{4}{3} = \dfrac{6}{AC}$

$\Rightarrow \quad 4\,AC = 18$

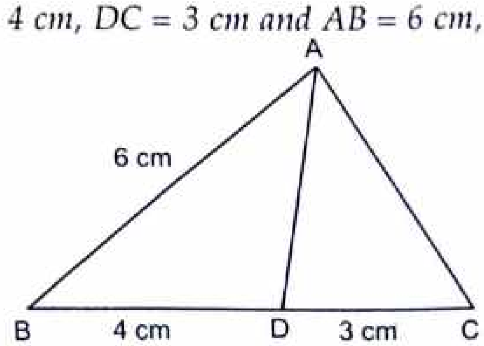


Fig. 7.41

$\Rightarrow \quad AC = \frac{9}{2}\text{ cm} = 4.5\text{ cm}.$

EXAMPLE 2 *In Fig. 7.42, AD is the bisector of $\angle BAC$. If $AB = 10$ cm. $AC = 14$ cm and $BC = 6$ cm, find BD and DC.*

SOLUTION Let $BD = x$ cm. Then, $DC = (6 - x)$ cm.

Since AD is the bisector of $\angle A$.

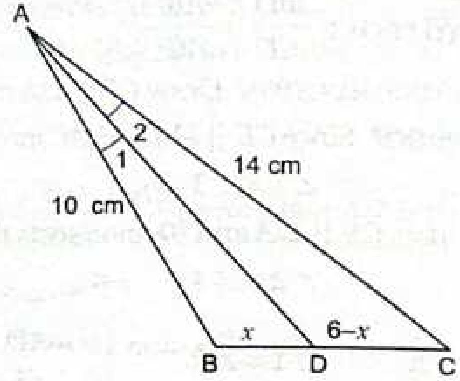


Fig. 7.42

$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$

$\Rightarrow \quad \frac{10}{14} = \frac{x}{6-x}$

$\Rightarrow \quad \frac{5}{7} = \frac{x}{6-x}$

$\Rightarrow \quad 30 - 5x = 7x \qquad [\because AC = 5.6]$

$\Rightarrow \quad 12x = 30$

$\Rightarrow \quad x = \frac{5}{2} = 2.5\text{ cm}$

$\Rightarrow \quad BD = 2.5\text{ cm and, } DC = (6 - x)\text{ cm} = (6 - 2.5)\text{ cm} = 3.5\text{ cm}$

EXAMPLE 3 *If the diagonal BD of a quadrilateral ABCD bisects both $\angle B$ and $\angle D$, show that $\frac{AB}{BC} = \frac{AD}{CD}$.*

GIVEN A quadrilateral $ABCD$ in which the diagonal BD bisects $\angle B$ and $\angle D$.

TO PROVE $\frac{AB}{BC} = \frac{AD}{CD}$

CONSTRUCTION Join AC intersecting BD in O.

PROOF In ΔABC, BO is the bisector of $\angle B$.

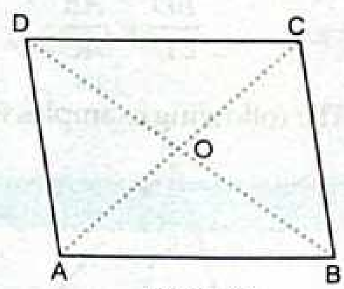


Fig. 7.43

$\therefore \quad \frac{AO}{OC} = \frac{BA}{BC}$

$\Rightarrow \quad \frac{OA}{OC} = \frac{AB}{BC} \qquad \ldots(i)$

In ΔADC, DO is the bisector of $\angle D$.

$\therefore \quad \frac{AO}{OC} = \frac{DA}{DC}$

$\Rightarrow \quad \frac{OA}{OC} = \frac{AD}{CD} \qquad \ldots(ii)$

From (i) and (ii), we get $\frac{AB}{BC} = \frac{AD}{CD}$

EXAMPLE 4 *If the bisector of an angle of a triangle bisects the opposite side, prove that the triangle is isosceles.* **[CBSE 2002]**

GIVEN In ΔABC, the bisector AD of $\angle A$ bisects the side BC.

TO PROVE $AB = AC$

PROOF In $\Delta\, ABC$, AD is the bisector of $\angle A$.

$$\therefore \qquad \frac{AB}{AC}=\frac{BD}{DC}$$

$$\Rightarrow \qquad \frac{AB}{AC}=1. \quad [D \text{ is the mid-point of } BC \therefore\ BD=DC]$$

$$\Rightarrow \qquad AB=AC$$

Hence, the triangle ABC is isosceles.

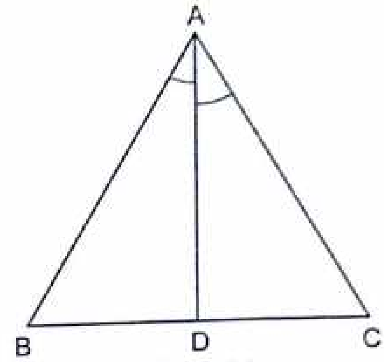


Fig. 7.44

EXAMPLE 5 *In $\Delta\, ABC$, the bisector of $\angle B$ meets AC at D. A line $PQ \parallel AC$ meets AB, BC and BD at P, Q and R respectively. Show that*

(i) $PR\,.\,BQ=QR\,.\,BP$ (ii) $AB\times CQ=BC\times AP$

GIVEN $\Delta\, ABC$ in which BD is the bisector of $\angle B$ and a line $PQ \parallel AC$ meets AB, BC and BD at P, Q and R respectively.

TO PROVE (i) $PR.BQ=QR.BP$ (ii) $AB\times CQ=BC\times AP$

PROOF (i) In $\Delta\, BQP$, BR is the bisector of $\angle B$.

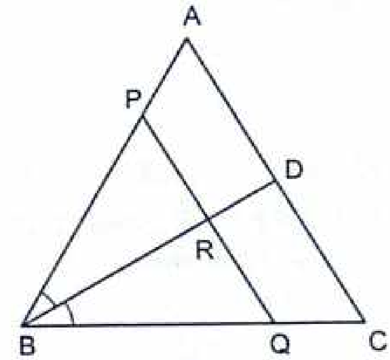


Fig. 7.45

$$\therefore \qquad \frac{BQ}{BP}=\frac{QR}{PR}$$

$$\Rightarrow \qquad BQ\,.\,PR=BP\,.\,QR$$

$$\Rightarrow \qquad PR\,.\,BQ=QR\,.\,BP$$

(i) In $\Delta\, ABC$, we have

$$PQ \parallel AC \qquad \text{[Given]}$$

$$\Rightarrow \qquad \frac{AB}{AP}=\frac{CB}{CQ} \qquad \text{[By Thale's Theorem]}$$

$$\Rightarrow \qquad AB\times CQ=BC\,.\,AP$$

EXAMPLE 6 *In $\Delta\, ABC$, if AD is the bisector of $\angle A$, prove that:*

$$\frac{\text{Area }(\Delta\, ABD)}{\text{Area }(\Delta\, ACD)}=\frac{AB}{AC}$$

SOLUTION In $\Delta\, ABC$, AD is the bisector of $\angle A$.

$$\therefore \qquad \frac{AB}{AC}=\frac{BD}{DC} \qquad \ldots\text{(i)}$$

From A, draw $AL\perp BC$.

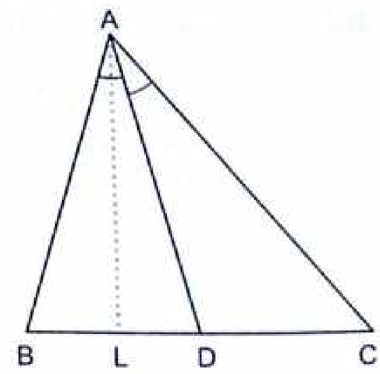


Fig. 7.46

$$\therefore \qquad \frac{\text{Area }(\Delta\, ABD)}{\text{Area }(\Delta\, ACD)}=\frac{(1/2)BD\,.\,AL}{(1/2)DC\,.\,AL}$$

$$\Rightarrow \qquad \frac{\text{Area }(\Delta\, ABD)}{\text{Area }(\Delta\, ACD)}=\frac{BD}{DC}$$

$$\Rightarrow \qquad \frac{\text{Area }(\Delta\, ABD)}{\text{Area }(\Delta\, ACD)}=\frac{AB}{AC} \qquad \text{[From (i)]}$$

EXAMPLE 7 *The bisectors of the angles B and C of a triangle ABC, meet the opposite sides in D and E respectively. If $DE \parallel BC$, prove that the triangle is isosceles.*

GIVEN A $\Delta\, ABC$ in which the bisectors of $\angle B$ and $\angle C$ meet the sides AC and AB at D and E respectively.

TO PROVE $AB = AC$

CONSTRUCTION Join DE

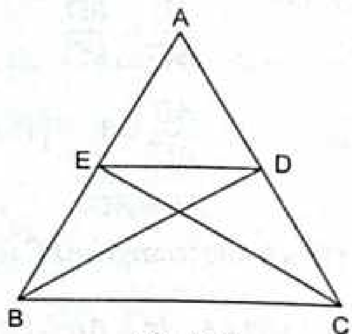


Fig. 7.47

PROOF In $\Delta\, ABC$, BD is the bisector of $\angle B$.

$$\therefore \quad \frac{AB}{BC} = \frac{AD}{DC} \quad \ldots(i)$$

In $\Delta\, ABC$, CE is the bisector of $\angle C$.

$$\therefore \quad \frac{AC}{BC} = \frac{AE}{BE} \quad \ldots(ii)$$

Now, $DE \parallel BC$

$$\Rightarrow \quad \frac{AE}{BE} = \frac{AD}{DC} \qquad \text{[By Thale's Theorem]} \quad \ldots(iii)$$

From (iii), we find the RHS of (i) and (ii) are equal. Therefore, their LHS are also equal i. e.,

$$\frac{AB}{BC} = \frac{AC}{BC}$$

$$\Rightarrow \quad AB = AC$$

Hence, $\Delta\, ABC$ is isosceles.

EXAMPLE 8 *BO and CO are respectively the bisectors of $\angle B$ and $\angle C$ of $\Delta\, ABC$. AO produced meets BC at P. Show that*

(i) $\dfrac{AB}{BP} = \dfrac{AO}{OP}$ (ii) $\dfrac{AC}{CP} = \dfrac{AO}{OP}$ (iii) $\dfrac{AB}{AC} = \dfrac{BP}{PC}$

(iv) *AP is the bisector of $\angle BAC$.*

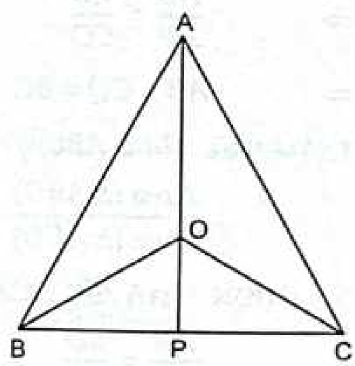


Fig. 7.48

SOLUTION (i) In $\Delta\, ABP$, BO is the bisector of $\angle B$

$$\therefore \quad \frac{AB}{BP} = \frac{AO}{OP}$$

(ii) In $\Delta\, ACP$, OC is the bisector of $\angle C$

$$\therefore \quad \frac{AC}{CP} = \frac{AO}{OP}$$

(iii) We have, proved that

$$\frac{AB}{BP} = \frac{AO}{OP} \text{ and } \frac{AC}{CP} = \frac{AO}{OP}$$

$$\Rightarrow \quad \frac{AB}{BP} = \frac{AC}{CP}$$

$$\Rightarrow \quad \frac{AB}{AC} = \frac{BP}{PC}$$

(iv) As proved above that in $\Delta\, ABC$, we have

$$\frac{AB}{AC} = \frac{BP}{CP} \Rightarrow AP \text{ is the bisector of } \angle BAC.$$

LEVEL-2

EXAMPLE 9 *The bisector of interior $\angle A$ of $\Delta\, ABC$ meets BC in D, and the bisector of exterior $\angle A$ meets BC produced in E. Prove that* $\dfrac{BD}{BE} = \dfrac{CD}{CE}$.

GIVEN In ΔABC, AD and AE are respectively the bisectors of the interior and exterior angles at A

TO PROVE $\dfrac{BD}{BE} = \dfrac{CD}{CE}$

PROOF Since AD is the internal bisector of $\angle A$ meeting BC at D.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC} \quad \text{...(i)}$$

Since AE is the external bisector of $\angle A$ meeting BC produced in E.

$$\therefore \quad \frac{AB}{AC} = \frac{BE}{CE} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{BD}{DC} = \frac{BE}{CE}$$

$$\Rightarrow \quad \frac{BD}{BE} = \frac{CD}{CE}$$

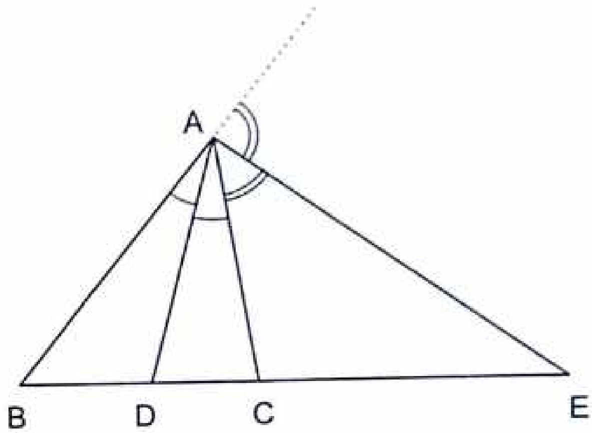


Fig. 7.49

EXAMPLE 10 *ABCD is a quadrilateral in which $AB = AD$. The bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the points E and F respectively. Prove that $EF \parallel BD$.*

GIVEN A quadrilateral $ABCD$ in which $AB = AD$ and the bisectors of $\angle BAC$ and $\angle CAD$ meet the sides BC and CD at E and F respectively.

TO PROVE $EF \parallel BD$

CONSTRUCTION Join AC, BD and EF.

PROOF In ΔCAB, AE is the bisector of $\angle BAC$.

$$\therefore \quad \frac{AC}{AB} = \frac{CE}{BE} \quad \text{...(i)}$$

In ΔACD, AF is the bisector of $\angle CAD$.

$$\therefore \quad \frac{AC}{AD} = \frac{CF}{DF}$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{CF}{DF} \quad [\because AD = AB] \quad \text{...(ii)}$$

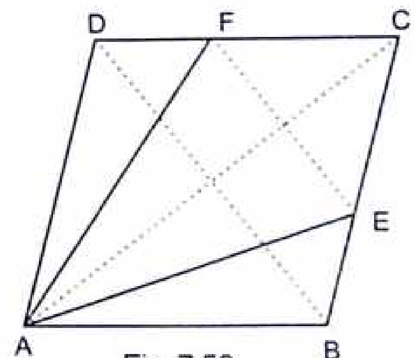


Fig. 7.50

From (i) and (ii), we get

$$\frac{CE}{BE} = \frac{CF}{DF}$$

$$\Rightarrow \quad \frac{CE}{EB} = \frac{CF}{FD}$$

Thus, in ΔCBD, E and F divide the sides CB and CD respectively in the same ratio. Therefore, by the converse of Thale's Theorem, we have $EF \parallel BD$.

EXAMPLE 11 *O is any point inside a triangle ABC. The bisector of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in point D, E and F respectively. Show that*

$$AD \times BE \times CF = DB \times EC \times FA$$

SOLUTION In ΔAOB, OD is the bisector of $\angle AOB$.

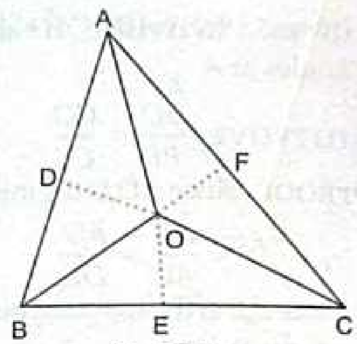


Fig. 7.51

$$\therefore \quad \frac{OA}{OB} = \frac{AD}{DB} \quad \ldots\text{(i)}$$

In ΔBOC, OE is the bisector of $\angle BOC$.

$$\therefore \quad \frac{OB}{OC} = \frac{BE}{EC} \quad \ldots\text{(ii)}$$

In ΔCOA, OF is the bisector of $\angle COA$.

$$\therefore \quad \frac{OC}{OA} = \frac{CF}{FA} \quad \ldots\text{(iii)}$$

Multiplying the corresponding sides of (i), (ii) and (iii), we get

$$\frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA} = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow \quad 1 = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow \quad DB \times EC \times FA = AD \times BE \times CF$$

$$\Rightarrow \quad AD \times BE \times CF = DB \times EC \times FA$$

EXAMPLE 12 *AD is a median of ΔABC. The bisector of $\angle ADB$ and $\angle ADC$ meet AB and AC in E and F respectively. Prove that $EF \parallel BC$.*

GIVEN In ΔABC, AD is the median and DE and DF are the bisectors of $\angle ADB$ and $\angle ADC$ respectively, meeting AB and AC in E and F respectively.

TO PROVE $EF \parallel BC$

PROOF In ΔADB, DE is the bisector of $\angle ADB$.

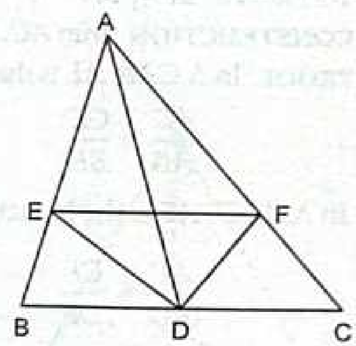


Fig. 7.52

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EB} \quad \ldots\text{(i)}$$

In ΔADC, DF is the bisector of $\angle ADC$.

$$\therefore \quad \frac{AD}{DC} = \frac{AF}{FC}$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AF}{FC} \quad \ldots\text{(ii)} \quad \left[\begin{array}{l} \because AD \text{ is the median} \\ \therefore BD = DC \end{array}\right]$$

From (i) and (ii), we get

$$\frac{AE}{EB} = \frac{AF}{FC}$$

Thus, in ΔABC, line segment EF divides the sides AB and AC in the same ratio.

Hence, EF is parallel to BC.

EXAMPLE 13 *In ΔABC, D is the mid-point of BC and ED is the bisector of the $\angle ADB$ and EF is drawn parallel to BC cutting AC in F. Prove that $\angle EDF$ is a right angle.*

GIVEN A ΔABC in which D is the mid-point of side BC and ED is the bisector of $\angle ADB$, meeting AB in E. EF is drawn parallel to BC meeting AC in F.

TO PROVE $\angle EDF$ is a right angle.

PROOF In ΔADB, DE is the bisector of $\angle ADB$.

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EB}$$

$$\Rightarrow \quad \frac{AD}{DC} = \frac{AE}{EB} \; [\because D \text{ is the mid-point of } BC \therefore DB = DC] \quad \ldots(i)$$

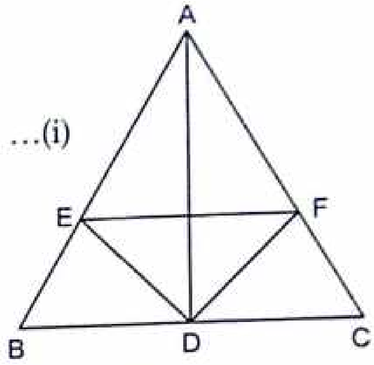


Fig. 7.53

In ΔABC, we have

$$EF \parallel BC$$

$$\Rightarrow \quad \frac{AE}{EB} = \frac{AF}{FC} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{AD}{DC} = \frac{AF}{FC}$$

$\Rightarrow$ In ΔADC, DF divides AC in the ratio $AD : DC$

$\Rightarrow$ DF is the bisector of $\angle ADC$

Thus, DE and DF are the bisectors of adjacent supplementary angles $\angle ADB$ and $\angle ADC$ respectively.

Hence, $\angle EDF$ is a right angle.

EXAMPLE 14 *In ΔABC, $\angle B = 2\angle C$ and the bisector of $\angle B$ intersects AC at D. Prove that* $\frac{BD}{DA} = \frac{BC}{BA}$.

SOLUTION In ΔABC, bisector of $\angle B$ meets AC at D.

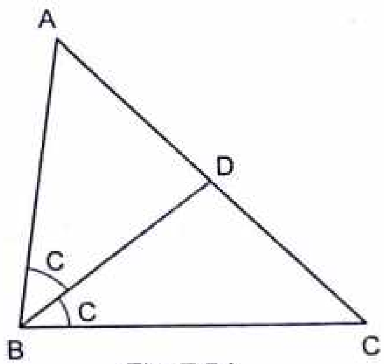


Fig. 7.54

$$\therefore \quad \frac{CD}{AD} = \frac{BC}{BA}$$

$$\Rightarrow \quad \frac{BD}{AD} = \frac{BC}{BA} \quad \left[\begin{array}{l}\text{In } \Delta BCD, \text{ we have} \\ \angle DBC = \angle BCD = C \therefore BD = CD\end{array}\right]$$

$$\Rightarrow \quad \frac{BD}{DA} = \frac{BC}{BA}$$

EXAMPLE 15 *In Fig. 7.55, $\angle BAC = 90°$, AD is its bisector. If $DE \perp AC$, prove that* $DE \times (AB + AC) = AB \times AC$.

SOLUTION It is given that AD is the bisector of $\angle A$ of ΔABC.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

$$\Rightarrow \quad \frac{AB}{AC} + 1 = \frac{BD}{DC} + 1 \quad [\text{Adding 1 on both sides}]$$

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{BD + DC}{DC}$$

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{BC}{DC} \quad \ldots(i)$$

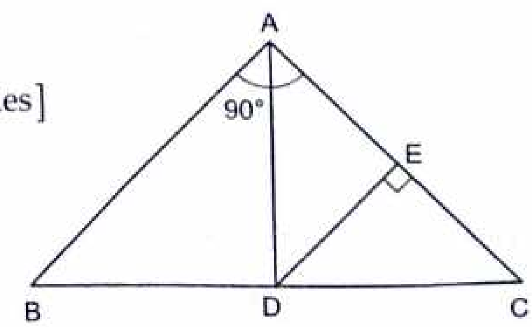


Fig. 7.55

In Δ's CDE and CBA, we have

$\angle DCE = \angle BCA = \angle C$ [Common]

$\angle BAC = \angle DEC$ [Each equal to 90°]

So, by AA-criterion of similarity, we have

$$\Delta CDE \sim \Delta CBA$$

$$\Rightarrow \quad \frac{CD}{CB} = \frac{DE}{BA}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{DC} \quad \ldots(ii)$$

From (i) and (ii), we obtain

$$\frac{AB + AC}{AC} = \frac{AB}{DE} \Rightarrow DE \times (AB + AC) = AB \times AC$$

LEVEL-2

EXAMPLE 16 *In a quadrilateral ABCD, if bisectors of the $\angle ABC$ and $\angle ADC$ meet on the diagonal AC, prove that the bisectors of $\angle BAD$ and $\angle BCD$ will meet on the diagonal BD.*

GIVEN $ABCD$ is a quadrilateral in which the bisectors of $\angle ABC$ and $\angle ADC$ meet on the diagonal AC at P.

TO PROVE Bisectors of $\angle BAD$ and $\angle BCD$ meet on the diagonal BD

CONSTRUCTION Join BP and DP. Let the bisector of $\angle BAD$ meet BD at Q. Join AQ and CQ.

PROOF In order to prove that the bisectors of $\angle BAD$ and $\angle BCD$ meet on the diagonal BD. It is sufficient to prove that CQ is the bisector of $\angle BCD$. For which we will prove that Q divides BD in the ratio $BC : DC$.

In ΔABC, BP is the bisector of $\angle ABC$.

$$\therefore \quad \frac{AB}{BC} = \frac{AP}{PC} \quad \ldots(i)$$

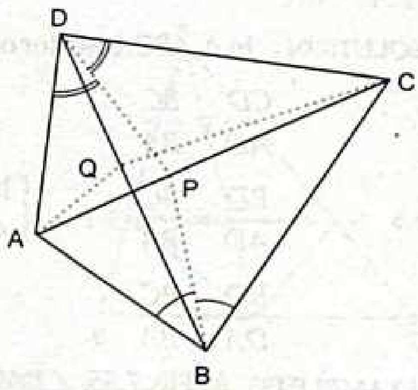


Fig. 7.56

In ΔACD, DP is the bisector of $\angle ADC$.

$$\therefore \quad \frac{AD}{DC} = \frac{AP}{PC} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{AB}{BC} = \frac{AD}{DC}$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{BC}{DC} \quad \ldots(iii)$$

In ΔABD, AQ is the bisector of $\angle BAD$. [By construction]

$$\therefore \quad \frac{AB}{AD} = \frac{BQ}{DQ} \quad \ldots(iv)$$

From (iii) and (iv), we get : $\frac{BC}{DC} = \frac{BQ}{DQ}$.

Thus, in ΔCBD, Q divides BD in the ratio $CB : CD$. Therefore, CQ is the bisectors of $\angle BCD$. Hence, bisectors of $\angle BAD$ and $\angle BCD$ meet on the diagonal BD.

EXERCISE 7.3

LEVEL-1

1. In a ΔABC, AD is the bisector of $\angle A$, meeting side BC at D.
   (i) If $BD = 2.5$ cm, $AB = 5$ cm and $AC = 4.2$ cm, find DC.
   (ii) If $BD = 2$ cm, $AB = 5$ cm and $DC = 3$ cm, find AC.
   (iii) If $AB = 3.5$ cm, $AC = 4.2$ cm and $DC = 2.8$ cm, find BD.
   (iv) If $AB = 10$ cm, $AC = 14$ cm and $BC = 6$ cm, find BD and DC.
   (v) If $AC = 4.2$ cm, $DC = 6$ cm and $BC = 10$ cm, find AB.
   (vi) If $AB = 5.6$ cm, $AC = 6$ cm and $DC = 3$ cm, find BC. [CBSE 2001C]
   (vii) If $AD = 5.6$ cm, $BC = 6$ cm and $BD = 3.2$ cm, find AC. [CBSE 2001C]
   (viii) If $AB = 10$ cm, $AC = 6$ cm and $BC = 12$ cm, find BD and DC. [CBSE 2001]
2. In Fig. 7.57, AE is the bisector of the exterior $\angle CAD$ meeting BC produced in E. If $AB = 10$ cm, $AC = 6$ cm and $BC = 12$ cm, find CE.

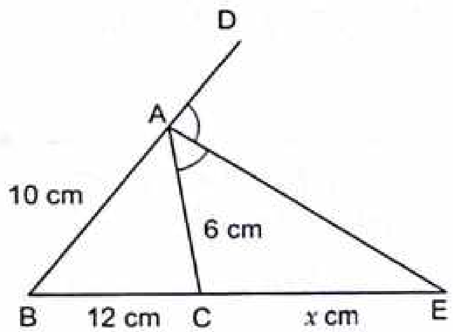


Fig. 7.57

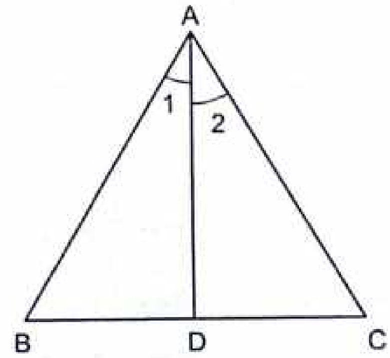


Fig. 7.58

3. In Fig. 7.58, ΔABC is a triangle such that $\frac{AB}{AC} = \frac{BD}{DC}$, $\angle B = 70°$, $\angle C = 50°$. Find $\angle BAD$.
4. In Fig. 7.59, check whether AD is the bisector of $\angle A$ of ΔABC in each of the following:
   (i) $AB = 5$ cm, $AC = 10$ cm, $BD = 1.5$ cm and $CD = 3.5$ cm
   (ii) $AB = 4$ cm, $AC = 6$ cm, $BD = 1.6$ cm and $CD = 2.4$ cm
   (iii) $AB = 8$ cm, $AC = 24$ cm, $BD = 6$ cm and $BC = 24$ cm
   (iv) $AB = 6$ cm, $AC = 8$ cm, $BD = 1.5$ cm and $CD = 2$ cm
   (v) $AB = 5$ cm, $AC = 12$ cm, $BD = 2.5$ cm and $BC = 9$ cm

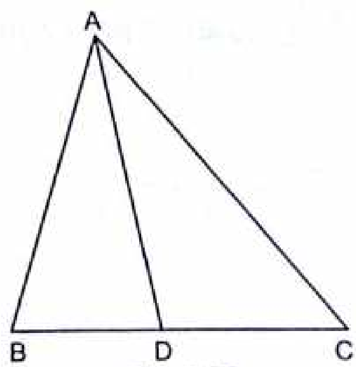


Fig. 7.59

5. In Fig. 7.60, AD bisects $\angle A$, $AB = 12$ cm, $AC = 420$ cm and $BD = 5$ cm, determine CD.

LEVEL-2

6. In $\Delta\, ABC$ (Fig. 7.60), if $\angle 1 = \angle 2$, prove that $\frac{AB}{AC} = \frac{BD}{DC}$.

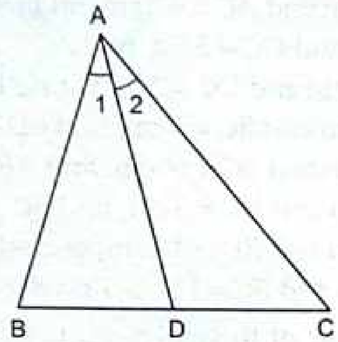


Fig. 7.60

7. D, E and F are the points on sides BC, CA and AB respectively of $\Delta\, ABC$ such that AD bisects $\angle A$, BE bisects $\angle B$ and CF bisects $\angle C$. If $AB = 5$ cm, $BC = 8$ cm and $CA = 4$ cm, determine AF, CE and BD.

ANSWERS

1. (i) 2.1 cm (ii) 7.5 cm (iii) 2.3 cm (iv) $BD = 2.5$ cm, $DC = 3.5$ cm
(v) 2.8 cm (vi) 5.8 cm (vii) 4.9 cm (viii) 7.5 cm, 4.5 cm
2. 18
3. 30°
4. (i) No (ii) Yes (iii) Yes (iv) Yes (v) No
5. 8.33 cm
7. $AF = \frac{5}{3}$ cm, $CE = \frac{32}{13}$ cm, $BD = \frac{40}{9}$ cm

HINTS TO SELECTED PROBLEMS

2. Since AE is the bisector of the exterior $\angle\, CAD$.

$$\therefore \quad \frac{BE}{CE} = \frac{AB}{AC} \Rightarrow \frac{12 + x}{x} = \frac{10}{6} \Rightarrow x = 18.$$

7. Since AD is the bisector of $\angle\, A$.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{CD}$$

$$\Rightarrow \quad \frac{5}{4} = \frac{BD}{BC - BD} \Rightarrow \frac{5}{4} = \frac{BD}{8 - BD} \Rightarrow 40 - 5\,BD = 4\,BD \Rightarrow 9\,BD = 40 \Rightarrow BD = \frac{40}{9}\text{ cm}.$$

Since BE is the bisector of $\angle\, B$.

$$\therefore \quad \frac{AB}{BC} = \frac{AE}{CE} \Rightarrow \frac{AB}{BC} = \frac{AC - CE}{CE} \Rightarrow \frac{5}{8} = \frac{4 - CE}{CE} \Rightarrow 13\,CE = 32 \Rightarrow CE = \frac{32}{13}\text{ cm}.$$

Since CF is the bisector of the $\angle\, C$.

$$\therefore \quad \frac{BC}{CA} = \frac{BF}{AF}$$

$$\Rightarrow \quad \frac{8}{4} = \frac{AB - AF}{AF} \Rightarrow 2 = \frac{5 - AF}{AF} \Rightarrow 3\,AF = 5 \Rightarrow AF = \frac{5}{3}\text{ cm}.$$

7.6 MORE ON BASIC PROPORTIONALITY THEOREM

In this section, we shall discuss some more applications of basic proportionality theorem.

THEOREM 1 *The line drawn from the mid-point of one side of a triangle parallel to another side bisects the third side.* [NCERT]

GIVEN A Δ ABC in which D is the mid-point of side AB and the line DE is drawn parallel to BC, meeting AC in E.

TO PROVE E is the mid-point of AC i. e., $AE = EC$.

PROOF In Δ ABC, we have

$$DE \parallel BC$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC} \qquad \text{[By Thale's Theorem]} \quad \ldots\text{(i)}$$

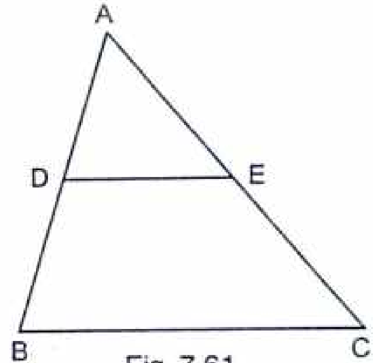


Fig. 7.61

But, D is the mid-point of AB.

$$\Rightarrow \quad AD = DB$$

$$\Rightarrow \quad \frac{AD}{DB} = 1 \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{AE}{EC} = 1 \Rightarrow AE = EC.$$

Hence, E bisects AC.

Q.E.D.

THEOREM 2 *The line joining the mid-points of two sides of a triangle is parallel to the third side.* [NCERT]

GIVEN A Δ ABC in which D and E are mid-points of sides AB and AC respectively.

TO PROVE $DE \parallel BC$.

PROOF Since D and E are mid-points of AB and AC respectively.

$$\therefore \quad AD = DB \text{ and } AE = EC$$

$$\Rightarrow \quad \frac{AD}{DB} = 1 \text{ and } \frac{AE}{EC} = 1$$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

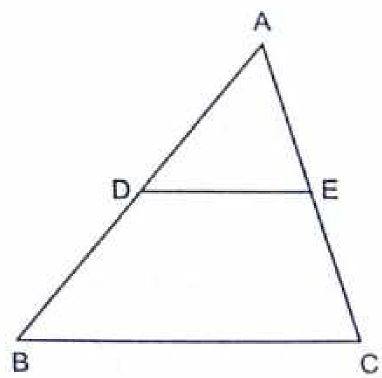


Fig. 7.62

Thus, the line DE divides the sides AB and AC of Δ ABC in the same ratio. Therefore, by the converse of Basic Proportionality Theorem, we obtain $DE \parallel BC$.

Q.E.D.

THEOREM 3 *Prove that the diagonals of a trapezium divide each other proportionally.* [NCERT]

GIVEN A trapezium $ABCD$ in which the diagonals AC and BD intersect at E.

TO PROVE $\dfrac{DE}{EB} = \dfrac{CE}{EA}$

CONSTRUCTION Draw $EF \parallel BA \parallel CD$, meeting AD in F.

PROOF In ΔABD, we have

$$FE \parallel AB$$

$$\Rightarrow \quad \frac{DE}{EB} = \frac{DF}{FA} \quad \text{[By Thale's Theorem] ...(i)}$$

In ΔCDA, we have

$$FE \parallel DC$$

$$\Rightarrow \quad \frac{CE}{EA} = \frac{DF}{FA} \quad \text{[By Thale's Theorem] ...(ii)}$$

From (i) and (ii), we get

$$\frac{DE}{EB} = \frac{CE}{EA}$$

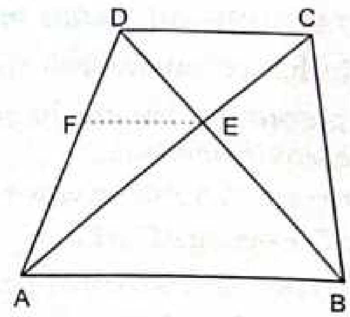


Fig. 7.63

Q.E.D.

THEOREM 4 *If the diagonals of a quadrilateral divide each other proportionally, then it is a trapezium.* **[NCERT, CBSE 2005]**

GIVEN A quadrilateral $ABCD$ whose diagonals AC and BD intersect at E such that $\frac{DE}{EB} = \frac{CE}{EA}$.

TO PROVE Quadrilateral $ABCD$ is a trapezium. For this it is sufficient to prove that $AB \parallel DC$.

CONSTRUCTION Draw $EF \parallel BA$, meeting AD in F.

PROOF In ΔABD, we have

$$EF \parallel BA$$

$$\Rightarrow \quad \frac{DF}{FA} = \frac{DE}{EB} \quad \text{[By Thale's Theorem] ...(i)}$$

But, $$\frac{DE}{EB} = \frac{CE}{EA} \quad \text{[Given] ...(ii)}$$

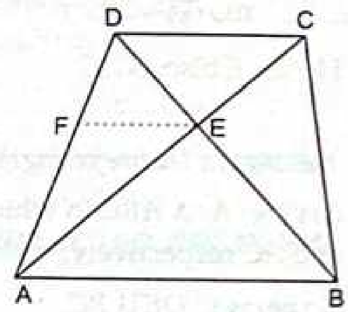


Fig. 7.64

From (i) and (ii), we get

$$\frac{DF}{FA} = \frac{CE}{EA}$$

Thus, in ΔDCA, E and F are points on CA and DA respectively such that

$$\frac{DF}{FA} = \frac{CE}{EA}$$

Therefore, by the converse of Basic Proportionality Theorem, we have

$$FE \parallel DC$$

But, $FE \parallel BA$ [By construction]

$\therefore \quad DC \parallel BA \Rightarrow AB \parallel DC$

Hence, $ABCD$, is a trapezium.

Q.E.D.

THEOREM 5 *Any line parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.*

GIVEN A trapezium $ABCD$ in which $DC \parallel AB$ and EF is a line parallel to DC and AB.

TO PROVE $\frac{AE}{ED} = \frac{BF}{FC}$

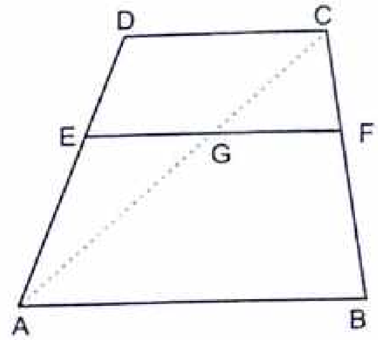


Fig. 7.65

CONSTRUCTION Join AC, meeting EF in G.

PROOF In $\Delta\ ADC$, we have

$EG \parallel DC$

$\Rightarrow \quad \frac{AE}{ED} = \frac{AG}{GC}$ [By Thale's Theorem] ...(i)

In $\Delta\ ABC$, we have

$GF \parallel AB$

$\Rightarrow \quad \frac{AG}{GC} = \frac{BF}{FC}$ [By Thale's Theorem] ...(ii)

From (i) and (ii), we get : $\frac{AE}{ED} = \frac{BF}{FC}$

Q.E.D.

THEOREM 6 *If three or more parallel lines are intersected by two transversals, prove that the intercepts made by them on the transversals are proportional.*

GIVEN Three parallel lines l, m, n which are cut by the transversals AB and CD in P, Q, R and E, F, G respectively.

TO PROVE $\frac{PQ}{QR} = \frac{EF}{FG}$

CONSTRUCTION Draw $PL \parallel CD$ meeting the lines m and n in M and L respectively.

PROOF Since $PE \parallel MF$ and $PM \parallel EF$.

$\therefore \quad PMFE$ is a parallelogram

$\Rightarrow \quad PM = EF$ ...(i)

Also, $\quad MF \parallel LG$ and $ML \parallel FG$.

$\therefore \quad MLGF$ is a parallelogram

$\Rightarrow \quad ML = FG$ ...(ii)

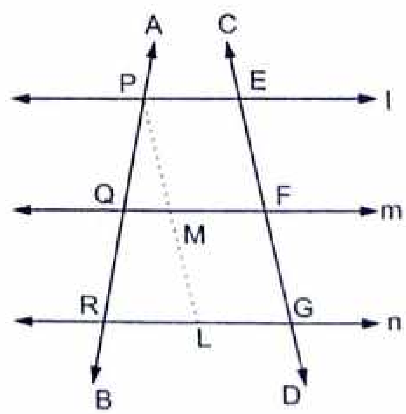


Fig. 7.66

Now, in $\Delta\ PRL$, we have

$QM \parallel RL$

$\Rightarrow \quad \frac{PQ}{QR} = \frac{PM}{ML}$ [By Thale's Theorem]

$\Rightarrow \quad \frac{PQ}{QR} = \frac{EF}{FG}$ [Using (i) and (ii)]

Hence, $\quad \frac{PQ}{QR} = \frac{EF}{FG}$. **Q.E.D.**

COROLLARY *If three or more parallel straight lines make equal intercepts on a given transversal, prove that they will make equal intercepts on any other transversal.*

PROOF Let, l, m, n be three parallel lines which make equal intercepts PQ and QR on a transversal AB (see Fig. 7.66). Let CD be any other transversal cutting l, m and n at E, F and G respectively. Then,

$$\frac{PQ}{QR} = \frac{EF}{FG}$$ [By Theorem 6] ...(i)

But, $PQ = QR. \Rightarrow \frac{PQ}{QR} = 1$ ...(ii)

From (i) and (ii), we get

$$\frac{EF}{FG} = 1 \Rightarrow EF = FG$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.*

GIVEN A quadrilateral $ABCD$ in which P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively.

TO PROVE $PQRS$ is a parallelogram.

CONSTRUCTION Join AC.

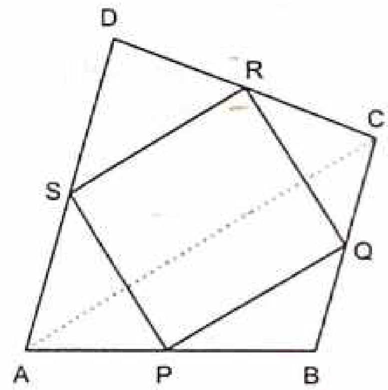


Fig. 7.67

PROOF In ΔABC, P and Q are the mid-points of AB and BC respectively.

$\therefore$ $PQ \parallel AC$ ...(i)

In ΔACD, R and S are the mid-points of CD and DA respectively.

$\therefore$ $SR \parallel AC$ ...(ii)

From (i) and (ii), we have

$PQ \parallel AC$ and $SR \parallel AC$

$\Rightarrow$ $PQ \parallel SR$

Similarly, by considering triangles ABD and BCD, we can prove that

$PS \parallel QR$

Hence, $PQRS$ is a parallelogram.

LEVEL-2

EXAMPLE 2 *In Fig. 7.68, P is the mid-point of BC and Q is the mid-point of AP. If BQ when produced meets AC at R, prove that* $RA = \frac{1}{3} CA$. **[CBSE 2006C]**

GIVEN A ΔABC in which P is the mid-point of BC, Q is the mid-point of BR and, Q is also the mid-point of AP such that BQ produced meets AC at R.

TO PROVE $RA = \frac{1}{3} CA$.

CONSTRUCTION Draw $PS \parallel BR$, meeting AC at S.

PROOF In ΔBCR, P is the mid-point of BC and $PS \parallel BR$.

$\therefore$ S is the mid-point of CR.

$\Rightarrow$ $CS = SR$ ...(i)

In ΔAPS, Q is the mid-point of AP and $QR \parallel PS$.

$\therefore$ R is the mid-point of AS.

$\Rightarrow$ $AR = RS$ ...(ii)

From (i) and (ii), we get

$AR = RS = SC$

$\Rightarrow$ $AC = AR + RS + SC = 3\,AR$

$\Rightarrow$ $AR = \frac{1}{3} AC = \frac{1}{3} CA$

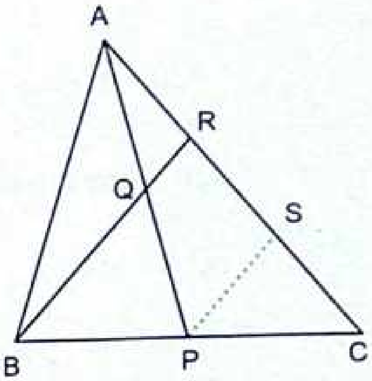


Fig. 7.68

EXAMPLE 3 *In Fig. 7. 69, AB || DC. Find the value of x.*

SOLUTION Since the diagonals of a trapezium divide each other proportionally.

$\therefore$ $\frac{AO}{OC} = \frac{BO}{OD}$ ...(iii)

$\Rightarrow$ $\frac{3x-19}{x-5} = \frac{x-3}{3}$

$\Rightarrow$ $3(3x-19) = (x-5)(x-3)$

$\Rightarrow$ $9x - 57 = x^2 - 8x + 15$

$\Rightarrow$ $x^2 - 17x + 72 = 0$

$\Rightarrow$ $(x-8)(x-9) = 0$

$\Rightarrow$ $x - 8 = 0$ or, $x - 9 = 0 \Rightarrow x = 8$ or, $x = 9$

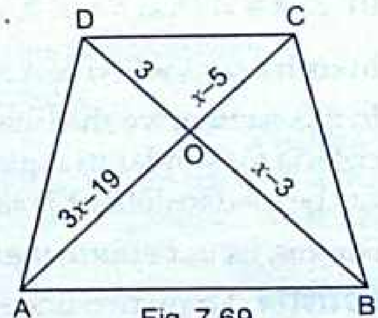


Fig. 7.69

EXERCISE 7.4

LEVEL-1

1. (i) In Fig. 7. 70, if $AB \parallel CD$, find the value of x.

(ii) In Fig. 7. 71, if $AB \parallel CD$, find the value of x.

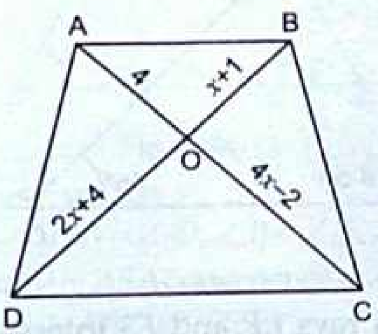


Fig. 7.70

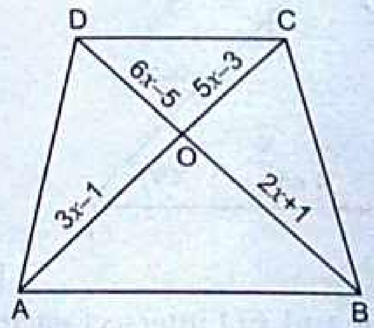


Fig. 7.71

(iii) In Fig. 7.72, $AB \parallel CD$. If $OA = 3x - 19$, $OB = x - 4$, $OC = x - 3$ and $OD = 4$, find x. **[CBSE 2000C]**

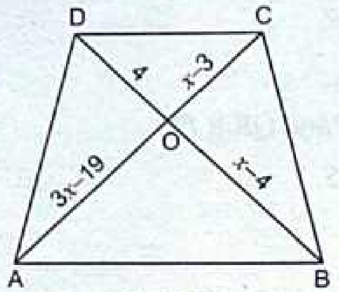


Fig. 7.72

ANSWERS

1. (i) 3 (ii) 2 (iii) 11 or, 8

7.7 CRITERIA FOR SIMILARITY OF TRIANGLES

In section 7. 3, we have defined similarity of two triangles. Let us recall that two triangles are similar iff (i) their corresponding angles are equal and (ii) their corresponding sides are proportional. In other words, two triangles ABC and DEF are similar, if

(i) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and, (ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

In such a case, we write $\Delta\, ABC \sim \Delta\, DEF$

In this section, we shall make use of the theorems discussed in earlier sections to derive some criteria for similar triangles which in turn will imply that either of the above two conditions can be used to define the similarity of two triangles.

For this, let us perform the following activity:

ACTIVITY Draw two line segments BC and EF of two different lengths, say 6 cm and 4 cm respectively. At B and C construct angles of some measures say 65° and 45° respectively. Also, construct angles of 65° and 45° at E and F respectively.

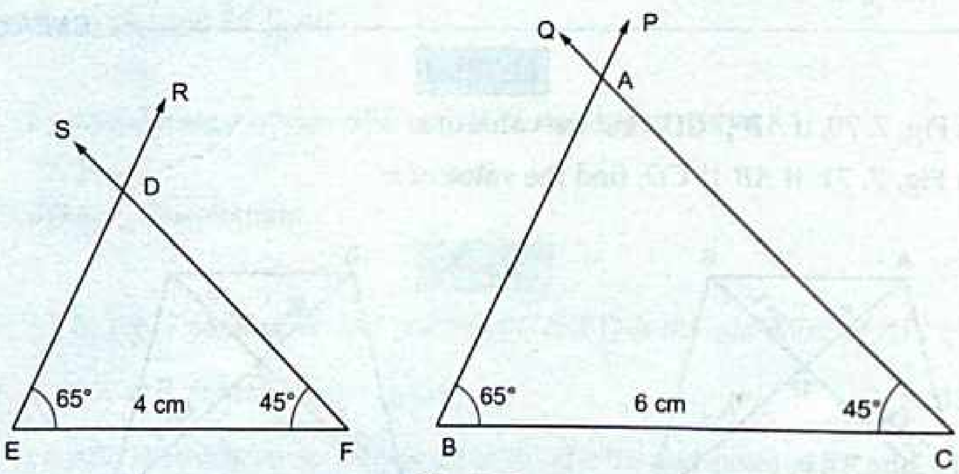


Fig. 7.73

Suppose rays BP and CQ intersect each other at A and rays ER and FS intersect each other at D.

We have,

$$\angle A = 180° - (\angle B + \angle C) = 180° - 110° = 70°$$

and, $\angle D = 180° - (\angle E + \angle F) = 180° - 110° = 70°$

In triangles ABC and DEF, we observe that

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$

That is, corresponding angles of these two triangles are equal. We also observe that

$$\frac{BC}{EF} = \frac{6}{4} = \frac{3}{2} = 1.5$$

Now, measure AB, DE, CA and FD and compute $\frac{AB}{DE}$ and $\frac{CA}{FD}$

You will find that $\frac{AB}{DE} = \frac{CA}{FD} = 1.5$

Thus, $$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

It follows from this activity that if corresponding angles of two triangles are equal, then their corresponding sides are in the same ratio.

Thus, we have following criterion for similarity of two triangles.

EQUIANGULAR TRIANGLES *Two triangles are said to be equiangular, if their corresponding angles are equal.*

THEOREM 1 *(AAA Similarity Criterion) If two triangles are equiangular, then they are similar.*

GIVEN Two triangles ABC and DEF such that $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$.

TO PROVE $\Delta ABC \sim \Delta DEF$ [NCERT]

PROOF Recall that two triangles are similar iff their corresponding angles are equal and the corresponding sides are proportional. Since corresponding angles are given equal, we must prove that the corresponding sides are proportional i.e.,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

For this purpose we divide the proof into three parts.

CASE I *When* $AB = DE$.

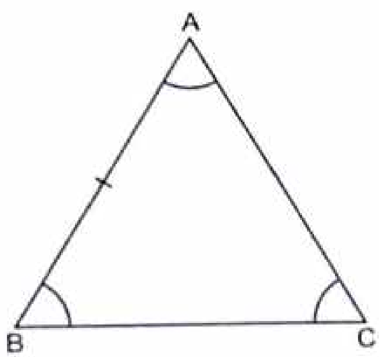


Fig. 7.74

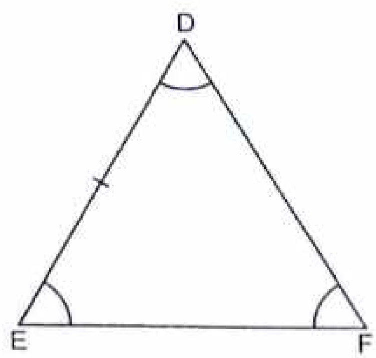


Fig. 7.75

In this case, we have

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F \text{ and } AB = DE$$

Therefore, by ASA congruence criterion, we have

$$\Delta ABC \cong \Delta DEF$$

$\Rightarrow$ $AB = DE, BC = EF$ and $AC = DF$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Hence, $\Delta ABC \sim \Delta DEF$.

CASE II *When* $AB < DE$.

Mark a point P on the line DE and Q on the line DF such that $AB = DP$ and $AC = DQ$. Join PQ.
In triangles ABC and DPQ, we have

$AB = DP$, $\angle A = \angle D$ and $AC = DQ$

$\therefore$ $\Delta ABC \cong \Delta DPQ$ [By SAS criterion of congruence]

$\Rightarrow$ $\angle B = \angle DPQ$

But, $\angle B = \angle E = \angle DEF$

$\therefore$ $\angle DPQ = \angle DEF$

$\Rightarrow$ $PQ \parallel EF$ [$\because$ Corresponding angles are equal]

$\Rightarrow$ $\dfrac{DP}{DE} = \dfrac{DQ}{DF}$ [By corollary of Thale's Theorem]

$\Rightarrow$ $\dfrac{AB}{DE} = \dfrac{AC}{DF}$

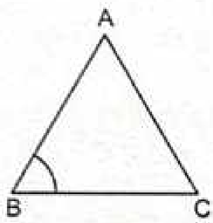


Fig. 7.76

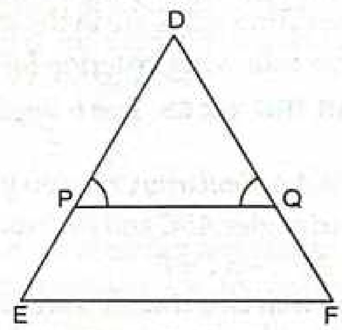


Fig. 7.77

Similarly, we can prove that

$$\frac{AB}{DE} = \frac{BC}{EF}$$

$\therefore$ $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

Hence, $\Delta ABC \sim \Delta DEF$.

CASE III *When* $AB > DE$.

Mark a point P on the line DE produced and Q on the line DF produced such that $DP = AB$ and $DQ = AC$. Join PQ.

In triangles ABC and DPQ, we have

$AB = DP$, $AC = DQ$ and $\angle A = \angle D$.

$\therefore$ $\Delta ABC \cong \Delta DPQ$ [By SAS criterion of congruence]

$\Rightarrow$ $\angle B = \angle DPQ$

But, $\angle B = \angle E = \angle DEF$

$\therefore$ $\angle DPQ = \angle DEF$

$\Rightarrow$ $PQ \parallel EF$ [$\because$ Corresponding angles are equal]

$\Rightarrow$ $\dfrac{DE}{DP} = \dfrac{DF}{DQ}$ [By corollary of Thale's Theorem]

$\Rightarrow$ $\dfrac{DE}{AB} = \dfrac{DF}{AC}$ [$\because AB = DP$ and $AC = DQ$]

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AC}{DF}$$

Similarly, we can prove that

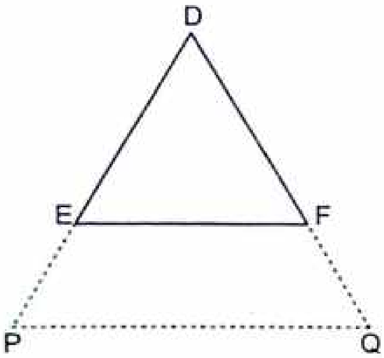


Fig. 7.78

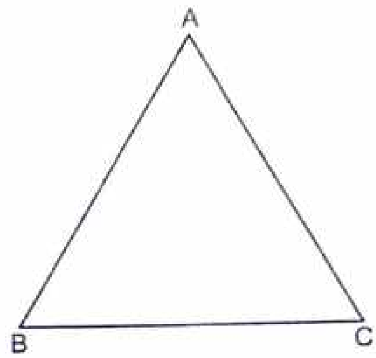


Fig. 7.79

$$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Hence, $\Delta ABC \sim \Delta DEF$.

Q.E.D.

REMARK *It follows from the above theorem that : Two triangles are similar $\Leftrightarrow$ They are equiangular.*

COROLLARY *(AA Similarity) If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.*

PROOF Let ΔABC and ΔDEF be two triangles such that $\angle A = \angle D$ and $\angle B = \angle E$.

In triangles ABC and DEF, we have

$$\angle A + \angle B + \angle C = 180^\circ \text{ and } \angle D + \angle E + \angle F = 180^\circ$$

$$\Rightarrow \quad \angle A + \angle B + \angle C = \angle D + \angle E + \angle F$$

$$\Rightarrow \quad \angle D + \angle E + \angle C = \angle D + \angle E + \angle F \qquad [\because \angle A = \angle D \text{ and } \angle B = \angle E]$$

$$\Rightarrow \quad \angle C = \angle F.$$

$$\therefore \quad \angle A = \angle D, \angle B = \angle E \text{ and } \angle C = \angle F.$$

Thus, the two triangles are equiangular and hence they are similar.

Q.E.D.

In the above discussion we have seen that if three angles of one triangle are respectively equal to three angles of another triangle, their corresponding sides are proportional and hence the triangles are similar. Now a natural question arises. Is the converse of this statement true? In other words, if the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal? For this, let us perform the following activity:

ACTIVITY Draw two triangles ABC and DEF such that $AB = 4.5$ cm, $BC = 9$ cm, $CA = 12$ cm, $DE = 3$ cm, $EF = 6$ cm and $FD = 8$ cm as shown in Fig. 7.80 (i) and (ii).

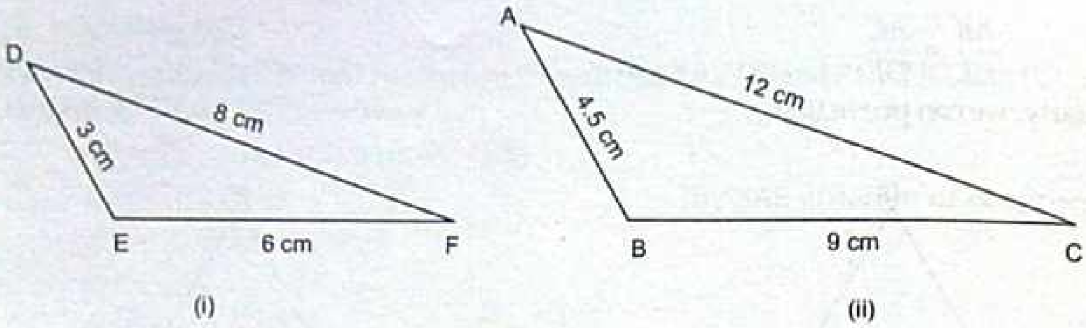


Fig. 7.80

We have,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{3}{2}$$

That is the corresponding sides of triangles ABC and DEF are proportional.

Now, measure $\angle A, \angle B, \angle C, \angle D, \angle E$ and $\angle F$. You will observe that $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ i.e., the corresponding angles of two triangles are equal and hence they are similar. Let us now prove this result as a criterion of similarity of two triangles as a theorem.

THEOREM 2 *(SSS Similarity Criterion) If the corresponding sides of two triangles are proportional, then they are similar.* **[NCERT]**

GIVEN Two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

TO PROVE $\Delta ABC \sim \Delta DEF$

CONSTRUCTION Let P and Q be points on DE and DF respectively such that $DP = AB$ and $DQ = AC$. Join PQ.

PROOF We have,

$$\frac{AB}{DE} = \frac{AC}{DF}$$

$\Rightarrow \quad \frac{DP}{DE} = \frac{DQ}{DF}$ [$\because AB = DP$ and $AC = DQ$]

$\Rightarrow \quad PQ \parallel EF$ [By the converse of Thale's Theorem]

$\Rightarrow \quad \angle DPQ = \angle E$ and $\angle DQP = \angle F$ [Corresponding angles]

Thus, in triangles DPQ and DEF, we have

$$\angle DPQ = \angle E \text{ and } \angle DQP = \angle F$$

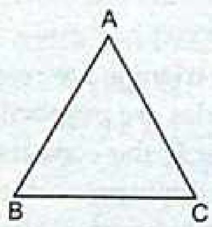


Fig. 7.81

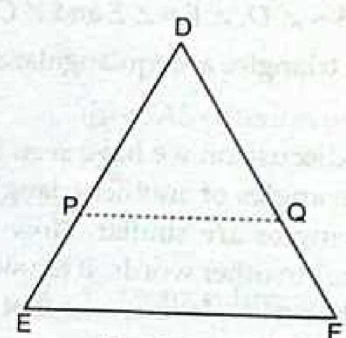


Fig. 7.82

Therefore, by AA-criterion of similarity, we have

$$\Delta DPQ \sim \Delta DEF \quad \text{...(i)}$$

$\Rightarrow \quad \frac{DP}{DE} = \frac{PQ}{EF}$ [By def. of similarity]

$\Rightarrow \quad \frac{AB}{DE} = \frac{PQ}{EF}$ [$\because DP = AB$]

But, $\quad \frac{AB}{DE} = \frac{BC}{EF}$

$\therefore \quad \frac{PQ}{EF} = \frac{BC}{EF}$

$\Rightarrow \quad PQ = BC$

Thus, in triangles ABC and DPQ, we have

$AB = DP, AC = DQ$ and $BC = PQ$

Therefore, by SSS criterion of congruence, we have

$\Delta ABC \cong \Delta DPQ$ ...(ii)

From (i) and (ii), we have

$\Delta ABC \cong \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

$\Rightarrow \quad \Delta ABC \sim \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$ [$\because \Delta ABC \cong \Delta DPQ \Leftrightarrow \Delta ABC \sim \Delta DPQ$]

$\Rightarrow \quad \Delta ABC \sim \Delta DEF$ **Q.E.D.**

In view of the above two theorems, we can also give the following definitions of the similarity of two triangles.

DEFINITION 1 *Two triangles are similar if their corresponding angles are equal i.e. they are equiangular.*

DEFINITION 2 *Two triangles are similar if their corresponding sides are proportional.*

In class IX, we have learnt about various criteria for congruency of two triangles. We observe that corresponding to SSS congruence criterion there is SSS similarity criterion. This suggests us to look for a similarity criterion corresponding to SAS congruency criterion of triangles. To check the existence of such criterion, let us perform the following activity:

ACTIVITY Draw two triangles ABC and DEF such that $AB = 6$ cm, $\angle A = 60°$, $AC = 12$ cm, $DE = 4$ cm, $\angle D = 60°$ and $DF = 8$ cm as shown in Fig. 7.83.

We observe that $\frac{AB}{DE} = \frac{AC}{DF}$ (each equal to 3/2 and $\angle A$ (included between the sides AB and AC) is equal to $\angle D$ (included between the sides DE and DF). That is, one angle of a triangle is equal to one angle of another triangle and sides including those angles are in the same ratio.

Now, measure $\angle B$, $\angle C$, $\angle E$ and $\angle F$. You will find that $\angle B = \angle E$ and $\angle C = \angle F$. So, by AAA similarity criterion, we obtain $\Delta ABC \sim \Delta DEF$.

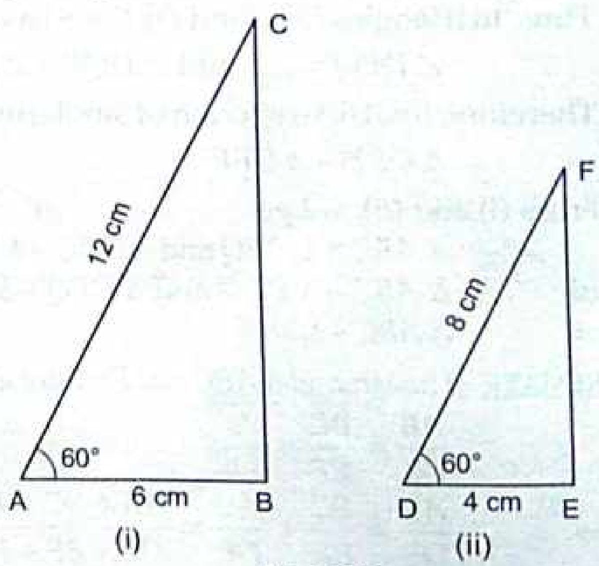


Fig. 7.83

It follows from the above activity that, if one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar. We prove the above observation as a theorem given below.

THEOREM 3 *(SAS Similarity Criterion) If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.* **[NCERT]**

GIVEN Two triangles ABC and DEF such that $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$

TO PROVE $\Delta ABC \sim \Delta DEF$

CONSTRUCTION Mark points P and Q on DE and DF respectively such that $DP = AB$ and $DQ = AC$. Join PQ.

PROOF In triangles ABC and DPQ, we have

$$AB = DP, \angle A = \angle D \text{ and } AC = DQ$$

Therefore, by SAS Criterion of Congruence, we have

$$\Delta ABC \cong \Delta DPQ \quad \ldots(i)$$

Now, $\frac{AB}{DE} = \frac{AC}{DF}$

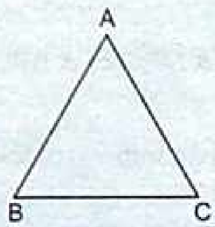


Fig. 7.84

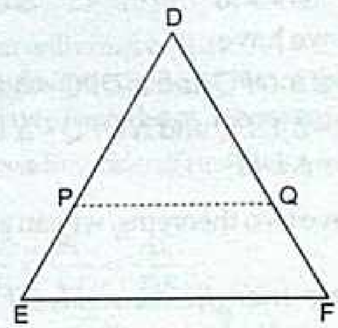


Fig. 7.85

$\Rightarrow \quad \frac{DP}{DE} = \frac{DQ}{DF}$ [$\because AB = DP$ and $AC = DQ$]

$\Rightarrow \quad PQ \parallel EF$ [By the converse of Thale's Theorem]

$\Rightarrow \quad \angle DPQ = \angle E$ and $\angle DQP = \angle F$[Corresponding angles]

Thus, in triangles DPQ and DEF, we have

$$\angle DPQ = \angle E \text{ and } \angle DQP = \angle F$$

Therefore, by AAA-criterion of similarity, we have

$$\Delta DPQ \sim \Delta DEF \quad \ldots(ii)$$

From (i) and (ii), we get

$\Delta ABC \cong \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

$\Rightarrow \quad \Delta ABC \sim \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

$\Rightarrow \quad \Delta ABC \sim \Delta DEF$

Q.E.D.

REMARK *If two triangles ABC and DEF are similar, then*

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AB + BC + AC}{DE + EF + DF}$ [Using ratio and proportion]

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta DEF}$

Thus, if two triangles are similar, then their corresponding sides are proportional and they are proportional to the corresponding perimerers.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Examine each pair of triangles in Fig. 7.86 and state which pair of triangles are similar. Also, state the similarity criterion used by you for answering the question and write the similarity relation in symbolic form.* **[NCERT]**

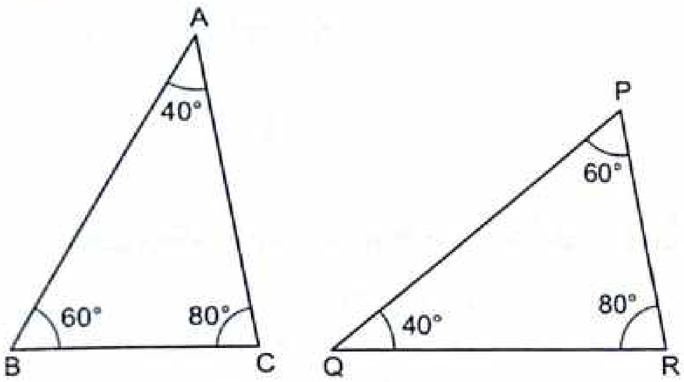


Fig. 7.86 (i)

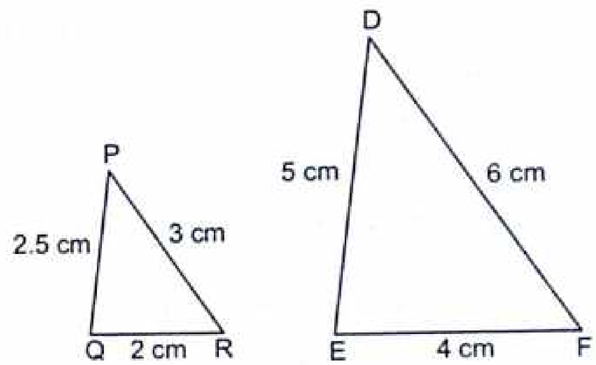


Fig. 7.86 (ii)

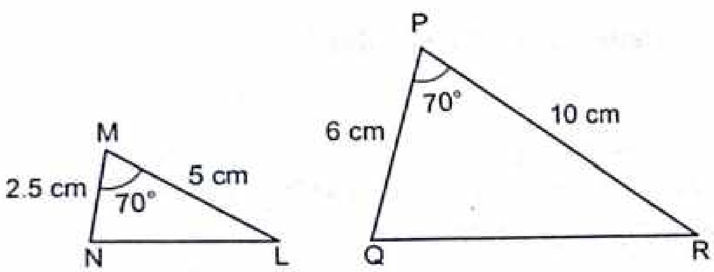


Fig. 7.86 (iii)

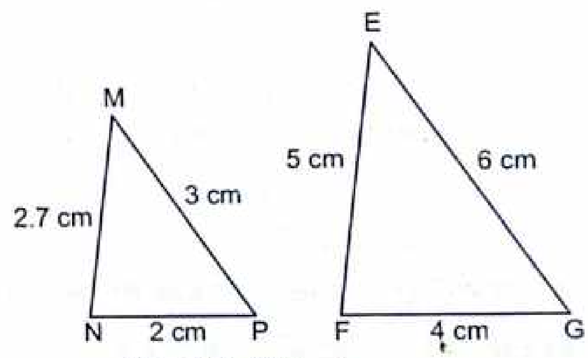


Fig. 7.86 (iv)

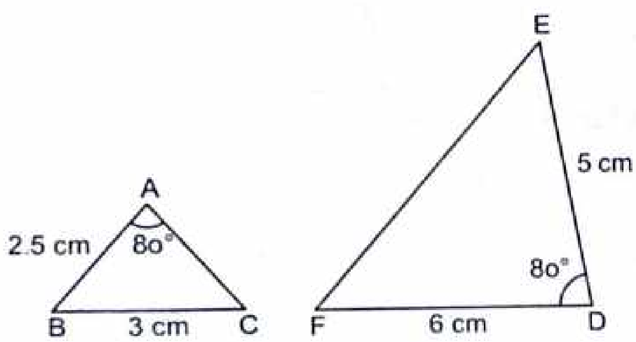


Fig. 7.86 (v)

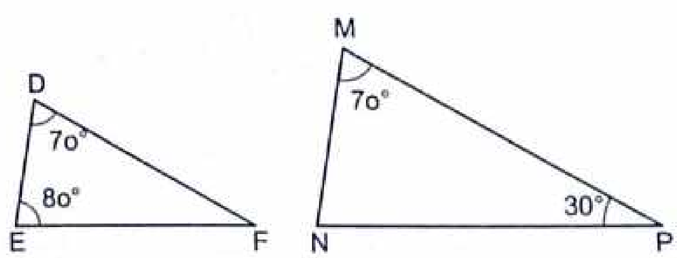


Fig. 7.86 (vi)

SOLUTION (i) In triangles ABC and PQR, we observe that

$$\angle A = \angle Q = 40^\circ, \angle B = \angle P = 60^\circ \text{ and } \angle C = \angle R = 80^\circ$$

Therefore, by AAA-criterion of similarity

$$\Delta ABC \sim \Delta QPR \text{ or, } \Delta PQR \sim \Delta BAC \text{ or, } \Delta ACB \sim \Delta QRP$$

(ii) In triangle PQR and DEF, we observe that

$$\frac{PQ}{DE} = \frac{QR}{EF} = \frac{PR}{DF} = \frac{1}{2}$$

Therefore, by *SSS*-criterion of similarity, we have

$$\Delta PQR \sim \Delta DEF$$

(iii) In triangles *LMN* and *PQR*, we have

$$\angle M = \angle P = 70^\circ$$

But, $\dfrac{MN}{PQ} \neq \dfrac{ML}{PR}$

Therefore, these two triangles are not similar as they do not satisfy SAS criterion of similarity.

(iv) In Δ's *MNP* and *EFG*, we observe that

$$\frac{NP}{FG} = \frac{MP}{EG} \neq \frac{MN}{EF}$$

Therefore, these two triangles are not similarly as they do not satisfy *SSS*-criterion of similarity.

(v) In Δ's *ABC* and *DEF*, we have

$$\angle A = \angle D = 80^\circ$$

But, $\dfrac{AB}{DE} \neq \dfrac{AC}{DF}$ [$\because$ *AC* is not given]

So, by *SAS*-criterion of similarity these two triangles are not similar.

(vi) In Δ's *DEF* and *MNP*, we have

$$\angle D = \angle M = 70^\circ$$

$$\angle E = \angle N = 80^\circ \quad [\because \angle N = 180^\circ - \angle M - \angle P = 180^\circ - 70^\circ - 30^\circ = 80^\circ]$$

So, by *AA*-criterion of similarity, we obtain $\Delta DEF \sim \Delta MNP$.

EXAMPLE 2 *In Fig. 7.87, find $\angle F$.*

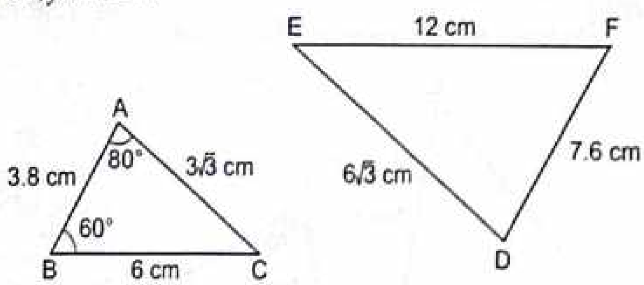


Fig. 7.87

SOLUTION In triangles *ABC* and *DEF*, we have

$$\frac{AB}{DF} = \frac{BC}{FE} = \frac{CA}{ED} = \frac{1}{2}$$

Therefore, by *SSS*-criterion of similarity, we have

$$\Delta ABC \sim \Delta DFE$$

$\Rightarrow$ $\angle A = \angle D, \angle B = \angle F$ and $\angle C = \angle E$

$\Rightarrow$ $\angle D = 80^\circ, \angle F = 60^\circ$

Hence, $\angle F = 60^\circ$.

EXAMPLE 3 *In Fig. 7.88, $\Delta ACB \sim \Delta APQ$. If $BC = 8$ cm, $PQ = 4$ cm, $BA = 6.5$ cm, $AP = 2.8$ cm, find CA and AQ.*

SOLUTION We have,

$$\Delta ACB \sim \Delta APQ$$

$$\Rightarrow \quad \frac{AC}{AP} = \frac{CB}{PQ} = \frac{AB}{AQ}$$

$$\Rightarrow \quad \frac{AC}{AP} = \frac{CB}{PQ} \text{ and } \frac{CB}{PQ} = \frac{AB}{AQ}$$

$$\Rightarrow \quad \frac{AC}{2.8} = \frac{8}{4} \text{ and } \frac{8}{4} = \frac{6.5}{AQ}$$

$$\Rightarrow \quad \frac{AC}{2.8} = 2 \text{ and } \frac{6.5}{AQ} = 2 \Rightarrow AC = (2 \times 2.8) \text{ cm} = 5.6 \text{ cm and } AQ = \frac{6.5}{2} \text{ cm} = 3.25 \text{ cm}$$

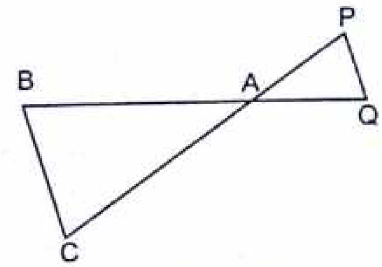


Fig. 7.88

EXAMPLE 4 *In Fig. 7.89, if $\Delta EDC \sim \Delta EBA$, $\angle BEC = 115°$ and $\angle EDC = 70°$. Find $\angle DEC$, $\angle DCE$, $\angle EAB$, $\angle AEB$ and $\angle EBA$.* **[NCERT]**

SOLUTION Since BD is a line and EC is a ray on it.

$\therefore \quad \angle DEC + \angle BEC = 180°$

$\Rightarrow \quad \angle DEC + 115° = 180°$

$\Rightarrow \quad \angle DEC = 180° - 115° = 65°$

But, $\quad \angle AEB = \angle DEC$ [Vertically opposite angles]

$\therefore \quad \angle AEB = 65°$

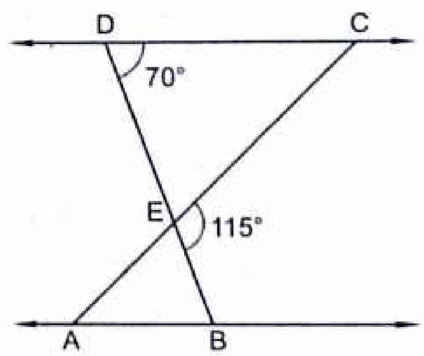


Fig. 7.89

In ΔCDE, we have

$\quad \angle CDE + \angle DEC + \angle DCE = 180°$

$\Rightarrow \quad 70° + 65° + \angle DCE = 180°$

$\Rightarrow \quad \angle DCE = 180° - 135° = 45°$

It is given that $\Delta EDC \sim \Delta EBA$

$\therefore \quad \angle EBA = \angle EDC, \angle EAB = \angle ECD$

$\Rightarrow \quad \angle EBA = 70°$ and $\angle EAB = 45°$ $\quad [\because \angle ECD = \angle DCE = 45°]$

Hence, $\angle DEC = 65°$, $\angle DCE = 45°$, $\angle EAB = 45°$, $\angle AEB = 65°$ and $\angle EBA = 70°$.

EXAMPLE 5 *In Fig. 7.90, if $\Delta POS \sim \Delta ROQ$, prove that $PS \parallel QR$.* **[NCERT]**

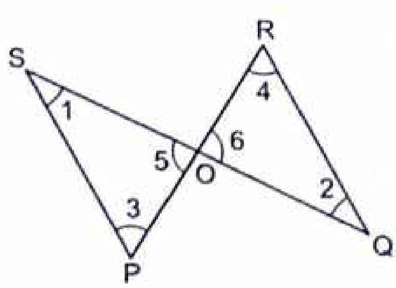


Fig. 7.90

SOLUTION We have,

$\Delta POS \sim \Delta ROQ$

$\Rightarrow \quad \angle 3 = \angle 4$ and $\angle 1 = \angle 2$

Thus, PS and QR are two lines and the transversal PR cuts them in such a way that $\angle 3 = \angle 4$ i.e., alternate angles are equal. Hence, $PS \parallel QR$.

EXAMPLE 6 *In Fig. 7.90, if $PS \parallel QR$, prove that $\Delta POS \sim \Delta ROQ$.*

SOLUTION It is given that $PS \parallel QR$ and transversal PR cuts them at P and R.

$\therefore \quad \angle 3 = \angle 4$

Again, $PS \parallel QR$ and transversal SQ cuts them at S and Q

$\therefore \quad \angle 1 = \angle 2$

Also, $\quad \angle 5 = \angle 6$ [Vertically opposite angles]

Thus, in Δ POS and QOR, we have

$\angle 1 = \angle 2$ i. e., $\angle S = \angle Q$

$\angle 3 = \angle 4$ i.e., $\angle P = \angle R$

and, $\quad \angle 5 = \angle 6$ i.e., $\angle POS = \angle QOR$

Therefore, by AAA-criterion of similarity, we obtain $\Delta POS \sim \Delta ROQ$.

EXAMPLE 7 *In Fig. 7.91, QA and PB are perpendiculars to AB. If AO = 10 cm, BO = 6 cm and PB = 9 cm. Find AQ.*

SOLUTION In triangles AOQ and BOP, we have

$\angle OAQ = \angle OBP$ [Each equal to 90°]

$\angle AOQ = \angle BOP$ [Vertically opposite angles]

Therefore, by AA- criterion of similarity, we obtain

$\Delta AOQ \sim \Delta BOP$

$$\Rightarrow \quad \frac{AO}{BO} = \frac{OQ}{OP} = \frac{AQ}{BP}$$

$$\Rightarrow \quad \frac{AO}{BO} = \frac{AQ}{BP}$$

$$\Rightarrow \quad \frac{10}{6} = \frac{AQ}{9}$$

$$\Rightarrow \quad AQ = \frac{10 \times 9}{6} = 15 \text{ cm}$$

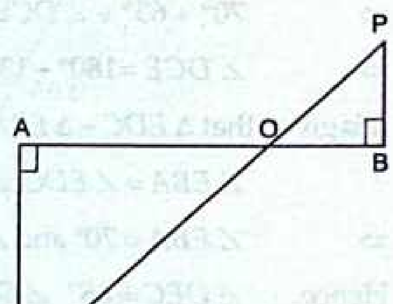


Fig. 7.91

EXAMPLE 8 *In Fig. 7.92, if $\angle ADE = \angle B$ show that $\Delta ADE \sim \Delta ABC$. If AD = 3.8 cm, AE = 3.6 cm, BE = 2.1 cm and BC = 4.2 cm, find DE.*

SOLUTION In triangles ADE and ABC, we have

$\angle ADE = \angle B$ (Given) and $\angle A = \angle A$ (Common)

So, by AA-criterion of similarity, we have

$$\Delta ADE \sim \Delta ABC$$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$$

$$\Rightarrow \quad \frac{AD}{AE + EB} = \frac{DE}{BC}$$

$$\Rightarrow \quad \frac{3.8}{3.6 + 2.1} = \frac{DE}{4.2}$$

$$\Rightarrow \quad DE = \frac{3.8 \times 4.2}{3.6 + 2.1} \text{ cm} = 2.8 \text{ cm}$$

Hence, $DE = 2.8$ cm

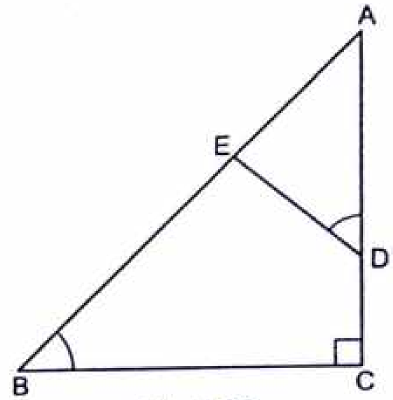


Fig. 7.92

EXAMPLE 9 *In Fig. 7.93, $\frac{AO}{OC} = \frac{BO}{OD} = \frac{1}{2}$ and $AB = 5$ cm. Find the value of DC.*

SOLUTION In $\Delta\, AOB$ and $\Delta\, COD$, we have

$\angle AOB = \angle COD$ [Vertically opposite angles]

$\frac{AO}{OC} = \frac{OB}{OD}$ [Given]

So, by SAS-criterion of similarity, we have

$$\Delta AOB \sim \Delta COD$$

$$\Rightarrow \quad \frac{AO}{OC} = \frac{BO}{OD} = \frac{AB}{DC}$$

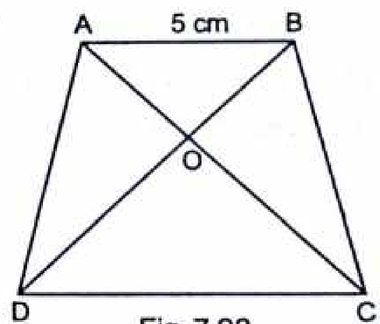


Fig. 7.93

$\Rightarrow \quad \frac{1}{2} = \frac{5}{DC}$ [$\because AB = 5$ cm]

$\Rightarrow \quad DC = 10$ cm

EXAMPLE 10 *In Fig. 7.94, if $\angle A = \angle C$, then prove that $\Delta\, AOB \sim \Delta\, COD$.*

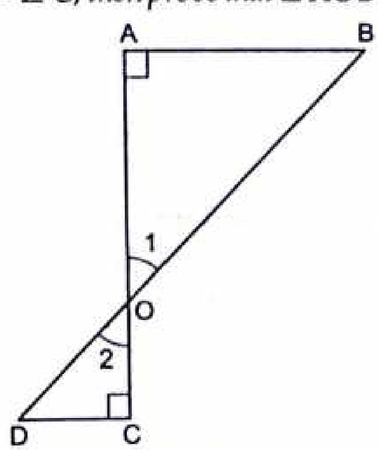


Fig. 7.94

SOLUTION In triangles AOB and COD, we obtain

$\angle A = \angle C$ (Given)

and, $\angle 1 = \angle 2$ [Vertically opposite angles]

Therefore, by AA-criterion of similarity, we obtain

$$\Delta AOB \sim \Delta COD$$

EXAMPLE 11 *In Fig. 7.95, if $AB \perp BC$ and $DE \perp AC$. Prove that $\Delta ABC \sim \Delta AED$.* **[CBSE 2009]**

SOLUTION In Δ's ABC and AED, we have

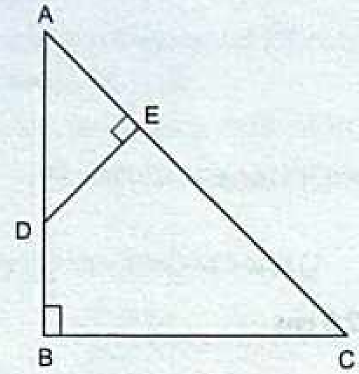


Fig. 7.95

$\angle ABC = \angle AED = 90°$

$\angle BAC = \angle EAD$ [Each equal to $\angle A$]

Therefore, by AA-criterion of similarity, we obtain $\Delta ABC \sim \Delta AED$.

EXAMPLE 12 *In Fig. 7.96, if $\angle P = \angle RTS$, prove that $\Delta RPQ \sim \Delta RTS$.*

SOLUTION In triangles RPQ and RTS, we have

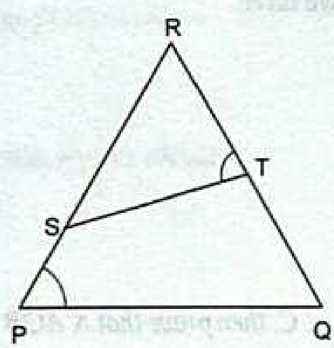


Fig. 7.96

$\angle RPQ = \angle RTS$ [Given]

$\angle PRQ = \angle TRS$ [Each equal to R]

Therefore, by AA-criterion of similarity, we obtain $\Delta RPQ \sim \Delta RTS$.

EXAMPLE 13 *In Fig. 7.97, if $\dfrac{QT}{PR} = \dfrac{QR}{QS}$ and $\angle 1 = \angle 2$. Prove that $\Delta PQS \sim \Delta TQR$.* **[NCERT]**

SOLUTION We have,

$$\frac{QT}{PR} = \frac{QR}{QS} \qquad \text{[Given]}$$

$$\Rightarrow \quad \frac{QT}{QR} = \frac{PR}{QS} \qquad \ldots(i)$$

We also have,

$\angle 1 = \angle 2$ [Given]

$\Rightarrow \quad PR = PQ$ $\left[\begin{array}{l}\text{Sides opposite to equal}\\ \text{angles are equal}\end{array}\right]$ ...(ii)

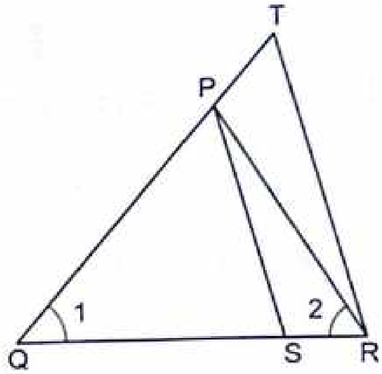


Fig. 7.97

From (i) and (ii), we get

$$\frac{QT}{QR} = \frac{PQ}{QS}$$

$$\Rightarrow \quad \frac{PQ}{QT} = \frac{QS}{QR} \quad \text{...(iii)}$$

Thus, in triangles PQS and TQR, we have

$$\frac{PQ}{QT} = \frac{QS}{QR} \text{ and } \angle PQS = \angle TQR = \angle Q$$

So, by SAS-criterion of similarity, we obtain $\Delta PQS \sim \Delta TQR$.

EXAMPLE 14 *In Fig. 7.98, AD and CE are two altitudes of ΔABC. Prove that*

(i) $\Delta AEF \sim \Delta CDF$ (ii) $\Delta ABD \sim \Delta CBE$

(iii) $\Delta AEF \sim \Delta ADB$ (iv) $\Delta FDC \sim \Delta BEC$ **[NCERT]**

SOLUTION (i) In triangles AEF and CDF, we have

$\angle AEF = \angle CDF = 90^\circ$ [$\because CE \perp AB$ and $AD \perp BC$]

$\angle AFE = \angle CFD$ [Vertically opposite angles]

Thus, by AA-criterion of similarity, we have

$\Delta AEF \sim \Delta CDF$

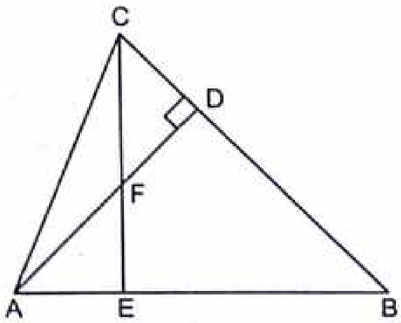


Fig. 7.98

(ii) In Δ's ABD and CBE, we have

$\angle ABD = \angle CBE = \angle B$ [Common angle]

$\angle ADB = \angle CEB = 90^\circ$ [$\because AD \perp BC$ and $CE \perp AB$]

Thus, by AA-criterion of similarity, we have

$\Delta ABD \sim \Delta CBE$

(iii) In Δ's AEF and ADB, we have

$\angle AEF = \angle ADB = 90^\circ$ [$\because AD \perp BC$ and $CE \perp AB$]

$\angle FAE = \angle DAB$ [Common angle]

Thus, by AA-criterion of similarity, we have

$\Delta AEF \sim \Delta ADB$

(iv) In Δ's FDC and BEC, we have

$\angle FDC = \angle BEC = 90^\circ$ [$\because AD \perp BC$ and $CE \perp AB$]

$\angle FCD = \angle ECB$ [Common angle]

Thus, by *AA*-criterion of similarity, we obtain $\Delta FDC \sim \Delta BEC$.

EXAMPLE 15 *In Fig. 7.99(i) and (ii), if CD and GH (D and H lie on AB and FE) are respectively bisectors of $\angle ACB$ and $\angle EGF$ and $\Delta ABC \sim \Delta FEG$, prove that* **[NCERT]**

(i) $\Delta DCA \sim \Delta HGF$ (ii) $\dfrac{CD}{GH} = \dfrac{AC}{FG}$ (iii) $\Delta DCB \sim \Delta HGE$

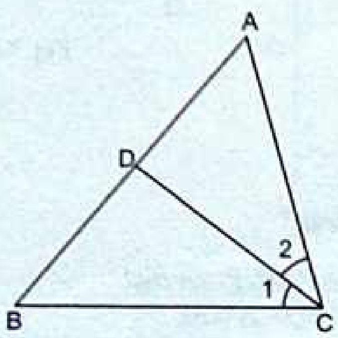


Fig. 7.99 (i)

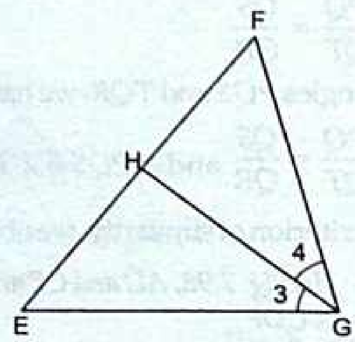


Fig. 7.99 (ii)

SOLUTION (i) We have,

$\Delta ABC \sim \Delta FEG$

$\Rightarrow \quad \angle A = \angle F$ ...(i)

and, $\quad \angle C = \angle G$

$\Rightarrow \quad \dfrac{1}{2}\angle C = \dfrac{1}{2}\angle G$

$\Rightarrow \quad \angle 1 = \angle 3$ and $\angle 2 = \angle 4$ $\left[\begin{array}{l}\therefore \text{ CD and GH are bisector of} \\ \angle C \text{ and } \angle G \text{ respectively}\end{array}\right]$ ...(ii)

Thus, in Δ's *ACD* and *FGH*, we have

$\angle A = \angle F$ [From (i)]

$\angle 2 = \angle 4$ [From (ii)]

Therefore, by *AA*-criterion of similarity, we obtain

$\Delta ACD \sim \Delta FGH$ or, $\Delta DCA \sim \Delta HGF$

(ii) We have,

$\Delta ACD \sim \Delta FGH \Rightarrow \dfrac{AC}{FG} = \dfrac{CD}{GH}$

(iii) In Δ's *DCB* and *HGE*, we have

$\angle 1 = \angle 3$ [From (ii)]

$\angle B = \angle E$ $[\because \Delta ABC \sim \Delta FEG]$

Thus, by *AA*-criterion of similarity, we obtain $\Delta DCB \sim \Delta HGE$.

EXAMPLE 16 *In Fig. 7.100, CD and GH are respectively the medians of ΔABC and ΔEFG. If $\Delta ABC \sim \Delta FEG$, prove that*

(i) $\Delta ADC \sim \Delta FHG$ (ii) $\dfrac{CD}{GH} = \dfrac{AB}{FE}$ (iii) $\Delta CDB \sim \Delta GHE$

[NCERT]

SOLUTION It is given that CD and GD are medians of Δ's ABC and EFG respectively.

$\therefore \quad 2AD = AB$ and $2FH = FE$ ...(i)

It is also given that $\Delta ABC \sim \Delta FEG$

$\therefore \quad \dfrac{AB}{FE} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ and, $\angle A = \angle F, \angle B = \angle E, \angle C = \angle G$ ...(ii)

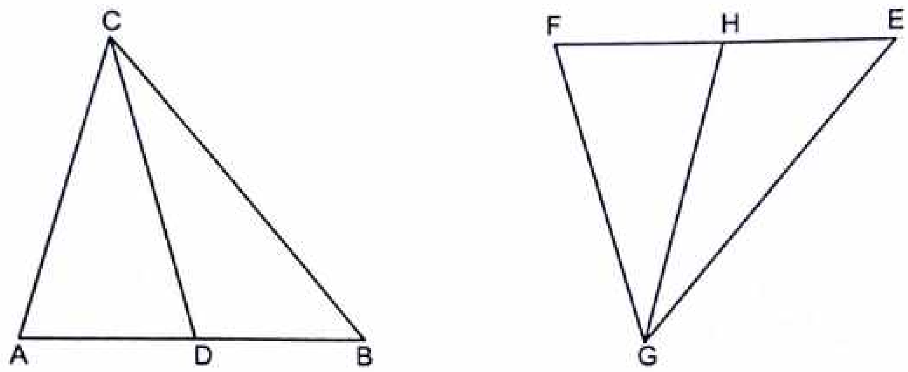


Fig. 7.100

Now, $\dfrac{AB}{FE} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$

$\Rightarrow \quad \dfrac{2AD}{2FH} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ [Using (i)]

$\Rightarrow \quad \dfrac{AD}{FH} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ ...(iii)

(i) In Δ's ADC and FHG, we have

$\dfrac{AD}{FH} = \dfrac{AC}{FG}$ [From (iii)]

and, $\angle A = \angle F$

So, by SAS criterion of similarity, we obtain $\Delta ADC \sim \Delta FHG$.

(ii) We have,

$\Delta ADC \sim \Delta FHG$ [Proved above]

$\Rightarrow \quad \dfrac{DC}{HG} = \dfrac{AD}{FH}$

$\Rightarrow \quad \dfrac{CD}{GH} = \dfrac{2AD}{2FH}$

$\Rightarrow \quad \dfrac{CD}{GH} = \dfrac{AB}{FE}$ [$\because AB = 2\,AD$ and $FE = 2\,FH$]

(iii) We have,

$\dfrac{AB}{FE} = \dfrac{AC}{FG} = \dfrac{BC}{EG}$ [From (i)]

Also, $\dfrac{CD}{GH} = \dfrac{AB}{FE}$ [As proved above]

$\therefore \quad \dfrac{CD}{GH} = \dfrac{BC}{EG}$ ...(iv)

Again, $\frac{AB}{FE} = \frac{AC}{FG} = \frac{BC}{EG}$

$\Rightarrow \quad \frac{2DB}{2HE} = \frac{BC}{EG}$ [$\because$ D and H are mid-points of AB and FE respectively]

$\Rightarrow \quad \frac{DB}{HE} = \frac{BC}{EG}$ ...(v)

From (iv) and (v), we have

$$\frac{CD}{GH} = \frac{BC}{EG} = \frac{DB}{HE}$$

$\Rightarrow \quad \frac{CD}{GH} = \frac{DB}{HE} = \frac{BC}{EG}$

$\Rightarrow \quad \Delta CDB \sim \Delta GHE$ [By SSS criterion of similarity]

EXAMPLE 17 *In Fig. 7.101, if $BD \perp AC$ and $CE \perp AB$, prove that*

(i) $\Delta AEC \sim \Delta ADB$ (ii) $\frac{CA}{AB} = \frac{CE}{DB}$ **[NCERT]**

SOLUTION (i) In Δ's AEC and ADB, we have

$\angle AEC = \angle ADB = 90°$ [$\because CE \perp AB$ and $BD \perp AC$]

and, $\angle EAC = \angle DAB$ [Each equal to $\angle A$]

Therefore, by AA-criterion of similarity, we obtain

$$\Delta AEC \sim \Delta ADB$$

(ii) We have,

$\Delta AEC \sim \Delta ADB$ [As proved above]

$\Rightarrow \quad \frac{CA}{BA} = \frac{EC}{DB}$

$\Rightarrow \quad \frac{CA}{AB} = \frac{CE}{DB}$

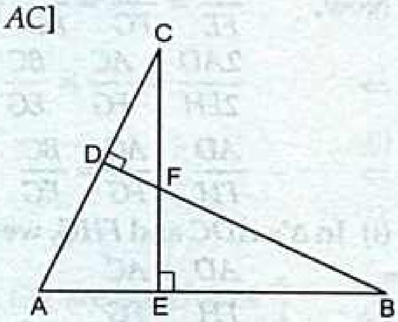


Fig. 7.101

EXAMPLE 18 *D is a point on the side BC of ΔABC such that $\angle ADC = \angle BAC$. Prove that $\frac{CA}{CD} = \frac{CB}{CA}$ or, $CA^2 = CB \times CD$.* **[NCERT, CBSE 2004]**

SOLUTION In ΔABC and ΔDAC, we have

$\angle ADC = \angle BAC$ and $\angle C = \angle C$

Therefore, by AA-criterion of similarity, we obtain

$$\Delta ABC \sim \Delta DAC$$

$\Rightarrow \quad \frac{AB}{DA} = \frac{BC}{AC} = \frac{AC}{DC}$

$\Rightarrow \quad \frac{CB}{CA} = \frac{CA}{CD}$

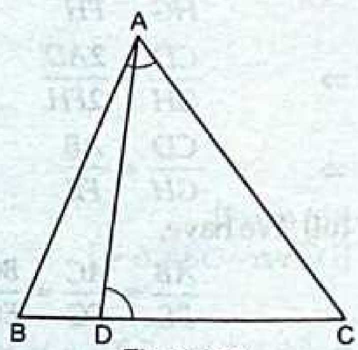


Fig. 7.102

EXAMPLE 19 *In Fig. 7.103, considering triangles BEP and CPD, prove that $BP \times PD = EP \times PC$.*

GIVEN A ΔABC in which $BD \perp AC$ and $CE \perp AB$ and BD and CE intersect at P.

TO PROVE $BP \times PD = EP \times PC$

PROOF In $\Delta\, EPB$ and $\Delta\, DPC$, we have

$\angle PEB = \angle PDC$ [Each equal to 90°]

$\angle EPB = \angle DPC$ [Vertically opposite angles]

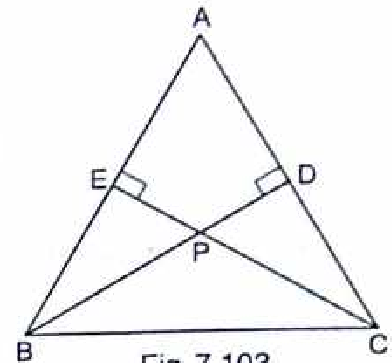


Fig. 7.103

Thus, by *AA*-criterion of similarity, we obtain

$$\Delta EPB \sim \Delta DPC$$

$$\frac{EP}{DP} = \frac{PB}{PC}$$

$\Rightarrow$ $BP \times PD = EP \times PC$

EXAMPLE 20 *P and Q are points on sides AB and AC respectively of* $\Delta\, ABC$. *If* $AP = 3$ *cm*, $PB = 6$ *cm*, $AQ = 5$ *cm and* $QC = 10$ *cm, show that* $BC = 3\,PQ$.

SOLUTION We have,

$AB = AP + PB = (3 + 6)$ cm $= 9$ cm and, $AC = AQ + QC = (5 + 10)$ cm $= 15$ cm.

$\therefore$ $\frac{AP}{AB} = \frac{3}{9} = \frac{1}{3}$ and $\frac{AQ}{AC} = \frac{5}{15} = \frac{1}{3}$

$\Rightarrow$ $\frac{AP}{AB} = \frac{AQ}{AC}$

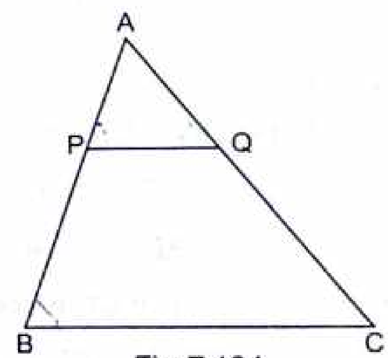


Fig. 7.104

Thus, in triangles APQ and ABC, we have

$\frac{AP}{AB} = \frac{AQ}{AC}$ and $\angle A = \angle A$ [Common]

Therefore, by *SAS*-criterion of similarity, we have

$\Delta APQ \sim \Delta ABC$

$\Rightarrow$ $\frac{AP}{AB} = \frac{PQ}{BC} = \frac{AQ}{AC}$

$\Rightarrow$ $\frac{PQ}{BC} = \frac{AQ}{AC}$

$\Rightarrow$ $\frac{PQ}{BC} = \frac{5}{15} \Rightarrow \frac{PQ}{BC} = \frac{1}{3} \Rightarrow BC = 3PQ$

EXAMPLE 21 *In Fig. 7.105, express x in terms of a, b and c.*

SOLUTION In $\Delta\, KPN$ and $\Delta\, KLM$, we have

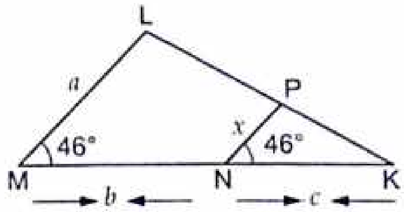


Fig. 7.105

$\angle KNP = \angle KML = 46°$ [Given]

$\angle K = \angle K$ [Common]

$\therefore$ $\Delta\, KNP \sim \Delta\, KML$ [By *AA*-criterion of similarity]

$\Rightarrow \quad \dfrac{KN}{KM} = \dfrac{NP}{ML}$ [$\because$ Corresponding sides of similar triangles are proportional]

$\Rightarrow \quad \dfrac{c}{b+c} = \dfrac{x}{a} \Rightarrow x = \dfrac{ac}{b+c}$

EXAMPLE 22 *The diagonal BD of a parallelogram ABCD intersects the segment AE at the point F, where E is any point on the side BC. Prove that DF × EF = FB × FA*

SOLUTION In ΔAFD and ΔBFE, we have

$\angle 1 = \angle 2$ [Vertically opposite angles]

$\angle 3 = \angle 4$ [Alternate angles]

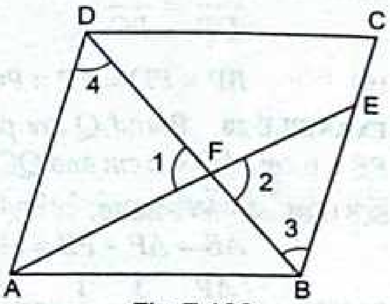


Fig. 7.106

So, by *AA*-criterion of similarity, we have

$\Delta FBE \sim \Delta FDA$

$\Rightarrow \quad \dfrac{FB}{FD} = \dfrac{FE}{FA}$

$\Rightarrow \quad \dfrac{FB}{DF} = \dfrac{EF}{FA}$

$\Rightarrow \quad DF \times EF = FB \times FA$

EXAMPLE 23 *In a ΔABC, BD and CE are the altitudes. Prove that ΔADB and ΔAEC are similar. Is $\Delta CDB \sim \Delta BEC$?*

SOLUTION In ΔABD and ΔAEC, we obtain

$\angle ADB = \angle AEC$ [Each equal to 90°]

$\angle BAD = \angle EAC$ [Common]

So, by *AA*-criterion of similarity, we have

$\Delta BDA \sim \Delta CEA$ or, $\Delta ADB \sim \Delta AEC$.

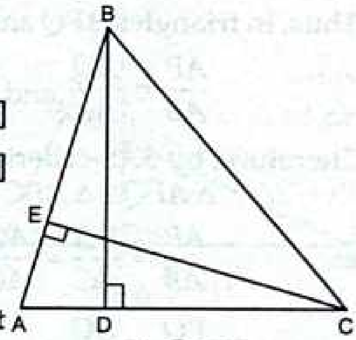


Fig. 7.107

Clearly, ΔCDB is not similar to ΔBEC, because they are not equiangular.

EXAMPLE 24 *E is a point on side AD produced of a parallelogram ABCD and BE intersects CD at F. Prove that $\Delta ABE \sim \Delta CFB$.* **[NCERT, CBSE 2008]**

SOLUTION In Δ's *ABE* and *CFB*, we have

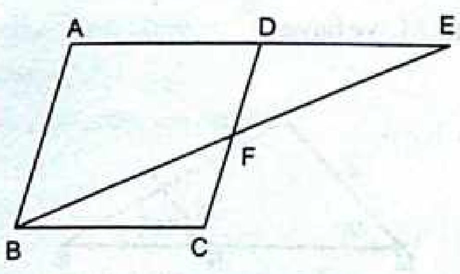


Fig. 7.108

$\angle AEB = \angle CBF$ [Alternate angles]

$\angle A = \angle C$ [Opposite angles of a parallelogram]

Thus, by *AA*-criterion of similarity, we have

$\Delta ABE \sim \Delta CFB$.

EXAMPLE 25 *In Fig. 7.109, AD and BE are respectively perpendiculars to BC and AC. Show that*

(i) $\Delta ADC \sim \Delta BEC$ (ii) $CA \times CE = CB \times CD$

(iii) $\Delta ABC \sim \Delta DEC$ (iv) $CD \times AB = CA \times DE$

SOLUTION (i) in Δ's ADC and BEC, we have

$\angle ADC = \angle BEC = 90°$ [Given]

$\angle ACD = \angle BCE$ [Common]

So, by AA-criterion of similarity, we obtain

$\Delta ADC \sim \Delta BEC$

(ii) We have,

$\Delta ADC \sim \Delta BEC$ [As proved above]

$\Rightarrow \quad \dfrac{AC}{BC} = \dfrac{DC}{EC}$ ...(i)

$\Rightarrow \quad CA \times CE = CB \times CD$

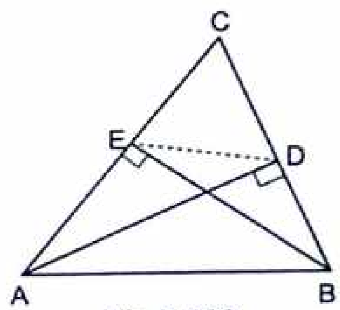


Fig. 7.109

(iii) In Δ's ABC and DEC, we have

$\dfrac{AC}{BC} = \dfrac{DC}{EC}$ [From(i)]

$\Rightarrow \quad \dfrac{AC}{DC} = \dfrac{BC}{EC}$

Also, $\angle ACB = \angle DCE$

So, by SAS-criterion of similarity, we obtain

$\Delta ABC \sim \Delta DEC$ [Common]

(iv) We have,

$\Delta ABC \sim \Delta DEC$ [As proved above]

$\Rightarrow \quad \dfrac{AB}{DE} = \dfrac{AC}{DC} \Rightarrow AB \times DC = AC \times DE \Rightarrow CD \times AB = CA \times DE$

EXAMPLE 26 *In Fig. 7.110, E is a point on side CB produced of an isosceles triangle ABC with $AB = AC$. If $AD \perp BC$ and $EF \perp AC$, prove that (i) $\Delta ABD \sim \Delta ECF$ (ii) $AB \times EF = AD \times EC$.*

[NCERT, CBSE 2010]

SOLUTION It is given that ΔABC is isosceles with

$AB = AC$

$\therefore \quad \angle B = \angle C$

Now, in Δ's ABD and ECF, we have

$\angle ABD = \angle ECF$ [$\because \angle B = \angle C$]

$\angle ADB = \angle EFC = 90°$ [$\because AD \perp BC$ and $EF \perp AC$]

So, by AA-criterion of similarity, we have

$\Delta ABD \sim \Delta ECF$

$\Rightarrow \quad \dfrac{AB}{EC} = \dfrac{AD}{EF}$

$\Rightarrow \quad AB \times EF = AD \times EC$

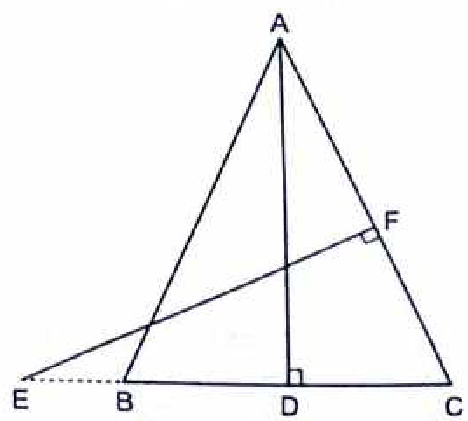


Fig. 7.110

EXAMPLE 27 *In Fig. 7.111, $\Delta FEC \cong \Delta GBD$ and $\angle 1 = \angle 2$. Prove that $\Delta ADE \sim \Delta ABC$.*

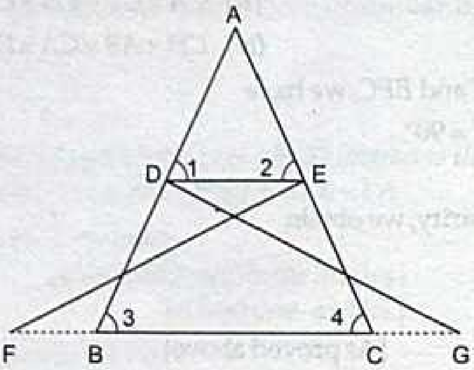


Fig. 7.111

SOLUTION We have,

$$\Delta FEC \cong \Delta GBD$$

$\Rightarrow \quad EC = BD$ ...(i)

It is given that

$$\angle 1 = \angle 2$$

$\Rightarrow \quad AD = AE$ [Sides opposite to equal angles are equal] ...(ii)

From (i) and (ii), we have

$$\frac{AE}{EC} = \frac{AD}{BD}$$

$\Rightarrow \quad DE \parallel BC$ [By the converse of basic proportionality theorem]

$\Rightarrow \quad \angle 1 = \angle 3$ and $\angle 2 = \angle 4$

Thus, in Δ's ADE and ABC, we have

$$\angle A = \angle A$$
$$\angle 1 = \angle 3$$
$$\angle 2 = \angle 4$$

So, by *AAA*-criterion of similarity, we have

$$\Delta ADE \sim \Delta ABC$$

EXAMPLE 28 *In Fig. 7.112, $\frac{OA}{OC} = \frac{OD}{OB}$. Prove that $\angle A = \angle C$ and $\angle B = \angle D$.* **[NCERT]**

SOLUTION in Δ's AOD and COB, we have

$$\frac{OA}{OC} = \frac{OD}{OB}$$

$\Rightarrow \quad \frac{OA}{OD} = \frac{OC}{OB}$

Also, $\angle 1 = \angle 2$ [Vertically opposite angles]

So, by *SAS*-criterion of similarity, we obtain

$$\Delta AOD \sim \Delta COB$$

$\Rightarrow \quad \angle A = \angle C$ and $\angle B = \angle D$

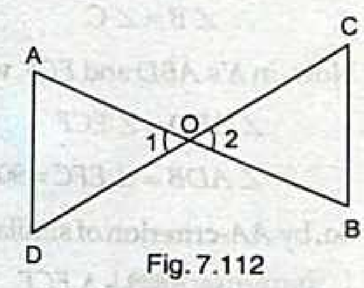


Fig. 7.112

EXAMPLE 29 *If a perpendicular is drawn from the vertex containing the right angle of a right triangle to the hypotenuse then prove that the triangle on each side of the perpendicular are similar to each other and to the original triangle. Also, prove that the square of the perpendicular is equal to the product of the lengths of the two parts of the hypotenuse.*

GIVEN A right triangle ABC right angled at B. $BD \perp AC$.

TO PROVE (i) $\Delta ADB \sim \Delta BDC$ (ii) $\Delta ADB \sim \Delta ABC$ (iii) $\Delta BDC \sim \Delta ABC$

(iv) $BD^2 = AD \times DC$ (v) $AB^2 = AD \times AC$ (vi) $BC^2 = CD \times AC$

[CBSE 2009]

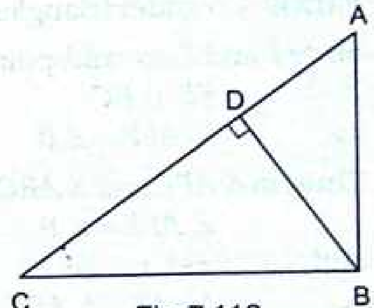


Fig. 7.113

PROOF (i) We have,

$\angle ABD + \angle DBC = 90°$

Also, $\angle C + \angle DBC + \angle BDC = 180°$

$\Rightarrow$ $\angle C + \angle DBC + 90° = 180°$

$\Rightarrow$ $\angle C + \angle DBC = 90°$

But, $\angle ABD + \angle DBC = 90°$

$\therefore$ $\angle ABD + \angle DBC = \angle C + \angle DBC$

$\Rightarrow$ $\angle ABD = \angle C$ ...(i)

Thus, in ΔADB and ΔBDC, we have

$\angle ABD = \angle C$ [From (i)]

and, $\angle ADB = \angle BDC$ [Each equal to 90°]

So, by AA-similarity criterion, we obtain $\Delta ADB \sim \Delta BDC$.

(ii) In ΔADB and ΔABC, we have

$\angle ADB = \angle ABC$ [Each equal to 90°]

and, $\angle A = \angle A$ [Common]

So, by AA-similarity criterion, we obtain $\Delta ADB \sim \Delta ABC$.

(iii) In ΔBDC and ΔABC, we have

$\angle BDC = \angle ABC$ [Each equal to 90°]

$\angle C = \angle C$ [Common]

So, by AA-similarity criterion, we obtain $\Delta BDC \sim \Delta ABC$.

(iv) From (i), we have

$\Delta ADB \sim \Delta BDC$

$$\Rightarrow \quad \frac{AD}{BD} = \frac{BD}{DC}$$

$$\Rightarrow \quad BD^2 = AD \times DC$$

(v) From (ii), we have

$\Delta ADB \sim \Delta ABC$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{AB}{AC}$$

$$\Rightarrow \quad AB^2 = AD \times AC$$

(vi) From (iii), we have

$\Delta BDC \sim \Delta ABC$

$$\Rightarrow \quad \frac{BC}{AC} = \frac{DC}{BC} \Rightarrow BC^2 = CD \times AC$$

EXAMPLE 30 *Prove that the line segments joining the mid-points of the sides of a triangle form four triangles, each of which is similar to the original triangle.*

SOLUTION

GIVEN ΔABC in which D, E, F are the mid-points of sides BC, CA and AB respectively.

TO PROVE Each of the triangles AFE, FBD, EDC and DEF is similar to ΔABC.

PROOF Consider triangles AFE and ABC.

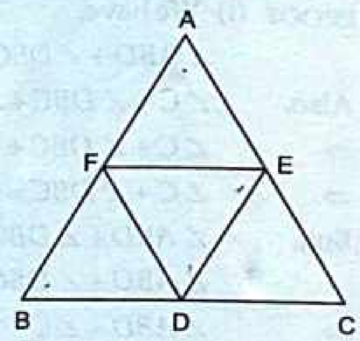


Fig. 7.114

Since F and E are mid-points of AB and AC respectively.

$\therefore \quad FE \parallel BC$

$\Rightarrow \quad \angle AFE = \angle B$ [Corresponding angles]

Thus, in ΔAFE and ΔABC, we have

$\angle AFE = \angle B$

and, $\angle A = \angle A$ [Common]

$\therefore \quad \Delta AFE \sim \Delta ABC.$

Similarly, we have

$\Delta FBD \sim \Delta ABC$ and $\Delta EDC \sim \Delta ABC$.

Now, we shall show that $\Delta DEF \sim \Delta ABC$.

Clearly, $ED \parallel AF$ and $DF \parallel EA$.

$\therefore \quad AFDE$ is a parallelogram.

$\Rightarrow \quad \angle EDF = \angle A$ [$\because$ Opposite angles of a parallelogram are equal]

Similarly, $BDEF$ is a parallelogram.

$\therefore \quad \angle DEF = \angle B$ [$\because$ Opposite angles of a parallelogram are equal]

Thus, in triangles DEF and ABC, we have

$\angle EDF = \angle A$ and $\angle DEF = \angle B$

So, by AA-criterion of similarity, we have

$\Delta DEF \sim \Delta ABC.$

Thus, each one of the triangles AFE, FBD, EDC and DEF is similar to ΔABC.

EXAMPLE 31 *In ΔABC, DE is parallel to base BC, with D on AB and E on AC. If $\frac{AD}{DB} = \frac{2}{3}$, find $\frac{BC}{DE}$.*

[CBSE 2002 C]

SOLUTION In ΔABC, we have

$DE \parallel BC$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE}$$

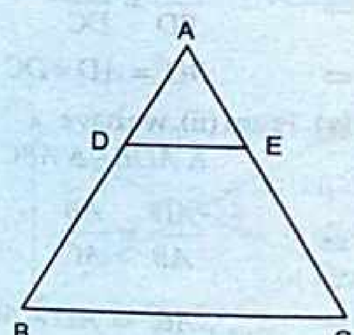


Fig. 7.115

Thus, in triangles ABC and ADE, we have

$$\frac{AB}{AD} = \frac{AC}{AE}$$

and, $\angle A = \angle A$

Therefore, by SAS-criterion of similarity, we have

$\Delta ABC \sim \Delta ADE$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{BC}{DE} \quad \ldots(i)$$

It is given that

$$\frac{AD}{DB} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{DB}{AD} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{DB}{AD} + 1 = \frac{3}{2} + 1 \Rightarrow \frac{DB + AD}{AD} = \frac{5}{2} \Rightarrow \frac{AB}{AD} = \frac{5}{2} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get $\frac{BC}{DE} = \frac{5}{2}$.

EXAMPLE 32 *In Fig. 7.116, if $\Delta ABE \cong \Delta ACD$, prove that $\Delta ADE \sim \Delta ABC$.* **[NCERT]**

SOLUTION It is given that $\Delta ABE \cong \Delta ACD$.

$\therefore \quad AB = AC$ [$\because$ Corresponding parts of congruent triangles are equal]

and, $\quad AE = AD$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE}$$

$$\Rightarrow \quad \frac{AB}{AC} = \frac{AD}{AE} \qquad \ldots\text{(i)}$$

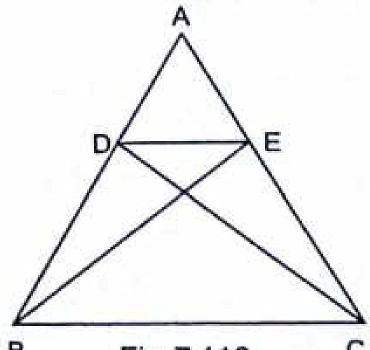


Fig. 7.116

Thus, in triangles ADE and ABC, we obtain

$$\frac{AB}{AC} = \frac{AD}{AE}$$

and, $\quad \angle BAC = \angle DAE$ [Common]

Hence, by SAS-criterion of similarity, we obtain $\Delta ADE \sim \Delta ABC$.

LEVEL-2

EXAMPLE 33 *A vertical stick 12 m long casts a shadow 8 m long on the ground. At the same time a tower casts the shadow 40 m long on the ground. Determine the height of the tower.*

SOLUTION Let AB be the vertical stick and AC be its shadow. Also, let DE be the vertical tower and DF be its shadow. Join BC and EF. Let $DE = x$ metres.

We have,

$$AB = 12 \text{ m}, AC = 8 \text{ m, and } DF = 40 \text{ m}.$$

In $\Delta\, ABC$ and $\Delta\, DEF$, we have

$\angle A = \angle D = 90°$ and $\angle C = \angle F$ [Angular elevation of the sum]

Therefore, by AA- criterion of similarity, we obtain $\Delta\, ABC \sim \Delta\, DEF$

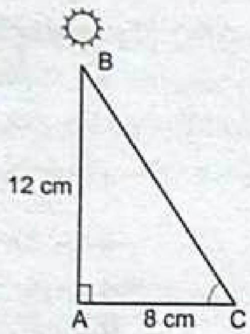


Fig. 7.117

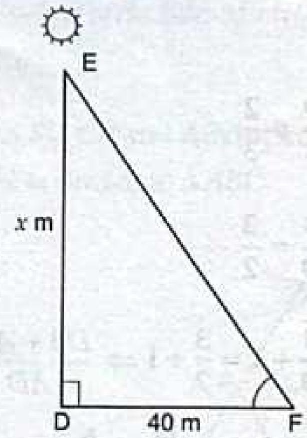


Fig. 7.118

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AC}{DF}$$

$$\Rightarrow \quad \frac{12}{x} = \frac{8}{40} \Rightarrow \frac{12}{x} = \frac{1}{5} \Rightarrow x = 60 \text{ metres}$$

EXAMPLE 34 *In Fig. 7.119, $\angle CAB = 90°$ and $AD \perp BC$. If $AC = 75$ cm, $AB = 1$ m and $BD = 1.25$ m, find AD.*

SOLUTION We have,

$$AB = 1 \text{ m} = 100 \text{ cm}, AC = 75 \text{ cm and } BD = 125 \text{ cm}.$$

In ΔBAC and ΔBDA, we have

$\angle BAC = \angle BDA$ [Each equal to 90°]

and, $\angle B = \angle B$

So, by AA-criterion of similarity, we obtain

$$\Delta BAC \sim \Delta BDA$$

$$\Rightarrow \quad \frac{BA}{BD} = \frac{AC}{AD}$$

$$\Rightarrow \quad \frac{100}{125} = \frac{75}{AD}$$

$$\Rightarrow \quad AD = \frac{125 \times 75}{100} \text{ cm} = 93.75 \text{ cm}$$

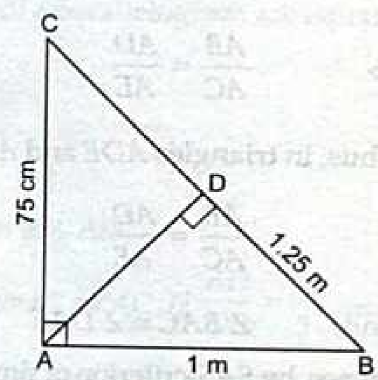


Fig. 7.119

EXAMPLE 35 *The perimeters of two similar triangles are 30 cm and 20 cm respectively. If one side of the first triangle is 12 cm, determine the corresponding side of the second triangle.*

SOLUTION Let ΔABC and ΔDEF be two similar triangles of perimeters P_1 and P_2 respectively. Also, let $AB = 12$ cm, $P_1 = 30$ cm and $P_2 = 20$ cm. Then,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{P_1}{P_2}$$ [∵ Ratio of corresponding sides of similar triangles is equal to the ratio of their perimeters]

$$\Rightarrow \quad \frac{AB}{DE} = \frac{P_1}{P_2}$$

$$\Rightarrow \quad \frac{12}{DE} = \frac{30}{20}$$

$$\Rightarrow \qquad DE = \frac{12 \times 20}{30} \text{ cm} = 8 \text{ cm}$$

Hence, the corresponding side of the second triangle is 8 cm.

EXAMPLE 36 *The perimeters of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. If PQ = 10 cm, find AB.*

SOLUTION Since the ratio of the corresponding sides of similar triangles is same as the ratio of their perimeters.

$$\therefore \qquad \Delta ABC \sim \Delta PQR$$

$$\Rightarrow \qquad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{36}{24}$$

$$\Rightarrow \qquad \frac{AB}{PQ} = \frac{36}{24}$$

$$\Rightarrow \qquad \frac{AB}{10} = \frac{36}{24}$$

$$\Rightarrow \qquad AB = \frac{36 \times 10}{24} \text{ cm} = 15 \text{ cm}$$

EXAMPLE 37 *Two triangles BAC and BDC, right angled at A and D respectively, are drawn on the same base BC and on the same side of BC. If AC and DB intersect at P, prove that $AP \times PC = DP \times PB$.* **[CBSE 2000C]**

SOLUTION In $\Delta\, APB$ and $\Delta\, DPC$, we have

$\angle A = \angle D = 90°$

and, $\angle APB = \angle DPC$ [Vertically opposite angles]

Thus, by AA-criterion of similarity, we obtain

$$\Delta\, APB \sim \Delta\, DPC$$

$$\Rightarrow \qquad \frac{AP}{DP} = \frac{PB}{PC}$$

$$\Rightarrow \qquad AP \times PC = DP \times PB$$

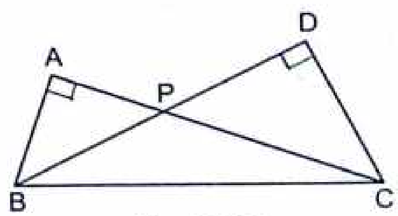


Fig. 7.120

EXAMPLE 38 *In Fig. 7.121, $\angle BAC = 90°$ and segment $AD \perp BC$. Prove that $AD^2 = BD \times DC$.*

SOLUTION In $\Delta\, ABD$ and $\Delta\, ACD$, we have

$\angle ADB = \angle ADC$ [Each equal to 90°]

and, $\angle DBA = \angle DAC$ $\left[\begin{array}{l}\text{Each equal to complement of} \\ \angle BAD \text{ i.e. } 90° - \angle BAD\end{array}\right]$

Therefore, by AA-criterion of similarity, we have

$$\Delta\, DBA \sim \Delta\, DAC \quad \left[\begin{array}{l}\therefore \angle D \leftrightarrow \angle D, \angle B \leftrightarrow \angle DAC \\ \text{and } \angle BAD \leftrightarrow \angle DCA\end{array}\right]$$

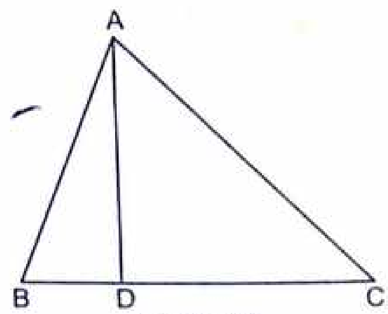


Fig. 7.121

$$\Rightarrow \qquad \frac{DB}{DA} = \frac{DA}{DC} \qquad \left[\text{In similar triangles corresponding sides are proportional}\right]$$

$$\Rightarrow \qquad \frac{BD}{AD} = \frac{AD}{DC}$$

$$\Rightarrow \qquad AD^2 = BD \times DC$$

EXAMPLE 39 *In ΔABC, if $AD \perp BC$ and $AD^2 = BD \times DC$, prove that $\angle BAC = 90°$.*

SOLUTION We have,

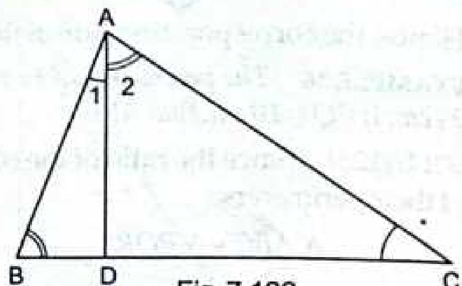


Fig. 7.122

$$AD^2 = BD \times DC$$

$\Rightarrow$ $AD \times AD = BD \times DC$

$\Rightarrow$ $\dfrac{AD}{DC} = \dfrac{BD}{AD}$

Thus, in ΔABD and ΔACD, we have

$$\frac{AD}{DC} = \frac{BD}{AD}$$

and, $\angle BDA = \angle CDA$ [Each equal to 90°]

So, by *SAS*-criterion of similarity, we get

$$\Delta DBA \sim \Delta DAC$$

$\Rightarrow$ ΔDBA and ΔDAC are equiangular

$\Rightarrow$ $\angle 1 = \angle C$ and $\angle 2 = \angle B$

$\Rightarrow$ $\angle 1 + \angle 2 = \angle B + \angle C$

$\Rightarrow$ $\angle A = \angle B + \angle C$ [$\because \angle 1 + \angle 2 = \angle A$]

But, $\angle A + \angle B + \angle C = 180°$

$\therefore$ $\angle A + \angle A = 180°$ [$\because \angle B + \angle C = \angle A$]

$\Rightarrow$ $2\angle A = 180° \Rightarrow \angle A = 90°$

Hence, $\angle BAC = 90°$.

EXAMPLE 40 *In Fig. 7.123, ABCD is a trapezium with $AB \parallel DC$. If ΔAED is similar to ΔBEC, prove that $AD = BC$.*

SOLUTION In ΔEDC and ΔEBA, we have

$\angle 1 = \angle 2$ [Alternate angles]

$\angle 3 = \angle 4$ [Alternate angles]

and, $\angle CED = \angle AEB$ [Vertically opposite angles]

$\therefore$ $\Delta EDC \sim \Delta EBA$

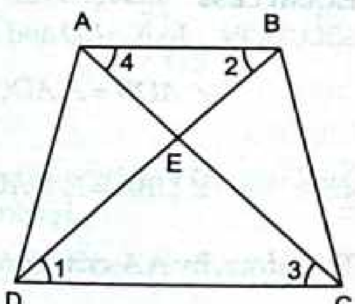


Fig. 7.123

$\Rightarrow$ $\dfrac{ED}{EB} = \dfrac{EC}{EA}$

$\Rightarrow$ $\dfrac{ED}{EC} = \dfrac{EB}{EA}$ ...(i)

It is given that $\Delta AED \sim \Delta BEC$

$\therefore$ $\dfrac{ED}{EC} = \dfrac{EA}{EB} = \dfrac{AD}{BC}$ ...(ii)

From (i) and (ii), we get

$$\frac{EB}{EA} = \frac{EA}{EB}$$

$\Rightarrow$ $(EB)^2 = (EA)^2$

$\Rightarrow$ $EB = EA$

Substituting $EB = EA$ in (ii), we get

$$\frac{EA}{EA} = \frac{AD}{BC}$$

$\Rightarrow \quad \frac{AD}{BC} = 1$

$\Rightarrow \quad AD = BC$

EXAMPLE 41 *Through the mid-point M of the side CD of a parallelogram ABCD, the line BM is drawn intersecting AC in L and AD produced in E. Prove that EL = 2 BL.* **[CBSE 2009]**

SOLUTION In $\Delta\, BMC$ and $\Delta\, EMD$, we have

$MC = MD$ [$\because M$ is the mid-point of CD]

$\angle\, CMB = \angle\, EMD$ [Vertically opposite angles]

and, $\angle\, MBC = \angle\, MED$ [Alternate angles]

So, by AAS-criterion of congruence, we have

$\therefore \quad \Delta\, BMC \cong \Delta\, EMD$

$\Rightarrow \quad BC = DE$ ...(i)

Also, $AD = BC$ [$\because ABCD$ is a parallelogram] ...(ii)

$AD + DE = BC + BC$

$\Rightarrow \quad AE = 2\, BC$ ...(iii)

Now, in $\Delta\, AEL$ and $\Delta\, CBL$, we have

$\angle\, ALE = \angle\, CLB$ [Vertically opposite angles]

$\angle\, EAL = \angle\, BCL$ [Alternate angles]

So, by AA-criterion of similarity of triangles, we have

$\Delta\, AEL \sim \Delta\, CBL$

$\Rightarrow \quad \frac{EL}{BL} = \frac{AE}{CB}$

$\Rightarrow \quad \frac{EL}{BL} = \frac{2BC}{BC}$ [Using equations (iii)]

$\Rightarrow \quad \frac{EL}{BL} = 2$

$\Rightarrow \quad EL = 2\, BL$

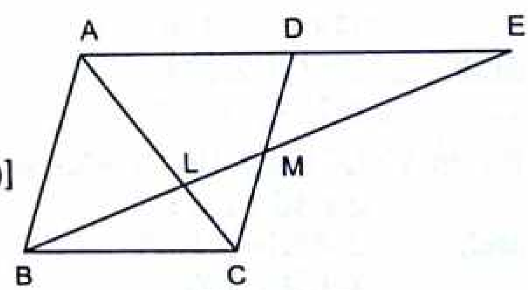


Fig. 7.124

EXAMPLE 42 *In a $\Delta\, ABC$, let P and Q be points on AB and AC respectively such that PQ || BC. Prove that the median AD bisects PQ.*

SOLUTION Suppose the median AD intersects PQ at E.

Now, $PQ \parallel BC$

$\Rightarrow \quad \angle\, APE = \angle\, B$ and $\angle\, AQE = \angle\, C$ [Corresponding angles]

So, in Δ's APE and ABD, we have

$\angle\, APE = \angle\, ABD$

and, $\angle\, PAE = \angle\, BAD$ [Common]

$\therefore \quad \Delta\, APE \sim \Delta\, ABD$

$\Rightarrow \quad \frac{PE}{BD} = \frac{AE}{AD}$ ...(i)

Similarly, we have

$\Delta\, AQE \sim \Delta\, ACD$

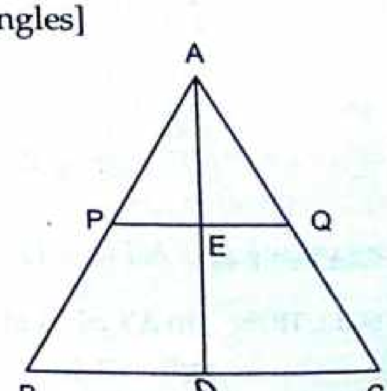


Fig. 7.125

$\therefore \quad \frac{QE}{CD} = \frac{AE}{AD}$ ...(ii)

From (i) and (ii), we get

$$\frac{PE}{BD} = \frac{QE}{CD}$$

$\Rightarrow \quad \frac{PE}{BD} = \frac{QE}{BD}$ [$\because$ AD is the median $\therefore BD = CD$]

$\Rightarrow \quad PE = QE$

Hence, AD bisects PQ.

EXAMPLE 43 *In Fig. 7.126, DEFG is a square and $\angle BAC = 90°$. Prove that*

(i) $\Delta AGF \sim \Delta DBG$ (ii) $\Delta AGF \sim \Delta EFC$ (iii) $\Delta DBG \sim \Delta EFC$ (iv) $DE^2 = BD \times EC$

[CBSE 2009]

SOLUTION (i) In ΔAGF and ΔDBG, we have

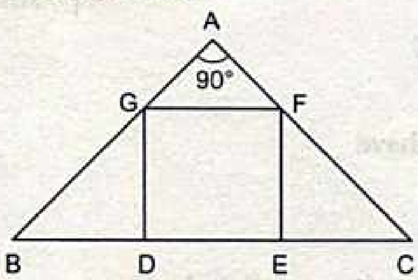


Fig. 7.126

$\angle GAF = \angle BDG$ [Each equal to 90°]

and, $\angle AGF = \angle DBG$ [Corresponding angles]

$\therefore \quad \Delta AGF \sim \Delta DBG$ [By AA-criterion of similarity]

(ii) In ΔAGF and ΔEFC, we have

$\angle FAG = \angle CEF$ [Each equal to 90°]

and, $\angle AFG = \angle ECF$ [Corresponding angles]

$\therefore \quad \Delta AGF \sim \Delta EFC$ [By AA-criterion of similarity]

(iii) Since $\Delta AGF \sim \Delta DBG$ and $\Delta AGF \sim \Delta EFC$

$\therefore \quad \Delta DBG \sim \Delta EFC$

(iv) We have,

$\Delta DBG \sim \Delta EFC$ [Using (iii)]

$\therefore \quad \frac{BD}{EF} = \frac{DG}{EC}$

$\Rightarrow \quad \frac{BD}{DE} = \frac{DE}{EC}$ [$\because$ $DEFG$ is a square $\therefore EF = DE, DG = DE$]

$\Rightarrow \quad DE^2 = BD \times EC$

EXAMPLE 44 *In Fig. 7.127, if $AD \perp BC$ and $\frac{BD}{DA} = \frac{DA}{DC}$, prove that ΔABC is a right triangle.*

SOLUTION In Δ's BDA and ADC, we have

$\frac{DB}{DA} = \frac{DA}{DC}$ [Given]

and, $\angle BDA = \angle ADC$ [Each equal to 90°]

So, by SAS-criterion of similarity, we have

$$\Delta BDA \sim \Delta ADC$$

$\Rightarrow \quad \angle ABD = \angle CAD$ and $\angle BAD = \angle ACD$

$\Rightarrow \quad \angle ABD + \angle ACD = \angle CAD + \angle BAD$

$\Rightarrow \quad \angle B + \angle C = \angle A$

$\Rightarrow \quad \angle A + \angle B + \angle C = 2\angle A$ [Adding $\angle A$ on both sides]

$\Rightarrow \quad 2\angle A = 180°$

$\Rightarrow \quad \angle A = 90°$

$\Rightarrow \quad \Delta ABC$ is a right triangle.

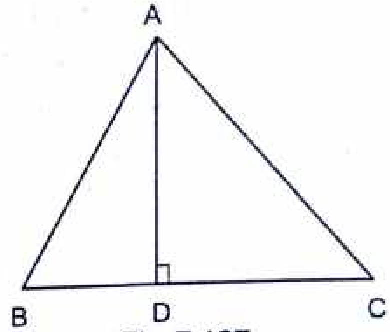


Fig. 7.127

EXAMPLE 45 *In Fig. 7.128, $\angle ACB = 90°$ and $CD \perp AB$. Prove that $\dfrac{CB^2}{CA^2} = \dfrac{BD}{AD}$.*

SOLUTION In triangles ACD and ABC, we have

$\angle ADC = \angle ACB$ [Each equal to 90°]

and, $\angle DAC = \angle BAC$ [Common]

So, by AA-criterion of similarity, we obtain

$$\Delta ACD \sim \Delta ABC$$

$\Rightarrow \quad \dfrac{AC}{AB} = \dfrac{AD}{AC}$

$\Rightarrow \quad AC^2 = AB \times AD$ ...(i)

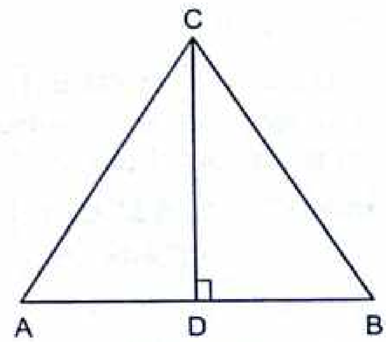


Fig. 7.128

In Δ's BCD and BAC, we have

$\angle BDC = \angle BCA$

and, $\angle DBC = \angle ABC$

So, by AA-criterion of similarity, we obtain

$$BCD \sim BAC$$

$\Rightarrow \quad \dfrac{BC}{BD} = \dfrac{BA}{BC}$

$\Rightarrow \quad BC^2 = AB \times BD$ ...(ii)

Dividing (ii) by (i), we get

$$\frac{BC^2}{AC^2} = \frac{AB \times BD}{AB \times AD} \Rightarrow \frac{BC^2}{AC^2} = \frac{BD}{AD}$$

EXAMPLE 46 *Through the mid-point M of the side CD of a parallelogram ABCD, the line BM is drawn intersecting AC at L and AD produced at E. Prove that EL = 2 BL.*

SOLUTION In D's BMC and EMD, we have

$\angle BMC = \angle EMD$ [Vertically opposite angles]

$MC = MD$ [$\because M$ is the mid-point of CD]

$\angle MCB = \angle MDE$ [Alternate angles]

So, by AAS-congruence criterion, we have

$$\Delta BMC \cong \Delta EMD$$

$\Rightarrow \quad BC = ED$ [$\because$ Corresponding parts of congruent triangle are equal]

In Δ's AEL and CBL, we have

$\angle ALE = \angle CLB$ [Vertically opposite angles]

$\angle EAL = \angle BCL$ [Alternate angles]

So, by AA-criterion of similarity, we have

$\Delta\, AEL \sim \Delta\, CBL$

$\Rightarrow \quad \frac{AE}{BC} = \frac{EL}{BL} = \frac{AL}{CL}$

$\Rightarrow \quad \frac{EL}{BL} = \frac{AE}{BC}$

$\Rightarrow \quad \frac{EL}{BL} = \frac{AD + DE}{BC} = \frac{BC + DE}{BC} = \frac{2BC}{BC} = 2$ [$\because$ $AD = BC$ and $DE = BC$]

$\Rightarrow \quad EL = 2\,BL.$

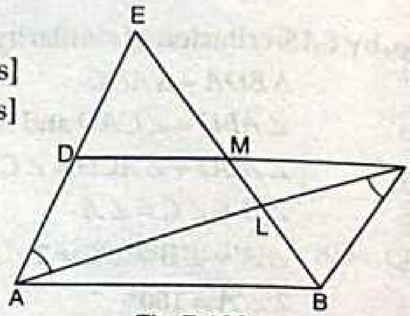


Fig. 7.129

LEVEL-3

EXAMPLE 47 *Two poles of height a metres and b metres are p metres apart. Prove that the height of the point of intersection of the lines joining the top of each pole to the foot of the opposite pole is given by* $\frac{ab}{a+b}$ *metres.*

SOLUTION Let AB and CD be two poles of heights a metres and b metres respectively such that the poles are p metres apart i.e. $AC = p$ metres. Suppose the lines AD and BC meet at O such that $OL = h$ metres. Let $CL = x$ and $LA = y$. Then, $x + y = p$.

In $\Delta\, ABC$ and $\Delta\, LOC$, we have

$\angle CAB = \angle CLO$ [Each equal to 90°]

$\angle C = \angle C$ [Common]

$\therefore \quad \Delta\, CAB \sim \Delta\, CLO$ [By AA-criterion of similarity]

$\Rightarrow \quad \frac{CA}{CL} = \frac{AB}{LO}$

$\Rightarrow \quad \frac{p}{x} = \frac{a}{h}$

$\Rightarrow \quad x = \frac{ph}{a}$ ...(i)

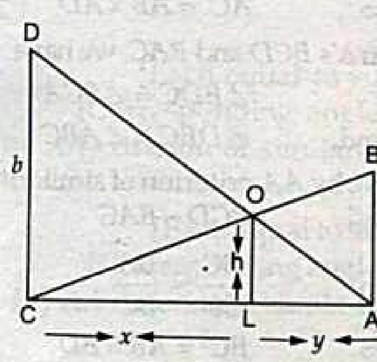


Fig. 7.130

In $\Delta\, ALO$ and $\Delta\, ACD$, we have

$\angle ALO = \angle ACD$ [Each equal to 90°]

$\angle A = \angle A$ [Common]

$\therefore \quad \Delta\, ALO \sim \Delta\, ACD$ [By AA-criterion of similarity]

$\Rightarrow \quad \frac{AL}{AC} = \frac{OL}{DC}$

$\Rightarrow \quad \frac{y}{p} = \frac{h}{b}$

$\Rightarrow \quad y = \frac{ph}{b}$ [$\because AC = x + y = p$] ...(ii)

From (i) and (ii), we have

$$x + y = \frac{ph}{a} + \frac{ph}{b}$$

$$\Rightarrow \quad p = ph\left(\frac{1}{a} + \frac{1}{b}\right) \qquad [\because x + y = p]$$

$$\Rightarrow \quad 1 = h\left(\frac{a+b}{ab}\right) \Rightarrow h = \frac{ab}{a+b} \text{ metres}$$

Hence, the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is $\frac{ab}{a+b}$ metres.

EXAMPLE 48 *ABC is a triangle in which $AB = AC$ and D is a point on AC such that $BC^2 = AC \times CD$. Prove that $BD = BC$.*

GIVEN ΔABC in which $AB = AC$ and D is a point on the side AC such that

$$BC^2 = AC \times CD$$

TO PROVE $BD = BC$

CONSTRUCTION Join BD

PROOF We have,

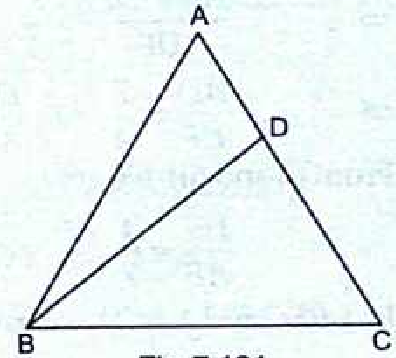


Fig. 7.131

$$BC^2 = AC \times CD$$

$$\Rightarrow \quad \frac{BC}{CD} = \frac{AC}{BC} \qquad \ldots(i)$$

Thus, in ΔABC and ΔBDC, we have

$$\frac{AC}{BC} = \frac{BC}{CD} \qquad \text{[From (i)]}$$

and, $\angle C = \angle C$ [Common]

$\therefore \quad \Delta ABC \sim \Delta BDC$ [By SAS criterion of similarity]

$$\Rightarrow \quad \frac{AB}{BD} = \frac{BC}{DC}$$

$$\Rightarrow \quad \frac{AC}{BD} = \frac{BC}{CD} \qquad [\because AB = AC]$$

$$\Rightarrow \quad \frac{AC}{BC} = \frac{BD}{CD} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{BC}{CD} = \frac{BD}{CD} \Rightarrow BD = BC$$

EXAMPLE 49 *In trapezium ABCD, $AB \parallel DC$ and $DC = 2AB$. EF drawn parallel to AB cuts AD in F and BC in E such that $\frac{BE}{EC} = \frac{3}{4}$. Diagonal DB intersects EF at G. Prove that $7\,FE = 10\,AB$.*

SOLUTION In ΔDFG and ΔDAB, we have

$\angle 1 = \angle 2$ $[\because AB \parallel DC \parallel EF \;\therefore \angle 1$ and $\angle 2$ are corresponding angles$]$

$\angle FDG = \angle ADB$ [Common]

So, by AA-criterion of similarity, we have

$\therefore \quad \Delta DFG \sim \Delta DAB$

$\Rightarrow \quad \dfrac{DF}{DA} = \dfrac{FG}{AB}$ ...(i)

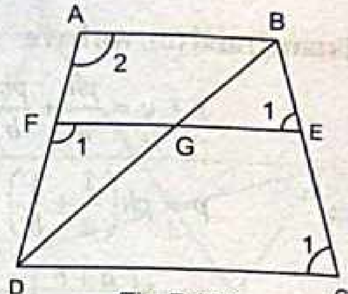


Fig. 7.132

In trapezium $ABCD$, we have

$EF \parallel AB \parallel DC$

$\therefore \quad \dfrac{AF}{DF} = \dfrac{BE}{EC}$

$\Rightarrow \quad \dfrac{AF}{DF} = \dfrac{3}{4}$ $\left[\because \dfrac{BE}{EC} = \dfrac{3}{4} \text{ (Given)}\right]$

$\Rightarrow \quad \dfrac{AF}{DF} + 1 = \dfrac{3}{4} + 1$ [Adding 1 on both sides]

$\Rightarrow \quad \dfrac{AF + DF}{DF} = \dfrac{7}{4}$

$\Rightarrow \quad \dfrac{AD}{DF} = \dfrac{7}{4} \Rightarrow \dfrac{DF}{AD} = \dfrac{4}{7}$ ...(ii)

From (i) and (ii), we get

$\dfrac{FG}{AB} = \dfrac{4}{7} \Rightarrow FG = \dfrac{4}{7} AB$ ...(iii)

In ΔBEG and ΔBCD, we have

$\angle BEG = \angle BCD$ [Corresponding angles]

$\angle B = \angle B$ [Common]

$\therefore \quad \Delta BEG \sim \Delta BCD$ [By AA-criterion of similarity]

$\Rightarrow \quad \dfrac{BE}{BC} = \dfrac{EG}{CD}$

$\Rightarrow \quad \dfrac{3}{7} = \dfrac{EG}{CD}$ $\left[\because \dfrac{BE}{EC} = \dfrac{3}{4} \Rightarrow \dfrac{EC}{BE} = \dfrac{4}{3} \Rightarrow \dfrac{EC}{BE} + 1 = \dfrac{4}{3} + 1 \Rightarrow \dfrac{BC}{BE} = \dfrac{7}{3}\right]$

$\Rightarrow \quad EG = \dfrac{3}{7} CD$

$\Rightarrow \quad EG = \dfrac{3}{7} \times 2\, AB$ $[\because CD = 2\, AB \text{ (given)}]$

$\Rightarrow \quad EG = \dfrac{6}{7} AB$ ...(iv)

Adding (iii) and (iv), we get

$$FG + EG = \frac{4}{7} AB + \frac{6}{7} AB \Rightarrow EF = \frac{10}{7} AB \Rightarrow 7\, EF = 10\, AB$$

EXAMPLE 50 *Through the vertex D of a parallelogram ABCD, a line is drawn to intersect the sides BA and BC produced at E and F respectively. Prove that*

$$\frac{DA}{AE} = \frac{FB}{BE} = \frac{FC}{CD}$$

SOLUTION In Δ's EAD and DCF, we have

$\angle 1 = \angle 2$ [$\because AB \parallel DC$ $\therefore$ Corresponding angles are equal]

$\angle 3 = \angle 4$ [$\because AD \parallel BC$ $\therefore$ Corresponding angles are equal]

Therefore, by AA-criterion of similarity, we have

$\Delta EAD \sim \Delta DCF$

$$\Rightarrow \quad \frac{EA}{DC} = \frac{AD}{CF} = \frac{DE}{FD}$$

$$\Rightarrow \quad \frac{EA}{DC} = \frac{AD}{CF}$$

$$\Rightarrow \quad \frac{AD}{AE} = \frac{CF}{CD} \qquad \ldots(i)$$

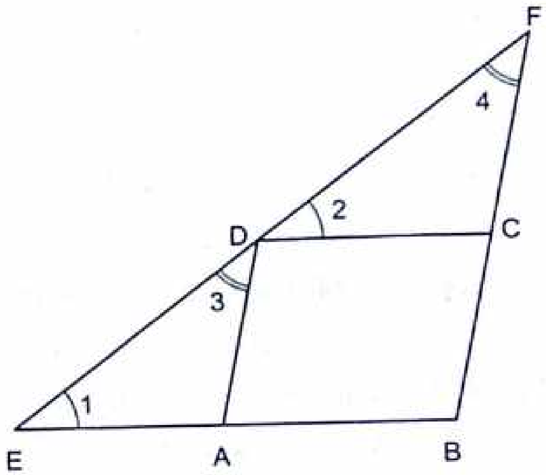


Fig. 7.133

Now, in Δ's EAD and EBF, we have

$\angle 1 = \angle 1$ [Common angle]

$\angle 3 = \angle 4$

So, by AA-criterion of similarity, we have

$\Delta EAD \sim \Delta EBF$

$$\Rightarrow \quad \frac{EA}{EB} = \frac{AD}{BF} = \frac{ED}{EF}$$

$$\Rightarrow \quad \frac{EA}{EB} = \frac{AD}{BF}$$

$$\Rightarrow \quad \frac{AD}{AE} = \frac{FB}{BE} \qquad \ldots(ii)$$

From (i) and (ii), we obtain: $\dfrac{AD}{AE} = \dfrac{FB}{BE} = \dfrac{CF}{CD}$

EXAMPLE 51 *In Fig. 7.134, ABC is a right triangle right angled at B and D is the foot of the perpendicular drawn from B on AC. If $DM \perp BC$ and $DN \perp AB$, prove that*

(i) $DM^2 = DN \times MC$ (ii) $DN^2 = DM \times AN$ **[NCERT]**

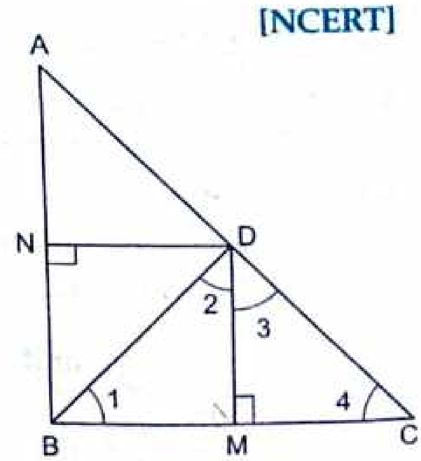


Fig. 7.134

SOLUTION We have,

$AB \perp BC$ and $DM \perp BC$

$\Rightarrow \quad AB \parallel DM$

Similarly, we have

$CB \perp AB$ and $DN \perp AB$

$\Rightarrow \quad CB \parallel DN$

Hence, quadrilateral $BMDN$ is a rectangle.

$\therefore \quad BM = ND$

(i) In ΔBMD, we have

$$\angle 1 + \angle BMD + \angle 2 = 180^\circ$$

$\Rightarrow$ $\angle 1 + 90^\circ + \angle 2 = 180^\circ$

$\Rightarrow$ $\angle 1 + \angle 2 = 90^\circ$

Similarly, in ΔDMC, we have

$$\angle 3 + \angle 4 = 90^\circ$$

Since $BD \perp AC$. Therefore,

$$\angle 2 + \angle 3 = 90^\circ$$

Now, $\angle 1 + \angle 2 = 90^\circ$ and $\angle 2 + \angle 3 = 90^\circ$

$\Rightarrow$ $\angle 1 + \angle 2 = \angle 2 + \angle 3$

$\Rightarrow$ $\angle 1 = \angle 3$

Also, $\angle 3 + \angle 4 = 90^\circ$ and $\angle 2 + \angle 3 = 90^\circ$

$\Rightarrow$ $\angle 3 + \angle 4 = \angle 2 + \angle 3 \Rightarrow \angle 2 = \angle 4$

Thus, in Δ's BMD and DMC, we have

$$\angle 1 = \angle 3 \text{ and } \angle 2 = \angle 4$$

So, by AA-criterion of similarity, we obtain

$$\Delta BMD \sim \Delta DMC$$

$\Rightarrow$ $\dfrac{BM}{DM} = \dfrac{MD}{MC}$

$\Rightarrow$ $\dfrac{DN}{DM} = \dfrac{DM}{MC}$ $[\because BM = ND]$

$\Rightarrow$ $DM^2 = DN \times MC$

(ii) Proceeding as in (i), we can prove that

$$\Delta BND \sim \Delta DNA$$

$\Rightarrow$ $\dfrac{BN}{DN} = \dfrac{ND}{NA}$

$\Rightarrow$ $\dfrac{DM}{DN} = \dfrac{DN}{AN}$ $[\because BN = DM]$

$\Rightarrow$ $DN^2 = DM \times AN$

EXAMPLE 52 *ABC is an isosceles triangle with $AB = AC$ and D is a point on AC such that $BC^2 = AC \times CD$. Prove that $BD = BC$.*

SOLUTION We have,

$$BC^2 = AC \times CD \text{ and } AB = AC$$

$\Rightarrow$ $BC \times BC = AC \times CD$ and $\angle B = \angle C$

$\Rightarrow$ $\dfrac{BC}{AC} = \dfrac{CD}{BC}$ and $\angle B = \angle C$

$\Rightarrow$ $\dfrac{BC}{CA} = \dfrac{DC}{CB}$ and $\angle B = \angle C$

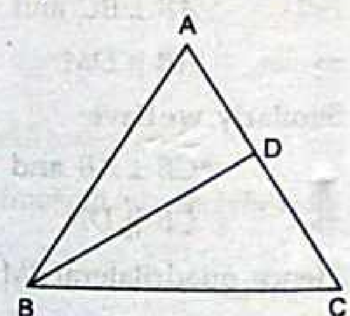


Fig. 7.135

So, by *SAS*-criterion of similarity, we obtain

$$\Delta BCA \sim \Delta DCB$$

$$\Rightarrow \quad \frac{BC}{DC} = \frac{CA}{CB} = \frac{BA}{DB}$$

$$\Rightarrow \quad \frac{CA}{CB} = \frac{BA}{DB}$$

$$\Rightarrow \quad \frac{BA}{CA} = \frac{DB}{CB}$$

$$\Rightarrow \quad 1 = \frac{DB}{CB} \qquad [\because AB = AC]$$

$$\Rightarrow \quad DB = CB \Rightarrow BD = BC$$

EXERCISE 7.5

LEVEL-1

1. In Fig. 7.136, $\Delta ACB \sim \Delta APQ$. If $BC = 8$ cm, $PQ = 4$ cm, $BA = 6.5$ cm and $AP = 2.8$ cm, find CA and AQ.

$\frac{BC}{PQ} = \frac{AB}{AQ}$

$\frac{8}{4} = \frac{6.5}{AQ}$

$AQ = \frac{6.5}{2}$ 3.25

$AQ = 3.25$ cm

$\frac{AC}{AP} = \frac{BC}{PQ}$

$\Rightarrow \frac{AC}{2.8} = \frac{8}{4}$

$\Rightarrow AC = 2.8 \times 2$

$AC = 5.6$ cm

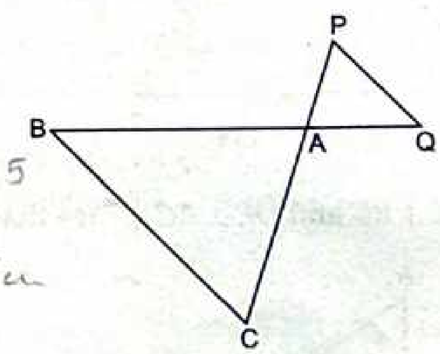


Fig. 7.136

2. In Fig. 7.137, $AB \parallel QR$. Find the length of PB.

$\frac{AB}{QR} = \frac{PB}{PR}$

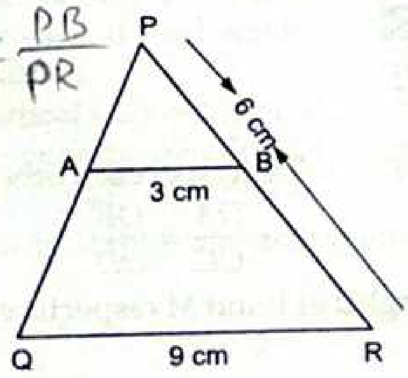


Fig. 7.137

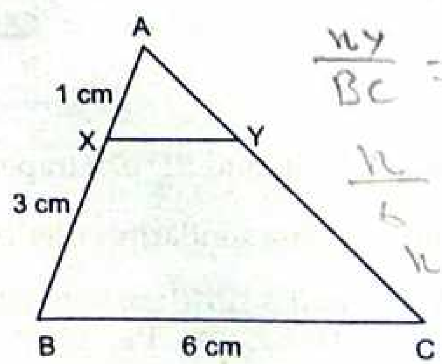


Fig. 7.138

$\frac{xy}{BC} = \frac{AX}{AB}$

$\frac{x}{6} = \frac{1}{4}$

$x = \frac{6}{4}$ 1.5

$XY = 1.5$ cm

3. In Fig. 7.138, $XY \parallel BC$. Find the length of XY.
4. In a right angled triangle with sides a and b and hypotenuse c, the altitude drawn on the hypotenuse is x. Prove that $ab = cx$.

5. In Fig. 7.139, $\angle ABC = 90^\circ$ and $BD \perp AC$. If $BD = 8$ cm and $AD = 4$ cm, find CD.

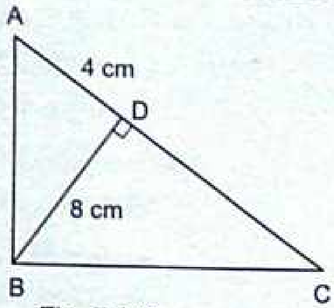


Fig. 7.139

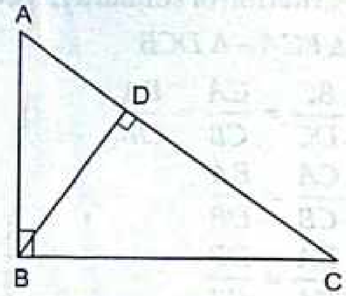


Fig. 7.140

6. In Fig. 7.140, $\angle ABC = 90^\circ$ and $BD \perp AC$. If $AB = 5.7$ cm, $BD = 3.8$ cm and $CD = 5.4$ cm, find BC.

7. In Fig. 7.141, $DE \parallel BC$ such that $AE = (1/4)\,AC$. If $AB = 6$ cm, find AD.

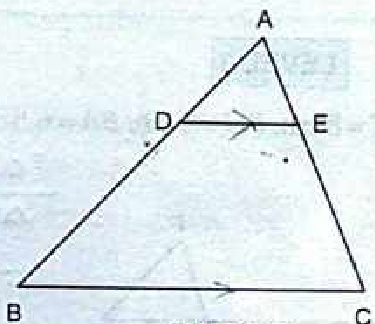


Fig. 7.141

8. In Fig. 7.142, if $AB \perp BC$, $DC \perp BC$ and $DE \perp AC$, prove that $\Delta\, CED \sim \Delta\, ABC$.

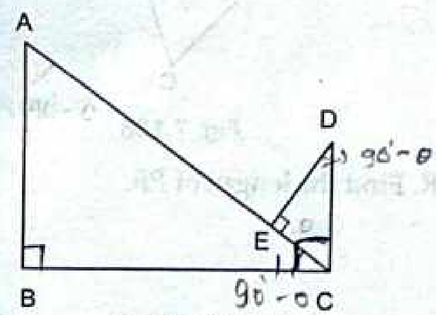


Fig. 7.142

9. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. Using similarity criterion for two triangles, show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.

10. If $\Delta\, ABC$ and $\Delta\, AMP$ are two right triangles, right angled at B and M respectively such that $\angle MAP = \angle BAC$. Prove that

(i) $\Delta\, ABC \sim \Delta\, AMP$ (ii) $\dfrac{CA}{PA} = \dfrac{BC}{MP}$

LEVEL-2

11. A vertical stick 10 cm long casts a shadow 8 cm long. At the same time a tower casts a shadow 30 m long. Determine the height of the tower.

12. In Fig. 7.143, $\angle A = \angle CED$, prove that $\Delta CAB \sim \Delta CED$. Also, find the value of x.

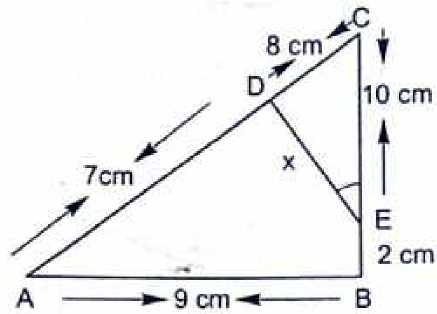


Fig. 7.143

13. The perimeters of two similar triangles are 25 cm and 15 cm respectively. If one side of first triangle is 9 cm, what is the corresponding side of the other triangle? **[CBSE 2002 C]**

14. In ΔABC and ΔDEF, it is being given that: $AB = 5$ cm, $BC = 4$ cm and $CA = 4.2$ cm; $DE = 10$ cm, $EF = 8$ cm and $FD = 8.4$ cm. If $AL \perp BC$ and $DM \perp EF$, find $AL : DM$.

15. D and E are the points on the sides AB and AC respectively of a ΔABC such that: $AD = 8$ cm, $DB = 12$ cm, $AE = 6$ cm and $CE = 9$ cm. Prove that $BC = 5/2\, DE$.

16. D is the mid-point of side BC of a ΔABC. AD is bisected at the point E and BE produced cuts AC at the point X. Prove that $BE : EX = 3 : 1$

17. $ABCD$ is a parallelogram and APQ is a straight line meeting BC at P and DC produced at Q. prove that the rectangle obtained by BP and DQ is equal to the rectangle contained by AB and BC.

18. In ΔABC, AL and CM are the perpendiculars from the vertices A and C to BC and AB respectively. If AL and CM intersect at O, prove that :

    (i) $\Delta OMA \sim \Delta OLC$ (ii) $\dfrac{OA}{OC} = \dfrac{OM}{OL}$

19. $ABCD$ is a quadrilateral in which $AD = BC$. If P, Q, R, S be the mid-points of AB, AC, CD and BD respectively, show that $PQRS$ is a rhombus.

20. In an isosceles ΔABC, the base AB is produced both the ways to P and Q such that $AP \times BQ = AC^2$. Prove that $\Delta APC \sim \Delta BCQ$.

21. A girl of heigh 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/sec. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

22. A vertical stick of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower. **[NCERT]**

23. In Fig. 7.144, ΔABC is right angled at C and $DE \perp AB$. Prove that $\Delta ABC \sim \Delta ADE$ and hence find the lengths of AE and DE.

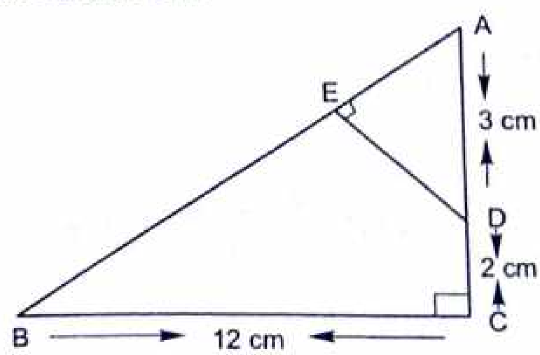


Fig. 7.144

LEVEL-3

24. In Fig. 7.145, PA, QB and RC are each perpendicular to AC. Prove that $\frac{1}{x}+\frac{1}{z}=\frac{1}{y}$.

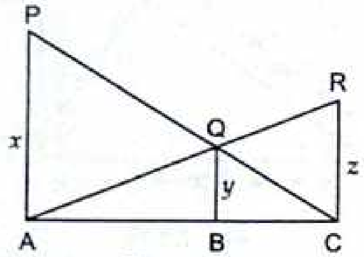


Fig. 7.145

25. In Fig. 7.146, we have $AB \parallel CD \parallel EF$. If $AB = 6$ cm, $CD = x$ cm, $EF = 10$ cm, $BD = 4$ cm and $DE = y$ cm, calculate the values of x and y.

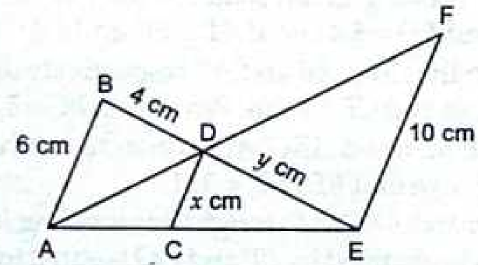


Fig. 7.146

ANSWERS

1. $CA = 5.6$ cm, $AQ = 3.25$ cm **2.** 2 cm **3.** 1.5 cm **5.** 16 cm

6. 8.1 cm **7.** 1.5 cm **11.** 37.5 m **12.** 6

13. 5.4 cm **14.** 1 : 2 **21.** 1.6 m **22.** 42 m

23. $DE = \frac{36}{13}$ cm and $AE = \frac{15}{13}$ cm **25.** $x = 3.75$ cm; $y = 6.67$ cm

HINT TO SELECTED PROBLEMS

2. Use: $\Delta PAB \sim \Delta PQR$

5. In ΔDBA and ΔDCB, we have

$\angle DBA = \angle DCB$ [Each equal to $90° - \angle DBC$]

and, $\angle D = \angle D = 90°$ [Common]

$\therefore \quad \Delta DBA \sim \Delta DCB$

$$\Rightarrow \quad \frac{BD}{CD} = \frac{AD}{BD} \Rightarrow CD = \frac{BD^2}{AD}$$

6. Clearly, $\Delta ABC \sim \Delta BDC$

$$\Rightarrow \quad \frac{AB}{BD} = \frac{BC}{DC} \Rightarrow \frac{5.7}{3.8} = \frac{BC}{5.4} \Rightarrow BC = \frac{5.7 \times 5.4}{3.8} = 8.1 \text{ cm}$$

7. Clearly, $\Delta ADE \sim \Delta ABC$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{AE}{AC} \Rightarrow \frac{AD}{6} = \frac{1}{4} \Rightarrow AD = \frac{6}{4} = 1.5 \text{ cm}.$$

12. In $\Delta\, CAB$ and $\Delta\, CED$, we have

$\angle A = \angle CED$ and $\angle C = \angle C$ [Common]

$\therefore \quad \Delta\, CAB \sim \Delta\, CED$

$$\Rightarrow \quad \frac{CA}{CE} = \frac{AB}{DE} = \frac{CB}{CD} \Rightarrow \frac{AB}{DE} = \frac{CB}{CD} \Rightarrow \frac{9}{x} = \frac{10+2}{8} \Rightarrow x = 6 \text{ cm}$$

13. Ratio of the corresponding sides = Ratio of perimeters.

$$\Rightarrow \quad \frac{2}{x} = \frac{25}{15} \Rightarrow x = \frac{27}{5} = 5.4 \text{ cm}$$

14. Clearly, $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF} = \dfrac{1}{2}$

$\therefore \quad \Delta\, ABC \sim \Delta\, DEF$

Now, use the result that in similar triangles the ratio of corresponding altitudes is same as the ratio of corresponding sides.

15. Clearly, $\dfrac{AD}{AB} = \dfrac{AE}{AC} = \dfrac{2}{3}$ and $\angle A$ is common in $\Delta\, ABC$ and $\Delta\, ADE$

$\therefore \quad \Delta\, ADE \sim \Delta\, ABC \Rightarrow \dfrac{BC}{DE} = \dfrac{AB}{AD}$

17. Use the result $\Delta\, ABP \sim \Delta\, QDA$ to prove that $AB \times BC = BP \times DQ$.

21. Let AB be the lamp-post and CD be the girl after walking for 4 seconds. Let DE be the length of her shadow such that $DE = x$ metres, $BD = 1.2 \times 4 = 4.8$ m.

In Δ's ABE and CDE, we have

$\angle B = \angle D$ and $\angle E = \angle E$

So, by AA-similarity criterion, we obtain $\Delta\, ABE \sim \Delta\, CDE$

$$\therefore \quad \frac{BE}{DE} = \frac{AB}{CD} \Rightarrow \frac{4.8 + x}{x} = \frac{3.6}{0.9} \Rightarrow x = 1.6 \text{ m}$$

23. In triangles ABC and ADE, we have

$$\angle ACB = \angle AED = 90^\circ \quad \text{and,} \quad \angle BAC = \angle DAE$$

So, by AA similarity criterion, we obtain

$$\Delta\, ABC \sim \Delta ADE$$

$$\Rightarrow \quad \frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$$

$$\Rightarrow \quad \frac{13}{3} = \frac{12}{DE} = \frac{5}{AE} \qquad [\because AB^2 = AC^2 + BC^2 = 5^2 + 12^2]$$

$\Rightarrow \quad DE = \frac{36}{13}$ and $AE = \frac{15}{13}$

24. In $\Delta\, PAC$, we have

$$BQ \parallel AP \Rightarrow \frac{BQ}{AP} = \frac{CB}{CA} \Rightarrow \frac{y}{x} = \frac{CB}{CA} \quad \text{...(i)}$$

In $\Delta\, ACR$, we have

$$BQ \parallel CR \Rightarrow \frac{BQ}{CR} = \frac{AB}{AC} \Rightarrow \frac{y}{z} = \frac{AB}{AC} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\frac{y}{x} + \frac{y}{z} = \frac{CB}{AC} + \frac{AB}{AC}$$

$$\Rightarrow \quad \frac{y}{x} + \frac{y}{z} = \frac{AB + BC}{AC} \Rightarrow \frac{y}{x} + \frac{y}{z} = \frac{AC}{AC} \Rightarrow \frac{y}{x} + \frac{y}{z} = 1 \Rightarrow \frac{1}{x} + \frac{1}{z} = \frac{1}{y}$$

7.8 MORE ON CHARACTERISTIC PROPERTIES

In the previous section, we have learnt about characteristic properties of similar triangles and their applications. In this section, we shall discuss some more results as theorems derived from the characteristic properties of similar triangles.

THEOREM 1 *If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding medians.* **[NCERT]**

GIVEN Two triangles ABC and DEF in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, AP and DQ are their medians.

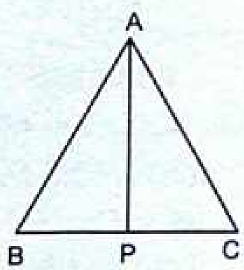


Fig. 7.147

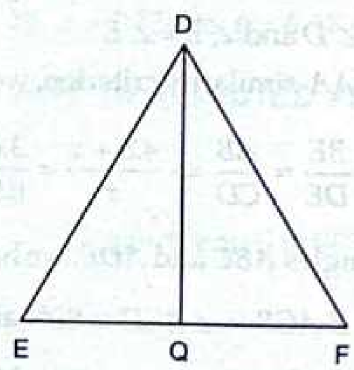


Fig. 7.148

TO PROVE $\frac{BC}{EF} = \frac{AP}{DQ}$

PROOF Since equiangular triangles are similar.

$\therefore \quad \Delta\, ABC \sim \Delta\, DEF$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{2BP}{2EQ} \qquad \left[\begin{array}{l}\because P \text{ and } Q \text{ are mid-points of } BC \text{ and } EF \text{ respectively} \\ \therefore BC = 2BP \text{ and } EF = 2\,EQ\end{array}\right]$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BP}{EQ} \qquad \text{...(ii)}$$

Now, in $\Delta\, ABP$ and $\Delta\, DFQ$, we have

$$\frac{AB}{DE} = \frac{BP}{EQ} \qquad \text{[From (ii)]}$$

and, $\angle B = \angle E$ [Given]

So, by *SAS*-criterion of similarity, we have

$$\Delta\, ABP \sim \Delta\, DEQ$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AP}{DQ} \qquad \text{...(iii)}$$

From (i) and (iii), we get : $\dfrac{BC}{EF} = \dfrac{AP}{DQ}$

Hence, the ratio of the corresponding sides is same as the ratio of corresponding medians.

Q.E.D.

THEOREM 2 *If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.*

GIVEN Two triangles ABC and DEF in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$; and AX, DY are the bisectors of $\angle A$ and $\angle D$ respectively.

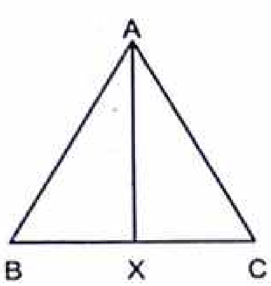


Fig. 7.149

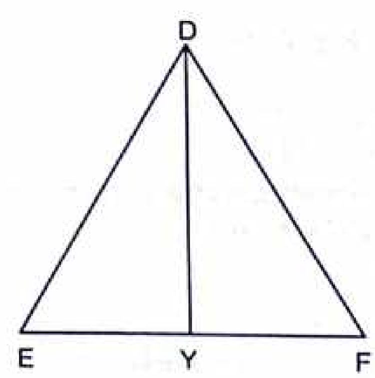


Fig. 7.150

TO PROVE $\dfrac{BC}{EF} = \dfrac{AX}{DY}$

PROOF Since equiangular triangles are similar.

$$\Delta\, ABC \sim \Delta\, DEF$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} \qquad \text{...(i)}$$

In $\Delta\, ABX$ and DEY, we have

$$\angle B = \angle E \qquad \text{[Given]}$$

and, $\angle BAX = \angle EDY$ $\left[\because \angle A = \angle D \Rightarrow \frac{1}{2}\angle A = \frac{1}{2}\angle D \Rightarrow \angle BAX = \angle EDY\right]$

So, by *AA*-criterion of similarity, we have

$$\Delta ABX \sim \Delta DEY$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AX}{DY} \quad \text{...(ii)}$$

From (i) and (ii), we get : $\frac{BC}{EF} = \frac{AX}{DY}$

Q.E.D.

THEOREM 3 *If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.*

GIVEN Two triangles *ABC* and *DEF* in which

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F \text{ and } AL \perp BC, DM \perp EF$$

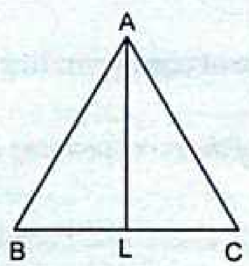


Fig. 7.151

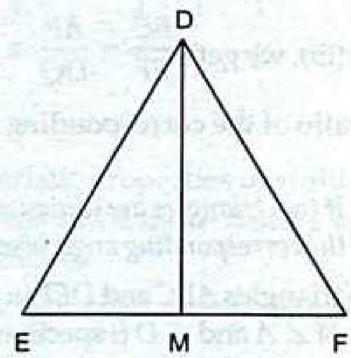


Fig. 7.152

TO PROVE $\frac{BC}{EF} = \frac{AL}{DM}$

PROOF Since equiangular triangles are similar.

$$\therefore \quad \Delta ABC \sim \Delta DEF$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF} \quad \text{...(I)}$$

In triangle *ALB* and *DME*, we have

$\angle ALB = \angle DME$ [Each equal to 90°]

$\angle B = \angle E$ [Given]

So, by *AA*-criterion of similarity, we have

$$\Delta ALB \sim \Delta DME$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AL}{DM} \quad \text{...(ii)}$$

From (i) and (ii), we get : $\frac{BC}{EF} = \frac{AL}{DM}$

Q.E.D.

THEOREM 4 *If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, prove that the triangles are similar.*

GIVEN Two triangles ABC and DEF in which $\angle A = \angle D$. The bisectors AP and DQ or $\angle A$ and $\angle D$ intersect BC and EF in P and Q respectively such that $\frac{BP}{PC} = \frac{EQ}{QF}$.

TO PROVE $\Delta ABC \sim \Delta DEF$

PROOF We know that the bisectors of an angle of a triangle intersects the opposite side in the ratio of the sides containing the angle.

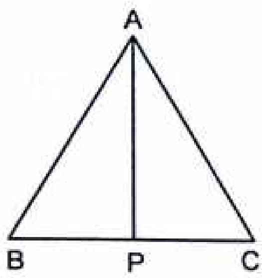


Fig. 7.153

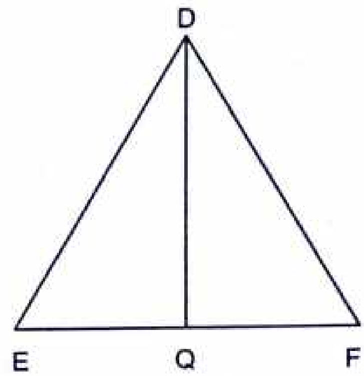


Fig. 7.154

$\therefore$ AP is the bisector of $\angle A$

$$\Rightarrow \quad \frac{BP}{PC} = \frac{AB}{AC} \quad \ldots\text{(i)}$$

DQ is the bisector of $\angle D$

$$\Rightarrow \quad \frac{EQ}{QF} = \frac{DE}{DF} \quad \ldots\text{(ii)}$$

But, $\frac{BP}{PC} = \frac{EQ}{QF}$ [Given]

Therefore, from (i) and (ii), we get

$$\frac{AB}{AC} = \frac{DE}{DF}$$

Thus, in triangles ABC and DEF, we have

$$\frac{AB}{AC} = \frac{DE}{DF}$$

and, $\angle A = \angle D$ [Given]

So, by SAS-criterion of similarity, we obtain : $\Delta ABC \sim \Delta DEF$.

Q.E.D.

THEOREM 5 *If two sides and a median bisecting one of these sides of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.* **[NCERT]**

GIVEN ΔABC and ΔDEF in which AP and DQ are the medians such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AP}{DQ}$$

TO PROVE $\Delta ABC \sim \Delta DEF$

PROOF We have,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AP}{DQ}$$

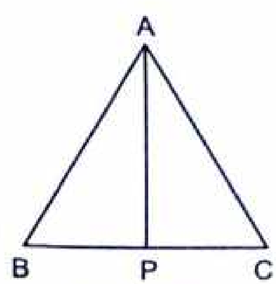


Fig. 7.155

$$\Rightarrow \quad \frac{AB}{DE} = \frac{\frac{1}{2}BC}{\frac{1}{2}EF} = \frac{AP}{DQ}$$

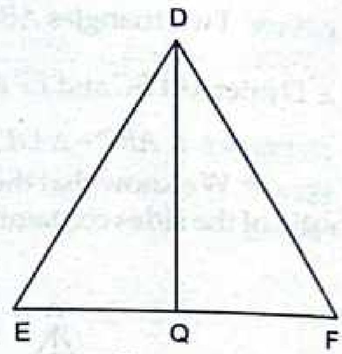


Fig. 7.156

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BP}{EQ} = \frac{AP}{DQ}$$

$\Rightarrow \quad \Delta\, ABP \sim \Delta\, DEQ$ [By *SSS*-similarity]

$\Rightarrow \quad \angle B = \angle E$

Now, in $\Delta\, ABC$ and $\Delta\, DEF$, we have

$\frac{AB}{DE} = \frac{BC}{EF}$ [Given]

and, $\angle B = \angle E$

So, by *SAS*-criterion of similarity, we obtain $\Delta\, ABC \sim \Delta\, DEF$

Q.E.D.

THEOREM 6 *If two sides and a median bisecting the third side of a triangle are respectively proportional to the corresponding sides and the median of another triangle, then the two triangles are similar.*

GIVEN Two triangles *ABC* and *DEF* in which *AP* and *DQ* are the medians such that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{AP}{DQ}$

TO PROVE $\Delta\, ABC \sim \Delta\, DEF$

CONSTRUCTION Produce *AP* to *G* so that $PG = AP$. Join *CG*. Also, produce *DQ* to *H* so that $QH = DQ$. Join *FH*.

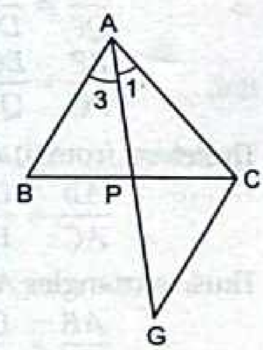


Fig. 7.157

PROOF In $\Delta\, APB$ and $\Delta\, GPC$, we have

$BP = CP$ [$\because$ *AP* is the median]

$AP = GP$ [By construction]

and, $\angle APB = \angle CPG$ [Vertically opposite angles]

So, by *SAS*-criterion of congruence, we have

$\Delta\, APB \cong \Delta\, GPC$

$\Rightarrow \quad AB = GC$ ...(i)

Again, In $\Delta\, DQE$ and $\Delta\, HQF$, we have

$EQ = FQ$ [$\because$ *DQ* is the median]

$DQ = HQ$ [By construction]

and, $\angle DQE = \angle HQF$ [Vertically opposite angles]

So, by *SAS*-criterion of congruence, we have

$\Delta\, DQE \cong \Delta\, HQF$

$\Rightarrow \quad DE = HF$ ...(ii)

Now, $\frac{AB}{DE} = \frac{AC}{DF} = \frac{AP}{DQ}$ [Given]

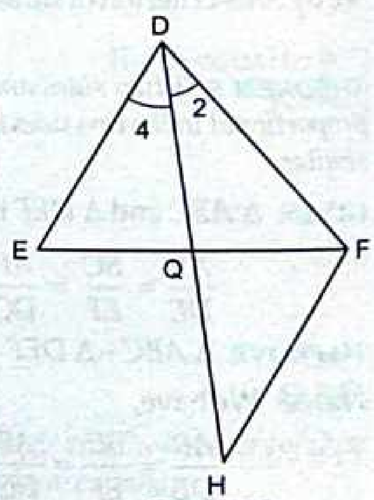


Fig. 7.158

$$\Rightarrow \quad \frac{GC}{HF} = \frac{AC}{DF} = \frac{AP}{DQ} \qquad \left[\begin{array}{l} \because AB = GC \text{ and } DE = HF \\ \text{(From (i) and (ii))} \end{array}\right]$$

$$\Rightarrow \quad \frac{GC}{HF} = \frac{AC}{DF} = \frac{2AP}{2DQ}$$

$$\Rightarrow \quad \frac{GC}{HF} = \frac{AC}{DF} = \frac{AG}{DH} \qquad [\because 2AP = AG \text{ and } 2DQ = DH]$$

$\Rightarrow \quad \Delta AGC \sim \Delta DHF$ [By SSS-criterion of similarity]

$\Rightarrow \quad \angle 1 = \angle 2$

Similarly, we have

$$\angle 3 = \angle 4 \Rightarrow \angle 1 + \angle 3 = \angle 2 + \angle 4 \Rightarrow \angle A = \angle D \qquad \ldots\text{(iii)}$$

Thus, in ΔABC and ΔDEF, we have

$\angle A = \angle D$ [From (iii)]

and, $\frac{AB}{DE} = \frac{AC}{DF}$ [Given]

So, by SAS-criterion of similarity, we obtain $\Delta ABC \sim \Delta DEF$

Q.E.D.

7.9 AREAS OF TWO SIMILAR TRIANGLES

In this section, we will discuss some theorems concerning the ratio of areas of similar triangles.

THEOREM 1 *The ratio of the areas of two similar triangles are equal to the ratio of the squares of any two corresponding sides.* **[NCERT, CBSE 2000, 2002, 2004, 2006C, 2008, 2010]**

GIVEN Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$.

TO PROVE $\frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$

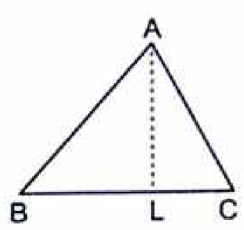


Fig. 7.159

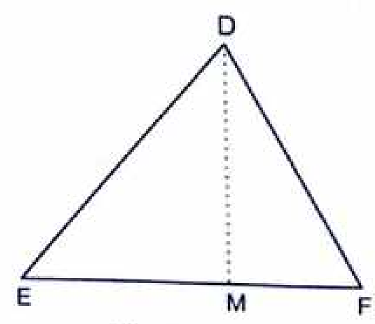


Fig. 7.160

CONSTRUCTION Draw $AL \perp BC$ and $DM \perp EF$.

PROOF Since similar triangles are equiangular and their corresponding sides are proportional. Therefore,

$\Delta ABC \sim \Delta DEF$

$\Rightarrow \quad \angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$ ...(i)

Thus, in $\Delta\, ALB$ and $\Delta\, DME$, we have

$\Rightarrow \quad \angle ALB = \angle DME$ [Each equal to 90°]

and, $\quad \angle B = \angle E$ [From (i)]

So, by AA-criterion of similarity, we have

$$\Delta\, ALB \sim \Delta\, DME$$

$$\Rightarrow \quad \frac{AL}{DM} = \frac{AB}{DE} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AL}{DM} \quad \text{...(iii)}$$

Now,

$$\frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{\frac{1}{2}(BC \times AL)}{\frac{1}{2}(EF \times DM)}$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{BC}{EF} \times \frac{AL}{DM}$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{BC}{EF} \times \frac{BC}{EF} \quad \left[\text{From (iii)}, \frac{BC}{EF} = \frac{AL}{DM}\right]$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{BC^2}{EF^2}$$

But, $\quad \dfrac{BC}{EF} = \dfrac{AB}{DE} = \dfrac{AC}{DF}$

$$\Rightarrow \quad \frac{BC^2}{EF^2} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2}$$

Hence, $\quad \dfrac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \dfrac{AB^2}{DE^2} = \dfrac{BC^2}{EF^2} = \dfrac{AC^2}{DF^2}$

Q.E.D.

THEOREM 2 *The areas of two similar triangles are in the ratio of the squares of the corresponding altitudes.*

GIVEN Two triangles ABC and DEF such that $\Delta\, ABC \sim \Delta\, DEF$ and $AL \perp BC$, $DM \perp EF$.

TO PROVE $\dfrac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \dfrac{AL^2}{DM^2}$

PROOF Since the ratio of the areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

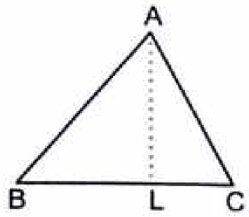


Fig. 7.161

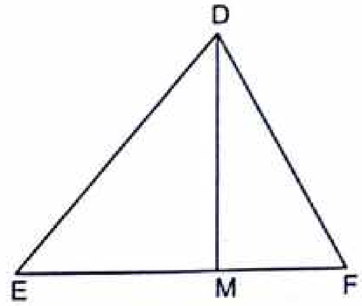


Fig. 7.162

$\therefore \qquad \dfrac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \dfrac{AB^2}{DE^2}$ ...(i)

Now, in ΔALB and ΔDME, we have

$\angle ALB = \angle DME$ [Each equal to 90°]

and, $\angle B = \angle E$ [$\because \Delta ABC \sim \Delta DEF \therefore \angle A = \angle D, \angle B = \angle E, \angle C = \angle F$]

So, by *AA*-criterion of similarity, we have

$\Delta ALB \sim \Delta DME$

$\Rightarrow \qquad \dfrac{AB}{DE} = \dfrac{AL}{DM}$

$\Rightarrow \qquad \dfrac{AB^2}{DE^2} = \dfrac{AL^2}{DM^2}$ ...(ii)

From (i) and (ii), we get

$\dfrac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \dfrac{AL^2}{DM^2}$ **Q.E.D.**

THEOREM 3 *The areas of two similar triangles are in the ratio of the squares of the corresponding medians.* **[NCERT]**

GIVEN Two triangles *ABC* and *DEF* such that $\Delta ABC \sim \Delta DEF$ and *AP*, *DQ* are their medians.

TO PROVE $\dfrac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \dfrac{AP^2}{DQ^2}$

PROOF Since the ratio of the areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

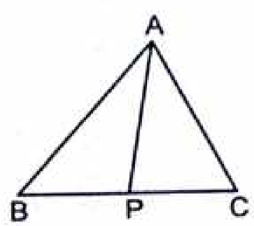


Fig. 7.163

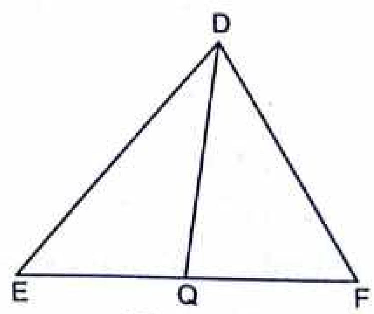


Fig. 7.164

$\therefore \qquad \dfrac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \dfrac{AB^2}{DE^2}$ ...(i)

Now, $\qquad \Delta ABC \sim \Delta DEF$

$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{EF}$

$\Rightarrow \quad \frac{AB}{DE} = \frac{2BP}{2EQ} = \frac{BP}{EQ}$ ...(ii)

Thus, in triangles APB and DQE, we have

$\frac{AB}{DE} = \frac{BP}{EQ}$ and $\angle B = \angle E$ $[\because \Delta\, ABC \sim \Delta\, DEF]$

So, by SAS-criterion of similarity, we have

$\Delta\, APB \sim \Delta\, DQE$

$\Rightarrow \quad \frac{BP}{EQ} = \frac{AP}{DQ}$ ...(iii)

From (ii) and (iii), we get

$$\frac{AB}{DE} = \frac{AP}{DQ}$$

$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{AP^2}{DQ^2}$ ...(iv)

From (i) and (iv), we get

$$\frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{AP^2}{DQ^2}$$

Q.E.D.

THEOREM 4 *The areas of two similar triangles are in the ratio of the squares of the corresponding angle bisector segments.*

GIVEN $\Delta\, ABC \sim \Delta\, DEF$ and AX and DY are bisector of $\angle A$ and $\angle D$ respectively.

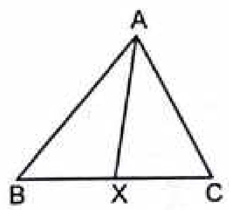


Fig. 7.165

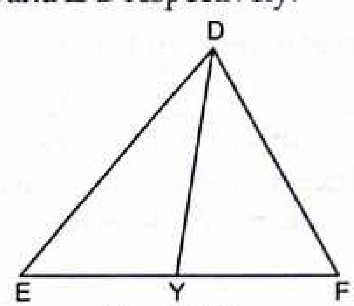


Fig. 7.166

TO PROVE $\frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{AX^2}{DY^2}$

PROOF Since the ratio of the areas of two similar triangles are equal to the ratio of the squares of any two corresponding sides.

$\therefore \quad \frac{\text{Area}\,(\Delta\, ABC)}{\text{Area}\,(\Delta\, DEF)} = \frac{AB^2}{DE^2}$ ...(i)

Now, $\quad \Delta\, ABC \sim \Delta\, DEF$

$\Rightarrow \quad \angle A = \angle D$

$\Rightarrow \quad \frac{1}{2}\angle A = \frac{1}{2}\angle D$

$\Rightarrow \quad \angle BAX = \angle EDY$

Thus, in triangles ABX and DEY, we have

$\angle BAX = \angle EDY$ and $\angle B = \angle E$ $\quad [\because \Delta ABC \sim \Delta DEF]$

So, by AA-similarity criterion, we have

$\Delta ABX \sim \Delta DEY$

$\Rightarrow \quad \frac{AB}{DE} = \frac{AX}{DY}$

$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{AX^2}{DY^2}$ ...(ii)

From (i) and (ii), we get

$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AP^2}{DY^2}$ **Q.E.D.**

THEOREM 5 *If the areas of two similar triangles are equal, then the triangles are congruent i.e. equal and similar triangles are congruent.* **[NCERT, CBSE 2002C, 2018]**

GIVEN Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$ and Area (ΔABC) = Area (ΔDEF).

TO PROVE $\Delta ABC \cong \Delta DEF$

PROOF We have,

$\Delta ABC \sim \Delta DEF \Rightarrow \angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

In order to prove that $\Delta ABC \cong \Delta DEF$, it is sufficient to show that $AB = DE$, $BC = EF$ and $AC = DF$.

It is given that: Area (ΔABC) = Area (ΔDEF)

$\Rightarrow \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = 1$

$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2} = 1$ $\quad \left[\because \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}\right]$

$\Rightarrow \quad AB^2 = DE^2,\ BC^2 = EF^2$ and $AC^2 = DF^2$

$\Rightarrow \quad AB = DE, BC = EF$ and $AC = DF$

Hence, $\quad \Delta ABC \cong \Delta DEF$. **Q.E.D.**

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If $\Delta ABC \sim \Delta DEF$ such that $AB = 1.2$ cm and $DE = 1.4$ cm. Find the ratio of areas of ΔABC and ΔDEF.*

SOLUTION We know that the ratio of areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{AB^2}{DE^2} \Rightarrow \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{(1.2)^2}{(1.4)^2} = \left(\frac{12}{14}\right)^2 = \frac{36}{49}$$

EXAMPLE 2 *In two similar triangles ABC and PQR, if their corresponding altitudes AD and PS are in the ratio 4 : 9, find the ratio of the areas of Δ ABC and Δ PQR.*

SOLUTION Since the areas of two similar triangles are in the ratio of the squares of the corresponding altitudes.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AD^2}{PS^2} \Rightarrow \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \left(\frac{4}{9}\right)^2 = \frac{16}{81} \qquad [\because AD : PS = 4 : 9]$$

Hence, Area (Δ ABC) : Area (Δ PQR) = 16 : 81

EXAMPLE 3 *If Δ ABC is similar to Δ DEF such that BC = 3 cm, EF = 4 cm and area of Δ ABC = 54 cm^2. Determine the area of Δ DEF.*

SOLUTION Since the ratio of areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

$$\therefore \quad \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \quad \frac{54}{\text{Area}(\Delta DEF)} = \frac{3^2}{4^2} \Rightarrow \text{Area}(\Delta DEF) = \frac{54 \times 16}{9} = 96 \text{ cm}^2$$

EXAMPLE 4 *If Δ ABC ~ Δ DEF such that area of Δ ABC is 9 cm^2 and the area of Δ DEF is 16 cm^2 and BC = 2.1 cm. Find the length of EF.*

SOLUTION We have,

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \quad \frac{9}{16} = \frac{(2.1)^2}{EF^2} \Rightarrow \frac{3}{4} = \frac{2.1}{EF} \Rightarrow EF = \frac{4 \times 2.1}{3} \text{ cm} = 2.8 \text{ cm}$$

EXAMPLE 5 *In Fig. 7.166, PB and QA are perpendiculars to segment AB. If PO = 5 cm, QO = 7 cm and Area Δ POB = 150 cm^2 find the area of Δ QOA.*

SOLUTION In Δ OAQ and Δ OBP, we have

$\angle A = \angle B$ [Each equal to 90°]

$\angle AOQ = \angle BOP$

So, by AA-criterion of similarity, we have

$\Delta AOQ \sim \Delta BOP$

$$\Rightarrow \quad \frac{\text{Area }(\Delta\, AOQ)}{\text{Area }(\Delta\, BOP)} = \frac{OQ^2}{OP^2}$$

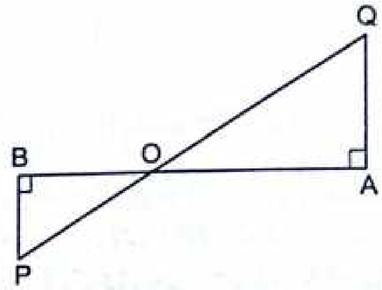


Fig. 7.167

$$\Rightarrow \quad \frac{\text{Area }(\Delta\, AOQ)}{150} = \frac{7^2}{5^2}$$

$$\Rightarrow \quad \text{Area }(\Delta\, AOQ) = \frac{49}{25} \times 150 \text{ cm}^2 = 294 \text{ cm}^2$$

EXAMPLE 6 *In Fig. 7.168, ABCD is a trapezium in which AB || DC and AB = 2 DC. Determine the ratio of the areas of Δ AOB and Δ COD.* **[NCERT]**

SOLUTION In triangle AOB and COD, we have

$\angle AOB = \angle COD$ [Vertically opposite angles]

and, $\angle OAB = \angle OCD$ [Alternate angles]

So, by AA-criterion of similarity, we have

$$\Delta\, AOB \sim \Delta\, COD$$

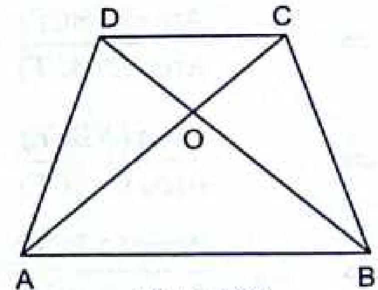


Fig. 7.168

$$\Rightarrow \quad \frac{\text{Area }(\Delta\, AOB)}{\text{Area }(\Delta\, COD)} = \frac{AB^2}{DC^2}$$

$$\Rightarrow \quad \frac{\text{Area }(\Delta\, AOB)}{\text{Area }(\Delta\, COD)} = \frac{(2\,DC)^2}{(DC)^2} = \frac{4}{1}$$

Hence, Area $(\Delta\, AOB)$: Area $(\Delta\, COD) = 4:1$.

EXAMPLE 7 *In the trapezium ABCD, AB || CD and AB = 2 CD. If the area of Δ AOB = 84 cm², find the area of Δ COD.* **[CBSE 2005]**

SOLUTION From example 6, we have

$$\frac{\text{Area }(\Delta\, AOB)}{\text{Area }(\Delta\, COD)} = \frac{4}{1}$$

$$\Rightarrow \quad \frac{84}{\text{Area }(\Delta\, COD)} = \frac{4}{1} \Rightarrow \text{Area }(\Delta\, COD) = 21 \text{ cm}^2$$

LEVEL-2

EXAMPLE 8 *Prove that the area of the triangle BCE described on one side BC of a square ABCD as base is one half the area of the similar triangle ACF described on the diagonal AC as base.*

SOLUTION $ABCD$ is a square. $\Delta\, BCE$ is described on side BC is similar to $\Delta\, ACF$ described on diagonal AC.

Since $ABCD$ is a square. Therefore,

$AB = BC = CD = DA$ and, $AC = \sqrt{2}\, BC$ [$\because$ Diagonal $= \sqrt{2}$ Side)]

Now, $\Delta\, BCE \sim \Delta\, ACF$

$$\Rightarrow \quad \frac{\text{Area }(\Delta\, BCE)}{\text{Area }(\Delta\, ACF)} = \frac{BC^2}{AC^2}$$

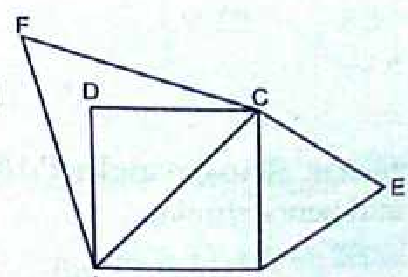


Fig. 7.169

$$\Rightarrow \qquad \frac{\text{Area}(\Delta\, BCE)}{\text{Area}(\Delta\, ACF)} = \frac{BC^2}{(\sqrt{2}BC)^2} = \frac{1}{2}$$

$$\Rightarrow \qquad \text{Area}(\Delta\, BCE) = \frac{1}{2}\,\text{Area}(\Delta\, ACF)$$

EXAMPLE 9 *Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.* **[NCERT, CBSE 2018]**

GIVEN A square $ABCD$. Equilateral triangles $\Delta\, BCE$ and $\Delta\, ACF$ have been described on side BC and diagonal AC respectively.

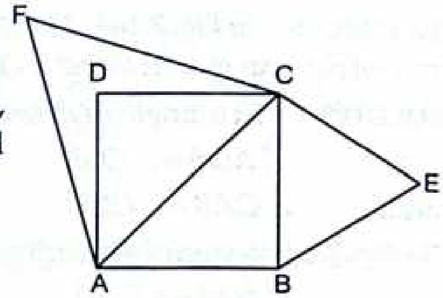


Fig. 7.170

TO PROVE Area ($\Delta\, BCE$) = $\frac{1}{2}$. Area ($\Delta\, ACF$)

PROOF Since $\Delta\, BCE$ and $\Delta\, ACF$ are equilateral. Therefore, they are equiangular (each angle being equal to 60°) and hence

$$\Delta\, BCE \sim \Delta\, ACF$$

$$\Rightarrow \qquad \frac{\text{Area}(\Delta\, BCE)}{\text{Area}(\Delta\, ACF)} = \frac{BC^2}{AC^2}$$

$$\Rightarrow \qquad \frac{\text{Area}(\Delta\, BCE)}{\text{Area}(\Delta\, ACF)} = \frac{BC^2}{(\sqrt{2}BC)^2} \qquad \left[\begin{array}{l}\because ABCD \text{ is a square} \\ \therefore \text{diagonal} = \sqrt{2}(\text{side}) \Rightarrow AC = \sqrt{2}BC\end{array}\right]$$

$$\Rightarrow \qquad \frac{\text{Area}(\Delta\, BCE)}{\text{Area}(\Delta\, ACF)} = \frac{1}{2}$$

EXAMPLE 10 *Equilateral triangles are drawn on the sides of a right triangle. Show that the area of the triangle on the hypotenuse is equal to the sum of the areas of triangles on the other two sides.*

[CBSE 2002]

GIVEN A right angled triangle ABC with right angle at B. Equilateral triangles PAB, QBC and RAC are described on sides AB, BC and CA respectively.

TO PROVE Area ($\Delta\, PAB$) + Area ($\Delta\, QBC$) = Area ($\Delta\, RAC$)

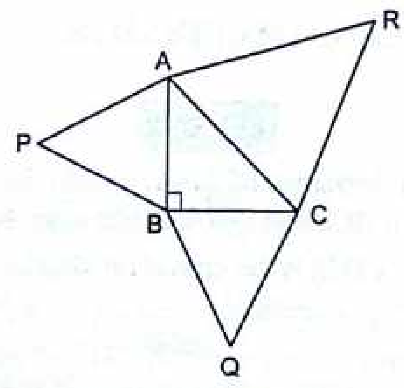


Fig. 7.171

PROOF Since triangles PAB, QBC and RAC are equilateral. Therefore, they are equiangular and hence similar.

$$\therefore \qquad \frac{\text{Area}(\Delta\, PAB)}{\text{Area}(\Delta\, RAC)} + \frac{\text{Area}(\Delta\, QBC)}{\text{Area}(\Delta\, RAC)} = \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta PAB)}{\text{Area}(\Delta RAC)} + \frac{\text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = \frac{AB^2 + BC^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta PAB)}{\text{Area}(\Delta RAC)} + \frac{\text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = \frac{AC^2}{AC^2} = 1 \quad \left[\begin{array}{l}\because \Delta ABC \text{ is a right angled triangle} \\ \text{with } \angle B = 90^\circ \therefore AC^2 = AB^2 + BC^2\end{array}\right]$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta PAB) + \text{Area}(\Delta QBC)}{\text{Area}(\Delta RAC)} = 1$$

$$\Rightarrow \quad \text{Area}(\Delta PAB) + \text{Area}(\Delta QBC) = \text{Area}(\Delta RAC)$$

EXAMPLE 11 *D, E, F are the mid-points of the sides BC, CA and AB respectively of a ΔABC. Determine the ratio of the areas of ΔDEF and ΔABC.* **[NCERT]**

SOLUTION Since D and E are the mid-points of the sides BC and AB respectively of ΔABC.

$\therefore \quad DE \parallel BA \Rightarrow DE \parallel FA$ ...(i)

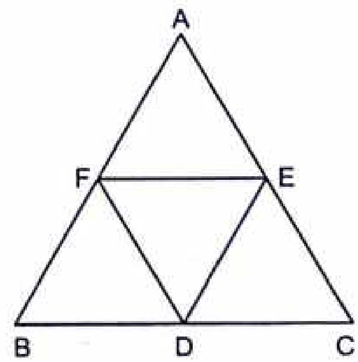


Fig. 7.172

Since D and F are mid-points of the sides BC and AB respectively of ΔABC. Therefore,

$DF \parallel CA \Rightarrow DF \parallel AE$ ...(ii)

From (i), and (ii), we conclude that $AFDE$ is a parallelogram.

Similarly, $BDEF$ is a parallelogram.

In ΔDEF and ΔABC, we have

$\angle FDE = \angle A$ [Opposite angles of parallelogram $AFDE$]

and, $\angle DEF = \angle B$ [Opposite angles of parallelogram $BDEF$]

So, by AA-similarity criterion, we have

$\Delta DEF \sim \Delta ABC$

$$\Rightarrow \quad \frac{\text{Area}(\Delta DEF)}{\text{Area}(\Delta ABC)} = \frac{DE^2}{AB^2} = \frac{(1/2AB)^2}{AB^2} = \frac{1}{4} \quad \left[\because DE = \frac{1}{2}AB\right]$$

Hence, $\text{Area}(\Delta DEF) : \text{Area}(\Delta ABC) = 1 : 4$

EXAMPLE 12 *D and E are points on the sides AB and AC respectively of a ΔABC such that $DE \parallel BC$ and divides ΔABC into two parts, equal in area, Find $\frac{BD}{AB}$.* **[NCERT, CBSE 2000]**

SOLUTION We have,

$$\text{Area}(\Delta ADE) = \text{Area (trapezium } BCED)$$

$$\Rightarrow \quad \text{Area}(\Delta ADE) + \text{Area}(\Delta ADE) = \text{Area (trapezium } BCED) + \text{Area}(\Delta ADE)$$

$$\Rightarrow \quad 2\,\text{Area}(\Delta ADE) = \text{Area}(\Delta ABC) \qquad \ldots(i)$$

In ΔADE and ΔABC, we have

$\angle ADE = \angle B$ [$\because DE \parallel BC \therefore \angle ADE = \angle B$ (Corresponding angles)]

and, $\angle A = \angle A$ [Common]

$\therefore$ $\Delta ADE \sim \Delta ABC$

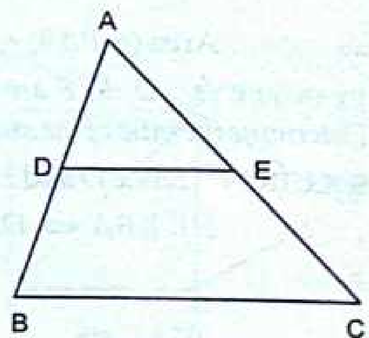


Fig. 7.173

$$\Rightarrow \quad \frac{\text{Area}(\Delta ADE)}{\text{Area}(\Delta ABC)} = \frac{AD^2}{AB^2}$$

$$\Rightarrow \quad \frac{\text{Area}(\Delta ADE)}{2\,\text{Area}(\Delta ADE)} = \frac{AD^2}{AB^2}$$

$$\Rightarrow \quad \frac{1}{2} = \left(\frac{AD}{AB}\right)^2$$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad AB = \sqrt{2}AD$$

$$\Rightarrow \quad AB = \sqrt{2}(AB - BD)$$

$$\Rightarrow \quad (\sqrt{2}-1)AB = \sqrt{2}BD \Rightarrow \frac{BD}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$$

EXAMPLE 13 *Two isosceles triangles have equal vertical angles and their areas are in the ratio 16 : 25. Find the ratio of their corresponding heights.* **[CBSE 2000]**

SOLUTION Let ΔABC and ΔDEF be the given triangles such that $AB = AC$ and $DE = DF$, $\angle A = \angle D$

and, $$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DEF)} = \frac{16}{25} \qquad \ldots(i)$$

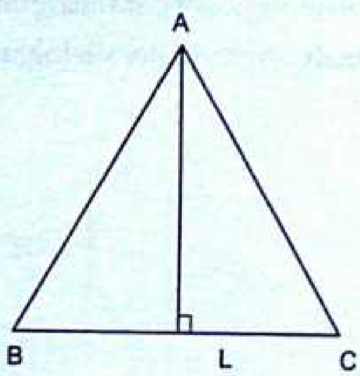


Fig. 7.174

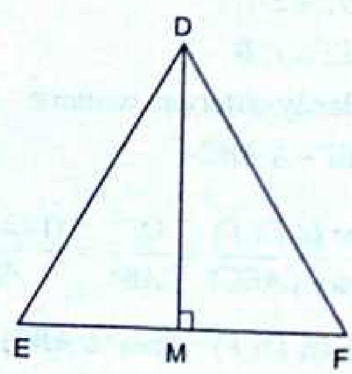


Fig. 7.175

Draw $AL \perp BC$ and $DM \perp EF$.

Now, $AB = AC, DE = DF$

$$\Rightarrow \quad \frac{AB}{AC} = 1 \text{ and } \frac{DE}{DF} = 1$$

$$\Rightarrow \quad \frac{AB}{AC} = \frac{DE}{DF}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AC}{DF}$$

Thus, in triangles ABC and DEF, we have

$$\frac{AB}{DE} = \frac{AC}{DF} \text{ and } \angle A = \angle D \qquad \text{[Given]}$$

So, by SAS-similarity criterion, we have

$$\Delta ABC \sim \Delta DEF$$

$$\Rightarrow \quad \frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)} = \frac{AL^2}{DM^2}$$

$$\Rightarrow \quad \frac{16}{25} = \frac{AL^2}{DM^2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{AL}{DM} = \frac{4}{5}$$

Hence, $\quad AL : DM = 4 : 5.$

EXAMPLE 14 *In Fig 7.176, $DE \parallel BC$ and $AD : DB = 5 : 4$. Find* $\frac{\text{Area }(\Delta DEF)}{\text{Area }(\Delta CFB)}$. **[CBSE 2000]**

SOLUTION In ΔABC, we have

$$DE \parallel BC$$

$$\Rightarrow \quad \angle ADE = \angle ABC \text{ and } \angle AED = \angle ACB \qquad \text{[Corresponding angles]}$$

Thus, in triangles ADE and ABC, we have

$$\angle A = \angle A \qquad \text{[Common]}$$

$$\angle ADE = \angle ABC$$

and, $\quad \angle AED = \angle ACB$

$$\therefore \quad \Delta AED \sim \Delta ABC \qquad \text{[By AAA similarity]}$$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$$

We have,

$$\frac{AD}{DB} = \frac{5}{4}$$

$$\Rightarrow \quad \frac{DB}{AD} = \frac{4}{5}$$

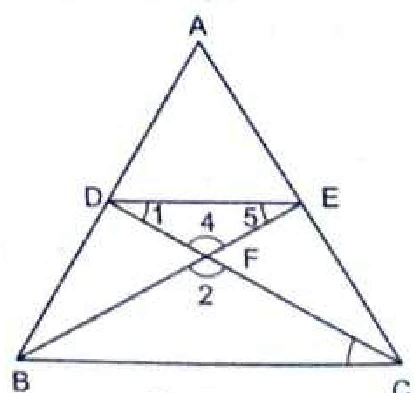


Fig. 7.176

$$\Rightarrow \quad \frac{DB}{AD} + 1 = \frac{4}{5} + 1$$

$$\Rightarrow \quad \frac{DB + AD}{AD} = \frac{9}{5} \Rightarrow \frac{AB}{AD} = \frac{9}{5} \Rightarrow \frac{AD}{AB} = \frac{5}{9}$$

$\therefore \quad \frac{DE}{BC} = \frac{5}{9}$

In $\Delta\, DFE$ and $\Delta\, CFB$, we have

$\angle 1 = \angle 3$ [Alternate interior angles]

$\angle 2 = \angle 4$ [Vertically opposite angles]

Therefore, by AA-similarity criterion, we have

$\Delta\, DFE \sim \Delta\, CFB$

$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, DFE)}{\text{Area}\,(\Delta\, CFB)} = \frac{DE^2}{BC^2}$

$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, DFE)}{\text{Area}\,(\Delta\, CFB)} = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$ [Using (i)]

EXAMPLE 15 *In Fig. 7.177, $XY \parallel AC$ and XY divides triangular region ABC into two parts equal in area. Determine $\frac{AX}{AB}$.* **[CBSE 2008]**

SOLUTION We have,

$XY \parallel AC$

and, $\text{Area}\,(\Delta\, BXY) = \text{Area}\,(\text{quad. } XYCA)$

$\Rightarrow \quad \text{Area}\,(\Delta\, ABC) = 2\,\text{Area}\,(\Delta\, BXY)$ ...(i)

Now, $XY \parallel AC$ and BA is a transversal.

$\Rightarrow \quad \angle BXY = \angle BAC$ ...(ii)

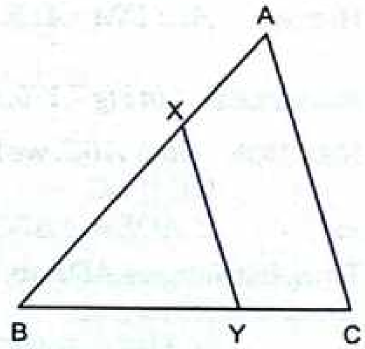


Fig. 7.177

Thus, in Δ's BAC and BXY, we have

$\angle XBY = \angle ABC$ [Common]

$\angle BXY = \angle BAC$ [From (ii)]

Therefore, AA-criterion of similarity, we have

$\Delta\, BAC \sim \Delta\, BXY$

$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, BAC)}{\text{Area}\,(\Delta\, BXY)} = \frac{BA^2}{BX^2}$

$\Rightarrow \quad 2 = \frac{BA^2}{BX^2}$ [Using (i)]

$\Rightarrow \quad BA = \sqrt{2} BX$

$\Rightarrow \quad BA = \sqrt{2}\,(BA - AX) \Rightarrow (\sqrt{2} - 1)\,BA = \sqrt{2}\,AX \Rightarrow \frac{AX}{AB} = \frac{\sqrt{2} - 1}{\sqrt{2}}$

EXERCISE 7.6

LEVEL-1

1. Triangles ABC and DEF are similar.
   (i) If area $(\Delta\, ABC) = 16\text{ cm}^2$, area $(\Delta\, DEF) = 25\text{ cm}^2$ and $BC = 2.3$ cm, find EF.
   (ii) If area $(\Delta\, ABC) = 9\text{ cm}^2$, area $(\Delta\, DEF) = 64\text{ cm}^2$ and $DE = 5.1$ cm, find AB.

(iii) If $AC = 19$ cm and $DF = 8$ cm, find the ratio of the area of two triangles.

(iv) If area $(\Delta ABC) = 36\text{ cm}^2$, area $(\Delta DEF) = 64\text{ cm}^2$ and $DE = 6.2$ cm, find AB.

(v) If $AB = 1.2$ cm and $DE = 1.4$ cm, find the ratio of the areas of ΔABC and ΔDEF.

2. In Fig. 7.178, $\Delta ACB \sim \Delta APQ$. If $BC = 10$ cm, $PQ = 5$ cm, $BA = 6.5$ cm and $AP = 2.8$ cm, find CA and AQ. Also, find the area (ΔACB) : area (ΔAPQ).

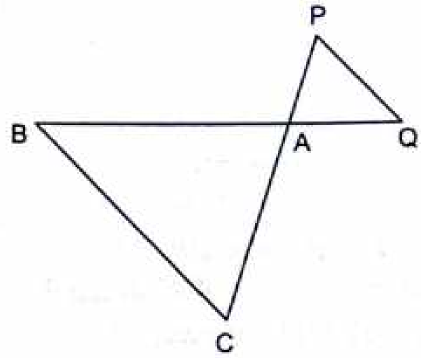


Fig. 7.178

3. The areas of two similar triangles are 81 cm^2 and 49 cm^2 respectively. Find the ratio of their corresponding heights. What is the ratio of their corresponding medians?
4. The areas of two similar triangles are 169 cm^2 and 121 cm^2 respectively. If the longest side of the larger triangle is 26 cm, find the longest side of the smaller triangle.
5. The areas of two similar triangles are 25 cm^2 and 36 cm^2 respectively. If the altitude of the first triangle is 2.4 cm, find the corresponding altitude of the other.
6. The corresponding altitudes of two similar triangles are 6 cm and 9 cm respectively. Find the ratio of their areas.
7. ABC is a triangle in which $\angle A = 90°$, $AN \perp BC$, $BC = 12$ cm and $AC = 5$ cm. Find the ratio of the areas of ΔANC and ΔABC.
8. In Fig. 7.179, $DE \parallel BC$

(i) If $DE = 4$ cm, $BC = 6$ cm and Area $(\Delta ADE) = 16\text{ cm}^2$, find the area of ΔABC.

(ii) If $DE = 4$ cm, $BC = 8$ cm and Area $(\Delta ADE) = 25\text{ cm}^2$, find the area of ΔABC.

(iii) If $DE : BC = 3 : 5$. Calculate the ratio of the areas of ΔADE and the trapezium $BCED$.

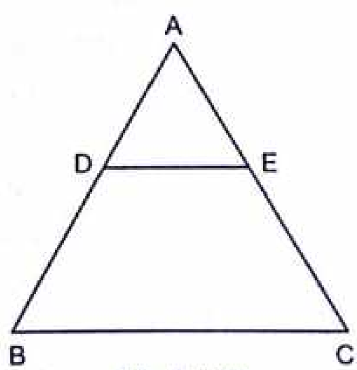


Fig. 7.179

9. In ΔABC, D and E are the mid-points of AB and AC respectively. Find the ratio of the areas of ΔADE and ΔABC.
10. The areas of two similar triangles are 100 cm^2 and 49 cm^2 respectively. If the altitude of the bigger triangle is 5 cm, find the corresponding altitude of the other. **[CBSE 2002]**

11. The areas of two similar triangles are 121 cm^2 and 64 cm^2 respectively. If the median of the first triangle is 12.1 cm, find the corresponding median of the other. **[CBSE 2001]**
12. If $\Delta ABC \sim \Delta DEF$ such that $AB = 5$ cm, area $(\Delta ABC) = 20$ cm^2 and area $(\Delta DEF) = 45$ cm^2, determine DE.
13. In ΔABC, PQ is a line segment intersecting AB at P and AC at Q such that $PQ \parallel BC$ and PQ divides ΔABC into two parts equal in area. Find $\dfrac{BP}{AB}$.
14. The areas of two similar triangles ABC and PQR are in the ratio $9:16$. If $BC = 4.5$ cm, find the length of QR. **[CBSE 2004]**
15. ABC is a triangle and PQ is a straight line meeting AB in P and AC in Q. If $AP = 1$ cm, $PB = 3$ cm, $AQ = 1.5$ cm, $QC = 4.5$ m, prove that area of ΔAPQ is one-sixteenth of the area of ΔABC. **[CBSE 2005]**
16. If D is a point on the side AB of ΔABC such that $AD : DB = 3.2$ and E is a point on BC such that $DE \parallel AC$. Find the ratio of areas of ΔABC and ΔBDE. **[CBSE 2006 C]**
17. If ΔABC and ΔBDE are equilateral triangles, where D is the mid point of BC, find the ratio of areas of ΔABC and ΔBDE. **[CBSE 2010]**

LEVEL-2

18. Two isosceles triangles have equal vertical angles and their areas are in the ratio 36 : 25. Find the ratio of their corresponding heights.
19. In Fig. 7.180, ΔABC and ΔDBC are on the same base BC. If AD and BC intersect at O, prove that **[NCERT, CBSE 2000, 2005]**

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta DBC)} = \frac{AO}{DO}$$

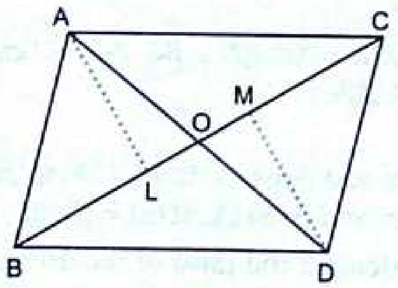


Fig. 7.180

20. $ABCD$ is a trapezium in which $AB \parallel CD$. The diagonals AC and BD intersect at O. Prove that: (i) $\Delta AOB \sim \Delta COD$ (ii) If $OA = 6$ cm, $OC = 8$ cm,
Find:

(a) $\dfrac{\text{Area}(\Delta AOB)}{\text{Area}(\Delta COD)}$ (b) $\dfrac{\text{Area}(\Delta AOD)}{\text{Area}(\Delta COD)}$

21. In ΔABC, P divides the side AB such that $AP : PB = 1 : 2$. Q is a point in AC such that $PQ \parallel BC$. Find the ratio of the areas of ΔAPQ and trapezium $BPQC$.
22. AD is an altitude of an equilateral triangle ABC. On AD as base, another equilateral triangle ADE is constructed. Prove that Area (ΔADE) : Area (ΔABC) = 3 : 4. **[CBSE 2010]**

ANSWERS

1. (i) 2.875 cm (ii) 1.9125 cm (iii) 361 : 64 (iv) 4.65 (v) 36 : 49
2. 5.6 cm, 3.25 cm, 4 : 1
3. 9 : 7 ; 9 : 7
4. 22 cm

5. 2.88 cm 6. $4:9$ 7. $25:144$
8. (i) 36 cm^2, (ii) 100 cm^2, (iii) $9:16$ 9. $1:4$
10. 3.5 cm 11. 8.8 cm 12. 7.5 cm
13. $\dfrac{\sqrt{2}-1}{\sqrt{2}}$ 14. 6 cm 16. $25:4$ 17. 4:1
18. $6:5$ 20. (a) $\dfrac{9}{16}$ (b) $\dfrac{3}{4}$ 21. $1:8$

HINT TO SELECTED PROBLEMS

3. Since the ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding altitudes and is also equal to the ratio of the squares of their corresponding medians. Hence,

ratio of altitudes $= 9:7 =$ ratio of medians.

4. We have,

$$\frac{169}{121} = \frac{26^2}{(\text{Side of the smaller triangle})^2}$$

$$\Rightarrow \frac{13}{11} = \frac{26}{\text{Side of the smaller triangle}} \Rightarrow \text{Side} = 22\text{ cm}$$

7. We have,

$$\Delta ANC \sim \Delta ABC \Rightarrow \frac{\text{Area }(\Delta ANC)}{\text{Area }(\Delta ABC)} = \frac{AC^2}{BC^2} = \frac{25}{144}$$

8. (i) In ΔADE and ΔABC, we have

$\angle ADE = \angle B$ [Corresponding angles ($\because DE \parallel BC$)]

$\angle A = \angle A$ [Common]

$$\therefore \quad \Delta ADE \sim \Delta ABC \Rightarrow \frac{\text{Area }(\Delta ADE)}{\text{Area }(\Delta ABC)} = \frac{DE^2}{BC^2}$$

(iii) $$\frac{\text{Area }(\Delta ADE)}{\text{Area }(\Delta ABC)} = \frac{DE^2}{BC^2} = \frac{3^2}{5^2} = \frac{9}{25}$$

Let Area $(\Delta ADE) = 9x$ sq. units and Area $(\Delta ABC) = 25x$ sq. units.

$\therefore$ Area (trap.$BCED$) $=$ Area $(\Delta ABC) -$ Area $(\Delta ADE) = 25x - 9x = 16x$

9. Since D and E are the mid-points of AB and AC respectively. Therefore, $DE \parallel BC$. Consequently, we have

$$\Delta ADE \sim \Delta ABC \Rightarrow \frac{\text{Area }(\Delta ADE)}{\text{Area }(\Delta ABC)} = \frac{AD^2}{AB^2} = \frac{AD^2}{(2AD)^2} = \frac{1}{4}$$

19. Draw $AL \perp BC$ and $DM \perp BC$. In ΔALO and ΔDMO, we have

$\angle ALO = \angle DMO = 90^\circ$ and, $\angle AOL = \angle DOM$

$\therefore \quad \Delta ALO \sim \Delta DMO$

$$\Rightarrow \quad \frac{AL}{DM} = \frac{AO}{DO} \quad \text{...(i)}$$

$$\therefore \quad \frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DBC)} = \frac{(1/2)\,BC \times AL}{(1/2)\,BC \times DM} = \frac{AL}{DM} = \frac{AO}{DO} \quad \text{[Using (i)]}$$

7.10 PYTHAGORAS THEOREM

In this section, we shall prove an important theorem known as Pythagoras Theorem. This Theorem is also known as Baudhayan Theorem.

THEOREM 1 *In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.* **[NCERT, CBSE 2001, 2002, 2004, 2005, 2006C, 2009, 2010, 2018]**

GIVEN A right-angled triangle ABC in which $\angle B = 90°$.

TO PROVE (Hypotenuse)^2 = (Base)^2 + (Perpendicular)^2 i.e. $AC^2 = AB^2 + BC^2$.

CONSTRUCTION From B draw $BD \perp AC$.

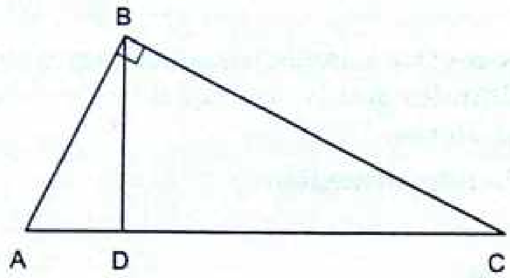


Fig. 7.181

PROOF In triangles ADB and ABC, we have

$\angle ADB = \angle ABC$ [Each equal to 90°]

and, $\angle A = \angle A$ [Common]

So, by AA-similarity criterion, we have

$\Delta ADB \sim \Delta ABC$

$\Rightarrow \dfrac{AD}{AB} = \dfrac{AB}{AC}$ [$\because$ In similar triangles corresponding sides are proportional]

$\Rightarrow AB^2 = AD \times AC$ ...(i)

In triangles BDC and ABC, we have

$\angle CDB = \angle ABC$ [Each equal to 90°]

and, $\angle C = \angle C$ [Common]

So, by AA-similarity criterion, we have

$\Delta BDC \sim \Delta ABC$

$\Rightarrow \dfrac{DC}{BC} = \dfrac{BC}{AC}$ [$\because$ In similar triangles corresponding sides are proportional]

$\Rightarrow BC^2 = AC \times DC$ ...(ii)

Adding equations (i) and (ii), we get

$AB^2 + BC^2 = AD \times AC + AC \times DC$

$\Rightarrow AB^2 + BC^2 = AC(AD + DC)$

$\Rightarrow AB^2 + BC^2 = AC \times AC$

$\Rightarrow AB^2 + BC^2 = AC^2$

Hence, $AC^2 = AB^2 + BC^2$

Q.E.D.

The converse of the above theorem is also true as proved below.

THEOREM 2 *(Converse of Pythagoras Theorem) In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the side is a right angle.*

[NCERT, CBSE 2000C, 2006C, 2009, 2010]

GIVEN A triangle ABC such that $AC^2 = AB^2 + BC^2$.

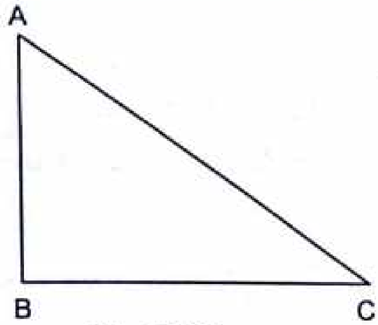


Fig. 7.182

Fig. 7.183

CONSTRUCTION Construct a triangle DEF such that $DE = AB$, $EF = BC$ and $\angle E = 90°$.

PROOF In order to prove that $\angle B = 90°$, it is sufficient to show that $\Delta ABC \sim \Delta DEF$. For this we proceed as follows:

Since ΔDEF is a right angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$$DF^2 = DE^2 + EF^2$$

$\Rightarrow \quad DF^2 = AB^2 + BC^2$ $\quad$ [$\because DE = AB$ and $EF = BC$ (By construction)]

$\Rightarrow \quad DF^2 = AC^2$ $\quad$ [$\because AB^2 + BC^2 = AC^2$ (Given)]

$\Rightarrow \quad DF = AC$ $\quad$ ...(i)

Thus, in ΔABC and ΔDEF, we have

$AB = DE, BC = EF$ $\quad$ [By construction]

and, $\quad AC = DF$ $\quad$ [From equation (i)]

$\therefore \quad \Delta ABC \cong \Delta DEF$

$\Rightarrow \quad \angle B = \angle E = 90°$

Hence, $\quad \Delta ABC$ is a right triangle right-angled at B. $\quad$ **Q.E.D.**

7.10.1 SOME IMPORTANT RESULTS BASED UPON PYTHAGORAS THEOREM

THEOREM 1 *(Result on obtuse triangle) In Fig. 7.184, ΔABC is an obtuse triangle, obtuse-angled at B. If $AD \perp CB$, prove that $AC^2 = AB^2 + BC^2 + 2\,BC \times BD$.* **[NCERT]**

GIVEN An obtuse triangle ABC, obtuse-angled at B and AD is perpendicular to CB produced.

TO PROVE $AC^2 = AB^2 + BC^2 + 2\,BC \times BD$.

PROOF Since ΔADB is a right triangle right-angled at D. Therefore, by Pythagoras theorem, we have

$$AB^2 = AD^2 + DB^2 \quad \text{...(i)}$$

Again, ΔADC is a right triangle right-angled at D.

Therefore, by Pythagoras theorem, we have

$$AC^2 = AD^2 + DC^2$$

$\Rightarrow \quad AC^2 = AD^2 + (DB + BC)^2$

$\Rightarrow \quad AC^2 = AD^2 + DB^2 + BC^2 + 2\,BC \cdot BD$

$\Rightarrow \quad AC^2 = (AD^2 + DB^2) + BC^2 + 2\,BC \cdot BD$

$\Rightarrow \quad AC^2 = AB^2 + BC^2 + 2\,BC \cdot BD$ $\quad$ [Using (i)]

Hence, $\quad AC^2 = AB^2 + BC^2 + 2\,BC \cdot BD$

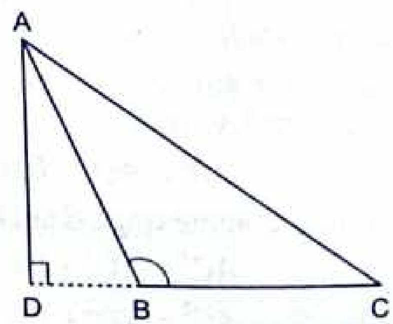


Fig. 7.184

REMARK *In the above theorem BD is known as the projection of AB on BC and the theorem can also be stated as:*

In an obtuse triangle, the square of the side opposite to obtuse angle is equal to the sum of the squares of other two sides plus twice the product of one side and the projection of other on first.

THEOREM 2 *(Result on acute triangle) In Fig. 7.185, $\angle B$ of ΔABC is an acute angle and $AD \perp BC$, prove that $AC^2 = AB^2 + BC^2 - 2\,BC \times BD$* **[NCERT]**

GIVEN A ΔABC in which $\angle B$ is an acute angle and $AD \perp BC$

TO PROVE $AC^2 = AB^2 + BC^2 - 2BC \times BD$.

PROOF Since ΔADB is a right triangle right-angled at D. So, by Pythagoras theorem, we have

$$AB^2 = AD^2 + BD^2 \quad \text{...(i)}$$

Again, ΔADC is a right triangle right-angled at D. Applying Pythagoras theorem, we obtain

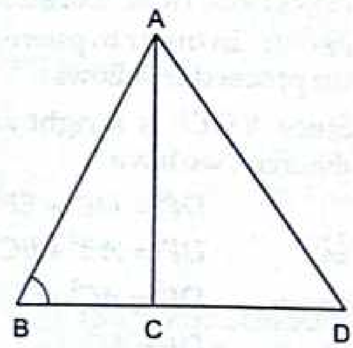


Fig. 7.185

$$AC^2 = AD^2 + DC^2$$

$\Rightarrow$ $$AC^2 = AD^2 + (BC - BD)^2$$

$\Rightarrow$ $$AC^2 = AD^2 + (BC^2 + BD^2 - 2\,BC \cdot BD)$$

$\Rightarrow$ $$AC^2 = (AD^2 + BD^2) + BC^2 - 2\,BC \cdot BD$$

$\Rightarrow$ $$AC^2 = AB^2 + BC^2 - 2\,BC \cdot BD \quad \text{[Using (i)]}$$

Hence, $$AC^2 = AB^2 + BC^2 - 2\,BC \cdot BD$$

Q.E.D.

REMARK *In the above theorem BD is known as the projection of AB on BC and the theorem can also be stated as:*

In an acute triangle, the square of the side opposite to an acute angle is equal to the sum of the squares of other two sides minus twice the product of one side and the projection of other on first.

THEOREM 3 *Prove that in any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.*

GIVEN A ΔABC in which AD is a median.

TO PROVE $AB^2 + AC^2 = 2AD^2 + 2\left(\frac{1}{2}BC\right)^2$ or, $AB^2 + AC^2 = 2(AD^2 + BD^2)$

CONSTRUCTION Draw $AE \perp BC$.

PROOF Since $\angle AED = 90°$. Therefore, in ΔADE, we have

$$\angle ADE < 90° \Rightarrow \angle ADB > 90°$$

Thus, ΔADB is an obtuse-angled triangle and ΔADC is an acute-angled triangle.

ΔABD is obtuse-angled at D and $AE \perp BD$ produced. Therefore, by theorem 1, we have

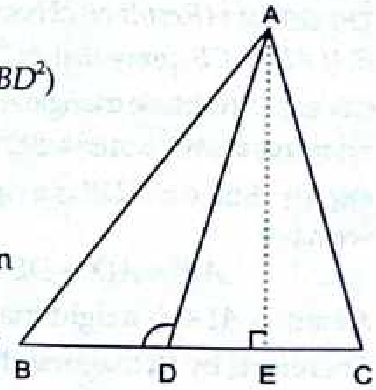


Fig. 7.186

$$AB^2 = AD^2 + BD^2 + 2\,BD \times DE \quad \text{...(i)}$$

ΔACD is acute-angled at D and $AE \perp CD$. Therefore, by theorem 2, we have

$$AC^2 = AD^2 + DC^2 - 2\,DC \times DE$$

$\Rightarrow$ $$AC^2 = AD^2 + BD^2 - 2\,BD \times DE \quad [\because CD = BD] \quad \text{...(ii)}$$

Adding equations (i) and (ii), we get

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\left\{AD^2 + \left(\frac{BC}{2}\right)^2\right\}$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\,AD^2 + 2\left(\frac{1}{2}BC\right)^2$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\,(AD^2 + BD^2)$$

Q.E.D.

THEOREM 4 *Prove that three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians of the triangle.*

GIVEN A $\Delta\, ABC$ in which AD, BE and CF are three medians.

TO PROVE $3\,(AB^2 + BC^2 + CA^2) = 4\,(AD^2 + BE^2 + CF^2)$

PROOF Since in any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median bisecting it.

Therefore, taking AD as the median bisecting side BC, we have

$$AB^2 + AC^2 = 2\,(AD^2 + BD^2)$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\left\{AD^2 + \left(\frac{BC}{2}\right)^2\right\}$$

$$\Rightarrow \quad AB^2 + AC^2 = 2\left\{AD^2 + \frac{BC^2}{4}\right\}$$

$$\Rightarrow \quad 2\,(AB^2 + AC^2) = (4\,AD^2 + BC^2) \quad \text{...(i)}$$

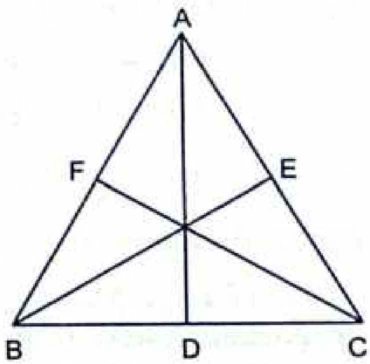


Fig. 7.187

Similarly, by taking BE and CF respectively as the medians, we get

$$2\,(AB^2 + BC^2) = (4\,BE^2 + AC^2) \quad \text{...(ii)}$$

and, $$2\,(AC^2 + BC^2) = (4\,CF^2 + AB^2) \quad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$4\,(AB^2 + BC^2 + AC^2) = 4\,(AD^2 + BE^2 + CF^2) + (BC^2 + AC^2 + AB^2)$$

$$\Rightarrow \quad 3\,(AB^2 + BC^2 + AC^2) = 4\,(AD^2 + BE^2 + CF^2)$$

Hence, $$3\,(AB^2 + BC^2 + AC^2) = 4\,(AD^2 + BE^2 + CF^2)$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A right triangle has hypotenuse of length p cm and one side of length q cm. If $p - q = 1$, find the length of the third side of the triangle.*

SOLUTION Let the third side be x cm. Then, by Pythagoras theorem, we have

$$p^2 = q^2 + x^2$$

$$\Rightarrow \quad x^2 = p^2 - q^2 = (p-q)(p+q) = p + q \qquad [\because p - q = 1]$$

$$\Rightarrow \quad x = \sqrt{p+q} = \sqrt{2q+1} \qquad [\because p - q = 1 \therefore p = q + 1]$$

Hence, the length of the third side is $\sqrt{2q+1}$ cm.

EXAMPLE 2 *The sides of certain triangles are given below. Determine which of them are right triangles :*

(i) $a = 6\ cm, b = 8\ cm$ and $c = 10\ cm$ (ii) $a = 5\ cm, b = 8\ cm$ and $c = 11\ cm$.

SOLUTION (i) We have,

$a = 6$ cm, $b = 8$ cm and $c = 10$ cm

Here, the larger side is $c = 10$ cm.

We have, $a^2 + b^2 = 6^2 + 8^2 = 36 + 64 = 100 = c^2$

So, the triangle with the given sides is a right triangle.

(ii) Here, the larger side is $c = 11$ cm.

Clearly, $a^2 + b^2 = 25 + 64 = 89 \neq c^2$

So, the triangle with the given sides is not a right triangle.

EXAMPLE 3 *A man goes 10 m due east and then 24 m due north. Find the distance from the starting point.*

SOLUTION Let the initial position of the man be O and his final position be B. Since the man goes 10 m due east and then 24 m due north. Therefore, $\Delta\, AOB$ is a right triangle right-angled at A such that $OA = 10$ m and $AB = 24$ m.

By Pythagoras theorem, we have

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad OB^2 = 10^2 + 24^2 = 100 + 576 = 676$$

$$\Rightarrow \quad OB = \sqrt{676} = 26 \text{ m}$$

Hence, the man is at a distance of 26 m from the starting point.

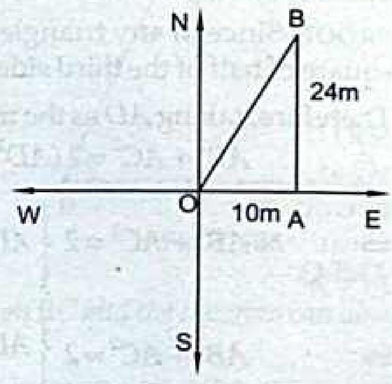


Fig. 7.188

EXAMPLE 4 *A ladder is placed in such a way that its foot is at a distance of 5 m from a wall and its tip reaches a window 12 m above the ground. Determine the length of the ladder.*

SOLUTION Let AB be the ladder and B be the window. Then,

$$BC = 12 \text{ m and } AC = 5 \text{ m}$$

Since $\Delta\, ABC$ is a right triangle right-angled at C.

$$\therefore \quad AB^2 = AC^2 + BC^2$$

$$\Rightarrow \quad AB^2 = 5^2 + 12^2 = 169$$

$$\Rightarrow \quad AB^2 = 13 \text{ m}$$

Hence, the length of the ladder is 13 m.

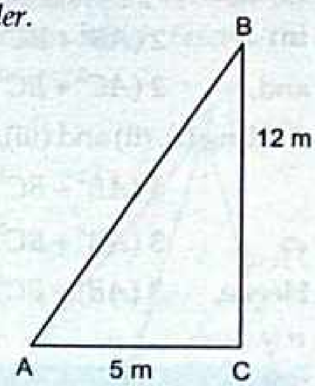


Fig. 7.189

EXAMPLE 5 *A ladder 25 m long reaches a window of a building 20 m above the ground. Determine the distance of the foot of the ladder from the building.*

SOLUTION Suppose that AB is the ladder, B is the window and CB is the building. Then, triangle ABC is a right triangle with right-angle at C.

$$\therefore \quad AB^2 = AC^2 + BC^2$$

$$\Rightarrow \quad 25^2 = AC^2 + 20^2$$

$$\Rightarrow \quad AC^2 = 625 - 400 = 225$$

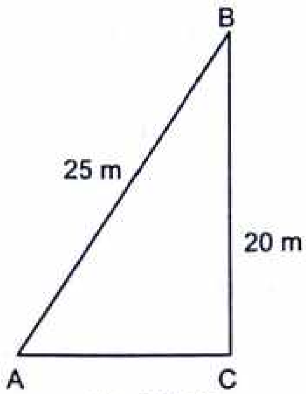


Fig. 7.190

$$\Rightarrow \quad AC = \sqrt{225} \text{ m} = 15 \text{ m}$$

Hence, the foot of the ladder is at a distance 15 m from the building.

EXAMPLE 6 *A ladder 15 m long reaches a window which is 9 m above the ground on one side of a street. Keeping its foot at the same point, the ladder is turned to other side of the street to reach a window 12 m high. Find the width of the street.*

SOLUTION Let AB be the width of the street and C be the foot of the ladder. Let D and E be the windows at heights of 9 m and 12 m respectively from the ground. Then, CD and EF are the two positions of the ladder.

Clearly, $AD = 9$ m, $BE = 12$ m, $CD = CE = 15$ m.

In ΔACD, we have

$$CD^2 = AC^2 + AD^2$$

$$\Rightarrow \quad 15^2 = AC^2 + 9^2$$

$$\Rightarrow \quad AC^2 = 225 - 81 = 144$$

$$\Rightarrow \quad AC = 12 \text{ m}$$

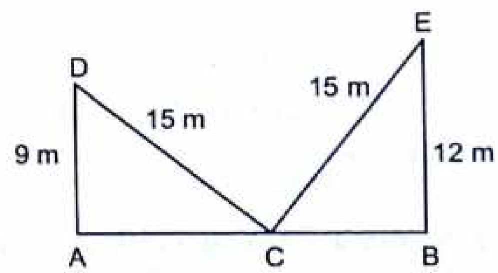


Fig. 7.191

In ΔBCE, we have

$$CE^2 = BC^2 + BE^2$$

$$\Rightarrow \quad 15^2 = BC^2 + 12^2$$

$$\Rightarrow \quad BC^2 = 225 - 144 = 81$$

$$\Rightarrow \quad BC = 9 \text{ m}$$

Hence, width of the street $= AB = AC + CB = (12 + 9)$ m $= 21$ m.

EXAMPLE 7 *The hypotenuse of a right triangle is 6 m more than the twice of the shortest side. If the third side is 2 m less than the hypotenuse, find the sides of the triangle.*

SOLUTION Let the shortest side be x metres in length. Then,

Hypotenuse $= (2x + 6)$ m and, Third side $= (2x + 4)$ m

By Pythagoras theorem, we have

$$(2x + 6)^2 = x^2 + (2x + 4)^2$$

$$\Rightarrow \quad 4x^2 + 24x + 36 = x^2 + 4x^2 + 16x + 16$$

$$\Rightarrow \quad x^2 - 8x - 20 = 0$$

$$\Rightarrow \quad (x - 10)(x + 2) = 0$$

$$\Rightarrow \quad x = 10 \text{ or}, x = -2$$

$$\Rightarrow \quad x = 10 \qquad [\because x \text{ cannot be negative}]$$

Hence, the sides of the triangle are 10 m, 26 m and 24 m.

EXAMPLE 8 *P and Q are the mid-points of the sides CA and CB respectively of a Δ ABC, right angled at C. Prove that :*

(i) $4AQ^2 = 4AC^2 + BC^2$ **[CBSE 2010]**

(ii) $4BP^2 = 4BC^2 + AC^2$ **[CBSE 2001]**

(iii) $4(AQ^2 + BP^2) = 5AB^2$ **[NCERT, CBSE 2001, 2006C]**

SOLUTION (i) Since ΔAQC is a right triangle right-angled at C.

$\therefore \quad AQ^2 = AC^2 + QC^2$

$\Rightarrow \quad 4AQ^2 = 4AC^2 + 4QC^2$ [Multiplying both sides by 4]

$\Rightarrow \quad 4AQ^2 = 4AC^2 + (2QC)^2$

$\Rightarrow \quad 4AQ^2 = 4AC^2 + BC^2 \quad [\because BC = 2QC]$

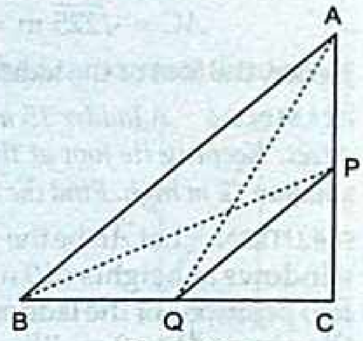


Fig. 7.192

(ii) Since ΔBPC is a right triangle right-angled at C.

$\therefore \quad BP^2 = BC^2 + CP^2$

$\Rightarrow \quad 4BP^2 = 4BC^2 + 4CP^2$ [Multiplying both sides by 4]

$\Rightarrow \quad 4BP^2 = 4BC^2 + (2CP)^2$

$\Rightarrow \quad 4BP^2 = 4BC^2 + AC^2 \quad [\because AC = 2CP]$

(iii) From (i) and (ii), we have

$4AQ^2 = 4AC^2 + BC^2$ and, $4BP^2 = 4BC^2 + AC^2$

$\therefore \quad 4AQ^2 + 4BP^2 = (4AC^2 + BC^2) + (4BC^2 + AC^2)$

$\Rightarrow \quad 4(AQ^2 + BP^2) = 5(AC^2 + BC^2)$

$\Rightarrow \quad 4(AQ^2 + BP^2) = 5AB^2 \quad \left[\text{In } \Delta ABC\text{, we have } AB^2 = AC^2 + BC^2\right]$

EXAMPLE 9 *ABC is a right triangle right-angled at B. Let D and E be any points on AB and BC respectively. Prove that $AE^2 + CD^2 = AC^2 + DE^2$* **[CBSE 2002C, 2007]**

SOLUTION Since ΔABE is right triangle, right-angled at B.

$\therefore \quad AE^2 = AB^2 + BE^2 \quad \ldots(i)$

Again, ΔDBC is right triangle right-angled at B.

$\therefore \quad CD^2 = BD^2 + BC^2 \quad \ldots(ii)$

Adding (i) and (ii), we get

$AE^2 + CD^2 = (AB^2 + BE^2) + (BD^2 + BC^2)$

$\Rightarrow \quad AE^2 + CD^2 = (AB^2 + BC^2) + (BE^2 + BD^2)$

Using Pythagoras theorem in ΔABC and ΔDBE, we have

$AC^2 = AB^2 + BC^2$ and $DE^2 = BE^2 + BD^2$

$\therefore \quad AE^2 + CD^2 = AC^2 + DE^2$

Hence, $\quad AE^2 + CD^2 = AC^2 + DE^2$

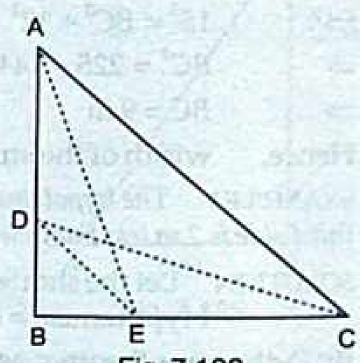


Fig. 7.193

EXAMPLE 10 *Prove that three times the square of any side of an equilateral-triangle is equal to four times the square of the altitude.* **[CBSE 2002]**

SOLUTION Let ABC be an equilateral triangle and let $AD \perp BC$.

In ΔADB and ΔADC, we have

$AB = AC$ [Given]

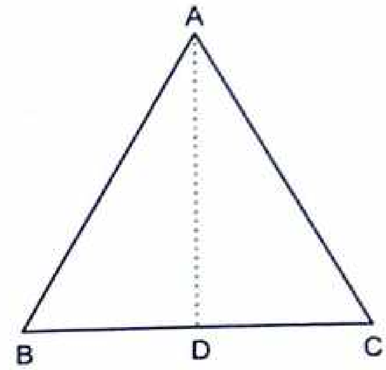


Fig. 7.194

$\angle B = \angle C$ [Each equal to 60°]

and, $\angle ADB = \angle ADC$ [Each equal to 90°]

$\therefore$ $\Delta ADB \cong \Delta ADC$

$\Rightarrow$ $BD = DC$

$\Rightarrow$ $BD = DC = \frac{1}{2} BC$

Since ΔADB is a right triangle right-angled at D.

$\therefore$ $AB^2 = AD^2 + BD^2$

$\Rightarrow$ $AB^2 = AD^2 + \left(\frac{1}{2} BC\right)^2$

$\Rightarrow$ $AB^2 = AD^2 + \frac{BC^2}{4}$

$\Rightarrow$ $AB^2 = AD^2 + \frac{AB^2}{4}$ $[\because BC = AB]$

$\Rightarrow$ $\frac{3}{4} AB^2 = AD^2 \Rightarrow 3AB^2 = 4AD^2$

EXAMPLE 11 *In an equilateral triangle with side a, prove that*

(i) Altitude $= \frac{a\sqrt{3}}{2}$ **[CBSE 2001C]** (ii) Area $= \frac{\sqrt{3}}{4} a^2$ **[NCERT, CBSE 2002]**

SOLUTION Let ABC be an equilateral triangle the length of whose each side is a units. Draw $AD \perp BC$. Then, D is the mid-point of BC.

$\Rightarrow$ $AB = a, BD = \frac{1}{2} BC = \frac{a}{2}$

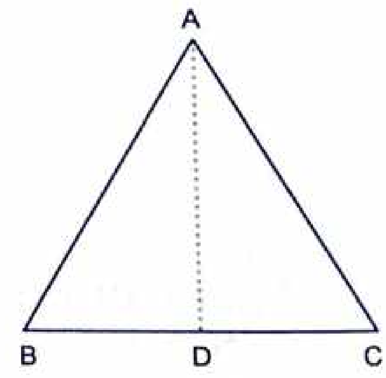


Fig. 7.195

Since ΔABD is a right triangle right-angled at D.

$\therefore$ $AB^2 = AD^2 + BD^2$

$\Rightarrow$ $a^2 = AD^2 + \left(\frac{a}{2}\right)^2$

$\Rightarrow$ $AD^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$

$\Rightarrow$ $AD = \frac{\sqrt{3a}}{2}$

$\therefore$ Altitude $\frac{\sqrt{3}}{2} a$

Now,

Area of $\Delta ABC = (1/2)$ (Base × Height)

$\Rightarrow$ Area of $\Delta ABC = \frac{1}{2}(BC \times AD) = \frac{1}{2} \times a \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{4} a^2$

EXAMPLE 12 *ABC is an isosceles right triangle right-angled at C. Prove that $AB^2 = 2AC^2$.* **[NCERT]**

SOLUTION Since Δ ABC is a right triangle right-angled at C.

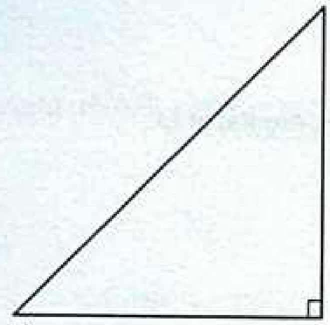


Fig. 7.196

$\therefore$ $AB^2 = AC^2 + BC^2$

$\Rightarrow$ $AB^2 = AC^2 + AC^2$ $[\because BC = AC]$

$\Rightarrow$ $AB^2 = 2AC^2$

EXAMPLE 13 *In Δ ABC, AD is perpendicular to BC. Prove that:*

(i) $AB^2 + CD^2 = AC^2 + BD^2$ **[CBSE 2008, 2009]** (ii) $AB^2 - BD^2 = AC^2 - CD^2$

SOLUTION Since triangles ABD and ACD are right triangles right-angled at D.

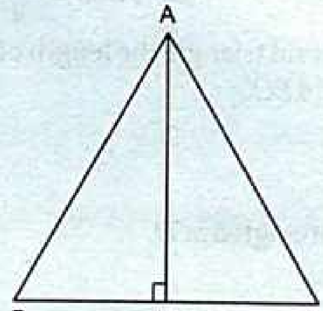


Fig. 7.197

$\therefore$ $AB^2 = AD^2 + BD^2$ ...(i)

and, $AC^2 = AD^2 + CD^2$ ...(ii)

Subtracting (ii) from (i), we get

$$AB^2 - AC^2 = BD^2 - CD^2$$

$\Rightarrow$ $AB^2 + CD^2 = AC^2 + BD^2$ and $AB^2 - BD^2 = AC^2 - CD^2$

EXAMPLE 14 *P and Q are points on the sides CA and CB respectively of Δ ABC right angled at C. Prove that $AQ^2 + BP^2 = AB^2 + PQ^2$* **[NCERT, CBSE 2002]**

SOLUTION In right-angled triangles ACQ and PCB, we have

$$AQ^2 = AC^2 + CQ^2 \text{ and } PB^2 = PC^2 + CB^2$$

$\Rightarrow$ $AQ^2 + BP^2 = (AC^2 + CQ^2) + (PC^2 + CB^2)$

$\Rightarrow$ $AQ^2 + BP^2 = (AC^2 + BC^2) + (PC^2 + QC^2)$

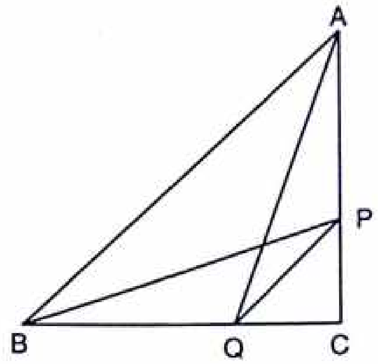


Fig. 7.198

$\Rightarrow \quad AQ^2 + BP^2 = AB^2 + PQ^2 \quad \left[\begin{array}{l}\text{By Pythagoras theorem, we obtain}\\ AC^2+BC^2=AB^2 \text{ and } PC^2+QC^2=PQ^2\end{array}\right]$

EXAMPLE 15 *ABC is an isosceles triangle with $AC = BC$. If $AB^2 = 2AC^2$, prove that ΔABC is right triangle.* **[NCERT, CBSE 2000]**

SOLUTION We have,

$AC = BC$ and $AB^2 = 2\,AC^2$

Now, $AB^2 = 2\,AC^2$

$\Rightarrow \quad AB^2 = AC^2 + AC^2$

$\Rightarrow \quad AB^2 = AC^2 + BC^2 \quad [\because AC = BC \text{ (Given)}]$

$\Rightarrow \quad \Delta\, ABC$ is a right triangle right-angled at C.

LEVEL-2

EXAMPLE 16 *In Fig. 7.199, ABC is a right triangle right-angled at B. AD and CE are the two medians drawn from A and C respectively. If $AC = 5$ cm and $AD = \frac{3\sqrt{5}}{2}$ cm, find the length of CE.*

SOLUTION Since $\Delta\, ABD$ is a right triangle right-angled at B. Therefore,

$$AD^2 = AB^2 + BD^2$$

$$\Rightarrow \quad AD^2 = AB^2 + \left(\frac{BC}{2}\right)^2 \quad [\because BD = DC]$$

$$\Rightarrow \quad AD^2 = AB^2 + \frac{1}{4}\cdot BC^2 \quad \ldots(i)$$

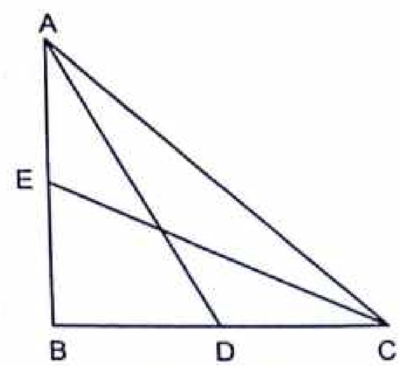


Fig. 7.199

Again, $\Delta\, BCE$ is a right triangle right angled at B.

$$\therefore \quad CE^2 = BC^2 + BE^2$$

$$\Rightarrow \quad CE^2 = BC^2 + \left(\frac{AB}{2}\right)^2 \quad [\because BE = EA]$$

$$\Rightarrow \quad CE^2 = BC^2 + \frac{1}{4}\cdot AB^2 \quad \ldots(ii)$$

Adding (i) and (ii), we get

$$AD^2 + CE^2 = AB^2 + \frac{1}{4}\,BC^2 + BC^2 + \frac{1}{4}\,AB^2$$

$\Rightarrow \quad AD^2 + CE^2 = \frac{5}{4}(AB^2 + BC^2)$

$\Rightarrow \quad AD^2 + CE^2 = \frac{5}{4} AC^2$ $\quad$ [$\because \Delta ABC$ is right triangle $\therefore AC^2 = AB^2 + BC^2$]

$\Rightarrow \quad \left(\frac{3\sqrt{5}}{2}\right)^2 + CE^2 = \frac{5}{4} \times 25$

$\Rightarrow \quad CE^2 = \frac{125}{4} - \frac{45}{4} = 20$

$\Rightarrow \quad CE = \sqrt{20}$ cm $= 2\sqrt{5}$ cm

EXAMPLE 17 *The perpendicular AD on the base BC of a ΔABC intersects BC at D so that $DB = 3CD$. Prove that $2AB^2 = 2AC^2 + BC^2$.* **[NCERT, CBSE 2005, 2009]**

SOLUTION We have,

$DB = 3CD$

$\therefore \quad BC = BD + DC$

$\Rightarrow \quad BC = 3CD + CD$

$\Rightarrow \quad BC = 4CD$

$\Rightarrow \quad CD = \frac{1}{4} BC$

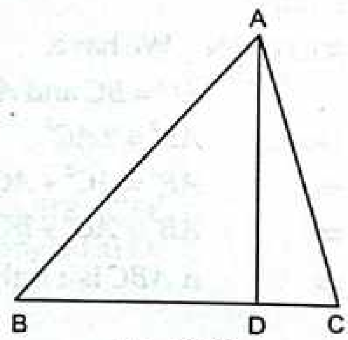


Fig. 7.200

$\therefore \quad CD = \frac{1}{4} BC$ and $BD = 3CD = \frac{3}{4} BC$ $\quad$ ...(i)

Since ΔABD is a right triangle right-angled at D.

$\therefore \quad AB^2 = AD^2 + BD^2$ $\quad$ ...(ii)

Similarly, ΔACD is a right triangle right angled at D.

$\therefore \quad AC^2 = AD^2 + CD^2$ $\quad$ ...(iii)

Subtracting equation (iii) from equation (ii) we get

$AB^2 - AC^2 = BD^2 - CD^2$

$\Rightarrow \quad AB^2 - AC^2 = \left(\frac{3}{4}BC\right)^2 - \left(\frac{1}{4}BC\right)^2$ $\quad$ $\left[\text{From (i) } CD = \frac{1}{4}BC,\ BD = \frac{3}{4}BC\right]$

$\Rightarrow \quad AB^2 - AC^2 = \frac{9}{16}BC^2 - \frac{1}{16}BC^2$

$\Rightarrow \quad AB^2 - AC^2 = \frac{1}{2}BC^2$

$\Rightarrow \quad 2(AB^2 - AC^2) = BC^2 \Rightarrow 2AB^2 = 2AC^2 + BC^2$

EXAMPLE 18 *ABC is a right triangle right-angled at C. Let $BC = a$, $CA = b$, $AB = c$ and let p be the length of perpendicular from C on AB, prove that*

(i) $cp = ab$ $\quad$ **[CBSE 2002]** $\quad$ (ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

SOLUTION Let $CD \perp AB$. Then, $CD = p$.

$\therefore$ Area of $\Delta ABC = \frac{1}{2}$(Base× Height)

$\Rightarrow$ Area of $\Delta ABC = \frac{1}{2}(AB \times CD) = \frac{1}{2}cp$

Also,

Area of $\Delta ABC = \frac{1}{2}(BC \times AC) = \frac{1}{2}ab$

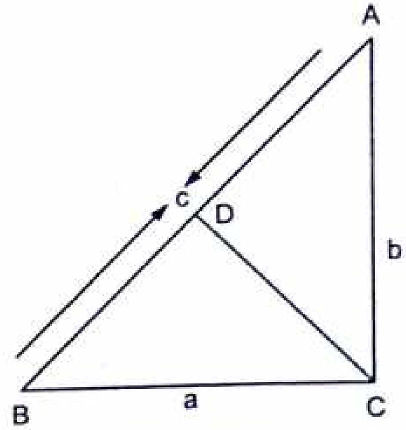


Fig. 7.201

$\therefore \quad \frac{1}{2}cp = \frac{1}{2}ab$

$\Rightarrow \quad cp = ab$

(ii) Since ΔABC is a right triangle right-angled at C.

$\therefore \quad AB^2 = BC^2 + AC^2$

$\Rightarrow \quad c^2 = a^2 + b^2$

$\Rightarrow \quad \left(\frac{ab}{p}\right)^2 = a^2 + b^2 \qquad \left[\because cp = ab \therefore c = \frac{ab}{p}\right]$

$\Rightarrow \quad \frac{a^2b^2}{p^2} = a^2 + b^2$

$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

EXAMPLE 19 *In an isosceles triangle ABC with AB = AC, BD is perpendicular from B to the side AC. Prove that* $BD^2 - CD^2 = 2CD \cdot AD$

SOLUTION Since ΔADB is a right triangle right-angled at D.

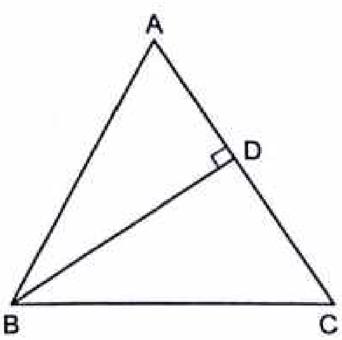


Fig. 7.202

$\therefore \quad AB^2 = AD^2 + BD^2$

$\Rightarrow \quad AC^2 = AD^2 + BD^2 \qquad [\because AB = AC]$

$\Rightarrow \quad (AD + CD)^2 = AD^2 + BD^2$

$\Rightarrow \quad AD^2 + CD^2 + 2\,AD \cdot CD = AD^2 + BD^2$

$\Rightarrow \quad BD^2 - CD^2 = 2\,CD \cdot AD$

EXAMPLE 20 *ABC is a triangle in which $AB = AC$ and D is any point in BC. Prove that $AB^2 - AD^2 = BD \cdot CD$* **[CBSE 2005]**

SOLUTION Draw $AE \perp BC$

In ΔAEB and ΔAEC, we have

$AB = AC$,

$AE = AE$ [Common]

and, $\angle B = \angle C$ [$\because AB = AC$]

$\therefore$ $\Delta AEB \cong \Delta AEC$

$\Rightarrow$ $BE = CE$

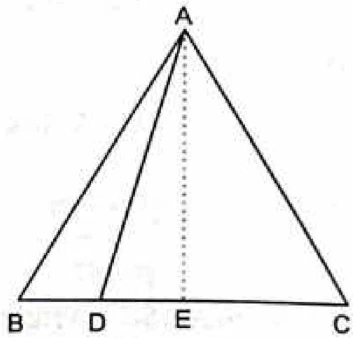


Fig. 7.203

Since ΔAED and ΔABE are right triangles right-angled at E. Therefore,

$AD^2 = AE^2 + DE^2$ and $AB^2 = AE^2 + BE^2$

$\Rightarrow$ $AB^2 - AD^2 = BE^2 - DE^2$

$\Rightarrow$ $AB^2 - AD^2 = (BE + DE)(BE - DE)$

$\Rightarrow$ $AB^2 - AD^2 = (CE + DE)(BE - DE)$ [$\because BE = CE$]

$\Rightarrow$ $AB^2 - AD^2 = CD \cdot BD$

Hence, $AB^2 - AD^2 = BD \cdot CD$

EXAMPLE 21 *From a point O in the interior of a ΔABC, perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB respectively. Prove that:*

(i) $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$ **[NCERT]**

SOLUTION Let O be a point in the interior of ΔABC and let $OD \perp BC, OE \perp CA$ and $OF \perp AB$.

(i) In right triangles ΔOFA, ΔODB and ΔOEC, we have

$OA^2 = AF^2 + OF^2$

$OB^2 = BD^2 + OD^2$

and, $OC^2 = CE^2 + OE^2$

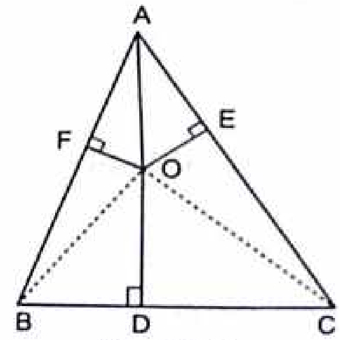


Fig. 7.204

Adding all these results, we get

$OA^2 + OB^2 + OC^2 = AF^2 + BD^2 + CE^2 + OF^2 + OD^2 + OE^2$

$\Rightarrow$ $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(ii) In right triangles ΔODB and ΔODC, we have

$OB^2 = OD^2 + BD^2$

and, $OC^2 = OD^2 + CD^2$

$\therefore$ $OB^2 - OC^2 = (OD^2 + BD^2) - (OD^2 + CD^2)$

$\Rightarrow$ $OB^2 - OC^2 = BD^2 - CD^2$ ...(i)

Similarly, we have

$OC^2 - OA^2 = CE^2 - AE^2$ ...(ii)

and, $OA^2 - OB^2 = AF^2 - BF^2$ ...(iii)

Adding (i), (ii) and (iii), we get

$$(OB^2 - OC^2) + (OC^2 - OA^2) + (OA^2 - OB^2) = (BD^2 - CD^2) + (CE^2 - AE^2) + (AF^2 - BF^2)$$

$\Rightarrow \quad (BD^2 + CE^2 + AF^2) - (AE^2 + CD^2 + BF^2) = 0$

$\Rightarrow \quad AF^2 + BD^2 + CE^2 = AE^2 + BF^2 + CD^2$

EXAMPLE 22 *A point O in the interior of a rectangle ABCD is joined with each of the vertices A, B, C and D. Prove that* $OB^2 + OD^2 = OC^2 + OA^2$ **[NCERT, CBSE 2006C]**

SOLUTION Let $ABCD$ be the given rectangle and let O be a point within it. Join OA, OB, OC and OD.

Through O, draw $EOF \parallel AB$. Then, $ABFE$ is a rectangle.

In right triangles $\Delta\, OEA$ and $\Delta\, OFC$, we have

$OA^2 = OE^2 + AE^2$ and $OC^2 = OF^2 + CF^2$

$\Rightarrow \quad OA^2 + OC^2 = (OE^2 + AE^2) + (OF^2 + CF^2)$

$\Rightarrow \quad OA^2 + OC^2 = OE^2 + OF^2 + AE^2 + CF^2 \quad \ldots(i)$

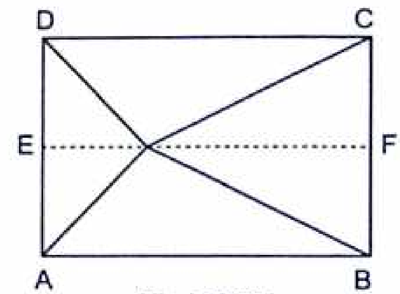


Fig. 7.205

Now, in right triangles $\Delta\, OFB$ and $\Delta\, ODE$, we have

$OB^2 = OF^2 + FB^2$ and $OD^2 = OE^2 + DE^2$

$\Rightarrow \quad OB^2 + OD^2 = (OF^2 + FB^2) + (OE^2 + DE^2)$

$\Rightarrow \quad OB^2 + OD^2 = OE^2 + OF^2 + DE^2 + BF^2$

$\Rightarrow \quad OB^2 + OD^2 = OE^2 + OF^2 + CF^2 + AE^2 \quad [\because DE = CF \text{ and } AE = BF] \quad \ldots(ii)$

From (i) and (ii), we get

$OA^2 + OC^2 = OB^2 + OD^2$

EXAMPLE 23 *ABCD is a rhombus. Prove that* $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

[NCERT, CBSE 2005]

SOLUTION Let the diagonals AC and BD of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus bisect each other at right angles.

$\therefore \quad \angle AOB = \angle BOC = \angle COD = \angle DOA = 90°$ and $AO = CO$, $BO = OD$.

Since $\Delta\, AOB$ is a right triangle right-angled at O.

$\therefore \quad AB^2 = OA^2 + OB^2$

$\Rightarrow \quad AB^2 = \left(\frac{1}{2} AC\right)^2 + \left(\frac{1}{2} BD\right)^2 \quad [\because OA = OC \text{ and } OB = OD]$

$\Rightarrow \quad 4\,AB^2 = AC^2 + BD^2 \quad \ldots(i)$

Similarly, we have

$4\,BC^2 = AC^2 + BD^2 \quad \ldots(ii)$

$4\,CD^2 = AC^2 + BD^2 \quad \ldots(iii)$

and, $\quad 4\,AD^2 = AC^2 + BD^2 \quad \ldots(iv)$

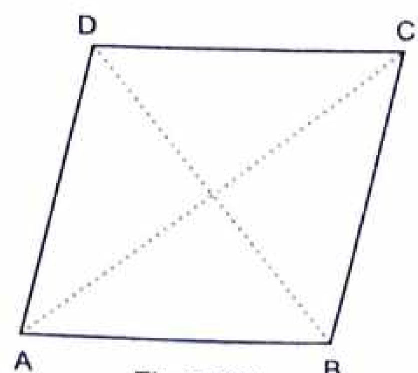


Fig. 7.206

Adding all these results, we get

$4\,(AB^2 + BC^2 + CD^2 + AD^2) = 4\,(AC^2 + BD^2)$

$\Rightarrow \quad AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

EXAMPLE 24 *In a triangle ABC, AC > AB, D is the mid-point of BC and AE ⊥ BC. Prove that :*

(i) $AC^2 = AD^2 + BC \cdot DE + \frac{1}{4} BC^2$ **[NCERT]**

(ii) $AB^2 = AD^2 - BC \cdot DE + \frac{1}{4} BC^2$ **[CBSE 2006C]**

(iii) $AB^2 + AC^2 = 2\,AD^2 + \frac{1}{2} BC^2$

SOLUTION We have, $\angle AED = 90°$,

$\therefore \quad \angle ADE < 90°$ and $\angle ADC > 90°$.

i.e., $\angle ADE$ is acute and $\angle ADC$ is obtuse.

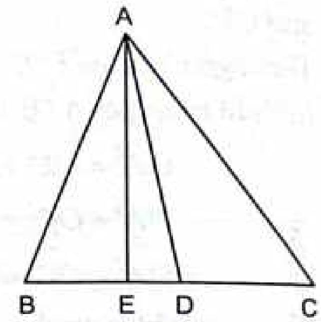


Fig. 7.207

(i) In ΔADC, $\angle ADC$ is an obtuse angle.

$$\therefore \quad AC^2 = AD^2 + DC^2 + 2\,DC \cdot DE$$

$$\Rightarrow \quad AC^2 = AD^2 + \left(\frac{1}{2} BC\right)^2 + 2 \cdot \frac{1}{2} BC \cdot DE$$

$$\Rightarrow \quad AC^2 = AD^2 + \frac{1}{4} BC^2 + BC \cdot DE$$

$$\Rightarrow \quad AC^2 = AD^2 + BC \cdot DE + \frac{1}{4} BC^2 \qquad \text{...(i)}$$

(ii) In ΔABD, $\angle ADE$ is an acute angle.

$$\therefore \quad AB^2 = AD^2 + BD^2 - 2\,BD \cdot DE$$

$$\Rightarrow \quad AB^2 = AD^2 + \left(\frac{1}{2} BC\right)^2 + 2 \cdot \frac{1}{2} BC \cdot DE$$

$$\Rightarrow \quad AB^2 = AD^2 + \frac{1}{4} BC^2 - BC \cdot DE$$

$$\Rightarrow \quad AB^2 = AD^2 - BC \cdot DE + \frac{1}{4} BC^2 \qquad \text{...(ii)}$$

(iii) From (i) and (ii), we get

$$AB^2 + AC^2 = 2AD^2 + \frac{1}{2} BC^2$$

EXAMPLE 25 *In an equilateral triangle ABC the side BC is trisected at D. Prove that* $9\,AD^2 = 7\,AB^2$

SOLUTION Let ABC be an equilateral triangle and let D be a point on BC such that $BD = \frac{1}{3} BC$. Draw $AE \perp BC$. Join AD. **[NCERT, CBSE 2018]**

In $\Delta\,AEB$, and $\Delta\,AEC$, we have $AB = AC$,

$\angle AEB = \angle AEC = 90°$

and, $AE = AE$

So, by RHS-criterion of similarity, we have

$\Delta\,AEB \sim \Delta\,AEC$

$\Rightarrow \quad BE = EC$

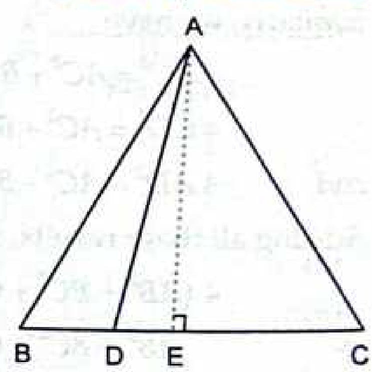


Fig. 7.208

Thus, we have

$$BD = \frac{1}{3} BC, DC = \frac{2}{3} BC \text{ and } BE = EC = \frac{1}{2} BC \quad \ldots(i)$$

Since $\angle C = 60°$. Therefore, ΔADC is an acute triangle.

$\therefore \quad AD^2 = AC^2 + DC^2 - 2\,DC \times EC$

$\Rightarrow \quad AD^2 = AC^2 + \left(\frac{2}{3} BC\right)^2 - 2 \times \frac{2}{3} BC \times \frac{1}{2} BC$ [Using (i)]

$\Rightarrow \quad AD^2 = AC^2 + \frac{4}{9} BC^2 - \frac{2}{3} BC^2$

$\Rightarrow \quad AD^2 = AB^2 + \frac{4}{9} AB^2 - \frac{2}{3} AB^2$ $[\because AB = BC = AC]$

$\Rightarrow \quad AD^2 = \frac{9AB^2 + 4AB^2 - 6AB^2}{9} = \frac{7}{9} AB^2$

$\Rightarrow \quad 9\,AD^2 = 7\,AB^2$

ALITER Draw $AE \perp BC$. Triangle ABC is equilateral. Therefore, E is the mid-point of BC.

$\therefore \quad BE = CE = \frac{1}{2} BC.$

Applying Pythagoras theorem in right triangles AEB and AED, we obtain

$AB^2 = AE^2 + BE^2$ and $AD^2 = AE^2 + DE^2$

$\Rightarrow \quad AB^2 - AD^2 = (AE^2 + BE^2) - (AE^2 + DE^2)$

$\Rightarrow \quad AB^2 - AD^2 = BE^2 - DE^2$

$\Rightarrow \quad AB^2 - AD^2 = \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{6}AB\right)^2 \quad \left[\because DE = BE - BD = \frac{1}{2}AB - \frac{1}{3}AB = \frac{1}{6}AB\right]$

$\Rightarrow \quad AB^2 - AD^2 = \frac{2}{9}AB^2 \Rightarrow \frac{7}{9}AB^2 = AD^2 \Rightarrow 9AD^2 = 7AB^2$

EXAMPLE 26 *In a $\Delta\, ABC$, $AD \perp BC$ and $AD^2 = BD \times CD$. Prove that $\Delta\, ABC$ is a right triangle.*

[CBSE 2006C]

SOLUTION In right triangles ADB and ADC, we have

$AB^2 = AD^2 + BD^2 \quad \ldots(i)$

and, $AC^2 = AD^2 + DC^2 \quad \ldots(ii)$

Adding (i) and (ii), we get

$AB^2 + AC^2 = 2\,AD^2 + BD^2 + DC^2$

$\Rightarrow \quad AB^2 + AC^2 = 2\,BD \times CD + BD^2 + CD^2$ $[\because AD^2 = BD \times CD$ (Given)$]$

$\Rightarrow \quad AB^2 + AC^2 = (BD + CD)^2 = BC^2$

Thus, in $\Delta\, ABC$, we obtain

$AB^2 + AC^2 = BC^2$

Hence, $\Delta\, ABC$ is a right triangle right-angled at A.

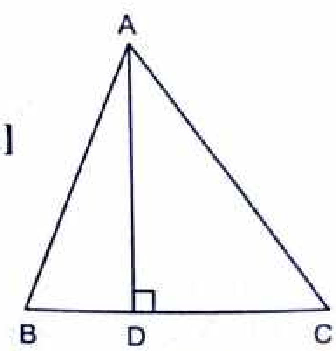


Fig. 7.209

EXAMPLE 27 *In Fig. 7.210, ABC is a right triangle right angled at B and points D and E trisect BC. Prove that* $8AE^2 = 3AC^2 + 5AD^2$. **[CBSE 2006 C]**

SOLUTION Since D and E are the points of trisection of BC. Therefore,

$$BD = DE = CE.$$

Let $BD = DE = CE = x$. Then, $BE = 2x$ and $BC = 3x$.

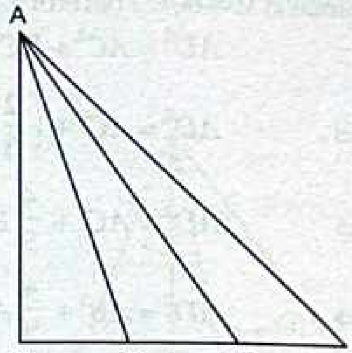


Fig. 7.210

In right triangles ABD, ABE and ABC, we have

$\Rightarrow \quad AD^2 = AB^2 + BD^2$

$\Rightarrow \quad AD^2 = AB^2 + x^2 \quad$ ...(i)

$\quad AE^2 = AB^2 + BE^2$

$\Rightarrow \quad AE^2 = AB^2 + 4x^2 \quad$ ...(ii)

and, $\quad AC^2 = AB^2 + BC^2$

$\Rightarrow \quad AC^2 = AB^2 + 9x^2 \quad$ ...(iii)

Now, $\quad 8AE^2 - 3AC^2 - 5AD^2 = 8(AB^2 + 4x^2) - 3(AB^2 + 9x^2) - 5(AB^2 + x^2)$

$\Rightarrow \quad 8AE^2 - 3AC^2 - 5AD^2 = 0$

$\Rightarrow \quad 8AE^2 = 3AC^2 + 5AD^2$

EXAMPLE 28 *ABC is a right triangle right-angled at C and* $AC = \sqrt{3}\,BC$. *Prove that* $\angle ABC = 60°$.

SOLUTION Let D be the mid-point of AB. Join CD. Since ABC is a right triangle right-angled at C.

$\therefore \quad AB^2 = AC^2 + BC^2$

$\Rightarrow \quad AB^2 = (\sqrt{3}BC)^2 + BC^2 \quad [\because AC = \sqrt{3}\,BC \text{ (Given)}]$

$\Rightarrow \quad AB^2 = 4BC^2$

$\Rightarrow \quad AB = 2BC$

But, $\quad BD = \frac{1}{2}AB$ or, $AB = 2BD$

$\therefore \quad BD = BC$

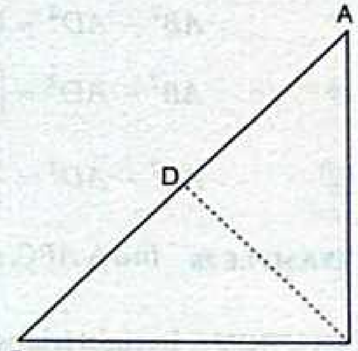


Fig. 7.211

We know that the mid-point of the hypotenuse of a right triangle is equidistant from the vertices.

$\therefore \quad CD = AD = BD$

$\Rightarrow \quad CD = BC \quad [\because BD = BC]$

Thus, in ΔBCD, we have

$\quad BD = CD = BC$

$\Rightarrow \quad \Delta BCD$ is equilateral

$\Rightarrow \quad \angle ABC = 60°$

EXAMPLE 29 *In a right triangle if a perpendicular is drawn from the right angle to the hypotenuse, prove that the square of the perpendicular is equal to the rectangle contained by the two segments of the hypotenuse.*

GIVEN A right triangle ABC right-angled at A, $AD \perp BC$.

TO PROVE $AD^2 = BD \times CD$

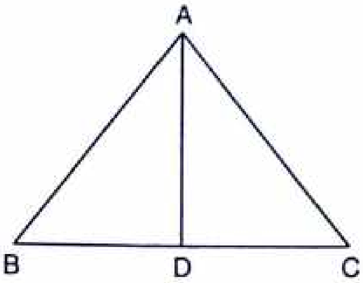


Fig. 7.212

PROOF Since $\Delta\, ABD$ and $\Delta\, ACD$ are right triangles.

$\therefore \qquad AB^2 = AD^2 + BD^2 \qquad \ldots(i)$

and, $\qquad AC^2 = AD^2 + CD^2 \qquad \ldots(ii)$

Adding equations (i) and (ii), we get

$$AB^2 + AC^2 = 2\,AD^2 + BD^2 + CD^2$$

$$\Rightarrow \quad BC^2 = 2\,AD^2 + BD^2 + CD^2 \quad \left[\because \Delta\, ABC \text{ is right-angled at } A \therefore AB^2 + AC^2 = BC^2\right]$$

$$\Rightarrow \quad (BD + CD)^2 = 2\,AD^2 + BD^2 + CD^2$$

$$\Rightarrow \quad BD^2 + CD^2 + 2\,BD \times CD = 2\,AD^2 + BD^2 + CD^2$$

$$\Rightarrow \quad 2\,BD \times CD = 2\,AD^2$$

$$\Rightarrow \quad AD^2 = BD \times CD$$

Hence, $\quad AD^2 = BD \times CD.$

EXAMPLE 30 *ABC is an isosceles triangle right-angled at B. Similar triangles ACD and ABE are constructed on sides AC and AB. Find the ratio between the areas of Δ ABE and Δ ACD.*

[CBSE 2001, 2002]

SOLUTION Let $AB = BC = x$.

It is given that $\Delta\, ABC$ is right-angled at B.

$$\therefore \quad AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC^2 = x^2 + x^2$$

$$\Rightarrow \quad AC = \sqrt{2}\, x$$

It is given that

$$\Delta\, ABE \sim \Delta\, ACD$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABE)}{\text{Area}\,(\Delta\, ACD)} = \frac{AB^2}{AC^2}$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABE)}{\text{Area}\,(\Delta\, ACD)} = \frac{x^2}{(\sqrt{2}x)^2}$$

$$\Rightarrow \quad \frac{\text{Area}\,(\Delta\, ABE)}{\text{Area}\,(\Delta\, ACD)} = \frac{1}{2}$$

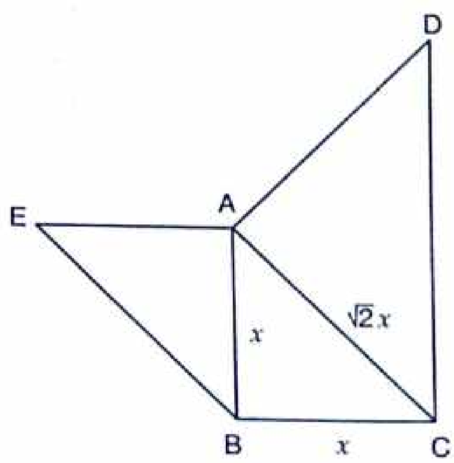


Fig. 7.213

EXAMPLE 31 *ABC is a right-angled triangle right angled at A. A circle is inscribed in it the lengths of the two sides containing the right angle are 6 cm and 8 cm. Find the radius of the circle.*

[CBSE 2002]

SOLUTION Using Pythagoras theorem in $\Delta\, BAC$, we have

$$BC^2 = AB^2 + AC^2$$

$$\Rightarrow \quad BC^2 = 6^2 + 8^2 = 100$$

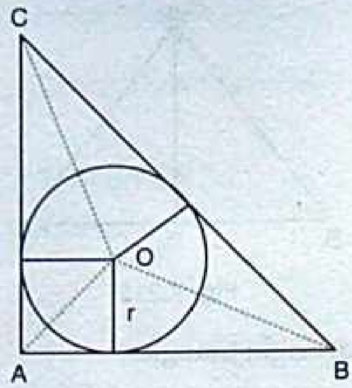


Fig. 7.214

$\Rightarrow \quad BC = 10$ cm

Now,

Area of $\Delta\, ABC$ = Area of $\Delta\, OAB$ + Area of $\Delta\, OBC$ + Area of $\Delta\, OCA$

$$\Rightarrow \quad \frac{1}{2} AB \times AC = \frac{1}{2} AB \times r + \frac{1}{2} BC \times r + \frac{1}{2} CA \times r$$

$$\Rightarrow \quad \frac{1}{2} \times 6 \times 8 = \frac{1}{2}(6 \times r) + \frac{1}{2}(10 \times r) + \frac{1}{2}(8 \times r)$$

$$\Rightarrow \quad 48 = 24\, r \Rightarrow r = 2 \text{ cm}$$

EXAMPLE 32 *In $\Delta\, PQR$, $QM \perp PR$ and $PR^2 - PQ^2 = QR^2$. Prove that $QM^2 = PM \times MR$* **[NCERT]**

SOLUTION In $\Delta\, PQR$, we have

$PR^2 - PQ^2 = QR^2$

$\therefore \quad PR^2 = PQ^2 + QR^2$

$\Rightarrow \quad \Delta\, PQR$ is a right triangle right-angled at Q.

$\Rightarrow \quad \angle 2 + \angle 3 = 90°$

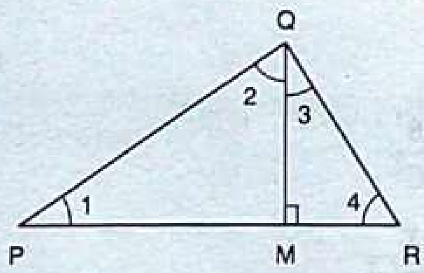


Fig. 7.215

Also, $\quad \angle 1 + \angle 2 = 90°$ $\quad [\because \angle PMQ = 90°]$

$\therefore \quad \angle 1 = \angle 3$

Similarly, we have

$\angle 2 = \angle 4$

Thus, in Δ's PMQ and QMR, we have

$\angle 1 = \angle 3$ and $\angle 2 = \angle 4$

So, by AA-criterion for similarity, we obtain

$$\Delta PMQ \sim \Delta QMR$$

$$\Rightarrow \quad \frac{PM}{QM} = \frac{MQ}{MR}$$

$$\Rightarrow \quad QM^2 = PM \times MR.$$

EXAMPLE 33 *Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.* **[NCERT]**

SOLUTION We know that if AD is a median of ΔABC, then

$$AB^2 + AC^2 = 2\,AD^2 + \frac{1}{2}\,BC^2 \qquad \text{[See Example 24 (iii)]}$$

Since diagonals of a parallelogram bisect each other. Therefore, BO and DO are medians of triangles ABC and ADC respectively.

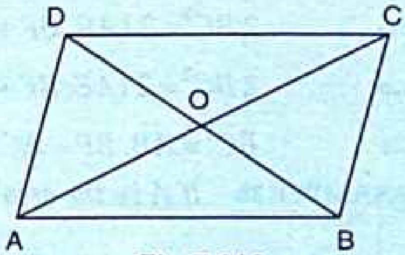


Fig. 7.216

$$\therefore \quad AB^2 + BC^2 = 2\,BO^2 + \frac{1}{2}\,AC^2 \qquad \ldots(i)$$

$$\text{and,} \quad AD^2 + CD^2 = 2\,DO^2 + \frac{1}{2}\,AC^2 \qquad \ldots(ii)$$

Adding (i) and (ii), we have

$$AB^2 + BC^2 + CD^2 + AD^2 = 2\,(BO^2 + DO^2) + AC^2$$

$$\Rightarrow \quad AB^2 + BC^2 + CD^2 + AD^2 = 2\left(\frac{1}{4}\,BD^2 + \frac{1}{4}\,BD^2\right) + AC^2 \qquad \left[\because DO = \frac{1}{2}\,BD\right]$$

$$\Rightarrow \quad AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$$

EXAMPLE 34 *In a right triangle ABC right-aangled at C, P and Q are the points on the sides CA and CB respectively, which divide these sides in the ratio 2 : 1. Prove that*

(i) $9\,AQ^2 = 9\,AC^2 + 4\,BC^2$ (ii) $9\,BP^2 = 9\,BC^2 + 4\,AC^2$ (iii) $9\,(AQ^2 + BP^2) = 13\,AB^2$

SOLUTION It is given that P divides CA in the ratio $2:1$. Therefore,

$$CP = \frac{2}{3}\,AC \qquad \ldots(i)$$

Also, Q divides CB in the ratio $2:1$.

$$\therefore \quad QC = \frac{2}{3}\,BC \qquad \ldots(ii)$$

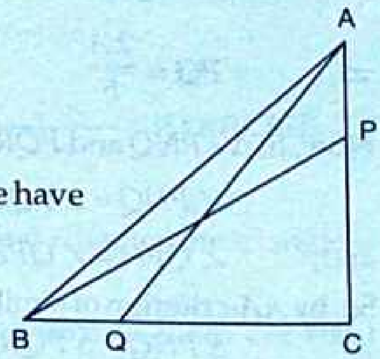


Fig. 7.217

(i) Applying Pythagoras theorem in right-angled triangle ACQ, we have

$$AQ^2 = QC^2 + AC^2$$

$$\Rightarrow \quad AQ^2 = \frac{4}{9}\,BC^2 + AC^2 \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad 9\,AQ^2 = 4\,BC^2 + 9\,AC^2 \qquad \ldots(iii)$$

(ii) Applying Pythagoras theorem in right triangle BCP, we have

$$BP^2 = BC^2 + CP^2$$

$$\Rightarrow \quad BP^2 = BC^2 + \frac{4}{9}\,AC^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 9\,BP^2 = 9\,BC^2 + 4\,AC^2 \qquad \ldots(iv)$$

Adding (iii) and (iv), we get

$$9(AQ^2 + BP^2) = 13(BC^2 + AC^2)$$

$$\Rightarrow \quad 9(AQ^2 + BP^2) = 13AB^2 \qquad [\because BC^2 = AC^2 + AB^2]$$

LEVEL-3

EXAMPLE 35 *In a Δ ABC, the angles at B and C are acute. If BE and CF be drawn perpendiculars on AC and AB respectively, prove that $BC^2 = AB \times BF + AC \times CE$.*

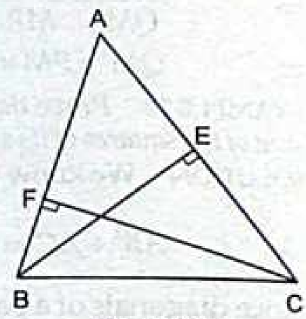


Fig. 7.218

SOLUTION In ΔABC, $\angle B$ is acute and $CF \perp AB$.

$\therefore \quad AC^2 = AB^2 + BC^2 - 2\,AB.\,BF \quad \ldots(i)$

Similarly in ΔABC, $\angle B$ is acute and $BE \perp AC$.

$\therefore \quad AB^2 = BC^2 + AC^2 - 2\,AC.\,CE \quad \ldots(ii)$

Adding (i) and (ii), we get

$$AC^2 + AB^2 = AB^2 + BC^2 - 2\,AB.\,BF + BC^2 + AC^2 - 2\,AC.\,CE$$

$\Rightarrow \quad 2\,BC^2 - 2\,(AB.\,BF + AC.CE) = 0$

$\Rightarrow \quad 2\,BC^2 = 2\,(AB.\,BF + AC.CE)$

$\Rightarrow \quad BC^2 = AB.\,BF + AC.CE$

EXAMPLE 36 *If A be the area of a right triangle and b one of the sides containing the right angle, prove that the length of the altitude on the hypotenuse is $\dfrac{2Ab}{\sqrt{b^4 + 4A^2}}$.*

SOLUTION Let PQR be a right triangle right-angled at Q such that $QR = b$ and $A =$ Area of $\Delta\, PQR$. Draw QN perpendicular to PR.

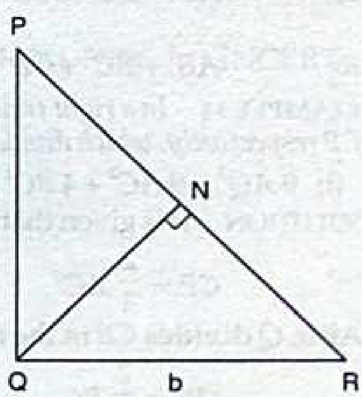


Fig. 7.219

Now,

$A =$ Area of $\Delta\, PQR$

$\Rightarrow \quad A = \dfrac{1}{2}\,(QR \times PQ)$

$\Rightarrow \quad A = \dfrac{1}{2}\,(b \times PQ)$

$\Rightarrow \quad PQ = \dfrac{2A}{b} \quad \ldots(i)$

Now, in Δ's PNQ and PQR, we have

$\angle PNQ = \angle PQR$ [Each equal to 90°]

and, $\angle QPN = \angle QPR$ [Common]

So, by AA-criterion of similarity, we obtain

$\Delta\, PNQ \sim \Delta\, PQR$

$\Rightarrow \quad \dfrac{PQ}{PR} = \dfrac{NQ}{QR} \quad \ldots(ii)$

Applying Pythagoras theorem in $\Delta\, PQR$, we obtain

$PQ^2 + QR^2 = PR^2$

$\Rightarrow \quad \dfrac{4A^2}{b^2} + b^2 = PR^2$

$\Rightarrow \quad PR = \sqrt{\dfrac{4A^2 + b^4}{b^2}} = \dfrac{\sqrt{4A^2 + b^4}}{b}$

From (i) and (ii), we have

$$\frac{2A}{b \times PR} = \frac{NQ}{b}$$

$$\Rightarrow \quad NQ = \frac{2A}{PR} \Rightarrow NQ = \frac{2Ab}{\sqrt{4A^2 + b^4}} \qquad \left[\because PR = \frac{\sqrt{4A^2 + b^4}}{b}\right]$$

EXAMPLE 37 *The perimeter of a right triangle is 60 cm. Its hypotenuse is 25 cm. Find the area of the triangle.* **[CBSE 2016]**

SOLUTION Let ABC be a right triangle right angled at B. It is given that $AC = 25$ cm. Let $AB = x$ and $BC = y$.

Using Pythagoras theorem, we obtain

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \quad x^2 + y^2 = 25^2 \qquad ...(i)$$

It is given that the perimeter of triangle ABC is 60 cm.

$$\therefore \quad AB + BC + CA = 60$$

$$\Rightarrow \quad x + y + 25 = 60$$

$$\Rightarrow \quad x + y = 35$$

$$\Rightarrow \quad (x + y)^2 = 35^2 \qquad \text{[On squaring both sides]}$$

$$\Rightarrow \quad x^2 + y^2 + 2xy = 35^2$$

$$\Rightarrow \quad 25^2 + 2xy = 35^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2xy = 35^2 - 25^2$$

$$\Rightarrow \quad 2xy = (35 + 25)(35 - 25)$$

$$\Rightarrow \quad 2xy = 60 \times 10$$

$$\Rightarrow \quad xy = 300$$

$$\Rightarrow \quad \frac{1}{2}xy = 150$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = 150 \text{ cm}^2$$

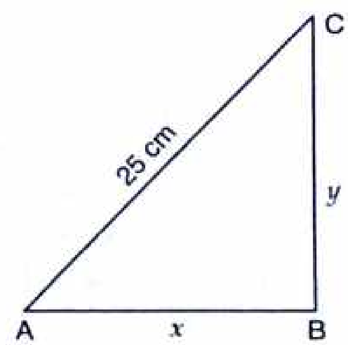


Fig. 7.220

EXERCISE 7.7

LEVEL-1

1. If the sides of a triangle are 3 cm, 4 cm and 6 cm long, determine whether the triangle is a right-angled triangle.
2. The sides of certain triangles are given below. Determine which of them are right triangles.
   (i) $a = 7$ cm, $b = 24$ cm and $c = 25$ cm
   (ii) $a = 9$ cm, $b = 16$ cm and $c = 18$ cm
   (iii) $a = 1.6$ cm, $b = 3.8$ cm and $c = 4$ cm
   (iv) $a = 8$ cm, $b = 10$ cm and $c = 6$ cm
3. A man goes 15 metres due west and then 8 metres due north. How far is he from the starting point?
4. A ladder 17 m long reaches a window of a building 15 m above the ground. Find the distance of the foot of the ladder from the building.
5. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between their feet is 12 m, find the distance between their tops. [NCERT, CBSE 2002C]

6. In an isosceles triangle ABC, $AB = AC = 25$ cm, $BC = 14$ cm. Calculate the altitude from A on BC.
7. The foot of a ladder is 6 m away from a wall and its top reaches a window 8 m above the ground. If the ladder is shifted in such a way that its foot is 8 m away from the wall, to what height does its tip reach?
8. Two poles of height 9 m and 14 m stand on a plane ground. If the distance between their feet is 12 m, find the distance between their tops.
9. Using Pythagoras theorem determine the length of AD in terms of b and c shown in Fig. 7. 221.

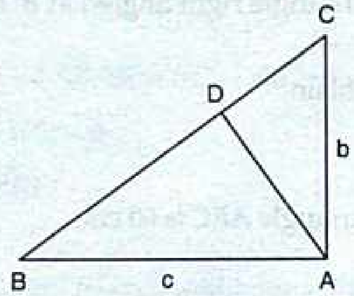


Fig. 7.221

10. A triangle has sides 5 cm, 12 cm and 13 cm. Find the length to one decimal place, of the perpendicular from the opposite vertex to the side whose length is 13 cm.
11. $ABCD$ is a square. F is the mid-point of AB. BE is one third of BC. If the area of $\Delta FBE = 108 \text{ cm}^2$, find the length of AC.
12. In an isosceles triangle ABC, if $AB = AC = 13$ cm and the altitude from A on BC is 5 cm, find BC.
13. In a ΔABC, $AB = BC = CA = 2a$ and $AD \perp BC$. Prove that
    (i) $AD = a\sqrt{3}$ (ii) Area $(\Delta ABC) = \sqrt{3}\, a^2$
14. The lengths of the diagonals of a rhombus are 24 cm and 10 cm. Find each side of the rhombus.
15. Each side of a rhombus is 10 cm. If one of its diagonals is 16 cm find the length of the other diagonal.
16. Calculate the height of an equilateral triangle each of whose sides measures 12 cm.
17. In Fig. 7.222, $\angle B < 90°$ and segment $AD \perp BC$, show that
    (i) $b^2 = h^2 + a^2 + x^2 - 2ax$ (ii) $b^2 = a^2 + c^2 - 2ax$

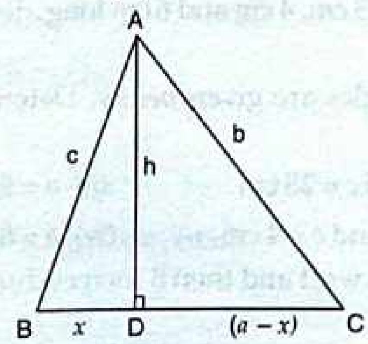


Fig. 7.222

18. In an equilateral ΔABC, $AD \perp BC$, prove that $AD^2 = 3\, BD^2$ **[CBSE 2002C]**

19. ΔABD is a right triangle right-angled at A and $AC \perp BD$. Show that

(i) $AB^2 = BC \cdot BD$ (ii) $AC^2 = BC \cdot DC$ (iii) $AD^2 = BD \cdot CD$ (iv) $\frac{AB^2}{AC^2} = \frac{BD}{DC}$ [NCERT]

20. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut? [NCERT]

21. Determine whether the triangle having sides $(a-1)$ cm, $2\sqrt{a}$ cm and $(a+1)$ cm is a right angled triangle. [CBSE 2010]

LEVEL-2

22. In an acute-angled triangle, express a median in terms of its sides.

23. In right-angled triangle ABC in which $\angle C = 90°$, if D is the mid-point of BC, prove that $AB^2 = 4AD^2 - 3AC^2$. [CBSE 2010]

24. In Fig. 7.223, D is the mid-point of side BC and $AE \perp BC$. If $BC = a$ $AC = b$, $AB = c$, $ED = x$, $AD = p$ and $AE = h$, prove that:

(i) $b^2 = p^2 + ax + \frac{a^2}{4}$ (ii) $c^2 = p^2 - ax + \frac{a^2}{4}$ (iii) $b^2 + c^2 = 2p^2 + \frac{a^2}{2}$

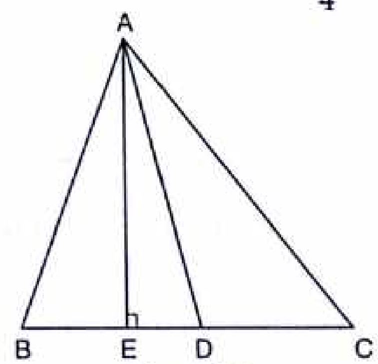


Fig. 7.223

25. In ΔABC, $\angle A$ is obtuse, $PB \perp AC$ and $QC \perp AB$. Prove that:

(i) $AB \times AQ = AC \times AP$ (ii) $BC^2 = (AC \times CP + AB \times BQ)$

26. In a right ΔABC right-angled at C, if D is the mid-point of BC, prove that $BC^2 = 4(AD^2 - AC^2)$.

27. In a quadrilateral $ABCD$, $\angle B = 90°$, $AD^2 = AB^2 + BC^2 + CD^2$, prove that $\angle ACD = 90°$

28. An aeroplane leaves an airport and flies due north at a speed of 1000 km/hr. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km/hr. How far apart will be the two planes after $1\frac{1}{2}$ hours? [NCERT]

ANSWERS

1. No
2. (i), (iv)
3. 17 m
4. 8 m
5. 13 m
6. 24 m
7. 6 m
8. 13 m
9. $\frac{bc}{\sqrt{b^2 + c^2}}$
10. 4.6 cm
11. 50.904 cm
12. 24 cm
14. 13 cm
15. 12 cm
16. 10.39 cm
20. $6\sqrt{7}$ m
21. yes
22. $\frac{2AB^2 + 2AC^2 - BC^2}{4}$
28. 2343 km (Approx)

HINT TO SELECTED PROBLEMS

5. Find the hypotenuse of a right triangle having two sides $(11-6)$ m $= 5$ m and 12 m.
6. Let D be the foot of the perpendicular from A on BC. Then,

$$\Delta\, ABD \cong ACD \Rightarrow BD = CD = 7 \text{ cm}.$$

Now, apply Pythagoras theorem in $\Delta\, ABD$.

9. Area of $\Delta\, ABC = \frac{1}{2}(AB \times AC) = \frac{1}{2}bc.$

Also, Area of $\Delta\, ABC = \frac{1}{2}(BC \times AD) = \frac{1}{2}\sqrt{b^2+c^2} \times AD$

$$\therefore \quad \frac{1}{2}\sqrt{b^2+c^2} \times AD = \frac{1}{2}bc \Rightarrow AD = \frac{bc}{\sqrt{b^2+c^2}}$$

10. Let $AB = 5$ cm, $BC = 12$ cm and $AC = 13$ cm. Then, $AC^2 = AB^2 + BC^2$. This proves that $\Delta\, ABC$ is a right triangle, right-angled at B. Let BD be the length of perpendicular from B on AC.

Now,

Area $\Delta\, ABC = \frac{1}{2}(BC \times BA) = \frac{1}{2}(12 \times 5) = 30 \text{ cm}^2$

Also, Area of $\Delta\, ABC = \frac{1}{2}AC \times BD = \frac{1}{2}(13 \times BD) \Rightarrow (13 \times BD) = 30 \Rightarrow BD = \frac{60}{13}$ cm.

13. First prove that $\Delta\, ABD \cong \Delta\, ACD$ and then use Phythagoras theorem in $\Delta\, ABD$ to find AD.
14. Let $ABCD$ be a rhombus in which $AC = 24$ cm and $BD = 10$ cm. Suppose the diagonals intersect at O. Since the diagonals of a rhombus bisect each other at right angles. Therefore, $\Delta\, OAB$ is a right triangle, right-angled at O such that

$$OA = \frac{1}{2}AC = 12 \text{ cm and } OB = \frac{1}{2}BD = 5 \text{ cm}.$$

Using Pythagoras theorem, we obtain

$$AB^2 = OA^2 + OB^2 = 12^2 + 5^2 = 169 \Rightarrow AB = 13 \text{ cm}.$$

22. Let ABC be an acute angled triangle and let AD be a median. Then,

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$ [See Theorem 3 on page 7.100]

$$\Rightarrow \quad AD^2 = \frac{2AB^2 + 2AC^2 - BC^2}{4}$$

23. In right triangles ABC and ADC, we have

$$AB^2 = AC^2 + BC^2 \quad \ldots(i)$$

and, $AD^2 = AC^2 + CD^2 \quad \ldots(ii)$

Now,

$$AB^2 = AC^2 + BC^2$$

$\Rightarrow \quad AB^2 = AC^2 + 4CD^2 \qquad \left[\because CD = BD = \frac{1}{2}BC\right]$

$\Rightarrow \quad AB^2 = AC^2 + 4(AD^2 - AC^2) \qquad$ [Using (ii)]

$\Rightarrow \quad AB^2 = 4AD^2 - 3AC^2$

REVISION EXERCISE

1. In each of the figures [7.224 (i)- (iv)] given below, a line segment is drawn parallel to one side of the triangle and the lengths of certain line-segments are marked. Find the value of x in each of the following :

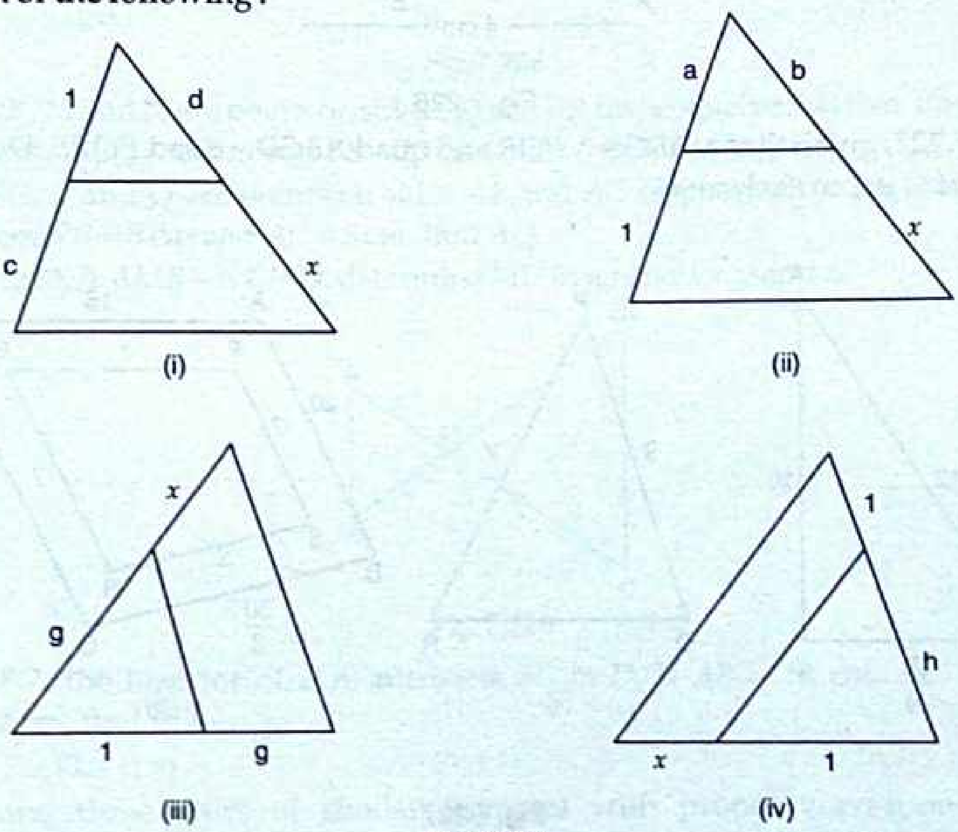


Fig. 7.224

2. What values of x will make $DE \parallel AB$ in the Fig. 7.225?

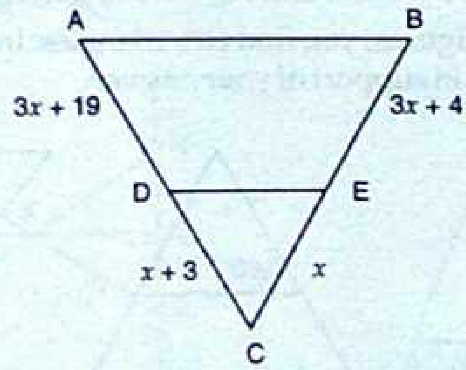


Fig. 7.225

3. In ΔABC, points P and Q are on CA and CB, respectively such that $CA = 16$ cm, $CP = 10$ cm, $CB = 30$ cm and $CQ = 25$ cm. Is $PQ \parallel AB$?
4. In Fig. 7.226, $DE \parallel CB$. Determine AC and AE.

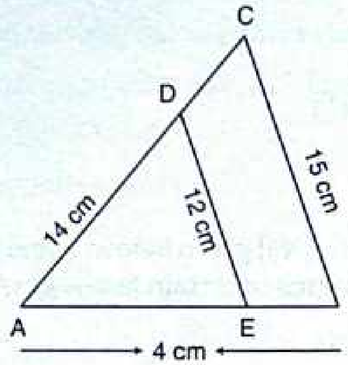


Fig. 7.226

5. In Fig. 7.227, given that $\Delta ABC \sim \Delta PQR$ and quad $ABCD \sim$ quad $PQRS$. Determine the values of x, y, z in each case.

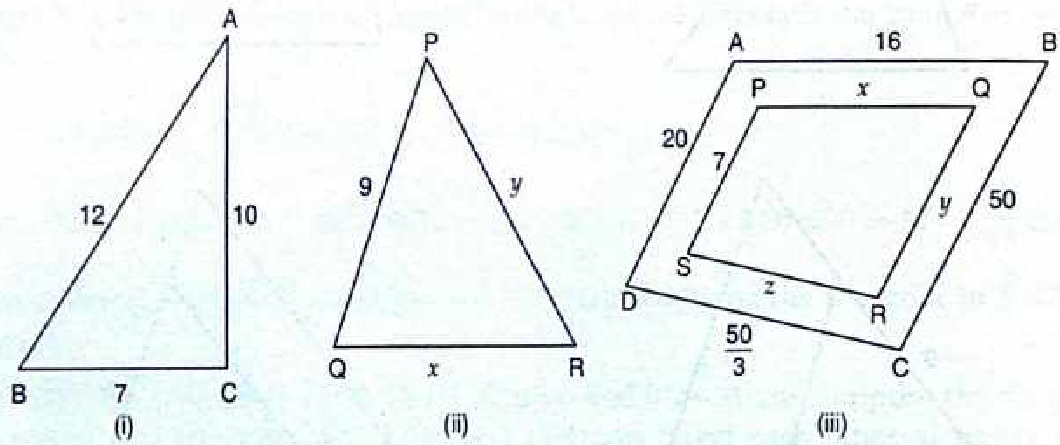
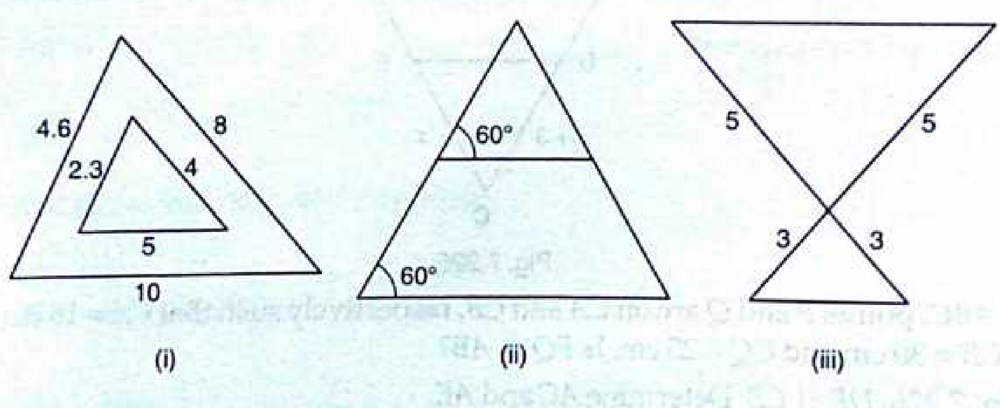
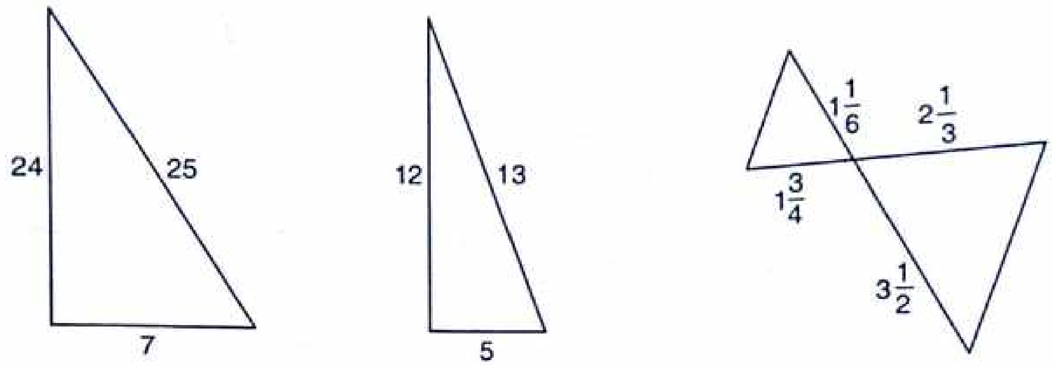


Fig. 7.227

6. In ΔABC, P and Q are points on sides AB and AC respectively such that $PQ \parallel BC$. If $AP = 4$ cm, $PB = 6$ cm and $PQ = 3$ cm, determine BC.

7. In each of the following figures, you find two triangles. Indicate whether the triangles are similar. Give reasons in support of your answer.





(iv) (v) (vi)

Fig. 7.228

8. In $\Delta\ PQR$, M and N are points on sides PQ and PR respectively such that $PM = 15$ cm and $NR = 8$ cm. If $PQ = 25$ cm and $PR = 20$ cm state whether $MN \parallel QR$.
9. In $\Delta\ ABC$, P and Q are points on sides AB and AC respectively such that $PQ \parallel BC$. If $AP = 3$ cm, $PB = 5$ cm and $AC = 8$ cm, find AQ.
10. In Fig.7.229, $\Delta\ AMB \sim \Delta\ CMD$; determine MD in terms of x, y and z.

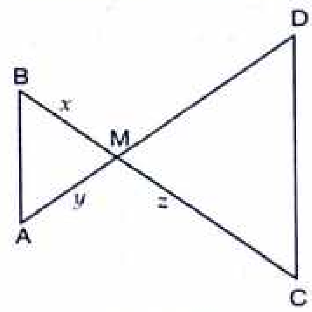


Fig. 7.229

11. In $\Delta\ ABC$, the bisector of $\angle A$ intersects BC in D. If $AB = 18$ cm, $AC = 15$ cm and $BC = 22$ cm, find BD
12. In Fig. 7.230, $l \parallel m$
   (i) Name three pairs of similar triangles with proper correspondence; write similarities.
   (ii) Prove that $\dfrac{AB}{PQ} = \dfrac{AC}{PR} = \dfrac{BC}{RQ}$

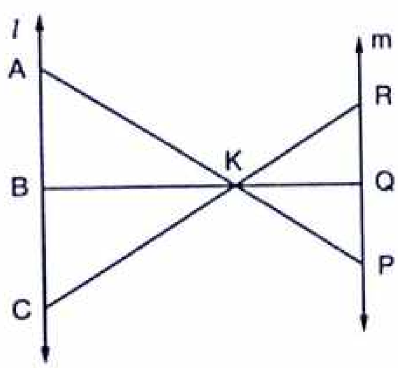


Fig.7.230

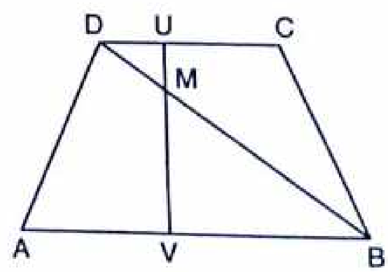


Fig. 7.231

13. In Fig. 7.231, $AB \parallel DC$ Prove that
   (i) $\Delta\ DMU \sim \Delta\ BMV$ (ii) $DM \times BV = BM \times DU$

14. $ABCD$ is a trapezium in which $AB \parallel DC$. P and Q are points on sides AD and BC such that $PQ \parallel AB$. If $PD = 18$, $BQ = 35$ and $QC = 15$, find AD.

15. In ΔABC, D and E are points on sides AB and AC respectively such that
$AD \times EC = AE \times DB$. Prove that $DE \parallel BC$.

16. $ABCD$ is a trapezium having $AB \parallel DC$. Prove that O, the point of intersection of diagonals, divides the two diagonals in the same ratio. Also prove that
$$\frac{ar(\Delta OCD)}{ar(\Delta OAB)} = \frac{1}{9}, \text{ if } AB = 3\,CD.$$

17. Corresponding sides of two triangles are in the ratio 2 : 3. If the area of the smaller triangle is 48 cm^2, determine the area of the larger triangle.

18. The areas of two similar triangles are 36 cm^2 and 100 cm^2. If the length of a side of the smaller triangle in 3 cm, find the length of the corresponding side of the larger triangle.

19. Corresponding sides of two similar triangles are in the ratio 1 : 3. If the area of the smaller triangle in 40 cm^2, find the area of the larger triangle.

20. In Fig. 7.232, each of PA, QB, RC and SD is perpendicular to l. If $AB = 6$ cm, $BC = 9$ cm, $CD = 12$ cm and $PS = 36$ cm, then determine PQ, QR and RS.

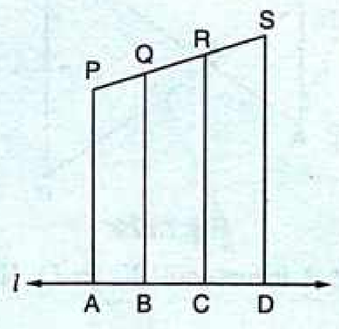


Fig. 7.232

21. In each of the figures given below, an altitude is drawn to the hypotenuse by a right-angled triangle. The length of different line-segments are marked in each figure. Determine x, y, z in each case

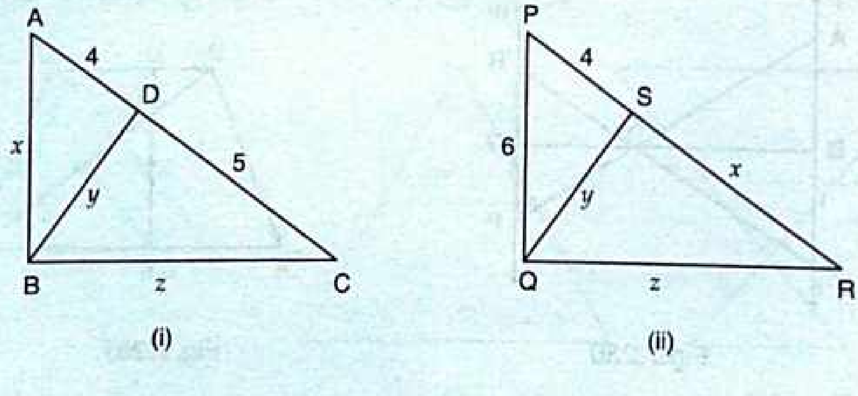


Fig. 7.233

22. Prove that in an equilateral triangle, three times the square of a side is equal to four times the square of its altitudes.

LEVEL-2

23. In $\Delta\, ABC$, AD and BE are altitudes. Prove that $\dfrac{ar\,(\Delta\, DEC)}{ar\,(\Delta\, ABC)} = \dfrac{DC^2}{AC^2}$.

24. The diagonals of quadrilateral $ABCD$ intersect at O. Prove that $\dfrac{ar\,(\Delta\, ACB)}{ar\,(\Delta\, ACD)} = \dfrac{BO}{DO}$.

25. In $\Delta\, ABC$, ray AD bisects $\angle A$ and intersects BC in D. If $BC = a$, $AC = b$ and $AB = c$, prove that

    (i) $BD = \dfrac{ac}{b+c}$ (ii) $DC = \dfrac{ab}{b+c}$

26. There is a staircase as shown in Fig. 7.234, connecting points A and B. Measurements of steps are marked in the figure. Find the straight line distance between A and B.

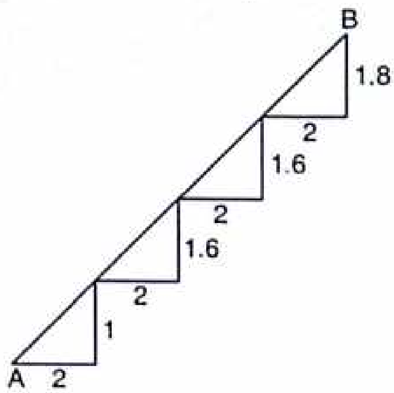


Fig. 7.234

27. In $\Delta\, ABC$, $\angle A = 60°$. Prove that $BC^2 = AB^2 + AC^2 - AB \cdot AC$.

28. In $\Delta\, ABC$, $\angle C$ is an obtuse angle. $AD \perp BC$ and $AB^2 = AC^2 + 3\,BC^2$. Prove that $BC = CD$.

29. A point D is on the side BC of an equilateral triangle ABC such that $DC = \dfrac{1}{4}\,BC$. Prove that $AD^2 = 13\,CD^2$.

30. In $\Delta\, ABC$, if $BD \perp AC$ and $BC^2 = 2\,AC \cdot CD$, then prove that $AB = AC$.

31. In a quadrilateral $ABCD$, given that $\angle A + \angle D = 90°$. Prove that $AC^2 + BD^2 = AD^2 + BC^2$.

32. In $\Delta\, ABC$, given that $AB = AC$ and $BD \perp AC$. Prove that $BC^2 = 2\,AC \cdot CD$

33. $ABCD$ is a rectangle. Points M and N are on BD such that $AM \perp BD$ and $CN \perp BD$. Prove that $BM^2 + BN^2 = DM^2 + DN^2$.

34. In $\Delta\, ABC$, AD is a median. Prove that $AB^2 + AC^2 = 2\,AD^2 + 2\,DC^2$.

35. In $\Delta\, ABC$, $\angle ABC = 135°$. Prove that $AC^2 = AB^2 + BC^2 + 4\,ar\,(\Delta\, ABC)$

36. In a quadrilateral $ABCD$, $\angle B = 90°$. If $AD^2 = AB^2 + BC^2 + CD^2$ then prove that $\angle ACD = 90°$

37. In a triangle ABC, N is a point on AC such that $BN \perp AC$ $\cdot$ If $BN^2 = AN \cdot NC$, prove that $\angle B = 90°$.

38. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out (See Fig. 7.235)? If she pulls the string at the rate of 5 cm per second, what will the horizontal distance of the fly from her after 12 seconds.

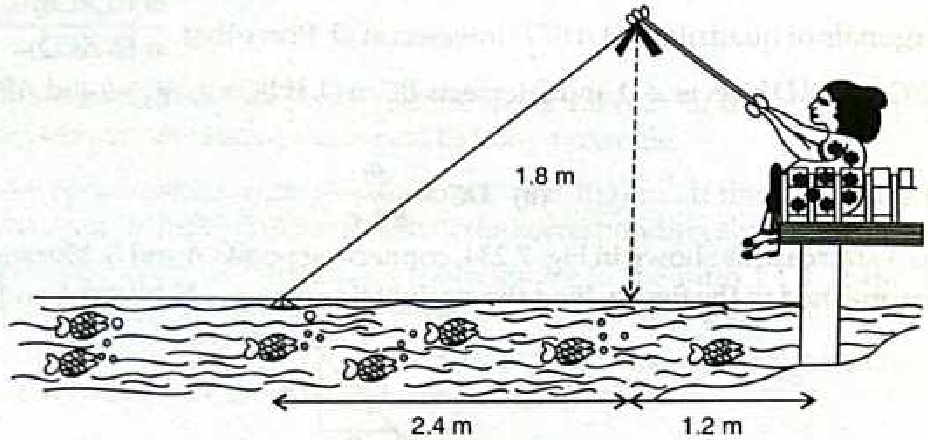


Fig. 7.235

ANSWERS

1. (i) cd (ii) $\frac{b}{a}$ (iii) g^2 (iv) $\frac{1}{h}$ 2. 2 3. No

4. $\frac{35}{2}, \frac{16}{5}$ 5. (i) $x = \frac{21}{4}, y = \frac{15}{2}$ (ii) $x = \frac{28}{5}, y = \frac{35}{2}, z = \frac{35}{6}$

6. 7.5 cm 7. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) Yes (vi) No

8. Yes 9. 3 cm 10. $\frac{xz}{y}$ 11. 12 14. 60 17. 108 cm^2

18. 5 cm 19. 360 cm^2 20. 8 cm, 12 cm, 16 cm

21. (i) $x = 6, y = 2\sqrt{5}, z = 3\sqrt{5}$, (ii) $x = 5, y = 2\sqrt{5}, z = 3\sqrt{5}$ 26. 10.

HINT TO SELECTED PROBLEM

31. Extend AB and CD to intersect at O.

Now, $\angle AOD = 90°$

$\Rightarrow AC^2 = OA^2 + OC^2$ and $BD^2 = OB^2 + OD^2$

$\Rightarrow AC^2 + BD^2 = (OA^2 + OD^2) + (OB^2 + OC^2) = AD^2 + BC^2$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. State basic proportionality theorem and its converse.
2. In Fig. 7.236, find AC.

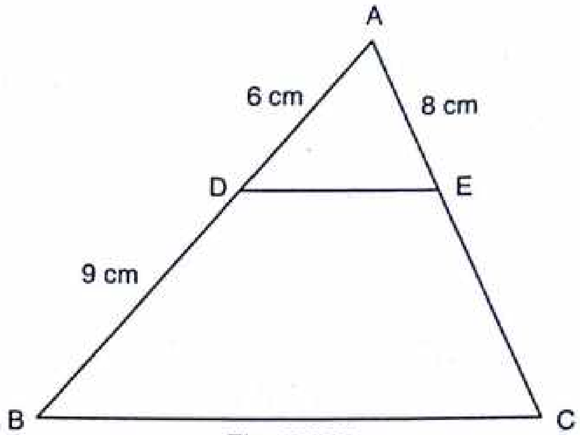


Fig. 7.236

3. In Fig. 7.237, if AD is the bisector of $\angle A$, what is AC?

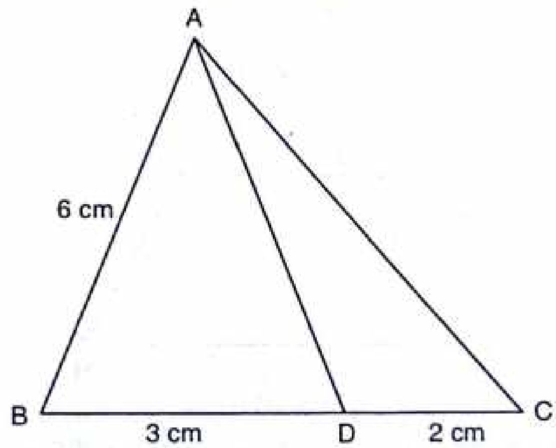


Fig. 7.237

4. Given $\Delta ABC \sim \Delta PQR$, if $\frac{AB}{PQ} = \frac{1}{3}$, then find $\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)}$. **[CBSE 2018]**
5. State SSS similarity criterion.
6. State SAS similarity criterion.
7. In the adjoining figure, DE is parallel to BC and $AD = 1$ cm, $BD = 2$ cm. What is the ratio of the area of $\Delta\ ABC$ to the area of ΔADE?
8. In the figure given below $DE \parallel BC$. If $AD = 2.4$ cm, $DB = 3.6$ cm and $AC = 5$ cm. Find AE.
9. If the areas of two similar triangles ABC and PQR are in the ratio 9 : 16 and $BC = 4.5$ cm, what is the length of QR?
10. The areas of two similar triangles are 169 cm^2 and 121 cm^2 respectively. If the longest side of the larger triangle is 26 cm, what is the length of the longest side of the smaller triangle?
11. If ABC and DEF are similar triangles such that $\angle A = 57°$ and $\angle E = 73°$, what is the measure of $\angle C$?
12. If the altitude of two similar triangles are in the ratio 2 : 3, what is the ratio of their areas?
13. If ΔABC and ΔDEF are two triangles such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{3}{4}$, then write Area ($\Delta ABC$): Area ($\Delta DEF$).
14. If ΔABC and ΔDEF are similar triangles such that $AB = 3$ cm, $BC = 2$ cm $CA = 2.5$ cm and $EF = 4$ cm, write the perimeter of ΔDEF.
15. State Pythagoras theorem and its converse.
16. The lengths of the diagonals of a rhombus are 30 cm and 40 cm. Find the side of the rhombus. **[CBSE 2008]**

17. In Fig. 7.238, $PQ \parallel BC$ and $AP : PB = 1 : 2$. Find $\dfrac{\text{area }(\Delta\ APQ)}{\text{area }(\Delta\ ABC)}$. **[CBSE 2008]**

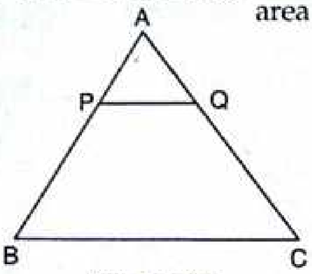


Fig. 7.238

18. In Fig. 7.239, S and T are points on the sides PQ and PR respectively of ΔPQR such that $PT = 2$ cm, $TR = 4$ cm and ST is parallel to QR. Find the ratio of the areas of ΔPST *and* ΔPQR. **[CBSE 2010]**

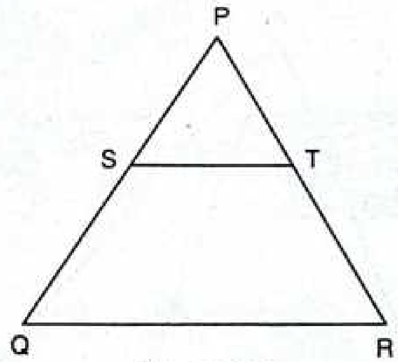


Fig. 7.239

19. In Fig. 7.240, ΔAHK is similar to ΔABC. If $AK = 10$ cm, $BC = 3.5$ cm and $HK = 7$ cm, find AC. **[CBSE 2010]**

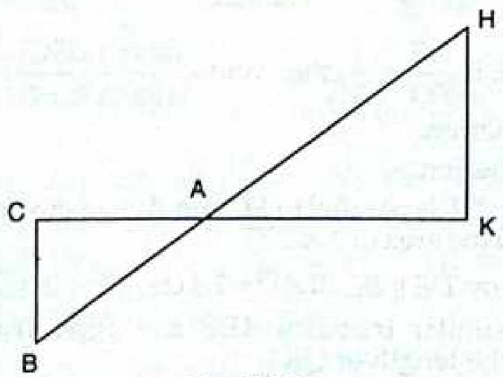


Fig. 7.240

20. In Fig. 7.241, $DE \parallel BC$ in ΔABC such that $BC = 8$ cm, $AB = 6$ cm and $DA = 1.5$ cm. Find DE.

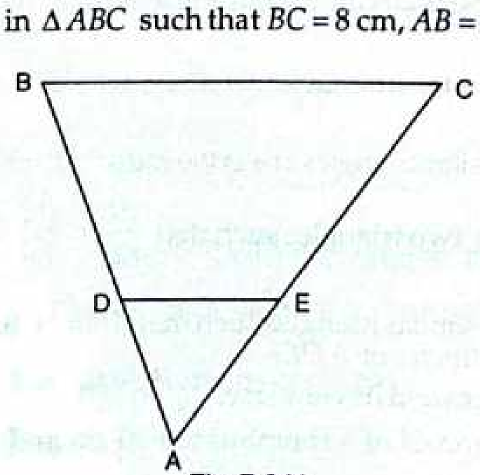


Fig. 7.241

21. In Fig. 7.242, $DE \parallel BC$ and $AD = \frac{1}{2} BD$. If $BC = 4.5$ cm, find DE. [CBSE 2010]

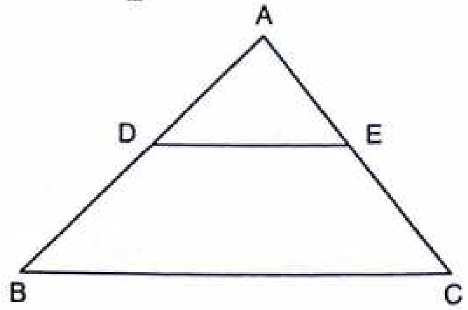


Fig. 7.242

LEVEL-2

22. In Fig. 7.243, $LM = LN = 46°$. Express x in terms of a, b and c where a, b, c are lengths of LM, MN and and NK respectively.

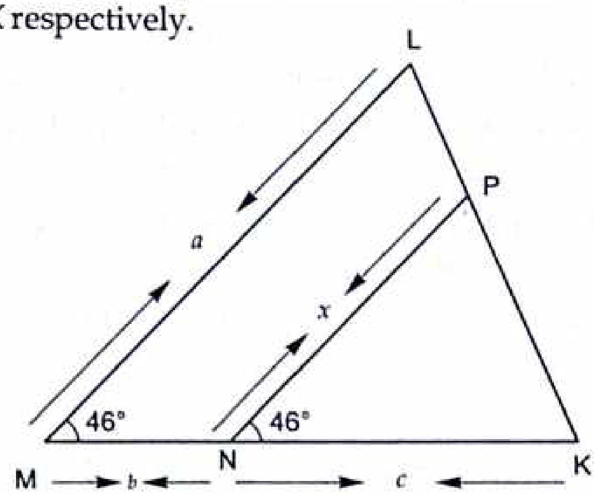


Fig. 7.243

ANSWERS

1. 20 cm	2. 4 cm	4. $\frac{1}{9}$	7. $9:1$	8. 2 cm	9. 6 cm
10. 22 cm	11. 50°	12. $4:9$	13. $9:16$	14. 15 cm	16. 25 cm
17. $1:4$	18. $1:9$	19. 5 cm	20. 2 cm	21. 1.5 cm	22. $\frac{ac}{b+c}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

LEVEL-1

1. Sides of two similar triangles are in the ratio $4:9$. Areas of these triangles are in the ratio.
   (a) $2:3$ (b) $4:9$ (c) $81:16$ (d) $16:81$
2. The areas of two similar triangles are in respectively 9 cm^2 and 16 cm^2. The ratio of their corresponding sides is
   (a) $3:4$ (b) $4:3$ (c) $2:3$ (d) $4:5$
3. The areas of two similar triangles $\Delta\, ABC$ and $\Delta\, DEF$ are 144 cm^2 and 81 cm^2 respectively. If the longest side of larger $\Delta\, ABC$ be 36 cm, then. the longest side of the smaller triangle $\Delta\, DEF$ is

(a) 20 cm (b) 26 cm (c) 27 cm (d) 30 cm

4. ΔABC and ΔBDE are two equilateral triangles such that D is the mid-point of BC. The ratio of the areas of triangles ABC and BDE is

(a) 2 : 1 (b) 1 : 2 (c) 4 : 1 (d) 1 : 4

5. If ΔABC and ΔDEF are similar such that $2\,AB = DE$ and $BC = 8$ cm, then $EF =$

(a) 16 cm (b) 12 cm (c) 8 cm (d) 4 cm.

6. If ΔABC and ΔDEF are two triangles such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{2}{5}$, then Area (ΔABC) : Area $(\Delta DEF) =$

(a) 2 : 5 (b) 4 : 25 (c) 4 : 15 (d) 8 : 125

7. XY is drawn parallel to the base BC of a ΔABC cutting AB at X and AC at Y. If $AB = 4\,BX$ and $YC = 2$ cm, then $AY =$

(a) 2 cm (b) 4 cm (c) 6 cm (d) 8 cm.

8. Two poles of height 6 m and 11 m stand vertically upright on a plane ground. If the distance between their foot is 12 m, the distance between their tops is

(a) 12 m (b) 14 m (c) 13 m. (d) 11 m

9. In ΔABC, D and E are points on side AB and AC respectively such that $DE \parallel BC$ and $AD : DB = 3 : 1$. If $EA = 3.3$ cm, then $AC =$

(a) 1.1 cm (b) 4 cm (c) 4.4 cm (d) 5.5 cm

10. In triangles ABC and DEF, $\angle A = \angle E = 40°$, $AB : ED = AC : EF$ and $\angle F = 65°$, then $\angle B =$

(a) 35° (b) 65° (c) 75° (d) 85°

11. If ABC and DEF are similar triangles such that $\angle A = 47°$ and $\angle E = 83°$, then $\angle C =$

(a) 50° (b) 60° (c) 70° (d) 80°

12. If D, E, F are the mid-points of sides BC, CA and AB respectively of ΔABC, then the ratio of the areas of triangles DEF and ABC is

(a) 1 : 4 (b) 1 : 2 (c) 2 : 3 (d) 4 : 5

13. In an equilateral triangle ABC, if $AD \perp BC$, then

(a) $2\,AB^2 = 3\,AD^2$ (b) $4\,AB^2 = 3\,AD^2$ (c) $3\,AB^2 = 4\,AD^2$ (d) $3\,AB^2 = 2\,AD^2$

14. If ΔABC is an equilateral triangle such that $AD \perp BC$, then $AD^2 =$

(a) $\frac{3}{2}DC^2$ (b) $2\,DC^2$ (c) $3\,CD^2$ (d) $4\,DC^2$

15. In a ΔABC, AD is the bisector of $\angle BAC$. If $AB = 6$ cm, $AC = 5$ cm and $BD = 3$ cm, then $DC =$

(a) 11.3 cm (b) 2.5 cm (c) 3 : 5 cm (d) None of these.

16. In a ΔABC, AD is the bisector of $\angle BAC$. If $AB = 8$ cm, $BD = 6$ cm and $DC = 3$ cm. Find AC

(a) 4 cm (b) 6 cm (c) 3 cm (d) 8 cm

17. $ABCD$ is a trapezium such that $BC \parallel AD$ and $AB = 4$ cm. If the diagonals AC and BD intersect at O such that $\frac{AO}{OC} = \frac{DO}{OB} = \frac{1}{2}$, then $BC =$

(a) 7 cm (b) 8 cm (c) 9 cm (d) 6 cm

18. If ABC is a right triangle right-angled at B and M, N are the mid-points of AB and BC respectively, then $4(AN^2 + CM^2) =$

(a) $4\,AC^2$ (b) $5\,AC^2$ (c) $\frac{5}{4}\,AC^2$ (d) $6\,AC^2$

19. If in ΔABC and ΔDEF, $\frac{AB}{DE} = \frac{BC}{FD}$, then $\Delta ABC \sim \Delta DEF$ *when*

(a) $\angle A = \angle F$ (b) $\angle A = \angle D$ (c) $\angle B = \angle D$ (d) $\angle B = \angle E$

20. If in two triangles ABC and DEF, $\frac{AB}{DE} = \frac{BC}{FE} = \frac{CA}{FD}$, then

(a) $\Delta FDE \sim \Delta CAB$ (b) $\Delta FDE \sim \Delta ABC$ (c) $\Delta CBA \sim \Delta FDE$ (d) $\Delta BCA \sim \Delta FDE$

21. $\Delta ABC \sim \Delta DEF$, ar $(\Delta ABC) = 9\text{ cm}^2$, ar $(\Delta DEF) = 16\text{ cm}^2$. If $BC = 2.1$ cm, then the measure of EF is

(a) 2.8 cm (b) 4.2 cm (c) 2.5 cm (d) 4.1 cm

22. The length of the hypotenuse of an isosceles right triangle whose one side is $4\sqrt{2}$ cm is

(a) 12 cm (b) 8 cm (c) $8\sqrt{2}$ cm (d) $12\sqrt{2}$ cm

23. A man goes 24 m due west and then 7 m due north. How far is he from the starting point?

(a) 31 m (b) 17 m (c) 25 m (d) 26 m

24. $\Delta ABC \sim \Delta DEF$. If $BC = 3$ cm, $EF = 4$ cm and ar $(\Delta ABC) = 54\text{ cm}^2$, then ar $(\Delta DEF) =$

(a) 108 cm^2 (b) 96 cm^2 (c) 48 cm^2 (d) 100 cm^2

25. $\Delta ABC \sim \Delta PQR$ such that ar $(\Delta ABC) = 4$ ar (ΔPQR). If $BC = 12$ cm, then $QR =$

(a) 9 cm (b) 10 cm (c) 6 cm (d) 8 cm

26. The areas of two similar triangles are 121 cm^2 and 64 cm^2 respectively. If the median of the first triangle is 12.1 cm, then the corresponding median of the other triangle is

(a) 11 cm (b) 8.8 cm (c) 11.1 cm (d) 8.1 cm

27. In an equilateral triangle ABC if $AD \perp BC$, then $AD^2 =$

(a) CD^2 (b) $2CD^2$ (c) $3CD^2$ (d) $4CD^2$

28. In an equilateral triangle ABC if $AD \perp BC$, then

(a) $5AB^2 = 4AD^2$ (b) $3AB^2 = 4AD^2$ (c) $4AB^2 = 3AD^2$ (d) $2AB^2 = 3AD^2$

29. In an isosceles triangle ABC if $AC = BC$ and $AB^2 = 2AC^2$, then $\angle C =$

(a) 30° (b) 45° (c) 90° (d) 60°

30. ΔABC is an isosceles triangle in which $\angle C = 90°$. If $AC = 6$ cm, then $AB =$

(a) $6\sqrt{2}$ cm (b) 6 cm (c) $2\sqrt{6}$ cm (d) $4\sqrt{2}$ cm

31. If in two triangles ABC and DEF, $\angle A = \angle E$, $\angle B = \angle F$, then which of the following is not true?

(a) $\frac{BC}{DF} = \frac{AC}{DE}$ (b) $\frac{AB}{DE} = \frac{BC}{DF}$ (c) $\frac{AB}{EF} = \frac{AC}{DE}$ (d) $\frac{BC}{DF} = \frac{AB}{EF}$

32. In Fig. 7.244 the measures of $\angle D$ and $\angle F$ are respectively

(a) $50°, 40°$ (b) $20°, 30°$ (c) $40°, 50°$ (d) $30°, 20°$

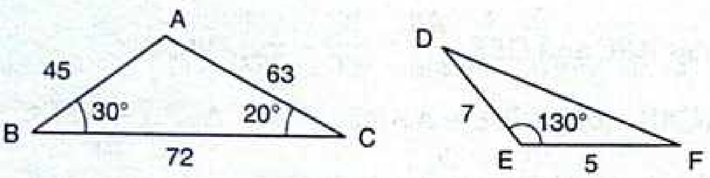


Fig. 7.244

33. In Fig. 7.245, the value of x for which $DE \parallel AB$ is

(a) 4 (b) 1 (c) 3 (d) 2

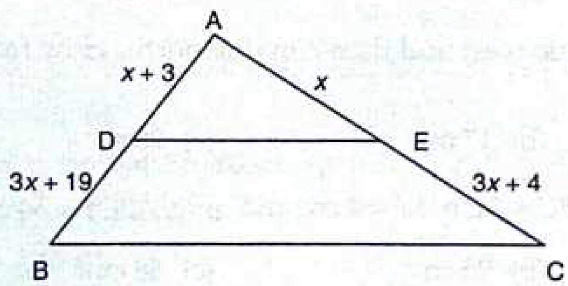


Fig. 7.245

34. In Fig. 7.246, if $\angle ADE = \angle ABC$, then $CE =$

(a) 2 (b) 5 (c) 9/2 (d) 3

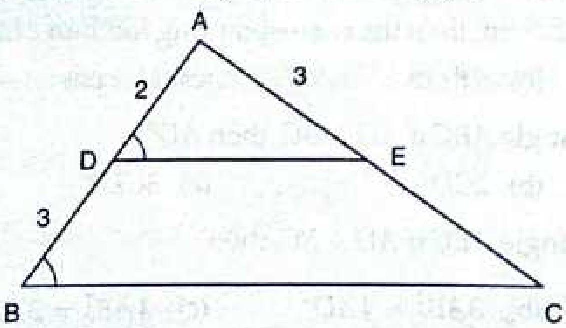


Fig. 7.246

35. In Fig. 7.247, $RS \parallel DB \parallel PQ$. If $CP = PD = 11$ cm and $DR = RA = 3$ cm. Then the values of x and y are respectively

(a) 12, 10 (b) 14, 6 (c) 10, 7 (d) 16, 8

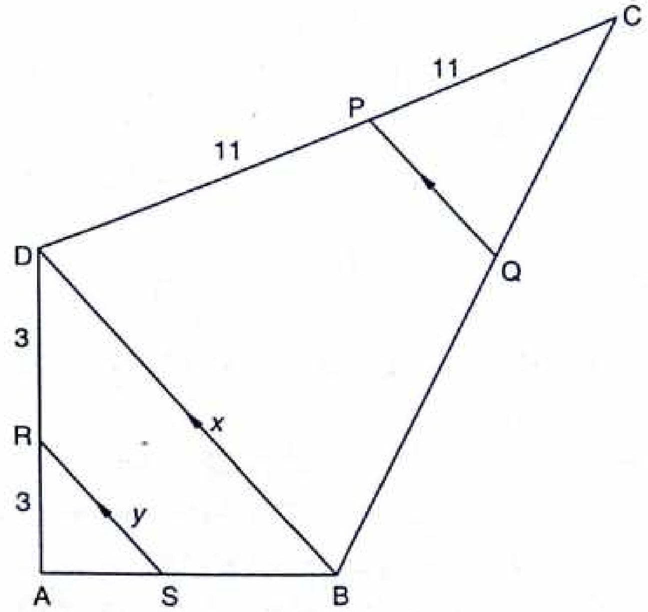


Fig. 7.247

36. In Fig. 7.248, if $PB \parallel CF$ and $DP \parallel EF$, then $\frac{AD}{DE} =$

(a) $\frac{3}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

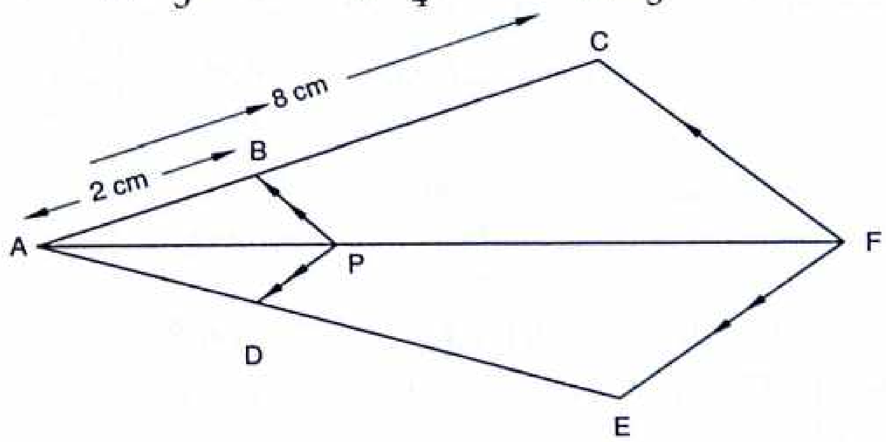


Fig. 7.248

37. A chord of a circle of radius 10 cm subtends a right angle at the centre. The length of the chord (in cm) is

(a) $5\sqrt{2}$ (b) $10\sqrt{2}$ (c) $\frac{5}{\sqrt{2}}$ (d) $10\sqrt{3}$ **[CBSE 2014]**

LEVEL-2

38. A vertical stick 20 m long casts a shadow 10 m long on the ground. At the same time, a tower casts a shadow 50 m long on the ground. The height of the tower is

(a) 100 m (b) 120 m (c) 25 m (d) 200 m

39. Two isosceles triangles have equal angles and their areas are in the ratio 16 : 25. The ratio of their corresponding heights is

(a) 4 : 5 (b) 5 : 4 (c) 3 : 2 (d) 5 : 7

40. $\Delta\, ABC$ is such that $AB = 3$ cm, $BC = 2$ cm and $CA = 2.5$ cm. If $\Delta\, DEF \sim \Delta\, ABC$ and $EF = 4$ cm, then perimeter of $\Delta\, DEF$ is

(a) 7.5 cm (b) 15 cm (c) 22.5 cm (d) 30 cm.

41. In ΔABC, a line XY parallel to BC cuts AB at X and AC at Y. If BY bisects $\angle XYC$, then

(a) $BC = CY$ (b) $BC = BY$ (c) $BC \neq CY$ (d) $BC \neq BY$

42. In a ΔABC, $\angle A = 90°$, $AB = 5$ cm and $AC = 12$ cm. If $AD \perp BC$, then $AD =$

(a) $\frac{13}{2}$ cm (b) $\frac{60}{13}$ cm (c) $\frac{13}{60}$ cm (d) $\frac{2\sqrt{15}}{13}$ cm

43. In a ΔABC, perpendicular AD from A on BC meets BC at D. If $BD = 8$ cm, $DC = 2$ cm and $AD = 4$ cm, then

(a) ΔABC is isosceles (b) ΔABC is equilateral

(c) $AC = 2AB$ (d) ΔABC is right-angled at A.

44. In a ΔABC, point D is on side AB and point E is on side AC, such that $BCED$ is a trapezium. If $DE : BC = 3:5$, then Area (ΔADE) : Area ($\square BCED$) =

(a) $3:4$ (b) $9:16$ (c) $3:5$ (d) $9:25$

45. If ABC is an isosceles triangle and D is a point on BC such that $AD \perp BC$, then

(a) $AB^2 - AD^2 = BD.DC$ (b) $AB^2 - AD^2 = BD^2 - DC^2$

(c) $AB^2 + AD^2 = BD.DC$ (d) $AB^2 + AD^2 = BD^2 - DC^2$

46. ΔABC is a right triangle right-angled at A and $AD \perp BC$. Then, $\frac{BD}{DC} =$

(a) $\left(\frac{AB}{AC}\right)^2$ (b) $\frac{AB}{AC}$ (c) $\left(\frac{AB}{AD}\right)^2$ (d) $\frac{AB}{AD}$

47. If E is a point on side CA of an equilateral triangle ABC such that $BE \perp CA$, then $AB^2 + BC^2 + CA^2 =$

(a) $2\,BE^2$ (b) $3\,BE^2$ (c) $4\,BE^2$ (d) $6\,BE^2$

48. In a right triangle ABC right-angled at B, if P and Q are points on the sides AB and AC respectively, then

(a) $AQ^2 + CP^2 = 2(AC^2 + PQ^2)$ (b) $2(AQ^2 + CP^2) = AC^2 + PQ^2$

(c) $AQ^2 + CP^2 = AC^2 + PQ^2$ (d) $AQ + CP = \frac{1}{2}(AC + PQ)$.

49. If $\Delta ABC \sim \Delta DEF$ such that $DE = 3$ cm, $EF = 2$ cm, $DF = 2.5$ cm, $BC = 4$ cm, then perimeter of ΔABC is

(a) 18 cm (b) 20 cm (c) 12 cm (d) 15 cm

50. If $\Delta ABC \sim \Delta DEF$ such that $AB = 9.1$ cm and $DE = 6.5$ cm. If the perimeter of ΔDEF is 25 cm, then the perimeter of ΔABC is

(a) 36 cm (b) 30 cm (c) 34 cm (d) 35 cm

51. In an isosceles triangle ABC, if $AB = AC = 25$ cm and $BC = 14$ cm, then the measure of altitude from A on BC is

(a) 20 cm (b) 22 cm (c) 18 cm (d) 24 cm

ANSWERS

1. (d)	2. (a)	3. (c)	4. (c)	5. (d)	6. (b)
7. (c)	8. (c)	9. (c)	10. (c)	11. (a)	12. (a)
13. (c)	14. (c)	15. (b)	16. (a)	17. (b)	18. (b)
19. (c)	20. (a)	21. (a)	22. (b)	23. (c)	24. (b)
25. (c)	26. (b)	27. (c)	28. (b)	29. (c)	30. (a)
31. (b)	32. (b)	33. (d)	34. (c)	35. (d)	36. (b)
37. (b)	38. (a)	39. (a)	40. (b)	41. (a)	42. (b)
43. (d)	44. (d)	45. (a)	46. (b)	47. (c)	48. (c)
49. (d)	50. (d)	51. (d)			

SUMMARY

1. Two figures having the same shape but not necessarily the same size are called similar figures.
2. All congruent figures are similar but the converse is not true.
3. Two polygons having the same number of sides are similar, if
   (i) their corresponding angles are equal and
   (ii) their corresponding sides are proportional (i.e., in the same ratio).
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.
5. If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side of the triangle.
6. The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.
7. If a line through one vertex of a triangle divides the opposite side in the ratio of other two sides, then the line bisects the angle at the vertex.
8. The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.
9. The line drawn from the mid-point of one side of a triangle is parallel of another side bisects the third side.
10. The line joining the mid-points of two sides of a triangle is parallel to the third side.
11. The diagonals of a trapezium divide each other proportionally.
12. If the diagonals of a quadrilateral divide each other proportionally, then it is a trapezium.
13. Any line parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.
14. If three or more parallel lines are intersected by two transversals, then the intercepts made by them on the transversals are proportional.
15. *AAA Similarity criterion:* If in two triangles, corresponding angles are equal, then the triangles are similar.
16. *AA Similarity criterion:* If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar.
17. *SSS Similarity criterion:* If in two triangles, corresponding sides are in the same ratio, then the two triangles are similar.

18. If one angle of a triangles is equal to one angle of another triangle and the sides including these angles are in the same ratio, then the triangles are similar.
19. If two triangles are equiangular, then
    (i) the ratio of the corresponding sides is same as the ratio of corresponding medians.
    (ii) the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.
    (iii) the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.
20. If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, then the triangles are similar.
21. If two sides and a median bisecting one of these sides of a triangle áre respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.
22. If two sides and a median bisecting the third side of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.
23. The ratio of the areas of two similar triangles is equal to the ratio of
    (i) the squares of any two corresponding sides.
    (ii) the squares of the corresponding altitudes.
    (iii) the squares of the corresponding medians.
    (iv) the squares of the corresponding angle bisector segments.
24. If the areas of two similar triangles are equal, then the triangles are congruent i.e., equal and similar triangles congruent.
25. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole. triangle and also to each other.
26. *Pythagoras Theorem:* In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
27. *Converse of Pythagoras Theorem:* If in a triangle, square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to first side is a right angle.
28. In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with the twice of the square of the median which bisects the third side.
29. Three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians of the triangle.
30. Three times the square of any side of an equilateral triangle is equal to four times the square of the altitude.

CHAPTER 8

CIRCLES

8.1 SECANT AND TANGENT

In Class IX, we have studied that a circle is a collection of all points in a plane which are at a constant distance from a fixed point. The fixed point is called the centre and the constant distance is known as the radius. We have also studied various terms related to a circle like chord, segment, sector, arc etc. Let us now see, how many different positions a line can take with respect to a given circle.

Consider a circle $C(O, r)$ and a line l in a plane. We find that there are three different situations as shown in Fig. 8.1.

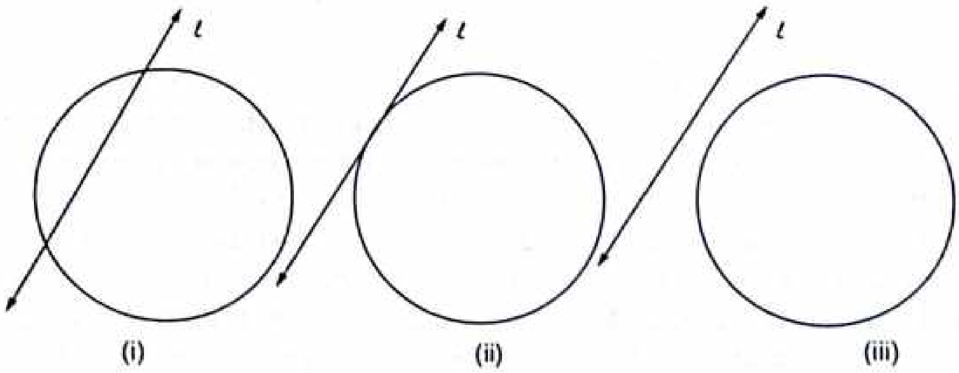


Fig. 8.1

(i) *When the line l intersects the circle in two distinct points* : Since a circle cannot pass through three collinear points. So, the line l intersects the circle in two points only. In such a case the line l is called a *secant* of the circle. Thus, we have the following definition:

SECANT *A line which intersects a circle in two distinct points is called a secant of the circle.*

(ii) *When the line l intersects the circle in only one point* : In this case the line is said to be a tangent to the circle and point is called the point of contact of the tangent. The concept of tangent can also be understood as follows:

Consider a secant PQ of a circle intersecting it in two points P and Q. If we rotate the secant about the point P so that the point Q comes nearer and nearer to the point P, we find that ultimately the point Q will coincide with P. In this case the two points of intersection will coincide and the secant will touch the circle at P. Thus, we have the following definition:

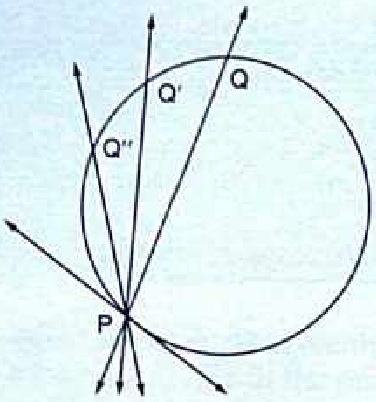


Fig. 8.2

TANGENT *A tangent to a circle is a line that intersects the circle in exactly one point.*

The point is called the point of contact of the tangent and the line is said to touch the circle at this point.

The word tangent is originated from the Latin word TANGERE which means 'to touch'.

NOTE *The point of contact is the only point which is common to the tangent and the circle and every other point on the tangent lies outside the circle. Thus, of all the points on a tangent to a circle, the point of contact is nearest to the centre of the circle.*

In order to understand the existence of the tangent to a circle at a point let us perform the following activities:

ACTIVITY 1 Draw a circle and a secant AB of the circle on a paper. Now, draw various lines parallel to the secant AB on both sides of it. You will find that the lengths of chords cut by the lines will decrease as we go away from the secant AB as shown in Fig. 8.3. You will also notice that on two sides of secant AB, $A_1 B_1$ and $A_2 B_2$ are the limiting positions of secants when the lengths of the chords cut become zero. These are the tangents to the circle parallel to the secant AB. It is evident from the above activity that there cannot be more than two tangents parallel to the given secant or in general, to a given line. It also establishes that a tangent is the secant when both of the end points of the corresponding chord coincide.

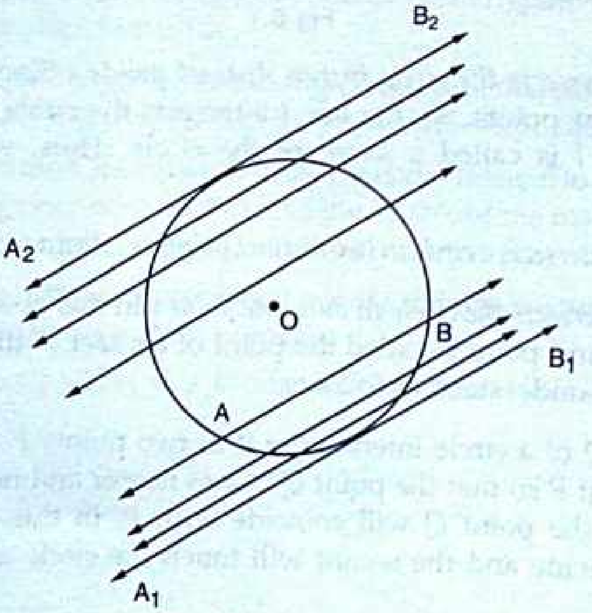


Fig. 8.3

ACTIVITY 2 Take a circular wire and a straight wire AB. Attach the straight wire AB at a point P of the circular wire in such a way that AB can rotate about the point P in a plane. Put the system on a table and rotate wire AB about point P to get different positions of the wire as shown in Fig. 8.4. You will see that the wire AB intersects the circular wire at P and at another point Q_1 or Q_2 etc. when it is rotated in clockwise sense about point P. And in one position $A_1 B_1$ it will intersect the circle just at point P only. In fact, it becomes tangent to the circular wire at P. This shows that a tangent exists at a point P on the circle.

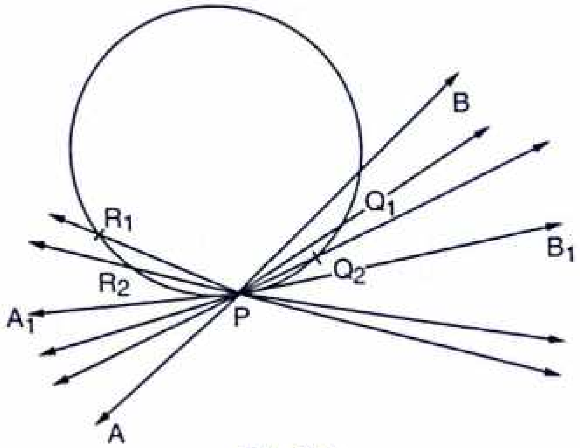


Fig. 8.4

Now, if we rotate wire AB in anticlockwise sense about point P, it will intersect the circle at P and an another point R_1 or R_2 etc. In position $A_1 B_1$ it intersects the circle at P only. Thus, there is only one tangent at point P.

This activity establishes that the tangent at a point P to a circle is the limiting position of secants PQ_1, PQ_2, PR_1, PR_2 etc. through P when Q_1, Q_2, R_1, R_2 etc. coincide with P.

8.2 SOME PROPERTIES OF TANGENT TO A CIRCLE

If you look at the wheels of a bicycle you will find that its all spokes are along its radii. When a bicycle runs, its wheels move along the lines which are tangents at the points where they touch the ground. You will also notice that in all positions, the radii through the points of contact with the ground appear to be at right angle to the tangent. We shall now prove this property and some other properties of tangents to a circle as theorems.

THEOREM 1 *A tangent to a circle is perpendicular to the radius through the point of contact.*

[NCERT, CBSE 2009, CBSE 2012, 2014, 2015, 2016]

GIVEN A circle $C(O, r)$ and a tangent AB at a point P.

TO PROVE $OP \perp AB$.

CONSTRUCTION Take any point Q, other than P, on the tangent AB. Join OQ. Suppose OQ meets the circle at R.

PROOF We know that among all line segments joining the point O to a point on AB, the shortest one is perpendicular to AB. So, to prove that $OP \perp AB$, it is sufficient to prove that OP is shorter than any other segment joining O to any point of AB.

Cleraly, $OP = OR$ [Radii of the same circle]

Now, $OQ = OR + RQ$

$\Rightarrow$ $OQ > OR$

$\Rightarrow$ $OQ > OP$ [$\because OP = OR$]

$\Rightarrow$ $OP < OQ$

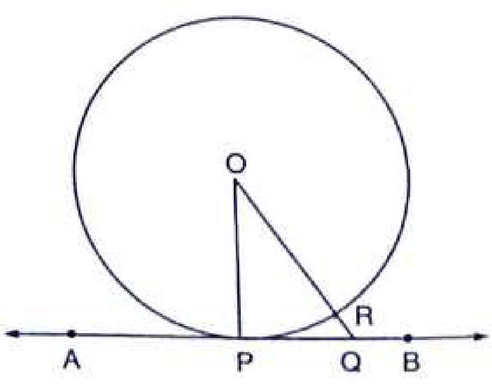


Fig. 8.5

Thus, OP is shorter than any other segment joining O to any point of AB.
Hence, $OP \perp AB$.

Q.E.D.

REMARK *The converse of the above theorem is also true as given below.*

THEOREM 2 *A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.* [NCERT, CBSE 2012, 2013]

GIVEN A radius OP of a circle $C(O, r)$ and a line APB, perpendicular to OP.

TO PROVE AB is a tangent to the circle at the point P.

PROOF Take a point Q, different from P, on the line AB.

Now, $OP \perp AB$.

$\Rightarrow$ Among all the line segments joining O to a point on AB, OP is the shortest.

$\Rightarrow$ $OP < OQ$

$\Rightarrow$ $OQ > OP$

$\Rightarrow$ Q lies outside the circle.

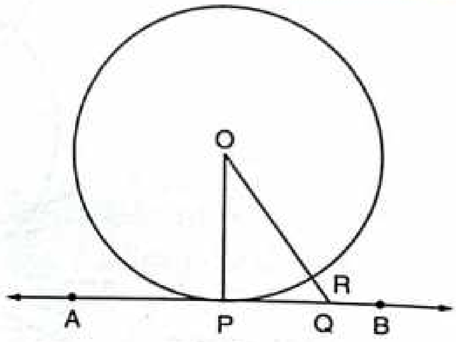


Fig. 8.6

Thus, every point on AB, other than P, lies outside the circle. This shows that AB meets the circle only at the point P.

Hence, AB is a tangent to the circle at P.

Q. E. D.

REMARK *The above theorem provides us a method of constructing a tangent at a given point to a given circle. For this we draw a line through the given point perpendicular to the radius at the given point.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that OQ = 13 cm. Find the length of PQ.* [NCERT]

SOLUTION Since tangent at a point is perpendicular to the radius through that point. Therefore, OP is perpendicular to PQ. In right triangle OPQ, we have

$$OQ^2 = OP^2 + PQ^2 \quad [\text{Using Pythagoras theorem}]$$

$$\Rightarrow \quad 13^2 = 5^2 + PQ^2$$

$$\Rightarrow \quad PQ^2 = 169 - 25 = 144$$

$$\Rightarrow \quad PQ = 12 \text{ cm}.$$

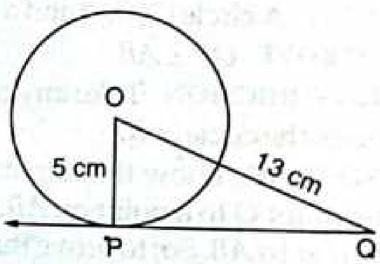


Fig. 8.7

EXAMPLE 2 *A line through the centre O of a circle of radius 7 cm cuts the tangent, at a point P on the circle, at Q such that PQ = 24 cm. Find OQ.*

SOLUTION Since tangent at a point on a circle is perpendicular to the radius through the point. Therefore, OP is perpendicular to PQ.

In right triangle OPQ, we have

$$OQ^2 = OP^2 + PQ^2$$

$\Rightarrow$ $$OQ^2 = 7^2 + 24^2$$

$\Rightarrow$ $$OQ^2 = 49 + 576 \Rightarrow OQ^2 = 625$$

$\Rightarrow$ $$OQ = 25 \text{ cm}.$$

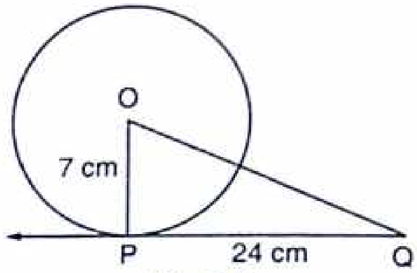


Fig. 8.8

EXERCISE 8.1

LEVEL-1

1. Fill in the blanks:
   (i) The common point of a tangent and the circle is called .... [NCERT]
   (ii) A circle may have .... parallel tangents. [NCERT]
   (iii) A tangent to a circle intersects it in .... point(s). [NCERT]
   (iv) A line intersecting a circle in two points is called a .... [NCERT]
   (v) The angle between tangent at a point on a circle and the radius through the point is ....
2. How many tangents can a circle have? [NCERT]
3. O is the centre of a circle of radius 8 cm. The tangent at a point A on the circle cuts a line through O at B such that $AB = 15$ cm. Find OB.
4. If the tangent at a point P to a circle with centre O cuts a line through O at Q such that $PQ = 24$ cm and $OQ = 25$ cm. Find the radius of the circle.

ANSWERS

1. (i) Point of contact (ii) two (iii) one (iv) secant (v) 90°.
2. Infinitely many 3. 17 cm. 4. 7 cm.

8.3 TANGENT FROM A POINT ON A CIRCLE

In this section, we will learn about the number of tangents drawn from a point to a circle. We will also learn some properties of these tangents. In order to have an idea of the number of tangents drawn from a point to a circle, let us perform the following activity:

ACTIVITY Draw a circle on a paper and take a point P inside it. Let us now try to draw a tangent to the circle through this point P. We observe that all the lines through point P intersect the circle in two distinct points. Therefore, none of them can be a tangent to the circle as shown in Fig. 8.9.

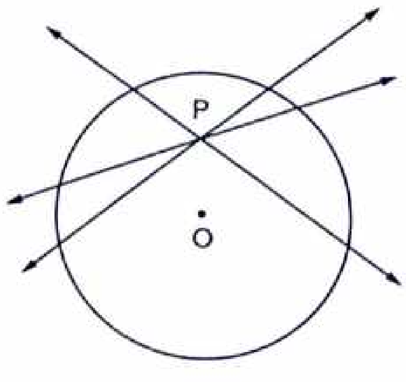


Fig. 8.9

Now, take a point P on the circle. We have discussed in section 8.2 that there exists one and only one tangent to a circle at a given point on it. So, there is only one tangent to the circle at point P as shown in Fig. 8.10(i).

Finally, let us take a point P outside the circle and try to draw tangents to the circle from point P. We observe that we can draw exactly two tangents to the circle through point P as shown in Fig. 8.10(ii).

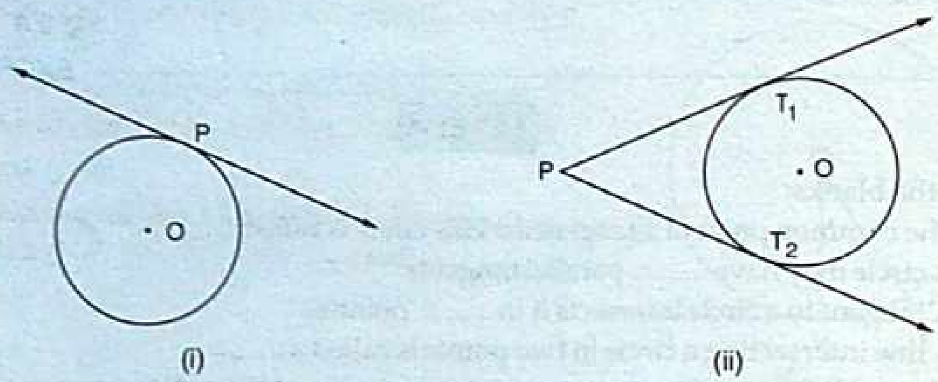


Fig. 8.10

These facts can be summarized as follows:

(i) No tangent can be drawn to a circle from a point lying inside it.

(ii) One and only one tangent can be drawn to a circle at a point on the circle.

(iii) Two tangents can be drawn to a circle from a point lying outside it.

In Fig. 8.10(ii), PT_1 and PT_2 are two tangents drawn from a point P lying outside the circle. These tangents touch the circle at T_1 and T_2 respectively. So, T_1 and T_2 are known as the points of contact of tangents PT_1 and PT_2 respectively.

LENGTH OF TANGENT *The length of the segment of the tangent between the point and the given point of contact with the circle is called the length of the tangent from the point to the circle.*

In Fig. 8.10(i), PT_1 and PT_2 are the lengths of tangents from point P to the circle.

In the following theorem, we will prove that these two lengths are equal.

THEOREM 3 *The lengths of two tangents drawn from an external point to a circle are equal.*

[NCERT, CBSE 2008, 2009, 2010, 2013, 2014, 2015, 2016, 2017, 2018]

GIVEN AP and AQ are two tangents from a point A to a circle $C(O, r)$.

TO PROVE $AP = AQ$

CONSTRUCTION Join OP, OQ and OA.

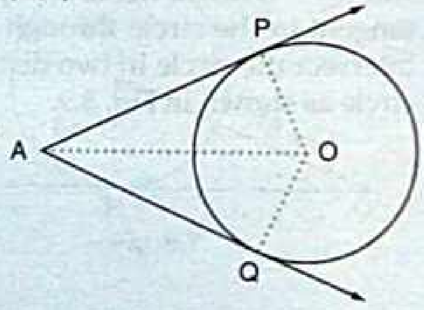


Fig. 8.11

PROOF In order to prove that $AP = AQ$, we shall first prove that $\Delta OPA \cong \Delta OQA$.

Since a tangent at any point of a circle is perpendicular to the radius through the point of contact.

$\therefore \quad OP \perp AP$ and $OQ \perp AQ$.

$\Rightarrow \quad \angle OPA = \angle OQA = 90^\circ \quad \ldots$(i)

Now, in right triangles OPA and OQA, we have

$OP = OQ$ [Radii of a circle]

$\angle OPA = \angle OQA$ [From (i)]

and, $OA = OA$ [Common]

So, by *RHS*-criterion of congruence, we get

$\Delta OPA \cong \Delta OQA$

$\Rightarrow$ $AP = AQ$

Q. E. D.

THEOREM 4 *If two tangents are drawn to a circle from an external point, then:*

(i) *they subtend equal angles at the centre,*

(ii) *they are equally inclined to the segment, joining the centre to that point.*

GIVEN A circle $C(O, r)$ and a point A outside the circle such that AP and AQ are the tangents drawn to the circle from point A.

TO PROVE (i) $\angle AOP = \angle AOQ$ (ii) $\angle OAP = \angle OAQ$.

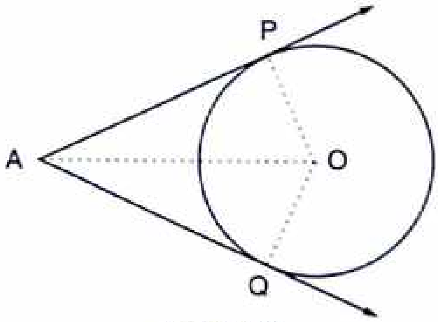


Fig. 8.12

PROOF In right triangles OAP and OAQ, we have

$AP = AQ$ [Tangents from an external point are equal]

$OP = OQ$ [Radii of a circle]

and, $OA = OA$ [Common]

So, by *SSS*-criterion of congruence, we have

$\Delta OAP \cong \Delta OAQ$

$\Rightarrow$ $\angle AOP = \angle AOQ$ and $\angle OAP = \angle OAQ$.

Q.E.D.

REMARK *It follows from the above theorem that the centre of the circle lies on the angle bisector of $\angle PAQ$. This fact can be used in drawing circles touching two intersecting lines. In particular, a circle can be drawn to touch all the three sides of a triangle as discussed below:*

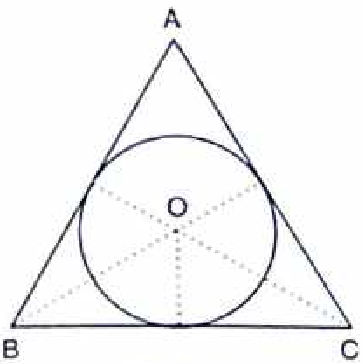


Fig. 8.13

Let ABC be a triangle. Draw angle bisectors of any two angles, say $\angle B$ and $\angle C$. Suppose they intersect at O. Then, as O lies on the angle bisectors of $\angle B$ and $\angle C$, a circle can be drawn with *centre O to touch AB, BC and AC. The radius of the circle will be the length of the perpendicular from O on any side. This circle is called the incircle of Δ ABC and its centre is called the incentre of Δ ABC.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON FINDING THE LENGTH OF THE TANGENT OR RADIUS OF A CIRCLE

EXAMPLE 1 *A point P is 13 cm from the centre of the circle. The length of the tangent drawn from P to the circle is 12 cm. Find the radius of the circle.*

SOLUTION Since tangent to a circle is perpendicular to the radius through the point of contact.

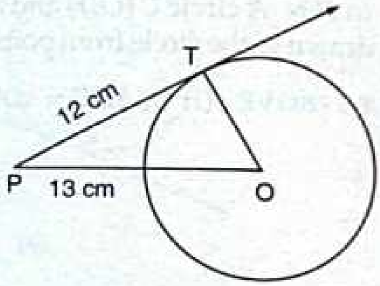


Fig. 8.14

$\therefore \quad \angle OTP = 90^\circ$

In right triangle OTP, we have

$$OP^2 = OT^2 + PT^2$$

$$\Rightarrow \quad 13^2 = OT^2 + 12^2$$

$$\Rightarrow \quad OT^2 = 13^2 - 12^2 = (13-12)(13+12) = 25$$

$$\Rightarrow \quad OT = 5.$$

Hence, radius of the circle is 5 cm.

EXAMPLE 2 *Find the length of the tangent drawn from a point whose distance from the centre of a circle is 25 cm. Given that the radius of the circle is 7 cm.*

SOLUTION Let P be the given point, O be the centre of the circle and PT be the length of tangent from P. Then, $OP = 25$ cm and $OT = 7$ cm.

Since tangent to a circle is always perpendicular to the radius through the point of contact.

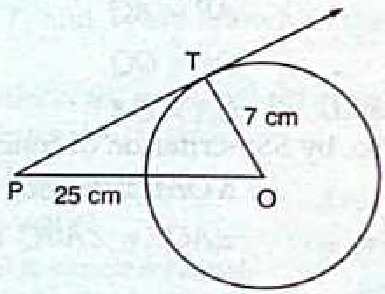


Fig. 8.15

$\therefore \quad \angle OTP = 90^\circ$

in right triangle OTP, we have

$$OP^2 = OT^2 + PT^2$$

$$\Rightarrow \quad 25^2 = 7^2 + PT^2$$

$$\Rightarrow \quad PT^2 = 25^2 - 7^2 = (25-7)(25+7) = 576$$

$$\Rightarrow \quad PT = 24 \text{ cm}.$$

Hence, length of tangent from $P = 24$ cm.

Type II BASED ON THE RESULT THAT THE TANGENTS DRAWN FROM AN EXTERIOR POINT TO A CIRCLE ARE EQUAL IN LENGTH.

EXAMPLE 3 *In Fig. 8.16, if AB = AC, prove that BE = EC.*

OR

ABC is an isosceles triangle in which AB = AC, circumscribed about a circle, as shown in Fig. 8.16. Prove that the base is bisected by the point of contact.

[CBSE 2008, 2012, 2014]

SOLUTION Since tangents from an exterior point to a circle are equal in length.

$\therefore \quad AD = AF$ [Tangents from A] ...(i)

$BD = BE$ [Tangents from B] ...(ii)

$CE = CF$ [Tangents from C] ...(iii)

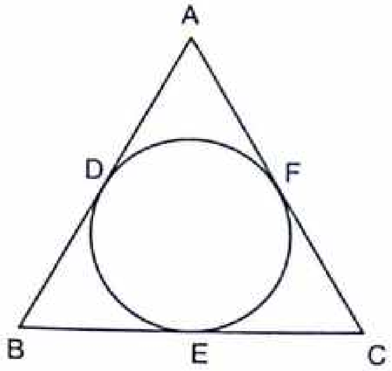


Fig. 8.16

Now,

$$AB = AC$$

$\Rightarrow$ $AB - AD = AC - AD$ [Subtracting AD from both sides]

$\Rightarrow$ $AB - AD = AC - AF$ [Using (i)]

$\Rightarrow$ $BD = CF$

$\Rightarrow$ $BE = CF$ [Using (ii)]

$\Rightarrow$ $BE = CE$ [Using (iii)]

EXAMPLE 4 *A circle is touching the side BC of ΔABC at P and touching AB and AC produced at Q and R respectively. Prove that*

$$AQ = \frac{1}{2}\text{ (Perimeter of } \Delta ABC\text{)}.$$ **[CBSE 2000, 2001, 2002, NCERT EXEMPLAR]**

SOLUTION Since tangents from an exterior point to a circle are equal in length.

$\therefore$ $BP = BQ$ [From B] ...(i)

$CP = CR$ [From C] ...(ii)

and, $AQ = AR$ [From A] ...(iii)

From (iii), we have

$$AQ = AR$$

$\Rightarrow$ $AB + BQ = AC + CR$

$\Rightarrow$ $AB + BP = AC + CP$ [Using (i) and (ii)] ...(iv)

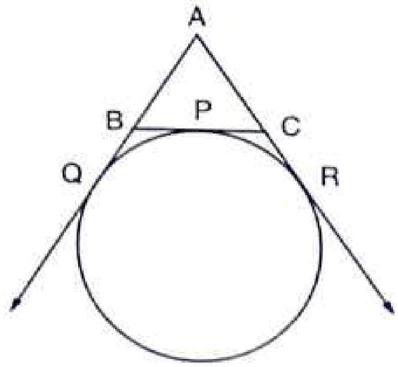


Fig. 8.17

Now,

Perimeter of $\Delta ABC = AB + BC + AC$

$\Rightarrow$ Perimeter of $\Delta ABC = AB + (BP + PC) + AC$

$\Rightarrow$ Perimeter of $\Delta ABC = (AB + BP) + (AC + PC)$

$\Rightarrow$ Perimeter of $\Delta ABC = 2(AB + BP)$ [Using (iv)]

$\Rightarrow$ Perimeter of $\Delta ABC = 2(AB + BQ)$ [Using (i)]

$\Rightarrow$ Perimeter of $\Delta ABC = 2AQ$

$\therefore$ $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

EXAMPLE 5 *In Fig. 8.18, XP and XQ are tangents from X to the circle with centre O. R is a point on the circle. Prove that, $XA + AR = XB + BR$.* **[CBSE 2014]**

SOLUTION Since lengths of tangents from an exterior point to a circle are equal.

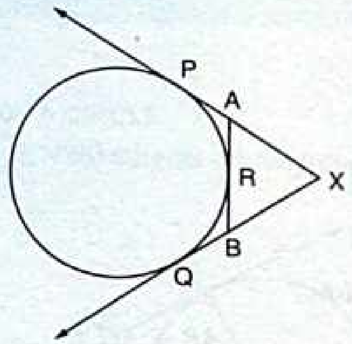


Fig. 8.18

$\therefore$ $XP = XQ$ [From X] ...(i)

$AP = AR$ [From A] ...(ii)

$BQ = BR$ [From B] ...(iii)

Now, $XP = XQ$

$\Rightarrow$ $XA + AP = XB + BQ$

$\Rightarrow$ $XA + AR = XB + BR$ [Using (i) and (ii)]

EXAMPLE 6 *In Fig. 8.19, the incircle of $\Delta\ ABC$ touches the sides BC, CA and AB at D, E and F respectively. Show that*

$$AF + BD + CE = AE + BF + CD = \frac{1}{2}(\textit{Perimeter of } \Delta\ ABC)$$

SOLUTION Since lengths of the tangents from an exterior point to a circle are equal.

$\therefore$ $AF = AE$ [From A] ...(i)

$BD = BF$ [From B] ...(ii)

and, $CE = CD$ [From C] ...(iii)

Adding equations (i), (ii) and (iii), we get

$$AF + BD + CE = AE + BF + CD$$

Now,

Perimeter of $\Delta\,ABC = AB + BC + AC$

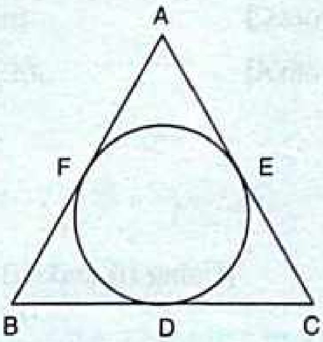


Fig. 8.19

$\Rightarrow$ Perimeter of $\Delta\,ABC = (AF + FB) + (BD + CD) + (AE + EC)$

$\Rightarrow$ Perimeter of $\Delta\,ABC = (AF + AE) + (BF + BD) + (CD + CE)$

$\Rightarrow$ Perimeter of $\Delta\,ABC = 2\,AF + 2\,BD + 2\,CE$

$\Rightarrow$ Perimeter of $\Delta\,ABC = 2(AF + BD + CE)$ $\left[\begin{array}{l}\text{From (i), (ii) and (iii), we get}\\ AE = AF, BD = BF \text{ and } CD = CE\end{array}\right]$

$\Rightarrow$ $AF + BD + CE = \frac{1}{2}(\text{Perimeter of } \Delta\ ABC)$

Hence, $AF + BD + CE = AE + BF + CD = \frac{1}{2}(\text{Perimeter of } \Delta\ ABC)$

EXAMPLE 7 *In Fig. 8.20, the sides AB, BC and CA of triangle ABC touch a circle with centre O and radius r at P, Q and R respectively.*

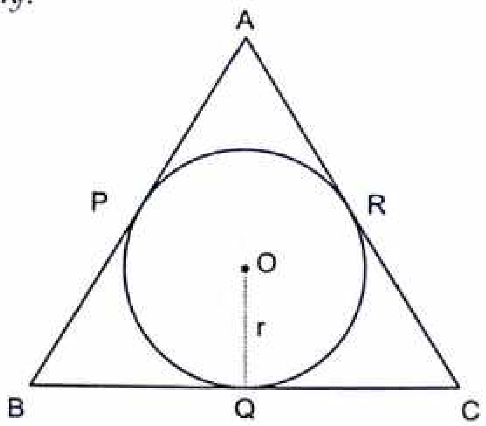


Fig. 8.20

Prove that:

(i) $AB + CQ = AC + BQ$

(ii) $Area\,(\Delta ABC) = \frac{1}{2}(Perimeter\ of\ \Delta ABC) \times r$ **[CBSE 2013]**

SOLUTION We know that lengths of tangents drawn from an external point to a circle are equal.

$\therefore \quad AP = AR, BP = BQ$ and $CQ = CR$

(i) $AB + CQ = AP + PB + CQ$

$= AR + BQ + CQ$ $\quad [\because AP = AR$ and $PB = BQ]$

$= (AR + CR) + BQ$ $\quad [\because CQ = CR]$

$= AC + BQ$ $\quad [\because AR + CR = AC]$

(ii) Area (ΔABC) = Area (ΔOBC) + Area (ΔOAB) + Area (ΔOAC)

$$= \frac{1}{2}(BC \times OQ) + \frac{1}{2}(AB \times OP) + \frac{1}{2}(AC \times OR)$$

$$= \frac{1}{2}(BC \times r) + \frac{1}{2}(AB \times r) + \frac{1}{2}(AC \times r)$$

$$= \frac{1}{2}(BC + AB + AC) \times r$$

$$= \frac{1}{2}(\text{Perimeter of } \Delta ABC) \times r$$

EXAMPLE 8 *In Fig. 8.21, two circles touch each other at the point C. Prove that the common tangent to the circles at C, bisects the common tangent at P and Q.*

[CBSE 2013]

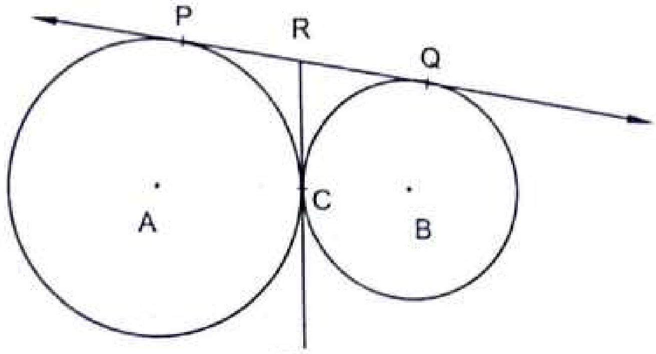


Fig. 8.21

SOLUTION We know that the tangents drawn from an external point to a circle are equal.

$\therefore \quad RP = RC$ and $RC = RQ$

$\Rightarrow \quad RP = RQ$

$\Rightarrow \quad R$ is the mid-point of PQ.

EXAMPLE 9 *A circle touches all the four sides of a quadrilateral ABCD. Prove that:*

$AB + CD = BC + DA$. **[NCERT, CBSE 2008, 2009, 2012, 2013, 2014, 2015, 2017]**

SOLUTION Since tangents drawn from an exterior point to a circle are equal in length.

$\therefore \quad AP = AS$ [From A] ...(i)

$BP = BQ$ [From B] ...(ii)

$CR = CQ$ [From C] ...(iii)

and, $DR = DS$ [From D] ...(iv)

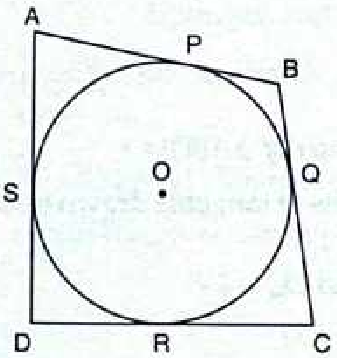


Fig. 8.22

Adding (i), (ii), (iii) and (iv), we get

$$AP + BP + CR + DR = AS + BQ + CQ + DS$$

$$\Rightarrow \quad (AP + BP) + (CR + DR) = (AS + DS) + (BQ + CQ)$$

$$\Rightarrow \quad AB + CD = AD + BC$$

Hence, $AB + CD = BC + DA$

EXAMPLE 10 *If a hexagon ABCDEF circumscribes a circle, prove that*

$AB + CD + EF = BC + DE + FA$. **[NCERT EXEMPLAR]**

SOLUTION Let O be the centre of the circle touching sides AB, BC, CD, DE, EF and FA at P, Q, R, S, T and U respectively. The lengths of tangents drawn from an external point to a circle are equal.

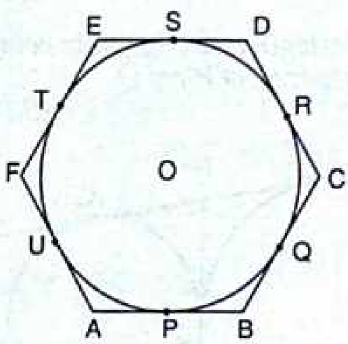


Fig. 8.23

$\therefore \quad AP = AU, BP = BQ, CQ = CR, DR = DS, ES = ET$ and $FU = FT$

Now,

$$\begin{aligned} & AB + CD + EF \\ &= (AP + PB) + (CR + DR) + (ET + TF) \\ &= (AU + PB) + (CQ + DS) + (ES + FU) \qquad \left[\begin{array}{c} \because AP = AU, PB = BQ, CR = CQ, \\ DR = DS, ET = ES, FT = FU \end{array}\right] \\ &= (AU + FU) + (BQ + CQ) + (DS + ES) \\ &= AF + BC + DE = BC + DE + FA \end{aligned}$$

EXAMPLE 11 *Let s denote the semi perimeter of a triangle ABC in which BC = a, CA = b and AB = c. If a circle touches the sides BC, CA, AB at D, E, F respectively, prove that*

$$AF = AE = s - a, BD = BF = s - b \text{ and } CD = CE = s - c.$$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$s = \frac{AB + BC + CA}{2} = \frac{a + b + c}{2}$$

$\Rightarrow$ $a + b + c = 2s$

$\Rightarrow$ $b + c = 2s - a, c + a = 2s - b$ and $a + b = 2s - c$

$\Rightarrow$ $b + c - a = 2(s - a), c + a - b = 2(s - b)$ and $a + b - c = 2(s - c)$

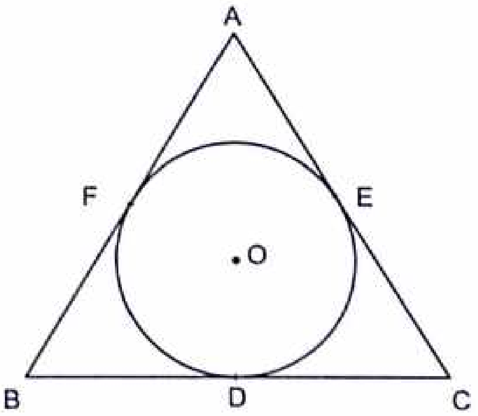


Fig. 8.24

The lengths of tangents drawn from an external point to a circle are equal.

$\therefore$ $AF = AE, BD = BF$ and $CD = CE$

Now,

$2s = BC + CA + AB$

$\Rightarrow$ $2s = (BD + DC) + (CE + AE) + (AF + BF)$

$\Rightarrow$ $2s = (BD + DC) + (CD + AF) + (AF + BD)$

$\Rightarrow$ $2s = 2(BD + DC) + 2AF$

$\Rightarrow$ $2s = 2BC + 2AF$

$\Rightarrow$ $2s = 2a + 2AF$

$\Rightarrow$ $AF = s - a \Rightarrow AF = AE = s - a$

Again,

$2s = BC + CA + AB$

$\Rightarrow$ $2s = (BD + CD) + (CE + AE) + (AF + FB)$

$\Rightarrow \quad 2s = (BF + CE) + (CE + AE) + (AE + FB)$

$\Rightarrow \quad 2s = 2BF + 2(AE + CE)$

$\Rightarrow \quad 2s = 2BF + 2AC$

$\Rightarrow \quad 2s = 2BF + 2b$

$\Rightarrow \quad BF = s - b$

$\Rightarrow \quad BD = BF = s - b$

Similarly, we can prove that $CD = CE = s - c$.

EXAMPLE 12 *If a, b, c are the sides of a right triangle where c is the hypotenuse, prove that the radius r of the circle which touches the sides of the triangle is given by* $r = \frac{a+b-c}{2}$ *or, $r = s - c$, where s is the semi-perimeter of the triangle.* **[NCERT EXEMPLAR]**

SOLUTION Let the circle touches the sides BC, CA and AB of the right triangle ABC at D, E and F respectively.

We have,

$$BC = a, CA = b \text{ and } AB = c$$

It is given that the triangle ABC is right angled at C.

$\therefore \quad AB^2 = BC^2 + AC^2$

$\Rightarrow \quad c^2 = a^2 + b^2$

The length of tangents drawn from a point to a circle are equal.

$\therefore \quad AE = AF, CD = CE \text{ and } BD = BF$

We observe that $CD = OE$ and $CE = OD$

$\therefore \quad CD = r \text{ and } CE = r$

Now,

$AF = AE \quad \text{and} \quad BD = BF$

$\Rightarrow \quad AF = AC - CE \quad \text{and} \quad BF = BC - CD$

$\Rightarrow \quad AF = b - r \quad \text{and} \quad BF = a - r$

$\Rightarrow \quad AF + BF = (b - r) + (a - r)$

$\Rightarrow \quad AB = a + b - 2r$

$\Rightarrow \quad c = a + b - 2r$

$\Rightarrow \quad r = \frac{a + b - c}{2}$

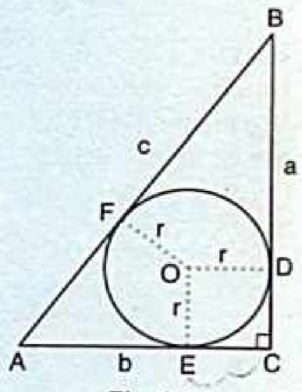


Fig. 8.25

ALITER From example 11, we obtain

$CD = CE = s - c$

$\Rightarrow \quad r = s - c$

$\Rightarrow \quad r = \frac{a + b + c}{2} - c = \frac{a + b - c}{2}$

EXAMPLE 13 *In Fig. 8.26, AB and CD are common tangents to two circles of unequal radii. Prove that AB = CD.* **[NCERT EXEMPLAR]**

SOLUTION Produce AB and CD to meet at P. The lengths of tangents drawn from P to the two circles are equal.

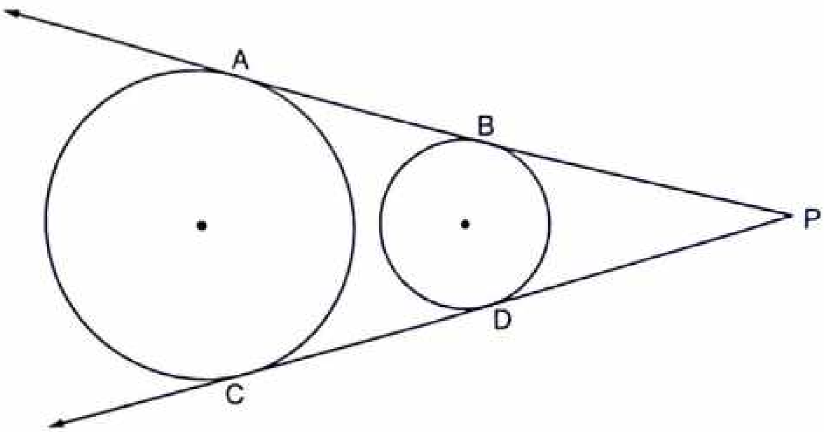


Fig. 8.26

$\therefore \quad PA = PC$ and $PB = PD$

$\Rightarrow \quad PA - PB = PC - PD$

$\Rightarrow \quad AB = CD$

EXAMPLE 14 *If all the side of a parallelogram touch a circle, show that the parallelogram is a rhombus.*

OR

Prove that a parallelogram circumscribing a circle is a rhombus.
[NCERT, CBSE 2000C, 2002, 2008, 2012, 2013, 2014]

SOLUTION Let $ABCD$ be a parallelogram such that its sides touch a circle with centre O.

We know that the tangents to a circle from an exterior point are equal in length.

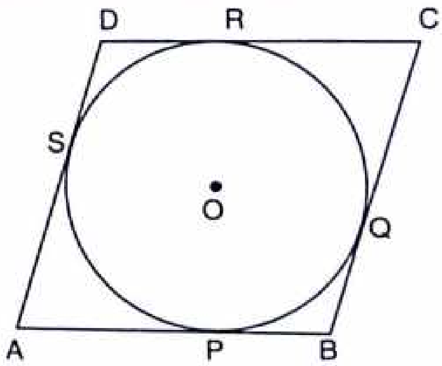


Fig. 8.27

$\therefore \quad AP = AS$ [From A] ...(i)

$BP = BQ$ [From B] ...(ii)

$CR = CQ$ [From C] ...(iii)

and, $DR = DS$ [From D] ...(iv)

Adding (i), (ii), (iii) and (iv), we get

$AP + BP + CR + DR = AS + BQ + CQ + DS$

$\Rightarrow \quad (AP + BP) + (CR + DR) = (AS + DS) + (BQ + CQ)$

$\Rightarrow \quad AB + CD = AD + BC$

$\Rightarrow \quad 2\,AB = 2\,BC$ $\quad [\because ABCD$ is a parallelogram $\therefore AB = CD$ and $BC = AD]$

$\Rightarrow \quad AB = BC$

Thus, $AB = BC = CD = AD$

Hence, $ABCD$ is a rhombus.

EXAMPLE 15 *PA and PB are tangents from P to the circle with centre O. At point M, a tangent is drawn cutting PA at K and PB at N. Prove that KN = AK + BN.*

SOLUTION We know that the tangents drawn from an external point to a circle are equal in length.

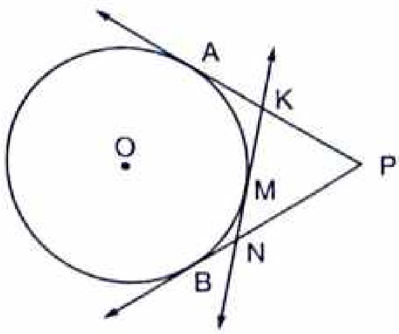


Fig. 8.28

$\therefore \quad PA = PB$ [From P] ...(i)

$KA = KM$ [From K] ...(ii)

and, $NB = NM$ [From N] ...(iii)

Adding (ii) and (iii), we get

$KA + NB = KM + NM$

$\Rightarrow \quad AK + BN = KM + MN$

$\Rightarrow \quad AK + BN = KN$

EXAMPLE 16 *ABCD is a quadrilateral such that $\angle D = 90°$. A circle C (O, r) touches the sides AB, BC, CD and DA at P, Q, R and S respectively. If BC = 38 cm, CD = 25 cm and BP = 27 cm, find r.*

SOLUTION Since tangents to a circle is perpendicular to the radius through the point.

$\therefore \quad \angle ORD = \angle OSD = 90°$

It is given that $\angle D = 90°$. Also, $OR = OS$. Therefore, $ORDS$ is a square.

Since tangents from an exterior point to a circle are equal in length.

$\therefore \quad BP = BQ$

$CQ = CR$

and, $\quad DR = DS$.

Now,

$BP = BQ$

$\Rightarrow \quad BQ = 27 \quad [\because BP = 27 \text{ cm (Given)}]$

$\Rightarrow \quad BC - CQ = 27$

$\Rightarrow \quad 38 - CQ = 27 \quad [\because BC = 38 \text{ cm}]$

$\Rightarrow \quad CQ = 11 \text{ cm}$

$\Rightarrow \quad CR = 11 \text{ cm} \quad [\because CR = CQ]$

$\Rightarrow \quad CD - DR = 11$

$\Rightarrow \quad 25 - DR = 11 \quad [\because CD = 25 \text{ cm}]$

$\Rightarrow \quad DR = 14 \text{ cm}$

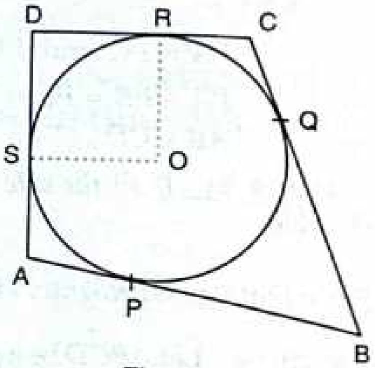


Fig. 8.29

But, $ORDS$ is a square. Therefore, $OR = DR = 14$ cm.

Hence, $r = 14$ cm.

EXAMPLE 17 *Prove that the tangents at the extremities of any chord make equal angles with the chord.* **[NCERT EXEMPLAR]**

SOLUTION Let AB be a chord of a circle with centre O, and let AP and BP be the tangents at A and B respectively. Suppose the tangents meet at P. Join OP. Suppose OP meets AB at C. We have to prove that $\angle PAC = \angle PBC$.

In triangles PCA and PCB, we have

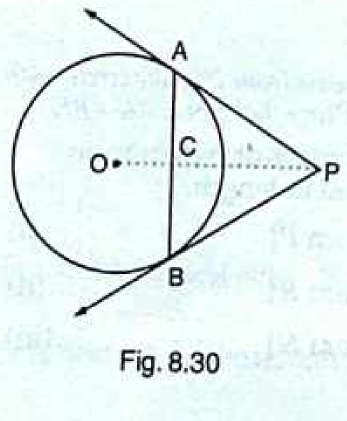


Fig. 8.30

$PA = PB$ [$\because$ Tangents from an external point are equal]

$\angle APC = \angle BPC$ [$\because$ PA and PB are equally inclined to OP]

and, $PC = PC$ [Common]

So, by SAS-criterion of congruence, we obtain

$$\Delta PAC \cong \Delta PBC$$

$\Rightarrow$ $\angle PAC = \angle PBC$

EXAMPLE 18 *From an external point P, two tangents PA and PB are drawn to the circle with centre O. Prove that OP is the perpendicular bisector of AB.*

[CBSE 2015]

SOLUTION Suppose OP intersects AB at C.
In triangles PAC and PBC, we have

$PA = PB$ [$\because$ Tangents from an external point are equal]

$\angle APC = \angle BPC$ [$\because$ PA and PB are equally inclined to OP]

and, $PC = PC$ [Common]

So, by SAS-criterion of similaritry, we obtain

$$\Delta PAC \cong \Delta PBC$$

$\Rightarrow$ $AC = BC$ and $\angle ACP = \angle BCP$

But, $\angle ACP + \angle BCP = 180°$

$\therefore$ $\angle ACP = \angle BCP = 90°$

Hence, $OP \perp AB$.

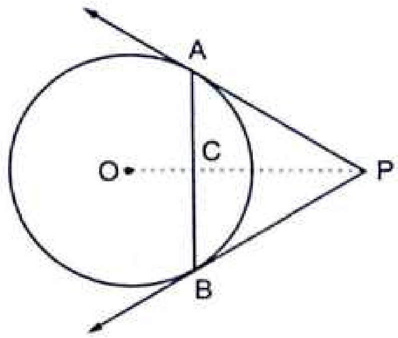


Fig. 8.31

EXAMPLE 19 *Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that* $\angle PTQ = 2\angle OPQ$. **[NCERT, CBSE 2009, 2017]**

SOLUTION We know that lengths of tangents drawn from an external point to a circle are equal.

$\therefore$ $TP = TQ$

$\Rightarrow$ ΔTPQ is an isosceles triangle.

$\Rightarrow$ $\angle TPQ = \angle TQP$

In ΔTPQ, we have

$$\angle TPQ + \angle TQP + \angle PTQ = 180°$$

$\Rightarrow$ $2\angle TPQ = 180° - \angle PTQ$

$\Rightarrow$ $\angle TPQ = 90° - \frac{1}{2}\angle PTQ$

$\Rightarrow$ $\frac{1}{2}\angle PTQ = 90° - \angle TPQ$ ... (i)

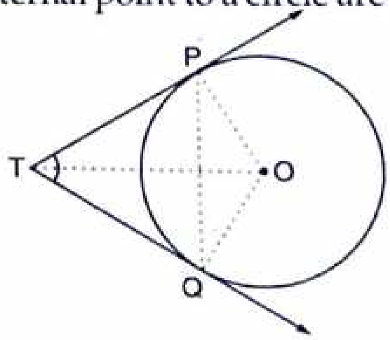


Fig. 8.32

Since, $OP \perp TP$.

$\therefore$ $\angle OPT = 90°$

$\Rightarrow \quad \angle OPQ + \angle TPQ = 90°$

$\Rightarrow \quad \angle OPQ = 90° - \angle TPQ$ ...(ii)

From (i) and (ii), we get

$$\frac{1}{2}\angle PTQ = \angle OPQ \Rightarrow \angle PTQ = 2\angle OPQ$$

EXAMPLE 20 *PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length TP.* **[CBSE 2014, NCERT, CBSE 2016]**

SOLUTION Let $TR = y$.

Since OT is perpendicular bisector of PQ.

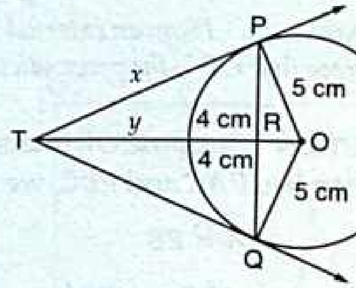


Fig. 8.33

$\therefore \quad PR = QR = 4$ cm ...(i)

In right triangle ORP, we have

$$OP^2 = OR^2 + PR^2$$

$\Rightarrow \quad OR^2 = OP^2 - PR^2 = 5^2 - 4^2 = 9$

$\Rightarrow \quad OR = 3$ cm.

In right triangles PRT and OPT, we have

$$TP^2 = TR^2 + PR^2$$

and, $\quad OT^2 = TP^2 + OP^2$ ...(ii)

$\Rightarrow \quad OT^2 = (TR^2 + PR^2) + OP^2$ [Substituting the value of TP^2]

$\Rightarrow \quad (y+3)^2 = y^2 + 16 + 25$

$\Rightarrow \quad 6y = 32$

$\Rightarrow \quad y = \frac{16}{3}$

$\Rightarrow \quad TR = \frac{16}{3}$

$\therefore \quad TP^2 = TR^2 + PR^2$

$\Rightarrow \quad TP^2 = \left(\frac{16}{3}\right)^2 + 4^2 = \frac{256}{9} + 16 = \frac{400}{9}$ [Using (i) and (ii)]

$\Rightarrow \quad TP = \frac{20}{3}$ cm

ALITER Since ΔTPQ is isosceles and TO is the angle bisector of $\angle PTQ$. Therefore, $OT \perp PQ$ and OT bisects PQ.

$\therefore \quad PR = QR = 4$ cm.

In right triangle ORP, we have

$$OP^2 = OR^2 + PR^2$$

$\Rightarrow \quad OR^2 = OP^2 - PR^2 = 25 - 16 = 9$

$\Rightarrow \quad OR = 3$ cm.

In triangles TRP and PRO, we have

$$\angle TPR + \angle PTR = 90° \text{ and } \angle TPR + \angle RPO = 90°$$

$\Rightarrow \quad \angle TPR + \angle PTR = \angle TPR + \angle RPO$

$\Rightarrow \quad \angle PTR = \angle RPO$

Also, $\quad \angle TRP = \angle ORP = 90°$

$PR = PR$ [Common]

$\therefore \quad \Delta TRP \sim \Delta PRO$

$\Rightarrow \quad \frac{TP}{PO} = \frac{RP}{RO} \Rightarrow \frac{TP}{5} = \frac{4}{3} \Rightarrow TP = \frac{20}{3}\text{cm}$

EXAMPLE 21 *In Fig. 8.34, l and m are two parallel tangents at A and B. The tangent at C makes an intercept DE between l and m. Prove that* $\angle DFE = 90°$. **[NCERT, CBSE 2000, 2013]**

SOLUTION Since tangents drawn from an external point to a circle are equal. Therefore, $DA = DC$.

Thus, in triangles ADF and DFC, we have

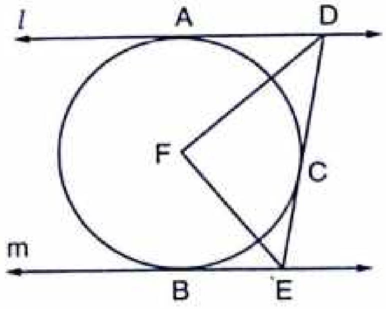


Fig. 8.34

$DA = DC$

$DF = DF$ Common]

$AF = CF$ [Radii of the same circle]

So, by SSS-criterion of congruence, we obtain

$\Delta ADF \cong \Delta DFC$

$\Rightarrow \quad \angle ADF = \angle CDF$

$\Rightarrow \quad \angle ADC = 2\angle CDF$ ...(i)

Similarly, we can prove that

$\angle BEF = \angle CEF$

$\Rightarrow \quad \angle CEB = 2\angle CEF$ ...(ii)

Now, $\quad \angle ADC + \angle CEB = 180°$ [Sum of the interior angles on the same side of transversal is 180°]

$\Rightarrow \quad 2\angle CDF + 2\angle CEF = 180°$ [Using equations (i) and (ii)]

$\Rightarrow \quad \angle CDF + \angle CEF = 90°$

$\Rightarrow \quad 180° - \angle DFE = 90°$ $\left[\because \angle CDF, \angle CEF \text{ and } \angle DFE \text{ are angles of a triangle} \atop \therefore \angle CDF + \angle CEF + \angle DFE = 180°\right]$

$\Rightarrow \quad \angle DFE = 90°$

EXAMPLE 22 *Prove that the angle between two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line segments joining the points of contact at the centre.* **[NCERT]**

SOLUTION Let PA and PB be two tangents drawn from an external point P to a circle with centre O. We have to prove that angles $\angle AOB$ and $\angle APB$ are supplementary i.e. $\angle AOB + \angle APB = 180°$.

In right triangles OAP and OBP, we have

$PA = PB$ [Tangents drawn from an external point are equal]

$OA = OB$ [Each equal to radius]

and, $\quad OP = OP$ [Common]

So, by SSS-criterion of congruence, we obtain

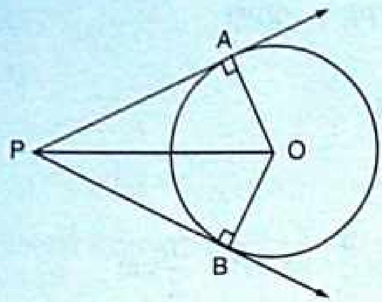


Fig. 8.35

$$\Delta OAP \cong \Delta OBP$$

$\Rightarrow$ $\angle OPA = \angle OPB$ and $\angle AOP = \angle BOP$

$\Rightarrow$ $\angle APB = 2\angle OPA$ and $\angle AOB = 2\angle AOP$ ...(i)

But, $\angle AOP = 90° - \angle OPA$ [$\because$ ΔOAP is right triangle]

$\therefore$ $2\angle AOP = 180° - 2\angle OPA$

$\Rightarrow$ $\angle AOB = 180° - \angle APB$ [Using (i)]

$\Rightarrow$ $\angle AOB + \angle APB = 180°$

Type III **BASED ON THE RESULT THAT THE TANGENT TO A CIRCLE AT A POINT IS PERPENDICULAR TO THE RADIUS THROUGH THE POINT**

EXAMPLE 23 *Show that tangent lines at the end points of a diameter of a circle are parallel.*

[CBSE 2014, 2017, NCERT]

SOLUTION Let AB be a diameter of a given circle, and let PQ and RS be the tangent lines drawn to the circle at points A and B respectively. Since tangent at a point to a circle is perpendicular to the radius through the point.

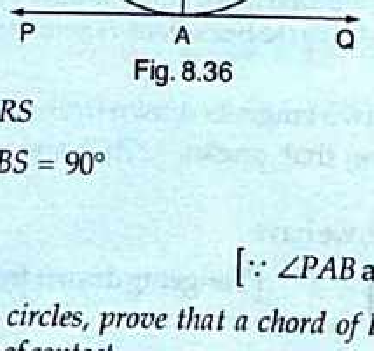


Fig. 8.36

$\therefore$ $AB \perp PQ$ and $AB \perp RS$

$\Rightarrow$ $\angle PAB = 90°$ and $\angle ABS = 90°$

$\Rightarrow$ $\angle PAB = \angle ABS$

$\Rightarrow$ $PQ \parallel RS$ [$\because$ $\angle PAB$ and $\angle ABS$ are alternate angles]

EXAMPLE 24 *In two concentric circles, prove that a chord of larger circle which is tangent to smaller circle is bisected at the point of contact.*

[CBSE 2012]

SOLUTION Let O be the common centre of two con-centric circles, and let AB be a chord of the larger circle touching the smaller circle at P.

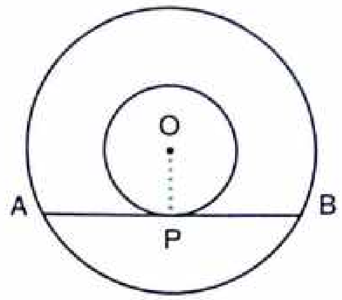


Fig. 8.37

Join OP.

Since OP is the radius of the smaller circle and AB is a tangent to this circle at a point P.

$\therefore \quad OP \perp AB$

We know that the perpendicular drawn from the centre of a circle to any chord of the circle, bisects the chord. So,

$$OP \perp AB$$

$\Rightarrow \quad AP = BP$

Hence, AB is bisected at P.

EXAMPLE 25 *In two concentric circles, a chord of length 24 cm of larger circle becomes a tangent to the smaller circle whose radius is 5 cm. Find the radius of the larger circle.*

SOLUTION Let O be the centre of concentric circles and APB be the chord of length 24 cm, of the larger circle touching the smaller circle at P. Then, $OP \perp AB$ and P is the mid-point of AB.

$\therefore \quad AP = PB = 12$ cm

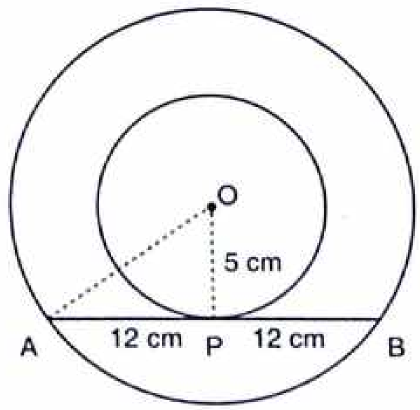


Fig. 8.38

In ΔOPA, we have

$$OA^2 = OP^2 + AP^2 \quad \text{[By Pythagoras Theorem]}$$

$\Rightarrow \quad OA^2 = 5^2 + 12^2 = 169$

$\Rightarrow \quad OA = 13$ cm

Hence, the radius of the smaller circle is 13 cm.

EXAMPLE 26 *Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.* [NCERT]

SOLUTION Let O be the centre of the concentric circles of radii 5 cm and 3 cm respectively. Let AB be a chord of the larger circle touching the smaller circle at P. Then

$$AP = PB \text{ and } OP \perp AB$$

Applying Pythagoras theorem in ΔOPA, we have

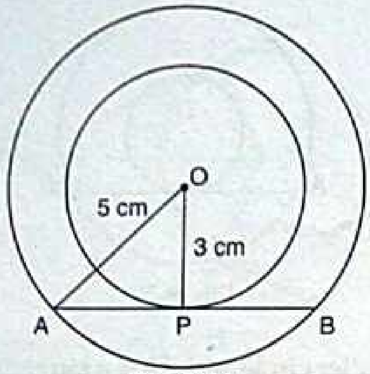


Fig. 8.39

$$OA^2 = OP^2 + AP^2$$

$\Rightarrow \quad 25 = 9 + AP^2$

$\Rightarrow \quad AP^2 = 16 \Rightarrow AP = 4$ cm

$\therefore \quad AB = 2AP = 8$ cm

EXAMPLE 27 *The radii of two concentric circles are 13 cm and 8 cm. AB is a diameter of the bigger circle. BD is a tangent to the smaller circle touching it at D. Find the length AD.*

SOLUTION Produce BD to meet the bigger circle at E. Join AE. Then,

$\angle AEB = 90$ [Angle in a semicircle]

$OD \perp BE$ [$\because$ BE is tangent to the smaller circle at D and OD is its radius]

and, $BD = DE$ [$\because$ BE is a chord of the circle and $OD \perp BE$]

$\therefore \quad OD \parallel AE$ [$\because \angle AEB = \angle ODB = 90°$]

In ΔAEB, O and D are mid-points of AB and BE. Therefore, by mid-point theorem, we have

$$OD = \frac{1}{2}AE$$

$\Rightarrow \quad AE = 2 \times 8 = 16$ cm [$\because OD = 8$ cm]

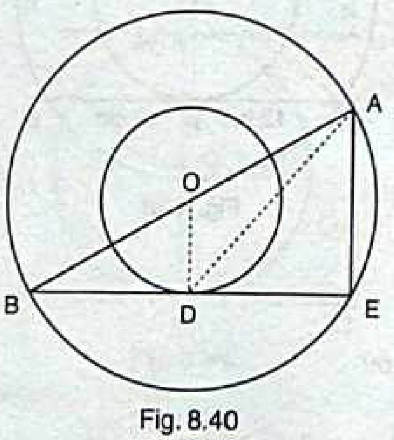


Fig. 8.40

In ΔODB, we have

$OB^2 = OD^2 + BD^2$ [By Pythagoras Theorem]

$\Rightarrow \quad 13^2 = 8^2 + BD^2$

$\Rightarrow \quad BD^2 = 169 - 64 = 105$

$\Rightarrow \quad BD = \sqrt{105}$ cm

$\Rightarrow \quad DE = \sqrt{105}$ cm [$\because BD = DE$]

In ΔAED, we have

$$AD^2 = AE^2 + ED^2 \quad \text{[By Pythagoras Theorem]}$$

$$\Rightarrow \quad AD^2 = 16^2 + (\sqrt{105})^2 = 256 + 105 = 361$$

$$\Rightarrow \quad AD = 19 \text{ cm}$$

EXAMPLE 28 *In Fig. 8.41, O is the centre of the circle. PA and PB are tangent segments. Show that the quadrilateral AOBP is cyclic.* **[NCERT EXEMPLAR]**

SOLUTION Since tangent at a point to a circle is perpendi-cular to the radius through the point.

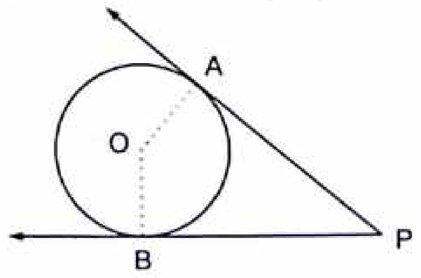


Fig. 8.41

$$\therefore \quad OA \perp AP \text{ and } OB \perp BP$$

$$\Rightarrow \quad \angle OAP = 90^\circ \text{ and } \angle OBP = 90^\circ$$

$$\Rightarrow \quad \angle OAP + \angle OBP = 90^\circ + 90^\circ = 180^\circ \quad \ldots\text{(i)}$$

In quadrilateral $OAPB$, we have

$$\angle OAP + \angle APB + \angle AOB + \angle OBP = 360^\circ$$

$$\Rightarrow \quad (\angle APB + \angle AOB) + (\angle OAP + \angle OBP) = 360^\circ$$

$$\Rightarrow \quad \angle APB + \angle AOB + 180^\circ = 360^\circ \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \angle APB + \angle AOB = 180^\circ \quad \ldots\text{(ii)}$$

From (i) and (ii), we can say that the quadrilateral $AOBP$ is cyclic.

Type IV BASED ON THE RESULT THAT THE TANGENTS DRAWN FROM AN EXTERNAL POINT OF A CIRCLE SUBTEND EQUAL ANGLES AT THE CENTRE

EXAMPLE 29 *A circle touches the sides of a quadrilateral ABCD at P, Q, R, S respectively. Show that the angles subtended at the centre by a pair of opposite sides are supplementary.*

[NCERT, CBSE 2012, 2014]

GIVEN A circle with centre O touches the sides AB, BC, CD and DA of a quadrilateral $ABCD$ at the points P, Q, R and S respectively.

TO PROVE $\angle AOB + \angle COD = 180^\circ$ and, $\angle AOD + \angle BOC = 180^\circ$

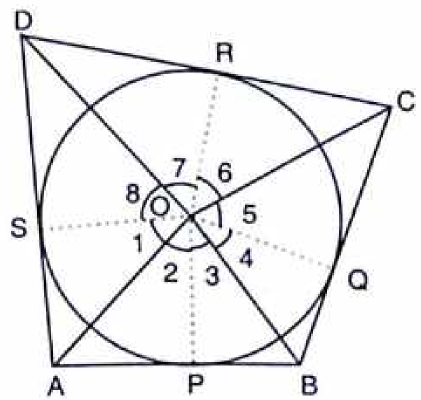


Fig. 8.42

CONSTRUCTION Join OP, OQ, OR and OS.

PROOF Since the two tangents drawn from an external point to a circle subtend equal angles at the centre.

$\therefore$ $\angle 1 = \angle 2, \angle 3 = \angle 4, \angle 5 = \angle 6$ and $\angle 7 = \angle 8$

Now, $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 + \angle 7 + \angle 8 = 360°$ $\left[\begin{array}{l}\text{Sum of all the angles}\\ \text{subtended at a point is } 360°\end{array}\right]$

$\Rightarrow$ $2(\angle 2 + \angle 3 + \angle 6 + \angle 7) = 360°$ and $2(\angle 1 + \angle 8 + \angle 4 + \angle 5) = 360°$

$\Rightarrow$ $(\angle 2 + \angle 3) + (\angle 6 + \angle 7) = 180°$ and $(\angle 1 + \angle 8) + (\angle 4 + \angle 5) = 180°$

$\Rightarrow$ $\angle AOB + \angle COD = 180°$ $\left[\begin{array}{l}\because \angle 2 + \angle 3 = \angle AOB, \angle 6 + \angle 7 = \angle COD\\ \angle 1 + \angle 8 = \angle AOD \text{ and } \angle 4 + \angle 5 = \angle BOC\end{array}\right]$

and, $\angle AOD + \angle BOC = 180°$

Type V MISCELLANEOUS PROBLEMS

EXAMPLE 30 *Prove that the segment joining the points of contact of two parallel tangents passes through the centre.* **[CBSE 2014]**

SOLUTION Let PAQ and RBS be two parallel tangents to a circle with centre O. Join OA and OB. Draw $OC \parallel PQ$.

Now, $PA \parallel CO$

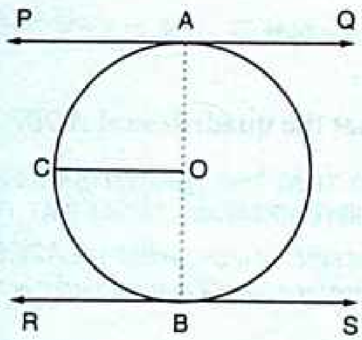


Fig. 8.43

$\Rightarrow$ $\angle PAO + \angle COA = 180°$ $\left[\begin{array}{l}\text{Sum of the angles on the same}\\ \text{side of a transversal is } 180°\end{array}\right]$

$\Rightarrow$ $90° + \angle COA = 180°$ $[\because \angle PAO = \text{angle between a tangent and radius} = 90°]$

$\Rightarrow$ $\angle COA = 90°$

Similarly, $\angle COB = 90°$

$\therefore$ $\angle COA + \angle COB = 90° + 90° = 180°$

Hence, AOB is a straight line passing through O.

EXAMPLE 31 *O is the centre of a circle of radius 5 cm. T is a point such that $OT = 13$ cm and OT intersects the circle at E. If AB is the tangent to the circle at E, find length of AB.*

[CBSE 2016, NCERT EXEMPLAR]

SOLUTION Clearly $\angle OPT = 90°$

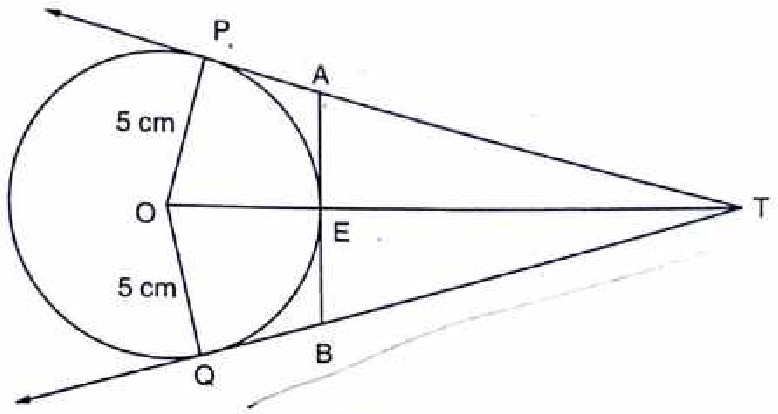


Fig. 8.44

Applying Pythagoras in ΔOPT, we have

$$OT^2 = OP^2 + PT^2$$
$$\Rightarrow \quad 13^2 = 5^2 + PT^2$$
$$\Rightarrow \quad PT^2 = 169 - 25 = 144$$
$$\Rightarrow \quad PT = 12 \text{ cm}$$

Since lengths of tangents drawn from a point to a circle are equal. Therefore,

$$AP = AE = x \text{ (say)}$$
$$\Rightarrow \quad AT = PT - AP = (12 - x) \text{ cm}$$

Since AB is the tangent to the circle E. Therefore, $OE \perp AB$

$$\Rightarrow \quad \angle OEA = 90°$$
$$\Rightarrow \quad \angle AET = 90°$$
$$\Rightarrow \quad AT^2 = AE^2 + ET^2 \qquad \text{[Applying Pythagoras Theorem in } \Delta AET\text{]}$$
$$\Rightarrow \quad (12 - x)^2 = x^2 + (13 - 5)^2$$
$$\Rightarrow \quad 144 - 24x + x^2 = x^2 + 64$$
$$\Rightarrow \quad 24x = 80$$
$$\Rightarrow \quad x = \frac{10}{3} \text{ cm}$$

Similarly, $BE = \frac{10}{3}$ cm

$$\therefore \quad AB = AE + BE = \left(\frac{10}{3} + \frac{10}{3}\right) \text{cm} = \frac{20}{3} \text{ cm}$$

LEVEL-2

EXAMPLE 32 *The radius of the incircle of a triangle is 4 cm and the segments into which one side is divided by the point of contact are 6 cm and 8 cm. Determine the other two sides of the triangle.*

[CBSE 2014, NCERT]

SOLUTION Let I be the incentre of ΔABC such that in-radius $= IL = IM = IN = 4$ cm.

Also, $AM = 6$ cm, and $CM = 8$ cm.

Let $BL = BN = x$ cm

We have,

$AM = 6$ cm and $AM = AN$

$\therefore$ $AN = 6$ cm

Similarly, $CL = CM = 8$ cm

$\therefore$ $a = BC = BL + CL = (x+8)$ cm

$b = AC = AM + CM = (6+8)$ cm $= 14$ cm

and, $c = AB = AN + BN = (x+6)$ cm

$\therefore$ $2s = a + b + c$

$\Rightarrow$ $2s = x + 8 + 14 + x + 6$

$\Rightarrow$ $s = x + 14$

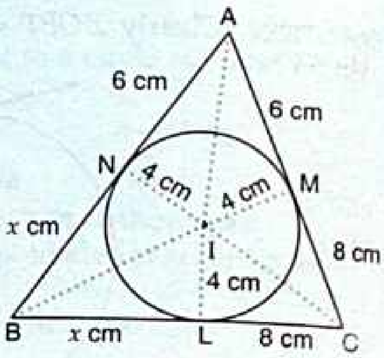


Fig. 8.45

Now,

$$\text{Area of } \Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(x+14)(x+14-x-8)(x+14-14)(x+14-x-6)}$$

$$= \sqrt{(x+14)\times(6)\times(x)\times 8}$$

$$= \sqrt{48x(x+14)} \qquad \ldots\text{(i)}$$

Also,

Area of ΔABC = Area of ΔIBC + Area of ΔICA + Area of ΔIAB

$$= \frac{1}{2}\times BC \times IL + \frac{1}{2}\times CA \times IM + \frac{1}{2}\times AB \times IN$$

$$= \frac{1}{2}\times(x+8)\times 4 + \frac{1}{2}\times 14 \times 4 + \frac{1}{2}\times(x+6)\times 4$$

$$= 2(x+8) + 28 + 2(x+6)\text{ cm}^2 \quad = 4x + 56\text{ cm}^2 \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$\sqrt{48x(x+14)} = (4x+56)$

$\Rightarrow$ $48x(x+14) = (4x+56)^2$

$\Rightarrow$ $48x(x+14) = 16(x+14)^2$

$\Rightarrow$ $3x(x+14) = (x+14)^2$

$\Rightarrow$ $3x(x+14) - (x+14)^2 = 0$

$\Rightarrow$ $(x+14)(3x - x - 14) = 0$

$\Rightarrow$ $2(x+14)(x-7) = 0$

$\Rightarrow$ $x - 7 = 0$ $\qquad [\because x > 0 \therefore x + 14 \neq 0]$

$\Rightarrow$ $x = 7$

$\therefore$ $BC = (x+8)$ cm $= 15$cm and $AB = (x+6)$ cm $= 13$ cm.

EXAMPLE 33 *A circle is inscribed in a ΔABC having sides 8 cm, 10 cm and 12 cm as shown in Fig. 8.46. Find AD, BE and CF.* **[CBSE 2001, 2013, 2015, 2016]**

SOLUTION We know that the tangents drawn from an external point to a circle are equal. Therefore,

$$AD = AF = x, \text{ say}$$
$$BD = BE = y, \text{ say}$$
and, $$CE = CF = z, \text{ say}$$

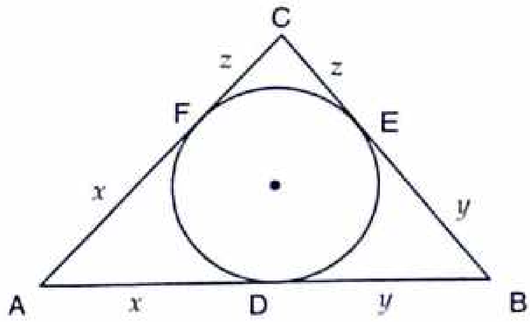


Fig. 8.46

Now,
$$AB = 12\text{cm}, BC = 8\text{cm} \text{ and, } CA = 10\text{cm}$$
$$\Rightarrow \quad x + y = 12, y + z = 8 \text{ and } z + x = 10$$
$$\Rightarrow \quad (x+y)+(y+z)+(z+x) = 12+8+10$$
$$\Rightarrow \quad 2(x+y+z) = 30$$
$$\Rightarrow \quad x+y+z = 15$$

Now,
$$x + y = 12 \text{ and } x + y + z = 15 \Rightarrow 12 + z = 15 \Rightarrow z = 3.$$
$$y + z = 8 \text{ and } x + y + z = 15 \Rightarrow x + 8 = 15 \Rightarrow x = 7$$
and, $$z + x = 10 \text{ and } x + y + z = 15 \Rightarrow y + 10 = 15 \Rightarrow y = 5.$$
Hence, $AD = x = 7$ cm, $BE = y = 5$ cm and $CF = z = 3$ cm.

EXAMPLE 34 *Find the locus of the centres of circles which touch a given line at a given point.*

SOLUTION Let APB be the given line, and let a circle with centre O touch APB at P. Then, $\angle OPB = 90°$. Let there be another circle with centre O' which touches the line APB at P. Then, $\angle O'PB = 90°$.

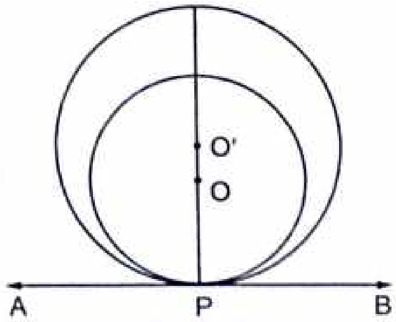


Fig. 8.47

This is possible only when O and O' lie on the same line $O'OP$. Hence, the required locus is a line perpendicular to the given line at the point of contact.

EXAMPLE 35 *In Fig. 8.48, circles C (O, r) and C (O' r/2) touch internally at a point A and AB is a chord of the circle C (O, r) intersecting C (O', r/2) at C. Prove that AC = CB.*

SOLUTION Join OA, OC and OB. Clearly, $\angle OCA$ is the angle in a semi-circle.

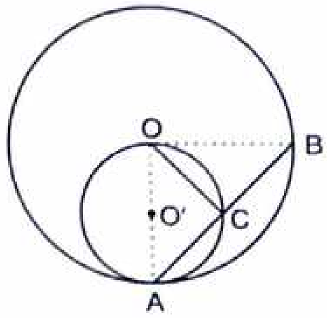


Fig. 8.48

$\therefore \quad \angle OCA = 90°$

In right triangles OCA and OCB, we have

$$OA = OB = r$$

$$\angle OCA = \angle OCB = 90°$$

and, $\quad OC = OC$

So, by *RHS*-criterion of congruence, we get

$$\Delta OCA \cong \Delta OCB$$

$\Rightarrow \quad AC = CB$

EXAMPLE 36 *In two concentric circles, prove that all chords of the outer circle which touch the inner are of equal length.*

SOLUTION Let AB and CD be two chords of the circle which touch the inner circle at M and N respectively.

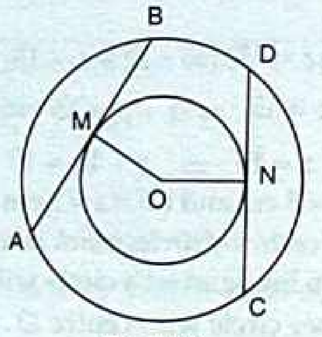


Fig. 8.49

Then, we have to prove that $AB = CD$.

Since AB and CD are tangents to the smaller circle.

$\therefore \quad OM = ON =$ Radius of the smaller circle.

Thus, AB and CD are two chords of the larger circle such that they are equidistant from the centre.

Hence, $\quad AB = CD$.

EXAMPLE 37 *Find the locus of centres of circles which touch two intersecting lines.*

SOLUTION Let l_1 and l_2 be two intersecting lines which intersect at point P. Let O be the centre of the circle which touches both l_1 and l_2.

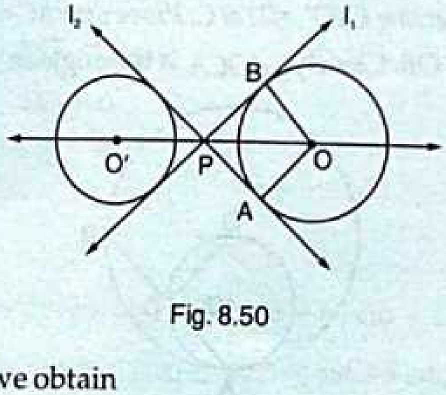


Fig. 8.50

In triangles OAP and OBP, we obtain

$OA = OB$ [Each equal to radius]

$PA = PB$ [Tangents drawn from an external point to a circle are equal]

and, $OP = OP$ [Common]

So, by SSS-congruence criterion, we obtain

$\Delta OAP \cong \Delta OBP$

$\Rightarrow$ $\angle APO = \angle BPO$

$\Rightarrow$ OP is the bisector of $\angle APB$

$\Rightarrow$ O lies on the bisector of the angle between l_1 and l_2.

Hence, the required locus is the line bisecting the angle between the given lines.

EXAMPLE 38 *Let A be one point of intersection of two intersecting circles with centres O and Q. The tangents at A to the two circles meet the circles again at B and C, respectively. Let the point P be located so that AOPQ is a parallelogram. Prove that P is the circumcentre of the triangle ABC.*

SOLUTION In order to prove that P is the circumcentre of ΔABC, it is sufficient to show that P is the point of intersection of perpendicular bisectors of the sides of

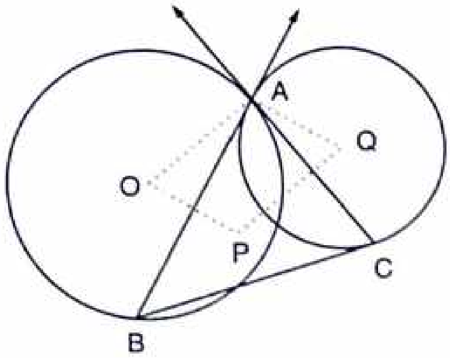


Fig. 8.51

ΔABC, i. e. OP and PQ are perpendicular bisectors of sides AB and AC respectively. Now, AC is tangent at A to the circle with centre at O and OA is its radius.

$\therefore$ $OA \perp AC$

$\Rightarrow$ $PQ \perp AC$ [$\because$ $OAQP$ is a parallelogram $\therefore$ $OA \parallel PQ$]

$\Rightarrow$ PQ is the perpendicular bisector of AC. [$\because$ Q is the centre of the circle]

Similarly, BA is the tangent to the circle at A and AQ is its radius through A.

$\therefore$ $BA \perp AQ$

$\therefore$ $BA \perp OP$ [$\because$ $AQPO$ is parallelogram $\therefore$ $OP \parallel AQ$]

$\Rightarrow$ OP is the perpendicular bisector of AB.

Thus, P is the point of intersection of perpendicular bisectors PQ and PO of sides AC and AB respectively

Hence, P is the circumcentre of ΔABC.

EXAMPLE 39 *Two circles with centres A and B of radii 3 cm and 4 cm respectively intersect at two points C and D such that AC and BC are tangents to the two circles. Find the length of the common chord CD.* **[NCERT EXEMPLAR]**

SOLUTION Since tangent at a point to a circle is perpendicular to the radius through the point of contact. Therefore, $\angle ACB = 90°$

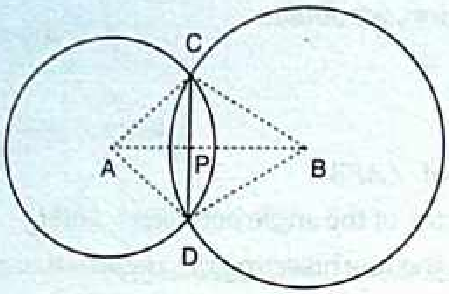


Fig. 8.52

In ΔACB, we have

$$AB^2 = AC^2 + BC^2 \quad \text{[By Pythagoras Theorem]}$$

$\Rightarrow \quad AB^2 = 3^2 + 4^2 = 9 + 16 = 25$

$\Rightarrow \quad AB = 5 \text{ cm}$

Since the line joining the centres of two intersecting circles is perpendicular bisector of their common chord.

$\therefore \quad AP \perp CD$ and $CP = PD$

Let $AP = x$. Then, $BP = 5 - x$ $\quad [\because AB = 5 \text{ cm}]$

Further, let $CP = DP = y$ cm.

In ΔAPC and ΔBPC applying Pythagoras theorem, we have

$$AC^2 = AP^2 + PC^2, \; BC^2 = PB^2 + PC^2$$

$\Rightarrow \quad 3^2 = x^2 + y^2$ and $4^2 = (5 - x)^2 + y^2$

$\Rightarrow \quad 4^2 - 3^2 = \left\{(5 - x)^2 + y^2\right\} - \left\{x^2 + y^2\right\}$ [On subtracting first from second]

$\Rightarrow \quad 7 = 25 - 10x$

$\Rightarrow \quad 10x = 18 \Rightarrow x = 1.8 \text{ cm}$

$\therefore \quad 3^2 = x^2 + y^2 \Rightarrow y = \sqrt{9 - (1.8)^2} = \sqrt{5.76} = 2.4 \text{ cm}$

Hence, $CD = 2CP = 2y = 4.8$ cm

EXAMPLE 40 *If an isosceles triangle ABC in which $AB = AC = 6$ cm is inscribed in a circle of radius 9 cm, find the area of the triangle.* **[NCERT EXEMPLAR]**

SOLUTION Let O be the centre of the circle and let P be the mid-point of BC. Then, $OP \perp BC$. Since ΔABC is isosceles and P is the mid-point of BC. Therefore, $AP \perp BC$ as median from the vertex in an isosceles triangle is perpendicular to the base.

Let $AP = x$ and $PB = CP = y$.

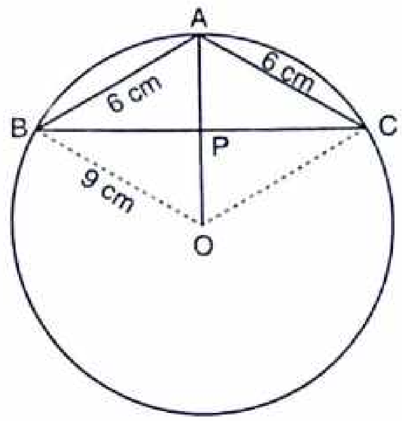


Fig. 8.53

Applying Pythagoras theorem in Δ's APB and OPB, we have

$$AB^2 = BP^2 + AP^2 \text{ and } OB^2 = OP^2 + BP^2$$

$\Rightarrow \quad 36 = y^2 + x^2 \quad \ldots\text{(i)} \quad$ and, $81 = (9 - x)^2 + y^2 \quad \ldots\text{(ii)}$

$\Rightarrow \quad 81 - 36 = \{(9 - x)^2 + y^2\} - \{y^2 + x^2\}$ [Subtracting (i) from (ii)]

$\Rightarrow \quad 45 = 81 - 18x$

$\Rightarrow \quad x = 2 \text{ cm}$

Putting $x = 2$ in (i), we get

$$36 = y^2 + 4 \Rightarrow y^2 = 32 \Rightarrow y = 4\sqrt{2} \text{ cm}$$

$\therefore \quad BC = 2BP = 2y = 8\sqrt{2} \text{ cm}$

Hence, Area of $\Delta ABC = \frac{1}{2}(BC \times AP = \frac{1}{2} \times 8\sqrt{2} \times 2 \text{ cm}^2 = 8\sqrt{2} \text{ cm}^2$

EXAMPLE 41 *AB is a diameter of a circle. P is a point on the semi-circle APB. AH and BK are perpendiculars from A and B respectively to the tangent at P. Prove that AH + BK = AB.*

SOLUTION Clearly, $\angle MPO = 90°$

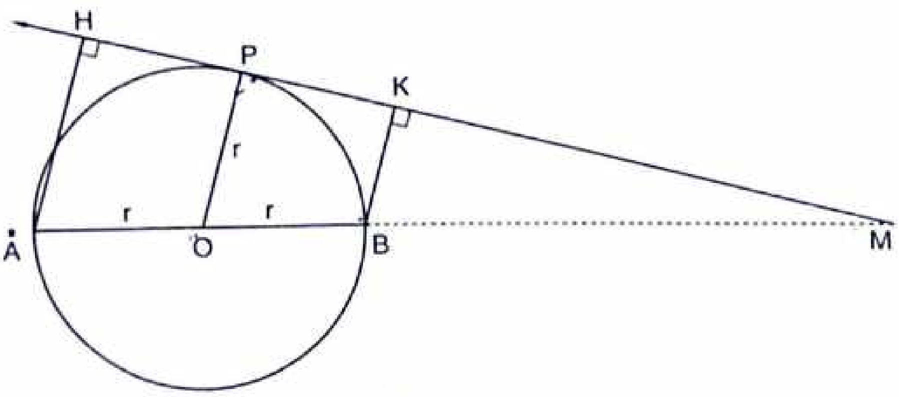


Fig. 8.54

Since $BK \perp HM$, $AH \perp HM$ and $OP \perp HM$. Therefore, $AH \parallel OP \parallel BK$.

Let $AH = x$, $BK = y$ and $OP = r$. Further, let $BM = z$.

In ΔMKB and ΔMHA, we have

$$\angle MKB = \angle MHA = 90^\circ$$

and $\angle BMK = \angle AMH$ [Common]

So, by AA criterion of similarity, we have

$$\Delta MKB \sim \Delta MHA$$

$$\Rightarrow \frac{BK}{AH} = \frac{MB}{MA} \Rightarrow \frac{y}{x} = \frac{z}{2r+z} \Rightarrow 2ry + yz = zx \Rightarrow z = \frac{2ry}{x-y} \quad \ldots(i)$$

In ΔMKB and ΔMPO, we have

$$\angle MKB = \angle MPO = 90^\circ$$

$$\angle BMK = \angle OMP \quad \text{[Common]}$$

So, by AA criterion of similarity, we obtain

$$\Delta MKB \sim \Delta MPO$$

$$\Rightarrow \frac{BK}{OP} = \frac{BM}{OM} \Rightarrow \frac{y}{r} = \frac{z}{r+z} \Rightarrow ry + yz = rz \Rightarrow z = \frac{ry}{r-y} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{ry}{r-y} = \frac{2ry}{x-y} \Rightarrow 2r - 2y = x - y \Rightarrow 2r = x + y \Rightarrow AB = AH + BK$$

EXAMPLE 42 *From an external point P, a tangent PT and a line segment PAB is drawn to a circle with centre O. ON is perpendicular on the chord AB. Prove that*

(i) $PA \cdot PB = PN^2 - AN^2$ (ii) $PN^2 - AN^2 = OP^2 - OT^2$ (iii) $\boxed{PA \cdot PB = PT^2}$

[NCERT EXEMPLAR]

SOLUTION (i) $PA \cdot PB = (PN - AN)(PN + BN)$

$= (PN - AN)(PN + AN)$ $\left[\begin{array}{l}\because ON \perp AB \\ \therefore N \text{ is the mid-point of } AB \\ \Rightarrow AN = BN\end{array}\right]$

$= PN^2 - AN^2$

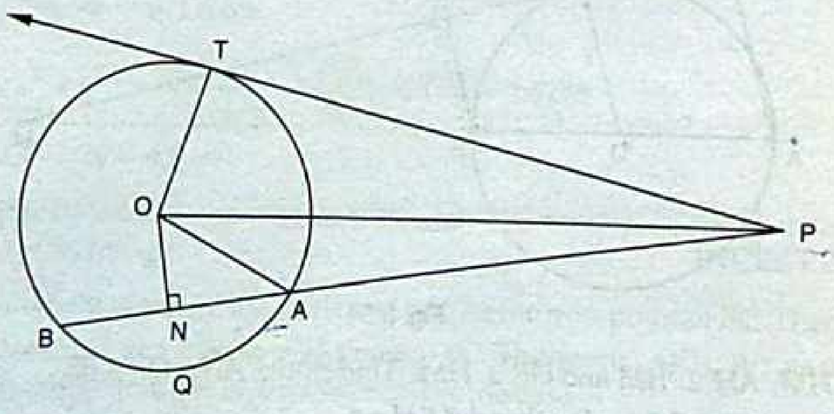


Fig. 8.55

(ii) Applying Pythagoras theorem in right triangle PNO, we obtain

$$OP^2 = ON^2 + PN^2 \Rightarrow PN^2 = OP^2 - ON^2$$

$$\therefore \quad PN^2 - AN^2 = (OP^2 - ON^2) - AN^2$$

$$= OP^2 - (ON^2 + AN^2)$$

$$= OP^2 - OA^2 \qquad \text{[Using Pythagoras theorem in } \Delta ONA\text{]}$$

$$= OP^2 - OT^2 \qquad [\because OA = OT = \text{radius}]$$

(iii) From (i) and (ii), we obtain

$$PA \cdot PB = PN^2 - AN^2 \text{ and } PN^2 - AN^2 = OP^2 - OT^2$$

$$\Rightarrow \quad PA \cdot PB = OP^2 - OT^2$$

Applying Pythagoras theorem in ΔOTP, we obtain

$$OP^2 = OT^2 + PT^2 \Rightarrow OP^2 - OT^2 = PT^2$$

Thus, we obtain

$$PA \,.\, PB = OP^2 - OT^2 \text{ and } OP^2 - OT^2 = PT^2$$

Hence, $PA \,.\, PB = PT^2$.

EXERCISE 8.2

LEVEL-1

1. If PT is a tangent at T to a circle whose centre is O and $OP = 17$ cm, $OT = 8$ cm, Find the length of the tangent segment PT.
2. Find the length of a tangent drawn to a circle with radius 5 cm, from a point 13 cm from the centre of the circle.
3. A point P is 26 cm away from the centre O of a circle and the length PT of the tangent drawn from P to the circle is 10 cm. Find the radius of the circle.
4. If from any point on the common chord of two intersecting circles, tangents be drawn to the circles, prove that they are equal.
5. If the sides of a quadrilateral touch a circle, prove that the sum of a pair of opposite sides is equal to the sum of the other pair.
6. Out of the two concentric circles, the radius of the outer circle is 5 cm and the chord AC of length 8 cm is a tangent to the inner circle. Find the radius of the inner circle. [NCERT EXEMPLAR]
7. A chord PQ of a circle is parallel to the tangent drawn at a point R of the circle. Prove that R bisects the arc PRQ. [NCERT EXEMPLAR]
8. Prove that a diameter AB of a circle bisects all those chords which are parallel to the tangent at the point A. [NCERT EXEMPLAR]
9. If AB, AC, PQ are tangents in Fig. 8.56 and $AB = 5$ cm, find the perimeter of ΔAPQ. [CBSE 2000]

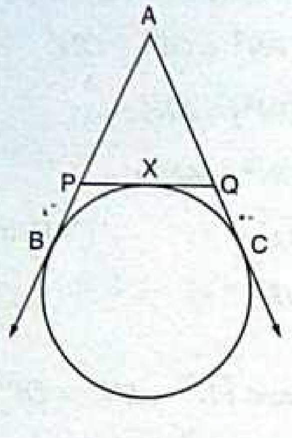


Fig. 8.56

10. Prove that the intercept of a tangent between two parallel tangents to a circle subtends a right angle at the centre.

11. In Fig. 8.57, PQ is tangent at a point R of the circle with centre O. If $\angle TRQ = 30°$, find $m\angle PRS$.

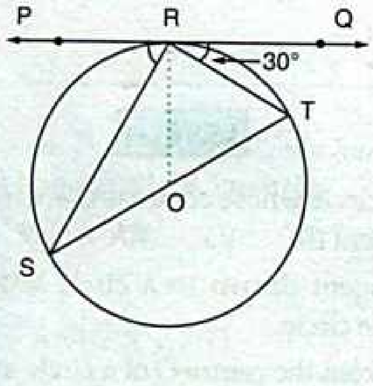


Fig. 8.57

12. If PA and PB are tangents from an outside point P. such that $PA = 10$ cm and $\angle APB = 60°$. Find the length of chord AB. **[CBSE 2016]**

13. In a right triangle ABC in which $\angle B = 90°$, a circle is drawn with AB as diameter intersecting the hypotenuse AC at P. Prove that the tangent to the circle at P bisects BC. **[NCERT EXEMPLAR]**

14. From an external point P, tangents PA and PB are drawn to a circle with centre O. At one point E on the circle tangent is drawn, which intersects PA and PB at C and D respectively. If $PA = 14$ cm, find the perimeter of ΔPCD. **[NCERT EXEMPLAR]**

15. In Fig. 8.58, ABC is a right triangle right-angled at B such that $BC = 6$ cm and $AB = 8$ cm. Find the radius of its incircle. **[CBSE 2002]**

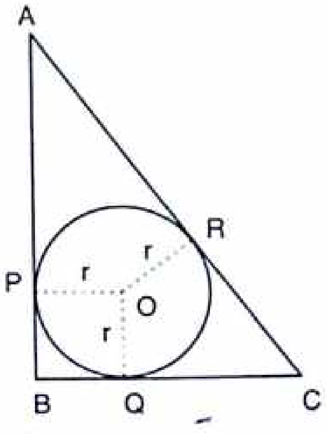


Fig. 8.58

16. Prove that the tangent drawn at the mid-point of an arc of a circle is parallel to the chord joining the end points of the arc. **[NCERT EXEMPLAR]**

17. From a point P, two tangents PA and PB are drawn to a circle with centre O. If OP = diameter of the circle, show that ΔAPB is equilateral.

18. Two tangent segments PA and PB are drawn to a circle with centre O such that $\angle APB = 120°$. Prove that $OP = 2\,AP$. **[CBSE 2014]**

19. If ΔABC is isosceles with $AB = AC$ and $C(O, r)$ is the incircle of the ΔABC touching BC at L, prove that L bisects BC.

20. AB is a diameter and AC is a chord of a circle with centre O such that $\angle BAC = 30°$. The tangent at C intersects AB at a point D. Prove that $BC = BD$. **[NCERT EXEMPLAR]**

21. In Fig. 8.59, a circle touches all the four sides of a quadrilateral $ABCD$ with $AB = 6$ cm, $BC = 7$ cm and $CD = 4$ cm. Find AD. **[CBSE 2002]**

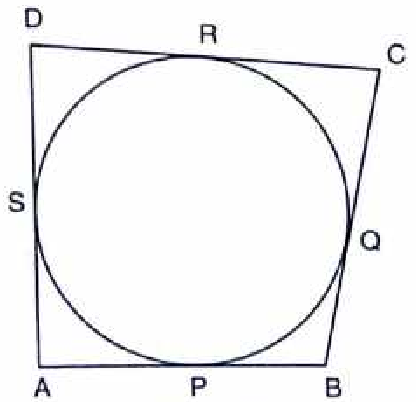


Fig. 8.59

22. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre of the circle.

23. Two circles touch externally at a point P. From a point T on the tangent at P, tangents TQ and TR are drawn to the circles with points of contact Q and R respectively. Prove that $TQ = TR$.

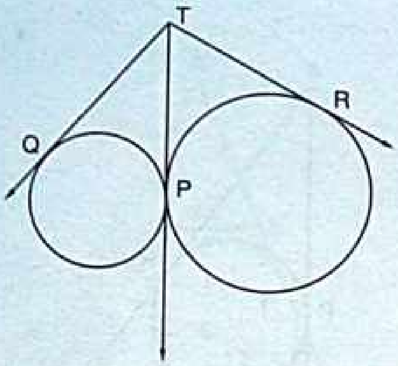


Fig. 8.60

24. A is a point at a distance 13 cm from the centre O of a circle of radius 5 cm. AP and AQ are the tangents to the circle at P and Q. If a tangent BC is drawn at a point R lying on the minor arc PQ to intersect AP at B and AQ at C, find the perimeter of the ΔABC.
[NCERT EXEMPLAR]

25. In Fig. 8.61, a circle is inscribed in a quadrilateral $ABCD$ in which $\angle B = 90°$. If $AD = 23$ cm, $AB = 29$ cm and $DS = 5$ cm, find the radius r of the circle.

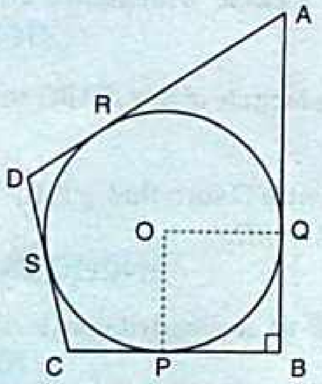


Fig. 8.61

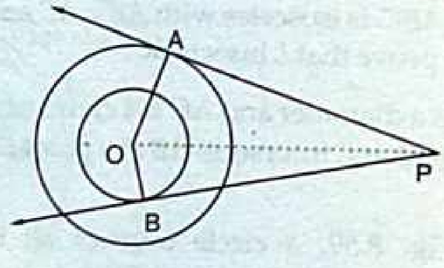


Fig. 8.62

26. In Fig. 8.62, there are two concentric circles with centre O of radii 5 cm and 3 cm. From an external point P, tangents PA and PB are drawn to these circles. If $AP = 12$ cm, find the length of BP.
[CBSE 2010, 2012, 2016]

27. In Fig. 8.63, AB is a chord of length 16 cm of a circle of radius 10 cm. The tangents at A and B intersect at a point P. Find the length of PA.
[CBSE 2010]

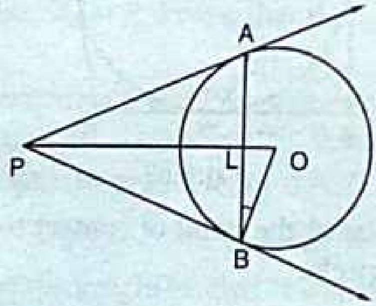


Fig. 8.63

28. In Fig. 8.64, PA and PB are tangents from an external point P to a circle with centre O. LN touches the circle at M. Prove that $PL + LM = PN + MN$. **[CBSE 2010]**

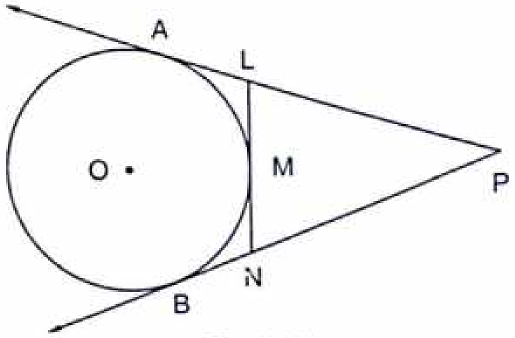


Fig. 8.64

29. In Fig. 8.65, BDC is a tangent to the given circle at point D such that $BD = 30$ cm and $CD = 7$ cm. The other tangents BE and CF are drawn respectively from B and C to the circle and meet when produced at A making BAC a right angle triangle. Calculate (i) AF (ii) radius of the circle.

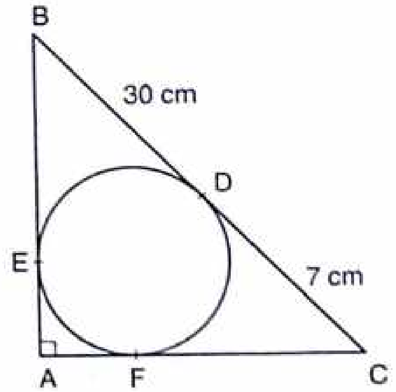


Fig. 8.65

30. If $d_1, d_2\ (d_2 > d_1)$ be the diameters of two concentric circles and c be the length of a chord of a circle which is tangent to the other circle, prove that $d_2^2 = c^2 + d_1^2$. **[NCERT EXEMPLAR]**

31. In Fig. 8.66, tangents PQ and PR are drawn from an external point P to a circle with centre O, such that $\angle RPQ = 30°$. A chord RS is drawn parallel to the tangent PQ. Find $\angle RQS$. **[CBSE 2015, NCERT EXEMPLAR]**

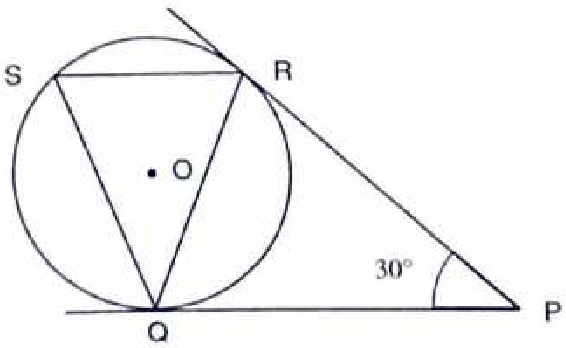


Fig. 8.66

32. From an external point P, tangents $PA = PB$ are drawn to a circle with centre O. If $\angle PAB = 50°$, then find $\angle AOB$. **[CBSE 2016]**

33. In Fig. 8.67, two tangents AB and AC are drawn to a circle with centre O such that $\angle BAC = 120°$. Prove that $OA = 2AB$. **[CBSE 2016]**

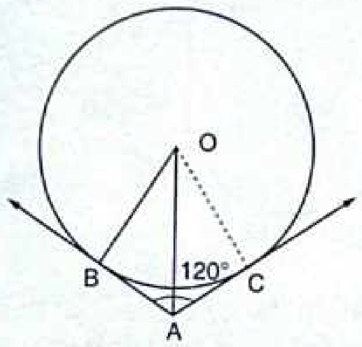


Fig. 8.67

34. The lengths of three consecutive sides of a quadrilateral circumscribing a circle are 4 cm, 5 cm, and 7 cm respectively. Determine the length of the fourth side.

35. The common tangents AB and CD to two circles with centres O and O' intersect at E between their centres. Prove that the points O, E and O' are collinear. **[NCERT EXEMPLAR]**

36. In Fig. 8.68, common tangents PQ and RS to two circles intersect at A. Prove that $PQ = RS$. **[CBSE 2014, NCERT EXEMPLAR]**

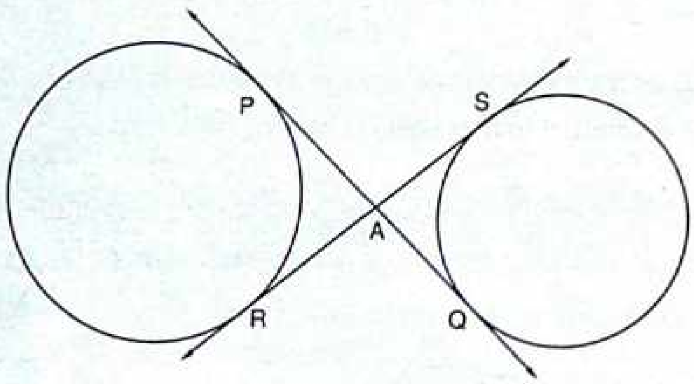


Fig. 8.68

37. Two concentric circles are of diameters 30 cm and 18 cm. Find the length of the chord of the larger circle which touches the smaller circle. **[CBSE 2014]**

38. AB and CD are common tangents to two circles of equal radii. Prove that $AB = CD$. **[NCERT EXEMPLAR]**

39. A triangle PQR is drawn to circumscribe a circle of radius 8 cm such that the segments QT and TR, into which QR is divided by the point of contact T, are of lengths 14 cm and 16 cm respectively. If area of ΔPQR is 336 cm^2, find the sides PQ and PR.

[CBSE 2014]

40. In Fig. 8.69, the tangent at a point C of a circle and a diameter AB when extended intersect at P. If $\angle PCA = 110°$, find $\angle CBA$. **[NCERT EXEMPLAR]**

[Hint: Join CO.]

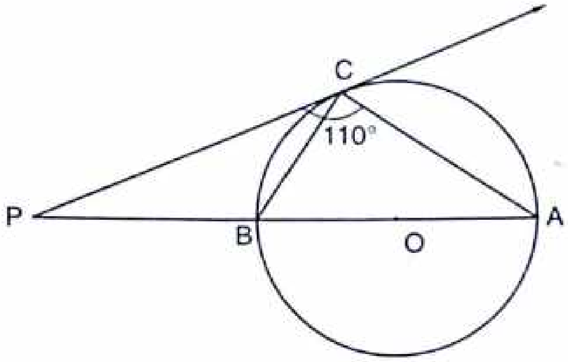


Fig. 8.69

41. AB is a chord of a circle with centre O, AOC is a diameter and AT is the tangent at A as shown in Fig. 8.70. Prove that $\angle BAT = \angle ACB$. **[NCERT EXEMPLAR]**

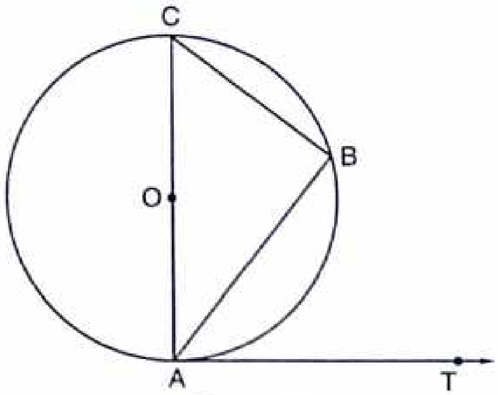


Fig. 8.70

LEVEL-2

42. In Fig. 8.71, a ΔABC is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC are of lengths 8 cm and 6 cm respectively. Find the lengths of sides AB and AC, when area of ΔABC is 84 cm^2. **[CBSE 2015]**

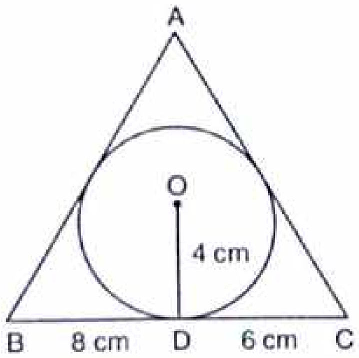


Fig. 8.71

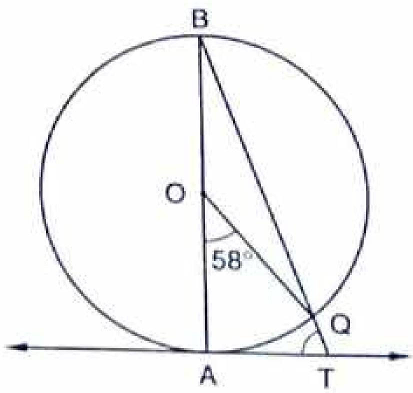


Fig. 8.72

43. In Fig. 8.72, AB is a diameter of a circle with centre O and AT is a tangent. If $\angle AOQ = 58°$, find $\angle ATQ$. **[CBSE 2015]**

44. In Fig. 8.73, $OQ : PQ = 3 : 4$ and perimeter of $\Delta POQ = 60$ cm. Determine PQ, QR and OP.

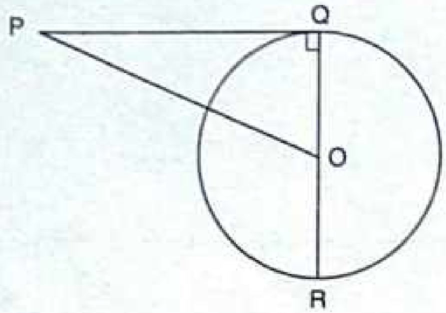


Fig. 8.73

45. Equal circles with centres O and O' touch each other at X. OO' produced to meet a circle with centre O', at A. AC is a tangent to the circle whose centre is O. $O'D$ is perpendicular to AC. Find the value of $\frac{DO'}{CO}$.

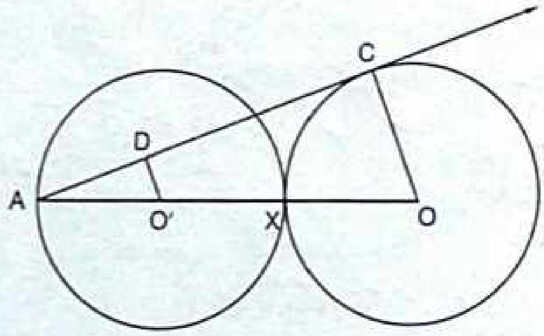


Fig. 8.74

46. In Fig. 8.75, BC is a tangent to the circle with centre O. OE bisects AP. Prove that $\Delta AEO \sim \Delta ABC$.

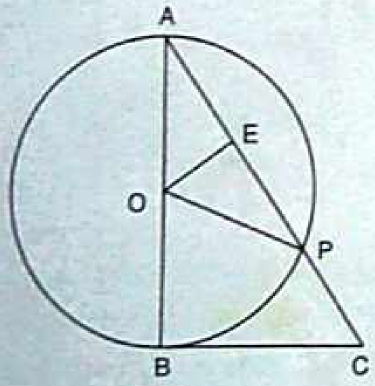


Fig. 8.75

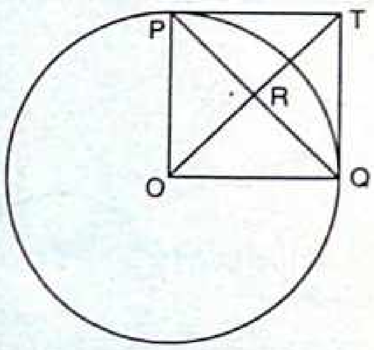


Fig. 8.76

47. In Fig. 8.76, $PO \perp QO$. The tangents to the circle at P and Q intersect at a point T. Prove that PQ and OT are right bisectors of each other.

48. In Fig. 8.77, O is the centre of the circle and BCD is tangent to it at C. Prove that $\angle BAC + \angle ACD = 90°$.

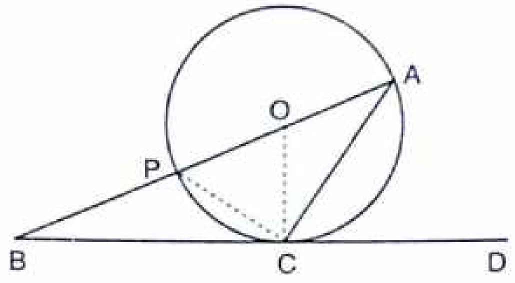


Fig. 8.77

49. Prove that the centre of a circle touching two intersecting lines lies on the angle bisector of the lines. **[NCERT EXEMPLAR]**

50. In Fig. 8.78, there are two concentric circles with centre O. PRT and PQS are tangents to the inner circle from a point P lying on the outer circle. If $PR = 5$ cm, find the length of PS. **[CBSE 2017]**

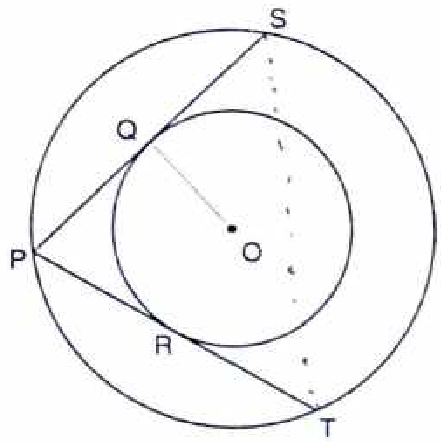


Fig. 8.78

51. In Fig. 8.79, PQ is a tangent from an external point P to a circle with centre O and OP cuts the circle at T and QOR is a diameter. If $\angle POR = 130°$ and S is a point on the circle, find $\angle 1 + \angle 2$. **[CBSE 2017]**

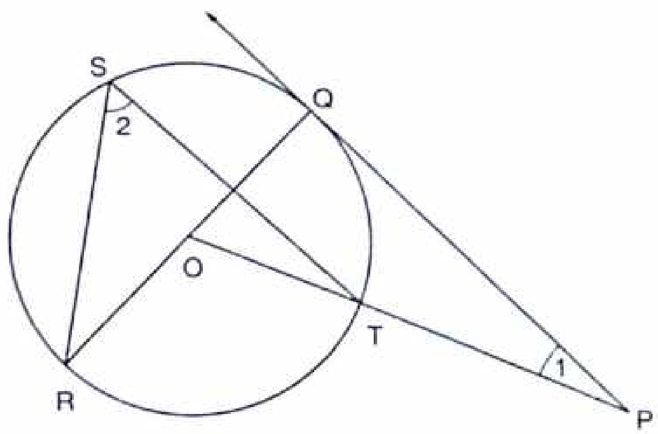


Fig. 8.79

52. In Fig. 8.80, PA and PB are tangents to the circle from an external point P. CD is another tangent touching the circle at Q. If $PA = 12$ cm, $QC = QD = 3$ cm, then find $PC + PD$. **[CBSE 2017]**

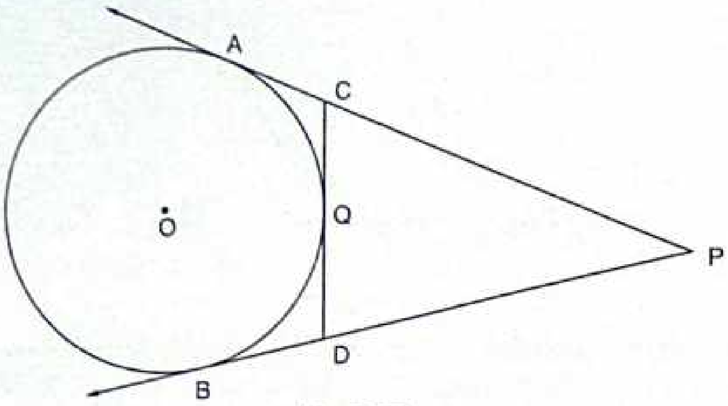


Fig. 8.80

ANSWERS

1. 15 cm	2. 12 cm	3. 24 cm	6. 3 cm	9. 10 cm	11. 60°
12. 10 cm	14. 28 cm	15. 2 cm	21. 3 cm	24. 24 cm	25. 11. cm
26. $4\sqrt{10}$ cm	27. $\frac{40}{3}$ cm	29. (i) 5 cm (ii) 5 cm		31. 75°	32. 100°
34. 6 cm	37. 24 cm	40. 70	42. 13 cm, 15 cm		43. 61°
44. $PQ = 20$ cm, $QR = 30$ cm, $OP = 25$ cm			45. $\frac{1}{3}$	50. 10 cm	52. 18 cm

HINTS TO SELECTED PROBLEMS

1. Since ΔOTP is a right triangle. Therefore,

$$OT^2 + PT^2 = OP^2 \Rightarrow PT^2 = OP^2 - OT^2 \Rightarrow PT^2 = 17^2 - 8^2 \Rightarrow PT = 15 \text{ cm}.$$

9. We have, $PB = PX$, $QC = QX$ and $AB = AC$

$\therefore$ Perimeter of $\Delta APQ = AP + PQ + AQ$

$$= AP + (PX + XQ) + AQ$$
$$= (AP + PX) + (AQ + XQ)$$
$$= (AP + PB) + (AQ + QC) = AB + AC = 10 \text{ cm}.$$

11. We have, $\angle TRQ = 30°$. Since ST is a diameter and angle in a semi-circle is a right angle. Therefore, $\angle SRT = 90°$.

Now, $\angle TRQ + \angle SRT + \angle PRS = 180° \Rightarrow 30° + 90° + \angle PRS = 180° \Rightarrow \angle PRS = 60°$

12. Tangents from an external point are equal in length. Therefore,

$PA = PB \Rightarrow \Delta PAB$ is isosceles $\Rightarrow \angle PAB = \angle PBA = 60° \Rightarrow \Delta PAB$ is equilateral
Hence, $AB = 10$ cm.

15. We have, $AR = AP = AB - BP = (8 - r)$ cm and, $CR = CQ = CB - BQ = (6 - r)$ cm.

$\therefore$ $AC = AR + CR = (8 - r + 6 - r)\text{cm} = (14 - 2r)$ cm

Now, $AC^2 = AB^2 + BC^2 \Rightarrow (14 - 2r)^2 = 8^2 + 6^2 \Rightarrow r = 2$ cm

17. Join OP. Suppose OP meets the circle at Q. join AQ.
Now,

$OP =$ Diameter

$\Rightarrow \quad OQ + PQ =$ Diameter

$\Rightarrow \quad PQ =$ Diameter – Radius $\qquad [\because OQ =$ radius$]$

$\Rightarrow \quad PQ =$ Radius.

$\therefore \quad OQ = PQ =$ Radius.

Thus, OP is the hypotenuse of right triangle OAP and Q is the mid-point of OP. The mid-point of hypotenuse of a right triangle is equidistant from the vertices

$\therefore \quad OA = AQ = OQ$

$\Rightarrow \quad \Delta OAQ$ is equilateral

$\Rightarrow \quad \angle AOQ = 60°$

So, $\angle APO = 30°$.

$\therefore \quad \angle APB = 2\angle APO = 60°$

Also, $PA = PB \Rightarrow \angle PAB = \angle PBA$.

But, $\angle APB = 60°$. Therefore, $\angle PAB = \angle PBA = 60°$

Hence, ΔAPB is equilateral

21. In example 9 on page 8.12, we have proved that if a circle touches all the four sides of quadrilateral $ABCD$. Then,

$$AB + CD = AD + BC \Rightarrow 6 + 4 = AD + 7 \Rightarrow AD = 3\text{cm}.$$

23. The tangents drawn from an external point to a circle are equal.

$\therefore \quad TP = TQ$ and $TP = TR \Rightarrow TQ = TR$

25. Proceed as in Example 16.

26. Join OA, OB and OP. In ΔOAP, we have

$$OP^2 = OA^2 + AP^2 \Rightarrow OP^2 = 5^2 + 12^2 \Rightarrow OP = 13\text{ cm}$$

In ΔOBP, we have

$$OP^2 = OB^2 + BP^2 \Rightarrow 13^2 = 3^2 + BP^2 \Rightarrow BP^2 = 169 - 9 = 160 \Rightarrow BP = \sqrt{160}\text{ cm} = 4\sqrt{10}\text{ cm}$$

27. We have, $AB = 16$ cm. Therefore, $AL = BL = 8$ cm.

In ΔOLB, we have

$$OB^2 = OL^2 + LB^2 \Rightarrow 10^2 = OL^2 + 8^2 \Rightarrow OL^2 = 100 - 64 = 36 \Rightarrow OL = 6\text{ cm}$$

Let $PL = x$ and $PB = y$. Then, $OP = (x + 6)$ cm.

In Δ's PLB and OBP, we have

$$PB^2 = PL^2 + BL^2 \text{ and } OP^2 = OB^2 + PB^2$$

$\Rightarrow \quad y^2 = x^2 + 64$ and $(x+6)^2 = 100 + y^2$ [Substituting the value of y^2 in second equation]

$$\Rightarrow \quad (x+6)^2 = 100 + x^2 + 64 \Rightarrow 12x = 128 \Rightarrow x = \frac{32}{3}\text{cm}$$

$\therefore \quad y^2 = x^2 + 64 \Rightarrow y^2 = \left(\frac{32}{3}\right)^2 + 64 = \frac{1600}{9} \Rightarrow y = \frac{40}{3}$ cm

Hence, $PA = PB = \frac{40}{3}$ cm

28. We have,

$PA = PB$ [Tangents drawn from P]

$\Rightarrow \quad PL + AL = PN + BN \Rightarrow PL + LM = PN + MN$ [$\because AL = LM$ and $BN = MN$]

33. In Δ's OAB and OAC, we have

$\angle OBA = \angle OCA = 90°$ and, $OA = OA$ [Common]

So, by RHS congruence criterion, we obtain

$$\Delta OBA \cong \Delta OCA \Rightarrow \angle OAB = \angle OAC = \frac{1}{2} \times 120° = 60°$$

In ΔOBA, we have

$$\cos 60° = \frac{AB}{OA} \Rightarrow \frac{1}{2} = \frac{AB}{OA} \Rightarrow OA = 2AB$$

36. $AP = AR$ and $AS = AQ \Rightarrow AP + AQ = AS + AR \Rightarrow PQ = SR$

39. Since length of tangents drawn from a point to a circle are equal. Therefore, $QS = AT = 14$ cm, $RU = RT = 16$ cm and, $PS = PU = x$.

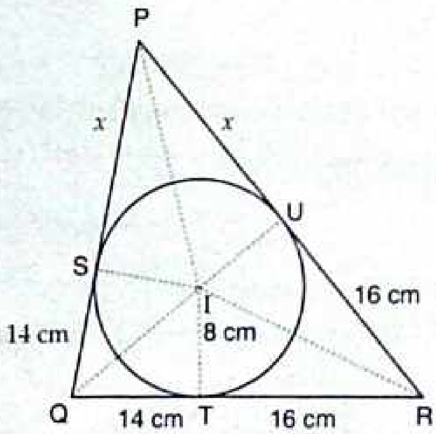


Fig. 8.81

Thus, $PQ = x + 14$, $PR = x + 16$ and $QR = 30$

Now, Area of ΔPQR = Area of ΔIQR + Area of ΔIQP + Area of ΔIPR

$$\Rightarrow \quad 336 = \frac{1}{2}(QR \times 8) + \frac{1}{2}(14 + 2) \times 8 + \frac{1}{2}(16 + x) \times 8$$

$$\Rightarrow \quad 84 = 30 + 14 + x + 16 + x$$

$$\Rightarrow \quad 24 = 2x \Rightarrow x = 12$$

Hence, $PQ = 26$ cm and $PR = 28$ cm

47. In Δ's TPO and TQO, we have

$$\angle TPO = \angle TQO \qquad \text{[Eqch equal to } 90^\circ]$$

So, by RHS congruence criterion, we obtain

$$\Delta TPO \cong \Delta TQO \Rightarrow \angle PTO = \angle QTO$$

Also, $\Delta PTR \cong \Delta QTR$ [By SAS congruence criterion]

$\therefore \quad PR = QR$ and $\angle TRP = \angle TRQ$

But, $\angle TRP + \angle TRQ = 180^\circ$

$\therefore \quad \angle TRP = \angle TRQ = 90^\circ$

Hence, PQ and OT are right bisectors of each other.

48. $OA = OC$ [Each equal to radius]

$\Rightarrow \quad \angle OAC = \angle OCA$ ...(i)

Clearly, $\angle OCD = 90^\circ$

$\Rightarrow \quad \angle ACD + \angle OCA = 90^\circ$

$\Rightarrow \quad \angle ACD + \angle OAC = 90^\circ$ [From (i)]

$\Rightarrow \quad \angle ACD + \angle BAC = 90^\circ$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. In Fig. 8.82, PA and PB are tangents to the circle drawn from an external point P. CD is a third tangent touching the circle at Q. If $PB = 10$ cm and $CQ = 2$ cm, what is the length PC?

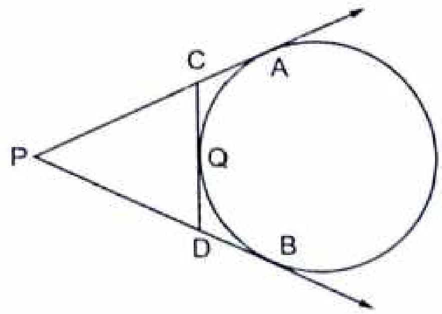


Fig. 8.82

2. What is the distance between two parallel tangents of a circle of radius 4 cm?
3. The length of tangent from a point A at a distance of 5 cm from the centre of the circle is 4 cm. What is the radius of the circle?
4. Two tangents TP and TQ are drawn from an external point T to a circle with centre O as shown in Fig. 8.83. If they are inclined to each other at an angle of 100°, then what is the value of $\angle POQ$?

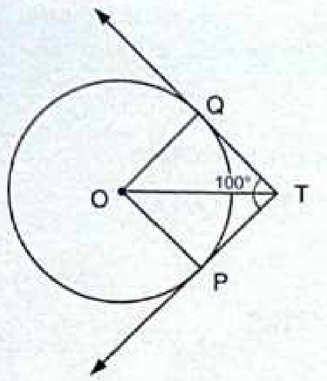


Fig. 8.83

5. What the distance between two parallel tangents to a circle of radius 5 cm?
6. In Q. No. 1, if $PB = 10$ cm, what is the perimeter of ΔPCD?
7. In Fig. 8.84, CP and CQ are tangents to a circle with centre O. ARB is another tangent touching the circle at R. If $CP = 11$ cm and $BC = 7$ cm, then find the length of BR. **[CBSE 2009]**

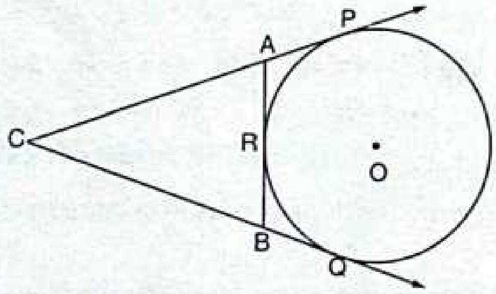


Fig. 8.84

8. In Fig. 8.85, $\Delta\, ABC$ is circumscribing a circle. Find the length of BC. **[CBSE 2009]**

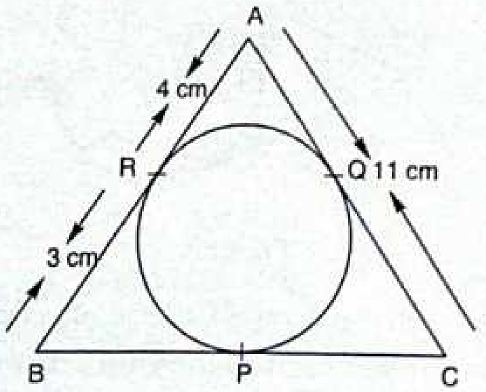


Fig. 8.85

9. In Fig. 8.86, CP and CQ are tangents from an external point C to a circle with centre O. AB is another tangent which touches the circle at R. If $CP = 11$ cm and $BR = 4$ cm, find the length of BC. **[CBSE 2010]**

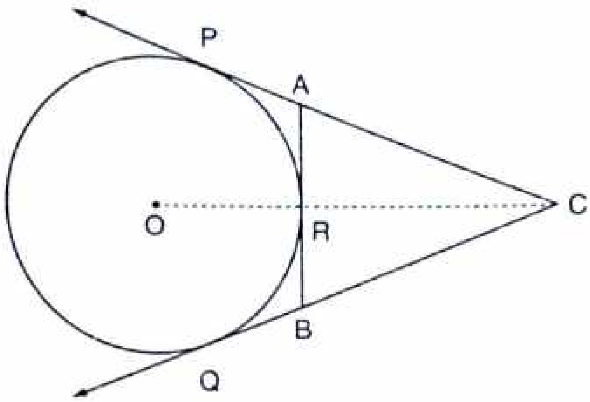


Fig. 8.86

[**Hint:** We have, $CP = 11$ cm

$\therefore \quad CP = CQ \Rightarrow CQ = 11$ cm

Now, $BR = BQ$ [Tangents drawn from B]

$\Rightarrow \quad BQ = 4$ cm

$\therefore \quad BC = CQ - BQ = (11-4)\text{cm} = 7\,\text{cm}$

10. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

11. In Fig. 8.87, PA and PB are tangents to the circle with centre O such that $\angle APB = 50°$. Write the measure of $\angle OAB$. **[CBSE 2015]**

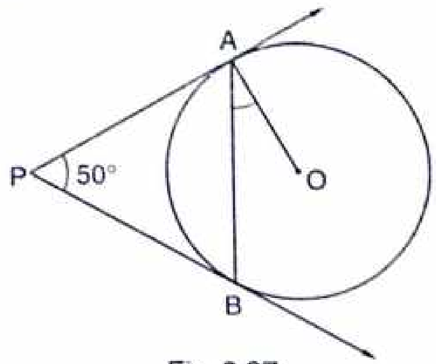


Fig. 8.87

12. In Fig. 8.88, PQ is a chord of a circle and PT is the tangent at P such that $\angle QPT = 60°$. Then, find $\angle PRQ$. **[NCERT EXEMPLAR]**

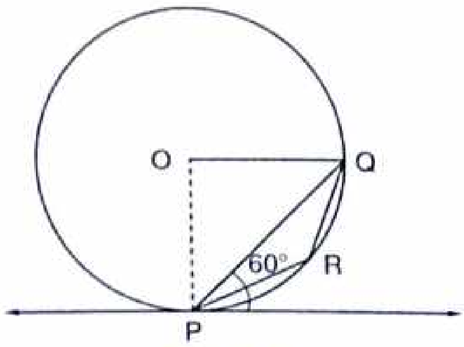


Fig. 8.88

13. In Fig. 8.89, PQL and PRM are tangents to the circle with centre O at the points Q and R respectively and S is a point on the circle such that $\angle SQL = 50°$ and $\angle SRM = 60°$. Then, find $\angle QSR$. **[NCERT EXEMPLAR]**

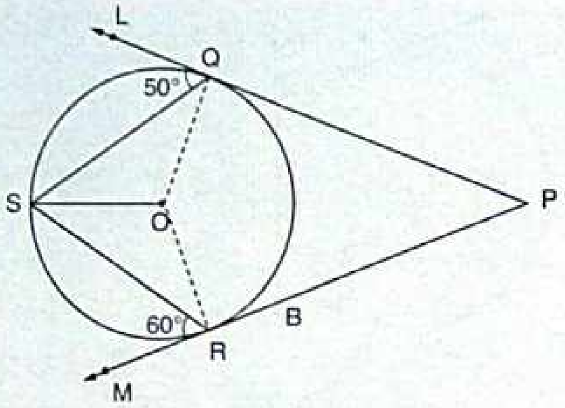


Fig. 8.89

14. In Fig. 8.90, BOA is a diameter of a circle and the tangent at a point P meets BA produced at T. If $\angle PBO = 30°$, then find $\angle PTA$. [NCERT EXEMPLAR]

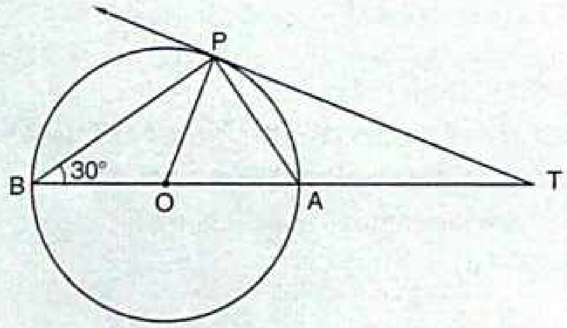


Fig. 8.90

ANSWERS

1. 8 cm	3. 3 cm	4. 80°	5. 10 cm	6. 20 cm	7. 4 cm	8. 10 cm
9. 7 cm	10. 8 cm	11. 25°	12. 120°	13. 70°	14. 30°	

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q such that $OQ = 12$ cm. Length PQ is
   (a) 12 cm (b) 13 cm (c) 8.5 cm (d) $\sqrt{119}$ cm
2. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. The radius of the circle is
   (a) 7 cm (b) 12 cm (c) 15 cm (d) 24.5 cm. [NCERT]
3. The length of the tangent from a point A at a circle, of radius 3 cm, is 4 cm. The distance of A from the centre of the circle is
   (a) $\sqrt{7}$ cm (b) 7 cm (c) 5 cm (d) 25 cm
4. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at an angle of 80° then $\angle POA$ is equal to
   (a) 50° (b) 60° (c) 70° (d) 80° [NCERT]

5. If TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110°$, then, $\angle PTQ$ is equal to
   (a) $60°$ (b) $70°$ (c) $80°$ (d) $90°$ [NCERT]

6. PQ is a tangent to a circle with centre O at the point P. If ΔOPQ is an isosceles triangle, then $\angle OQP$ is equal to
   (a) $30°$ (b) $45°$ (c) $60°$ (d) $90°$

7. Two equal circles touch each other externally at C and AB is a common tangent to the circles. Then, $\angle ACB =$
   (a) $60°$ (b) $45°$ (c) $30°$ (d) $90°$

8. ABC is a right angled triangle, right angled at B such that $BC = 6$ cm and $AB = 8$ cm. A circle with centre O is inscribed in ΔABC. The radius of the circle is
   (a) 1 cm (b) 2 cm (c) 3 cm (d) 4 cm

9. PQ is a tangent drawn from a point P to a circle with centre O and QOR is a diameter of the circle such that $\angle POR = 120°$, then $\angle OPQ$ is
   (a) $60°$ (b) $45°$ (c) $30°$ (d) $90°$

10. If four sides of a quadrilateral $ABCD$ are tangential to a circle, then
    (a) $AC + AD = BD + CD$ (b) $AB + CD = BC + AD$
    (c) $AB + CD = AC + BC$ (d) $AC + AD = BC + DB$

11. The length of the tangent drawn from a point 8 cm away from the centre of a circle of radius 6 cm is
    (a) $\sqrt{7}$ cm (b) $2\sqrt{7}$ cm (c) 10 cm (d) 5 cm

12. AB and CD are two common tangents to circles which touch each other at C. If D lies on AB such that $CD = 4$ cm, then AB is equal to
    (a) 4 cm (b) 6 cm (c) 8 cm (d) 12 cm

13. In Fig. 8.91, if AD, AE and BC are tangents to the circle at D, E and F respectively. Then,

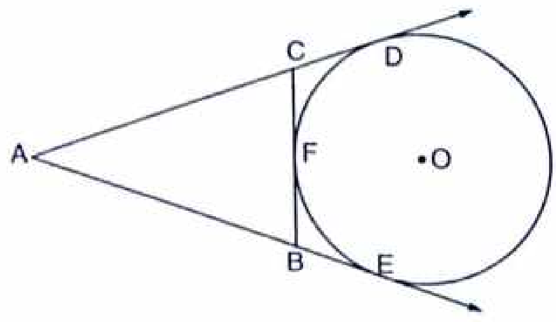


Fig. 8.91

    (a) $AD = AB + BC + CA$ (b) $2AD = AB + BC + CA$
    (c) $3AD = AB + BC + CA$ (d) $4AD = AB + BC + CA$

14. In Fig. 8.92, RQ is a tangent to the circle with centre O. If $SQ = 6$ cm and $QR = 4$ cm, then $OR =$
    (a) 8 cm (b) 3 cm (c) 2.5 cm (d) 5 cm

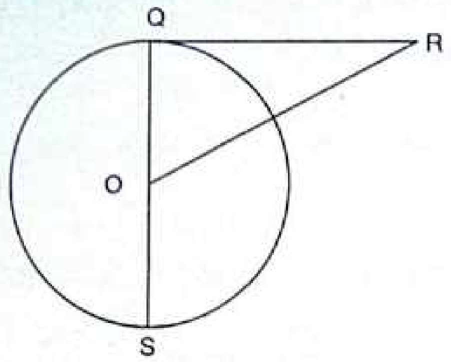


Fig. 8.92

15. In Fig. 8.93, the perimeter of ΔABC is
(a) 30 cm (b) 60 cm (c) 45 cm (d) 15 cm

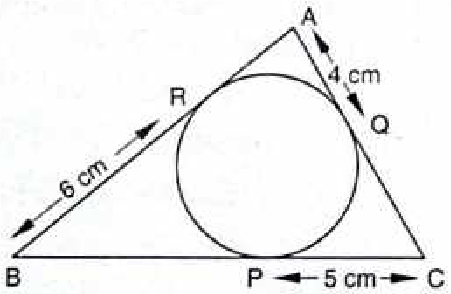


Fig. 8.93

16. In Fig. 8.94, AP is a tangent to the circle with centre O such that $OP = 4$ cm and $\angle OPA = 30°$. Then, $AP =$

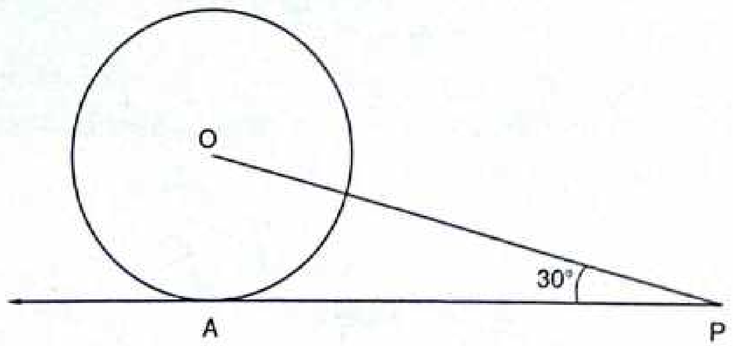


Fig. 8.94

(a) $2\sqrt{2}$ cm (b) 2 cm (c) $2\sqrt{3}$ cm (d) $3\sqrt{2}$ cm

17. AP and PQ are tangents drawn from a point A to a circle with centre O and radius 9 cm. If $OA = 15$ cm, then $AP + AQ =$
(a) 12 cm (b) 18 cm (c) 24 cm (d) 36 cm

18. At one end of a diameter PQ of a circle of radius 5 cm, tangent XPY is drawn to the circle. The length of chord AB parallel to XY and at a distance of 8 cm from P is
(a) 5 cm (b) 6 cm (c) 7 cm (d) 8 cm

19. If PT is tangent drawn from a point P to a circle touching it at T and O is the centre of the circle, then $\angle OPT + \angle POT =$
(a) 30° (b) 60° (c) 90° (d) 180°

20. In Fig. 8.95, if $AB = 12$ cm, $BC = 8$ cm and $AC = 10$ cm, then $AD =$

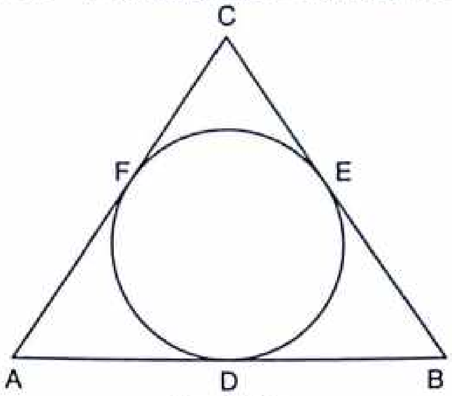


Fig. 8.95

(a) 5 cm (b) 4 cm (c) 6 cm (d) 7 cm

21. In Fig. 8.96, if $AP = PB$, then

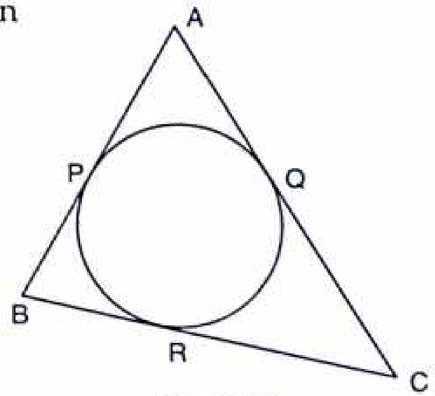


Fig. 8.96

(a) $AC = AB$ (b) $AC = BC$ (c) $AQ = QC$ (d) $AB = BC$

22. In Fig. 8.97, if $AP = 10$ cm, then $BP =$

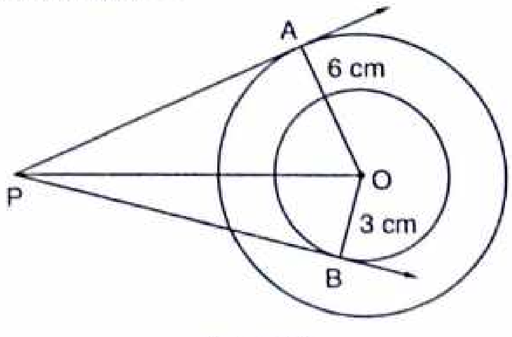


Fig. 8.97

(a) $\sqrt{91}$ cm (b) $\sqrt{127}$ cm (c) $\sqrt{119}$ cm (d) $\sqrt{109}$ cm

23. In Fig. 8.98, if PR is tangent to the circle at P and Q is the centre of the circle, then $\angle POQ =$

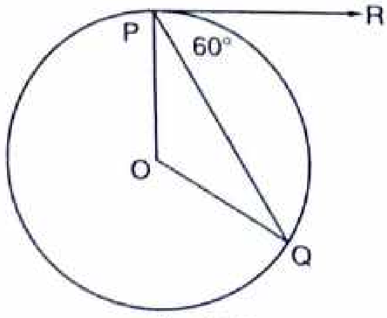


Fig. 8.98

(a) 110° (b) 100° (c) 120° (d) 90°

24. In Fig. 8.99, if quadrilateral $PQRS$ circumscribes a circle, then $PD + QB =$

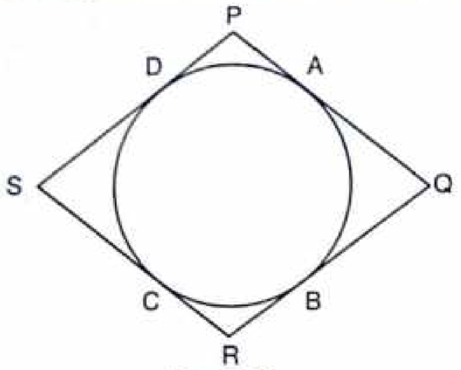


Fig. 8.99

(a) PQ (b) QR (c) PR (d) PS

25. In Fig. 8.100, two equal circles touch each other at T, if $QP = 4.5$ cm, then $QR =$

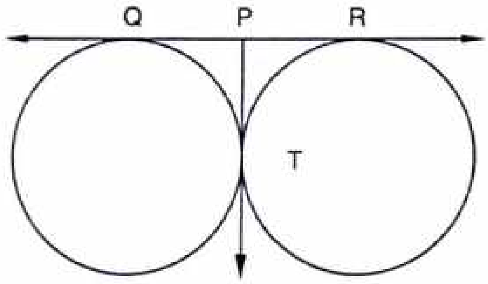


Fig. 8.100

(a) 9 cm (b) 18 cm (c) 15 cm (d) 13.5 cm

26. In Fig. 8.101, APB is a tangent to a circle with centre O at point P. If $\angle QPB = 50°$, then the measure of $\angle POQ$ is

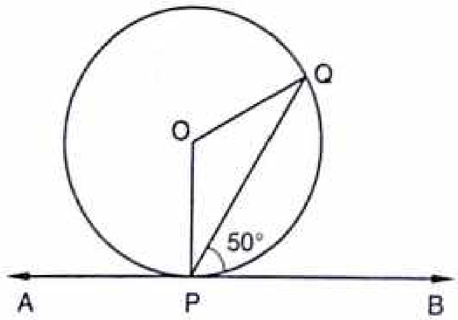


Fig. 8.101

(a) 100° (b) 120° (c) 140° (d) 150°

27. In Fig. 8.102, if tangents PA and PB are drawn to a circle such that $\angle APB = 30°$ and chord AC is drawn parallel to the tangent PB, then $\angle ABC =$ **[NCERT EXEMPLAR]**

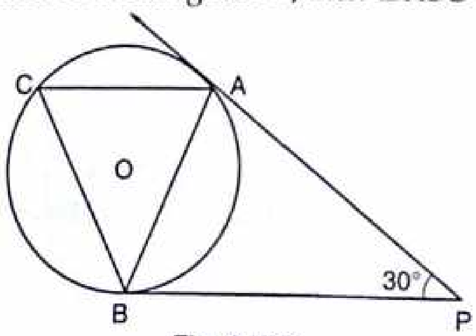


Fig. 8.102

(a) 60° (b) 90° (c) 30° (d) None of these

28. In Fig. 8.103, $PR =$

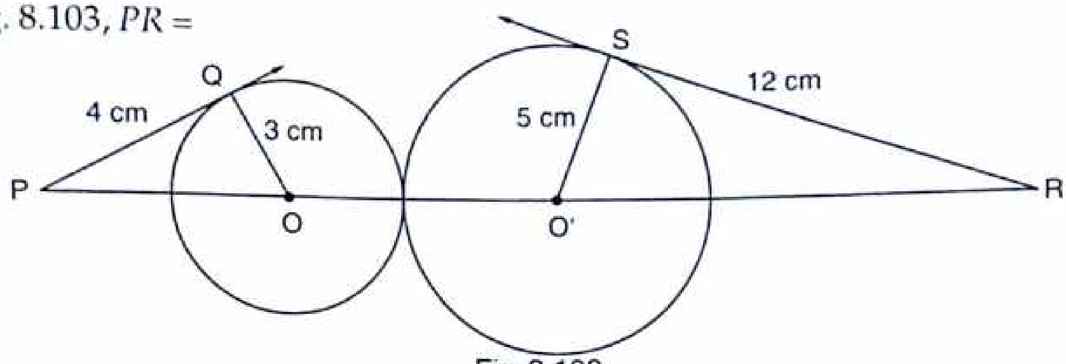


Fig. 8.103

(a) 20 cm (b) 26 cm (c) 24 cm (d) 28 cm

29. Two circles of same radii r and centres O and O' touch each other at P as shown in Fig. 8.104. If OO' is produced to meet the circle $C(O', r)$ at A and AT is a tangent to the circle $C(O, r)$ such that $O'Q \perp AT$. Then $AO : AO' =$

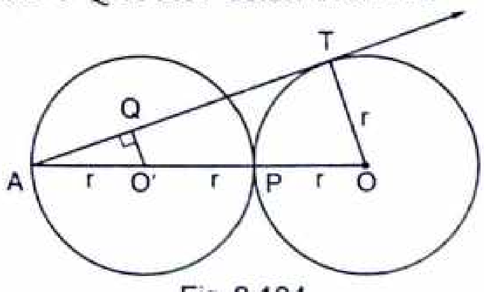


Fig. 8.104

(a) 3/2 (b) 2 (c) 3 (d) 1/4

30. Two concentric circles of radii 3 cm and 5 cm are given. Then length of chord BC which touches the inner circle at P is equal to **[CBSE 2014]**

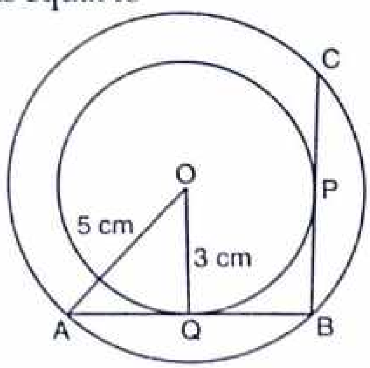


Fig. 8.105

(a) 4 cm (b) 6 cm (c) 8 cm (d) 10 cm

31. In Fig. 8.106, there are two concentric circles with centre O. PR and PQS are tangents to the inner circle from point plying on the outer circle. If $PR = 7.5$ cm, then PS is equal to

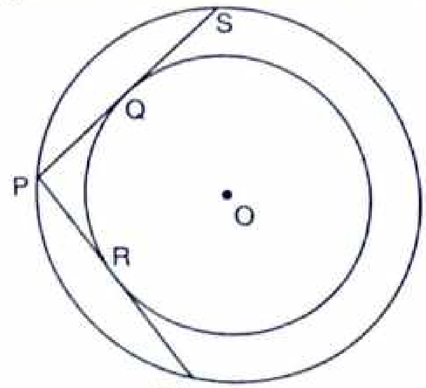


Fig. 8.106

(a) 10 cm (b) 12 cm (c) 15 cm (d) 18 cm

32. In Fig. 8.107, if $AB = 8$ cm and $PE = 3$ cm, then $AE =$

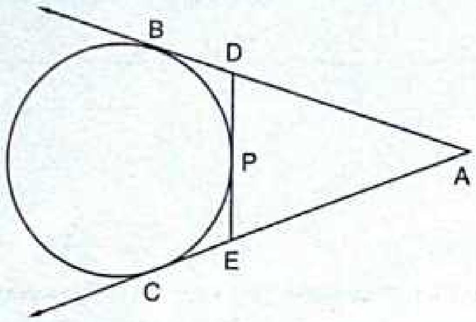


Fig. 8.107

(a) 11 cm (b) 7 cm (c) 5 cm (d) 3 cm

33. In Fig. 8.108, PQ and PR are tangents drawn from P to a circle with centre O. If $\angle OPQ = 35°$, then

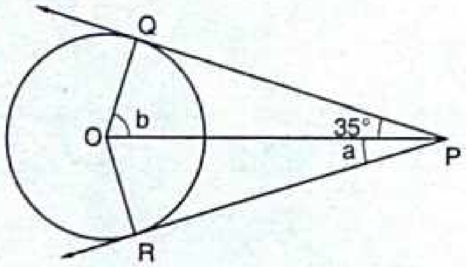


Fig. 8.108

(a) $a = 30°, b = 60°$ (b) $a = 35°, b = 55°$

(c) $a = 40°, b = 50°$ (d) $a = 45°, b = 45°$

34. In Fig. 8.109, if TP and TQ are tangents drawn from an external point T to a circle with centre O such that $\angle TQP = 60°$, then $\angle OPQ =$

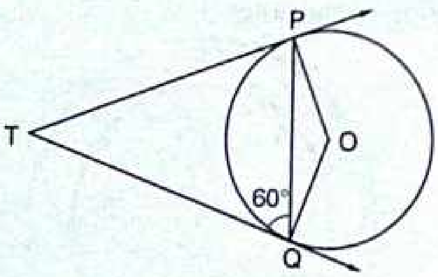


Fig. 8.109

(a) 25° (b) 30° (c) 40° (d) 60°

35. In Fig. 8.110, the sides AB, BC and CA of triangle ABC, touch a circle at P, Q and R respectively. If $PA = 4\,\text{cm}$, $BP = 3\,\text{cm}$ and $AC = 11\,\text{cm}$, then length of BC is

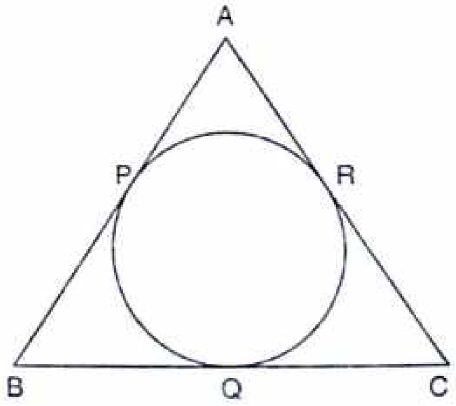


Fig. 8.110

(a) 11 cm (b) 10 cm (c) 14 cm (d) 15 cm [CBSE 2012]

36. In Fig. 8.111, a circle touches the side DF of ΔEDF at H and touches ED and EF produced at K and M respectively. If $EK = 9\,\text{cm}$, then the perimeter of ΔEDF is

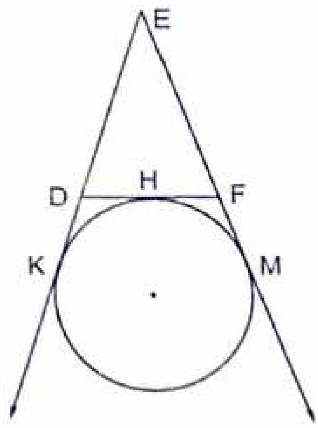


Fig. 8.111

(a) 18 cm (b) 13.5 cm (c) 12 cm (d) 9 cm [CBSE 2012]

37. In Fig. 8.112, DE and DF are tangents from an external point D to a circle with centre A. If $DE = 5\,\text{cm}$ and $DE \perp DF$, then the radius of the circle is

(a) 3 cm (b) 5 cm (c) 4 cm (d) 6 cm [CBSE 2013]

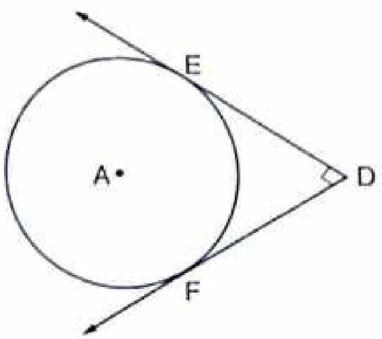


Fig. 8.112

38. In Fig. 8.113, a circle with centre O is inscribed in a quadrilateral $ABCD$ such that, it touches sides BC, AB, AD and CD at points P, Q, R and S respectively. If $AB = 29$ cm, $AD = 23$ cm, $\angle B = 90°$ and $DS = 5$ cm, then the radius of the circle (in cm) is

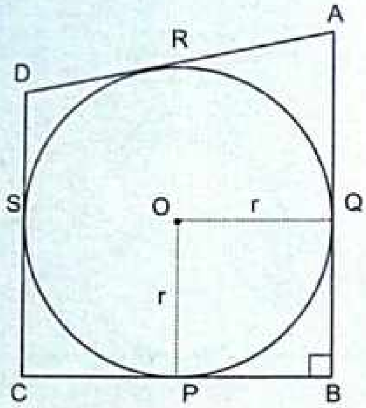


Fig. 8.113

(a) 11 (b) 18 (c) 6 (d) 15 [CBSE 2013]

39. In a right triangle ABC, right angled at B, $BC = 12$ cm and $AB = 5$ cm. The radius of the circle inscribed in the triangle (in cm) is

(a) 4 (b) 3 (c) 2 (d) 1 [CBSE 2014]

40. Two circles touch each other externally at P. AB is a common tangent to the circle touching them at A and B. The value of $\angle APB$ is

(a) 30° (b) 45° (c) 60° (d) 90° [CBSE 2014]

41. In Fig. 8.114, PQ and PR are two tangents to a circle with centre O. If $\angle QPR = 46°$, then $\angle QOR$ equals

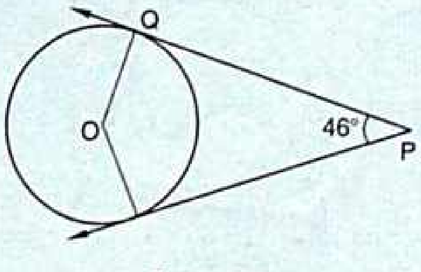


Fig. 8.114

(a) 67° (b) 134° (c) 44° (d) 46° [CBSE 2014]

42. In Fig. 8.115, QR is a common tangent to the given circles touching externally at the point T. The tangent at T meets QR at P. If $PT = 3.8$ cm, then the length of QR (in cm) is

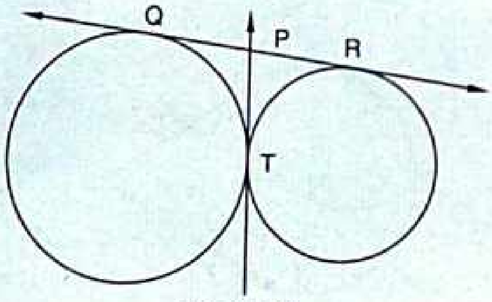


Fig. 8.115

(a) 3.8 (b) 7.6 (c) 5.7 (d) 1.9 [CBSE 2014]

43. In Fig. 8.116, a quadrilateral $ABCD$ is drawn to circumscribe a circle such that its sides AB, BC, CD and AD touch the circle at P, Q, R and S respectively. If $AB = x$ cm, $BC = 7$ cm, $CR = 3$ cm and $AS = 5$ cm, then $x =$

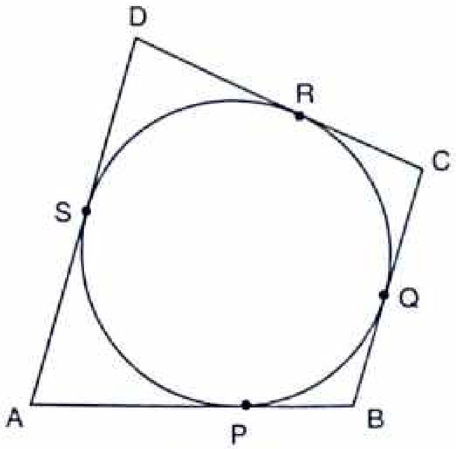


Fig. 8.116

(a) 10 (b) 9 (c) 8 (d) 7 [CBSE 2014]

44. If angle between two radii of a circle is 130°, the angle between the tangents at the ends of radii is

(a) 90° (b) 50° (c) 70° (d) 40°

[NCERT EXEMPLAR]

45. If two tangents inclined at an angle of 60° are drawn to a circle of radius 3 cm, then length of each tangent is equal to

(a) $\frac{3\sqrt{3}}{2}$ cm (b) 6 cm (c) 3 cm (d) $3\sqrt{3}$ cm

[NCERT EXEMPLAR]

46. If radii of two concentric circles are 4 cm and 5 cm, then the length of each chord of one circle which is tangent to the other circle is

(a) 3 cm (b) 6 cm (c) 9 cm (d) 1 cm

[NCERT EXEMPLAR]

47. At one end A of a diameter AB of a circle of radius 5 cm, tangent XAY is drawn to the circle. The length of the chord CD parallel to XY and at a distance 8 cm from A is

(a) 4 cm (b) 5 cm (c) 6 cm (d) 8 cm

[NCERT EXEMPLAR]

48. From a point P which is at a distance 13 cm from the centre O of a circle of radius 5 cm, the pair of tangents PQ and PR to the circle are drawn. Then the area of the quadrilateral $PQOR$ is

(a) 60 cm^2 (b) 65 cm^2 (c) 30 cm^2 (d) 32.5 cm^2

[NCERT EXEMPLAR]

49. If PA and PB are tangents to the circle with centre O such that $\angle APB = 50°$, then $\angle OAB$ is equal to

(a) 25° (b) 30° (c) 40° (d) 50°

50. The pair of tangents AP and AQ drawn from an external point to a circle with centre O are prependicular to each other and length of each tangent is 5 cm. The radius of the circle is

(a) 10 cm (b) 7.5 cm (c) 5 cm (d) 2.5 cm

[NCERT EXEMPLAR]

51. In Fig. 8.117, if $\angle AOB = 125°$, then $\angle COD$ is equal to

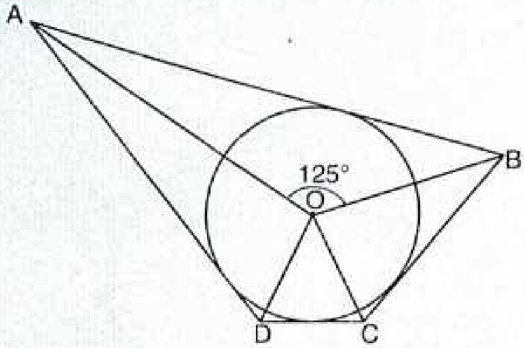


Fig. 8.117

(a) 45° (b) 35° (c) 55° (d) $62\frac{1}{2}°$

[NCERT EXEMPLAR]

52. In Fig. 8.118, if PQR is the tangent to a circle at Q whose centre is O, AB is a chord parallel to PR and $\angle BQR = 70°$, then $\angle AQB$ is equal to

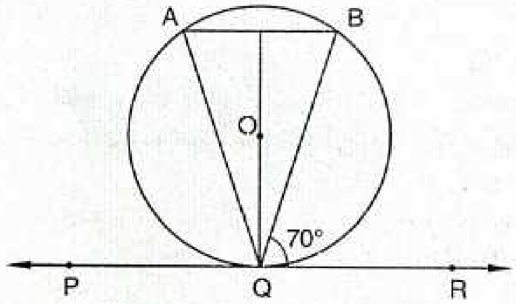


Fig. 8.118

(a) 20° (b) 40° (c) 35° (d) 45°

[NCERT EXEMPLAR]

ANSWERS

1. (d)	2. (a)	3. (c)	4. (a)	5. (b)	6. (b)
7. (d)	8. (b)	9. (c)	10. (b)	11. (b)	12. (c)
13. (b)	14. (d)	15. (a)	16. (c)	17. (c)	18. (d)
19. (c)	20. (d)	21. (b)	22. (b)	23. (c)	24. (a)
25. (a)	26. (a)	27. (c)	28. (b)	29. (c)	30. (c)
31. (c)	32. (c)	33. (b)	34. (b)	35. (b)	36. (a)
38. (a)	39. (c)	40. (d)	41. (b)	42. (b)	43. (b)
44. (b)	45. (d)	46. (b)	47. (d)	48. (a)	49. (a)
50. (c)	51. (c)	52. (b)			

SUMMARY

1. Tangent to a circle at a point is perpendicular to the radius through the point of contact.
2. From a point, lying outside a circle, two and only two tangents can be drawn to it.
3. The lengths of the two tangents drawn from an external point to a circle are equal.

CHAPTER 9

CONSTRUCTIONS

9.1 INTRODUCTION

In class IX, we have done some constructions, namely drawing perpendicular bisector of a line segment, bisecting on angle, construction of some standard angles and construction of some triangles. We have used ruler and a pair of compasses in these constructions. We have also given their justifications. In this chapter, we shall learn some more constructions by using the knowledge of constructions learnt in earlier classes. We will also give their justifications by using various concepts of geometry which we have learnt so far.

9.2 DIVISION OF A LINE SEGMENT

In this section, we will learn the method of dividing a line segment internally in a given ratio. The justification of the method will be given by using the Basic Proportionality Theorem and the concept of similar triangles.

In order to divide a line segment internally in a given ratio $m : n$, where both m and n are positive integers, we follow the following steps:

Steps of construction

STEP I Draw a line segement AB of given length by using a ruler.

STEP II Draw any ray AX making an acute angle with AB.

STEP III Along AX mark off $(m+n)$ points $A_1, A_2, ..., A_m, A_{m+1}, \ldots, A_{m+n}$ such that $AA_1 = A_1A_2 = A_{m+n-1}A_{m+n}$.

STEP IV Join $B\ A_{m+n}$.

STEP V Through the point A_m draw a line parallel to $A_{m+n}\ B$ by making an angle equal to $\angle A\ A_{m+n}\ B$ *at* A_m. Suppose this line meets AB at a point P.

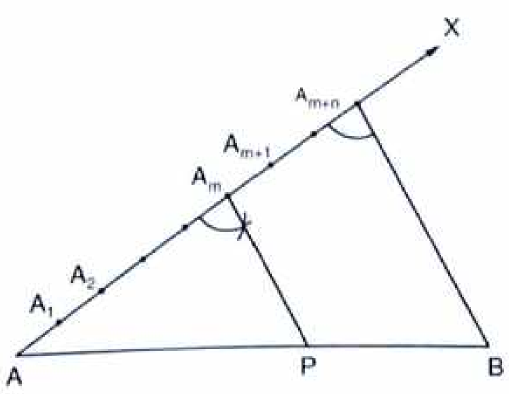


Fig. 9.1

The point P so obtained is the required point which divides AB internally in the ratio $m : n$.

Justification: In $\Delta\, AB\, A_{m+n}$, we observe that $A_m P$ is parallel to $A_{m+n}\, B$. Therefore, by basic proportionality theorem, we have

$$\frac{AA_m}{A_m\, A_{m+n}} = \frac{AP}{PB}$$

$$\Rightarrow \quad \frac{AP}{PB} = \frac{m}{n} \qquad \left[\because \frac{AA_m}{A_m\, A_{m+n}} = \frac{m}{n} \text{ (By construction)}\right]$$

$$\Rightarrow \quad AP : PB = m : n$$

Hence, P divides AB in the ratio $m : n$.

ILLUSTRATION 1 *Divide a line segment of length 10 cm internally in the ratio 3 : 2.*

SOLUTION We follow the following steps of construction.

Steps of construction

STEP I Draw a line segment $AB = 10$ cm by using a ruler.

STEP II Draw any ray making an acute angle $\angle BAX$ with AB.

STEP III Along AX, mark-off $5\,(= 3 + 2)$ points A_1, A_2, A_3, A_4 and A_5 such that $AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$.

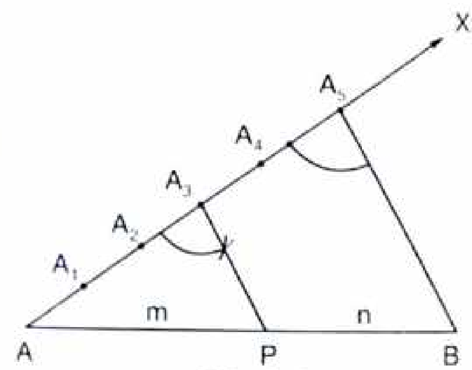


Fig. 9.2

STEP IV Join BA_5

STEP V Through A_3 draw a line $A_3 P$ parallel to $A_5 B$ by making an angle equal to $\angle AA_5 B$ at A_3 intersecting AB at a point P.

The point P so obtained is the required point, which divides AB internally in the ratio 3 : 2.

ALTERNATIVE METHOD FOR DIVISION OF A LINE SEGMENT INTERNALLY IN A GIVEN RATIO

We may use the following steps to divide a given line segment AB internally in a given ratio $m : n$, where m and n are natural numbers.

Steps of construction

STEP I Draw line segment AB of given length.

STEP II Draw any ray AX making an acute angle $\angle BAX$ with AB.

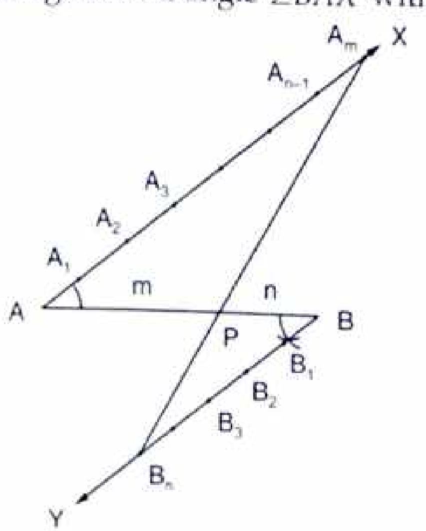


Fig. 9.3

STEP III Draw a ray BY, on opposite side of AX, parallel to AX by making an angle $\angle ABY$ equal to $\angle BAX$.

STEP IV Mark off m points $A_1, A_2, \ldots, A_m$, on AX and n points $B_1, B_2, \ldots, B_n$ on BY such that $AA_1 = A_1 A_2 = \ldots = A_{m-1} A_m$

$$= BB_1 = B_1 B_2 = \ldots = B_{n-1} B_n.$$

STEP V Join $A_m B_n$. Suppose it intersects AB at P.

The point P is the required point dividing AB in the ratio $m : n$.

Justification: In triangles $AA_m P$ and $B B_n P$, we have

$\angle A_m AP = \angle PBB_n$ $[\because \angle XAB = \angle ABY]$

and, $\angle APA_m = \angle BPB_n$ [Vertically opposite angles]

So, by AA similarity criterion, we obtain

$$\Delta A A_m P \sim \Delta B B_n P$$

$$\Rightarrow \quad \frac{AA_m}{BB_n} = \frac{AP}{BP} \Rightarrow \frac{AP}{BP} = \frac{m}{n} \qquad \left[\because \frac{AA_m}{BB_n} = \frac{m}{n}\right]$$

ILLUSTRATION 2 *Divide a line segment of length 8 cm internally in the ratio 3 : 4.*

SOLUTION We follow the following steps:

Steps of construction

STEP I Draw the line segment AB of length 8 cm.

STEP II Draw any ray AX making an acute angle $\angle BAX$ with AB.

STEP III Draw a ray BY parallel to AX by making $\angle ABY$ equal to $\angle BAX$.

STEP IV Mark of three point A_1, A_2, A_3 on AX and 4 points B_1, B_2, B_3, B_4 on BY such that

$$AA_1 = A_1A_2 = A_2A_3 = BB_1 = B_1B_2 = B_2B_3 = B_3B_4.$$

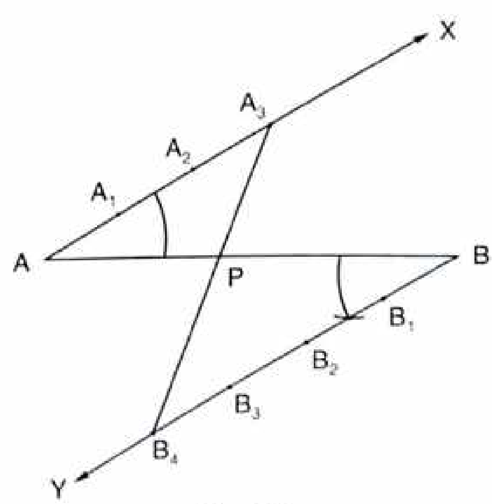


Fig. 9.4

STEP V Join $A_3 B_4$. Suppose it intersects AB at a point P.

Then, P is the point dividing AB internally in the ratio 3 : 4.

EXERCISE 9.1

LEVEL-1

1. Determine a point which divides a line segment of length 12 cm internally in the ratio 2 : 3. Also, justify your construction.
2. Divide a line segment of length 9 cm internally in the ratio 4 : 3. Also, give justification of the construction.
3. Divide a line segment of length 14 cm internally in the ratio 2 : 5. Also, justify your construction.
4. Draw a line segment of length 8 cm and divide it internally in the ratio 4 : 5.

[CBSE 2017]

9.3 CONSTRUCTION OF A TRIANGLE SIMILAR TO A GIVEN TRIANGLE

In this section, we will learn about the construction of a triangle similar to the given triangle. The triangle to be constructed may be smaller or larger than the given triangle. So, we define the following term.

SCALE FACTOR *The ratio of the sides of the triangle to be constructed with the corresponding sides of the given triangle is known as their scale factor.*

Suppose we are given a triangle ABC and we have to construct a triangle PQR similar to ΔABC whose sides are $\frac{3}{4}$ of the corresponding sides of ΔABC. Then, we say that the scale factor is $\frac{3}{4}$. If the sides of ΔPQR are $\frac{5}{4}$ of the corresponding sides of ΔABC, then we say that the scale factor is $\frac{5}{4}$.

Let ABC be the given triangle and we want to construct a triangle similar to ΔABC such that each of its sides is $\left(\frac{m}{n}\right)^{th}$ of the corresponding sides of ΔABC. We follow the following steps to construct the same.

Steps of construction when $m < n$:

STEP I Construct the given triangle ABC by using the given data.

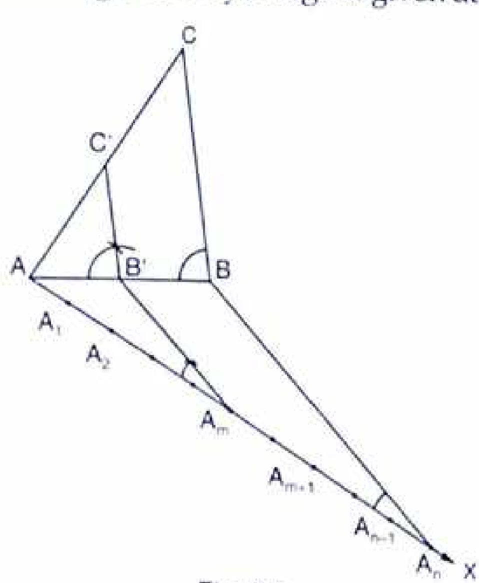


Fig. 9.5

STEP II Take any one of the three sides of the given triangle as base. Let AB be the base of the given triangle.

STEP III At one end, say A, of base AB construct an acute angle $\angle BAX$ below the base AB.

STEP IV Along AX mark off n points $A_1, A_2, A_3, \ldots, A_n$ such that
$AA_1 = A_1 A_2 = \ldots = A_{n-1} A_n$.

STEP V Join $A_n B$.

STEP VI Start from A and reach to point A_m on AX. Draw $A_m B'$ parallel to $A_n B$ which meets AB at B'.

STEP VII From B' draw $B'C' \parallel CB$ meeting AC at C'.

Triangle $AB'C'$ is the required triangle, each of whose sides is $\left(\frac{m}{n}\right)^{th}$ of the corresponding sides of ΔABC.

Justification: We shall now see how this construction gives the triangle similar to the given triangle.

Since $A_m B' \parallel A_n B$. Therefore,

$$\frac{AB'}{B'B} = \frac{AA_m}{A_m A_n} \qquad \text{[By basic proportionality theorem]}$$

$$\Rightarrow \quad \frac{AB'}{B'B} = \frac{m}{n-m}$$

$$\Rightarrow \quad \frac{B'B}{AB'} = \frac{n-m}{m}$$

Now, $$\frac{AB}{AB'} = \frac{AB' + B'B}{AB'}$$

$$\Rightarrow \quad \frac{AB}{AB'} = 1 + \frac{B'B}{AB'} = 1 + \frac{n-m}{m} = \frac{n}{m}$$

$$\Rightarrow \quad \frac{AB'}{AB} = \frac{m}{n}$$

In triangles ABC and $AB'C'$, we have

$$\angle BAC = \angle B'AC'$$

and, $$\angle ABC = \angle AB'C' \qquad [\because B'C' \parallel BC]$$

So, by AA similarity criterion, we obtain

$$\Delta AB'C' \sim \Delta ABC$$

$$\Rightarrow \quad \frac{AB'}{AB} = \frac{B'C'}{BC} = \frac{AC'}{AC}$$

$$\Rightarrow \quad \frac{AB'}{AB} = \frac{B'C'}{BC} = \frac{AC'}{AC} = \frac{m}{n} \qquad \left[\because \frac{AB'}{AB} = \frac{m}{n}\right]$$

LEVEL-1

ILLUSTRATION 1 *Construct a ΔABC in which $AB = 4$ cm, $BC = 5$ cm and $AC = 6$ cm. Now, construct a triangle similar to ΔABC such that each of its sides is two-third of the corresponding sides of ΔABC. Also, prove your assertion.*

SOLUTION *Steps of construction*

STEP I Draw a line segment $AB = 4$ cm.

STEP II With A as centre and radius $= AC = 6$ cm, draw an arc.

STEP III With B as centre and radius $= BC = 5$ cm, draw another arc, intersecting the arc drawn in step II at C.

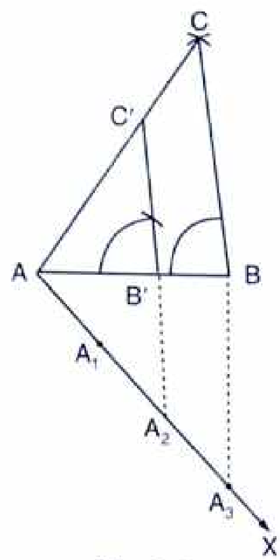


Fig. 9.6

STEP IV Join AC and BC to obtain ΔABC.

STEP V Below AB, make an acute angle $\angle BAX$.

STEP VI Along AX, mark off three points (greater of 2 and 3 in $\frac{2}{3}$) A_1, A_2, A_3 such that $AA_1 = A_1A_2 = A_2A_3$.

STEP VII Join A_3B.

STEP VIII Since we have to construct a triangle each of whose sides is two-third of the corresponding sides of ΔABC. So, take two parts out of three equal parts on AX i.e. from point A_2, draw $A_2B' \parallel A_3B$, meeting AB at B'.

STEP IX From B', draw $B'C' \parallel BC$, meeting AC at C'. $AB'C'$ is the required triangle, each of the whose sides is two-third of the corresponding sides of ΔABC.

Justification: Since $B'C' \parallel BC$. So, $\Delta ABC \sim \Delta AB'C'$

$$\therefore \quad \frac{B'C'}{BC} = \frac{AC'}{AC} = \frac{AB'}{AB} = \frac{2}{3} \qquad \left[\because \frac{AB'}{AB} = \frac{2}{3}\right]$$

Let ABC be the given triangle and we want to construct a triangle similar to ΔABC such that each of its sides is $\left(\frac{m}{n}\right)^{th}$ of the corresponding sides of ΔABC such that $m < n$. We follow the following steps to construct the same.

Steps of construction when $m > n$:

STEP I Construct the given triangle by using the given data.

STEP II Take any of the three sides of the given triangle and consider it as the base. Let AB be the base of the given triangle.

STEP III At one end, say A, of base AB construct an acute angle $\angle BAX$ below base AB i.e. on the opposite side of the vertex C.

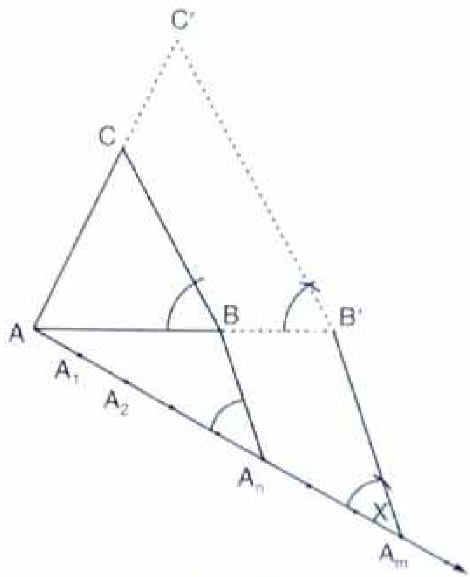


Fig. 9.7

STEP IV Along AX, mark-off m (large of m and n) points $A_1, A_2, \dots, A_m$ on AX such that $AA_1 = A_1A_2 = \dots = A_{m-1}A_m$.

STEP V Join A_n to B and draw a line through A_m parallel to $A_n B$, intersecting the extended line segment AB at B'.

STEP VI Draw a line through B' parallel to BC intersecting the extended line segment AC at C'.

STEP VII $\Delta AB'C'$ so obtained is the required triangle.

Justification: For justification of the above construction consider triangles ABC and $AB'C'$. In these two triangles, we have

$$\angle BAC = \angle B'AC'$$

$$\angle ABC = \angle AB'C' \qquad [\because B'C' \parallel BC]$$

So, by AA similarity criterion, we have

$$\Delta ABC \sim \Delta AB'C'$$

$$\Rightarrow \quad \frac{AB}{AB'} = \frac{BC}{B'C'} = \frac{AC}{AC'} \qquad \text{...(i)}$$

In $\Delta A A_m B'$, $A_n B \parallel A_m B'$.

$$\therefore \quad \frac{AB}{BB'} = \frac{AA_n}{A_n A_m}$$

$$\Rightarrow \quad \frac{BB'}{AB} = \frac{A_n A_m}{AA_n}$$

$$\Rightarrow \quad \frac{BB'}{AB} = \frac{m-n}{n}$$

EXERCISE 9.2

LEVEL-1

1. Construct a triangle of sides 4 cm, 5 cm and 6 cm and then a triangle similar to it whose sides are (2/3) of the corresponding sides of it. [CBSE 2013, 2017]
2. Construct a triangle similar to a given ΔABC such that each of its sides is $(5/7)^{th}$ of the corresponding sides of ΔABC. It is given that $AB = 5$ cm, $BC = 7$ cm and $\angle ABC = 50°$.
3. Construct a triangle similar to a given ΔABC such that each of its sides is $(2/3)^{rd}$ of the corresponding sides of ΔABC. It is given that $BC = 6$ cm, $\angle B = 50°$ and $\angle C = 60°$.
4. Draw a ΔABC in which $BC = 6$ cm, $AB = 4$ cm and $AC = 5$ cm. Draw a triangle similar to ΔABC with its sides equal to $(3/4)^{th}$ of the corresponding sides of ΔABC.
5. Construct a triangle with sides 5 cm, 6 cm and 7 cm and then another triangle whose sides are 7/5 of the corresponding sides of the first triangle.
6. Draw a right triangle ABC in which $AC = AB = 4.5$ cm and $\angle A = 90°$. Draw a triangle similar to ΔABC with its sides equal to (5/4)th of the corresponding sides of ΔABC.
7. Draw a right triangle in which the sides (other than hypotenuse) are of lengths 5 cm and 4 cm. Then construct another triangle whose sides are 5/3 times the corresponding sides of the given triangle. [CBSE 2008]
8. Construct an isosceles triangle whose base is 8 cm and altitude 4 cm and then another triangle whose sides are 3/2 times the corresponding sides of the isosceles triangle. [CBSE 2014, 2015, 2017]
9. Draw a ΔABC with side $BC = 6$ cm, $AB = 5$ cm and $\angle ABC = 60°$. Then, construct a triangle whose sides are $(3/4)^{th}$ of the corresponding sides of the ΔABC. [CBSE 2009, 2013, 2016]
10. Construct a triangle similar to ΔABC in which $AB = 4.6$ cm, BC = 5.1 cm, $\angle A = 60°$ with scale factor 4 : 5.
11. Construct a triangle similar to a given ΔXYZ with its sides equal to $(3/4)^{th}$ of the corresponding sides of ΔXYZ. Write the steps of construction.
12. Draw a right triangle in which sides (other than the hypotenuse) are of lengths 8 cm and 6 cm. Then construct another triangle whose sides are 3/4 times the corresponding sides of the first triangle. [CBSE 2009, 2012, 2016]
13. Construct a triangle with sides 5 cm, 5.5 cm and 6.5 cm. Now, construct another triangle whose sides are 3/5 times the corresponding sides of the given triangle. [CBSE 2014, 2016]
14. Construct a triangle PQR with side $QR = 7$ cm, $PQ = 6$ cm and $\angle PQR = 60°$. Then construct another triangle whose sides are 3/5 of the corresponding sides of ΔPQR. [CBSE 2014, 2015]
15. Draw a ΔABC in which base $BC = 6$ cm, $AB = 5$ cm and $\angle ABC = 60°$. Then construct another triangle whose sides are $\frac{3}{4}$ of the corresponding sides of ΔABC. [CBSE 2017, 2018]
16. Draw a right triangle in which the sides (other than the hypotenuse) are of lengths 4 cm and 3 cm. Now, construct another triangle whose sides are $\frac{3}{5}$ times the corresponding sides of the given triangle. [CBSE 2017]

LEVEL-2

17. Construct a ΔABC in which $AB = 5$ cm. $\angle B = 60°$ altitude $CD = 3$ cm. Construct a ΔAQR similar to ΔABC such that side of ΔAQR is 1.5 times that of the corresponding sides of ΔACB.

9.4 CONSTRUCTION OF TANGENTS TO A CIRCLE

In the previous chapter, we have studied that if a point lies inside a circle, there cannot be a tangent to the circle through this point. However, if a point lies on the circle, then there is only one tangent to the circle at this point and it is perpendicular to the radius through this point. We have also learnt that if the point lies outside the circle there will be two tangents to the circle from this point.

In this section, we shall learn the construction of tangent(s) to a circle when the centre is known and also when its centre is not known.

9.4.1 CONSTRUCTION OF A TANGENT TO A CIRCLE AT A GIVEN POINT

Type I **CONSTRUCTION OF A TANGENT TO A CIRCLE WHEN ITS CENTRE IS KNOWN**

Steps of construction

STEP I Take a point O on the plane of the paper and draw a circle of given radius.

STEP II Take a point P on the circle.

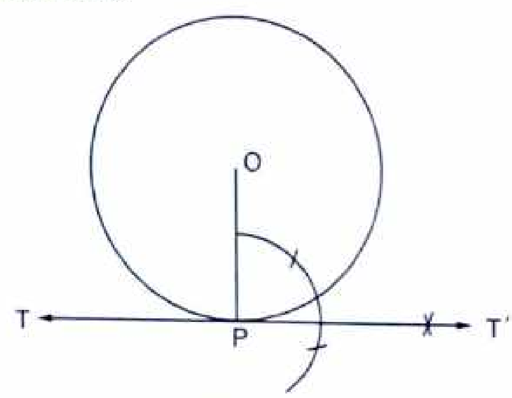


Fig. 9.9

STEP III Join OP.

STEP IV Construct $\angle OPT = 90°$.

STEP V Produce TP to T' to get TPT' as the required tangent.

ILLUSTRATION 1 *Take a point O on the plane of the paper. With O as centre, draw a circle of radius 3 cm. Take a point P on this circle and draw a tangent at P.*

SOLUTION We follow the following steps:

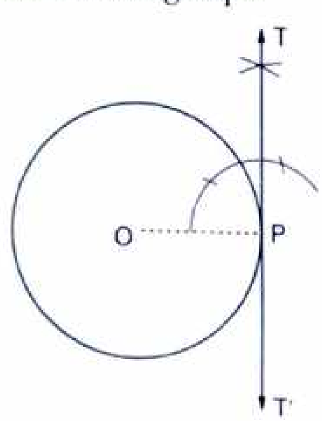


Fig. 9.10

Steps of construction

STEP I Take a point O on the plane of the paper and draw a circle of given radius 3 cm.

STEP II Take a point P on the circle and join OP.

STEP III Construct $\angle OPT = 90°$.

STEP IV Produce TP to T' to obtain the required tangent TPT'.

ILLUSTRATION 2 *Draw a circle of radius 4 cm with centre O. Draw a diameter POQ. Through P or Q draw tangent to the circle.*

SOLUTION We follow the following steps:

Steps of construction

STEP I Taking O as centre and radius equal to 4 cm draw a circle.

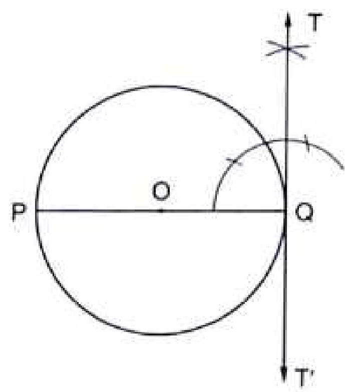


Fig. 9.11

STEP II Draw diameter of POQ.

STEP III Construct $\angle PQT = 90°$.

STEP IV Produce TQ to T' to obtain the required tangent TQT'.

Type II **CONSTRUCTION OF A TANGENT TO A CIRCLE AT A GIVEN POINT WHEN ITS CENTRE IS NOT KNOWN**

Steps of construction

STEP I Draw any chord PQ through the given point P on the circle.

STEP II Join P and Q to a point R either in the major arc or in the minor arc.

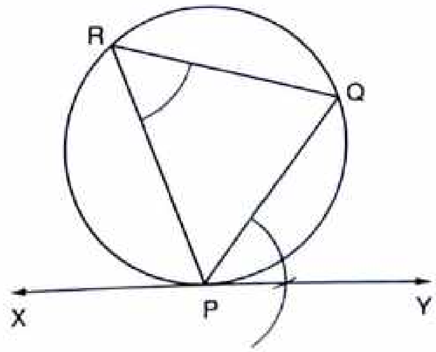


Fig. 9.12

STEP III Construct $\angle QPY$ equal to $\angle PRQ$ and on the opposite side of the chord PQ.

STEP IV Produce YP to X to get YPX as the required tangent.

ILLUSTRATION 3 *Draw a circle of radius 4 cm. Take a point P on it. Without using the cent circle, draw a tangent to the circle at point P.*

SOLUTION We follow the steps of constructions:

Steps of construction

STEP I Draw any chord PQ through the given point P on the circle.

STEP II Take a point R on the circle and join P and Q to a point R.

STEP III Construct $\angle QPY = \angle PRQ$ and on the opposite side of the chord PQ.

STEP IV Produce YP to X to get YPX as the required tangent.

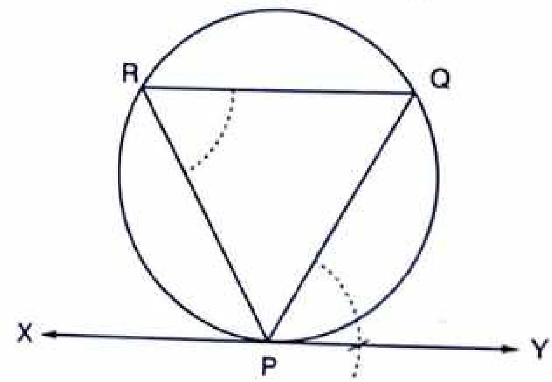


Fig. 9.13

9.4.2 CONSTRUCTION OF TANGENTS TO A CIRCLE FROM AN EXTERNAL POINT

In this section, we shall study the construction of tangents to a circle from an external when its centre is (i) known (ii) unknown.

Type I **CONSTRUCTION OF TANGENTS TO A CIRCLE FROM AN EXTERNAL POINT WHEN CENTRE IS KNOWN**

Steps of construction

STEP I Join the centre O of the circle to the given external point P i.e. Join OP.

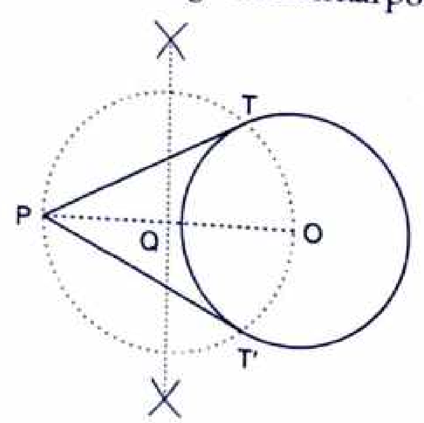


Fig. 9.14

STEP II Draw right bisector of OP, intersecting OP at Q.

STEP III Taking Q as centre and $OQ = PQ$ as radius, draw a circle to intersect the g circle at T and T'.

STEP IV Join PT and PT' to get the required tangents as PT and PT'.

LEVEL-1

ILLUSTRATION 1 *Draw a circle of radius 3 cm. Take a point at a distance of 5.5 cm from the centre of the circle. From point P, draw two tangents to the circle.*

SOLUTION In order to construct the required tangents, we follow the following steps:

Steps of construction

STEP I Take a point O in the plane of the paper and draw a circle of radius 3 cm.

STEP II Mark a point P at a distance of 5.5 cm from the centre O and join OP.

STEP III Draw the right bisector of OP, intersecting OP at Q.

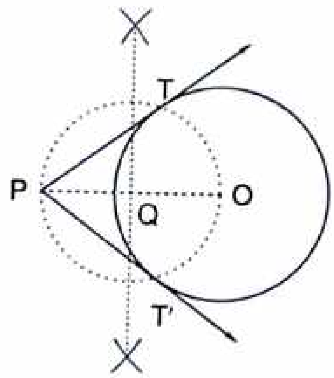


Fig. 9.15

STEP IV Taking Q as centre and $OQ = PQ$ as radius, draw a circle to intersect the given circle at T and T'

STEP V Join PT and PT' to get the required tangents.

ILLUSTRATION 2 *Construct a tangent to a circle of radius 4 cm from a point on the concentric circle of radius 6 cm and measure its length. Also verify the measurement by actual calculation.*

SOLUTION In order to do the desired construction, we follow the following steps:

Steps of construction

STEP I Take a point O on the plane of the paper and draw a circle of radius $OA = 4$ cm. Also, draw a concentric circle of radius $OB = 6$ cm.

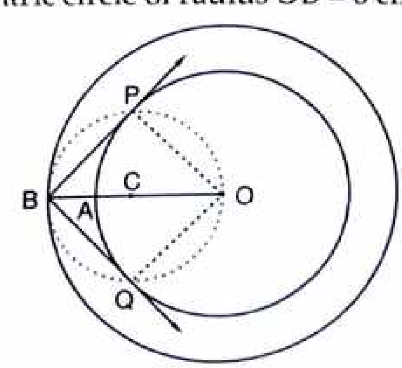


Fig. 9.16

STEP II Find the mid-point C of OB and draw a circle of radius $OC = BC$. Suppose this circle intersects the circle of radius 4 cm at P and Q.

STEP III Join BP and BQ to get the desired tangents from a point B on the circle of radius 6 cm. By actual measurement, we find the $BP = BQ = 4.5$ cm

Justification: In ΔBPO, we have

$$OB = 6 \text{ cm and } OP = 4 \text{cm}$$

Applying Pythagoras theorem in BPO, we obtain

$$OB^2 = BP^2 + OP^2$$

$$\Rightarrow \quad BP = \sqrt{OB^2 - OP^2} = \sqrt{36 - 16} = \sqrt{20} \text{ cm} = 4.47 \text{ cm} \simeq 5 \text{cm}$$

Similarly, $BQ = 4.47 \text{ cm} \simeq 4.5 \text{ cm}$.

LEVEL-2

Type II ON CONSTRUCTIONS OF TANGENTS TO A CIRCLE FROM AN EXTERNAL POINT WHEN ITS CENTRE IS KNOWN

ILLUSTRATION 3 *Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60°.* **[CBSE 2013]**

SOLUTION In order to draw the pair of tangents, we follow the following steps.

Steps of construction

STEP I Take a point O on the plane of the paper and draw a circle of radius $OA = 5$ cm.

STEP II Produce OA to B such that $OA = AB = 5$ cm.

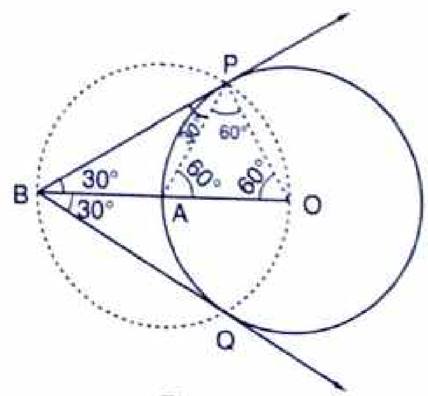


Fig. 9.17

STEP III Taking A as the centre draw a circle of radius $AO = AB = 5$ cm. Suppose it cuts the circle drawn in step I at P and Q.

STEP IV Join BP and BQ to get the desired tangents.

Justification: In OAP, we have

$$OA = OP = 5 \text{cm} \ (= \text{Radius})$$

Also, $\quad AP = 5\text{cm} \ (= \text{Radius of circle with centre } A)$

$\therefore \quad \Delta OAP$ is equilateral. $\Rightarrow \angle PAO = 60° \Rightarrow \angle BAP = 120°$

In ΔBAP, we have

$$BA = AP \text{ and } \angle BAP = 120°$$

$$\therefore \quad \angle ABP = \angle APB = 30°$$

$$\Rightarrow \quad \angle PBQ = 60°$$

ALITER

Steps of construction

STEP I Take a point O on the plane of the paper and draw a circle with centre O and radius $OA = 5$ cm.

STEP II At O construct radii OA and OB such that to $\angle AOB$ equal $120°$ i.e. supple-ment of the angle between the tangents.

STEP III Draw perpendiculars to OA and OB at A and B respectively suppose these perpendiculars intersect at P. Then, PA and PB are required tangents.

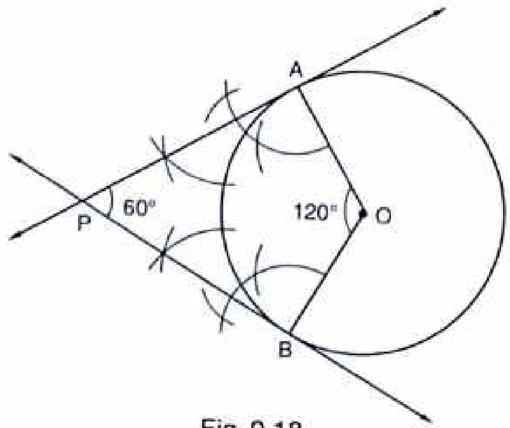


Fig. 9.18

Justification: In quadrilateral $OAPB$, we have

$$\angle OAP = \angle OBP = 90° \text{ and } \angle AOB = 120°$$

$\therefore$ $$\angle OAP + \angle OBP + \angle AOB + \angle APB = 360°$$

$\Rightarrow$ $$90° + 90° + 120° + \angle APB = 360°$$

$\Rightarrow$ $$\angle APB = 360° - (90° + 90° + 120°) = 60°$$

ILLUSTRATION 4 *Let ABC be a right triangle in which $AB = 3$ cm, $BC = 4$ cm and $\angle B = 90°$. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.*

SOLUTION We follow the following steps.

Steps of construction

STEP I Draw ΔABC and perpendicular BD from B on AC.

STEP II Draw a circle with BC as a diameter. This circle will pass through D.

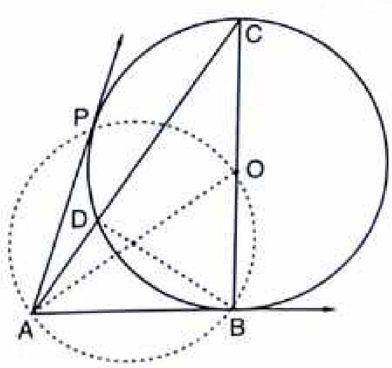


Fig. 9.19

STEP III Let O be the mid-point of BC. Join AO.

STEP IV Draw a circle with AO as diameter. This circle cuts the circle drawn in step II at B and P.

STEP V Join AP. AP and AB are desired tangents drawn from A to the circle passing through B, C and D.

Type III **CONSTRUCTION OF TANGENTS TO A CIRCLE FROM AN EXETERNAL POINT ITS CENTRE IS NOT KNOWN**

Steps of construction

STEP I Let P be the external point from where the tangents are to be drawn to the given Through P draw a secant PAB to intersect the circle at A and B (say).

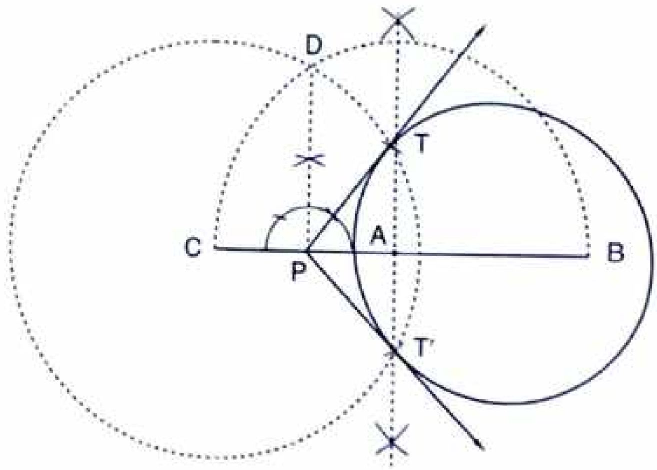


Fig. 9.20

STEP II Produce AP to a point C such that $AP = PC$ i.e, P is the mid-point of AC.

STEP III Draw a semi-circle with BC as diameter.

STEP IV Draw $PD \perp CB$, intersecting the semi-circle at D.

STEP V With P as centre and PD as radius draw arcs to intersect the given circle at T an

STEP VI Join PT and PT'. Then PT and PT' are the required tangents.

ILLUSTRATION 5 *Draw a circle of radius 4 cm. Take a point P outside the circle. Without using centre of the circle, draw two tangents to the circle from point P.*

SOLUTION *Steps of construction*

STEP I Draw a line segment 4 cm.

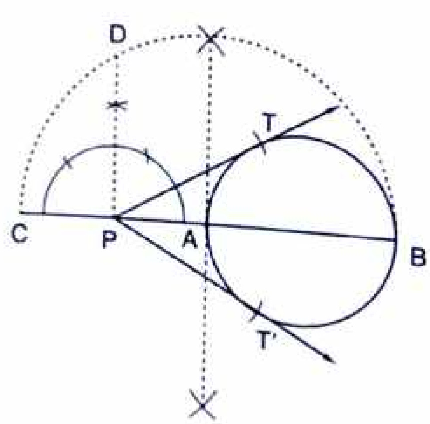


Fig. 9.21

STEP II Take a point P outside the circle and draw a secant PAB, intersecting the circle at A and B.

STEP III Produce AP to C such that $AP = CP$.

STEP IV Draw a semi-circle with CB as diameter.

STEP V Draw $PD \perp CB$, intersecting the semi-circle at D.

STEP VI With P as centre and PD as radius draw arcs to intersect the given circle at T and T'.

STEP VII Join PT and PT'. Then, PT and PT' are the required tangents.

ILLUSTRATION 6 *Draw a circle of radius 6 cm. Draw a tangent to this circle making an angle of 30° with a line passing through the centre.*

SOLUTION *Steps of construction*

STEP I Draw a circle with centre O and radius 3 cm.

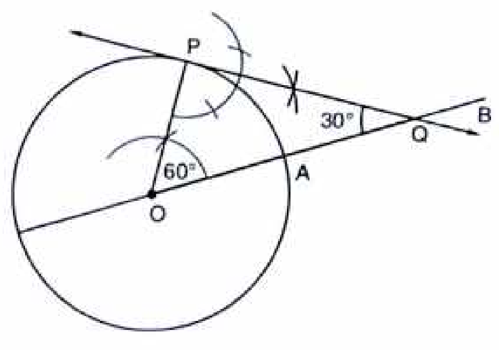


Fig. 9.22

STEP II Draw a radius OA of this circle and produce it to B.

STEP III Construct an angle $\angle AOP$ equal to the complement of 30° i.e. equal to 60°.

STEP IV Draw perpendicular to OP at P which intersects OA produced at Q

Clearly, PQ is the desired tangent such that $\angle OQP = 30°$

EXERCISE 9.3

LEVEL-1

1. Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.
2. Draw a circle of radius 3 cm. Take two points P and Q on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points P and Q.
3. Draw a line segment AB of length 8 cm. Taking A as centre, draw a circle of radius 4 cm and taking B as centre, draw another circle of radius 3 cm. Construct tangents to each circle from the centre of the other circle. **[CBSE 2015]**
4. Draw two tangents to a circle of radius 3.5 cm from a point P at a distance of 6.2 cm from its centre. **[CBSE 2013]**
5. Draw a pair of tangents to a circle of radius 4.5 cm, which are inclined to each other at an angle of 45°. **[CBSE 2013]**

LEVEL-2

6. Draw a right triangle ABC in which $AB = 6$ cm, $BC = 8$ cm and $\angle B = 90°$. Draw BD perpendicular from B on AC and draw a circle passing through the points B, C and D. Construct tangents from A to this circle. [CBSE 2014]
7. Draw two concentric circles of radii 3 cm and 5 cm. Construct a tangent to the smaller circle from a point on the larger circle. Also, measure its length. [CBSE 2016]

ANSWERS

1. 8 cm. 7. 4 cm

SUMMARY

In this chapter, we have done the following constructions:

1. To divide a line segment in a given ratio.
2. To constuct a triangle similar to a given triangle as per given scale factor which may be less than or may be greater than 1.
3. To construct tangent at a point on a given circle.
4. To construct the pair of tangents from an external point to a circle.

CHAPTER 10

TRIGONOMETRIC RATIOS

10.1 INTRODUCTION

In this chapter, we intend to study an important branch of mathematics called "Trigonometry". The word 'Trigonometry' is derived from the Greek words: (i) trigonon and, (ii) metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of mathematics which deals with the measurement of the sides and the angles of a triangle and the problems allied with angles.

10.2 ANGLE

Consider a ray OA. If this ray rotates about its end point O and takes the position OB, then we say that the angle $\angle AOB$ has been generated.

Thus, an angle is considered as the figure obtained by rotating a given ray about its end-point.

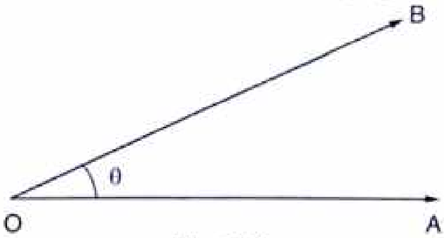


Fig. 10.1

The revolving ray is called the *generating line* of the angle. The initial position OA is called the *initial side* and the final position OB is called the *terminal side* of the angle. The end point O about which the ray rotates is called the vertex of the angle.

MEASURE OF AN ANGLE *The measure of an angle is the amount of rotation from the initial side to the terminal side.*

10.3 TRIGONOMETRIC RATIOS

The most important task of trigonometry is to find the remaining sides and angles of a triangle when some of its sides and angles are given. This problem is solved by using some ratios of the sides of a triangle with respect to its acute angles. These ratios of acute

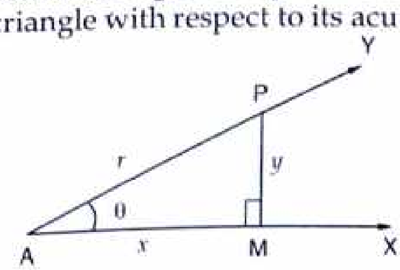


Fig. 10.2

angles are called *trigonometric ratios* of angles. Let us now define various tri ratios.

Consider an acute angle $\angle YAX = \theta$ with initial side AX and terminal side AY. I point on the terminal side AY. Draw PM perpendicular from P on AX to get the r triangle AMP in which $\angle PAM = \theta$.

In the right angled triangle AMP, Base $= AM = x$, Perpendicular $= PM$ Hypotenuse $= AP = r$.

We define the following six trigonometric ratios:

(i) $\text{Sin } \theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{y}{r}$, and is written as $\sin \theta$

(ii) $\text{Cosine } \theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{x}{r}$, and is written as $\cos \theta$

(iii) $\text{Tangent } \theta = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{y}{x}$, and is written as $\tan \theta$

(iv) $\text{Cosecant } \theta = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{r}{y}$, and is written as cossec θ

(v) $\text{Secant } \theta = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{r}{x}$, and is written as $\sec \theta$

(vi) $\text{Cotangent } \theta = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{x}{y}$, and is written as $\cot \theta$

NOTE *It should be noted that* $\sin \theta$ *is an abbreviation for "sine of angle* θ*", it is not the sin and* θ*. Similar is the case for other trigonometric ratios.*

REMARK 1 *The above trigonometric ratios are defined for an acute angle* θ*.*

REMARK 2 *The above trigonometric ratios depend only on the value of angle* θ *and are inde the position of the point P on the terminal side AY of the acute angle* $\angle XAY$*. This mean choose P somewhere else on AY, then the lengths PM, AM and AP will change but the trig ratios will remain same as proved in the following theorem.*

THEOREM *The trigonometric ratios are same for the same angle.*

PROOF Let $\angle XAY = \theta$ be an acute angle with initial side AX and terminal side and Q be two points, both different from A, on the terminal side AY. Draw prper PM and QN from P and Q respectively on AX. We have to prove that the trigo ratios of angle θ are same in both the triangles viz. ΔAMP and ΔANQ.

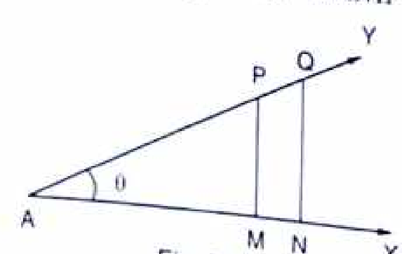


Fig. 10.3

In ΔAMP and ΔANQ, we have

$$\angle MAP = \angle XAY = \angle NAQ$$

and, $\angle AMP = \angle ANQ =$ One right angle.

Thus, the corresponding angles of triangles AMP and ANQ are equal and, therefore by AAA similarity criterion, we have

$$\frac{AP}{AQ} = \frac{PM}{QN} = \frac{AM}{AN}$$

$$\Rightarrow \quad \frac{PM}{AP} = \frac{QN}{AQ} \qquad \ldots(i)$$

In ΔAMP, we have

$$\sin\theta = \frac{PM}{AP}$$

$$\Rightarrow \quad \sin\theta = \frac{QN}{AQ} \qquad \text{[Using (i)]}$$

This shows that the value of $\sin\theta$ is independent of the position of point P.

Similarly, it can be proved that the other trigonometric ratios are independent of the position of point P.

Q.E.D.

10.4 RELATIONS BETWEEN TRIGONOMETRIC RATIOS

The trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ of an angle θ are very closely connected by a relation. If any one of them is known, the other two can be easily calculated.

From Fig. 10.2, we have

$$\sin\theta = \frac{PM}{AP}, \cos\theta = \frac{AM}{AP} \text{ and } \tan\theta = \frac{PM}{AM}$$

Now, $$\tan\theta = \frac{PM}{AM}$$

$$\Rightarrow \quad \tan\theta = \frac{\frac{PM}{AP}}{\frac{AM}{AP}}$$

$$\Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

REMARK *It is clear from the definitions of the trigonometric ratios that for any acute angle θ, we have*

(i) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ or, $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$

(ii) $\sec\theta = \dfrac{1}{\cos\theta}$ or, $\cos\theta = \dfrac{1}{\sec\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$ or, $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(v) $\tan\theta . \cot\theta = 1$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In a ΔABC, right angled at A, if $AB = 12$, $AC = 5$ and $BC = 13$, find all the six trigonometric ratios of angle B.*

SOLUTION With reference to $\angle B$, we have

$$\text{Base} = AB = 12, \text{ Perpendicular} = AC = 5 \text{ and, Hypotenuse} = BC = 13$$

Using the definitions of trigonometric ratios, we have

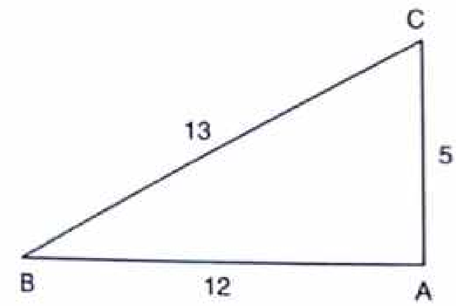


Fig. 10.4

$$\sin B = \frac{AC}{BC} = \frac{5}{13}$$

$$\cos B = \frac{AB}{BC} = \frac{12}{13}$$

$$\tan B = \frac{AC}{AB} = \frac{5}{12}$$

$$\operatorname{cosec} B = \frac{BC}{AC} = \frac{13}{5}$$

$$\sec B = \frac{BC}{AB} = \frac{13}{12}$$

and, $$\cot B = \frac{AB}{AC} = \frac{12}{5}$$

EXAMPLE 2 *In a ΔABC, right angled at A, if $AB = 5$, $AC = 12$ and $BC = 13$, find sin B, cos C and tan B.*

SOLUTION With reference to $\angle B$, we have

$$\text{Base} = AB = 5, \text{ Perpendicular} = AC = 12 \text{ and, Hypotenuse} = BC = 13$$

$\therefore$ $$\sin B = \frac{AC}{BC} = \frac{12}{13}$$

and, $$\tan B = \frac{AC}{AB} = \frac{12}{5}$$

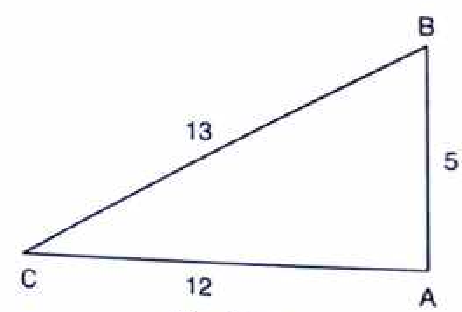


Fig. 10.5

With reference to $\angle C$, we have

Base $= AC = 12$, Perpendicular $= AB = 5$ and,

Hypotenuse $= BC = 13$

$\therefore$ $$\cos C = \frac{AC}{BC} = \frac{12}{13}$$

EXAMPLE 3 *In a ΔABC, right angled at B, if $AB = 4$ and $BC = 3$, find all the six trigonometric ratios of $\angle A$.*

SOLUTION We have, $AB = 4$ and $BC = 3$.

By Pythagoras theorem, we have

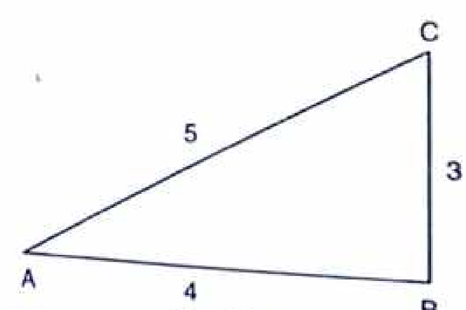


Fig. 10.6

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $$AC = \sqrt{AB^2 + BC^2}$$

$\Rightarrow$ $$AC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

When we consider the t-ratios of $\angle A$, we have

Base $= AB = 4$, Perpendicular $= BC = 3$ and, Hypotenuse $= AC = 5$.

$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{3}{5}, \ \cos A = \frac{AB}{AC} = \frac{4}{5}, \ \tan A = \frac{BC}{AB} = \frac{3}{4}$$

$$\operatorname{cosec} A = \frac{AC}{BC} = \frac{5}{3}, \ \sec A = \frac{AC}{AB} = \frac{5}{4} \text{ and, } \cot A = \frac{AB}{BC} = \frac{4}{3}$$

EXAMPLE 4 *In a ΔABC, right angled at B, if $AB = 12$ and $BC = 5$, find:*
(i) $\sin A$ *and* $\tan A$ (ii) $\cos C$ *and* $\cot C$

SOLUTION By Pythagoras theorem, we have

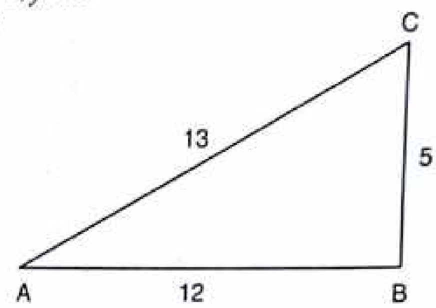


Fig. 10.7

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC^2 = 12^2 + 5^2$$

$$\Rightarrow \quad AC^2 = 169$$

$$\Rightarrow \quad AC = 13$$

(i) When we consider t-ratios of $\angle A$, we have

Base $= AB = 12$, Perpendicular $= BC = 5$ and, Hypotenuse $= AC = 13$

$$\therefore \quad \sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{5}{13} \text{ and, } \tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{5}{12}$$

(ii) When we consider t-ratios of $\angle C$, we have

Base $= BC = 5$, Perpendicular $= AB = 12$ and, Hypotenuse $= AC = 13$

$$\therefore \quad \cos C = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{5}{13} \text{ and, } \cot C = \frac{\text{Base}}{\text{Perpendicular}} = \frac{5}{12}$$

EXAMPLE 5 *If $\sin A = \frac{3}{5}$, find $\cos A$ and $\tan A$.*

SOLUTION We have,

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}$$

So, we draw a triangle ABC, right angled at B such that Perpendicular $= BC = 3$ units and, Hypotenuse $= AC = 5$ units.

By Pythagoras theorem, we have

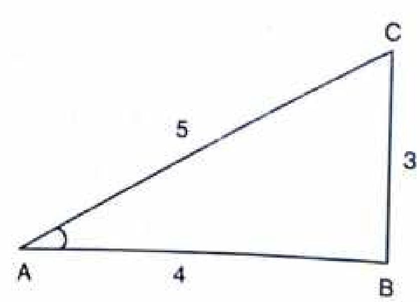


Fig. 10.8

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad 5^2 = AB^2 + 3^2$$

$$\Rightarrow \quad AB^2 = 5^2 - 3^2$$

$$\Rightarrow \quad AB^2 = 16$$

$$\Rightarrow \quad AB = 4$$

When we consider the t-ratios of $\angle A$, we have

$$\text{Base} = AB = 4, \text{Perpendicular} = BC = 3, \text{Hypotenuse} = AC = 5$$

$$\therefore \quad \cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{4}{5} \text{ and, } \tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{3}{4}$$

EXAMPLE 6 *If* $\cos B = \frac{1}{3}$, *find the other five trigonometric ratios.*

SOLUTION We have,

$$\cos B = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{1}{3}$$

So, we draw a triangle ABC, right angled at C such that

$$\text{Base} = BC = 1 \text{ unit and, Hypotenuse} = AB = 3 \text{ units.}$$

By Pythagoras theorem, we have

$$AB^2 = BC^2 + AC^2$$

$$\Rightarrow \quad 3^2 = 1^2 + AC^2$$

$$\Rightarrow \quad AC^2 = 9 - 1 = 8$$

$$\Rightarrow \quad AC = \sqrt{8} = 2\sqrt{2}$$

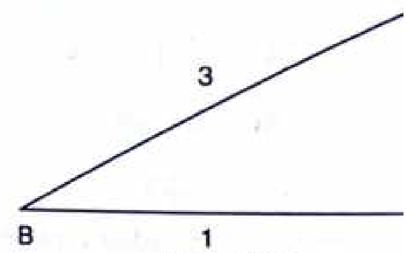


Fig. 10.9

When we consider the t-ratios of $\angle B$, we have

$$\text{Base} = BC = 1, \text{Perpendicular} = AC = 2\sqrt{2} \text{ and, Hypotenuse} = AB = 3$$

$$\therefore \quad \sin B = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{2\sqrt{2}}{3}, \quad \tan B = \frac{\text{Perpendicular}}{\text{Base}} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

$$\operatorname{cosec} B = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{3}{2\sqrt{2}}, \quad \sec B = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{3}{1} = 3$$

$$\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{1}{2\sqrt{2}}$$

EXAMPLE 7 *If* $\cos\theta = \frac{8}{17}$, *find the other five trigonometric ratios.*

SOLUTION We have,

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{8}{17}$$

So, we draw a triangle ABC right-angled at A such that

$$\text{Base} = AB = 8 \text{ units, Hypotenuse} = AC = 17 \text{ units and, } \angle BAC = \theta.$$

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad 17^2 = 8^2 + BC^2$$

$$\Rightarrow \quad BC^2 = 17^2 - 8^2$$

$$\Rightarrow \quad BC^2 = 289 - 64 = 225$$

$\Rightarrow \qquad BC = \sqrt{225} = 15$

Vhen we consider the t-ratios of $\angle BAC = \theta$, we have

Base $= AB = 8$, Perpendicular $= BC = 15$, and Hypotenuse $= AC = 17$

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{15}{17}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{15}{8}$$

$$\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{17}{15}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{17}{8}$$

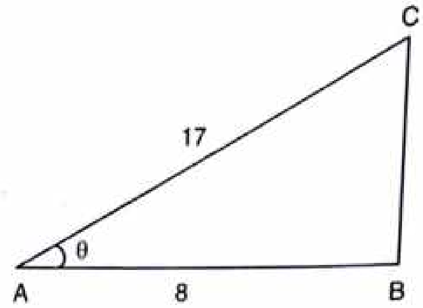


Fig. 10.10

nd, $\qquad \cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{8}{15}$

XAMPLE 8 *If* cosec $A = \sqrt{10}$, *find other five trigonometric ratios.*

OLUTION We have,

$$\text{cosec}\,A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{\sqrt{10}}{1}$$

o, we draw a right triangle ABC, right-angled at B such that

Perpendicular $= BC = 1$ unit and, Hypotenuse $= AC = \sqrt{10}$ units.

y Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \qquad (\sqrt{10})^2 = AB^2 + 1^2$

$\Rightarrow \qquad AB^2 = 10 - 1 = 9$

$\Rightarrow \qquad AB = \sqrt{9} = 3$

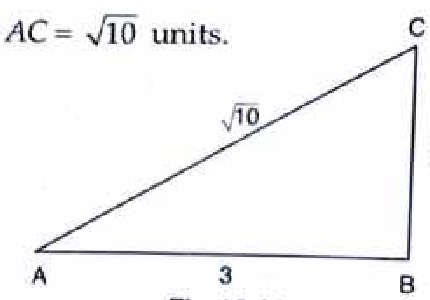


Fig. 10.11

hen we consider the trigonometric ratios of $\angle A$, we have

Base $= AB = 3$, Perpendicular $= BC = 1$ and, Hypotenuse $= AC = \sqrt{10}$

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{1}{\sqrt{10}}, \quad \cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{\sqrt{10}}$$

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{1}{3}, \quad \sec A = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{\sqrt{10}}{3}$$

d, $\qquad \cot A = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{3}{1} = 3$

AMPLE 9 *In a right triangle ABC right angled at B if* $\sin A = \frac{3}{5}$, *find all the six trigonometric tios of* $\angle C$.

SOLUTION We have,

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}$$

So, we draw a right triangle right angled at B such that Perpendicular $= BC = 3$ and, Hypotenuse $= AC = 5$

By Pythagoras theorem, we have

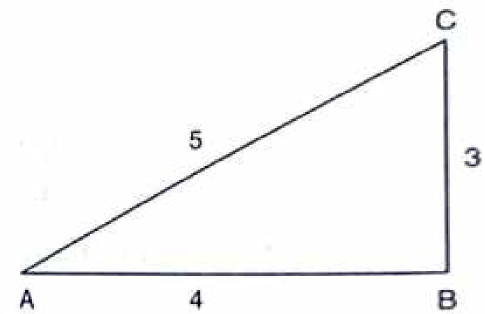


Fig. 10.12

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad 5^2 = AB^2 + 3^2$$

$$\Rightarrow \quad AB^2 = 25 - 9 = 16$$

$$\Rightarrow \quad AB = \sqrt{16} = 4$$

When we consider the trigonometric ratios of $\angle C$, we have

Base $= BC = 3$, Perpendicular $= AB = 4$ and, Hypotenuse $= AC = 4$.

$$\therefore \quad \sin C = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}, \quad \cos C = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{5}$$

$$\tan C = \frac{\text{Perpendicular}}{\text{Base}} = \frac{4}{3}, \quad \text{cosec } C = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{5}{4}$$

$$\sec C = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{3} \text{ and, } \cot C = \frac{\text{Base}}{\text{Perpendicular}} = \frac{3}{4}$$

EXAMPLE 10 *If* $\sin A = \frac{1}{3}$, *evaluate* $\cos A \text{ cosec } A + \tan A \sec A$.

SOLUTION We have,

$$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{1}{3}$$

So, we draw a right triangle, right angled at B such that Perpendicular $= BC = 1$ unit, Hypotenuse $= AC = 3$ units.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad 3^2 = AB^2 + 1^2$$

$$\Rightarrow \quad 9 - 1 = AB^2$$

$$\Rightarrow \quad AB = \sqrt{8} = 2\sqrt{2}$$

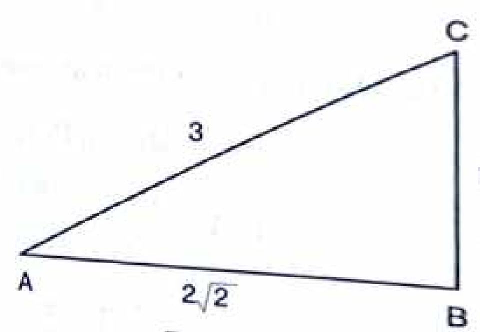


Fig. 10.13

Now, $$\cos A = \frac{AB}{AC} = \frac{2\sqrt{2}}{3}, \text{ cosec } A = \frac{AC}{BC} = \frac{3}{1} = 3,$$

$$\tan A = \frac{BC}{AB} = \frac{1}{2\sqrt{2}} \text{ and } \sec A = \frac{AC}{AB} = \frac{3}{2\sqrt{2}}$$

$$\therefore \quad \cos A \text{ cosec } A + \tan A \sec A = \frac{2\sqrt{2}}{3} \times 3 + \frac{1}{2\sqrt{2}} \times \frac{3}{2\sqrt{2}}$$

$$\Rightarrow \quad \cos A \text{ cosec } A + \tan A \sec A = 2\sqrt{2} + \frac{3}{8} = \frac{16\sqrt{2} + 3}{8}$$

EXAMPLE 11 *If* $\text{cosec } A = 2$, *find the value of* $\frac{1}{\tan A} + \frac{\sin A}{1+\cos A}$.

SOLUTION We have,

$$\text{cosec } A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{2}{1}$$

So, we draw a right triangle, right angled at B such that Perpendicular $= BC = 1$, Hypotenuse $= AC = 2$.

By Pythagoras theorem, we have

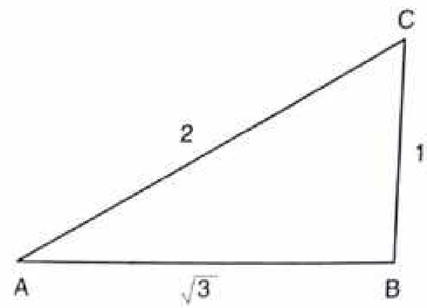


Fig. 10.14

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad 2^2 = 1^2 + AB^2$$

$$\Rightarrow \quad 4 - 1 = AB^2$$

$$\Rightarrow \quad AB = \sqrt{3}$$

Now,

$$\tan A = \frac{BC}{AB} = \frac{1}{\sqrt{3}}, \sin A = \frac{BC}{AC} = \frac{1}{2} \text{ and, } \cos A = \frac{AB}{AC} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \frac{1}{\tan A} + \frac{\sin A}{1+\cos A} = \frac{1}{\frac{1}{\sqrt{3}}} + \frac{1/2}{1+\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \quad \frac{1}{\tan A} + \frac{\sin A}{1+\cos A} = \frac{\sqrt{3}}{1} + \frac{1/2}{\frac{2+\sqrt{3}}{2}} = \frac{\sqrt{3}}{1} + \frac{1}{2+\sqrt{3}}$$

$$\Rightarrow \quad \frac{1}{\tan A} + \frac{\sin A}{1+\cos A} = \sqrt{3} + \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$\Rightarrow \quad \frac{1}{\tan A} + \frac{\sin A}{1+\cos A} = \sqrt{3} + \frac{2-\sqrt{3}}{2^2 - (\sqrt{3})^2} = \sqrt{3} + \frac{2-\sqrt{3}}{4-3} = \sqrt{3} + (2-\sqrt{3}) = 2$$

EXAMPLE 12 *If* $\tan A = \sqrt{2} - 1$ *show that* $\sin A \cos A = \frac{\sqrt{2}}{4}$

SOLUTION We have,

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\sqrt{2}-1}{1}$$

So, we draw a right triangle ABC, right angled at B such that

Base $= AB = 1$ and, Perpendicular $= BC = \sqrt{2} - 1$

By Pythagoras theorem, we have

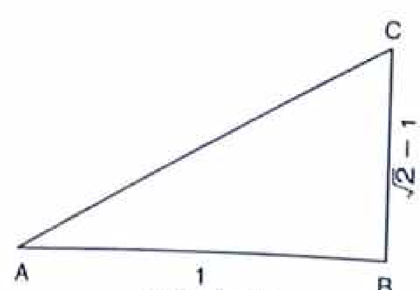


Fig. 10.15

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC^2 = 1^2 + (\sqrt{2}-1)^2$$

$$\Rightarrow \quad AC^2 = 1 + 2 + 1 - 2\sqrt{2}$$

$$\Rightarrow \quad AC^2 = 4 - 2\sqrt{2}$$

$$\Rightarrow \quad AC = \sqrt{4 - 2\sqrt{2}}$$

Now, $\sin A = \dfrac{BC}{AC} = \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}$ and, $\cos A = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

$$\therefore \quad \sin A \cos A = \frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \times \frac{1}{\sqrt{4-2\sqrt{2}}} = \frac{\sqrt{2}-1}{4-2\sqrt{2}} = \frac{\sqrt{2}-1}{2\sqrt{2}(\sqrt{2}-1)} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

EXAMPLE 13 *In a right triangle ABC, right angled at C, if* $\tan A = 1$, *then verify that* $2 \sin A \cos A = 1$. **[NCERT]**

SOLUTION In ΔABC, we have

$$\tan A = 1$$

$$\Rightarrow \quad \frac{BC}{AC} = 1$$

$$\Rightarrow \quad BC = x \text{ and } AC = x$$

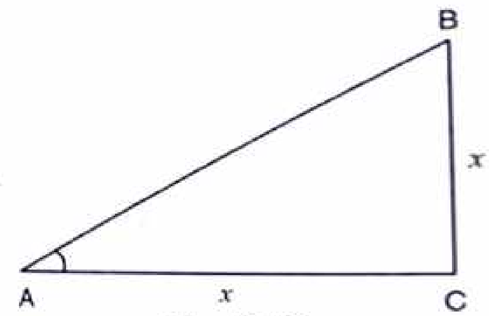


Fig. 10.16

By Pythagoras theorem, we have

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow \quad AB^2 = x^2 + x^2$$

$$\Rightarrow \quad AB = \sqrt{2}x$$

$$\therefore \quad \sin A = \frac{BC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \text{ and } \cos A = \frac{AC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1$$

EXAMPLE 14 *If* $\tan A = 1$ *and* $\tan B = \sqrt{3}$, *evaluate* $\cos A \cos B - \sin A \sin B$.

SOLUTION We have,

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{1}{1}$$

So, we draw a right angled triangle ABC, right angled at C such that Base $= AB = 1$ and, Perpendicular $= BC = 1$

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC^2 = 1^2 + 1^2$$

$$\Rightarrow \quad AC = \sqrt{2}$$

$$\therefore \quad \cos A = \frac{AB}{AC} = \frac{1}{\sqrt{2}} \text{ and, } \sin A = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

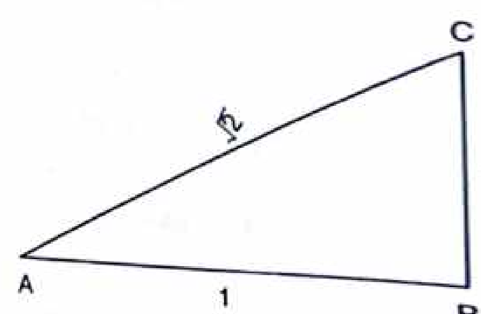


Fig. 10.17

We have,

$$\tan B = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\sqrt{3}}{1}$$

So, we draw a right triangle ABC, right angled at A such that

$$\text{Base} = AB = 1 \text{ and Perpendicular} = AC = \sqrt{3}$$

By Pythagoras theorem, we have

$$BC^2 = AB^2 + AC^2$$

$\Rightarrow \quad BC^2 = 1^2 + (\sqrt{3})^2 = 4$

$\Rightarrow \quad BC = 2$

$\therefore \quad \sin B = \dfrac{AC}{BC} = \dfrac{\sqrt{3}}{2}$, and $\cos B = \dfrac{AB}{BC} = \dfrac{1}{2}$

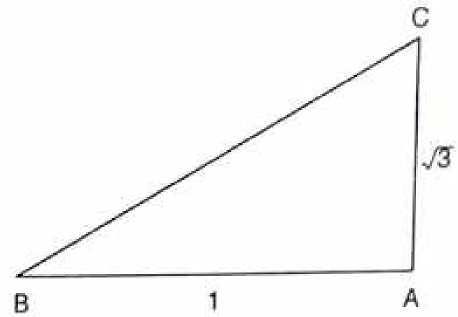


Fig. 10.18

Hence, $\quad \cos A \ \cos B - \sin A \ \sin B = \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

EXAMPLE 15 *In a ΔABC right angled at C, if $\tan A = \dfrac{1}{\sqrt{3}}$, find the value of* $\sin A \cos B + \cos A \sin B$. **[CBSE 2008]**

SOLUTION Let us draw a ΔABC, right angled at C in which $\tan A = \dfrac{1}{\sqrt{3}}$.

Now, $\quad \tan A = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \dfrac{BC}{AC} = \dfrac{1}{\sqrt{3}} \qquad \left[\because \tan A = \dfrac{BC}{AC}\right]$

$\Rightarrow \quad BC = x$ and $AC = \sqrt{3}x \qquad \ldots$(i)

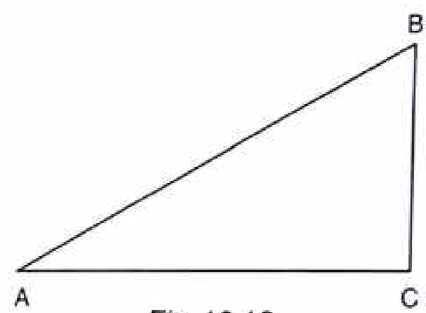


Fig. 10.19

By Pythagoras theorem, we have

$$AB^2 = AC^2 + BC^2$$

$\Rightarrow \quad AB^2 = \left(\sqrt{3}x\right)^2 + x^2$

$\Rightarrow \quad AB^2 = 3x^2 + x^2$

$\Rightarrow \quad AB^2 = 4x^2$

$\Rightarrow \quad AB = 2x$

To find the t-ratios of $\angle A$, we have

Base $= AC = \sqrt{3x}$, Perpendicular $= BC = x$ and Hypotenuse $= AB = 2x$

$\therefore \quad \sin A = \dfrac{BC}{AB} = \dfrac{x}{2x} = \dfrac{1}{2}$ and, $\cos A = \dfrac{AC}{AB} = \dfrac{\sqrt{3}x}{2x} = \dfrac{\sqrt{3}}{2}$

When we consider the t-ratios of $\angle B$, we have

Base $= BC = x$, Perpendicular $= AC = \sqrt{3x}$ and, Hypotenuse $= AB = 2x$

$\therefore \quad \cos B = \dfrac{BC}{AB} = \dfrac{x}{2x} = \dfrac{1}{2}$ and, $\sin B = \dfrac{AC}{AB} = \dfrac{\sqrt{3}x}{2x} = \dfrac{\sqrt{3}}{2}$

$\therefore \quad \sin A \ \cos B + \cos A \sin B = \dfrac{1}{2} \times \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{1}{4} + \dfrac{3}{4} = 1$

EXAMPLE 16 *If* $\sec\alpha = \frac{5}{4}$, *evaluate* $\frac{1-\tan\alpha}{1+\tan\alpha}$.

SOLUTION We have,

$$\sec\alpha = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{4}$$

So, we draw a right triangle ABC, right angled at B such that

Hypotenuse $= AC = 5$ units, Base $= AB = 4$ units, and $\angle BAC = \alpha$.

Applying Pythagoras theorem in ΔABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad 5^2 = 4^2 + BC^2$$

$$\Rightarrow \quad BC^2 = 5^2 - 4^2 = 9$$

$$\Rightarrow \quad BC = \sqrt{9} = 3$$

$$\therefore \quad \tan\alpha = \frac{BC}{AB} = \frac{3}{4}$$

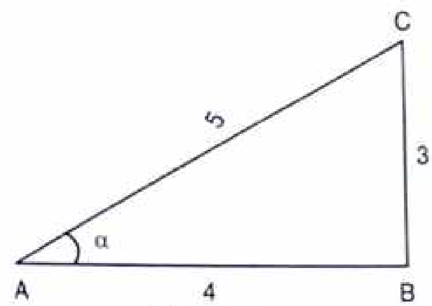


Fig. 10.20

Now, $$\frac{1-\tan\alpha}{1+\tan\alpha} = \frac{1-\frac{3}{4}}{1+\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}$$

NOTE *It should be noted that:* $\sin^2\theta = (\sin\theta)^2, \sin^3\theta = (\sin\theta)^3, \cos^3\theta = (\cos\theta)^3$ etc.

EXAMPLE 17 *If* $\sin B = \frac{1}{2}$, *show that* $3\cos B - 4\cos^3 B = 0$.

SOLUTION We have,

$$\sin B = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{1}{2}$$

So, we draw a right triangle ABC, right angled at C such that

Perpendicular $= AC = 1$ unit, Hypotenuse $= AB = 2$ units.

Applying Pythagoras theorem in BCA, we obtain

$$AB^2 = BC^2 + AC^2$$

$$\Rightarrow \quad 2^2 = BC^2 + 1^2$$

$$\Rightarrow \quad BC^2 = 3$$

$$\Rightarrow \quad BC = \sqrt{3}$$

$$\therefore \quad \cos B = \frac{BC}{AB} = \frac{\sqrt{3}}{2}$$

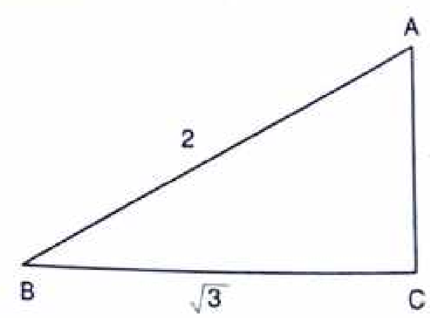


Fig. 10.21

Now,

$$3\cos B - 4\cos^3 B = 3\times\frac{\sqrt{3}}{2} - 4\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{2} - 4\times\frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$$

EXAMPLE 18 *If* $\tan A = 2$, *evaluate* $\sec A \sin A + \tan^2 A - \operatorname{cosec} A$.

SOLUTION We have,

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{2}{1}$$

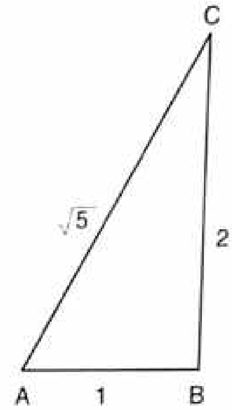


Fig. 10.22

So, we draw a right triangle ABC, right angled at C such that Perpendicular $= BC = 2$ unit and Base $= AB = 1$ unit.

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC^2 = 2^2 + 1^2 = 5$$

$$\Rightarrow \quad AC = \sqrt{5}$$

$$\therefore \quad \sec A = \frac{AC}{AB} = \sqrt{5},\ \tan A = \frac{BC}{AB} = \frac{2}{1} = \quad \sin A = \frac{BC}{AC} = \frac{2}{\sqrt{5}}$$

and, $$\operatorname{cosec} A = \frac{AC}{BC} = \frac{\sqrt{5}}{2}$$

Now, $$\sec A \sin A + \tan^2 A - \operatorname{cosec} A = \sqrt{5} \times \frac{2}{\sqrt{5}} + (2)^2 - \left(\frac{\sqrt{5}}{2}\right) = 2 + 4 - \frac{\sqrt{5}}{2} = 6 - \frac{\sqrt{5}}{2} = \frac{12 - \sqrt{5}}{2}$$

EXAMPLE 19 *Given* ΔACB *right angled at* C *in which* $AB = 29$ *units,* $BC = 21$ *units and* $\angle ABC = \theta$. *Determine the values of*

(i) $\cos^2\theta + \sin^2\theta$ (ii) $\cos^2\theta - \sin^2\theta$ **[NCERT]**

SOLUTION In ΔACB, we have

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow \quad AC = \sqrt{AB^2 - BC^2} = \sqrt{29^2 - 21^2} = \sqrt{(29+21)(29-21)} = \sqrt{400} = 20 \text{ units}$$

$$\therefore \quad \sin\theta = \frac{AC}{AB} = \frac{20}{29} \text{ and } \cos\theta = \frac{BC}{AB} = \frac{21}{29}$$

(i) Using the values of $\sin\theta$ and $\cos\theta$, we obtain

$$\cos^2\theta + \sin^2\theta = \left(\frac{21}{29}\right)^2 + \left(\frac{20}{29}\right)^2 = \frac{441 + 400}{841} = 1$$

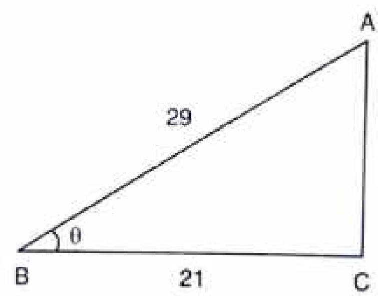


Fig. 10.23

(ii) Using the values of $\sin\theta$ and $\cos\theta$, we obtain

$$\cos^2\theta - \sin^2\theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{21^1 - 20^2}{29^2} = \frac{(21+20)(21-20)}{841} = \frac{41}{841}$$

EXAMPLE 20 *If* $\cot B = \frac{12}{5}$, *prove that* $\tan^2 B - \sin^2 B = \sin^4 B \sec^2 B$.

SOLUTION We have,

$$\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{12}{5}$$

So, we draw a right triangle ABC, right angled at C such that

Base $= BC = 12$ units and, Perpendicular $= AC = 5$ units.

Applying Pythagoras Theorem in ΔBCA, we obtain

$$AB^2 = BC^2 + AC^2$$

$\Rightarrow \quad AB^2 = 12^2 + 5^2 = 169$

$\Rightarrow \quad AB = \sqrt{169} = 13$

$\therefore \quad \sin B = \frac{AC}{AB} = \frac{5}{13}, \tan B = \frac{AC}{BC} = \frac{5}{12}$ and, $\sec B = \frac{AB}{BC} = \frac{13}{12}$

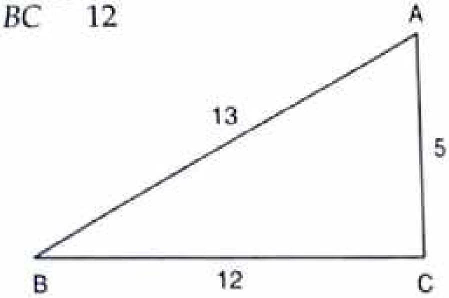


Fig. 10.24

Now, $\quad \text{LHS} = \tan^2 B - \sin^2 B$

$\Rightarrow \quad \text{LHS} = (\tan B)^2 - (\sin B)^2$

$\Rightarrow \quad \text{LHS} = \left(\frac{5}{12}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{25}{144} - \frac{25}{169}$

$\Rightarrow \quad \text{LHS} = 25\left(\frac{1}{144} - \frac{1}{169}\right) = 25\left(\frac{169-144}{144 \times 169}\right)$

$\Rightarrow \quad \text{LHS} = 25 \times \frac{25}{144 \times 169} = \frac{25 \times 25}{144 \times 169} = \frac{5^2 \times 5^2}{12^2 \times 13^2} \qquad \text{...(i)}$

and, $\quad \text{RHS} = \sin^4 B \sec^2 B$

$\Rightarrow \quad \text{RHS} = (\sin B)^4 (\sec B)^2 = \left(\frac{5}{13}\right)^4 \times \left(\frac{13}{12}\right)^2 = \frac{5^4 \times 13^2}{13^4 \times 12^2} = \frac{5^4}{13^2 \times 12^2} = \frac{5^2 \times 5^2}{13^2 \times 12^2}$

From (i) and (ii), we have

$$\tan^2 B - \sin^2 B = \sin^4 B \sec^2 B.$$

EXAMPLE 21 *If* $\sec \alpha = \frac{5}{4}$, *verify that* $\frac{\tan \alpha}{1 + \tan^2 \alpha} = \frac{\sin \alpha}{\sec \alpha}$

SOLUTION We have,

$$\sec \alpha = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{4}$$

So, we draw a right triangle ABC, right angled at B such that

$$\angle BAC = \alpha, \text{Base} = AB = 4 \text{ and Hypotenuse} = AC = 5.$$

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \quad 5^2 = 4^2 + BC^2$

$\Rightarrow \quad BC^2 = 25 - 16 = 9$

$\Rightarrow \quad BC = 3$

$\therefore \quad \tan \alpha = \frac{BC}{AB} = \frac{3}{4}$ and $\sin \alpha = \frac{BC}{AC} = \frac{3}{5}$

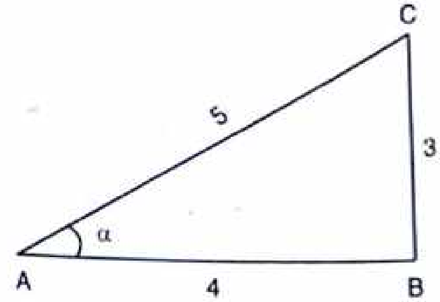


Fig. 10.25

Now, $\quad \frac{\tan \alpha}{1 + \tan^2 \alpha} = \frac{\frac{3}{4}}{1 + \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{4}}{1 + \frac{9}{16}} = \frac{\frac{3}{4}}{\frac{16+9}{16}} = \frac{\frac{3}{4}}{\frac{25}{16}} = \frac{3}{4} \times \frac{16}{25} = \frac{12}{25} \qquad \text{...(i)}$

and, $\dfrac{\sin\alpha}{\sec\alpha} = \dfrac{3/5}{5/4} = \dfrac{3}{5} \times \dfrac{4}{5} = \dfrac{12}{25}$ ...(ii)

From (i) and (ii), we obtain

$$\frac{\tan\alpha}{1+\tan^2\alpha} = \frac{\sin\alpha}{\sec\alpha}$$

EXAMPLE 22 *If* $\sin\theta = \dfrac{4}{5}$, *find the value of* $\dfrac{4\tan\theta - 5\cos\theta}{\sec\theta + 4\cot\theta}$.

SOLUTION We have,

$$\sin\theta = \frac{4}{5} \Rightarrow \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}$$

So, we construct a right triangle ABC, right angled at B such that $\angle BAC = \theta$, Perpendicular $= BC = 4$ and, Hypotenuse $= AC = 5$.

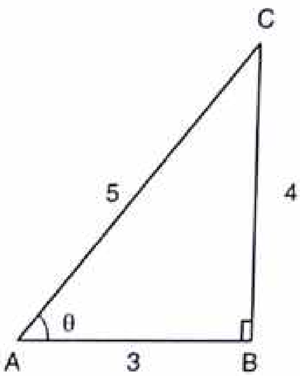


Fig. 10.26

Applying Pythagoras theorem in ΔABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $5^2 = AB^2 + 4^2$

$\Rightarrow$ $AB^2 = 25 - 16 = 9$

$\Rightarrow$ $AB = \sqrt{9} \Rightarrow AB = 3$

$\therefore$ $\cos\theta = \dfrac{AB}{AC} = \dfrac{3}{5}$, $\tan\theta = \dfrac{BC}{AB} = \dfrac{4}{3}$, $\cot\theta = \dfrac{AB}{BC} = \dfrac{3}{4}$ and, $\sec\theta = \dfrac{AC}{AB} = \dfrac{5}{3}$

$$\therefore \quad \frac{4\tan\theta - 5\cos\theta}{\sec\theta + 4\cot\theta} = \frac{4\times\frac{4}{3} - 5\times\frac{3}{5}}{\frac{5}{3} + 4\times\frac{3}{4}} = \frac{\frac{16}{3} - 3}{\frac{5}{3} + 3} = \frac{\frac{16-9}{3}}{\frac{5+9}{3}} = \frac{7}{14} = \frac{1}{2}$$

EXAMPLE 23 *In a right triangle ABC, right angled at B, the ratio of AB to AC is* $1 : \sqrt{2}$. *Find the values of*

(i) $\dfrac{2\tan A}{1+\tan^2 A}$ *and,* (ii) $\dfrac{2\tan A}{1-\tan^2 A}$

SOLUTION We have,

$$AB : AC = 1 : \sqrt{2} \Rightarrow \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$\therefore$ $AB = x$ and $AC = \sqrt{2}x$, for some x.

Applying Pythagores theorem in ΔABC, we obtain

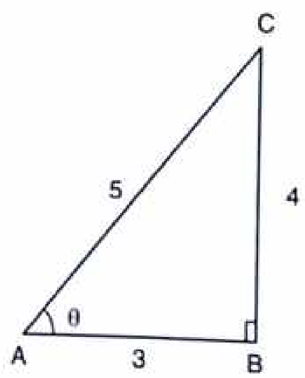


Fig. 10.27

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $(\sqrt{2}x)^2 = x^2 + BC^2$

$\Rightarrow$ $BC^2 = 2x^2 - x^2 = x^2$

$\Rightarrow$ $BC = x$

$\therefore \quad \tan A = \frac{BC}{AB} = \frac{x}{x} = 1$

(i) $\frac{2\tan A}{1+\tan^2 A} = \frac{2\times 1}{1+1^2} = \frac{2}{2} = 1$

(ii) $\frac{2\tan A}{1-\tan^2 A} = \frac{2\times 1}{1-1} = \frac{2}{0}$, which is undefined.

EXAMPLE 24 *In a right triangle ABC right angled at B, $\angle ACB = \theta$, $AB = 2\,cm$ and $BC = 1\,cm$. Find the value of $\sin^2\theta + \tan^2\theta$.*

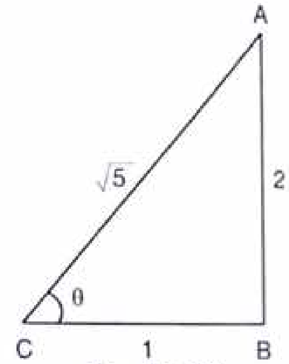


Fig. 10.28

SOLUTION We have,

$AB = 2$ cm and $BC = 1$ cm.

Applying Pythagoras theorem in ΔABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \quad AC^2 = 2^2 + 1 = 5$

$\Rightarrow \quad AC = \sqrt{5}$

Thus, Base $= BC = 1$ cm, Perpendicular $= AB = 2$ cm and, Hypotenuse $= AC = \sqrt{5}$ cm

$\therefore \quad \sin\theta = \frac{AB}{AC} = \frac{2}{\sqrt{5}}$ and, $\tan\theta = \frac{AB}{BC} = \frac{2}{1} = 2$

Now,

$\therefore \quad \sin^2\theta + \tan^2\theta = (\sin\theta)^2 + (\tan\theta)^2 = \left(\frac{2}{\sqrt{5}}\right)^2 + (2)^2 = \frac{4}{5} + 4 = \frac{24}{5}$

EXAMPLE 25 *In a ΔABC, right angled at C and $\angle A = \angle B$,*

(i) *Is* $\cos A = \cos B$? (ii) *Is* $\tan A = \tan B$?

What about the other trigonometric ratios for $\angle A$ and $\angle B$? Will they be equal?

SOLUTION We have,

$\angle A = \angle B$

$\Rightarrow \quad BC = AC$ [$\because$ Sides opposite to equal angles are equal]

$\Rightarrow \quad BC = AC = x$ (say)

Using Pythagoras theorem in ΔACB, we obtain

$$AB^2 = AC^2 + BC^2$$

$\Rightarrow \quad AB^2 = x^2 + x^2$

$\Rightarrow \quad AB = \sqrt{2}x$

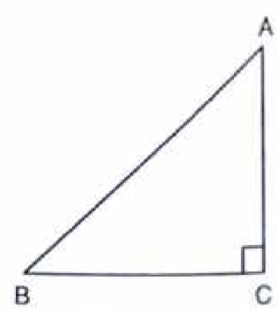


Fig. 10.29

(i) We have,

$\cos A = \frac{AC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$ and, $\cos B = \frac{BC}{AB} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos A = \cos B$

(ii) We have,

$\tan A = \frac{BC}{AC} = \frac{x}{x} = 1$ and, $\tan B = \frac{AC}{BC} = \frac{x}{x} = 1$

$\therefore \quad \tan A = \tan B$

Now, $\quad \sin A = \dfrac{BC}{AB} = \dfrac{x}{\sqrt{2}x} = \dfrac{1}{\sqrt{2}}$ and, $\sin B = \dfrac{AC}{AB} = \dfrac{x}{\sqrt{2}x} = \dfrac{1}{\sqrt{2}}$

$\therefore \quad \sin A = \sin B$

$\cot A = \dfrac{AC}{BC} = \dfrac{x}{x} = 1$ and $\cot B = \dfrac{BC}{AC} = \dfrac{x}{x} = 1$

$\therefore \quad \cot A = \cot B$

$\sec A = \dfrac{AB}{AC} = \dfrac{\sqrt{2}x}{x} = \sqrt{2}$ and $\sec B = \dfrac{AB}{BC} = \dfrac{\sqrt{2}x}{x} = \sqrt{2}$

$\therefore \quad \sec A = \sec B$

$\text{cosec } A = \dfrac{AB}{BC} = \dfrac{\sqrt{2}x}{x} = \sqrt{2}$ and $\text{cosec } B = \dfrac{AB}{AC} = \dfrac{\sqrt{2}x}{x} = \sqrt{2}$

$\therefore \quad \text{cosec } A = \text{cosec } B$

EXAMPLE 26 *In ΔABC, right angled at B, if $\tan A = \dfrac{1}{\sqrt{3}}$, find the value of*

(i) $\sin A \cos C + \cos A \sin C$ (ii) $\cos A \cos C - \sin A \sin C$ [NCERT]

SOLUTION We have,

$$\tan A = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{BC}{AB} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad BC = x \text{ and } AB = \sqrt{3}x$$

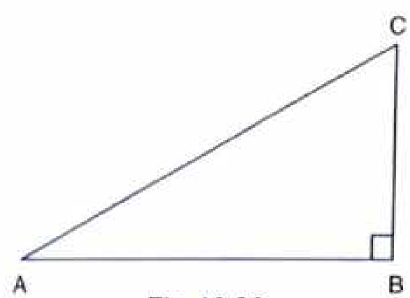


Fig. 10.30

Using Pythagoras theorem in ΔABC, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AC^2 = (\sqrt{3}x)^2 + x^2$$

$$\Rightarrow \quad AC^2 = 4x^2$$

$$\Rightarrow \quad AC = 2x$$

Now, $\quad \sin A = \dfrac{BC}{AC} = \dfrac{x}{2x} = \dfrac{1}{2}$, $\cos A = \dfrac{AB}{AC} = \dfrac{\sqrt{3}x}{2x} = \dfrac{\sqrt{3}}{2}$

$\sin C = \dfrac{AB}{AC} = \dfrac{\sqrt{3}x}{2x} = \dfrac{\sqrt{3}}{2}$ and, $\cos C = \dfrac{BC}{AC} = \dfrac{x}{2x} = \dfrac{1}{2}$

(i) $\quad \sin A \cos C + \cos A \sin C = \dfrac{1}{2} \times \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{1}{4} + \dfrac{3}{4} = 1$

(ii) $\quad \cos A \cos C - \sin A \sin C = \dfrac{\sqrt{3}}{2} \times \dfrac{1}{2} - \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} = 0$

EXAMPLE 27 *Given that $16 \cot A = 12$; find the value of* $\dfrac{\sin A + \cos A}{\sin A - \cos A}$.

SOLUTION We have, $16 \cot A = 12 \Rightarrow \cot A = \dfrac{12}{16} \Rightarrow \cot A = \dfrac{3}{4}$.

Now, $$\frac{\sin A+\cos A}{\sin A-\cos A}=\frac{\dfrac{\sin A+\cos A}{\sin A}}{\dfrac{\sin A-\cos A}{\sin A}}$$

[Dividing Numerator and Denominator by sin A]

$$=\frac{\dfrac{\sin A}{\sin A}+\dfrac{\cos A}{\sin A}}{\dfrac{\sin A}{\sin A}-\dfrac{\cos A}{\sin A}}$$

$$=\frac{1+\cot A}{1-\cot A}=\frac{1+\dfrac{12}{16}}{1-\dfrac{12}{16}} \qquad \left[\because 16\cot A=12 \Rightarrow \cot A=\frac{12}{16}\right]$$

$$=\frac{1+\dfrac{3}{4}}{1-\dfrac{3}{4}}=\frac{\dfrac{7}{4}}{\dfrac{1}{4}}=7$$

EXAMPLE 28 *If* $5\tan\alpha=4$, *show that* $\dfrac{5\sin\alpha-3\cos\alpha}{5\sin\alpha+2\cos\alpha}=\dfrac{1}{6}$

SOLUTION We have,

$$5\tan\alpha=4 \Rightarrow \tan\alpha=\frac{4}{5}$$

Now, $$\frac{5\sin\alpha-3\cos\alpha}{5\sin\alpha+2\cos\alpha}=\frac{\dfrac{5\sin\alpha-3\cos\alpha}{\cos\alpha}}{\dfrac{5\sin\alpha+2\cos\alpha}{\cos\alpha}} \qquad \left[\begin{matrix}\text{Dividing Numerator and}\\ \text{Denominator by }\cos\alpha\end{matrix}\right]$$

$$=\frac{\dfrac{5\sin\alpha}{\cos\alpha}-\dfrac{3\cos\alpha}{\cos\alpha}}{\dfrac{5\sin\alpha}{\cos\alpha}+\dfrac{2\cos\alpha}{\cos\alpha}}=\frac{5\tan\alpha-3}{5\tan\alpha+2}=\frac{5\times\dfrac{4}{5}-3}{5\times\dfrac{4}{5}+2} \qquad \left[\because \tan\alpha=\frac{4}{5}\right]$$

$$=\frac{4-3}{4+2}=\frac{1}{6}$$

EXAMPLE 29 *If* $\tan\theta=\dfrac{12}{13}$, *evaluate* $\dfrac{2\sin\theta\ \cos\theta}{\cos^2\theta-\sin^2\theta}$.

SOLUTION We have, $\tan\theta=\dfrac{12}{13}$

Now, $$\frac{2\sin\theta\ \cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{\dfrac{2\sin\theta\ \cos\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}} \qquad [\text{Dividing N}^{r}\text{ and D}^{r}\text{ by }\cos^2\theta]$$

$$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{12}{13}}{1-\left(\frac{12}{13}\right)^2} = \frac{\frac{24}{13}}{1-\frac{144}{169}} = \frac{\frac{24}{13}}{\frac{25}{169}} = \frac{24}{13}\times\frac{169}{25} = \frac{312}{25}$$

LEVEL-2

EXAMPLE 30 *If* $\sin\theta = \frac{a^2-b^2}{a^2+b^2}$, *find the values of other five trigonometric ratios.*

SOLUTION We have,

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a^2-b^2}{a^2+b^2}$$

So, draw a right triangle right angled at B such that

Perpendicular $= a^2 - b^2$, Hypotenuse $= a^2 + b^2$ and, $\angle BAC = \theta$

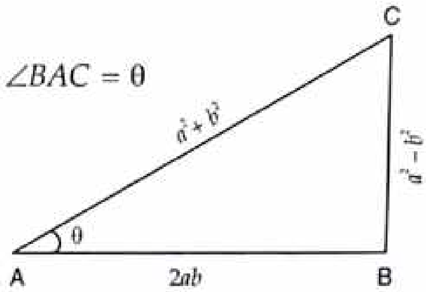


Fig. 10.31

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \quad (a^2+b^2)^2 = AB^2 + (a^2-b^2)^2$

$\Rightarrow \quad AB^2 = (a^2+b^2)^2 - (a^2-b^2)^2$

$\Rightarrow \quad AB^2 = (a^4+b^4+2a^2b^2) - (a^4+b^4-2a^2b^2)$

$\Rightarrow \quad AB^2 = 4a^2b^2 = (2ab)^2$

$\Rightarrow \quad AB = 2ab$

When we consider the trigonometric ratios of $\angle BAC = \theta$, we have
Base $= AB = 2ab$, Perpendicular $= BC = a^2 - b^2$ and, Hypotenuse $= AC = a^2 + b^2$

$\therefore \quad \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{2ab}{a^2+b^2}, \quad \tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a^2-b^2}{2ab}$

$\Rightarrow \quad \operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{a^2+b^2}{a^2-b^2}, \quad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{a^2+b^2}{2ab}$

and, $\quad \cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{2ab}{a^2-b^2}$

EXAMPLE 31 *From Fig. 10.32, write the values of:*

(i) $\sin A$ *(ii)* $\cot A$ *(iii)* $\tan B$ *(iv)* $\sin^2 B + \cos^2 B$

SOLUTION In ΔACD, we have

$$AC^2 = AD^2 + CD^2$$

$\Rightarrow \quad AC^2 = 3^2 + 4^2$

$\Rightarrow \quad AC^2 = 25$

$\Rightarrow \quad AC = \sqrt{25} = 5$

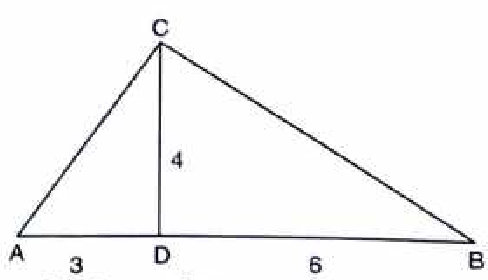


Fig. 10.32

In ΔBCD, we have

$$BC^2 = BD^2 + CD^2$$

$\Rightarrow \quad BC^2 = 6^2 + 4^2$

$$\Rightarrow \quad BC^2 = 36 + 16$$
$$\Rightarrow \quad BC^2 = 52$$
$$\Rightarrow \quad BC = \sqrt{52} = \sqrt{13 \times 4} = 2\sqrt{13}$$

Now, $\sin A = \frac{CD}{AC} = \frac{4}{5}$, $\cot A = \frac{AD}{CD} = \frac{3}{4}$

$\tan B = \frac{CD}{DB} = \frac{4}{6} = \frac{2}{3}$, $\sin B = \frac{CD}{BC} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$

and, $\cos B = \frac{BD}{BC} = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$

$$\therefore \quad \sin^2 B + \cos^2 B = \left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{3}{\sqrt{13}}\right)^2 = \frac{4}{13} + \frac{9}{13} = \frac{13}{13} = 1$$

EXAMPLE 32 *In Fig. 10.33, $AD = DB$ and $\angle B$ is a right angle. Determine:*

(i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$ (iv) $\sin^2\theta + \cos^2\theta$

SOLUTION We have,

$$AB = a$$
$$\Rightarrow \quad AD + DB = a$$
$$\Rightarrow \quad AD + AD = a \qquad [\because AD = DB]$$
$$\Rightarrow \quad 2\,AD = a$$
$$\Rightarrow \quad AD = \frac{a}{2}$$

Thus, $AD = DB = \frac{a}{2}$

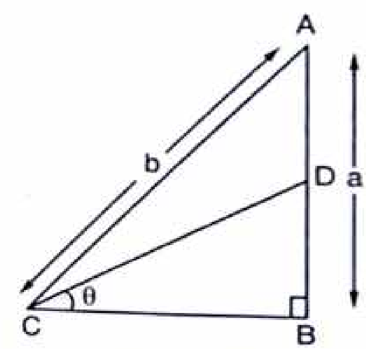


Fig. 10.33

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$
$$\Rightarrow \quad b^2 = a^2 + BC^2$$
$$\Rightarrow \quad BC^2 = b^2 - a^2$$
$$\Rightarrow \quad BC = \sqrt{b^2 - a^2}$$

Thus, in ΔBCD, we have

Base $= BC = \sqrt{b^2 - a^2}$ and Perpendicular $= BD = \frac{a}{2}$

Applying Pythagoras theorem in ΔBCD, we have

$$\Rightarrow \quad BC^2 + BD^2 = CD^2$$
$$\Rightarrow \quad \left(\sqrt{b^2 - a^2}\right)^2 + \left(\frac{a}{2}\right)^2 = CD^2$$
$$\Rightarrow \quad CD^2 = b^2 - a^2 + \frac{a^2}{4}$$
$$\Rightarrow \quad CD^2 = \frac{4b^2 - 4a^2 + a^2}{4}$$

$\Rightarrow \quad CD^2 = \dfrac{4b^2 - 3a^2}{4}$

$\Rightarrow \quad CD = \dfrac{\sqrt{4b^2 - 3a^2}}{2}$

Now,

(i) $\sin\theta = \dfrac{BD}{CD} \Rightarrow \sin\theta = \dfrac{a/2}{\dfrac{\sqrt{4b^2 - 3a^2}}{2}} = \dfrac{a}{\sqrt{4b^2 - 3a^2}}$

(ii) $\cos\theta = \dfrac{BC}{CD} \Rightarrow \cos\theta = \dfrac{\sqrt{b^2 - a^2}}{\dfrac{\sqrt{4b^2 - 3a^2}}{2}} = \dfrac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}$

(iii) $\tan\theta = \dfrac{BD}{BC} \Rightarrow \tan\theta = \dfrac{a/2}{\sqrt{b^2 - a^2}} = \dfrac{a}{2\sqrt{b^2 - a^2}}$

(iv) $\sin^2\theta + \cos^2\theta = \left(\dfrac{a}{\sqrt{4b^2 - 3a^2}}\right)^2 + \left(\dfrac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}\right)^2$

$$= \frac{a^2}{4b^2 - 3a^2} + \frac{4(b^2 - a^2)}{4b^2 - 3a^2} = \frac{4b^2 - 3a^2}{4b^2 - 3a^2} = 1$$

EXAMPLE 33 *In ΔOPQ right angled at P, $OP = 7$ cm, $OQ - PQ = 1$ cm. Determine the values of* $\sin Q$ *and* $\cos Q$. [NCERT]

SOLUTION In ΔOPQ, we have

$OQ^2 = OP^2 + PQ^2$

$\Rightarrow \quad (PQ + 1)^2 = OP^2 + PQ^2 \qquad [\because OQ - PQ = 1 \Rightarrow OQ = 1 + PQ]$

$\Rightarrow \quad PQ^2 + 2PQ + 1 = OP^2 + PQ^2$

$\Rightarrow \quad 2PQ + 1 = 49$

$\Rightarrow \quad 2PQ = 48$

$\Rightarrow \quad PQ = 24$ cm

$\therefore \quad OQ - PQ = 1$ cm

$\Rightarrow \quad OQ = (PQ + 1)$ cm $= 25$ cm

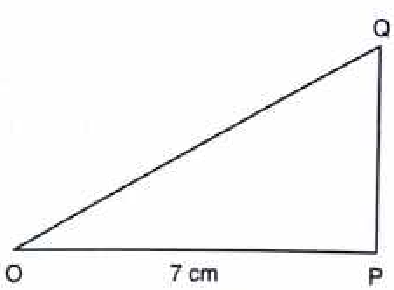


Fig. 10.34

Now $\quad \sin Q = \dfrac{OP}{OQ} = \dfrac{7}{25}$

and, $\quad \cos Q = \dfrac{PQ}{OQ} = \dfrac{24}{25}$

EXAMPLE 34 *In ΔPQR, right angled at Q, $PR + QR = 25$ cm and $PQ = 5$ cm. Determine the values of* $\sin P$, $\cos P$ *and* $\tan P$. [NCERT]

SOLUTION In ΔPQR, we have

$PR^2 = PQ^2 + QR^2$

$\Rightarrow \quad (25 - QR)^2 = 5^2 + QR^2 \qquad [\because PR + QR = 25 \text{ cm} \Rightarrow PR = 25 - QR]$

$\Rightarrow \quad 625 - 50\,QR + QR^2 = 25 + QR^2$

$\Rightarrow \quad 600 - 50\,QR = 0$

$\Rightarrow \quad QR = \dfrac{600}{50} = 12 \text{ cm}$

Now, $\quad PR + QR = 25$

$\Rightarrow \quad PR = 25 - QR = (25 - 12) = 13 \text{ cm}$

$\therefore \quad \sin P = \dfrac{QR}{PR} = \dfrac{12}{13}, \ \cos P = \dfrac{PQ}{PR} = \dfrac{5}{13}$ and, $\tan P = \dfrac{QR}{PQ} = \dfrac{12}{5}$

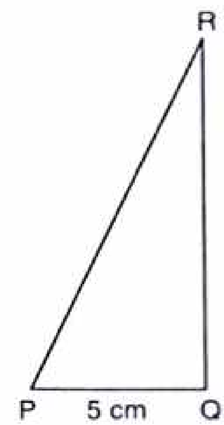


Fig. 10.35

EXAMPLE 35 *If* $\tan\theta + \dfrac{1}{\tan\theta} = 2$, *find the value of* $\tan^2\theta + \dfrac{1}{\tan^2\theta}$.

SOLUTION We have,

$$\tan\theta + \frac{1}{\tan\theta} = 2$$

$$\Rightarrow \quad \left(\tan\theta + \frac{1}{\tan\theta}\right)^2 = 2^2$$

$$\Rightarrow \quad \tan^2\theta + \frac{1}{\tan^2\theta} + 2 \times \tan\theta \times \frac{1}{\tan\theta} = 4$$

$$\Rightarrow \quad \tan^2\theta + \frac{1}{\tan^2\theta} + 2 = 4$$

$$\Rightarrow \quad \tan^2\theta + \frac{1}{\tan^2\theta} = 2$$

ALITER We have,

$$\tan\theta + \frac{1}{\tan\theta} = 2$$

$$\Rightarrow \quad \tan^2\theta + 1 = 2\tan\theta$$

$$\Rightarrow \quad \tan^2\theta - 2\tan\theta + 1 = 0$$

$$\Rightarrow \quad (\tan\theta - 1)^2 = 0$$

$$\Rightarrow \quad \tan\theta = 1$$

$$\therefore \quad \tan^2\theta + \frac{1}{\tan^2\theta} = 1 + 1 = 2$$

EXAMPLE 36 *If* $\angle B$ *and* $\angle Q$ *are acute angles such that* $\sin B = \sin Q$, *then prove that* $\angle B = \angle Q$. **[NCERT]**

SOLUTION Consider two right triangles ABC and PQR such that $\sin B = \sin Q$.

We have,

$$\sin B = \frac{AC}{AB} \text{ and, } \sin Q = \frac{PR}{PQ}$$

$$\therefore \quad \sin B = \sin Q$$

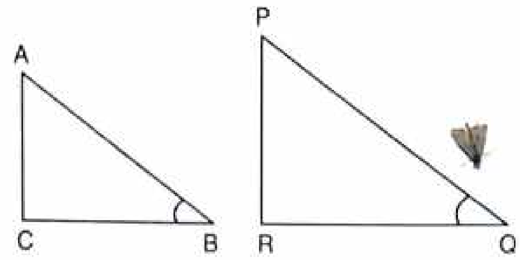


Fig. 10.36

$$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ}$$

$$\Rightarrow \quad \frac{AC}{PR} = \frac{AB}{PQ} = k, \text{(say)} \qquad \ldots(i)$$

$$\Rightarrow \quad AC = k\,PR \text{ and } AB = k\,PQ \qquad \ldots(ii)$$

Using Pythagoras theorem in triangles ABC and PQR, we obtain

$$AB^2 = AC^2 + BC^2 \text{ and } PQ^2 = PR^2 + QR^2$$

$$\Rightarrow \quad BC = \sqrt{AB^2 - AC^2} \text{ and } QR = \sqrt{PQ^2 - PR^2}$$

$$\Rightarrow \quad \frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}} \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad \frac{BC}{QR} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \qquad \ldots(iii)$$

From (i) and (iii), we have

$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

$$\Rightarrow \quad \Delta ACB \sim \Delta PRQ$$

$$\Rightarrow \quad \angle B = \angle Q$$

EXERCISE 10.1

LEVEL-1

1. In each of the following, one of the six trigonometric ratios is given. Find the values of the other trigonometric ratios.

(i) $\sin A = \frac{2}{3}$ (ii) $\cos A = \frac{4}{5}$ (iii) $\tan\theta = 11$ (iv) $\sin\theta = \frac{11}{15}$

(v) $\tan\alpha = \frac{5}{12}$ (vi) $\sin\theta = \frac{\sqrt{3}}{2}$ (vii) $\cos\theta = \frac{7}{25}$ (viii) $\tan\theta = \frac{8}{15}$

(ix) $\cot\theta = \frac{12}{5}$ (x) $\sec\theta = \frac{13}{5}$ (xi) $\operatorname{cosec}\theta = \sqrt{10}$ (xii) $\cos\theta = \frac{12}{15}$

2. In a ΔABC, right angled at B, $AB = 24$ cm, $BC = 7$ cm. Determine

(i) $\sin A, \cos A$ (ii) $\sin C, \cos C$ [NCERT]

3. In Fig. 10.37, find $\tan P$ and $\cot R$. Is $\tan P = \cot R$? [NCERT]

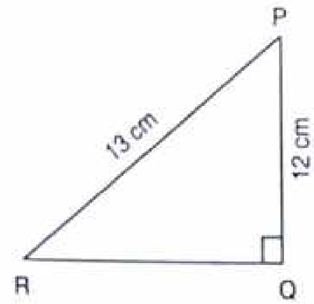


Fig. 10.37

4. If $\sin A = \frac{9}{41}$, compute $\cos A$ and $\tan A$.

5. Given $15 \cot A = 8$, find $\sin A$ and $\sec A$. [NCERT]

6. In ΔPQR, right angled at Q, $PQ = 4$ cm and $RQ = 3$ cm. Find the values of $\sin P$, $\sin R$, $\sec P$ and $\sec R$.

7. If $\cot \theta = \frac{7}{8}$, evaluate:

(i) $\frac{(1 + \sin \theta)(1 - \sin \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$ (ii) $\cot^2 \theta$ [NCERT]

8. If $3 \cot A = 4$, check whether $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A$ or not. [NCERT]

9. If $\tan \theta = \frac{a}{b}$, find the value of $\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}$

10. If $3 \tan \theta = 4$, find the value of $\frac{4 \cos \theta - \sin \theta}{2 \cos \theta + \sin \theta}$

11. If $3 \cot \theta = 2$, find the value of $\frac{4 \sin \theta - 3 \cos \theta}{2 \sin \theta + 6 \cos \theta}$

12. If $\tan \theta = \frac{a}{b}$, prove that $\frac{a \sin \theta - b \cos \theta}{a \sin \theta + b \cos \theta} = \frac{a^2 - b^2}{a^2 + b^2}$

13. If $\sec \theta = \frac{13}{5}$, show that $\frac{2 \sin \theta - 3 \cos \theta}{4 \sin \theta - 9 \cos \theta} = 3$

14. If $\cos \theta = \frac{12}{13}$, show that $\sin \theta (1 - \tan \theta) = \frac{35}{156}$

15. If $\cot \theta = \frac{1}{\sqrt{3}}$, show that $\frac{1 - \cos^2 \theta}{2 - \sin^2 \theta} = \frac{3}{5}$

16. If $\tan \theta = \frac{1}{\sqrt{7}}$, show that $\frac{\operatorname{cosec}^2 \theta - \sec^2 \theta}{\operatorname{cosec}^2 \theta + \sec^2 \theta} = \frac{3}{4}$

17. If $\sec \theta = \frac{5}{4}$, find the value of $\frac{\sin \theta - 2 \cos \theta}{\tan \theta - \cot \theta}$

18. If $\tan \theta = \frac{12}{13}$, find the value of $\frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$

19. If $\cos\theta = \frac{3}{5}$, find the value of $\dfrac{\sin\theta - \frac{1}{\tan\theta}}{2\tan\theta}$

20. If $\sin\theta = \frac{3}{5}$, evaluate $\dfrac{\cos\theta - \frac{1}{\tan\theta}}{2\cot\theta}$

21. If $\tan\theta = \frac{24}{7}$, find that $\sin\theta + \cos\theta$

22. If $\sin\theta = \frac{a}{b}$, find $\sec\theta + \tan\theta$ in terms of a and b.

23. If $8\tan A = 15$, find $\sin A - \cos A$

24. If $\tan\theta = \frac{20}{21}$, show that $\dfrac{1 - \sin\theta + \cos\theta}{1 + \sin\theta + \cos\theta} = \frac{3}{7}$

25. If cosec $A = 2$, find the value of $\dfrac{1}{\tan A} + \dfrac{\sin A}{1 + \cos A}$

26. If $\angle A$ and $\angle B$ are acute angles such that $\cos A = \cos B$, then show that $\angle A = \angle B$. **[NCERT]**

27. In a D ABC, right angled at A, if $\tan C = \sqrt{3}$, find the value of $\sin B \cos C + \cos B \sin C$. **[CBSE 2008]**

28. State whether the following are true or false. Justify your answer.
   (i) The value of $\tan A$ is always less than 1.
   (ii) $\sec A = \frac{12}{5}$ for some value of angle A.
   (iii) $\cos A$ is the abbreviation used for the cosecant of angle A.
   (iv) $\cot A$ is the product of cot and A.
   (v) $\sin\theta = \frac{4}{3}$ for some angle θ.

LEVEL-2

29. If $\sin\theta = \frac{12}{13}$, find the value of $\dfrac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta} \times \dfrac{1}{\tan^2\theta}$

30. If $\cos\theta = \frac{5}{13}$, find the value of $\dfrac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta} \times \dfrac{1}{\tan^2\theta}$

31. If $\sec A = \frac{5}{4}$, verify that $\dfrac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A} = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

32. If $\sin\theta = \frac{3}{4}$, prove that $\sqrt{\dfrac{\text{cosec}^2\theta - \cot^2\theta}{\sec^2\theta - 1}} = \dfrac{\sqrt{7}}{3}$

33. If $\sec A = \frac{17}{8}$, verify that $\dfrac{3 - 4\sin^2 A}{4\cos^2 A - 3} = \dfrac{3 - \tan^2 A}{1 - 3\tan^2 A}$

34. If $\cot\theta = \frac{3}{4}$, prove that $\sqrt{\frac{\sec\theta - \operatorname{cosec}\theta}{\sec\theta + \operatorname{cosec}\theta}} = \frac{1}{\sqrt{7}}$

35. If $3\cos\theta - 4\sin\theta = 2\cos\theta + \sin\theta$, find $\tan\theta$.

36. If $\angle A$ and $\angle P$ are acute angles such that $\tan A = \tan P$, then show that $\angle A = \angle P$.

ANSWERS

1. (i) $\cos A = \frac{\sqrt{5}}{3}$, $\tan A = \frac{2}{\sqrt{5}}$, $\sec A = \frac{3}{\sqrt{5}}$, $\operatorname{cosec} A = \frac{3}{2}$, $\cot A = \frac{\sqrt{5}}{2}$

(ii) $\sin A = \frac{3}{5}$, $\tan A = \frac{3}{4}$, $\sec A = \frac{5}{4}$, $\operatorname{cosec} A = \frac{5}{3}$, $\cot A = \frac{4}{3}$

(iii) $\sin\theta = \frac{11}{\sqrt{122}}$, $\cos\theta = \frac{1}{\sqrt{122}}$, $\operatorname{cosec}\theta = \frac{\sqrt{122}}{11}$, $\sec\theta = \sqrt{122}$, $\cot\theta = \frac{1}{11}$

(iv) $\cos\theta = \frac{2\sqrt{26}}{15}$, $\tan\theta = \frac{11}{2\sqrt{26}}$, $\sec\theta = \frac{15}{2\sqrt{26}}$, $\cot\theta = \frac{2\sqrt{26}}{11}$, $\operatorname{cosec}\theta = \frac{15}{11}$

(v) $\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$, $\cot\alpha = \frac{12}{5}$, $\operatorname{cosec}\alpha = \frac{13}{5}$, $\sec\alpha = \frac{13}{12}$

(vi) $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\sec\theta = 2$, $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$, $\cot\theta = \frac{1}{\sqrt{3}}$

(vii) $\sin\theta = \frac{24}{25}$, $\tan\theta = \frac{24}{7}$, $\sec\theta = \frac{25}{7}$, $\operatorname{cosec}\theta = \frac{25}{24}$, $\cot\theta = \frac{7}{24}$

(viii) $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\cot\theta = \frac{15}{8}$, $\operatorname{cosec}\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$

(ix) $\tan\theta = \frac{5}{12}$, $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\operatorname{cosec}\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$

(x) $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$, $\operatorname{cosec}\theta = \frac{13}{12}$, $\cot\theta = \frac{5}{12}$

(xi) $\sin\theta = \frac{1}{\sqrt{10}}$, $\cos\theta = \frac{3}{\sqrt{10}}$, $\tan\theta = \frac{1}{3}$, $\sec\theta = \frac{\sqrt{10}}{3}$, $\cot\theta = 3$

(xii) $\sin\theta = \frac{9}{15}$, $\tan\theta = \frac{9}{12}$, $\operatorname{cosec}\theta = \frac{15}{9}$, $\cot\theta = \frac{12}{9}$, $\sec\theta = \frac{15}{12}$

2. (i) $\frac{7}{25}, \frac{24}{25}$ (ii) $\frac{24}{25}, \frac{7}{25}$

3. $\tan P = \frac{5}{12}$, $\cot R = \frac{5}{12}$. Yes.

4. $\cos A = \frac{40}{41}$, $\tan A = \frac{9}{40}$

5. $\sin A = \frac{15}{17}$, $\sec A = \frac{17}{8}$

6. $\sin P = \frac{3}{5}$, $\sec P = \frac{5}{4}$, $\sec R = \frac{5}{3}$, $\sin R = \frac{4}{5}$

7. (i) $\frac{49}{64}$ (ii) $\frac{49}{64}$

8. Yes.

9. $\frac{b+a}{b-a}$

10. $\frac{4}{5}$

11. $\frac{1}{3}$

17. $\frac{12}{7}$

18. $\frac{312}{25}$

19. $\frac{3}{160}$

20. $\frac{-1}{5}$

21. $\frac{31}{25}$

22. $\sqrt{\frac{b+a}{b-a}}$

23. $\frac{7}{17}$

25. 2

27. 1

28. (i) False (ii) True (iii) False (iv) False (v) False

29. $\frac{595}{3456}$

30. $\frac{595}{3456}$

35. $\frac{1}{5}$

10.5 TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES

In this section we shall find the trigonometric ratios of some standard angles by using some elementary knowledge of geometry. For other angles, we can make use of the trigonometric tables.

10.5.1 TRIGONOMETRIC RATIOS OF 0° AND 90°

Let $\angle XAY = \theta$ be an acute angle and let P be a point on its terminal side AY. Draw perpendicular PM from P on AX.

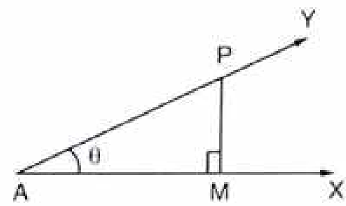


Fig. 10.38

In ΔAMP, we have

$$\sin\theta = \frac{PM}{AP}, \ \cos\theta = \frac{AM}{AP} \text{ and } \tan\theta = \frac{PM}{AM}$$

It is evident from ΔAMP that as θ becomes smaller and smaller, line segment PM also becomes smaller and smaller; and finally when θ becomes $0°$; the point P will coincide with M. Consequently, we have

$$PM = 0 \text{ and } AP = AM$$

$\therefore$ $$\sin 0° = \frac{PM}{AP} = \frac{0}{AP} = 0, \cos 0° = \frac{AM}{AP} = \frac{AP}{AP} = 1$$

and, $$\tan 0° = \frac{PM}{AP} = \frac{0}{AP} = 0$$

Thus, we have

$$\sin 0° = 0, \ \cos 0° = 1 \text{ and } \tan 0° = 0$$

From ΔAMP, it is evident that as θ increase, line segment AM becomes smaller and smaller and finally when θ becomes $90°$ the point M will coincide with A. Consequently, we have

$$AM = 0, \ AP = PM$$

$\therefore$ $$\sin 90° = \frac{PM}{AP} = \frac{PM}{PM} = 1 \text{ and } \cos 90° = \frac{AM}{AP} = \frac{0}{AP} = 0$$

Thus, we have

$$\sin 90° = 1 \text{ and } \cos 90° = 0$$

REMARK *It is evident from the above discussion that* $\tan 90° = \frac{PM}{AM} = \frac{PM}{0}$ *is not defined. Similarly, sec 90°, cosec 0°, cot 0° are not defined.*

10.5.2 TRIGONOMETRIC RATIOS OF 30° AND 60°

Consider an equilateral triangle ABC with each side of length $2a$. Since each angle of an equilateral triangle is of $60°$. Therefore, each angle of ΔABC is of $60°$. Let AD be perpendicular from A on BC. Since the triangle is equilateral. Therefore, AD is the bisector of $\angle A$ and D is the mid-point of BC.

$\therefore$ $$BD = DC = a \text{ and } \angle BAD = 30°$$

Thus, in ΔABD, $\angle D$ is a right angle, hypotenuse $AB = 2a$ and $BD = a$.

So, by Pythagoras theorem, we have

$$AB^2 = AD^2 + BD^2$$
$$\Rightarrow \quad (2a)^2 = AD^2 + a^2$$
$$\Rightarrow \quad AD^2 = 4a^2 - a^2$$
$$\Rightarrow \quad AD = \sqrt{3}\,a$$

Trigonometric ratios of 30°:

In right triangle ADB, we have **[CBSE 2009, 2010]**

Base $= AD = \sqrt{3}a$, Perpendicular $= BD = a$, Hypotenuse $= AB = 2a$ and $\angle DAB = 30°$

$$\therefore \quad \sin 30° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$
$$\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2}$$
$$\tan 30° = \frac{BD}{AD} = \frac{a}{\sqrt{3}\,a} = \frac{1}{\sqrt{3}}$$
$$\text{cosec}\, 30° = \frac{1}{\sin 30°} = 2$$
$$\sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}} \text{ and, } \cot 30° = \frac{1}{\tan 30°} = \sqrt{3}$$

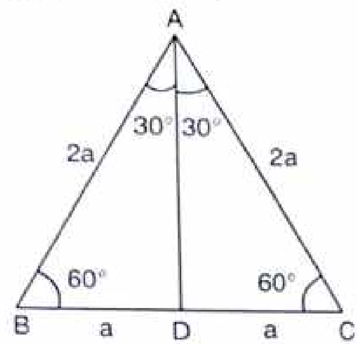


Fig. 10.39

Trigonometric ratios of 60°:

In right triangle ADB, we have **[CBSE 2009, 2010]**

Base $= BD = a$, Perpendicular $= AD = \sqrt{3}a$, Hypotenuse $= AB = 2a$ and $\angle ABD = 60°$

$$\therefore \quad \sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2}, \quad \cos 60° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$
$$\tan 60° = \frac{AD}{BD} = \frac{\sqrt{3}\,a}{a} = \sqrt{3}, \quad \text{cosec}\, 60° = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}$$
$$\sec 60° = \frac{1}{\cos 60°} = 2 \text{ and, } \cot 60° = \frac{1}{\tan 60°} = \frac{1}{\sqrt{3}}$$

10.5.3 TRIGONOMETRIC RATIOS OF 45°

Consider a right triangle ABC with right angle at B such that $\angle A = 45°$. Then,

$$\angle A + \angle B + \angle C = 180°$$
$$\Rightarrow \quad 45° + 90° + \angle C = 180°$$
$$\Rightarrow \quad \angle C = 45°$$
$$\therefore \quad \angle A = \angle C$$
$$\Rightarrow \quad AB = BC$$

Let $AB = BC = a$. Then, by Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$
$$\Rightarrow \quad AC^2 = a^2 + a^2$$
$$\Rightarrow \quad AC^2 = 2a^2$$
$$\Rightarrow \quad AC = \sqrt{2}\,a$$

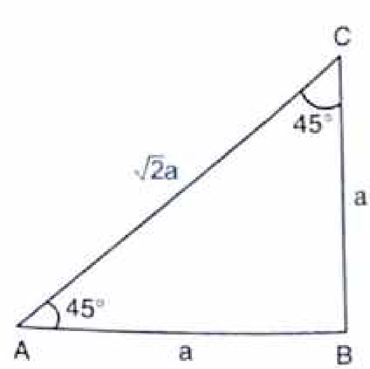


Fig. 10.40

Thus, in ΔABC, we have

$\angle A = 45°$, Base $= AB = a$, Perpendicular $= BC = a$, and Hypotenuse $= AC = \sqrt{2}a$.

$$\therefore \quad \sin 45° = \frac{BC}{AC} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}}, \quad \cos 45° = \frac{AB}{AC} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{BC}{AB} = \frac{a}{a} = 1, \quad \operatorname{cosec} 45° = \frac{1}{\sin 45°} = \sqrt{2}$$

$$\sec 45° = \frac{1}{\cos 45°} = \sqrt{2} \text{ and, } \cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

Following table gives the values of various trigonometric ratios of $0°, 30°, 45°, 60°$ and $90°$ for ready reference.

θ / T. ratios	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\operatorname{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Following examples will illustrate the use of the values of various trigonometric ratios given in the above table to find the values of trigonometric expressions.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate each of the following in the simplest form:*

(i) $\sin 60° \cos 30° + \cos 60° \sin 30°$ (ii) $\sin 60° \cos 45° + \cos 60° \sin 45°$

(iii) $\cos 60° \cos 30° + \sin 60° \sin 30°$ [NCERT] (iv) $\cos 60° \cos 30° - \sin 60° \sin 30°$

SOLUTION (i) We have,

$$\sin 60° \cos 30° + \cos 60° \sin 30° = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$$

(ii) We have,

$$\sin 60° \cos 45° + \cos 60° \sin 45° = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) We have,

$$\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 2\left(\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{2}$$

(iv) We have,

$$\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ = \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

EXAMPLE 2 *Evaluate each of the following in the simplest form:*

(i) $\operatorname{cosec} 30^\circ + \cot 45^\circ$ (ii) $\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$

(iii) $\tan 30^\circ \sec 45^\circ + \tan 60^\circ \sec 30^\circ$ (iv) $\sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

SOLUTION (i) We have,

$$\operatorname{cosec} 30^\circ + \cot 45^\circ = 2 + 1 = 3$$

(ii) We have,

$$\cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(iii) We have,

$$\tan 30^\circ \sec 45^\circ + \tan 60^\circ \sec 30^\circ = \frac{1}{\sqrt{3}} \times \sqrt{2} + \sqrt{3} \times \frac{2}{\sqrt{3}} = \frac{\sqrt{2} + 2\sqrt{3}}{\sqrt{3}}$$

(iv) We have,

$$\sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ = \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

EXAMPLE 3 *Evaluate the following expressions:*

(i) $2\sin^2 30^\circ \tan 60^\circ - 3\cos^2 60^\circ \sec^2 30^\circ$ (ii) $\operatorname{cosec}^2 30^\circ \sin^2 45^\circ - \sec^2 60^\circ$

(iii) $\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$ (iv) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$

(v) $3\cos^2 30^\circ + \sec^2 30^\circ + 2\cos 0^\circ + 3\sin 90^\circ - \tan^2 60^\circ$

SOLUTION (i) We have,

$$2\sin^2 30^\circ \tan 60^\circ - 3\cos^2 60^\circ \sec^2 30^\circ$$

$$= 2(\sin 30^\circ)^2 \tan 60^\circ - 3(\cos 60^\circ)^2 (\sec 30^\circ)^2$$

$$= 2 \times \left(\frac{1}{2}\right)^2 \times \sqrt{3} - 3 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 2 \times \frac{1}{4} \times \sqrt{3} - 3 \times \frac{1}{4} \times \frac{4}{3} = \frac{\sqrt{3}}{2} - 1 = \frac{\sqrt{3}-2}{2}$$

(ii) We have,

$$\operatorname{cosec}^2 30^\circ \sin^2 45^\circ - \sec^2 60^\circ$$

$$= (\operatorname{cosec} 30^\circ)^2 (\sin 45^\circ)^2 - (\sec 60^\circ)^2 = (2)^2 \times \left(\frac{1}{\sqrt{2}}\right)^2 - (2)^2 = 2 - 4 = -2$$

(iii) We have,

$$\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$$

$$= \tan 60^\circ (\operatorname{cosec} 45^\circ)^2 + (\sec 60^\circ)^2 \tan 45^\circ$$

$$= \sqrt{3} \times (\sqrt{2})^2 + (2)^2 \times 1 = \sqrt{3} \times 2 + 4 = 4 + 2\sqrt{3}$$

(iv) We have,

$$4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$$

$$= 4(\cot 45^\circ)^2 - (\sec 60^\circ)^2 + (\sin 60^\circ)^2 + (\cos 90^\circ)^2$$

$$= 4 \times (1)^2 - (2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 0 = 4 - 4 + \frac{3}{4} + 0 = \frac{3}{4}$$

(v) We have,

$$3\cos^2 30^\circ + \sec^2 30^\circ + 2\cos 0^\circ + 3\sin 90^\circ - \tan^2 60^\circ$$

$$= 3(\cos 30^\circ)^2 + (\sec 30^\circ)^2 + 2\cos 0^\circ + 3\sin 90^\circ - (\tan 60^\circ)^2$$

$$= 3 \times \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + 2 \times 1 + 3 \times 1 - (\sqrt{3})^2 = \frac{9}{4} + \frac{4}{3} + 2 + 3 - 3 = \frac{67}{12}$$

EXAMPLE 4 *Prove that:* $\dfrac{\cos 30^\circ + \sin 60^\circ}{1 + \cos 60^\circ + \sin 30^\circ} = \dfrac{\sqrt{3}}{2}$.

SOLUTION We have,

$$\frac{\cos 30^\circ + \sin 60^\circ}{1 + \cos 60^\circ + \sin 30^\circ} = \frac{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}{1 + \frac{1}{2} + \frac{1}{2}} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{2} = \frac{\sqrt{3}}{2}$$

EXAMPLE 5 *Evaluate each of the following:*

(i) $\dfrac{\sin^2 45^\circ + \cos^2 45^\circ}{\tan^2 60^\circ}$ (ii) $\dfrac{\sin 30^\circ - \sin 90^\circ + 2\cos 0^\circ}{\tan 30^\circ \tan 60^\circ}$

(iii) $\dfrac{\sin 60^\circ}{\cos^2 45^\circ} - \cot 30^\circ + 15\cos 90^\circ$ (iv) $\dfrac{5\sin^2 30^\circ + \cos^2 45^\circ - 4\tan^2 30^\circ}{2\sin 30^\circ \cos 30^\circ + \tan 45^\circ}$

SOLUTION (i) We have,

$$\frac{\sin^2 45^\circ + \cos^2 45^\circ}{\tan^2 60^\circ} = \frac{(\sin 45^\circ)^2 + (\cos 45^\circ)^2}{(\tan 60^\circ)^2} = \frac{(1/\sqrt{2})^2 + (1/\sqrt{2})^2}{(\sqrt{3})^2} = \frac{\frac{1}{2} + \frac{1}{2}}{3} = \frac{1}{3}$$

(ii) We have,

$$\frac{\sin 30^\circ - \sin 90^\circ + 2\cos 0^\circ}{\tan 30^\circ \tan 60^\circ} = \frac{\frac{1}{2} - 1 + 2 \times 1}{\frac{1}{\sqrt{3}} \times \sqrt{3}} = \frac{\frac{1}{2} - 1 + 2}{1} = \frac{3}{2}$$

(iii) We have,

$$\frac{\sin 60^\circ}{\cos^2 45^\circ} - \cot 30^\circ + 15\cos 90^\circ = \frac{\frac{\sqrt{3}}{2}}{(1/\sqrt{2})^2} - \sqrt{3} + 15 \times 0 = \sqrt{3} - \sqrt{3} + 0 = 0$$

(iv) We have,

$$\frac{5\sin^2 30^\circ + \cos^2 45^\circ - 4\tan^2 30^\circ}{2\sin 30^\circ \cos 30^\circ + \tan 45^\circ}$$

$$= \frac{5\times(1/2)^2 + (1/\sqrt{2})^2 - 4\times(1/\sqrt{3})^2}{2\times\frac{1}{2}\times\frac{\sqrt{3}}{2}+1}$$

$$= \frac{\frac{5}{4}+\frac{1}{2}-\frac{4}{3}}{\frac{\sqrt{3}}{2}+1} = \frac{\frac{5}{12}}{\frac{\sqrt{3}+2}{2}} = \frac{5}{12}\times\frac{2}{\sqrt{3}+2} = \frac{5}{6(\sqrt{3}+2)} = \frac{5}{6}(2-\sqrt{3})$$

EXAMPLE 6 *Show that:*

(i) $2(\cos^2 45^\circ + \tan^2 60^\circ) - 6(\sin^2 45^\circ - \tan^2 30^\circ) = 6$

(ii) $2(\cos^4 60^\circ + \sin^4 30^\circ) - (\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ = \frac{1}{4}$

SOLUTION (i) We have,

$$2(\cos^2 45^\circ + \tan^2 60^\circ) - 6(\sin^2 45^\circ - \tan^2 30^\circ)$$

$$= 2\left\{\left(\frac{1}{\sqrt{2}}\right)^2 + (\sqrt{3})^2\right\} - 6\left\{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right\}$$

$$= 2\left(\frac{1}{2}+3\right) - 6\left(\frac{1}{2}-\frac{1}{3}\right) = 2\left(\frac{1+6}{2}\right) - 6\left(\frac{3-2}{6}\right) = 7-1 = 6$$

(ii) We have,

$$2(\cos^4 60^\circ + \sin^4 30^\circ) - (\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ$$

$$= 2\left\{\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right\} - \{(\sqrt{3})^2 + (1)^2\} + 3\left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 2\left(\frac{1}{16}+\frac{1}{16}\right) - (3+1) + 3\times\frac{4}{3} = 2\times\frac{1}{8} - 4 + 4 = \frac{1}{4}$$

EXAMPLE 7 *Find the value of θ in each of the following:*

(i) $2\sin 2\theta = \sqrt{3}$ (ii) $2\cos 3\theta = 1$ (iii) $\sqrt{3}\tan 2\theta - 3 = 0$

SOLUTION (i) We have,

$$2\sin 2\theta = \sqrt{3}$$

$$\Rightarrow \quad \sin 2\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin 2\theta = \sin 60^\circ \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

(ii) We have,

$$2\cos 3\theta = 1$$

$$\Rightarrow \quad \cos 3\theta = \frac{1}{2} \Rightarrow \cos 3\theta = \cos 60° \Rightarrow 3\theta = 60° \Rightarrow \theta = 20°$$

(iii) We have,

$$\sqrt{3}\tan 2\theta - 3 = 0$$

$$\Rightarrow \quad \sqrt{3}\tan 2\theta = 3$$

$$\Rightarrow \quad \tan 2\theta = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \tan 2\theta = \tan 60° \Rightarrow 2\theta = 60° \Rightarrow \theta = 30°$$

EXAMPLE 8 *Find the value of x in each of the following:*

(i) $\tan 3x = \sin 45° \cos 45° + \sin 30°$ (ii) $\cos x = \cos 60° \cos 30° + \sin 60° \sin 30°$

(iii) $\sin 2x = \sin 60° \cos 30° - \cos 60° \sin 30°$

SOLUTION (i) We have,

$$\tan 3x = \sin 45° \cos 45° + \sin 30°$$

$$\Rightarrow \quad \tan 3x = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2}$$

$$\Rightarrow \quad \tan 3x = \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow \quad \tan 3x = 1 \Rightarrow \tan 3x = \tan 45° \Rightarrow 3x = 45° \Rightarrow x = 15°$$

(ii) We have,

$$\cos x = \cos 60° \cos 30° + \sin 60° \sin 30°$$

$$\Rightarrow \quad \cos x = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2}$$

$$\Rightarrow \quad \cos x = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$$

$$\Rightarrow \quad \cos x = \frac{\sqrt{3}}{2} \Rightarrow \cos x = \cos 30° \Rightarrow x = 30°$$

(iii) We have,

$$\sin 2x = \sin 60° \cos 30° - \cos 60° \sin 30°$$

$$\Rightarrow \quad \sin 2x = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$\Rightarrow \quad \sin 2x = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \quad \sin 2x = \sin 30° \Rightarrow 2x = 30° \Rightarrow x = 15$$

EXAMPLE 9 *Solve each of the following equations when $0° < \theta < 90°$.*

(i) $2\cos\theta = 1$ (ii) $2\cos^2\theta = \frac{1}{2}$ (iii) $2\sin^2\theta = \frac{1}{2}$

(iv) $3\tan^2\theta - 1 = 0$

SOLUTION (i) We have,

$$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

(ii) We have,

$$2\cos^2\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{4}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad [\because \cos\theta > 0 \text{ for } 0^\circ < \theta < 90^\circ]$$

$$\Rightarrow \quad \theta = 60^\circ$$

(iii) We have,

$$2\sin^2\theta = \frac{1}{2}$$

$$\Rightarrow \quad \sin^2\theta = \frac{1}{4}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \qquad [\because \sin\theta > 0 \text{ for } 0^\circ < \theta < 90^\circ]$$

$$\Rightarrow \quad \theta = 30^\circ$$

(iv) We have,

$$3\tan^2\theta = 1$$

$$\Rightarrow \quad \tan^2\theta = \frac{1}{3}$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} \qquad [\because \tan\theta > 0 \text{ for } 0^\circ < \theta^\circ < 90^\circ]$$

$$\Rightarrow \quad \theta = 30^\circ$$

EXAMPLE 10 *Solve each of the following equations for* $0^\circ < \theta < 90^\circ$.

(i) $2\cos 3\theta = 1$ (ii) $2\sin 2\theta = \sqrt{3}$ (iii) $\tan 5\theta = 1$

SOLUTION (i) We have,

$$2\cos 3\theta = 1 \Rightarrow \cos 3\theta = \frac{1}{2} \Rightarrow 3\theta = 60^\circ \Rightarrow \theta = 20^\circ$$

(ii) We have,

$$2\sin 2\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow 2\theta = 60^\circ \Rightarrow \theta^\circ = 30^\circ$$

(iii) We have,

$$\tan 5\theta = 1 \Rightarrow 5\theta = 45^\circ \Rightarrow \theta = 9^\circ$$

EXAMPLE 11 *If* $x = 30^\circ$, *verify that*

(i) $\sin 3x = 3\sin x - 4\sin^3 x$ (ii) $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$ (iv) $\sin x = \sqrt{\dfrac{1 - \cos 2x}{2}}$

SOLUTION (i) We have,

$$x = 30^\circ \Rightarrow 3x = 90^\circ$$

$$\therefore \quad \sin 3x = \sin 90^\circ = 1$$

and, $$3\sin x - 4\sin^3 x = 3\sin 30^\circ - 4\sin^3 30^\circ = 3 \times \frac{1}{2} - 4\left(\frac{1}{2}\right)^3 = \frac{3}{2} - \frac{1}{2} = 1.$$

$$\therefore \quad \sin 3x = 3\sin x - 4\sin^3 x$$

(ii) We have,

$$x = 30^\circ \Rightarrow 3x = 90^\circ$$

$$\therefore \quad \cos 3x = \cos 90^\circ = 0$$

and, $$4\cos^3 x - 3\cos x = 4\cos^3 30^\circ - 3\cos 30^\circ$$

$$\Rightarrow \quad 4\cos^3 x - 3\cos x = 4\left(\frac{\sqrt{3}}{2}\right)^3 - 3\left(\frac{\sqrt{3}}{2}\right) = 4 \times \frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$$

$$\therefore \quad \cos 3x = 4\cos^3 x - 3\cos x$$

(iii) We have,

$$x = 30^\circ \Rightarrow 2x = 60^\circ$$

$$\therefore \quad \tan 2x = \tan 60^\circ = \sqrt{3}$$

and, $$\frac{2\tan x}{1 - \tan^2 x} = \frac{2\tan 30^\circ}{1 - \tan^2 30^\circ} = \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2/\sqrt{3}}{1 - \frac{1}{3}} = \frac{2/\sqrt{3}}{2/3} = \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}$$

$$\therefore \quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

(iv) We have,

$$x = 30^\circ \Rightarrow 2x = 60^\circ$$

$$\therefore \quad \sqrt{\frac{1 - \cos 2x}{2}} = \sqrt{\frac{1 - \cos 60^\circ}{2}} = \sqrt{\frac{1 - \frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \text{ and, } \sin x = \sin 30^\circ = \frac{1}{2}$$

$$\therefore \quad \sin x = \sqrt{\frac{1 - \cos 2x}{2}}$$

EXAMPLE 12 *Verify that:*

(i) $\sin 60^\circ = \dfrac{2\tan 30^\circ}{1 + \tan^2 30^\circ} = \dfrac{\sqrt{3}}{2}$ (ii) $\cos 60^\circ = \dfrac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \dfrac{1}{2}$

(iii) $\cos 60^\circ = \cos^2 30^\circ - \sin^2 30^\circ = \dfrac{1}{2}$

(iv) $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ) = 2$

SOLUTION (i) We have,

$$\text{LHS} = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\text{RHS} = \frac{2\tan 30^\circ}{1+\tan^2 30^\circ} = \frac{2\times\frac{1}{\sqrt{3}}}{1+(1/\sqrt{3})^2} = \frac{2/\sqrt{3}}{1+\frac{1}{3}} = \frac{2/\sqrt{3}}{4/3} = \frac{2}{\sqrt{3}}\times\frac{3}{4} = \frac{\sqrt{3}}{2}$$

$\therefore$ LHS = RHS i.e., $\sin 60^\circ = \dfrac{2\tan 30^\circ}{1+\tan^2 30} = \dfrac{\sqrt{3}}{2}$.

(ii) We have,

$$\text{RHS} = \frac{1-\tan^2 30^\circ}{1+\tan^2 30^\circ} = \frac{1-(1/\sqrt{3})^2}{1+(1/\sqrt{3})^2} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{2/3}{4/3} = \frac{1}{2} \text{ and, LHS} = \text{cos}$$

$\therefore$ LHS = RHS i.e., $\cos 60^\circ = \dfrac{1-\tan^2 30^\circ}{1+\tan^2 30^\circ} = \dfrac{1}{2}$

(iii) We have,

$$\text{RHS} = \cos^2 30^\circ - \sin^2 30^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

and, LHS $= \cos 60^\circ = \dfrac{1}{2}$

$\therefore$ LHS = RHS i.e, $\cos 60^\circ = \cos^2 30^\circ - \sin^2 30^\circ = \dfrac{1}{2}$

(iv) We have,

$$\text{LHS} = 4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ)$$

$$= 4\left\{\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right\} - 3\left\{\left(\frac{1}{\sqrt{2}}\right)^2 - (1)^2\right\}$$

$$= 4\left(\frac{1}{16}+\frac{1}{16}\right) - 3\left(\frac{1}{2}-1\right) = 4\times\frac{2}{16} - 3\left(-\frac{1}{2}\right) = \frac{1}{2}+\frac{3}{2} = 2 = \text{RHS}$$

LEVEL-2

EXAMPLE 13 *If θ is an acute angle and $\tan\theta + \cot\theta = 2$, find the value of* $\tan^7\theta + \text{c}$

SOLUTION We have,

$$\tan\theta + \cot\theta = 2$$

$\Rightarrow$ $\tan\theta + \dfrac{1}{\tan\theta} = 2$

$\Rightarrow$ $\dfrac{\tan^2\theta + 1}{\tan\theta} = 2$

$\Rightarrow$ $\tan^2\theta - 2\tan\theta + 1 = 0$

$\Rightarrow$ $(\tan\theta - 1)^2 = 0$

$\Rightarrow$ $\tan\theta - 1 = 0 \Rightarrow \tan\theta = 1 \Rightarrow \tan\theta = \tan 45^\circ \Rightarrow \theta = 45$

$\therefore \quad \tan^7\theta + \cot^7\theta = \tan^7 45^\circ + \cot^7 45^\circ = (\tan 45^\circ)^7 + (\cot 45^\circ)^7 = (1)^7 + (1)^7 = 2$

EXAMPLE 14 *Find an acute angle θ, when* $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{1-\sqrt{3}}{1+\sqrt{3}}$

SOLUTION We have,

$$\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

$$\Rightarrow \quad \frac{(\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta) - (\cos\theta + \sin\theta)} = \frac{(1-\sqrt{3}) + (1+\sqrt{3})}{(1-\sqrt{3}) - (1+\sqrt{3})} \qquad \left[\text{Applying componendo and dividendo}\right]$$

$$\Rightarrow \quad \frac{2\cos\theta}{-2\sin\theta} = \frac{2}{-2\sqrt{3}}$$

$$\Rightarrow \quad \cot\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \tan\theta = \tan 60^\circ \Rightarrow \theta = 60^\circ$$

ALITER We have,

$$\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

$$\Rightarrow \quad \frac{\dfrac{\cos\theta - \sin\theta}{\cos\theta}}{\dfrac{\cos\theta + \sin\theta}{\cos\theta}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} \qquad \left[\text{Dividing Numerator and Denominator by } \cos\theta\right]$$

$$\Rightarrow \quad \frac{1-\tan\theta}{1+\tan\theta} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta = \sqrt{3}$$

$$\Rightarrow \quad \tan\theta = \tan 60^\circ \qquad [\text{On comparing two sides}]$$

$$\Rightarrow \quad \theta = 60^\circ$$

EXAMPLE 15 *If* $\sin(A+B) = 1$ *and* $\cos(A-B) = \dfrac{\sqrt{3}}{2}$, $0^\circ < A+B \le 90^\circ$, $A > B$ *then find* A *and* B.

SOLUTION We have,

$$\sin(A+B) = 1$$

$$\Rightarrow \quad \sin(A+B) = \sin 90^\circ$$

$$\Rightarrow \quad A + B = 90^\circ \qquad \text{...(i)}$$

and, $\cos(A-B) = \dfrac{\sqrt{3}}{2}$

$$\Rightarrow \quad \cos(A-B) = \cos 30^\circ$$

$$\Rightarrow \quad A - B = 30^\circ \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$(A+B) + (A-B) = 90^\circ + 30^\circ \Rightarrow 2A = 120^\circ \Rightarrow A = 60^\circ$$

Putting $A = 60°$ in (i), we get

$$60° + B = 90° \Rightarrow B = 30°$$

Hence, $A = 60°$ and $B = 30°$

EXAMPLE 16 *If θ is an acute angle and* $\sin\theta = \cos\theta$, *find the value of* $2\tan^2\theta + \sin^2\theta - 1$.

SOLUTION We have,

$$\sin\theta = \cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta} \quad \text{[Dividing both sides by } \cos\theta]$$

$$\Rightarrow \tan\theta = 1 \Rightarrow \tan\theta = \tan 45° \Rightarrow \theta = 45°$$

$$\therefore \quad 2\tan^2\theta + \sin^2\theta - 1$$

$$= 2\tan^2 45° + \sin^2 45° - 1 = 2(1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 1 = 2 + \frac{1}{2} - 1 = \frac{5}{2} - 1 = \frac{3}{2}$$

EXAMPLE 17 *Given that* $\sin(A+B) = \sin A\cos B + \cos A\sin B$, *find the value of* $\sin 75°$.

SOLUTION Putting $A = 45°$ and $B = 30°$ in $\sin(A + B) = \sin A\cos B + \cos A\sin B$, we get

$$\sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30°$$

$$\Rightarrow \quad \sin 75° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

EXAMPLE 18 *ABC is a right triangle, right angled at C. If $A = 30°$ and $AB = 40$ units, find the remaining two sides and $\angle B$ of ΔABC.*

SOLUTION We have,

$$\angle A + \angle B + \angle C = 180°$$

$$\Rightarrow 30° + \angle B + 90° = 180° \quad [\because \angle A = 30° \text{ and } \angle C = 90°]$$

$$\Rightarrow \angle B = 180° - 120° = 60°$$

Now, $\cos A = \frac{AC}{AB}$

$$\Rightarrow \cos 30° = \frac{AC}{40}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{AC}{40}$$

$$\Rightarrow AC = \frac{\sqrt{3}}{2} \times 40 \Rightarrow AC = 20\sqrt{3} \text{ units}$$

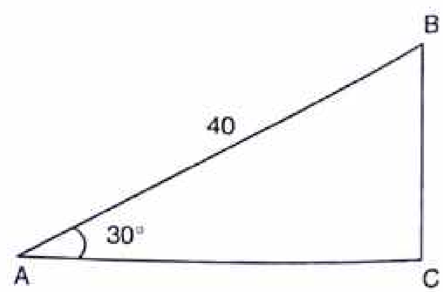


Fig. 10.41

and, $\sin A = \frac{BC}{AB}$

$$\Rightarrow \sin 30° = \frac{BC}{40}$$

$$\Rightarrow \frac{1}{2} = \frac{BC}{40} \Rightarrow BC = 40 \times \frac{1}{2} = 20 \text{ units.}$$

Hence, $AC = 20\sqrt{3}$ units, $BC = 20$ units and $\angle B = 60°$.

EXAMPLE 19 *In a rectangle ABCD, AB = 20 cm, ∠BAC = 60°. Calculate side BC.*

SOLUTION In $\Delta\, ABC$, we have

$$AB = 20,\ \angle BAC = 60^\circ$$

$\therefore$ $$\tan \angle BAC = \frac{BC}{AB}$$

$\Rightarrow$ $$\tan 60^\circ = \frac{BC}{20}$$

$\Rightarrow$ $$\sqrt{3} = \frac{BC}{20}$$

$\Rightarrow$ $$BC = 20\sqrt{3} \text{ cm}$$

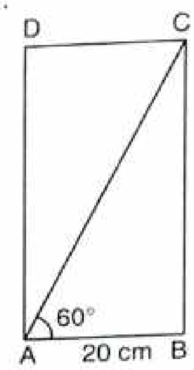


Fig. 10.42

EXAMPLE 20 *A rhombus of side 20 cm has two angles of 60° each. Find the length of the diagonals.*

SOLUTION Let $ABCD$ be a rhombus of side 20 cm and $\angle BAD = \angle BCD = 60^\circ$.
Note that the diagonals of a rhombus are perpendicular bisector of each other and diagonals AC and BD are bisectors of $\angle BAD$ and $\angle ABC$ respectively.
So, $\Delta\, AOB$ is a right triangle such that $\angle BAO = 30^\circ, \angle AOB = 90^\circ$ and $AB = 20$ cm.

$\therefore$ $$\cos \angle BAO = \frac{OA}{AB}$$

$\Rightarrow$ $$\cos 30^\circ = \frac{OA}{20}$$

$\Rightarrow$ $$\frac{\sqrt{3}}{2} = \frac{OA}{20}$$

$\Rightarrow$ $$OA = \frac{\sqrt{3}}{2} \times 20 = 10\sqrt{3}$$

Also, $$\sin \angle BAO = \frac{OB}{AB}$$

$\Rightarrow$ $$\sin 30^\circ = \frac{BO}{20}$$

$\Rightarrow$ $$\frac{1}{2} = \frac{BO}{20}$$

$\Rightarrow$ $$BO = \frac{20}{2} = 10$$

$\therefore$ $$AC = 2\ AO = 2 \times 10\sqrt{3} = 20\sqrt{3} \text{ cm},\ BD = 2\ BO = 2 \times 10 = 20 \text{ cm}$$

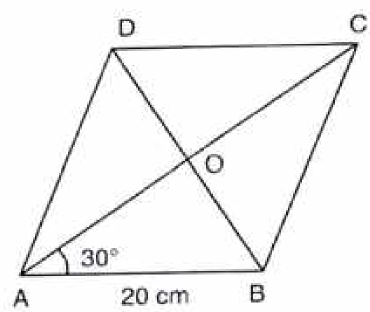


Fig. 10.43

EXAMPLE 21 *The altitude AD of a $\Delta\, ABC$, in which ∠A obtuse and, AD = 10 cm. If BD = 10 cm and CD = $10\sqrt{3}$ cm, determine ∠A.*

SOLUTION $\Delta\, ABC$ is a right triangle, right angled at D such that $AD = 10$ cm and $BD = 10$ cm.

$\therefore$ $$\tan \angle BAD = \frac{BD}{AD}$$

$\Rightarrow$ $$\tan \angle BAD = \frac{10}{10} = 1$$

$\Rightarrow$ $$\tan \angle BAD = \tan 45^\circ$$

$\Rightarrow \quad \angle BAD = 45^\circ$

ΔACD is a right triangle right angled at D such that $AD = 10$ cm and $DC = 10\sqrt{3}$ cm.

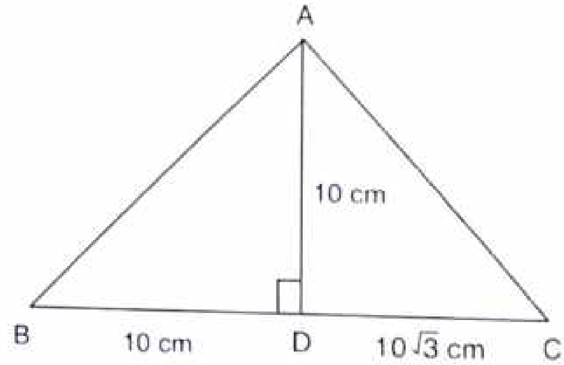


Fig. 10.44

$$\therefore \quad \tan \angle CAD = \frac{CD}{AD}$$

$$\Rightarrow \quad \tan \angle CAD = \frac{10\sqrt{3}}{10} = \sqrt{3}$$

$$\Rightarrow \quad \tan \angle CAD = \tan 60^\circ$$

$$\therefore \quad \angle CAD = 60^\circ$$

From (i) and (ii), we have

$$\angle BAC = \angle BAD + \angle CAD = 45^\circ + 60^\circ = 105^\circ$$

EXAMPLE 22 *An equilateral triangle is inscribed in a circle of radius 6 cm. Find its side.*

SOLUTION Let ABC be an equilateral triangle inscribed in a circle of radius 6 cm. Let O be the centre of the circle. Then,

$$OA = OB = OC = 6 \text{ cm}.$$

Let OD be perpendicular from O on side BC. Then, D is the mid-point of BC and OB and OC are bisectors of $\angle B$ and $\angle C$ respectively.

$$\therefore \quad \angle OBD = 30^\circ$$

In ΔOBD, right angled at D, we have

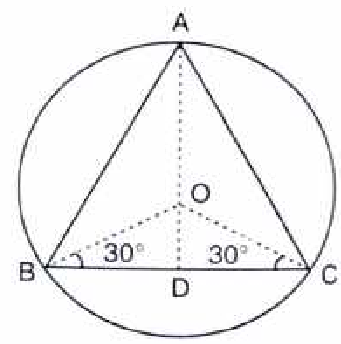


Fig. 10.45

$$\angle OBD = 30^\circ \text{ and } OB = 6 \text{ cm}.$$

$$\therefore \quad \cos \angle OBD = \frac{BD}{OB}$$

$$\Rightarrow \quad \cos 30^\circ = \frac{BD}{6}$$

$$\Rightarrow \quad BD = 6 \cos 30^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3} \text{ cm}$$

$$\Rightarrow \quad BC = 2\,BD = 2\,(3\sqrt{3}) = 6\sqrt{3} \text{ cm}$$

Hence, the side of the equilateral triangle is $6\sqrt{3}$ cm.

EXAMPLE 23 *If each of α, β and γ is a positive acute angle such that*

$$\sin(\alpha + \beta - \gamma) = \frac{1}{2}, \cos(\beta + \gamma - \alpha) = \frac{1}{2} \text{ and } \tan(\gamma + \alpha - \beta) = 1,$$

find the values of α, β and γ.

SOLUTION We have,

$$\sin(\alpha + \beta - \gamma) = \frac{1}{2}, \cos(\beta + \gamma - \alpha) = \frac{1}{2} \text{ and } \tan(\gamma + \alpha - \beta) = 1,$$

$$\Rightarrow \quad \sin(\alpha + \beta - \gamma) = \sin 30^\circ, \cos(\beta + \gamma - \alpha) = \cos 60^\circ \text{ and } \tan(\gamma + \alpha - \beta) = \tan 45^\circ$$

$$\Rightarrow \quad \alpha + \beta - \gamma = 30^\circ \quad \text{...(i)}$$

$$\beta + \gamma - \alpha = 60^\circ \quad \text{...(ii)}$$

$$\gamma + \alpha - \beta = 45^\circ \quad \text{...(iii)}$$

Adding (i) to (ii) and (iii) respectively, we get

$$2\beta = 90^\circ \text{ and } 2\alpha = 75^\circ \Rightarrow \beta = 45^\circ \text{ and } \alpha = 37\frac{1}{2}^\circ$$

Putting the values of α and β in (i), we get,

$$37\frac{1}{2}^\circ + 45 - \gamma = 30^\circ \Rightarrow \gamma = 52\frac{1}{2}^\circ$$

Hence, $\alpha = 37\frac{1}{2}^\circ$, $\beta = 45^\circ$ and $\gamma = 52\frac{1}{2}^\circ$

EXAMPLE 24 *In an acute angled triangle ABC, if* $\tan(A + B - C) = 1$ *and,* $\sec(B + C - A) = 2$, *find the value of A, B and C.*

SOLUTION We have,

$$\tan(A + B - C) = 1 \text{ and } \sec(B + C - A) = 2$$

$$\Rightarrow \tan(A + B - C) = \tan 45^\circ \text{ and } \sec(B + C - A) = \sec 60^\circ$$

$$\Rightarrow A + B - C = 45^\circ \text{ and } B + C - A = 60^\circ$$

$$\Rightarrow (A + B - C) + (B + C - A) = 45^\circ + 60^\circ$$

$$\Rightarrow 2B = 105^\circ$$

$$\Rightarrow B = 52\frac{1}{2}^\circ$$

Putting $B = 52\frac{1}{2}^\circ$ in $B + C - A = 60^\circ$, we get

$$52\frac{1}{2}^\circ + C - A = 60^\circ \Rightarrow C - A = 7\frac{1}{2}^\circ \quad ...(i)$$

Also, in ΔABC, we have

$$A + B + C = 180^\circ$$

$$\Rightarrow A + 52\frac{1}{2}^\circ + C = 180^\circ \quad \left[\because B = 52\frac{1}{2}^\circ\right]$$

$$\Rightarrow C + A = 127\frac{1}{2}^\circ \quad ...(ii)$$

Adding and subtracting (i) and (ii), we get

$$2C = 135^\circ \text{ and } 2A = 120^\circ \Rightarrow C = 67\frac{1}{2}^\circ \text{ and } A = 60^\circ$$

Hence, $A = 60^\circ$, $B = 52\frac{1}{2}^\circ$ and $C = 67\frac{1}{2}^\circ$.

EXERCISE 10.2

LEVEL-1

Evaluate each of the following (1-19):

1. $\sin 45^\circ \sin 30^\circ + \cos 45^\circ \cos 30^\circ$
2. $\sin 60 \cos 30^\circ + \cos 60^\circ \sin 30^\circ$
3. $\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$
4. $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ + \sin^2 90^\circ$
5. $\cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 90^\circ$
6. $\tan^2 30^\circ + \tan^2 60^\circ + \tan^2 45^\circ$
7. $2\sin^2 30^\circ - 3\cos^2 45^\circ + \tan^2 60^\circ$
8. $\sin^2 30^\circ \cos^2 45^\circ + 4\tan^2 30^\circ + \frac{1}{2}\sin^2 90^\circ - 2\cos^2 90^\circ + \frac{1}{24}\cos^2 0^\circ$

9. $4(\sin^4 60° + \cos^4 30°) - 3(\tan^2 60° - \tan^2 45°) + 5\cos^2 45°$

10. $(\operatorname{cosec}^2 45° \sec^2 30°)(\sin^2 30° + 4\cot^2 45° - \sec^2 60°)$

11. $\operatorname{cosec}^3 30° \cos 60° \tan^3 45° \sin^2 90° \sec^2 45° \cot 30°$

12. $\cot^2 30° - 2\cos^2 60° - \frac{3}{4}\sec^2 45° - 4\sec^2 30°$

13. $(\cos 0° + \sin 45° + \sin 30°)(\sin 90° - \cos 45° + \cos 60°)$

14. $\dfrac{\sin 30° - \sin 90° + 2\cos 0°}{\tan 30° \tan 60°}$

15. $\dfrac{4}{\cot^2 30°} + \dfrac{1}{\sin^2 60°} - \cos^2 45°$

16. $4(\sin^4 30° + \cos^2 60°) - 3(\cos^2 45° - \sin^2 90°) - \sin^2 60°$

17. $\dfrac{\tan^2 60° + 4\cos^2 45° + 3\sec^2 30° + 5\cos^2 90°}{\operatorname{cosec} 30° + \sec 60° - \cot^2 30°}$

18. $\dfrac{\sin 30°}{\sin 45°} + \dfrac{\tan 45°}{\sec 60°} - \dfrac{\sin 60°}{\cot 45°} - \dfrac{\cos 30°}{\sin 90°}$

19. $\dfrac{\tan 45°}{\operatorname{cosec} 30°} + \dfrac{\sec 60°}{\cot 45°} - \dfrac{5\sin 90°}{2\cos 0°}$

Find the value of x in each of the following: (20-25)

20. $2\sin 3x = \sqrt{3}$

21. $2\sin\frac{x}{2} = 1$

22. $\sqrt{3}\sin x = \cos x$

23. $\tan x = \sin 45° \cos 45° + \sin 30°$

24. $\sqrt{3}\tan 2x = \cos 60° + \sin 45° \cos 45°$

25. $\cos 2x = \cos 60° \cos 30° + \sin 60° \sin 30°$

26. If $\theta = 30°$, verify that:

(i) $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

(ii) $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

(iii) $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

(iii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

27. If $A = B = 60°$, verify that

(i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(iii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

28. If $A = 30°$ and $B = 60°$, verify that

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

29. If $\sin(A+B) = 1$ and $\cos(A-B) = 1$, $0^\circ < A+B \le 90^\circ$, $A \ge B$ find A and B.

30. If $\tan(A-B) = \dfrac{1}{\sqrt{3}}$ and $\tan(A+B) = \sqrt{3}$, $0^\circ < A+B \le 90^\circ$, $A > B$ find A and B. [NCERT]

31. If $\sin(A-B) = \dfrac{1}{2}$ and $\cos(A+B) = \dfrac{1}{2}$, $0^\circ < A+B \le 90^\circ$, $A < B$ find A and B. [NCERT]

32. In a ΔABC right angled at B, $\angle A = \angle C$. Find the values of
    (i) $\sin A \ \cos C + \cos A \ \sin C$ (ii) $\sin A \ \sin B + \cos A \ \cos B$

33. Find acute angles A and B, if $\sin(A+2B) = \dfrac{\sqrt{3}}{2}$ and $\cos(A+4B) = 0$, $A > B$.

34. In ΔPQR, right-angled at Q, $PQ = 3\,\text{cm}$ and $PR = 6\,\text{cm}$. Determine $\angle P$ and $\angle R$. [NCERT]

LEVEL-2

35. If $\sin(A-B) = \sin A\cos B - \cos A\sin B$ and $\cos(A-B) = \cos A\cos B + \sin A\sin B$, find the values of $\sin 15^\circ$ and $\cos 15^\circ$.

36. In a right triangle ABC, right angled at C, if $\angle B = 60^\circ$ and $AB = 15$ units. Find the remaining angles and sides.

37. If ΔABC is a right triangle such that $\angle C = 90^\circ$, $\angle A = 45^\circ$ and $BC = 7$ units. Find $\angle B$, AB and AC.

38. In a rectangle $ABCD$, $AB = 20\,\text{cm}$, $\angle BAC = 60^\circ$, calculate side BC and diagonals AC and BD.

39. If A and B are acute angles such that $\tan A = \dfrac{1}{2}$, $\tan B = \dfrac{1}{3}$ and

    $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \ \tan B}$, find $A+B$.

40. Prove that $(\sqrt{3}+1)(3 - \cot 30^\circ) = \tan^3 60^\circ - 2\sin 60^\circ$. [NCERT EXEMPLAR]

ANSWERS

1. $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$	2. 1	3. $\dfrac{1-\sqrt{3}}{2\sqrt{2}}$	4. $\dfrac{5}{2}$	5. $\dfrac{3}{2}$
6. $\dfrac{13}{3}$	7. 2	8. 2	9. 1	10. $\dfrac{2}{3}$
11. $8\sqrt{3}$	12. $\dfrac{-13}{3}$	13. $\dfrac{7}{4}$	14. $\dfrac{3}{2}$	15. $\dfrac{13}{6}$
16. 2	17. 9	18. $\dfrac{\sqrt{2}+1-2\sqrt{3}}{2}$	19. 0	20. 20°
21. 60°	22. 30°	23. 45°	24. 15°	25. 15°
29. $A = B = 45^\circ$	30. $A = 45^\circ$, $B = 15^\circ$	31. $A = 45^\circ$, $B = 115^\circ$	32. (i) 1,	(ii) $\dfrac{1}{\sqrt{2}}$

33. $A = 30^\circ, B = 15^\circ$ 34. $\angle P = 60^\circ, \angle R = 30^\circ$ 35. $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

36. $\angle A = 30^\circ$, $BC = 7.5$ units and $AC = \frac{15}{2}\sqrt{3}$ units

37. $\angle B = 45^\circ$, $AC = 7$, $AB = 7\sqrt{2}$ 38. $BC = 20\sqrt{3}$ cm, $AC = 40$ cm, $BD = 40$ cm 39. 45°

10.6 TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

In this section, we will obtain the trigonometric ratios of complementary angles in terms of the trigonometric ratios of the given angles.

COMPLEMENTARY ANGLE *Two angles are said to be complementary, if their sum is 90°.*

It follows from the above definition that θ and $(90^\circ - \theta)$ are complementary angles for an acute angle θ.

THEOREM *If θ is an acute angle, then prove that*

$\sin(90^\circ - \theta) = \cos\theta$, $\cos(90^\circ - \theta) = \sin\theta$,

$\tan(90^\circ - \theta) = \cot\theta$, $\cot(90^\circ - \theta) = \tan\theta$,

$\sec(90^\circ - \theta) = \text{cosec}\,\theta$ *and,* $\text{cosec}(90^\circ - \theta) = \sec\theta$

PROOF Consider a right triangle OPM, right angled at M as shown in Fig. 10.46.

Let $MOP = \theta$, then $\angle OPM = (90^\circ - \theta)$

For the reference angle θ, we have

$$\left.\begin{array}{l} \sin\theta = \dfrac{PM}{OP}, \cos\theta = \dfrac{OM}{OP}, \tan\theta = \dfrac{PM}{OM}, \\ \text{cosec}\,\theta = \dfrac{OP}{PM}, \cot\theta = \dfrac{OM}{PM} \text{ and, } \sec\theta = \dfrac{OP}{OM} \end{array}\right\} \quad ...(i)$$

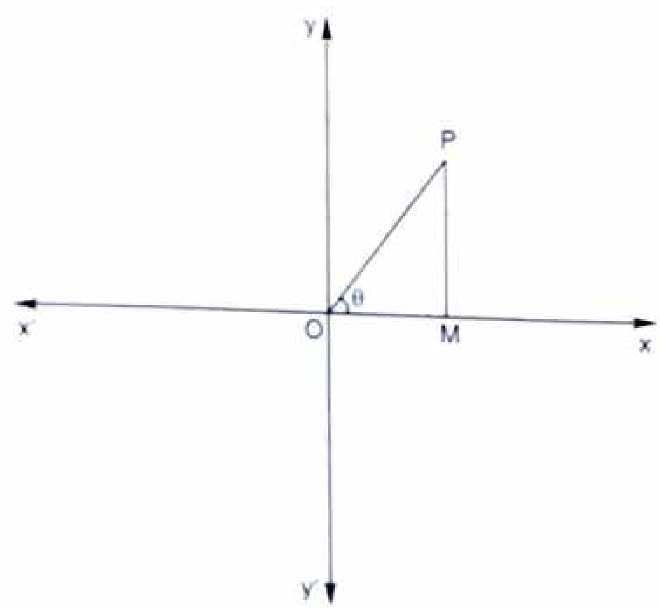


Fig. 10.46

For the reference angle $\angle OPM = (90^\circ - \theta)$, we have

Base = PM, Perpendicular = OM and, Hypotenuse = OP

$$\left.\begin{aligned} &\sin(90^\circ-\theta)=\frac{OM}{OP},\ \cos(90^\circ-\theta)=\frac{PM}{OP},\ \tan(90^\circ-\theta)=\frac{OM}{PM} \\ \therefore\ &\operatorname{cosec}(90^\circ-\theta)=\frac{OP}{OM},\ \sec(90^\circ-\theta)=\frac{OP}{PM} \text{ and, } \cot(90^\circ-\theta)=\frac{PM}{OM}\end{aligned}\right\} \quad \ldots(ii)$$

From (i) and (ii), we obtain

$$\sin(90^\circ-\theta)=\cos\theta,\ \cos(90^\circ-\theta)=\sin\theta,\ \tan(90^\circ-\theta)=\cot\theta$$

$$\operatorname{cosec}(90^\circ-\theta)=\sec\theta,\ \sec(90^\circ-\theta)=\operatorname{cosec}\theta \text{ and } \cot(90^\circ-\theta)=\tan\theta$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON EVALUATING EXPRESSIONS INVOLVING TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

EXAMPLE 1 *Evaluate the following:*

(i) $\frac{\cos 37^\circ}{\sin 53^\circ}$ (ii) $\frac{\sin 41^\circ}{\cos 49^\circ}$ (iii) $\frac{\tan 54^\circ}{\cot 36^\circ}$ (iv) $\frac{\operatorname{cosec} 32^\circ}{\sec 58^\circ}$

SOLUTION (i) We have,

$$\frac{\cos 37^\circ}{\sin 53^\circ}=\frac{\cos(90^\circ-53^\circ)}{\sin 53^\circ}=\frac{\sin 53^\circ}{\sin 53^\circ}=1 \quad [\because \cos(90^\circ-\theta)=\sin\theta]$$

(ii) We have,

$$\frac{\sin 41^\circ}{\cos 49^\circ}=\frac{\sin(90^\circ-49^\circ)}{\cos 49^\circ}=\frac{\cos 49^\circ}{\cos 49^\circ}=1 \quad [\because \sin(90^\circ-\theta)=\cos\theta]$$

(iii) We have,

$$\frac{\tan 54^\circ}{\cot 36^\circ}=\frac{\tan(90^\circ-36^\circ)}{\cot 36^\circ}=\frac{\cot 36^\circ}{\cot 36^\circ}=1 \quad [\because \tan(90^\circ-\theta)=\cot\theta]$$

(iv) We have,

$$\frac{\operatorname{cosec} 32^\circ}{\sec 58^\circ}=\frac{\operatorname{cosec}(90^\circ-58^\circ)}{\sec 58^\circ}=\frac{\sec 58^\circ}{\sec 58^\circ}=1 \quad [\because \operatorname{cosec}(90^\circ-\theta)=\sec\theta]$$

EXAMPLE 2 *Evaluate the following:*

(i) $\sin 39^\circ-\cos 51^\circ$ (ii) $\operatorname{cosec} 25^\circ-\sec 65^\circ$ (iii) $\cot 34^\circ-\tan 56^\circ$

(iv) $\frac{\sin 36^\circ}{\cos 54^\circ}-\frac{\sin 54^\circ}{\cos 36^\circ}$ (v) $\cos^2 13^\circ-\sin^2 77^\circ$

SOLUTION (i) We have,

$$\begin{aligned}\sin 39^\circ-\cos 51^\circ &= \sin(90^\circ-51^\circ)-\cos 51^\circ \\ &= \cos 51^\circ-\cos 51^\circ=0 \quad [\because \sin(90^\circ-\theta)=\cos\theta]\end{aligned}$$

(ii) We have,

$$\begin{aligned}\operatorname{cosec} 25^\circ-\sec 65^\circ &= \operatorname{cosec}(90^\circ-65^\circ)-\sec 65^\circ \\ &= \sec 65^\circ-\sec 65^\circ=0 \quad [\because \operatorname{cosec}(90^\circ-\theta)=\sec\theta]\end{aligned}$$

(iii) We have,

$$\begin{aligned}\cot 34^\circ-\tan 56^\circ &= \cot(90^\circ-56^\circ)-\tan 56^\circ \\ &= \tan 56^\circ-\tan 56^\circ=0 \quad [\because \cot(90^\circ-\theta)=\tan\theta]\end{aligned}$$

(iv) We have,

$$\frac{\sin 36^\circ}{\cos 54^\circ} - \frac{\sin 54^\circ}{\cos 36^\circ} = \frac{\sin (90^\circ - 54^\circ)}{\cos 54^\circ} - \frac{\sin (90^\circ - 36^\circ)}{\cos 36^\circ}$$

$$= \frac{\cos 54^\circ}{\cos 54^\circ} - \frac{\cos 36^\circ}{\cos 36^\circ} = 1 - 1 = 0 \qquad \left[\begin{array}{l} \because \sin(90^\circ - \theta) = \cos\theta, \\ \cos(90^\circ - \theta) = \sin\theta \end{array}\right]$$

(v) We have,

$$\cos^2 13^\circ - \sin^2 77^\circ = \cos^2 (90^\circ - 77^\circ) - \sin^2 77^\circ$$

$$= \sin^2 77^\circ - \sin^2 77^\circ = 0 \qquad [\because \cos(90^\circ - \theta) = \sin\theta]$$

EXAMPLE 3 *Evaluate the following:*

(i) $\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ$

(ii) $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2$

(iii) $\frac{2\tan 53^\circ}{\cot 37^\circ} - \frac{\cot 80^\circ}{\tan 10^\circ}$

(iv) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

(v) $\sec 70^\circ \sin 20^\circ - \cos 20^\circ \operatorname{cosec} 70^\circ$

SOLUTION (i) We have,

$$\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ$$

$$= \frac{\cos (90^\circ - 10^\circ)}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} (90^\circ - 59^\circ)$$

$$= \frac{\sin 10^\circ}{\sin 10^\circ} + \cos 59^\circ \sec 59^\circ = \frac{\sin 10^\circ}{\sin 10^\circ} + \frac{\cos 59^\circ}{\cos 59^\circ} = 1 + 1 = 2$$

(ii) We have,

$$\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2$$

$$= \frac{\cot (90^\circ - 36^\circ)}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot (90^\circ - 20^\circ)} - 2 = \frac{\tan 36^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\tan 20^\circ} - 2 = 1 + 1 - 2 = 0$$

(iii) We have,

$$\frac{2\tan 53^\circ}{\cot 37^\circ} - \frac{\cot 80^\circ}{\tan 10^\circ}$$

$$= \frac{2\tan 53^\circ}{\tan (90^\circ - 37^\circ)} - \frac{\cot (90^\circ - 10^\circ)}{\tan 10^\circ} = \frac{2\tan 53^\circ}{\tan 53^\circ} - \frac{\tan 10^\circ}{\tan 10^\circ} = 2 - 1 = 1$$

(iv) We have,

$$\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$$

$$= \sec (90^\circ - 40^\circ) \sin 40^\circ + \cos 40^\circ \operatorname{cosec} (90^\circ - 40^\circ)$$

$$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \sec 40^\circ = \frac{\sin 40^\circ}{\sin 40^\circ} + \frac{\cos 40^\circ}{\cos 40^\circ} = 1 + 1 = 2$$

(v) We have,

$$\sec 70^\circ \sin 20^\circ - \cos 20^\circ \operatorname{cosec} 70^\circ$$

$= \sec(90° - 20°) \sin 20° - \cos 20° \operatorname{cosec}(90° - 20°)$

$= \operatorname{cosec} 20° \sin 20° - \cos 20° \sec 20°$ $\left[\begin{array}{l}\because \operatorname{cosec}(90° - \theta) = \sec\theta \\ \text{and, } \sec(90° - \theta) = \operatorname{cosec}\theta\end{array}\right]$

$= \dfrac{\sin 20°}{\sin 20°} - \dfrac{\cos 20°}{\cos 20°} = 1 - 1 = 0$

EXAMPLE 4 *Prove that:*

(i) $\sin 35° \sin 55° - \cos 35° \cos 55° = 0$ (ii) $\dfrac{\cos 70°}{\sin 20°} + \dfrac{\cos 59°}{\sin 31°} - 8 \sin^2 30° = 0$

SOLUTION (i) We have,

$\text{LHS} = \sin 35° \sin 55° - \cos 35° \cos 55°$

$\Rightarrow \text{LHS} = \sin(90° - 55°) \sin(90° - 35°) - \cos 35° \cos 55°$

$\Rightarrow \text{LHS} = \cos 55° \cos 35° - \cos 35° \cos 55°$ $[\because \sin(90° - \theta) = \cos\theta]$

$\Rightarrow \text{LHS} = 0 = \text{RHS}$

(ii) We have,

$\text{LHS} = \dfrac{\cos 70°}{\sin 20°} + \dfrac{\cos 59°}{\sin 31°} - 8\sin^2 30°$

$\Rightarrow \text{LHS} = \dfrac{\cos(90° - 20°)}{\sin 20°} + \dfrac{\cos(90° - 31°)}{\sin 31°} - 8\sin^2 30°$

$\Rightarrow \text{LHS} = \dfrac{\sin 20°}{\sin 20°} + \dfrac{\sin 31°}{\sin 31°} - 8\sin^2 30°$ $[\because \cos(90° - \theta) = \sin\theta]$

$\Rightarrow = 1 + 1 - 8\left(\dfrac{1}{2}\right)^2 = 2 - 2 = 0 = \text{RHS}$ $\left[\because \sin 30° = \dfrac{1}{2}\right]$

EXAMPLE 5 *Express each of the following in terms of trigonometric ratios of angles between 0° and 45°:*

(i) $\sin 85° + \operatorname{cosec} 85°$ (ii) $\tan 68° + \sec 68°$ (iii) $\operatorname{cosec} 69° + \cot 69°$

(iv) $\sin 81° + \tan 81°$ (v) $\sin 72° + \cot 72°$

SOLUTION (i) We have,

$\sin 85° + \operatorname{cosec} 85°$

$= \sin(90° - 5°) + \operatorname{cosec}(90° - 5°)$

$= \cos 5° + \sec 5°$ $[\because \sin(90° - \theta) = \cos\theta \text{ and } \operatorname{cosec}(90° - \theta) = \sec\theta]$

(ii) We have,

$\tan 68° + \sec 68°$

$= \tan(90° - 22°) + \sec(90° - 22°)$

$= \cot 22° + \operatorname{cosec} 22°$ $[\because \tan(90 - \theta) = \cot\theta, \sec(90 - \theta) = \operatorname{cosec}\theta]$

(iii) We have,

$\operatorname{cosec} 69° + \cot 69°$

$= \operatorname{cosec}(90° - 21°) + \cot(90° - 21°)$

$= \sec 21° + \tan 21°$ $[\because \operatorname{cosec}(90° - \theta) = \sec\theta \text{ and } \cot(90° - \theta) = \tan\theta]$

(iv) We have,

$$\sin 81^\circ + \tan 81^\circ$$

$$= \sin (90^\circ - 9) + \tan (90^\circ - 9^\circ)$$

$$= \cos 9^\circ + \cot 9^\circ \qquad [\because \sin (90^\circ - \theta^\circ) = \cos \theta \text{ and, } \tan (90^\circ - 18^\circ) = \cot 90^\circ]$$

(v) We have,

$$\sin 72^\circ + \cot 72^\circ$$

$$= \sin (90^\circ - 18^\circ) + \cot (90^\circ - 18^\circ)$$

$$= \cos 18^\circ + \tan 18^\circ \qquad [\because \sin (90^\circ - 18^\circ) = \cos 18^\circ \text{and, } \tan (90^\circ - 18^\circ) = \cot 18^\circ]$$

EXAMPLE 6 *Evaluate the following:*

(i) $\left(\dfrac{\sin 35^\circ}{\cos 55^\circ}\right)^2 + \left(\dfrac{\cos 55^\circ}{\sin 35^\circ}\right)^2 - 2\cos 60^\circ$ (ii) $\dfrac{2\cos 67^\circ}{\sin 23^\circ} - \dfrac{\tan 40^\circ}{\cot 50^\circ} - \cos 0^\circ$

(iii) $\left(\dfrac{\sin 47^\circ}{\cos 43^\circ}\right)^2 + \left(\dfrac{\cos 43^\circ}{\sin 47^\circ}\right)^2 - 4\cos^2 45^\circ$

SOLUTION (i) We have,

$$\left\{\frac{\sin 35^\circ}{\cos 55^\circ}\right\}^2 + \left\{\frac{\cos 55^\circ}{\sin 35^\circ}\right\}^2 - 2\cos 60^\circ$$

$$= \left\{\frac{\sin (90^\circ - 55^\circ)}{\cos 55^\circ}\right\}^2 + \left\{\frac{\cos (90^\circ - 35^\circ)}{\sin 35^\circ}\right\}^2 - 2\cos 60^\circ$$

$$= \left(\frac{\cos 55^\circ}{\cos 55^\circ}\right)^2 + \left(\frac{\sin 35^\circ}{\sin 35^\circ}\right)^2 - 2\cos 60^\circ \qquad \left[\begin{array}{l}\because \cos (90^\circ - \theta) = \sin \theta \\ \text{and, } \sin (90^\circ - \theta) = \cos \theta\end{array}\right]$$

$$= 1^2 + 1^2 - 2 \times \frac{1}{2} = 1 \qquad \left[\because \cos 60^\circ = \frac{1}{2}\right]$$

(ii) We have,

$$\frac{2\cos 67^\circ}{\sin 23^\circ} - \frac{\tan 40^\circ}{\cot 50^\circ} - \cos 0^\circ$$

$$= \frac{2\cos (90^\circ - 23^\circ)}{\sin 23^\circ} - \frac{\tan (90^\circ - 50^\circ)}{\cot 50^\circ} - \cos 0^\circ$$

$$= \frac{2\sin 23^\circ}{\sin 23^\circ} - \frac{\cot 50^\circ}{\cot 50^\circ} - \cos 0^\circ \qquad \left[\begin{array}{l}\because \cos (90^\circ - \theta) = \sin \theta \\ \text{and, } \tan (90^\circ - \theta) = \cot \theta\end{array}\right]$$

$$= 2 \times 1 - 1 - 1 = 0 \qquad [\because \cos 0^\circ = 1]$$

(iii) We have,

$$\left\{\frac{\sin 47^\circ}{\cos 43^\circ}\right\}^2 + \left\{\frac{\cos 43^\circ}{\sin 47^\circ}\right\}^2 - 4\cos^2 45^\circ$$

$$= \left\{\frac{\sin (90^\circ - 45^\circ)}{\cos 43^\circ}\right\}^2 + \left\{\frac{\cos (90^\circ - 47^\circ)}{\sin 47^\circ}\right\}^2 - 4\cos^2 45^\circ$$

$$= \left(\frac{\cos 43^\circ}{\cos 43^\circ}\right)^2 + \left(\frac{\sin 47^\circ}{\sin 47^\circ}\right)^2 - 4\cos^2 45^\circ \qquad \left[\begin{array}{l}\because \sin(90^\circ - \theta) = \cos\theta \\ \text{and, } \cos(90^\circ - \theta) = \sin\theta\end{array}\right]$$

$$= 1^2 + 1^2 - 4\left(\frac{1}{\sqrt{2}}\right)^2 = 0 \qquad \left[\because \cos 45^\circ = \frac{1}{\sqrt{2}}\right]$$

EXAMPLE 7 *Evaluate each of the following:*

(i) $\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ$ **[CBSE 2001 C]**

(ii) $\tan 5^\circ \tan 25^\circ \tan 30^\circ \tan 65^\circ \tan 85^\circ$ **[CBSE 2001 C]**

SOLUTION (i) Grouping terms in such a way that the angles involved are complementary i.e., their sum is 90°, we have

$$\cot 12^\circ \cot 38^\circ \cot 52^\circ \cot 60^\circ \cot 78^\circ$$

$$= (\cot 12^\circ \cot 78^\circ)(\cot 38^\circ \cot 52^\circ)\cot 60^\circ$$

$$= (\cot 12^\circ \tan 12^\circ)(\cot 38^\circ \tan 38^\circ)(\cot 60^\circ)$$

$$[\because \cot 78^\circ = \cot(90^\circ - 12^\circ) = \tan 12^\circ \; \cot 52^\circ = \cot(50^\circ - 38^\circ) = \tan 38^\circ]$$

$$= 1 \times 1 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

(ii) We have,

$$\tan 5^\circ \tan 25^\circ \tan 30^\circ \tan 65^\circ \tan 85^\circ$$

$$= (\tan 5^\circ \tan 85^\circ)(\tan 25^\circ \tan 65^\circ)\tan 30^\circ$$

$$= (\tan 5^\circ \cot 5^\circ)(\tan 25^\circ \cot 25^\circ)\tan 30^\circ$$

$$= 1 \times 1 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \left[\begin{array}{l}\because \tan 85^\circ = \tan(90^\circ - 5^\circ) = \cot 5^\circ \\ \quad \tan 65^\circ = \tan(90^\circ - 25^\circ) = \cot 25^\circ\end{array}\right]$$

EXAMPLE 8 *If A, B, C are the interior angles of a triangle ABC, prove that:* $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}$.

SOLUTION In ΔABC, we have

$$A + B + C = 180^\circ$$

$$\Rightarrow \quad B + C = 180^\circ - A$$

$$\Rightarrow \quad \frac{B+C}{2} = 90^\circ - \frac{A}{2}$$

$$\Rightarrow \quad \tan\left(\frac{B+C}{2}\right) = \tan\left(90^\circ - \frac{A}{2}\right) \Rightarrow \tan\left(\frac{B+C}{2}\right) = \cot\frac{A}{2}$$

Type II **ON FINDING THE UNKNOWN VARIABLE**

EXAMPLE 9 *Find* θ, *if* $\sin(\theta + 36^\circ) = \cos\theta$, *where* $\theta + 36^\circ$ *is an acute angle*

SOLUTION We have,

$$\sin(\theta + 36^\circ) = \cos\theta$$

$$\Rightarrow \quad \cos\{90^\circ - (\theta + 36^\circ)\} = \cos\theta$$

$$\Rightarrow \quad 90^\circ - (\theta + 36^\circ) = \theta \Rightarrow 2\theta = 54^\circ \Rightarrow \theta = 27^\circ$$

EXAMPLE 10 *If* $\tan 2\theta = \cot(\theta + 6^\circ)$, *where* 2θ *and* $\theta + 6^\circ$ *are acute angles, find the value of* θ.

SOLUTION We have,

$$\tan 2\theta = \cot(\theta + 6^\circ)$$

$\Rightarrow$ $\cot(90^\circ - 2\theta) = \cot(\theta + 6^\circ)$

$\Rightarrow$ $90^\circ - 2\theta = \theta + 6^\circ \Rightarrow 3\theta = 84^\circ \Rightarrow \theta = 28^\circ$

EXAMPLE 11 *If* $\sin 5\theta = \cos 4\theta$, *where* 5θ *and* 4θ *are acute angles, find the value of* θ.

SOLUTION We have,

$$\sin 5\theta = \cos 4\theta$$

$\Rightarrow$ $\sin 5\theta = \sin(90^\circ - 4\theta)$

$\Rightarrow$ $5\theta = 90^\circ - 4\theta$

$\Rightarrow$ $9\theta = 90^\circ$

$\Rightarrow$ $\theta = 10^\circ$

EXAMPLE 12 *If* $\tan 2A = \cot(A - 18^\circ)$, *where* $2A$ *is an acute angle, find the value of* A.

SOLUTION We have, [NCERT]

$$\tan 2A = \cot(A - 18^\circ)$$

$\Rightarrow$ $\tan 2A = \tan\{90^\circ - (A - 18^\circ)\}$

$\Rightarrow$ $\tan 2A = \tan(108^\circ - A)$

$\Rightarrow$ $2A = 108^\circ - A \Rightarrow 3A = 108^\circ \Rightarrow A = 36^\circ$

EXAMPLE 13 *If* $\tan A = \cot B$, *Prove that* $A + B = 90^\circ$ [NCERT]

SOLUTION We have,

$$\tan A = \cot B$$

$\Rightarrow$ $\tan A = \tan(90^\circ - B) \Rightarrow A = 90^\circ - B \Rightarrow A + B = 90^\circ$

EXAMPLE 14 *If* $\sec 5A = \operatorname{cosec}(A + 36^\circ)$, *where* $5A$ *is an acute angle, find the value of* A

SOLUTION We have,

$$\sec 5A = \operatorname{cosec}(A + 36^\circ)$$

$\Rightarrow$ $\sec 5A = \sec(90^\circ - (A + 36^\circ))$

$\Rightarrow$ $\sec 5A = \sec(54^\circ - A)$

$\Rightarrow$ $5A = 54^\circ - A \Rightarrow 6A = 54^\circ \Rightarrow A = 9^\circ$

EXAMPLE 15 *If* $\sec 4A = \operatorname{cosec}(A - 20^\circ)$, *where* $4A$ *is an acute angle, find the value of* A.

SOLUTION We have, [NCERT, CBSE 2008]

$$\sec 4A = \operatorname{cosec}(A - 20^\circ)$$

$\Rightarrow$ $\sec 4A = \sec\{90^\circ - (A - 20^\circ)\}$

$\Rightarrow$ $\sec 4A = \sec(110^\circ - A)$

$\Rightarrow$ $4A = 110^\circ - A \Rightarrow 5A = 110^\circ \Rightarrow A = 22^\circ$

LEVEL-2

EXAMPLE 16 *Evaluate each of the following:*

(i) $\cos(40^\circ - \theta) - \sin(50^\circ + \theta) + \dfrac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$ [CBSE 2002]

(ii) $\dfrac{\cos 70^\circ}{\sin 20^\circ} + \dfrac{\cos 55^\circ \operatorname{cosec} 35^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ}$ [CBSE 2002]

(iii) $2\left(\dfrac{\cos 58^\circ}{\sin 32^\circ}\right) - \sqrt{3}\left(\dfrac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 15^\circ \tan 60^\circ \tan 75^\circ}\right)$ [CBSE 2008]

SOLUTION (i) We have,

$$\cos(40^\circ - \theta) - \sin(50^\circ + \theta) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$$

$$= \sin\{90^\circ - (40^\circ - \theta)\} - \sin(50^\circ + \theta) + \frac{\cos^2 40^\circ + \cos^2(90^\circ - 40^\circ)}{\sin^2 40^\circ + \sin^2(90^\circ - 40^\circ)}$$

$$[\because \cos\alpha = \sin(90^\circ - \alpha)]$$

$$= \sin(50^\circ + \theta) - \sin(50^\circ + \theta) + \frac{\cos^2 40^\circ + \sin^2 40^\circ}{\sin^2 40^\circ + \cos^2 40^\circ} = 0 + 1 = 1$$

(ii) We have,

$$\frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 55^\circ \operatorname{cosec} 35^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ}$$

$$= \frac{\cos(90^\circ - 20^\circ)}{\sin 20^\circ} + \frac{\cos 55^\circ \operatorname{cosec}(90^\circ - 55^\circ)}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan(90^\circ - 25^\circ)\tan(90^\circ - 5^\circ)}$$

$$= \frac{\sin 20^\circ}{\sin 20^\circ} + \frac{\cos 55^\circ \sec 55^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \cot 25^\circ \cot 5^\circ}$$

$$= 1 + \frac{1}{\tan 45^\circ} = 1 + 1 = 2 \qquad [\because \tan 5^\circ \cot 5^\circ = 1 \text{ and } \tan 25^\circ \cot 25^\circ = 1$$

(iii) We have,

$$2\left(\frac{\cos 58^\circ}{\sin 32^\circ}\right) - \sqrt{3}\left(\frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 15^\circ \tan 60^\circ \tan 75^\circ}\right)$$

$$= 2\left\{\frac{\cos(90^\circ - 32^\circ)}{\sin 32^\circ}\right\} - \sqrt{3}\left\{\frac{\cos 38^\circ \operatorname{cosec}(90^\circ - 38^\circ)}{\tan 15^\circ \tan 60^\circ \tan(90^\circ - 15^\circ)}\right\}$$

$$= 2\left(\frac{\sin 32^\circ}{\sin 32^\circ}\right) - \sqrt{3}\left\{\frac{\cos 38^\circ \sec 38^\circ}{\tan 15^\circ \times \sqrt{3} \times \cot 15^\circ}\right\}$$

$$= 2 - \sqrt{3}\left\{\frac{\cos 38^\circ \times \dfrac{1}{\cos 38^\circ}}{\tan 15^\circ \times \sqrt{3} \times \dfrac{1}{\tan 15^\circ}}\right\} = 2 - \frac{\sqrt{3}}{\sqrt{3}} = 2 - 1 = 1$$

EXAMPLE 17 *Prove that:*

(i) $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ = 1$ (ii) $\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ = 1$

(iii) $\cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \cos 180^\circ = 0$

SOLUTION (i) We have,

$$\text{LHS} = \tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ$$
$$= \tan(90^\circ - 80^\circ)\tan(90^\circ - 75^\circ)\tan 75^\circ \tan 80^\circ$$
$$= \cot 80^\circ \cot 75^\circ \tan 75^\circ \tan 80^\circ \qquad [\because \tan(90^\circ - \theta) = \cot\theta]$$
$$= (\cot 80^\circ \tan 80^\circ)(\cot 75^\circ \tan 75^\circ) = 1 \times 1 = 1 = \text{RHS} \qquad [\because \cot\theta \,.\, \tan\theta = 1]$$

(ii) We have,

$$\text{LHS} = \tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ$$
$$= \tan(90^\circ - 89^\circ)\tan(90^\circ - 88^\circ)\tan(90^\circ - 87^\circ)\ldots \tan 87^\circ \tan 88^\circ \tan 89^\circ$$
$$= \cot 89^\circ \cot 88^\circ \cot 87^\circ \ldots \tan 87^\circ \tan 88^\circ \tan 89^\circ$$
$$= (\cot 89^\circ \tan 89^\circ)(\cot 88^\circ \tan 88^\circ)(\cot 87^\circ \tan 87^\circ)\ldots(\cot 44^\circ \tan 44^\circ)\cdot \tan 45^\circ$$
$$= 1 \times 1 \times 1 \ldots \times 1 = 1 = \text{RHS} \qquad [\because \cot\theta \tan\theta = 1 \text{ and } \tan 45^\circ = 1]$$

(iii) We have,

$$\text{LHS} = \cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \cos 180^\circ$$
$$= \cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \cos 89^\circ \cos 90^\circ \cos 91^\circ \ldots \cos 180^\circ$$
$$= \cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \times \cos 89^\circ \times 0 \times \cos 91^\circ \times \ldots \cos 180^\circ = 0 = \text{RHS}$$
$$[\because \cos 90^\circ = 0]$$

EXAMPLE 18 *If $A + B = 90^\circ$, prove that*

$$\sqrt{\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}} = \tan A$$

SOLUTION We have,

$$A + B = 90^\circ \Rightarrow B = 90^\circ - A$$

$$\therefore \quad \text{LHS} = \sqrt{\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{\tan A \tan(90^\circ - A) + \tan A \cot(90^\circ - A)}{\sin A \sec(90^\circ - A)} - \frac{\sin^2(90^\circ - A)}{\cos^2 A}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{\tan A \cot A + \tan A \tan A}{\sin A \operatorname{cosec} A} - \frac{\cos^2 A}{\cos^2 A}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{1 + \tan^2 A - 1} = \sqrt{\tan^2 A} = \tan A = \text{RHS}$$

EXERCISE 10.3

LEVEL-1

1. Evaluate the following:

(i) $\dfrac{\sin 20^\circ}{\cos 70^\circ}$ (ii) $\dfrac{\cos 19^\circ}{\sin 71^\circ}$ (iii) $\dfrac{\sin 21^\circ}{\cos 69^\circ}$

(iv) $\frac{\tan 10^\circ}{\cot 80^\circ}$ (v) $\frac{\sec 11^\circ}{\operatorname{cosec} 79^\circ}$

2. Evaluate the following:

(i) $\left(\frac{\sin 49^\circ}{\cos 41^\circ}\right)^2 + \left(\frac{\cos 41^\circ}{\sin 49^\circ}\right)^2$ (ii) $\cos 48^\circ - \sin 42^\circ$

(iii) $\frac{\cot 40^\circ}{\tan 50^\circ} - \frac{1}{2}\left(\frac{\cos 35^\circ}{\sin 55^\circ}\right)$ (iv) $\left(\frac{\sin 27^\circ}{\cos 63^\circ}\right)^2 - \left(\frac{\cos 63^\circ}{\sin 27^\circ}\right)^2$

(v) $\frac{\tan 35^\circ}{\cot 55^\circ} + \frac{\cot 78^\circ}{\tan 12^\circ} - 1$ (vi) $\frac{\sec 70^\circ}{\operatorname{cosec} 20^\circ} + \frac{\sin 59^\circ}{\cos 31^\circ}$

(vii) $\operatorname{cosec} 31^\circ - \sec 59^\circ$ (viii) $(\sin 72^\circ + \cos 18^\circ)(\sin 72^\circ - \cos 18^\circ)$

(ix) $\sin 35^\circ \sin 55^\circ - \cos 35^\circ \cos 55^\circ$ (x) $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ$

(xi) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

3. Express each one of the following in terms of trigonometric ratios of angles lying between 0° and 45°

(i) $\sin 59^\circ + \cos 56^\circ$ (ii) $\tan 65^\circ + \cot 49^\circ$ (iii) $\sec 76^\circ + \operatorname{cosec} 52^\circ$

(iv) $\cos 78^\circ + \sec 78^\circ$ (v) $\operatorname{cosec} 54^\circ + \sin 72^\circ$ (vi) $\cot 85^\circ + \cos 75^\circ$ [NCERT]

(vii) $\sin 67^\circ + \cos 75^\circ$ [NCERT]

4. Express $\cos 75^\circ + \cot 75^\circ$ in terms of angles between 0° and 30°.

5. If $\sin 3A = \cos(A - 26^\circ)$, where $3A$ is an acute angle, find the value of A. [NCERT]

6. If A, B, C, are the interior angles of a triangle ABC, prove that

(i) $\tan\left(\frac{C+A}{2}\right) = \cot\frac{B}{2}$ (ii) $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ [NCERT]

7. Prove that:

(i) $\tan 20^\circ \tan 35^\circ \tan 45^\circ \tan 55^\circ \tan 70^\circ = 1$

(ii) $\sin 48^\circ \sec 42^\circ + \cos 48^\circ \operatorname{cosec} 42^\circ = 2$

(iii) $\frac{\sin 70^\circ}{\cos 20^\circ} + \frac{\operatorname{cosec} 20^\circ}{\sec 70^\circ} - 2\cos 70^\circ \operatorname{cosec} 20^\circ = 0$

(iv) $\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ = 2$

8. Prove the following:

(i) $\sin\theta \sin(90^\circ - \theta) - \cos\theta \cos(90^\circ - \theta) = 0$

(ii) $\frac{\cos(90^\circ - \theta)\sec(90^\circ - \theta)\tan\theta}{\operatorname{cosec}(90^\circ - \theta)\sin(90^\circ - \theta)\cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot\theta} = 2$

(iii) $\dfrac{\tan(90^\circ - A)\cot A}{\operatorname{cosec}^2 A} - \cos^2 A = 0$

(iv) $\dfrac{\cos(90^\circ - A)\sin(90^\circ - A)}{\tan(90^\circ - A)} = \sin^2 A$

(v) $\sin(50^\circ + \theta) - \cos(40^\circ - \theta) + \tan 1^\circ \tan 10^\circ \tan 20^\circ \tan 70^\circ \tan 80^\circ \tan 89^\circ = 1$

[CBSE 2002]

9. Evaluate:

(i) $\dfrac{2}{3}(\cos^4 30^\circ - \sin^4 45^\circ) - 3(\sin^2 60^\circ - \sec^2 45^\circ) + \dfrac{1}{4}\cot^2 30^\circ$ [CBSE 2001 C]

(ii) $4(\sin^4 30^\circ + \cos^4 60^\circ) - \dfrac{2}{3}(\sin^2 60^\circ - \cos^2 45^\circ) + \dfrac{1}{2}\tan^2 60^\circ$ [CBSE 2001C]

(iii) $\dfrac{\sin 50^\circ}{\cos 40^\circ} + \dfrac{\operatorname{cosec} 40^\circ}{\sec 50^\circ} - 4\cos 50^\circ \operatorname{cosec} 40^\circ$ [CBSE 2001]

(iv) $\tan 35^\circ \tan 40^\circ \tan 45^\circ \tan 50^\circ \tan 55^\circ$ [CBSE 2000]

(v) $\operatorname{cosec}(65^\circ + \theta) - \sec(25^\circ - \theta) - \tan(55^\circ - \theta) + \cot(35^\circ + \theta)$ [CBSE 2000]

(vi) $\tan 7^\circ \tan 23^\circ \tan 60^\circ \tan 67^\circ \tan 83^\circ$ [CBSE 2000]

(vii) $\dfrac{2\sin 68^\circ}{\cos 22^\circ} - \dfrac{2\cot 15^\circ}{5\tan 75^\circ} - \dfrac{3\tan 45^\circ \tan 20^\circ \tan 40^\circ \tan 50^\circ \tan 70^\circ}{5}$ [CBSE 2004]

(viii) $\dfrac{3\cos 55^\circ}{7\sin 35^\circ} - \dfrac{4(\cos 70^\circ \operatorname{cosec} 20^\circ)}{7(\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ)}$ [CBSE 2007]

(ix) $\dfrac{\sin 18^\circ}{\cos 72^\circ} + \sqrt{3}\{\tan 10^\circ \tan 30^\circ \tan 40^\circ \tan 50^\circ \tan 80^\circ\}$ [CBSE 2008]

(x) $\dfrac{\cos 58^\circ}{\sin 32^\circ} + \dfrac{\sin 22^\circ}{\cos 68^\circ} - \dfrac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 18^\circ \tan 35^\circ \tan 60^\circ \tan 72^\circ \tan 55^\circ}$ ~ 1 [CBSE 2009]

10. If $\sin\theta = \cos(\theta - 45^\circ)$, where θ and $\theta - 45^\circ$ are acute angles, find the degree measure of θ.

11. If A, B, C are the interior angles of a ΔABC, show that:

(i) $\sin\dfrac{B+C}{2} = \cos\dfrac{A}{2}$ (ii) $\cos\dfrac{B+C}{2} = \sin\dfrac{A}{2}$

12. If $2\theta + 45^\circ$ and $30^\circ - \theta$ are acute angles, find the degree measure of θ satisfying

$$\sin(2\theta + 45^\circ) = \cos(30^\circ - \theta)$$

13. If θ is a positive acute angle such that $\sec\theta = \operatorname{cosec} 60^\circ$, find the value of $2\cos^2\theta - 1$.

14. If $\cos 2\theta = \sin 4\theta$, where 2θ and 4θ are acute angles, find the value of θ.

15. If $\sin 3\theta = \cos(\theta - 6^\circ)$, where 3θ and $\theta - 6^\circ$ are acute angles, find the value of θ.

16. If $\sec 4A = \operatorname{cosec}(A - 20^\circ)$, where $4A$ is an acute angle, find the value of A.

17. If $\sec 2A = \operatorname{cosec}(A - 42^\circ)$, where $2A$ is an acute angle, find the value of A.

[CBSE 2008]

18. If $\tan 2a = \cot(A - 18^\circ)$, where $2A$ is an acute angle, find the value of A. [CBSE 2018]

ANSWERS

1. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 1
2. (i) 2 (ii) 0 (iii) 1/2 (iv) 0
(v) 1 (vi) 2 (vii) 0 (viii) 0
(ix) 0 (x) 1 (xi) 2
3. (i) $\cos 31^\circ + \sin 34^\circ$ (ii) $\cot 25^\circ + \tan 41^\circ$
(iii) $\operatorname{cosec} 14^\circ + \sec 38^\circ$ (iv) $\sin 12^\circ + \operatorname{cosec} 12^\circ$
(v) $\sec 36^\circ + \cos 18^\circ$ (vi) $\tan 5^\circ + \sin 15^\circ$
(vii) $\cos 23^\circ + \sin 15^\circ$
4. $\sin 15^\circ + \tan 15^\circ$
5. 29°
9. (i) $\frac{113}{24}$ (ii) $\frac{11}{6}$ (iii) -2 (iv) 1 (v) 0
(vi) $\sqrt{3}$ (vii) 1 (viii) $-\frac{1}{7}$ (ix) 2 (x) $\frac{6-\sqrt{3}}{3}$
10. $67\frac{1}{2}^\circ$
12. 15°
13. $\frac{1}{2}$
14. 15°
15. 24°
16. 22°
17. 44°
18. 36°

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Write the maximum and minimum values of $\sin\theta$.
2. Write the maximum and minimum values of $\cos\theta$.
3. What is the maximum value of $\frac{1}{\sec\theta}$?
4. What is the maximum value of $\frac{1}{\operatorname{cosec}\theta}$?
5. If $\tan\theta = \frac{4}{5}$, find the value of $\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$.
6. If $\cos\theta = \frac{2}{3}$, find the value of $\frac{\sec\theta - 1}{\sec\theta + 1}$.
7. If $3\cot\theta = 4$, find the value of $\frac{4\cos\theta - \sin\theta}{2\cos\theta + \sin\theta}$.
8. Given $\tan\theta = \frac{1}{\sqrt{5}}$, what is the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$?
9. If $\cot\theta = \frac{1}{\sqrt{3}}$, write the value of $\frac{1-\cos^2\theta}{2-\sin^2\theta}$.
10. If $\tan A = \frac{3}{4}$ and $A + B = 90^\circ$, then what is the value of $\cot B$?

11. If $A + B = 90°$ and $\cos B = \frac{3}{5}$, what is the value of $\sin A$?
12. Write the acute angle θ satisfying $\sqrt{3}\sin\theta = \cos\theta$.
13. Write the value of $\cos 1° \cos 2° \cos 3° ...... \cos 179° \cos 180°$.
14. Write the value of $\tan 10° \tan 15° \tan 75° \tan 80°$.
15. If $A + B = 90°$ and $\tan A = \frac{3}{4}$, what is $\cot B$?
16. If $\tan A = \frac{5}{12}$, find the value of $(\sin A + \cos A)\sec A$. [CBSE 2008]

ANSWERS

1. -1 and 1 2. -1 and 1 3. 1 4. 1 5. $\frac{1}{9}$

6. $\frac{1}{5}$ 7. $\frac{13}{11}$ 8. $\frac{2}{3}$ 9. $\frac{3}{5}$ 10. $\frac{3}{4}$ 11. $\frac{3}{5}$ 12. $30°$

13. 0 14. 1 15. $\frac{3}{4}$ 16. $\frac{17}{12}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If θ is an acute angle such that $\cos\theta = \frac{3}{5}$, then $\frac{\sin\theta\tan\theta - 1}{2\tan^2\theta} =$

   (a) $\frac{16}{625}$ (b) $\frac{1}{36}$ (c) $\frac{3}{160}$ (d) $\frac{160}{3}$

2. If $\tan\theta = \frac{a}{b}$, then $\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta}$ is equal to

   (a) $\frac{a^2+b^2}{a^2-b^2}$ (b) $\frac{a^2-b^2}{a^2+b^2}$ (c) $\frac{a+b}{a-b}$ (d) $\frac{a-b}{a+b}$

3. If $5\tan\theta - 4 = 0$, then the value of $\frac{5\sin\theta - 4\cos\theta}{5\sin\theta + 4\cos\theta}$ is

   (a) $\frac{5}{3}$ (b) $\frac{5}{6}$ (c) 0 (d) $\frac{1}{6}$

4. If $16\cot x = 12$, then $\frac{\sin x - \cos x}{\sin x + \cos x}$ equals

   (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) 0

5. If $8\tan x = 15$, then $\sin x - \cos x$ is equal to

   (a) $\frac{8}{17}$ (b) $\frac{17}{7}$ (c) $\frac{1}{17}$ (d) $\frac{7}{17}$

6. If $\tan\theta = \frac{1}{\sqrt{7}}$, then $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta} =$

(a) $\frac{5}{7}$ (b) $\frac{3}{7}$ (c) $\frac{1}{12}$ (d) $\frac{3}{4}$

7. If $\tan\theta = \frac{3}{4}$, then $\cos^2\theta - \sin^2\theta =$

(a) $\frac{7}{25}$ (b) 1 (c) $\frac{-7}{25}$ (d) $\frac{4}{25}$

8. If θ is an acute angle such that $\tan^2\theta = \frac{8}{7}$, then the value of $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$ is

(a) $\frac{7}{8}$ (b) $\frac{8}{7}$ (c) $\frac{7}{4}$ (d) $\frac{64}{49}$

9. If $3\cos\theta = 5\sin\theta$, then the value of $\frac{5\sin\theta - 2\sec^3\theta + 2\cos\theta}{5\sin\theta + 2\sec^3\theta - 2\cos\theta}$ is

(a) $\frac{271}{979}$ (b) $\frac{316}{2937}$ (c) $\frac{542}{2937}$ (d) None of these

10. If $\tan^2 45^\circ - \cos^2 30^\circ = x \sin 45^\circ \cos 45^\circ$, then $x =$

(a) 2 (b) -2 (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

11. The value of $\cos^2 17^\circ - \sin^2 73^\circ$ is

(a) 1 (b) $\frac{1}{3}$ (c) 0 (d) -1

12. The value of $\frac{\cos^3 20^\circ - \cos^3 70^\circ}{\sin^3 70^\circ - \sin^3 20^\circ}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) 2

13. If $\frac{x \operatorname{cosec}^2 30^\circ \sec^2 45^\circ}{8\cos^2 45^\circ \sin^2 60^\circ} = \tan^2 60^\circ - \tan^2 30^\circ$, then $x =$

(a) 1 (b) -1 (c) 2 (d) 0

14. If A and B are complementary angles, then

(a) $\sin A = \sin B$ (b) $\cos A = \cos B$ (c) $\tan A = \tan B$ (d) $\sec A = \operatorname{cosec} B$

15. If $x \sin(90^\circ - \theta)\cot(90^\circ - \theta) = \cos(90^\circ - \theta)$, then $x =$

(a) 0 (b) 1 (c) -1 (d) 2

16. If $x \tan 45^\circ \cos 60^\circ = \sin 60^\circ \cot 60^\circ$, then x is equal to

(a) 1 (b) $\sqrt{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

17. If angles A, B, C of a ΔABC form an increasing AP, then $\sin B =$

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\frac{1}{\sqrt{2}}$

18. If θ is an acute angle such that $\sec^2\theta = 3$, then the value of $\frac{\tan^2\theta - \operatorname{cosec}^2\theta}{\tan^2\theta + \operatorname{cosec}^2\theta}$ is

(a) $\frac{4}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) $\frac{1}{7}$

19. The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \cdots\cdots \tan 89^\circ$ is

(a) 1 (b) -1 (c) 0 (d) None of these

20. The value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \cdots\cdots \cos 180^\circ$ is

(a) 1 (b) 0 (c) -1 (d) None of these

21. The value of $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ$ is

(a) -1 (b) 0 (c) 1 (d) None of these

22. The value of $\dfrac{\cos(90^\circ-\theta)\sec(90^\circ-\theta)\tan\theta}{\operatorname{cosec}(90^\circ-\theta)\sin(90^\circ-\theta)\cot(90^\circ-\theta)}+\dfrac{\tan(90^\circ-\theta)}{\cot\theta}$ is

(a) 1 (b) -1 (c) 2 (d) -2

23. If θ and $2\theta-45^\circ$ are acute angles such that $\sin\theta=\cos(2\theta-45^\circ)$, then $\tan\theta$ is equal to

(a) 1 (b) -1 (c) $\sqrt{3}$ (d) $\dfrac{1}{\sqrt{3}}$

24. If 5θ and 4θ are acute angles satisfying $\sin 5\theta=\cos 4\theta$, then $2\sin 3\theta-\sqrt{3}\tan 3\theta$ is equal to

(a) 1 (b) 0 (c) -1 (d) $1+\sqrt{3}$

25. If $A+B=90^\circ$, then $\dfrac{\tan A\tan B+\tan A\cot B}{\sin A\sec B}-\dfrac{\sin^2 B}{\cos^2 A}$ is equal to

(a) $\cot^2 A$ (b) $\cot^2 B$ (c) $-\tan^2 A$ (d) $-\cot^2 A$

26. $\dfrac{2\tan 30^\circ}{1+\tan^2 30^\circ}$ is equal to

(a) $\sin 60^\circ$ (b) $\cos 60^\circ$ (c) $\tan 60^\circ$ (d) $\sin 30^\circ$ **[NCERT]**

27. $\dfrac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ}$ is equal to

(a) $\tan 90^\circ$ (b) 1 (c) $\sin 45^\circ$ (d) $\sin 0^\circ$ **[NCERT]**

28. $\sin 2A = 2\sin A$ is true when $A =$

(a) 0° (b) 30° (c) 45° (d) 60° **[NCERT]**

29. $\dfrac{2\tan 30^\circ}{1-\tan^2 30^\circ}$ is equal to

(a) $\cos 60^\circ$ (b) $\sin 60^\circ$ (c) $\tan 60^\circ$ (d) $\sin 30^\circ$ **[NCERT]**

30. If A, B and C are interior angles of a triangle ABC, then $\sin\left(\dfrac{B+C}{2}\right)=$

(a) $\sin\dfrac{A}{2}$ (b) $\cos\dfrac{A}{2}$ (c) $-\sin\dfrac{A}{2}$ (d) $-\cos\dfrac{A}{2}$

31. If $\cos\theta=\dfrac{2}{3}$, then $2\sec^2\theta+2\tan^2\theta-7$ is equal to

(a) 1 (b) 0 (c) 3 (d) 4

32. $\tan 5^\circ \times \tan 30^\circ \times 4\tan 85^\circ$ is equal to

(a) $\dfrac{4}{\sqrt{3}}$ (b) $4\sqrt{3}$ (c) 1 (d) 4

33. The value of $\dfrac{\tan 55^\circ}{\cot 35^\circ} + \cot 1^\circ \cot 2^\circ \cot 3^\circ .... \cot 90^\circ$, is

(a) -2 (b) 2 (c) 1 (d) 0

34. In Fig. 10.47, the value of $\cos\phi$ is

(a) $\dfrac{5}{4}$ (b) $\dfrac{5}{3}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{4}{5}$

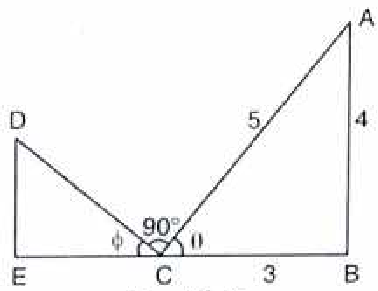


Fig. 10.47

35. In Fig. 10.48, *if* $AD = 4$ cm $BD = 3$ cm and $CB = 12$ cm, then $\cot\theta$ =

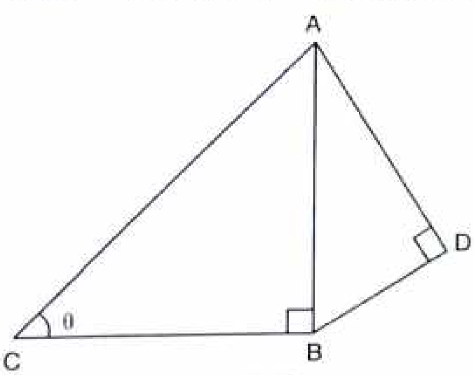


Fig. 10.48

(a) $\dfrac{12}{5}$ (b) $\dfrac{5}{12}$ (c) $\dfrac{13}{12}$ (d) $\dfrac{12}{13}$

[CBSE 2008]

ANSWERS

1. (c)	2. (a)	3. (c)	4. (a)	5. (d)	6. (d)
7. (a)	8. (a)	9. (a)	10. (d)	11. (c)	12. (c)
13. (a)	14. (d)	15. (b)	16. (a)	17. (b)	18. (d)
19. (a)	20. (b)	21. (c)	22. (c)	23. (a)	24. (b)
25. (b)	26. (a)	27. (d)	28. (a)	29. (c)	30. (b)
31. (b)	32. (a)	33. (c)	34. (d)	35. (a)	

SUMMARY

1. An angle is considered as the figure obtained by rotating a given ray about its end-point. The revolving ray is called the generating line of the angle. The initial position OA is called the initial side and the final position OB is called the terminal side of the angle.
2. The measure of an angle is the amount of rotation from the initial side to the terminal side.

3. If ABC is a right triangle right angled at B and $\angle BAC = \theta$, then with reference to angle θ, we have

Base $= AB$, Perpendicular $= BC$ and, Hypotenuse $= AC$

Also,

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}}$$

$$\cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$$

4. We have, $\text{cosec}\,\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$

Also, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

5. The trigonometric ratios for angles $0°, 30°, 45°, 60°$ and $90°$ are given in table given on page 10.30.

6. The values of $\sin\theta$ and $\cos\theta$ never exceed 1, whereas the values of $\sec\theta$ and $\text{cosec}\,\theta$ are always greater than or equal to 1.

7. If θ is an acute angle, then

$\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$

$\tan(90° - \theta) = \cot\theta$, $\cot(90° - \theta) = \tan\theta$

$\sec(90° - \theta) = \text{cosec}\,\theta$, $\text{cosec}(90° - \theta) = \sec\theta$

CHAPTER 11

TRIGONOMETRIC IDENTITIES

11.1 INTRODUCTION

In the previous chapter, we have learnt about various trigonometric ratios and relations between them. In this chapter, we will prove some trigonometric identities, and use them to prove other useful trigonometric identities.

11.2 TRIGONOMETRIC IDENTITIES

We know that an equation is called an identity if it is true for all values of the variable (s) involved. For example,

$$x^2 - 9 = (x-3)(x+3) \text{ and } \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$$

are algebraic identities as they are satisfied by every value of the variable x.

In this section, we will discuss some trigonometric identities. We may define the term trigonometric identity as follows.

DEFINITION *An equation involving trigonometric ratios of an angle θ (say) is said to be a trigonometric identity if it is satisfied for all values of θ for which the given trigonometric ratios are defined.*

For example, $\cos^2\theta - \frac{1}{2}\cos\theta = \cos\theta\left(\cos\theta - \frac{1}{2}\right)$ is a trigonometric identity, whereas $\cos\theta\left(\cos\theta - \frac{1}{2}\right) = 0$ is an equation.

Also, $\sec\theta = \frac{1}{\cos\theta}$ is a trigonometric identity, because it holds for all values of q except for which $\cos\theta = 0$. For $\cos\theta = 0$, $\sec\theta$ is not defined.

In this section, we shall establish some fundamental trigonometric identities and use them to obtain some more identities.

THEOREM 1 *Prove that* $\sin^2\theta + \cos^2\theta = 1$.

PROOF The following three cases arise :

CASE I *When* $\theta = 0$

In this case, we have

$$\sin\theta = \sin 0^\circ = 0 \text{ and } \cos\theta = \cos 0^\circ = 1$$

$$\therefore \quad \sin^2\theta + \cos^2\theta = 0 + 1 = 1$$

CASE II *When* $\theta = 90°$

In this case, we have

$$\sin\theta = \sin 90° = 1 \text{ and } \cos\theta = \cos 90° = 0$$

$$\therefore \quad \sin^2\theta + \cos^2\theta = 1 + 0 = 1$$

CASE III *When* θ *is an acute angle*

Let $XAY = \theta$ be the given acute angle. Let P be any point on the terminal side AY other than A. Draw perpendicular PM from P on the initial side AX.

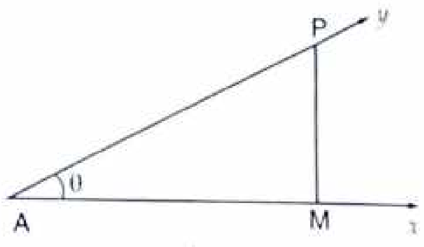


Fig. 11.1

In ΔAMP, we have

$$\sin\theta = \frac{PM}{AP} \text{ and } \cos\theta = \frac{AM}{AP}$$

$$\Rightarrow \quad (\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} \qquad [\because (\sin\theta)^2 = \sin^2\theta \text{ and } (\cos\theta)^2 = \cos^2\theta]$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{PM^2 + AM^2}{AP^2}$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{AP^2}{AP^2} \qquad \left[\begin{array}{l}\because \Delta AMP \text{ is a right angled triangle} \\ \therefore PM^2 + AM^2 = AP^2\end{array}\right]$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = 1$$

Thus, in all the three cases, we have

$$\sin^2\theta + \cos^2\theta = 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$ for all values of variable θ.

REMARK 1 *Note that* $\sin^2\theta$ *is the square of the sine of angle* θ. *Similarly* $\cos^2\theta$ *is the square of the cosine of the angle* θ.

REMARK 2 *It follows from the above identity that* $\sin^2\theta = 1 - \cos^2\theta$ *and* $\cos^2\theta = 1 - \sin^2\theta$

THEOREM 2 *Prove that* $\sec^2\theta = 1 + \tan^2\theta$

PROOF In the right angled triangle AMP (Fig. 11.1), we have

$$AM^2 + MP^2 = AP^2 \qquad \text{[By Pythagoras Theorem]}$$

$$\Rightarrow \quad \frac{AM^2 + MP^2}{AM^2} = \frac{AP^2}{AM^2} \qquad \left[\text{Dividing both sides by } AM^2\right]$$

$$\Rightarrow \quad \frac{AM^2}{AM^2} + \frac{MP^2}{AM^2} = \frac{AP^2}{AM^2}$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta \qquad \left[\because \tan\theta = \frac{MP}{AM} \text{ and } \sec\theta = \frac{AP}{AM}\right]$$

Hence, $1 + \tan^2\theta = \sec^2\theta$

Q.E.D.

REMARK 1 *It follows from the above identity that:*

$$\sec^2\theta - \tan^2\theta = 1 \text{ and, } \sec^2\theta - 1 = \tan^2\theta$$

REMARK 2 $\sec^2\theta - \tan^2\theta = 1 \Rightarrow (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$$\therefore \quad \sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta} \text{ and, } \sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta}$$

THEOREM 3 *Prove that* $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

PROOF In the right-angled triangle AMP (Fig. 11.1), we have

$$AM^2 + MP^2 = AP^2 \qquad \text{[Using Pythagoras Theorem]}$$

$$\Rightarrow \quad \frac{AM^2 + MP^2}{MP^2} = \frac{AP^2}{MP^2} \qquad \text{[Dividing both sides by } MP^2\text{]}$$

$$\Rightarrow \quad \frac{AM^2}{MP^2} + \frac{MP^2}{MP^2} = \frac{AP^2}{MP^2}$$

$$\Rightarrow \quad \left(\frac{AM}{MP}\right)^2 + 1 = \left(\frac{AP}{MP}\right)^2$$

$$\Rightarrow \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad \left[\because \cot\theta = \frac{AM}{MP} \text{ and } \operatorname{cosec}\theta = \frac{AP}{MP}\right]$$

Hence, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Q.E.D.

REMARK 3 *It follows from the above identity that:*

$$\operatorname{cosec}^2\theta - \cot^2\theta = 1 \text{ and, } \operatorname{cosec}^2\theta - 1 = \cot^2\theta$$

REMARK 4 $\operatorname{cosec}^2\theta - \cot^2\theta = 1 \Rightarrow (\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) = 1$

$$\therefore \quad \operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta} \text{ and, } \operatorname{cosec}\theta - \cot\theta = \frac{1}{\operatorname{cosec}\theta + \cot\theta}.$$

THEOREM 4 *For any acute angle* θ, *prove the following identities:*

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

PROOF From Fig. 11.1, we have

$$\sin\theta = \frac{PM}{AP}, \cos\theta = \frac{AM}{AP}, \tan\theta = \frac{PM}{AM} \text{ and } \cot\theta = \frac{AM}{PM}$$

(i) We have,

$$\tan\theta = \frac{PM}{AM}$$

$$\Rightarrow \quad \tan\theta = \frac{(PM/AP)}{(AM/AP)} \qquad \text{[Dividing Numerator and Denominator by } AP\text{]}$$

$$\Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \left[\because \sin\theta = \frac{PM}{AP} \text{ and } \cos\theta = \frac{AM}{AP}\right]$$

Q.E.D.

(ii) We have,

$$\cot\theta = \frac{AM}{PM}$$

$$\Rightarrow \quad \cot\theta = \frac{\left(\frac{AM}{PM}\right)}{\left(\frac{PM}{AP}\right)} \qquad \text{[Dividing Numerator and Denominator by } AP]$$

$$\Rightarrow \quad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \left[\because \cos\theta = \frac{AM}{AP} \text{ and } \sin\theta = \frac{PM}{AP}\right]$$

Hence, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Q.E.D

The above identities and some more identities obtained from the above identities by performing simple algebraic operations like addition, subtraction are listed below for ready reference.

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\cos^2\theta = 1 - \sin^2\theta$

(iii) $\sin^2\theta = 1 - \cos^2\theta$ (iv) $1 + \tan^2\theta = \sec^2\theta$

(v) $\sec^2\theta - \tan^2\theta = 1$ (vi) $\sec^2\theta - 1 = \tan^2\theta$

(vii) $1 + \cot^2\theta = \text{cosec}^2\theta$ (viii) $\text{cosec}^2\theta - \cot^2\theta = 1$

(ix) $\text{cosec}^2\theta - 1 = \cot^2\theta$ (x) $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

(xi) $\text{cosec}\,\theta + \cot\theta = \dfrac{1}{\text{cosec}\,\theta - \cot\theta}$

REMARK 5 *We have proved the above identities for an acute angle θ. But, these identities are true for any angle θ for which the trigonometric ratios are meaningful.*

REMARK 6 *In this book, we shall deal mainly with acute angles.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove the following trigonometric identiti4es:*

(i) $(1 - \sin^2\theta)\sec^2\theta = 1$ (ii) $\cos^2\theta(1 + \tan^2\theta) = 1$

(iii) $\cos^2\theta + \dfrac{1}{1 + \cot^2\theta} = 1$ (iv) $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$

(v) $\text{cosec}^2\theta + \sec^2\theta = \text{cosec}^2\theta\sec^2\theta$ [CBSE 2001]

(vi) $\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta$ [NCERT EXEMPLAR]

(vii) $(\sin^4\theta - \cos^4\theta + 1)\,\text{cosec}^2\theta = 2$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$\text{LHS} = (1 - \sin^2\theta)\sec^2\theta$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta\sec^2\theta \qquad [\because 1 - \sin^2\theta = \cos^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) = 1 = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta} \therefore \sec^2\theta = \frac{1}{\cos^2\theta}\right]$$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

(ii) We have,

$$\text{LHS} = \cos^2\theta(1+\tan^2\theta)$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta\sec^2\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) = 1 = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

(iii) We have,

$$\text{LHS} = \cos^2\theta + \frac{1}{1+\cot^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta + \frac{1}{\text{cosec}^2\theta} \qquad [\because 1+\cot^2\theta = \text{cosec}^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta + \sin^2\theta = 1 = \text{RHS} \qquad \left[\because \frac{1}{\text{cosec}\,\theta} = \sin\theta\right]$$

(iv) We have,

$$\text{LHS} = \frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)} \qquad \text{[On taking LCM]}$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{1-\sin^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{\cos^2\theta} \qquad [\because 1-\sin^2\theta = \cos^2\theta]$$

$$\Rightarrow \quad \text{LHS} = 2\sec^2\theta = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

(v) We have,

$$\text{LHS} = \text{cosec}^2\theta + \sec^2\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta\sin^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta}\times\frac{1}{\cos^2\theta} = \text{cosec}^2\theta\sec^2\theta = \text{RHS}$$

(vi) We have,

$$\text{LHS} = \sqrt{\sec^2\theta + \text{cosec}^2\theta}$$

$$= \sqrt{(1+\tan^2\theta)+(1+\cot^2\theta)}$$

$$= \sqrt{\tan^2 + \cot^2\theta + 2)} = \sqrt{\tan^2\theta+\cot^2\theta+2\tan\theta\cot\theta} \qquad [\because \tan\theta\cot\theta = 1]$$

$$= \sqrt{(\tan\theta+\cot\theta)^2} = \tan\theta + \cot\theta = \text{RHS}$$

(vii) We have,

$$\text{LHS} = (\sin^4\theta - \cos^4\theta + 1)\,\text{cosec}^2\theta$$
$$= \{(\sin^2\theta)^2 - (\cos^2\theta)^2 + 1\}\,\text{cosec}^2\theta$$
$$= \{(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) + 1\}\,\text{cosec}^2\theta$$
$$= (\sin^2\theta - \cos^2\theta + 1)\,\text{cosec}^2\theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$
$$= (\sin^2\theta + \sin^2\theta)\,\text{cosec}^2\theta \qquad [1 - \cos^2\theta = \sin^2\theta]$$
$$= 2\sin^2\theta - \text{cosec}^2\theta = 2\sin^2\theta \times \frac{1}{\sin^2\theta} = 2 = \text{RHS}$$

EXAMPLE 2 *Prove that following trigonometric identities :*

(i) $\cot^2\theta - \dfrac{1}{\sin^2\theta} = -1$

(ii) $(1 + \tan^2\theta)(1 + \sin\theta)(1 - \sin\theta) = 1$ **[NCERT EXEMPLAR]**

(iii) $(1 + \cot^2\theta)(1 - \cos\theta)(1 + \cos\theta) = 1$ (iv) $\tan^2\theta - \dfrac{1}{\cos^2\theta} = -1$

SOLUTION (i) We have,

$$\text{LHS} = \cot^2\theta - \frac{1}{\sin^2\theta}$$
$$\Rightarrow \quad \text{LHS} = \cot^2\theta - \text{cosec}^2\theta \qquad \left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta \;\therefore \frac{1}{\sin^2\theta} = \text{cosec}^2\theta\right]$$
$$\Rightarrow \quad \text{LHS} = -(\text{cosec}^2\theta - \cot^2\theta) = -1 = \text{RHS}$$

(ii) We have,

$$\text{LHS} = (1 + \tan^2\theta)(1 + \sin\theta)(1 - \sin\theta)$$
$$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta)\{(1 + \sin\theta)(1 - \sin\theta)\}$$
$$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta)(1 - \sin^2\theta)$$
$$\Rightarrow \quad \text{LHS} = \sec^2\theta\cos^2\theta \qquad [\because 1 + \tan^2\theta = \sec^2\theta \text{ and } 1 - \sin^2\theta = \cos^2\theta]$$
$$\Rightarrow \quad \text{LHS} = \frac{1}{\cos^2\theta} \times \cos^2\theta = 1 = \text{RHS} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta} \therefore \sec^2\theta = \frac{1}{\cos^2\theta}\right]$$

(iii) We have,

$$\text{LHS} = (1 + \cot^2\theta)(1 - \cos\theta)(1 + \cos\theta)$$
$$\Rightarrow \quad \text{LHS} = (1 + \cot^2\theta)(1 - \cos^2\theta)$$
$$\Rightarrow \quad \text{LHS} = \text{cosec}^2\theta\sin^2\theta \qquad [\because 1 + \cot^2\theta = \text{cosec}^2\theta \text{ and } 1 - \cos^2\theta = \sin^2\theta]$$
$$\Rightarrow \quad \text{LHS} = \text{cosec}^2\theta\sin^2\theta$$
$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin^2\theta} \times \sin^2\theta = 1 = \text{RHS} \qquad \left[\because \text{cosec}\,\theta = \frac{1}{\sin\theta} \therefore \text{cosec}^2\theta = \frac{1}{\sin\theta}\right]$$

(iv) We have,

$$\text{LHS} = \tan^2\theta - \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \tan^2\theta - \sec^2\theta \qquad \left[\because \frac{1}{\cos\theta} = \sec\theta \therefore \frac{1}{\cos^2\theta} = \sec^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = -(\sec^2\theta - \tan^2\theta) = -1 = \text{RHS}$$

EXAMPLE 3 *Prove the following trigonometric identities:*

(i) $\dfrac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$ (ii) $\dfrac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \dfrac{\sec\theta+1}{\sec\theta-1}$

(iii) $\cot\theta - \tan\theta = \dfrac{2\cos^2\theta-1}{\sin\theta\cos\theta}$ (iv) $\tan\theta - \cot\theta = \dfrac{2\sin^2\theta-1}{\sin\theta\cos\theta}$

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta}{(1-\cos\theta)} \times \frac{(1+\cos\theta)}{(1+\cos\theta)} \qquad \left[\begin{array}{l}\text{Multiplying numerator and}\\ \text{denominator by } (1+\cos\theta)\end{array}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} \qquad [\because (1+\cos\theta)(1-\cos\theta) = 1-\cos^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} \qquad [\because 1-\cos^2\theta = \sin^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \frac{1+\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec}\theta + \cot\theta = \text{RHS} \qquad \left[\because \frac{1}{\sin\theta} = \operatorname{cosec}\theta \text{ and, } \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$$

(ii) We have,

$$\text{LHS} = \frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\frac{\sin\theta}{\cos\theta}+\sin\theta}{\frac{\sin\theta}{\cos\theta}-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta\left(\frac{1}{\cos\theta}+1\right)}{\sin\theta\left(\frac{1}{\cos\theta}-1\right)} = \frac{\frac{1}{\cos\theta}+1}{\frac{1}{\cos\theta}-1} = \frac{\sec\theta+1}{\sec\theta-1} = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \cot\theta - \tan\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta - (1-\cos^2\theta)}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta = 1-\cos^2\theta]$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta - 1 + \cos^2\theta}{\sin\theta\cos\theta} = \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta} = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \tan\theta - \cot\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta - (1-\sin^2\theta)}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta - 1 + \sin^2\theta}{\sin\theta\cos\theta} = \frac{2\sin^2\theta - 1}{\sin\theta\cos\theta} = \text{RHS}$$

EXAMPLE 4 *Prove the following trigonometric identities:*

(i) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (ii) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

SOLUTION (i) We have,

$$\text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}} \qquad [\text{Multiplying and dividing by } (1-\sin\theta)]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2} = \frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}} \qquad \left[\begin{array}{l}\text{Multiplying and dividing within the} \\ \text{square root sign by } (1+\cos\theta)\end{array}\right]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{\left(\frac{1+\cos\theta}{\sin\theta}\right)^2} = \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta = \text{RHS}$$

EXAMPLE 5 *Prove the following identities:*

(i) $\dfrac{1-\sin\theta}{1+\sin\theta}=(\sec\theta-\tan\theta)^2$ (ii) $\dfrac{1-\cos\theta}{1+\cos\theta}=(\operatorname{cosec}\theta-\cot\theta)^2$

(iii) $\dfrac{\cos\theta}{1-\sin\theta}+\dfrac{\cos\theta}{1+\sin\theta}=2\sec\theta$

(iv) $\dfrac{\sin A+\cos A}{\sin A-\cos A}+\dfrac{\sin A-\cos A}{\sin A+\cos A}=\dfrac{2}{\sin^2 A-\cos^2 A}=\dfrac{2}{2\sin^2 A-1}=\dfrac{2}{1-2\cos^2 A}$

[CBSE 2000, 2000C]

(v) $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$

(vi) $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$ [NCERT, CBSE 2000]

SOLUTION (i) We have,

$$\text{LHS}=\frac{1-\sin\theta}{1+\sin\theta}$$

$\Rightarrow$ $\text{LHS}=\dfrac{1-\sin\theta}{1+\sin\theta}\times\dfrac{1-\sin\theta}{1-\sin\theta}$ [Multiplying numerator and denominator by $1-\sin\theta$]

$$\Rightarrow \text{LHS}=\frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$\Rightarrow$ $\text{LHS}=\dfrac{(1-\sin\theta)^2}{\cos^2\theta}$ [$\because 1-\sin^2\theta=\cos^2\theta$]

$$\Rightarrow \text{LHS}=\left(\frac{1-\sin\theta}{\cos\theta}\right)^2$$

$$\Rightarrow \text{LHS}=\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=(\sec\theta-\tan\theta)^2=\text{RHS}$$

(ii) We have,

$$\text{LHS}=\frac{1-\cos\theta}{1+\cos\theta}$$

$\Rightarrow$ $\text{LHS}=\dfrac{1-\cos\theta}{1+\cos\theta}\times\dfrac{1-\cos\theta}{1-\cos\theta}$ [Multiplying numerator and denominator by $1-\cos\theta$]

$\Rightarrow$ $\text{LHS}=\dfrac{(1-\cos\theta)^2}{1-\cos^2\theta}=\dfrac{(1-\cos\theta)^2}{\sin^2\theta}$ [$\because 1-\cos^2\theta=\sin^2\theta$]

$$\Rightarrow \text{LHS}=\left(\frac{1-\cos\theta}{\sin\theta}\right)^2=\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)^2=(\operatorname{cosec}\theta-\cot\theta)^2=\text{RHS}$$

(iii) We have,

$$\text{LHS}=\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos\theta(1+\sin\theta)+\cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos\theta+\cos\theta\sin\theta+\cos\theta-\cos\theta\sin\theta}{1-\sin^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\cos\theta}{\cos^2\theta} \qquad [\because 1-\sin^2\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{\cos\theta} = 2\sec\theta = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \frac{\sin A+\cos A}{\sin A-\cos A} + \frac{\sin A-\cos A}{\sin A+\cos A}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A+\cos A)^2+(\sin A-\cos A)^2}{(\sin A-\cos A)(\sin A+\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin^2 A+\cos^2 A+2\sin A\cos A)+(\sin^2 A+\cos^2 A-2\sin A\cos A)}{\sin^2 A-\cos^2 A}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1+2\sin A\cos A)+(1-2\sin A\cos A)}{\sin^2 A-\cos^2 A} \qquad [\because \sin^2 A+\cos$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{\sin^2 A-\cos^2 A} = \frac{2}{\sin^2 A-(1-\sin^2 A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{2}{2\sin^2 A-1} = \frac{2}{2(1-\cos^2 A)-1} = \frac{2}{1-2\cos^2 A} = \text{RHS}$$

(v) We have,

$$\text{LHS} = (\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta}{\sin\theta}\times\frac{\sin^2\theta}{\cos\theta}\times\frac{1}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta+\cos^2$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta} = 1 = \text{RHS}$$

(vi) We have,

$$\text{LHS} = \frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta-1)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta\{1-2(1-\cos^2\theta)\}}{\cos\theta(2\cos^2\theta-1)} = \frac{\sin\theta(2\cos^2\theta-1)}{\cos\theta(2\cos^2\theta-1)} = \tan\theta = \text{RHS}$$

EXAMPLE 6 *Prove the following identities:*

(i) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$ **[NCERT, CBSE 2000]**

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = (1 + \sec\theta\operatorname{cosec}\theta)^2$ **[CBSE 2000C]**

(iii) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$ **[NCERT, CBSE 2000C]**

(iv) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$ **[NCERT EXEMPLAR]**

(v) $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$ **[CBSE 2000]**

(vi) $(\sin\theta - \sec\theta)^2 + (\cos\theta - \operatorname{cosec}\theta)^2 = (1 - \sec\theta\operatorname{cosec}\theta)^2$

SOLUTION (i) We have,

$$\text{LHS} = (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta) + (\cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\frac{1}{\sin\theta}\right) + \left(\cos^2\theta + \sec^2\theta + 2\cos\theta\frac{1}{\cos\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \operatorname{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2)$$

$$\Rightarrow \quad \text{LHS} = \sin^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + 4$$

$$\Rightarrow \quad \text{LHS} = 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4 \qquad \left[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta,\ \sec^2\theta = 1 + \tan^2\theta\right]$$

$$\Rightarrow \quad \text{LHS} = 7 + \tan^2\theta + \cot^2\theta = \text{RHS}$$

(ii) We have,

$$\text{LHS} = (\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$$

$$\Rightarrow \quad \text{LHS} = \left(\sin\theta + \frac{1}{\cos\theta}\right)^2 + \left(\cos\theta + \frac{1}{\sin\theta}\right)^2$$

$$\Rightarrow \quad \text{LHS} = \sin^2\theta + \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + \cos^2\theta + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + 2\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \cos^2\theta) + \left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}\right) + \frac{2(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = 1 + \frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad LHS = \left(1 + \frac{1}{\sin\theta\cos\theta}\right)^2 = (1 + \sec\theta\,\text{cosec}\,\theta)^2 = RHS$$

(iii) We have,

$$LHS = (\text{cosec}\,\theta - \cot\theta)^2$$

$$\Rightarrow \quad LHS = \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$$

$$\Rightarrow \quad LHS = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$\Rightarrow \quad LHS = \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta} \qquad [\because \sin^2\theta = 1 -$$

$$\Rightarrow \quad LHS = \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} = RHS$$

(iv) We have,

$$LHS = \sec^4\theta - \sec^2\theta$$

$$\Rightarrow \quad LHS = \sec^2\theta(\sec^2\theta - 1)$$

$$\Rightarrow \quad LHS = (1+\tan^2\theta)(1+\tan^2\theta - 1)$$

$$\Rightarrow \quad LHS = (1+\tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta = RHS \qquad [\because \sec^2\theta = 1 +$$

(v) We have,

$$LHS = 2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta$$

$$\Rightarrow \quad LHS = (\text{cosec}^4\theta - 2\,\text{cosec}^2\theta) - (\sec^4\theta - 2\sec^2\theta)$$

$$\Rightarrow \quad LHS = (\text{cosec}^4\theta - 2\,\text{cosec}^2\theta + 1) - (\sec^4\theta - 2\sec^2\theta + 1)$$

$$\Rightarrow \quad LHS = (\text{cosec}^2\theta - 1)^2 - (\sec^2\theta - 1)^2$$

$$\Rightarrow \quad LHS = (\cot^2\theta)^2 - (\tan^2\theta)^2$$

$$\Rightarrow \quad LHS = \cot^4\theta - \tan^4\theta = RHS$$

(vi) Proceed as in (ii) part.

EXAMPLE 7 *Prove the following identities:*

(i) $\dfrac{\sin\theta}{1-\cos\theta} + \dfrac{\tan\theta}{1+\cos\theta} = \sec\theta\,\text{cosec}\,\theta + \cot\theta$

(ii) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\,\text{cosec}\,\theta$

[NCERT EXEMPLAR, CBSE 20

(iii) $\dfrac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \text{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

(iv) $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

(v) $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta}$$

$$\text{LHS} = \frac{\sin\theta(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} + \frac{\tan\theta(1-\cos\theta)}{(1+\cos\theta)\,(1-\cos\theta)}$$

$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} + \frac{\tan\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} + \frac{\tan\theta(1-\cos\theta)}{\sin^2\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} + \frac{\sin\theta(1-\cos\theta)}{\cos\theta\sin^2\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} + \frac{1-\cos\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta\sin\theta} - \frac{\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta\sin\theta} - \frac{1}{\sin\theta} = \cot\theta + \sec\theta\,\text{cosec}\,\theta = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin^2\theta + \cos^2\theta) + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\Rightarrow \quad \text{LHS} = \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta} = 2\,\text{cosec}\,\theta = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \frac{\dfrac{\sin\theta}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta} = \frac{\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\cos\theta\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} - \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \sec^2\theta - \text{cosec}^2\theta = (1+\tan^2\theta) - (1+\cot^2\theta) = \tan^2\theta - \cot^2\theta = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \frac{1}{\sec\theta - \tan\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sec\theta - \tan\theta} \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta} = \frac{\sec\theta + \tan\theta}{1} \qquad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$\Rightarrow \quad \text{LHS} = \sec\theta + \tan\theta = \text{RHS}$$

(v) We have,

$$\text{LHS} = \frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} \times \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta - \tan\theta)^2}{\sec^2\theta - \tan^2\theta} = \frac{(\sec\theta - \tan\theta)^2}{1} \qquad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$\Rightarrow \quad \text{LHS} = \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta$$

$$\Rightarrow \quad \text{LHS} = (1+\tan^2\theta) - 2\sec\theta\tan\theta + \tan^2\theta \qquad [\because \sec^2\theta = 1 + \tan^2\theta]$$

$$\Rightarrow \quad \text{LHS} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta = \text{RHS}$$

EXAMPLE 8 *Prove the following identities:*

(i) $(1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta) = 2$ **[CBSE 2008]**

(ii) $\tan^2\theta + \cot^2\theta + 2 = \sec^2\theta\,\text{cosec}^2\theta$

(iii) $\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\text{LHS} = (1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin\theta + \cos\theta)^2 - 1^2}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$$

$\Rightarrow \quad \text{LHS} = \dfrac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} = \dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$

(ii) We have,

$\text{LHS} = \tan^2\theta + \cot^2\theta + 2$

$\Rightarrow \quad \text{LHS} = \tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta \qquad [\because \tan\theta \cdot \cot\theta = 1]$

$\Rightarrow \quad \text{LHS} = (\tan\theta + \cot\theta)^2$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)^2$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{1}{\sin\theta\cos\theta}\right)^2 = \dfrac{1}{\sin^2\theta\cos^2\theta} = \operatorname{cosec}^2\theta\sec^2\theta = \text{RHS}$

ALITER 1 We have,

$\text{LHS} = \tan^2\theta + \cot^2\theta + 2 = (1 + \tan^2\theta) + (1 + \cot^2\theta)$

$\Rightarrow \quad \text{LHS} = \sec^2\theta + \operatorname{cosec}^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$

$\Rightarrow \quad \text{LHS} = \dfrac{1}{\cos^2\theta\sin^2\theta} = \operatorname{cosec}^2\theta\sec^2\theta = \text{RHS}$

ALITER 2 $\text{LHS} = \tan^2\theta + \cot^2\theta + 2$

$\Rightarrow \quad \text{LHS} = 1 + \tan^2\theta + \cot^2\theta + 1$

$\Rightarrow \quad \text{LHS} = 1 + \tan^2\theta + \cot^2\theta + \tan^2\theta\cot^2\theta \qquad [\because \tan^2\theta\cot^2\theta = 1]$

$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta) + \cot^2\theta(1 + \tan^2\theta)$

$\Rightarrow \quad \text{LHS} = (1 + \tan^2\theta)(1 + \cot^2\theta) = \sec^2\theta\operatorname{cosec}^2\theta = \text{RHS}$

ALITER 3 $\text{RHS} = \sec^2\theta\operatorname{cosec}^2\theta$

$\Rightarrow \quad \text{RHS} = (1 + \tan^2\theta)(1 + \cot^2\theta)$

$\Rightarrow \quad \text{RHS} = 1 + \tan^2\theta + \cot^2\theta + \tan^2\theta\cot^2\theta$

$\Rightarrow \quad \text{RHS} = 1 + \tan^2\theta + \cot^2\theta + 1 = \tan^2\theta + \cot^2\theta + 2 = \text{LHS}$

(iii) We have,

$\text{LHS} = \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}$

$\Rightarrow \quad \text{LHS} = \sqrt{(1 + \tan^2\theta) + (1 + \cot^2\theta)} = \sqrt{2 + \tan^2\theta + \cot^2\theta}$

$\Rightarrow \quad \text{LHS} = \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta} \qquad [\because \tan\theta\cot\theta = 1]$

$\Rightarrow \quad \text{LHS} = \sqrt{(\tan\theta + \cot\theta)^2} = \tan\theta + \cot\theta = \text{RHS}$

EXAMPLE 9 *Prove the following identities:*

(i) $\frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}$

(ii) $\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \cos A + \sin A$ [C

(iii) $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A = 1 + \sec A \operatorname{cosec} A$ [NCEI

SOLUTION (i) We have,

$$\text{LHS} = \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{(\operatorname{cosec} A - \cot A)} \times \frac{(\operatorname{cosec} A + \cot A)}{(\operatorname{cosec} A + \cot A)} - \frac{1}{\sin A}$$

$$\Rightarrow \quad \text{LHS} = \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec}^2 A - \cot^2 A} - \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A + \cot A - \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = \cot A \qquad [\because \operatorname{cosec}$$

and, $$\text{RHS} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}$$

$$\Rightarrow \quad \text{RHS} = \frac{1}{\sin A} - \frac{\operatorname{cosec} A - \cot A}{(\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)}$$

$$\Rightarrow \quad \text{RHS} = \frac{1}{\sin A} - \frac{\operatorname{cosec} A - \cot A}{\operatorname{cosec}^2 A - \cot^2 A}$$

$$\Rightarrow \quad \text{RHS} = \operatorname{cosec} A - (\operatorname{cosec} A - \cot A)$$

$$\Rightarrow \quad \text{RHS} = \cot A \qquad [\because \operatorname{cosec}^2$$

From (i) and (ii) we find that LHS = RHS.

ALITER $$\text{LHS} = \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A}$$

$$\Rightarrow \quad \text{LHS} = (\operatorname{cosec} A + \cot A) - \operatorname{cosec} A \qquad \left[\because \frac{1}{\operatorname{cosec} A - \cot A} = \text{co}\right.$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A + (\cot A - \operatorname{cosec} A)$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A - (\operatorname{cosec} A - \cot A)$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A - \frac{1}{\operatorname{cosec} A + \cot A} \qquad \left[\because \operatorname{cosec} A - \cot A = \frac{}{\text{cos}}\right.$$

$$\text{LHS} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A} = \text{RHS}$$

We have,

$$\text{LHS} = \frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A}$$

$$\text{LHS} = \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}}$$

$$\text{LHS} = \frac{\cos A}{\frac{\cos A - \sin A}{\cos A}} + \frac{\sin A}{\frac{\sin A - \cos A}{\sin A}}$$

$$\text{LHS} = \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A}$$

$$\text{LHS} = \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A}$$

$$\text{LHS} = \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} = \frac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A - \sin A} = \cos A + \sin A = \text{RHS}$$

We have,

$$\text{LHS} = \frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$$

$$\text{LHS} = \frac{\tan A}{1 - \frac{1}{\tan A}} + \frac{\frac{1}{\tan A}}{1 - \tan A}$$

$$\text{LHS} = \frac{\tan A}{\frac{\tan A - 1}{\tan A}} + \frac{1}{\tan A\,(1 - \tan A)}$$

$$\text{LHS} = \frac{\tan^2 A}{\tan A - 1} + \frac{1}{\tan A\,(1 - \tan A)}$$

$$\text{LHS} = \frac{\tan^2 A}{\tan A - 1} - \frac{1}{\tan A\,(\tan A - 1)}$$

$$\text{LHS} = \frac{\tan^3 A - 1}{\tan A\,(\tan A - 1)} \qquad \text{[Taking LCM]}$$

$$\text{LHS} = \frac{(\tan A - 1)(\tan^2 A + \tan A + 1)}{\tan A\,(\tan A - 1)} \qquad [\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$$

$\Rightarrow \quad LHS = \dfrac{\tan^2 A + \tan A + 1}{\tan A}$

$\Rightarrow \quad LHS = \dfrac{\tan^2 A}{\tan A} + \dfrac{\tan A}{\tan A} + \dfrac{1}{\tan A}$

$\Rightarrow \quad LHS = \tan A + 1 + \cot A = (1 + \tan A + \cot A)$

$\therefore \quad \dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A$

Now, $\quad 1 + \tan A + \cot A$

$= 1 + \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}$

$= 1 + \dfrac{\sin^2 A + \cos^2 A}{\sin A \cos A} = 1 + \dfrac{1}{\sin A \cos A} = 1 + \text{cosec}\, A \sec A$

From (i) and (ii), we obtain

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A = 1 + \text{cosec}\, A \sec A$$

EXAMPLE 10 *Prove the following identities:*

(i) $\cos^4 A - \cos^2 A = \sin^4 A - \sin^2 A$

(ii) $\cot^4 A - 1 = \text{cosec}^4 A - 2\text{cosec}^2 A$

(iii) $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A$

(iv) $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A = 2\sin^2 A - 1 = 1 - 2\cos^2 A$

(v) $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A \cos^2 A$

(vi) $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$

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SOLUTION (i) We have,

$LHS = \cos^4 A - \cos^2 A$

$\Rightarrow \quad LHS = \cos^2 A(\cos^2 A - 1)$

$\Rightarrow \quad LHS = -\cos^2 A(1 - \cos^2 A)$

$\Rightarrow \quad LHS = -\cos^2 A \sin^2 A = -(1 - \sin^2 A)\sin^2 A = -\sin^2 A + \sin^4 A$

$\Rightarrow \quad LHS = \sin^4 A - \sin^2 A = RHS$

(ii) We have,

$LHS = \cot^4 A - 1$

$\Rightarrow \quad LHS = (\text{cosec}^2 A - 1)^2 - 1 \quad \left[\because \cot^2 A = \text{cosec}^2 A - 1 \therefore \cot^4 A = (\text{cosec}^2 A - 1\right.$

$\Rightarrow \quad LHS = \text{cosec}^4 A - 2\,\text{cosec}^2 A + 1 - 1 = \text{cosec}^4 A - 2\text{cosec}^2 A = RHS$

(iii) We have,

$$\text{LHS} = \sin^4 A + \cos^4 A$$

$\Rightarrow$ $$\text{LHS} = (\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A\cos^2 A$$

[Adding and subtracting $2\sin^2 A\cos^2 A$]

$\Rightarrow$ $$\text{LHS} = (\sin^2 A + \cos^2 A)^2 - 2\sin^2 A\cos^2 A = 1 - 2\sin^2 A\cos^2 A = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \sin^4 A - \cos^4 A$$

$\Rightarrow$ $$\text{LHS} = (\sin^2 A)^2 - (\cos^2 A)^2$$

$\Rightarrow$ $$\text{LHS} = (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)$$

$\Rightarrow$ $$\text{LHS} = \sin^2 A - \cos^2 A \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$\Rightarrow$ $$\text{LHS} = \sin^2 A - (1 - \sin^2 A) = 2\sin^2 A - 1$$

$\Rightarrow$ $$\text{LHS} = 2(1 - \cos^2 A) - 1 = 1 - 2\cos^2 A = \text{RHS}$$

(v) We have,

$$\text{LHS} = \sin^6 A + \cos^6 A$$

$\Rightarrow$ $$\text{LHS} = (\sin^2 A)^3 + (\cos^2 A)^3$$

$\Rightarrow$ $$\text{LHS} = (\sin^2 A + \cos^2 A)\left\{(\sin^2 A)^2 + (\cos^2 A)^2 - \sin^2 A\cos^2 A\right\}$$

$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$

$\Rightarrow$ $$\text{LHS} = \left\{(\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A\cos^2 A - \sin^2 A\cos^2 A\right\}$$

$\Rightarrow$ $$\text{LHS} = \left\{(\sin^2 A + \cos^2 A)^2 - 3\sin^2 A\cos^2 A\right\} = 1 - 3\sin^2 A\cos^2 A = \text{RHS}$$

ALITER $$\text{LHS} = (\sin^2 A)^3 + (\cos^2 A)^3$$

$\Rightarrow$ $$\text{LHS} = (\sin^2 A + \cos^2 A) - 3\sin^2 A\cos^2 A(\sin^2 A + \cos^2 A)$$

$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$\Rightarrow$ $$\text{LHS} = 1 - 3\sin^2 A\cos^2 A = \text{RHS}$$

(vi) We have,

$\Rightarrow$ $$\text{LHS} = \sec^4 A - \sec^2 A = \sec^2 A(\sec^2 A - 1)$$

$\Rightarrow$ $$\text{LHS} = (1 + \tan^2 A)(1 + \tan^2 A - 1) = (1 + \tan^2 A)\tan^2 A$$

$\Rightarrow$ $$\text{LHS} = \tan^2 A + \tan^4 A = \text{RHS}$$

EXAMPLE 11 *Prove the following identities :*

(i) $$\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A\cos^2 A} - 2$$

(ii) $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin^2 A}{\sin A-\cos A}=\sin A+\cos A$

(iii) $\dfrac{(1+\sin\theta)^2+(1-\sin\theta)^2}{\cos^2\theta}=2\left(\dfrac{1+\sin^2\theta}{1-\sin^2\theta}\right)$

(iv) $\dfrac{\cos^2\theta}{1-\tan\theta}+\dfrac{\sin^3\theta}{\sin\theta-\cos\theta}=1+\sin\theta\cos\theta$ [C

(v) $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}+\sin\theta\cos\theta=1$ [C

SOLUTION (i) We have,

$$\text{LHS}=\frac{\sin^2 A}{\cos^2 A}+\frac{\cos^2 A}{\sin^2 A}=\frac{\sin^4 A+\cos^4 A}{\sin^2 A\cos^2 A}$$ [On t

$$\Rightarrow \quad \text{LHS}=\frac{(\sin^2 A)^2+(\cos^2 A)^2+2\sin^2 A\cos^2 A-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$\Rightarrow \quad \text{LHS}=\frac{(\sin^2 A+\cos^2 A)^2-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$\Rightarrow \quad \text{LHS}=\frac{1-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$\Rightarrow \quad \text{LHS}=\frac{1}{\sin^2 A\cos^2 A}-\frac{2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}=\frac{1}{\sin^2 A\cos^2 A}-2=\text{RHS}$$

(ii) We have,

$$\text{LHS}=\frac{\cos A}{1-\tan A}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos A}{1-\dfrac{\sin A}{\cos A}}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos A}{\dfrac{\cos A-\sin A}{\cos A}}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos^2 A}{\cos A-\sin A}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos^2 A}{\cos A-\sin A}-\frac{\sin^2 A}{\cos A-\sin A}$$

$$\Rightarrow \quad \text{LHS}=\frac{\cos^2 A-\sin^2 A}{\cos A-\sin A}$$

$$\Rightarrow \quad \text{LHS}=\frac{(\cos A+\sin A)(\cos A-\sin A)}{\cos A-\sin A}=\cos A+\sin A=\text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{(1+\sin\theta)^2 + (1-\sin\theta)^2}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1+2\sin\theta+\sin^2\theta)+(1-2\sin\theta+\sin^2\theta)}{\cos^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2+2\sin^2\theta}{\cos^2\theta} = \frac{2(1+\sin^2\theta)}{1-\sin^2\theta} = 2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right) = \text{RHS}$$

(iv) We have,

$$\text{LHS} = \frac{\cos^2\theta}{1-\tan\theta} + \frac{\sin^3\theta}{\sin\theta-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^3\theta}{\cos\theta-\sin\theta} - \frac{\sin^3\theta}{\cos\theta-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^3\theta-\sin^3\theta}{\cos\theta-\sin\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos\theta-\sin\theta)(\cos^2\theta+\sin^2\theta+\cos\theta\sin\theta)}{\cos\theta-\sin\theta} = 1+\sin\theta\cos\theta = \text{LHS}$$

(v) We have,

$$\text{LHS} = \frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} + \sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)}{\sin\theta+\cos\theta} + \sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = 1-\sin\theta\cos\theta+\sin\theta\cos\theta = 1 = \text{RHS}$$

EXAMPLE 12 *Prove the following identities :*

(i) $\tan^2 A - \tan^2 B = \dfrac{\cos^2 B - \cos^2 A}{\cos^2 B \cos^2 A} = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$ **[CBSE 2005]**

(ii) $\dfrac{\sin A - \sin B}{\cos A + \cos B} + \dfrac{\cos A - \cos B}{\sin A + \sin B} = 0$

SOLUTION We have,

$$\text{LHS} = \tan^2 A - \tan^2 B$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2 A\cos^2 B - \cos^2 A\sin^2 B}{\cos^2 A\ \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1-\cos^2 A)\cos^2 B - \cos^2 A(1-\cos^2 B)}{\cos^2 A \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2 B - \cos^2 A\cos^2 B - \cos^2 A + \cos^2 A\cos^2 B}{\cos^2 A \cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2 B - \cos^2 A}{\cos^2 A\cos^2 B}$$

$$\Rightarrow \quad \text{LHS} = \frac{(1-\sin^2 B)-(1-\sin^2 A)}{\cos^2 A \cos^2 B} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} = \text{RHS}$$

(ii) We have,

$$\Rightarrow \quad \text{LHS} = \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A - \sin B)(\sin A + \sin B) + (\cos A + \cos B)(\cos A - \cos B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1-1}{(\cos A + \cos B)(\sin A + \sin B)} = 0 = \text{RHS}$$

EXAMPLE 13 *Prove that:* $(1-\sin\theta+\cos\theta)^2 = 2(1+\cos\theta)(1-\sin\theta)$ [CBS

SOLUTION We know that $(a-b+c)^2 = a^2+b^2+c^2-2ab-2bc+2ac$

$$\text{LHS} = (1-\sin\theta+\cos\theta)^2$$

$$\Rightarrow \quad \text{LHS} = 1+\sin^2\theta+\cos^2\theta - 2\sin\theta + 2\cos\theta - 2\sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = 2 - 2\sin\theta + 2\cos\theta - 2\sin\theta\cos\theta$$

$$\Rightarrow \quad \text{LHS} = 2(1-\sin\theta) + 2\cos\theta(1-\sin\theta) = 2(1-\sin\theta)(1+\cos\theta) = \text{RHS}$$

EXAMPLE 14 *If* $\sin\theta + \sin^2\theta = 1$, *prove that* $\cos^2\theta + \cos^4\theta = 1$

[CBSE 2002 C, NCERT EXEM

SOLUTION We have,

$$\sin\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = 1-\sin^2\theta \Rightarrow \sin\theta = \cos^2\theta$$

Now, $\cos^2\theta + \cos^4\theta = \cos^2\theta + (\cos^2\theta)^2 = \cos^2\theta + \sin^2\theta = 1$ [$\because \cos^2\theta$

Type II **ON PROVING RESULTS INVOLVING TRIGONOMETRIC RATIOS**

EXAMPLE 15 *If* $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$, *show that* $m^2 - n^2 = 4\sqrt{mn}$

[CBS

SOLUTION We have, $m = \tan\theta + \sin\theta$ and, $n = \tan\theta - \sin\theta$.

$$\therefore \quad \text{LHS} = m^2 - n^2 = (m+n)(m-n)$$
$$= (\tan\theta + \sin\theta + \tan\theta - \sin\theta)(\tan\theta + \sin\theta - \tan\theta + \sin\theta)$$
$$= (2\tan\theta)(2\sin\theta) = 4\tan\theta\sin\theta = 4\sqrt{\tan^2\theta\sin^2\theta}$$
$$= 4\sqrt{\tan^2\theta(1-\cos^2\theta)} = 4\sqrt{\tan^2\theta - \tan^2\theta\cos^2\theta}$$
$$= 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)} = 4\sqrt{mn} = \text{RHS}$$

EXAMPLE 16 *If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *show that* $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$ **[CBSE 2002 C]**

SOLUTION We have,

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$$
$$\Rightarrow \quad (\cos\theta + \sin\theta)^2 = 2\cos^2\theta$$
$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\cos^2\theta$$
$$\Rightarrow \quad \cos^2\theta - 2\cos\theta\sin\theta = \sin^2\theta$$
$$\Rightarrow \quad \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta = 2\sin^2\theta$$
$$\Rightarrow \quad (\cos\theta - \sin\theta)^2 = 2\sin^2\theta$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$$

ALITER We have,

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$$
$$\Rightarrow \quad (\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$
$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$$
$$\Rightarrow \quad \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$
$$\Rightarrow \quad (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta}$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \qquad [\because \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$$

EXAMPLE 17 *If* $x = a\sin\theta$ *and* $y = b\tan\theta$, *then prove that* $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

SOLUTION We have, $x = a\sin\theta$ and $y = b\tan\theta$

$$\therefore \quad \text{LHS} = \frac{a^2}{x^2} - \frac{b^2}{y^2}$$

$\Rightarrow \quad \text{LHS} = \dfrac{a^2}{a^2 \sin^2\theta} - \dfrac{b^2}{b^2 \tan^2\theta}$ $\quad [\because x = a\sin\theta,\ y = b\tan\theta]$

$\Rightarrow \quad \text{LHS} = \dfrac{1}{\sin^2\theta} - \dfrac{1}{\tan^2\theta}$

$\Rightarrow \quad \text{LHS} = \operatorname{cosec}^2\theta - \cot^2\theta = 1 = \text{RHS}$ $\quad \left[\because 1+\cot^2\theta = \operatorname{cosec}^2\theta \therefore \operatorname{cosec}^2\theta - \cot^2\theta = 1\right]$

EXAMPLE 18 *If* $x = r\sin A\cos C$, $y = r\sin A\sin C$ *and* $z = r\cos A$, *prove that* $r^2 = x^2 + y^2 + z^2$.

SOLUTION We have,

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2\sin^2 A\cos^2 C + r^2\sin^2 A\sin^2 C + r^2\cos^2 A \\ &= r^2\sin^2 A(\cos^2 C + \sin^2 C) + r^2\cos^2 A \\ &= r^2\sin^2 A + r^2\cos^2 A \qquad [\because \cos^2 C + \sin^2 C = 1] \\ &= r^2(\sin^2 A + \cos^2 A) = r^2 \end{aligned}$$

Hence, $r^2 = x^2 + y^2 + z^2$

EXAMPLE 19 *If* $a\cos\theta + b\sin\theta = m$ *and* $a\sin\theta - b\cos\theta = n$, *prove that* $a^2 + b^2 = m^2 + n^2$.

[CBSE 2001 C]

SOLUTION We have, $m = a\cos\theta + b\sin\theta$ and $n = a\sin\theta - b\cos\theta$

$$\begin{aligned} \therefore \quad \text{RHS} &= m^2 + n^2 \\ &= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 \\ &= (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta) + (a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta) \\ &= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2 = \text{LHS} \end{aligned}$$

EXAMPLE 20 *If* $a\cos\theta - b\sin\theta = c$, *prove that* $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

[CBSE 2001 C]

SOLUTION $(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$$\begin{aligned} &= (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta) + (a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta) \\ &= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2 \end{aligned}$$

$\therefore \quad c^2 + (a\sin\theta + b\cos\theta)^2 = a^2 + b^2$ $\quad [\because a\cos\theta - b\sin\theta = c]$

$\Rightarrow \quad (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow \quad a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

ALITER We have,

$a\cos\theta - b\sin\theta = c$

$\Rightarrow \quad (a\cos\theta - b\sin\theta)^2 = c^2$ $\quad$ [Squaring both sides]

$\Rightarrow \quad a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2(1-\sin^2\theta)+b^2(1-\cos^2\theta)-2ab\sin\theta\cos\theta=c^2$

$\Rightarrow \quad a^2\sin^2\theta+b^2\cos^2\theta+2ab\sin\theta\cos\theta=a^2+b^2-c^2$

$\Rightarrow \quad (a\sin\theta+b\cos\theta)^2=a^2+b^2-c^2$

$\Rightarrow \quad a\sin\theta+b\cos\theta=\pm\sqrt{a^2+b^2-c^2}$

Type III ON PROVING TRIGONOMETRIC IDENTITIES INVOLVING TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

EXAMPLE 21 *Prove the following identities:*

(i) $\cos\theta\sin(90^\circ-\theta)+\sin\theta\cos(90^\circ-\theta)=1$ (ii) $\dfrac{\sin(90^\circ-\theta)\sin\theta}{\tan\theta}-1=-\sin^2\theta$

(iii) $\dfrac{\sin(90^\circ-\theta)\cos(90^\circ-\theta)}{\tan\theta}=1-\sin^2\theta$

SOLUTION (i) We know that $\sin(90^\circ-\theta)=\cos\theta$ and $\cos(90^\circ-\theta)=\sin\theta$

$$\text{LHS}=\cos\theta\sin(90^\circ-\theta)+\sin\theta\cos(90^\circ-\theta)$$

$$=\cos\theta\cos\theta+\sin\theta\sin\theta=\cos^2\theta+\sin^2\theta=1$$

(ii) We know that $\sin(90^\circ-\theta)=\cos\theta$

$$\therefore \quad \text{LHS}=\frac{\sin(90^\circ-\theta)\sin\theta}{\tan\theta}-1$$

$$=\frac{\cos\theta\sin\theta}{\sin\theta/\cos\theta}-1=\cos^2\theta-1=-(1-\cos^2\theta)=-\sin^2\theta=\text{RHS}$$

(iii) We know that $\sin(90^\circ-\theta)=\cos\theta$ and $\cos(90^\circ-\theta)=\sin\theta$

$$\therefore \quad \text{LHS}=\frac{\sin(90^\circ-\theta)\cos(90^\circ-\theta)}{\tan\theta}=\frac{\cos\theta\sin\theta}{\sin\theta/\cos\theta}=\cos^2\theta=1-\sin^2\theta=\text{RHS}$$

EXAMPLE 22 *Prove that*

(i) $\dfrac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sin(90^\circ-\theta)}+\dfrac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\cos(90^\circ-\theta)}=1$

(ii) $\operatorname{cosec}^2(90^\circ-\theta)-\tan^2\theta=\cos^2(90^\circ-\theta)+\cos^2\theta$

SOLUTION (i) We know that $\sin(90^\circ-\theta)=\cos\theta$ and $\cos(90^\circ-\theta)=\sin\theta$

$$\therefore \quad \text{LHS}=\frac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sin(90^\circ-\theta)}+\frac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\cos(90^\circ-\theta)}$$

$$=\frac{\sin\theta\sin\theta\cos\theta}{\cos\theta}+\frac{\cos\theta\cos\theta\sin\theta}{\sin\theta}=\sin^2\theta+\cos^2\theta=1=\text{RHS}$$

(ii) We have,

$$\text{LHS}=\operatorname{cosec}^2(90^\circ-\theta)-\tan^2\theta \qquad [\because \operatorname{cosec}(90^\circ-\theta)=\sec\theta]$$

$$=\sec^2\theta-\tan^2\theta=1 \qquad [\because \sec^2\theta-\tan^2\theta=1]$$

and,

$$\text{RHS} = \cos^2(90^\circ - \theta) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1 \qquad [\because \cos(90^\circ - \theta) = \sin\theta$$

Hence, $\quad \text{LHS} = \text{RHS}$

EXAMPLE 23 *Without using trigonometric tables, evaluate each of the following:*

(i) $\dfrac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \dfrac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \dfrac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$ **[CBSE 2002C**

(ii) $\cos(40^\circ + \theta) - \sin(50^\circ - \theta) + \dfrac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$ **[CBSE 2002**

SOLUTION (i) We have,

$$\frac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

$$= \frac{\sin^2 20^\circ + \sin^2(90^\circ - 20^\circ)}{\cos^2 20^\circ + \cos^2(90^\circ - 20^\circ)} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

$$= \frac{\sin^2 20^\circ + \cos^2 20^\circ}{\cos^2 20^\circ + \sin^2 20^\circ} + \frac{\cos\theta\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta\cos\theta}{\frac{\cos\theta}{\sin\theta}} \qquad \begin{bmatrix} \sin(90^\circ - \theta) = \cos\theta \\ \text{and} \\ \cos(90^\circ - \theta) = \sin\theta \end{bmatrix}$$

$$= \frac{1}{1} + \cos^2\theta + \sin^2\theta = 1 + 1 = 2$$

(ii) We have,

$$\cos(40^\circ + \theta) - \sin(50^\circ - \theta) + \frac{\cos^2 40^\circ + \cos^2 50^\circ}{\sin^2 40^\circ + \sin^2 50^\circ}$$

$$= \sin[90^\circ - (40^\circ + \theta)] - \sin(50^\circ - \theta) + \frac{\cos^2 40 + \cos^2(90^\circ - 40^\circ)}{\sin^2 40^\circ + \sin^2(90^\circ - 40^\circ)}$$

$$= \sin(50^\circ - \theta) - \sin(50^\circ - \theta) + \frac{\cos^2 40^\circ + \sin^2 40^\circ}{\sin^2 40^\circ + \cos^2 40^\circ} = 0 + \frac{1}{1} = 1$$

EXAMPLE 24 *Without using trigonometric tables, evaluate each of the following:*

(i) $\dfrac{\cos^2 20^\circ + \cos^2 70^\circ}{\sec^2 50^\circ - \cot^2 40^\circ} + 2\operatorname{cosec}^2 58^\circ - 2\cot 58^\circ \tan 32^\circ - 4\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$

[CBSE 2006C]

(ii) $\dfrac{\sec 39^\circ}{\operatorname{cosec} 51^\circ} + \dfrac{2}{\sqrt{3}}\tan 17^\circ \tan 38^\circ \tan 60^\circ \tan 52^\circ \tan 73^\circ - 3(\sin^2 31^\circ + \sin^2 59^\circ)$

[CBSE 2006C]

(iii) $\dfrac{-\tan\theta\cot(90^\circ - \theta) + \sec\theta\operatorname{cosec}(90^\circ - \theta) + \sin^2 35^\circ + \sin^2 55^\circ}{\tan 10^\circ \tan 20^\circ \tan 30^\circ \tan 70^\circ \tan 80^\circ}$

[CBSE 2005]

(iv) $\dfrac{\sec^2 54^\circ - \cot^2 36^\circ}{\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ} + 2\sin^2 38^\circ \sec^2 52^\circ - \sin^2 45^\circ$

[CBSE 2005]

(v) $\frac{2}{3}\operatorname{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan 32^\circ - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$

[CBSE 2009]

SOLUTION (i) We have,

$$\frac{\cos^2 20^\circ + \cos^2 70^\circ}{\sec^2 50^\circ - \cot^2 40^\circ} + 2\operatorname{cosec}^2 58^\circ - 2\cot 58^\circ \tan 32^\circ - 4\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$$

$$= \frac{\cos^2 20^\circ + \cos^2(90^\circ - 20^\circ)}{\sec^2 50^\circ - \cot^2(90^\circ - 50^\circ)} + 2\operatorname{cosec}^2 58^\circ - 2\cot 58^\circ \tan(90^\circ - 58^\circ)$$

$$- 4\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan(90^\circ - 37^\circ)\tan(90^\circ - 13^\circ)$$

$$= \frac{\cos^2 20^\circ + \sin^2 20^\circ}{\sec^2 50^\circ - \tan^2 50^\circ} + 2\operatorname{cosec}^2 58^\circ - 2\cot^2 58^\circ - 4\tan 13^\circ \tan 37^\circ \tan 45^\circ \cot 37^\circ \cot 13^\circ$$

$$= \frac{1}{1} + 2(\operatorname{cosec}^2 58^\circ - \cot^2 58^\circ) - 4(\tan 13^\circ \cot 13^\circ)(\tan 37^\circ \cot 37^\circ)\tan 45^\circ$$

$$= 1 + 2 - 4 \times 1 \times 1 \times 1 = 3 - 4 = -1$$

(ii) We have,

$$\frac{\sec 39^\circ}{\operatorname{cosec} 51^\circ} + \frac{2}{\sqrt{3}}\tan 17^\circ \tan 38^\circ \tan 60^\circ \tan 52^\circ \tan 73^\circ - 3(\sin^2 31^\circ + \sin^2 59^\circ)$$

$$= \frac{\sec 39^\circ}{\operatorname{cosec}(90^\circ - 39^\circ)} + \frac{2}{\sqrt{3}}\tan 17^\circ \tan 38^\circ \tan 60^\circ \tan(90^\circ - 38^\circ)\tan(90^\circ - 17^\circ)$$

$$-3\left(\sin^2 31^\circ + \sin^2(90^\circ - 31^\circ)\right)$$

$$= \frac{\sec 39^\circ}{\sec 39^\circ} + \frac{2}{\sqrt{3}}\tan 17^\circ \tan 38^\circ \times \sqrt{3} \times \cot 38^\circ \times \cot 17^\circ - 3(\sin^2 31^\circ + \cos^2 31^\circ)$$

$$= 1 + \frac{2}{\sqrt{3}} \times \sqrt{3} - 3 \times 1 = 1 + 2 - 3 = 0$$

(iii) We have,

$$\frac{-\tan\theta \cot(90^\circ - \theta) + \sec\theta \operatorname{cosec}(90^\circ - \theta) + \sin^2 35^\circ + \sin^2 55^\circ}{\tan 10^\circ \tan 20^\circ \tan 30^\circ \tan 70^\circ \tan 80^\circ}$$

$$= \frac{-\tan\theta \tan\theta + \sec\theta \sec\theta + \sin^2 35^\circ + \sin^2(90^\circ - 35^\circ)}{\tan 10^\circ \tan 20^\circ \tan 30^\circ \tan(90^\circ - 20^\circ)\tan(90^\circ - 10^\circ)}$$

$$= \frac{-\tan^2\theta + \sec^2\theta + \sin^2 35^\circ + \cos^2 35^\circ}{\tan 10^\circ \tan 20^\circ \tan 30^\circ \cot 20^\circ \cot 10^\circ}$$

$$= \frac{(\sec^2\theta - \tan^2\theta) + (\cos^2 35^\circ + \sin^2 35^\circ)}{(\tan 10^\circ \cot 10^\circ)(\tan 20^\circ \cot 20^\circ) \times \frac{1}{\sqrt{3}}} = \frac{1 + 1}{1 \times 1 \times \frac{1}{\sqrt{3}}} = 2\sqrt{3}$$

(iv) We have,

$$\frac{\sec^2 54^\circ - \cot^2 36^\circ}{\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ} + 2\sin^2 38^\circ \sec^2 52^\circ - \sin^2 45^\circ$$

$$= \frac{\sec^2 54° - \cot^2(90° - 54°)}{\text{cosec}^2 (90° - 33°) - \tan^2 33°} + 2\sin^2 38° \sec^2 (90° - 38°) - \sin^2 45°$$

$$= \frac{\sec^2 54° - \tan^2 54°}{\sec^2 33° - \tan^2 33°} + 2\sin^2 38° \text{cosec}^2 38° - \sin^2 45° = \frac{1}{1} + 2 \times 1 - \left(\frac{1}{\sqrt{2}}\right)^2 = 3 - \frac{1}{2}$$

(v) We have,

$$\frac{2}{3}\text{cosec}^2 58° - \frac{2}{3}\cot 58° \tan 32° - \frac{5}{3}\tan 13° \tan 37° \tan 45° \tan 53° \tan 77°$$

$$= \frac{2}{3}\text{cosec}^2 58° - \frac{2}{3}\cot 58° \tan (90° - 58°) - \frac{5}{3}\tan 13° \tan 37° \tan 45° \tan (90° - 37°) \tan (90° - 13°)$$

$$= \frac{2}{3}\text{cosec}^2 58 - \frac{2}{3}\cot^2 58 - \frac{5}{3}\tan 13° \tan 37° \tan 45° \cot 37° \cot 13°$$

$$= \frac{2}{3}(\text{cosec}^2 58° - \cot^2 58°) - \frac{5}{3}\tan 13° \tan 37° \times 1 \times \frac{1}{\tan 37°} \times \frac{1}{\tan 13°}$$

$$= \frac{2}{3} \times 1 - \frac{5}{3} = \frac{2}{3} - \frac{5}{3} = -1.$$

LEVEL-2

Type I **ON PROVING TRIGONOMETRIC IDENTITIES**

EXAMPLE 25 *Prove the following identities :*

(i) $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$ [NCERT EXEMPLAR, CBSE 20

(ii) $\frac{\cot A + \text{cosec} A - 1}{\cot A - \text{cosec} A + 1} = \frac{1 + \cos A}{\sin A}$

(iii) $\frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} = 2 + \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta}$ [CBSE 20

(iv) $(\text{cosec}\,\theta - \sin\theta)(\sec\theta - \cos\theta) = \frac{1}{\tan\theta + \cot\theta}$ [NCERT, CBSE 20

SOLUTION (i) We have,

$$\text{LHS} = \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1} \quad [\because \sec^2\theta - \tan^2\theta =$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)}{(\tan\theta - \sec\theta + 1)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{(\tan\theta - \sec\theta + 1)}$$

$$\Rightarrow \quad \text{LHS} = \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta} = \text{RHS}$$

ALITER We have,

$$\text{LHS} = \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec\theta + \tan\theta) - 1}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1}{\sec\theta - \tan\theta} - 1}{\tan\theta - \sec\theta + 1} \qquad \left[\because \sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1 - (\sec\theta - \tan\theta)}{\sec\theta - \tan\theta}}{\tan\theta - \sec\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{1 - \sec\theta + \tan\theta}{\tan\theta - \sec\theta + 1} \times \frac{1}{\sec\theta - \tan\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta \qquad \left[\because \frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cot A + \operatorname{cosec} A) - (\operatorname{cosec}^2 A - \cot^2 A)}{(\cot A - \operatorname{cosec} A) + 1} \qquad [\because \operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$\Rightarrow \quad \text{LHS} = \frac{(\operatorname{cosec} A + \cot A) - (\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\operatorname{cosec} A + \cot A)[1 - (\operatorname{cosec} A - \cot A)]}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\operatorname{cosec} A + \cot A)(\cot A - \operatorname{cosec} A + 1)}{(\cot A - \operatorname{cosec} A + 1)}$$

$$\Rightarrow \quad \text{LHS} = \operatorname{cosec} A + \cot A = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \frac{1 + \cos A}{\sin A} = \text{RHS}$$

ALITER We have,

$$\text{LHS} = \frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1}{\operatorname{cosec} A - \cot A} - 1}{\cot A - \operatorname{cosec} A + 1} \qquad \left[\because \operatorname{cosec} A + \cot A = \frac{1}{\operatorname{cosec} A - \cot A}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\dfrac{1-(\text{cosec } A - \cot A)}{\text{cosec } A - \cot A}}{\cot A - \text{cosec } A + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot A - \text{cosec } A + 1}{\cot A - \text{cosec } A + 1} \times \frac{1}{\text{cosec } A - \cot A}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\text{cosec } A - \cot A} = \text{cosec } A + \cot A \qquad \left[\because \frac{1}{\text{cosec } A - \cot A} = \text{cosec } A + \cot A\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \frac{1+\cos A}{\sin A} = \text{RHS}$$

(iii) We have to prove that

$$\frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} = 2 + \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta} \text{ or, } \frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} - \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta} = 2$$

Now,

$$\text{LHS} = \frac{\sin\theta}{\cot\theta + \text{cosec}\,\theta} - \frac{\sin\theta}{\cot\theta - \text{cosec}\,\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta}{\text{cosec}\,\theta + \cot\theta} + \frac{\sin\theta}{\text{cosec}\,\theta - \cot\theta}$$

$$\Rightarrow \quad \text{LHS} = \sin\theta\left\{\frac{1}{\text{cosec}\,\theta + \cot\theta} + \frac{1}{\text{cosec}\,\theta - \cot\theta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \sin\theta\left\{\frac{\text{cosec}\,\theta - \cot\theta + \text{cosec}\,\theta + \cot\theta}{\text{cosec}^2\theta - \cot^2\theta}\right\} = \sin\theta\left(\frac{2\,\text{cosec}\,\theta}{1}\right)$$

$$\Rightarrow \quad \text{LHS} = \sin\theta\,(2\,\text{cosec}\,\theta) = 2\sin\theta \times \frac{1}{\sin\theta} = 2 = \text{RHS}$$

ALITER $\quad \text{LHS} = \dfrac{\sin\theta}{\cot\theta + \text{cosec}\,\theta}$

$$\Rightarrow \quad \text{LHS} = \sin\theta\,(\text{cosec}\,\theta - \cot\theta) \qquad \left[\because \frac{1}{\text{cosec}\,\theta + \cot\theta} = \text{cosec}\,\theta - \cot\theta\right]$$

$$\Rightarrow \quad \text{LHS} = \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right) = \sin\theta\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \text{LHS} = 1 - \cos\theta$$

$$\Rightarrow \quad \text{LHS} = 2 - (1 + \cos\theta)$$

$$\Rightarrow \quad \text{LHS} = 2 - \frac{(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = 2 - \frac{(1-\cos^2\theta)}{1-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = 2 - \frac{\sin^2\theta}{1-\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = 2 - \frac{\sin\theta}{\frac{1-\cos\theta}{\sin\theta}} = 2 - \frac{\sin\theta}{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}} = 2 - \frac{\sin\theta}{\operatorname{cosec}\theta - \cot\theta} = \text{RHS}$$

(iv) We have,

$$\text{LHS} = (\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^2\theta}{\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\theta\cos\theta}{1} = \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}} = \frac{1}{\tan\theta + \cot\theta} = \text{RHS}$$

EXAMPLE 26 *Prove the following identities :*

(i) $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$

(ii) $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta = 1$

(iii) $(\sin^8\theta - \cos^8\theta) = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

SOLUTION (i) We have,

$$\text{LHS} = 2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$\Rightarrow \quad \text{LHS} = 2\left\{(\sin^2\theta)^3 + (\cos^2\theta)^3\right\} - 3(\sin^4\theta + \cos^4\theta) + 1$$

Using $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ and $a^2 + b^2 = (a+b)^2 - 2ab$, we obtain

$$\text{LHS} = 2\left\{(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)\right\}$$
$$-3\left\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta + 1\right\}$$

$$\Rightarrow \quad \text{LHS} = 2(1 - 3\sin^2\theta\cos^2\theta) - 3(1 - 2\sin^2\theta\cos^2\theta) + 1$$

$$\Rightarrow \quad \text{LHS} = 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1 = 0 = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta)^3 + (\cos^2\theta)^3 + 3\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \text{LHS} = (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) + 3\sin^2\theta\cos^2\theta$$

$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$\Rightarrow$ $\text{LHS} = 1 - 3\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos^2\theta = 1 = \text{RHS}$

(iii) We have,

$$\text{LHS} = \sin^8\theta - \cos^8\theta = (\sin^4\theta)^2 - (\cos^4\theta)^2 = (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)$$

$$\Rightarrow \text{LHS} = (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta)$$

$$\Rightarrow \text{LHS} = (\sin^2\theta - \cos^2\theta)\left\{(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta\right\}$$

$$\Rightarrow \text{LHS} = (\sin^2\theta - \cos^2\theta)\left\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta\right\}$$

$$\Rightarrow \text{LHS} = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta) = \text{RHS}$$

EXAMPLE 27 *Prove the following identities:*

(i) $(1 + \tan A\tan B)^2 + (\tan A - \tan B)^2 = \sec^2 A\sec^2 B$

(ii) $(\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2 = 2\tan A\cot B(\operatorname{cosec} A + \sec B)$

SOLUTION (i) We have,

$$\text{LHS} = (1 + \tan A\tan B)^2 + (\tan A - \tan B)^2$$

$$\Rightarrow \text{LHS} = 1 + 2\tan A\tan B + \tan^2 A\tan^2 B + \tan^2 A + \tan^2 B - 2\tan A\tan B$$

$$\Rightarrow \text{LHS} = 1 + \tan^2 A\tan^2 B + \tan^2 A + \tan^2 B$$

$$\Rightarrow \text{LHS} = (1 + \tan^2 A) + (\tan^2 B + \tan^2 A\tan^2 B)$$

$$\Rightarrow \text{LHS} = (1 + \tan^2 A) + \tan^2 B(1 + \tan^2 A)$$

$$\Rightarrow \text{LHS} = (1 + \tan^2 A)(1 + \tan^2 B) = \sec^2 A\sec^2 B = \text{RHS}$$

(ii) We have,

$$\text{LHS} = (\tan A + \operatorname{cosec} B)^2 - (\cot B - \sec A)^2$$

$$\Rightarrow \text{LHS} = (\tan^2 A + \operatorname{cosec}^2 B + 2\tan A\operatorname{cosec} B) - (\cot^2 B + \sec^2 A - 2\cot B\sec A)$$

$$\Rightarrow \text{LHS} = (\tan^2 A - \sec^2 A) + (\operatorname{cosec}^2 B - \cot^2 B) + 2\tan A\operatorname{cosec} B + 2\cot B\sec A$$

$$\Rightarrow \text{LHS} = -1 + 1 + 2\tan A\operatorname{cosec} B + 2\cot B\sec A$$

$$\Rightarrow \text{LHS} = 2(\tan A\operatorname{cosec} B + \cot B\sec A)$$

$$\Rightarrow \text{LHS} = 2\tan A\cot B\left(\frac{\operatorname{cosec} B}{\cot B} + \frac{\sec A}{\tan A}\right) \quad \text{[Dividing and multiplying by } \tan A\cot B]$$

$$\Rightarrow \text{LHS} = 2\tan A\cot B\left\{\frac{\frac{1}{\sin B}}{\frac{\cos B}{\sin B}} + \frac{\frac{1}{\cos A}}{\frac{\sin A}{\cos A}}\right\}$$

$$\Rightarrow \text{LHS} = 2\tan A\cot B\left(\frac{1}{\cos B} + \frac{1}{\sin A}\right) = 2\tan A\cot B(\sec B + \operatorname{cosec} A) = \text{RHS}$$

EXAMPLE 28 *Prove the following identities:*

(i) $(\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A \operatorname{cosec} A)^2$

(ii) $\cot^2 A\left(\dfrac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A\left(\dfrac{\sin A - 1}{1 + \sec A}\right) = 0$

SOLUTION (i) We have,

$$\text{LHS} = (\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2$$

$$\Rightarrow \quad \text{LHS} = \sin^2 A + \sec^2 A + 2\sin A \sec A + \cos^2 A + \operatorname{cosec}^2 A + 2\cos A \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = (\sin^2 A + \cos^2 A) + (\sec^2 A + \operatorname{cosec}^2 A) + 2\sin A \sec A + 2\cos A \operatorname{cosec} A$$

$$\Rightarrow \quad \text{LHS} = 1 + \left(\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}\right) + 2\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$

$$\Rightarrow \quad \text{LHS} = 1 + \left(\frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}\right) + 2\left(\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}\right)$$

$$\Rightarrow \quad \text{LHS} = 1 + \frac{1}{\sin^2 A \cos^2 A} + \frac{2}{\sin A \cos A}$$

$$\Rightarrow \quad \text{LHS} = 1 + \operatorname{cosec}^2 A \sec^2 A + 2\operatorname{cosec} A \sec A = (1 + \sec A \operatorname{cosec} A)^2 = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cot^2 A\left(\frac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A(\sec A - 1)(\sec A + 1) + \sec^2 A(\sin A - 1)(1 + \sin A)}{(1 + \sin A)(1 + \sec A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A(\sec^2 A - 1) + \sec^2 A(\sin^2 A - 1)}{(1 + \sin A)(1 + \sec A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A(\sec^2 A - 1) - \sec^2 A(1 - \sin^2 A)}{(1 + \sin A)(1 + \sec A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cot^2 A \tan^2 A - \sec^2 A \cos^2 A}{(1 + \sin A)(1 + \sec A)} = \frac{1 - 1}{(1 + \sin A)(1 + \sec A)} = 0 = \text{RHS}$$

EXAMPLE 29 *Prove the following identities:*

(i) $\dfrac{\cos A}{1 - \sin A} + \dfrac{\sin A}{1 - \cos A} + 1 = \dfrac{\sin A \cos A}{(1 - \sin A)(1 - \cos A)}$

(ii) $\dfrac{(1 + \cot A + \tan A)(\sin A - \cos A)}{\sec^3 A - \operatorname{cosec}^3 A} = \sin^2 A \cos^2 A$

SOLUTION (i) We have,

$$\Rightarrow \quad \text{LHS} = \frac{\cos A}{1-\sin A} + \frac{\sin A}{1-\cos A} + 1$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos A\,(1-\cos A) + \sin A\,(1-\sin A) + (1-\sin A)(1-\cos A)}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos A - \cos^2 A + \sin A - \sin^2 A + 1 - \sin A - \cos A + \sin A \cos A}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos A + \sin A) - (\cos^2 A + \sin^2 A) + 1 - (\cos A + \sin A) + \sin A \cos A}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos A + \sin A) - 1 + 1 - (\cos A + \sin A) + \sin A \cos A}{(1-\sin A)(1-\cos A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin A \cos A}{(1-\sin A)(1-\cos A)} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\left(1 + \dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\cos A}\right)(\sin A - \cos A)}{\left(\dfrac{1}{\cos^3 A} - \dfrac{1}{\sin^3 A}\right)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\left(1 + \dfrac{\cos^2 A + \sin^2 A}{\sin A \cos A}\right)(\sin A - \cos A)}{\left(\dfrac{\sin^3 A - \cos^3 A}{\sin^3 A \cos^3 A}\right)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\left(1 + \dfrac{1}{\sin A \cos A}\right)(\sin A - \cos A)\sin^3 A \cos^3 A}{(\sin^3 A - \cos^3 A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A \cos A + 1)(\sin A - \cos A)\sin^2 A \cos^2 A}{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A \cos A)}$$

$$\left[\because a^3 - b^3 = (a-b)(a^2 + b^2 + ab)\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin A \cos A + 1)\sin^2 A \cos^2 A}{(1 + \sin A \cos A)} = \sin^2 A \cos^2 A = \text{RHS}$$

EXAMPLE 30 *If*

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$$

Prove that each of the side is equal to ± 1.

SOLUTION We have,

$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$

Multiplying both sides by $(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$, we get

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) \times (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$$
$$= (\sec A - \tan A)^2 (\sec B - \tan B)^2 (\sec C - \tan C)^2$$

$$\Rightarrow \quad (\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C) = (\sec A - \tan A)^2 (\sec B - \tan B)^2 (\sec C - \tan C)^2$$

$$\Rightarrow \quad 1 = \left[(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)\right]^2$$

$$\Rightarrow \quad (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = \pm 1$$

Similarly, multiplying both sides by $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C)$, we get

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = \pm 1$$

EXAMPLE 31 *If* $\sin\theta + \cos\theta = p$ *and* $\sec\theta + \operatorname{cosec}\theta = q$, *show that* $q(p^2 - 1) = 2p$. [NCERT EXEMPLAR]

SOLUTION We have, $p = \sin\theta + \cos\theta$ and $q = \sec\theta + \operatorname{cosec}\theta$

$$\therefore \quad \text{LHS} = q(p^2 - 1)$$
$$= (\sec\theta + \operatorname{cosec}\theta)\left\{(\sin\theta + \cos\theta)^2 - 1\right\}$$
$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)\left\{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1\right\}$$
$$= \left(\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}\right)(1 + 2\sin\theta\cos\theta - 1)$$
$$= \left(\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}\right)(2\sin\theta\cos\theta) = 2(\sin\theta + \cos\theta) = 2p = \text{RHS}$$

ALITER We have, $\sec\theta + \operatorname{cosec}\theta = q$

$$\Rightarrow \quad \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = q \Rightarrow \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = q \Rightarrow \frac{p}{\sin\theta\cos\theta} = q \Rightarrow \sin\theta\cos\theta = \frac{p}{q}$$

$$\Rightarrow \quad 2\sin\theta\cos\theta = \frac{2p}{q} \Rightarrow 1 + 2\sin\theta\cos\theta = 1 + \frac{2p}{q} \Rightarrow (\sin\theta + \cos\theta)^2 = 1 + \frac{2p}{q}$$

$$\Rightarrow \quad p^2 = 1 + \frac{2p}{q} \Rightarrow (p^2 - 1) = \frac{2p}{q} \Rightarrow q(p^2 - 1) = 2p$$

EXAMPLE 32 *If* $\sec\theta + \tan\theta = p$, *obtain the values of* $\sec\theta$, $\tan\theta$ *and* $\sin\theta$ *in terms of* p. [NCERT EXEMPLAR]

SOLUTION We have,

$$\sec\theta + \tan\theta = p \quad \text{...(i)}$$

Now, $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \quad (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$\Rightarrow \quad p(\sec\theta - \tan\theta) = 1 \Rightarrow \sec\theta - \tan\theta = \dfrac{1}{p}$ ...(ii)

Adding and subtracting (i) and (ii), we get

$$(\sec\theta + \tan\theta) + (\sec\theta - \tan\theta) = p + \frac{1}{p} \text{ and, } (\sec\theta + \tan\theta) - (\sec\theta - \tan\theta) = p - \frac{1}{p}$$

$$\Rightarrow \quad 2\sec\theta = p + \frac{1}{p} \text{ and, } 2\tan\theta = p - \frac{1}{p}$$

$$\Rightarrow \quad \sec\theta = \frac{1}{2}\left(p + \frac{1}{p}\right) \text{ and, } \tan\theta = \frac{1}{2}\left(p - \frac{1}{p}\right)$$

Now, $$\sin\theta = \frac{\tan\theta}{\sec\theta} \Rightarrow \sin\theta = \frac{\frac{1}{2}\left(p - \frac{1}{p}\right)}{\frac{1}{2}\left(p + \frac{1}{p}\right)} = \frac{p^2 - 1}{p^2 + 1}$$

EXAMPLE 33 *If* $\sec\theta + \tan\theta = p$, *show that* $\dfrac{p^2 - 1}{p^2 + 1} = \sin\theta$. **[CBSE 2004]**

SOLUTION We have,

$$\text{LHS} = \frac{p^2 - 1}{p^2 + 1} = \frac{(\sec\theta + \tan\theta)^2 - 1}{(\sec\theta + \tan\theta)^2 + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sec^2\theta - 1) + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + 2\sec\theta\tan\theta + (1 + \tan^2\theta)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\tan^2\theta + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + 2\sec\theta\tan\theta + \sec^2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\tan^2\theta + 2\tan\theta\sec\theta}{2\sec^2\theta + 2\sec\theta\tan\theta} = \frac{2\tan\theta(\tan\theta + \sec\theta)}{2\sec\theta(\sec\theta + \tan\theta)} = \frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta . \sec\theta} = \sin\theta = \text{RHS}$$

ALITER We have, $\sec\theta + \tan\theta = p$ ...(i)

$\therefore \quad (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1 \Rightarrow (\sec\theta - \tan\theta)p = 1 \Rightarrow \sec\theta - \tan\theta = \dfrac{1}{p}$ ...(ii)

Adding (i) and (ii), we obtain

$$2\sec\theta = p + \frac{1}{p} \Rightarrow \sec\theta = \frac{p^2 + 1}{2p} \Rightarrow \cos\theta = \frac{2p}{p^2 + 1}$$

Subtracting (ii) from (i), we obtain

$$2\tan\theta = p - \frac{1}{p}$$

$$\Rightarrow \quad \tan\theta = \frac{p^2-1}{2p}$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{p^2-1}{2p} \Rightarrow \frac{\sin\theta}{\frac{2p}{p^2+1}} = \frac{p^2-1}{2p} \Rightarrow \sin\theta = \frac{p^2-1}{p^2+1}$$

EXAMPLE 34 *If* $\operatorname{cosec}\theta + \cot\theta = p$, *then prove that* $\cos\theta = \dfrac{p^2-1}{p^2+1}$.

SOLUTION We know that $\operatorname{cosec} - \cot\theta = \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$

$$\therefore \quad \operatorname{cosec}\theta + \cot\theta = p \qquad \text{...(i)}$$

$$\Rightarrow \quad \operatorname{cosec}\theta - \cot\theta = \frac{1}{p} \qquad \text{...(ii)}$$

Adding (i) and (ii), we obtain

$$2\operatorname{cosec}\theta = p + \frac{1}{p} \Rightarrow \operatorname{cosec}\theta = \frac{p^2+1}{2p} \Rightarrow \sin\theta = \frac{2p}{p^2+1} \qquad \text{...(iii)}$$

Subtracting (ii) from (i), we obtain

$$2\cot\theta = p - \frac{1}{p} \Rightarrow \cot\theta = \frac{p^2-1}{2p} \qquad \text{...(iv)}$$

Now, $\cos\theta = \cot\theta \times \sin\theta$

$$\Rightarrow \quad \cos\theta = \frac{p^2-1}{2p^2} \times \frac{2p}{p^2+1} = \frac{p^2-1}{p^2+1}.$$

EXAMPLE 35 *If* $\dfrac{\cos\alpha}{\cos\beta} = m$ *and* $\dfrac{\cos\alpha}{\sin\beta} = n$ *show that* $(m^2+n^2)\cos^2\beta = n^2$.

SOLUTION We have, $m = \dfrac{\cos\alpha}{\cos\beta}$ and $n = \dfrac{\cos\alpha}{\sin\beta}$.

$$\therefore \quad \text{LHS} = (m^2+n^2)\cos^2\beta$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta \qquad \left[\because m = \frac{\cos\alpha}{\cos\beta} \text{ and } n = \frac{\cos\alpha}{\sin\beta}\right]$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\cos^2\alpha \sin^2\beta + \cos^2\alpha \cos^2\beta}{\cos^2\beta \sin^2\beta} \right) \cos^2\beta$$

$$\Rightarrow \quad \text{LHS} = \cos^2\alpha \left(\frac{\sin^2\beta + \cos^2\beta}{\cos^2\beta \sin^2\beta} \right) \cos^2\beta$$

$$\Rightarrow \quad \text{LHS} = \cos^2\alpha \left(\frac{1}{\cos^2\beta \sin^2\beta} \right) \cos^2\beta = \frac{\cos^2\alpha}{\sin^2\beta} = \left(\frac{\cos\alpha}{\sin\beta} \right)^2 = n^2 = \text{RHS}$$

ALITER We have, $m = \dfrac{\cos\alpha}{\cos\beta}$ and $n = \dfrac{\cos\alpha}{\sin\beta}$.

$$\Rightarrow \quad \cos\beta = \frac{1}{m}\cos\alpha \text{ and } \sin\beta = \frac{1}{n}\cos\alpha$$

$$\Rightarrow \quad \cos^2\beta + \sin^2\beta = \frac{1}{m^2}\cos^2\alpha + \frac{1}{n^2}\cos^2\alpha$$

$$\Rightarrow \quad 1 = \cos^2\alpha \left(\frac{1}{m^2} + \frac{1}{n^2} \right)$$

$$\Rightarrow \quad 1 = \left(\frac{m^2 + n^2}{m^2 n^2} \right) \cos^2\alpha$$

$$\Rightarrow \quad 1 = \left(\frac{m^2 + n^2}{n^2} \right) \left(\frac{\cos\alpha}{m} \right)^2$$

$$\Rightarrow \quad 1 = \left(\frac{m^2 + n^2}{n^2} \right) (\cos\beta)^2 \qquad \left[\because m = \frac{\cos\alpha}{\cos\beta} \therefore \frac{\cos\alpha}{m} = \cos\beta \right]$$

$$\Rightarrow \quad (m^2 + n^2)\cos^2\beta = n^2.$$

EXAMPLE 36 *If* $\operatorname{cosec}\theta - \sin\theta = l$ *and* $\sec\theta - \cos\theta = m$, *prove that* $l^2 m^2 (l^2 + m^2 + 3) = 1$.

SOLUTION We have, $l = \operatorname{cosec}\theta - \sin\theta$ and $m = \sec\theta - \cos\theta$

$$\therefore \quad \text{LHS} = l^2 m^2 (l^2 + m^2 + 3)$$

$$\Rightarrow \quad \text{LHS} = (\operatorname{cosec}\theta - \sin\theta)^2 (\sec\theta - \cos\theta)^2 \left\{ (\operatorname{cosec}\theta - \sin\theta)^2 + (\sec\theta - \cos\theta)^2 + 3 \right\}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1}{\sin\theta} - \sin\theta \right)^2 \left(\frac{1}{\cos\theta} - \cos\theta \right)^2 \left\{ \left(\frac{1}{\sin\theta} - \sin\theta \right)^2 + \left(\frac{1}{\cos\theta} - \cos\theta \right)^2 + 3 \right\}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1 - \sin^2\theta}{\sin\theta} \right)^2 \left(\frac{1 - \cos^2\theta}{\cos\theta} \right)^2 \left\{ \left(\frac{1 - \sin^2\theta}{\sin\theta} \right)^2 + \left(\frac{1 - \cos^2\theta}{\cos\theta} \right)^2 + 3 \right\}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\left\{\left(\frac{\cos^2\theta}{\sin\theta}\right)^2 + \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos^4\theta}{\sin^2\theta}\times\frac{\sin^4\theta}{\cos^2\theta}\left\{\frac{\cos^4\theta}{\sin^2\theta} + \frac{\sin^4\theta}{\cos^2\theta} + 3\right\}$$

$$\Rightarrow \quad \text{LHS} = \cos^2\theta \times \sin^2\theta\left\{\frac{\cos^6\theta + \sin^6\theta + 3\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \cos^6\theta + \sin^6\theta + 3\cos^2\theta\sin^2\theta$$

$$\Rightarrow \quad \text{LHS} = \left\{(\cos^2\theta)^3 + (\sin^2\theta)^3\right\} + 3\cos^2\theta\sin^2\theta$$

$$\Rightarrow \quad \text{LHS} = \left\{(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)\right\} + 3\sin^2\theta\cos^2\theta$$

$$[\because a^3 + b^3 = (a+b)^3 - 3ab\,(a+b)]$$

$$\Rightarrow \quad \text{LHS} = \left\{1 - 3\cos^2\theta\sin^2\theta\right\} + 3\cos^2\theta\sin^2\theta = \text{RHS} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

EXAMPLE 37 *If* $\tan A = n\tan B$ and $\sin A = m\sin B$, *prove that* $\cos^2 A = \dfrac{m^2-1}{n^2-1}$.

SOLUTION We have to find $\cos^2 A$ in terms of m and n. This means that the angle B is to be eliminated from the given relations.
Now,

$$\tan A = n\tan B \Rightarrow \tan B = \frac{1}{n}\tan A \Rightarrow \cot B = \frac{n}{\tan A}$$

and,
$$\sin A = m\sin B \Rightarrow \sin B = \frac{1}{m}\sin A \Rightarrow \operatorname{cosec} B = \frac{m}{\sin A}$$

Substituting the values of cot B and cosec B in $\operatorname{cosec}^2 B - \cot^2 B = 1$, we get

$$\Rightarrow \quad \frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1$$

$$\Rightarrow \quad \frac{m^2}{\sin^2 A} - \frac{n^2\cos^2 A}{\sin^2 A} = 1$$

$$\Rightarrow \quad \frac{m^2 - n^2\cos^2 A}{\sin^2 A} = 1$$

$$\Rightarrow \quad m^2 - n^2\cos^2 A = \sin^2 A$$

$$\Rightarrow \quad m^2 - n^2\cos^2 A = 1 - \cos^2 A$$

$$\Rightarrow \quad m^2 - 1 = n^2\cos^2 A - \cos^2 A$$

$$\Rightarrow \quad m^2 - 1 = (n^2 - 1)\cos^2 A$$

$$\Rightarrow \quad \frac{m^2-1}{n^2-1} = \cos^2 A$$

EXAMPLE 38 *If* $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ *and* $x\sin\theta = y\cos\theta$, *prove that* $x^2 + y^2 = 1$.

SOLUTION We have,

$$x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$$

$\Rightarrow$ $(x\sin\theta)\sin^2\theta + (y\cos\theta)\cos^2\theta = \sin\theta\cos\theta$

$\Rightarrow$ $x\sin\theta(\sin^2\theta) + (x\sin\theta)\cos^2\theta = \sin\theta\cos\theta$ $[\because x\sin\theta = y\cos\theta]$

$\Rightarrow$ $x\sin\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta$

$\Rightarrow$ $x\sin\theta = \sin\theta\cos\theta$

$\Rightarrow$ $x = \cos\theta$

Now,

$$x\sin\theta = y\cos\theta$$

$\Rightarrow$ $\cos\theta\sin\theta = y\cos\theta$ $[\because x = \cos\theta]$

$\Rightarrow$ $y = \sin\theta$

Hence, $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

EXAMPLE 39 *If* $\operatorname{cosec}\theta - \sin\theta = m$ *and* $\sec\theta - \cos\theta = n$, *prove that* $(m^2n)^{2/3} + (mn^2)^{2/3} = 1$

SOLUTION We have,

$$\operatorname{cosec}\theta - \sin\theta = m \text{ and } \sec\theta - \cos\theta = n$$

$\Rightarrow$ $\dfrac{1}{\sin\theta} - \sin\theta = m$ and $\dfrac{1}{\cos\theta} - \cos\theta = n$

$\Rightarrow$ $\dfrac{1-\sin^2\theta}{\sin\theta} = m$ and $\dfrac{1-\cos^2\theta}{\cos\theta} = n$

$\Rightarrow$ $\dfrac{\cos^2\theta}{\sin\theta} = m$ and $\dfrac{\sin^2\theta}{\cos\theta} = n$

$\therefore$ $(m^2n)^{2/3} + (mn^2)^{2/3} = \left(\dfrac{\cos^4\theta}{\sin^2\theta} \times \dfrac{\sin^2\theta}{\cos\theta}\right)^{2/3} + \left(\dfrac{\cos^2\theta}{\sin\theta} \times \dfrac{\sin^4\theta}{\cos^2\theta}\right)^{2/3}$

$$= (\cos^3\theta)^{2/3} + (\sin^3\theta)^{2/3} = \cos^2\theta + \sin^2\theta = 1$$

Hence, $(m^2n)^{2/3} + (mn^2)^{2/3} = 1$

EXAMPLE 40 *If* $\cot\theta + \tan\theta = x$ *and* $\sec\theta - \cos\theta = y$, *prove that* $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$.

SOLUTION We have,

$$\cot\theta + \tan\theta = x \text{ and } \sec\theta - \cos\theta = y$$

$\Rightarrow$ $\dfrac{1}{\tan\theta} + \tan\theta = x$ and $\dfrac{1}{\cos\theta} - \cos\theta = y$

$\Rightarrow$ $\dfrac{1+\tan^2\theta}{\tan\theta} = x$ and $\dfrac{1-\cos^2\theta}{\cos\theta} = y$

$$\Rightarrow \quad \frac{\sec^2\theta}{\tan\theta} = x \text{ and } \frac{\sin^2\theta}{\cos\theta} = y$$

$$\Rightarrow \quad \frac{1}{\cos^2\theta \times \dfrac{\sin\theta}{\cos\theta}} = x \text{ and } \frac{\sin^2\theta}{\cos\theta} = y$$

$$\Rightarrow \quad \frac{1}{\cos\theta\sin\theta} = x \text{ and, } \frac{\sin^2\theta}{\cos\theta} = y$$

$$\therefore \quad (x^2y)^{2/3} - (xy^2)^{2/3} = \left\{\frac{1}{\cos^2\theta\sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right\}^{2/3} - \left\{\frac{1}{\cos\theta\sin\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right\}^{2/3}$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{2/3} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3} = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - \tan^2\theta = 1$$

Hence, $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$

EXAMPLE 41 *If* $\sin\theta + \sin^2\theta = 1$, *find the value of*

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$$

SOLUTION We have,

$$\sin\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = 1 - \sin^2\theta \Rightarrow \sin\theta = \cos^2\theta$$

$$\therefore \quad \cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$$

$$= (\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta) + 2(\cos^4\theta + \cos^2\theta - 1)$$

$$= (\cos^4\theta + \cos^2\theta)^3 + 2(\cos^4\theta + \cos^2\theta - 1)$$

$$= (\sin^2\theta + \cos^2\theta)^3 + 2(\sin^2\theta + \cos^2\theta - 1) \quad \left[\because \cos^2\theta = \sin\theta \therefore \cos^4\theta = \sin^2\theta\right]$$

$$= 1 + 2(1 - 1) = 1$$

EXAMPLE 42 *If* $a\sec\theta + b\tan\theta + c = 0$ *and* $p\sec\theta + q\tan\theta + r = 0$, *prove that*

$$(br - qc)^2 - (pc - ar)^2 = (aq - bp)^2$$

SOLUTION We have,

$$a\sec\theta + b\tan\theta + c = 0$$

and, $$p\sec\theta + q\tan\theta + r = 0$$

Solving these two equations by the cross-multiplication for sec θ and tan θ, we get

$$\frac{\sec\theta}{br - qc} = \frac{\tan\theta}{cp - ar} = \frac{1}{aq - bp} \Rightarrow \sec\theta = \frac{br - cq}{aq - bp} \text{ and } \tan\theta = \frac{cp - ar}{aq - bp}$$

$$\therefore \quad \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad \left(\frac{br - cq}{aq - bp}\right)^2 - \left(\frac{cp - ar}{aq - bp}\right)^2 = 1 \Rightarrow (br - cq)^2 - (cp - ar)^2 = (aq - bp)^2$$

EXAMPLE 43 *If* $\tan^2\theta = 1 - a^2$, *prove that* $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2 - a^2)^{3/2}$.

SOLUTION We have,

$$\sec\theta + \tan^3\theta \operatorname{cosec}\theta$$

$$= \sec\theta\left\{\frac{\sec\theta + \tan^3\theta \operatorname{cosec}\theta}{\sec\theta}\right\} \qquad \text{[Multiplying and dividing by } \sec\theta\text{]}$$

$$= \sec\theta\left\{1 + \tan^3\theta \cdot \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \sec\theta\left\{1 + \tan^3\theta \times \cot\theta\right\}$$

$$= \sqrt{1 + \tan^2\theta}\left\{1 + \tan^2\theta\right\}$$

$$= \left\{1 + \tan^2\theta\right\}^{3/2} = \left\{1 + (1 - a^2)\right\}^{3/2} = (2 - a^2)^{3/2} \qquad [\because \tan^2\theta = 1 - a^2]$$

EXAMPLE 44 *If* $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, *then prove that* $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

SOLUTION We have,

$$\sin\theta + \sin^2\theta + \sin^3\theta = 1$$

$$\Rightarrow \quad \sin\theta + \sin^3\theta = 1 - \sin^2\theta$$

$$\Rightarrow \quad \sin\theta\,(1 + \sin^2\theta) = \cos^2\theta$$

$$\Rightarrow \quad \sin^2\theta\,(1 + \sin^2\theta)^2 = \cos^4\theta$$

$$\Rightarrow \quad (1 - \cos^2\theta)\left\{1 + (1 - \cos^2\theta)\right\}^2 = \cos^4\theta$$

$$\Rightarrow \quad (1 - \cos^2\theta)\,(2 - \cos^2\theta)^2 = \cos^4\theta$$

$$\Rightarrow \quad (1 - \cos^2\theta)\,(4 - 4\cos^2\theta + \cos^4\theta) = \cos^4\theta$$

$$\Rightarrow \quad 4 - 4\cos^2\theta + \cos^4\theta - 4\cos^2\theta + 4\cos^4\theta - \cos^6\theta = \cos^4\theta$$

$$\Rightarrow \quad -\cos^6\theta + 4\cos^4\theta - 8\cos^2\theta + 4 = 0 \Rightarrow \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$$

EXAMPLE 45 *If* $\sin\theta + \cos\theta = \sqrt{2}$, *then prove that* $\tan\theta + \cot\theta = 2$.

SOLUTION We have, [NCERT EXEMPLAR]

$$\sin\theta + \cos\theta = \sqrt{2}$$

$$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = (\sqrt{2})^2$$

$$\Rightarrow \quad \sin^2 + \cos^2\theta + 2\sin\theta\cos\theta = 2$$

$$\Rightarrow \quad 1 + 2\sin\theta\cos\theta = 2$$

$$\Rightarrow \quad 2\sin\theta\cos\theta = 1$$

$$\Rightarrow \quad 2\sin\theta\cos\theta = \sin^2\theta + \cos^2\theta \qquad \left[\because 1 = \sin^2\theta + \cos^2\theta\right]$$

$$\Rightarrow \quad \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad 2 = \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \Rightarrow 2 = \tan\theta + \cot\theta$$

EXAMPLE 46 *If* $1 + \sin^2\theta = 3\sin\theta\cos\theta$, *prove that* $\tan\theta = 1$ *or* $\frac{1}{2}$.

[NCERT EXEMPLAR]

SOLUTION We have, $1 + \sin^2\theta = 3\sin\theta\cos\theta$

Dividing both sides by $\cos^2\theta$, we obtain

$$\frac{1+\sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta\cos\theta}{\cos^2\theta}$$

$$\Rightarrow \quad \frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{3\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \sec^2\theta + \tan^2\theta = 3\tan\theta$$

$$\Rightarrow \quad 1 + \tan^2\theta + \tan^2\theta = 3\tan\theta \qquad [\because \sec^2\theta = 1 + \tan^2\theta]$$

$$\Rightarrow \quad 2\tan^2\theta - 3\tan\theta + 1 = 0$$

$$\Rightarrow \quad 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$$

$$\Rightarrow \quad 2\tan\theta(\tan\theta - 1) - (\tan\theta - 1) = 0$$

$$\Rightarrow \quad (2\tan\theta - 1)(\tan\theta - 1) = 0$$

$$\Rightarrow \quad 2\tan\theta - 1 = 0 \text{ or } \tan\theta - 1 = 0 \Rightarrow 2\tan\theta = 1 \text{ or } \tan\theta = 1 \Rightarrow \tan\theta = \frac{1}{2} \text{ or } \tan\theta = 1$$

EXERCISE 11.1

$1 + \cot^2 A = \operatorname{cosec}^2 A$

LEVEL-1

Prove the following trigonometric identities:

1. $(1 - \cos^2 A)\operatorname{cosec}^2 A = 1$
2. $(1 + \cot^2 A)\sin^2 A = 1$ $\quad 1 + \tan^2\theta =$
3. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$
4. $\operatorname{cosec}\theta\sqrt{1 - \cos^2\theta} = 1$
5. $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) = 1$
6. $\tan\theta + \frac{1}{\tan\theta} = \sec\theta\operatorname{cosec}\theta$
7. $\frac{\cos\theta}{1 - \sin\theta} = \frac{1+\sin\theta}{\cos\theta}$
8. $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$
9. $\cos^2 A + \frac{1}{1+\cot^2 A} = 1$
10. $\sin^2 A + \frac{1}{1+\tan^2 A} = 1$
11. $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$
12. $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

[redacted]

13. $\frac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$

14. $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

15. $\frac{(1+\cot^2\theta)\tan\theta}{\sec^2\theta} = \cot\theta$

16. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

17. $(\sec\theta + \cos\theta)(\sec\theta - \cos\theta) = \tan^2\theta + \sin^2\theta$

18. $(\operatorname{cosec}\theta + \sin\theta)(\operatorname{cosec}\theta - \sin\theta) = \cot^2\theta + \cos^2\theta$

19. $\sec A(1-\sin A)(\sec A + \tan A) = 1$

20. $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$

21. $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta) = 1$

22. $\sin^2 A\cot^2 A + \cos^2 A\tan^2 A = 1$

23. (i) $\cot\theta - \tan\theta = \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$

(ii) $\tan\theta - \cot\theta = \frac{2\sin^2\theta - 1}{\sin\theta\cos\theta}$

(iii) $\frac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A$ [CBSE 2018]

24. $\frac{\cos^2\theta}{\sin\theta} - \operatorname{cosec}\theta + \sin\theta = 0$

25. $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = 2\sec^2 A$

26. $\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} = 2\sec\theta$ [NCERT]

27. $\frac{(1+\sin\theta)^2 + (1-\sin\theta)^2}{2\cos^2\theta} = \frac{1+\sin^2\theta}{1-\sin^2\theta}$

28. $\frac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta$ [NCERT]

29. $\frac{1+\sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1-\cos\theta}$ [NCERT]

30. $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta$

31. $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

32. $\operatorname{cosec}^6\theta = \cot^6\theta + 3\cot^2\theta\operatorname{cosec}^2\theta + 1$

33. $\frac{(1+\tan^2\theta)\cot\theta}{\operatorname{cosec}^2\theta} = \tan\theta$

34. $\frac{1+\cos A}{\sin^2 A} = \frac{1}{1-\cos A}$

35. $\frac{\sec A - \tan A}{\sec A + \tan A} = \frac{\cos^2 A}{(1+\sin A)^2}$

36. $\frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$

37. (i) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$ [NCERT]

(ii) $\sqrt{\frac{1-\cos A}{1+\cos A}} + \sqrt{\frac{1+\cos A}{1-\cos A}} = 2\operatorname{cosec} A$

38. Prove that :

(i) $\sqrt{\frac{\sec\theta - 1}{\sec\theta + 1}} + \sqrt{\frac{\sec\theta + 1}{\sec\theta - 1}} = 2\operatorname{cosec}\theta$ [CBSE 2001, 2006C]

(ii) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$ [CBSE 2001]

(iii) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = 2\operatorname{cosec}\theta$

(iv) $\frac{\sec\theta - 1}{\sec\theta + 1} = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2$ [CBSE 2001 C]

39. $(\sec A - \tan A)^2 = \dfrac{1-\sin A}{1+\sin A}$

40. $\dfrac{1-\cos A}{1+\cos A} = (\cot A - \operatorname{cosec} A)^2$

41. $\dfrac{1}{\sec A - 1} + \dfrac{1}{\sec A + 1} = 2 \operatorname{cosec} A \cot A$

42. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$

43. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A$

44. $\dfrac{\tan^2 A}{1+\tan^2 A} + \dfrac{\cot^2 A}{1+\cot^2 A} = 1$

45. $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$

[NCERT, CBSE 2008]

46. $\dfrac{1+\cos\theta - \sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$

47. (i) $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$

(ii) $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$

[CBSE 2001, NCERT]

(iii) $\dfrac{\cos\theta - \sin\theta + 1}{\cos\theta + \sin\theta - 1} = \operatorname{cosec}\theta + \cot\theta$

(iv) $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \operatorname{cosec}\theta$ [NCERT EXEMPLAR]

48. $\dfrac{1}{\sec A + \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A - \tan A}$ [CBSE 2005]

49. $\tan^2 A + \cot^2 A = \sec^2 A \operatorname{cosec}^2 A - 2$

50. $\dfrac{\tan A}{1+\sec A} - \dfrac{\tan A}{1-\sec A} = 2\operatorname{cosec} A$

[NCERT EXEMPLAR]

51. $1 + \dfrac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta$

[NCERT EXEMPLAR]

52. $\dfrac{\cos\theta}{\operatorname{cosec}\theta + 1} + \dfrac{\cos\theta}{\operatorname{cosec}\theta - 1} = 2\tan\theta$

53. $\left(1+\tan^2 A\right) + \left(1 + \dfrac{1}{\tan^2 A}\right) = \dfrac{1}{\sin^2 A - \sin^4 A}$ [CBSE 2006C]

54. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$

55. (i) $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B$

(ii) $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$

56. $\cot^2 A \operatorname{cosec}^2 B - \cot^2 B \operatorname{cosec}^2 A = \cot^2 A - \cot^2 B$

57. $\tan^2 A \sec^2 B - \sec^2 A \tan^2 B = \tan^2 A - \tan^2 B$

LEVEL-2

Prove the following identities: (58-75)

58. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, prove that $x^2 - y^2 = a^2 - b^2$

[CBSE 2001, 2002C]

59. If $3\sin\theta + 5\cos\theta = 5$, prove that $5\sin\theta - 3\cos\theta = \pm 3$.

60. If $\operatorname{cosec}\theta + \cot\theta = m$ and $\operatorname{cosec}\theta - \cot\theta = n$, prove that $mn = 1$

61. $\dfrac{\tan^3\theta}{1+\tan^2\theta} + \dfrac{\cot^3\theta}{1+\cot^2\theta} = \sec\theta\operatorname{cosec}\theta - 2\sin\theta\cos\theta$

62. If $T_n = \sin^n\theta + \cos^n\theta$, prove that $\dfrac{T_3 - T_5}{T_1} = \dfrac{T_5 - T_7}{T_3}$

63. $\left(\tan\theta + \dfrac{1}{\cos\theta}\right)^2 + \left(\tan\theta - \dfrac{1}{\cos\theta}\right)^2 = 2\left(\dfrac{1+\sin^2\theta}{1-\sin^2\theta}\right)$

64. $\left(\dfrac{1}{\sec^2\theta - \cos^2\theta} + \dfrac{1}{\operatorname{cosec}^2\theta - \sin^2\theta}\right)\sin^2\theta\cos^2\theta = \dfrac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}$

65. (i) $\left(\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$ (ii) $\dfrac{1+\sec\theta-\tan\theta}{1+\sec\theta+\tan\theta} = \dfrac{1-\sin\theta}{\cos\theta}$

[NCERT EXEMPLAR]

66. $(\sec A + \tan A - 1)(\sec A - \tan A + 1) = 2\tan A$

67. $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2$

68. $(\operatorname{cosec}\theta - \sec\theta)(\cot\theta - \tan\theta) = (\operatorname{cosec}\theta + \sec\theta)(\sec\theta\operatorname{cosec}\theta - 2)$

69. $(\sec A - \operatorname{cosec} A)(1 + \tan A + \cot A) = \tan A\sec A - \cot A\operatorname{cosec} A$

70. $\dfrac{\cos A\operatorname{cosec} A - \sin A\sec A}{\cos A + \sin A} = \operatorname{cosec} A - \sec A$

71. $\dfrac{\sin A}{\sec A + \tan A - 1} + \dfrac{\cos A}{\operatorname{cosec} A + \cot A - 1} = 1$

72. $\dfrac{\tan A}{(1+\tan^2 A)^2} + \dfrac{\cot A}{(1+\cot^2 A)^2} = \sin A\cos A$

73. $\sec^4 A(1 - \sin^4 A) - 2\tan^2 A = 1$

74. $\dfrac{\cot^2 A(\sec A - 1)}{1+\sin A} = \sec^2 A\left(\dfrac{1-\sin A}{1+\sec A}\right)$

75. $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A} = \sin A\tan A - \cot A\cos A$

[CBSE 2008]

76. If $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ and $\dfrac{x}{a}\sin\theta - \dfrac{y}{b}\cos\theta = 1$, prove that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$

77. If $\operatorname{cosec}\theta - \sin\theta = a^3$, $\sec\theta - \cos\theta = b^3$, prove that $a^2b^2(a^2+b^2) = 1$

78. If $a\cos^3\theta + 3a\cos\theta\sin^2\theta = m$, $a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$, prove that $(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$

79. If $x = a\cos^3\theta$, $y = b\sin^3\theta$, prove that $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

80. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, prove that $a^2 + b^2 = m^2 + n^2$

81. If $\cos A + \cos^2 A = 1$, prove that $\sin^2 A + \sin^4 A = 1$

82. If $\cos\theta + \cos^2\theta = 1$, prove that
$\sin^{12}\theta + 3\sin^{10}\theta + 3\sin^8\theta + \sin^6\theta + 2\sin^4\theta + 2\sin^2\theta - 2 = 1$

83. Given that: $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$
Show that one of the values of each member of this equality is $\sin\alpha\sin\beta\sin\gamma$

84. If $\sin\theta + \cos\theta = x$, prove that $\sin^6\theta + \cos^6\theta = \dfrac{4-3(x^2-1)^2}{4}$

85. If $x = a\sec\theta\cos\phi$, $y = b\sec\theta\sin\phi$ and $z = c\tan\theta$, show that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

86. If $\sin\theta + 2\cos\theta = 1$ prove that $2\sin\theta - \cos\theta = 2$. [NCERT EXEMPLAR]

11.3 VALUES OF TRIGONOMETRIC RATIOS IN TERMS OF THE VALUE OF ONE OF THEM

In this section, we will find the remaining trigonometric ratios of an angle when one of the trigonometric ratios of the same angle is given. We will also find the values of trigonometric expressions for the given value of one of the trigonometric ratios.

Finding all other trigonometric ratios of angle θ when the value of $\sin\theta$ is given:

Let $\sin\theta = x$. Then,

$$\cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-x^2}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}, \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x},$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x} \text{ and } \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}$$

Finding all other trigonometric ratios of angle θ when the value of $\cos\theta$ is given:

Let $\cos\theta = x$. Then,

$$\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-x^2}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}, \cot\theta = \frac{1}{\tan\theta} = \frac{x}{\sqrt{1-x^2}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\sqrt{1-x^2}},\ \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

Finding all other trigonometric ratios of an angle when the value of is given:

Let $\tan\theta = x$. Then,

$$\sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+x^2}$$

$\therefore$ $$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{x},\ \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+x^2}}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{1+x^2}} = \frac{x}{\sqrt{1+x^2}} \text{ and, } \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\sqrt{1+x^2}}{x}$$

If the values of cosec, are given. From the given values we obtain the values of respectively and then proceed as discussed above.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If* $\sin\theta = \frac{3}{5}$, *find the values of other trigonometric ratios.*

SOLUTION We have, $\sin\theta = \frac{3}{5}$

$\therefore$ $$\cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$\therefore$ $$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{5}{3},\ \sec\theta = \frac{1}{\cos\theta} = \frac{5}{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4} \text{ and } \cot\theta = \frac{1}{\tan\theta} = \frac{4}{3}$$

EXAMPLE 2 *If* $\cot\theta = \frac{9}{40}$, *find the values of* $\operatorname{cosec}\theta$ *and* $\sec\theta$.

SOLUTION We have, $\cot\theta = \frac{9}{40}$

$\therefore$ $$\operatorname{cosec}\theta = \sqrt{1+\cot^2\theta} \Rightarrow \operatorname{cosec}\theta = \sqrt{1+\left(\frac{9}{40}\right)^2} = \sqrt{1+\frac{81}{1600}} = \sqrt{\frac{1681}{1600}} = \frac{41}{40}$$

Again, $$\cot\theta = \frac{9}{40} \Rightarrow \tan\theta = \frac{1}{\cot\theta} = \frac{40}{9}$$

$\therefore$ $$\sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+\left(\frac{40}{9}\right)^2} = \sqrt{\frac{1681}{81}} = \frac{41}{9}$$

EXAMPLE 3 *If* $\cos\theta = \frac{1}{2}$, *find the value of* $\frac{2\sec\theta}{1+\tan^2\theta}$.

SOLUTION We have, $\cos\theta = \frac{1}{2}$

$$\therefore \quad \sec\theta = \frac{1}{\cos\theta} = 2$$

$$\therefore \quad \frac{2\sec\theta}{1+\tan^2\theta} = \frac{2\sec\theta}{\sec^2\theta} = \frac{2}{\sec\theta} = \frac{2}{2} = 1$$

EXAMPLE 4 *If* $\tan\theta = \frac{12}{5}$, *find the value of* $\frac{1+\sin\theta}{1-\sin\theta}$.

SOLUTION We have, $\tan\theta = \frac{12}{5}$

$$\therefore \quad \sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+\left(\frac{12}{5}\right)^2} = \sqrt{\frac{169}{25}} = \frac{13}{5}$$

$$\therefore \quad \cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \frac{5}{13}$$

Now, $\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

Hence, $\frac{1+\sin\theta}{1-\sin\theta} = \frac{1+\frac{12}{13}}{1-\frac{12}{13}} = \frac{\frac{25}{13}}{\frac{1}{13}} = 25$

EXAMPLE 5 *If* $\sin\theta = \frac{3}{5}$, *find the value of* $(\tan\theta + \sec\theta)^2$.

SOLUTION We have, $\sin\theta = \frac{3}{5}$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4} \text{ and, } \sec\theta = \frac{1}{\cos\theta} \Rightarrow \sec\theta = \frac{5}{4}$$

Hence, $(\tan\theta + \sec\theta)^2 = \left(\frac{3}{4}+\frac{5}{4}\right)^2 = \left(\frac{8}{4}\right)^2 = 4$

EXAMPLE 6 *If* $\tan\theta = \frac{3}{4}$, *find the value of* $\frac{1-\cos\theta}{1+\cos\theta}$.

SOLUTION We have, $\tan\theta = \frac{3}{4}$

$$\therefore \quad \sec\theta = \sqrt{1+\tan^2\theta} \Rightarrow \sec\theta = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$$

and, $$\cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \frac{4}{5}$$

$\therefore$ $$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{4}{5}}{1+\frac{4}{5}} = \frac{\frac{1}{5}}{\frac{9}{5}} = \frac{1}{9}$$

EXAMPLE 7 *If* $\cos\theta = \frac{3}{5}$, *find the value of* $\cot\theta + \operatorname{cosec}\theta$.

SOLUTION We have, $\cos\theta = \frac{3}{5}$

$\therefore$ $$\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

and, $$\cot\theta = \frac{\cos\theta}{\sin\theta} \Rightarrow \cot\theta = \frac{3/5}{4/5} = \frac{3}{4}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \Rightarrow \operatorname{cosec}\theta = \frac{5}{4}$$

$\therefore$ $$\cot\theta + \operatorname{cosec}\theta = \frac{3}{4} + \frac{5}{4} = \frac{8}{4} = 2$$

EXAMPLE 8 *If* $\tan\theta = \frac{1}{\sqrt{7}}$, *find the value of* $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$.

SOLUTION We have, $\tan\theta = \frac{1}{\sqrt{7}}$

$\therefore$ $$\cot\theta = \frac{1}{\tan\theta} = \sqrt{7}$$

Now,

$$\sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta = 1 + \left(\frac{1}{\sqrt{7}}\right)^2 = 1 + \frac{1}{7} = \frac{8}{7}$$

and, $$\operatorname{cosec}^2\theta = 1 + \cot^2\theta \Rightarrow \operatorname{cosec}^2\theta = 1 + (\sqrt{7})^2 = 1 + 7 = 8.$$

$\therefore$ $$\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta} = \frac{8-\frac{8}{7}}{8+\frac{8}{7}} = \frac{48/7}{64/7} = \frac{48}{64} = \frac{3}{4}$$

EXAMPLE 9 *If* $\operatorname{cosec} A = \sqrt{2}$, *find the value of* $\frac{2\sin^2 A + 3\cot^2 A}{4\tan^2 A - \cos^2 A}$.

SOLUTION We have, $\operatorname{cosec} A = \sqrt{2}$

$\therefore$ $$\sin A = \frac{1}{\operatorname{cosec} A} \Rightarrow \sin A = \frac{1}{\sqrt{2}}$$

$$\cos A = \sqrt{1-\sin^2 A} \Rightarrow \cos A = \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\tan A = \frac{\sin A}{\cos A} \Rightarrow \tan A = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1 \text{ and, } \cot A = \frac{1}{\tan A} \Rightarrow \cot A = \frac{1}{1} = 1$$

Hence, $$\frac{2\sin^2 A + 3\cot^2 A}{4\tan^2 A - \cos^2 A} = \frac{2 \times \left(\frac{1}{\sqrt{2}}\right)^2 + 3(1)^2}{4(1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{2 \times \frac{1}{2} + 3}{4 - \frac{1}{2}} = \frac{1+3}{7/2} = \frac{8}{7}$$

EXAMPLE 10 *If* $\cot\theta = \frac{15}{8}$, *then evaluate:* $\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)}$ [CBSE 2009]

SOLUTION We have,

$$\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)}$$

$$= \frac{2(1+\sin\theta)(1-\sin\theta)}{2(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{2(1-\sin^2\theta)}{2(1-\cos^2\theta)} = \frac{\cos^2\theta}{\sin^2\theta} = \left(\frac{\cos\theta}{\sin\theta}\right)^2 = (\cot\theta)^2$$

$\therefore$ $$\cot\theta = \frac{15}{8} \Rightarrow \frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)} = (\cot\theta)^2 = \left(\frac{15}{8}\right)^2 = \frac{225}{64}$$

EXAMPLE 11 *If* $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$, $0 < \theta < 90°$, *find the values of* $\cos\theta$ *and* $\tan\theta$.

SOLUTION We have, $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

$\therefore$ $$\cos\theta = \sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = \sqrt{1 - \frac{a^2}{a^2+b^2}} = \sqrt{\frac{b^2}{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}}$$

and, $$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{a/\sqrt{a^2+b^2}}{b/\sqrt{a^2+b^2}} = \frac{a}{b}$$

LEVEL-2

EXAMPLE 12 *If* $\sin\theta + \cos\theta = \sqrt{2}\sin(90° - \theta)$, *determine* $\cot\theta$.

SOLUTION We have,

$$\sin\theta + \cos\theta = \sqrt{2}\sin(90° - \theta)$$

$\Rightarrow$ $$\sin\theta + \cos\theta = \sqrt{2}\cos\theta$$

$\Rightarrow$ $$\sin\theta = \sqrt{2}\cos\theta - \cos\theta$$

$\Rightarrow$ $$\sin\theta = (\sqrt{2} - 1)\cos\theta$$

$\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{(\sqrt{2}-1)\cos\theta}{\cos\theta}$ [Dividing throughout by $\cos\theta$

$\Rightarrow \quad \tan\theta = (\sqrt{2}-1)$

$\Rightarrow \quad \cot\theta = \dfrac{1}{\sqrt{2}-1}$ $\left[\because \cot\theta = \dfrac{1}{\tan\theta}\right]$

$\Rightarrow \quad \cot\theta = \dfrac{(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \dfrac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$

EXAMPLE 13 *If* $\tan\theta + \cot\theta = 2$, *find the value of* $\tan^2\theta + \cot^2\theta$.

SOLUTION We have,

$\tan\theta + \cot\theta = 2$

$\Rightarrow \quad (\tan\theta + \cot\theta)^2 = 4$ [On squaring both sides]

$\Rightarrow \quad \tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta = 4$

$\Rightarrow \quad \tan^2\theta + \cot^2\theta + 2 = 4$ [$\because \tan\theta\cot\theta = 1$]

$\Rightarrow \quad \tan^2\theta + \cot^2\theta = 2$

EXAMPLE 14 *Prove that* $\tan\theta + \tan(90^\circ - \theta) = \sec\theta\sec(90^\circ - \theta)$. [NCERT EXEMPLAR

SOLUTION LHS $= \tan\theta + \tan(90^\circ - \theta)$

$= \tan\theta + \cot\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$

$= \sec\theta\operatorname{cosec}\theta = \sec\theta\sec(90^\circ - \theta) =$ RHS.

EXAMPLE 15 *Show that* $\dfrac{\cos^2(45^\circ + \theta) + \cos^2(45^\circ - \theta)}{\tan(60^\circ + \theta)\tan(30^\circ - \theta)} = 1$. [NCERT EXEMPLAR

SOLUTION LHS $= \dfrac{\cos^2(45^\circ + \theta) + \cos^2(45^\circ - \theta)}{\tan(60^\circ + \theta)\tan(30^\circ - \theta)}$

$= \dfrac{\cos^2(45^\circ + \theta) + \sin^2(90^\circ - (45^\circ - \theta))}{\tan(60^\circ + \theta)\cot(90^\circ - (30^\circ - \theta))}$

$= \dfrac{\cos^2(45^\circ + \theta) + \sin^2(45^\circ + \theta)}{\tan(60^\circ + \theta)\cot(60^\circ + \theta)} = \dfrac{1}{1} = 1 =$ RHS

EXAMPLE 16 *Given that* $\alpha + \beta = 90^\circ$, *show that* $\sqrt{\cos\alpha\operatorname{cosec}\beta - \cos\alpha\sin\beta} = \sin\alpha$.

[NCERT EXEMPLAR

SOLUTION We have, $\alpha + \beta = 90^\circ$. Therefore, $\beta = 90^\circ - \alpha$.

$\therefore \quad \operatorname{cosec}\beta = \operatorname{cosec}(90^\circ - \alpha) = \sec\alpha$ and, $\sin\beta = \sin(90^\circ - \alpha) = \cos\alpha$

Now, $\sqrt{\cos\alpha\operatorname{cosec}\beta - \cos\alpha\sin\beta}$

$= \sqrt{\cos\alpha\sec\alpha - \cos\alpha\cos\alpha} = \sqrt{1 - \cos^2\alpha} = \sqrt{\sin^2\alpha} = \sin\alpha$

EXAMPLE 17 *If* $\sec\theta = x + \frac{1}{4x}$, *prove that* : $\sec\theta + \tan\theta = 2x$ *or*, $\frac{1}{2x}$ **[CBSE 2001]**

SOLUTION We have,

$$\sec\theta = x + \frac{1}{4x}$$

$$\therefore \quad \tan^2\theta = \sec^2\theta - 1$$

$$\Rightarrow \quad \tan^2\theta = \left(x + \frac{1}{4x}\right)^2 - 1 = x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1 = x^2 + \frac{1}{16x^2} - \frac{1}{2} = \left(x - \frac{1}{4x}\right)^2$$

$$\Rightarrow \quad \tan\theta = \pm\left(x - \frac{1}{4x}\right)$$

$$\Rightarrow \quad \tan\theta = \left(x - \frac{1}{4x}\right) \text{ or, } \tan\theta = -\left(x - \frac{1}{4x}\right)$$

CASE I When $\tan\theta = -\left(x - \frac{1}{4x}\right)$: In this case,

$$\sec\theta + \tan\theta = x + \frac{1}{4x} + x - \frac{1}{4x} = 2x$$

CASE II When $\tan\theta = -\left(x - \frac{1}{4x}\right)$: In this case,

$$\sec\theta + \tan\theta = \left(x + \frac{1}{4x}\right) - \left(x - \frac{1}{4x}\right) = \frac{2}{4x} = \frac{1}{2x}$$

Hence, $\sec\theta + \tan\theta = 2x$ or, $\frac{1}{2x}$

ALITER Let $\sec\theta + \tan\theta = \lambda$ ...(i)

Then,

$$\Rightarrow \quad \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$$

$$\Rightarrow \quad \lambda(\sec\theta - \tan\theta) = 1 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \sec\theta - \tan\theta = \frac{1}{\lambda} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2\sec\theta = \lambda + \frac{1}{\lambda}$$

$$\Rightarrow \quad 2\left(x + \frac{1}{4x}\right) = \lambda + \frac{1}{\lambda} \quad \left[\because \sec\theta = x + \frac{1}{4x} \text{ (Given)}\right]$$

$$\Rightarrow \quad 2x + \frac{1}{2x} = \lambda + \frac{1}{\lambda}$$

$$\Rightarrow \quad \lambda = 2x \text{ or, } \lambda = \frac{1}{2x} \quad \text{[On comparing two sides]}$$

$$\Rightarrow \quad \sec\theta + \tan\theta = 2x, \frac{1}{2x} \quad \text{[Using (i)]}$$

EXERCISE 11.2

LEVEL-1

1. If $\cos\theta = \frac{4}{5}$, find all other trigonometric ratios of angle θ

2. If $\sin\theta = \frac{1}{\sqrt{2}}$, find all other trigonometric ratios of angle θ

3. If $\tan\theta = \frac{1}{\sqrt{2}}$, find the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \cot^2\theta}$

4. If $4\tan\theta = 3$, evaluate $\frac{4\sin\theta - \cos\theta + 1}{4\sin\theta + \cos\theta - 1}$ [CBSE 2018]

5. If $\tan\theta = \frac{12}{5}$, find the value of $\frac{1+\sin\theta}{1-\sin\theta}$

6. If $\cot\theta = \frac{1}{\sqrt{3}}$, find the value of $\frac{1-\cos^2\theta}{2-\sin^2\theta}$

7. If $\operatorname{cosec} A = \sqrt{2}$, find the value of $\frac{2\sin^2 A + 3\cot^2 A}{4(\tan^2 A - \cos^2 A)}$

8. If $\cot\theta = \sqrt{3}$, find the value of $\frac{\operatorname{cosec}^2\theta + \cot^2\theta}{\operatorname{cosec}^2\theta - \sec^2\theta}$

9. If $3\cos\theta = 1$, find the value of $\frac{6\sin^2\theta + \tan^2\theta}{4\cos\theta}$

10. If $\sqrt{3}\tan\theta = 3\sin\theta$, find the value of $\sin^2\theta - \cos^2\theta$ [CBSE 2001]

11. If $\operatorname{cosec}\theta = \frac{13}{12}$, find the value of $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta}$ [CBSE 2001]

LEVEL-2

12. If $\sin\theta + \cos\theta = \sqrt{2}\cos(90° - \theta)$, find $\cot\theta$.

13. If $2\sin^2\theta - \cos^2\theta = 2$, then find the value of θ. [NCERT EXEMPLAR]

13. If $\sqrt{3}\tan\theta - 1 = 0$, find the value of $\sin^2 - \cos^2\theta$. [NCERT EXEMPLAR]

ANSWERS

1. $\sin\theta = 3/5, \tan\theta = 3/4, \sec\theta = 5/4, \operatorname{cosec}\theta = 5/3, \cot\theta = 4/3$
2. $\cos\theta = \frac{1}{\sqrt{2}}, \tan\theta = 1, \sec\theta = \sqrt{2}, \operatorname{cosec}\theta = \sqrt{2}, \cot\theta = 1$
3. $\frac{3}{10}$ 4. $\frac{13}{11}$ 5. 25 6. $\frac{3}{5}$ 7. 2 8. $\frac{21}{8}$
9. 10 10. $\frac{1}{3}$ 11. 3 12. $\sqrt{2}-1$ 13. $90°$ 14. $-\frac{1}{2}$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. Define an identity.
2. What is the value of $(1-\cos^2\theta)\ \mathrm{cosec}^2\theta$?
3. What is the value of $(1+\cot^2\theta)\ \sin^2\theta$?
4. What is the value of $\sin^2\theta+\dfrac{1}{1+\tan^2\theta}$?
5. If $\sec\theta+\tan\theta=x$, write the value of $\sec\theta-\tan\theta$ in terms of x.
6. If $\mathrm{cosec}\,\theta-\cot\theta=\alpha$, write the value of $\mathrm{cosec}\,\theta+\cot\alpha$.
7. Write the value of $\mathrm{cosec}^2(90^\circ-\theta)-\tan^2\theta$.
8. Write the value of $\sin A\cos(90^\circ-A)+\cos A\sin(90^\circ-A)$.
9. Write the value of $\cot^2\theta-\dfrac{1}{\sin^2\theta}$.
10. If $x=a\sin\theta$ and $y=b\cos\theta$, what is the value of $b^2x^2+a^2y^2$?
11. If $\sin\theta=\dfrac{4}{5}$, what is the value of $\cot\theta+\mathrm{cosec}\,\theta$?
12. What is the value of $9\cot^2\theta-9\,\mathrm{cosec}^2\theta$?
13. What is the value of $6\tan^2\theta-\dfrac{6}{\cos^2\theta}$?
14. What is the value of $\dfrac{\tan^2\theta-\sec^2\theta}{\cot^2\theta-\mathrm{cosec}^2\theta}$?
15. What is the value of $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta)$?
16. If $\cos A=\dfrac{7}{25}$, find the value of $\tan A+\cot A$. [CBSE 2008]
17. If $\sin\theta=\dfrac{1}{3}$, then find the value of $2\cot^2\theta+2$. [CBSE 2009]
18. If $\cos\theta=\dfrac{3}{4}$, then find the value of $9\tan^2\theta+9$.
19. If $\sec^2\theta(1+\sin\theta)(1-\sin\theta)=k$, then find the value of k. [CBSE 2009]
20. If $\mathrm{cosec}^2\theta(1+\cos\theta)(1-\cos\theta)=\lambda$, then find the value of λ.
21. If $\sin^2\theta\cos^2\theta(1+\tan^2\theta)(1+\cot^2\theta)=\lambda$, then find the value of λ.
22. If $5x=\sec\theta$ and $\dfrac{5}{x}=\tan\theta$, find the value of $5\left(x^2-\dfrac{1}{x^2}\right)$. [CBSE 2010]
23. If $\mathrm{cosec}\,\theta=2x$ and $\cot\theta=\dfrac{2}{x}$, find the value of $2\left(x^2-\dfrac{1}{x^2}\right)$ [CBSE 2010]

24. Write 'True' or 'False' and justify your answer in each of the following:

(i) The value of $\sin\theta$ is $x + \frac{1}{x}$, where 'x' is a positive real number.

(ii) $\cos\theta = \frac{a^2 + b^2}{2ab}$, where a and b are two distinct numbers such that $ab > 0$.

(iii) The value of $\cos^2 23 - \sin^2 67$ is positive.

(iv) The value of the expression $\sin 80° - \cos 80°$ is negative.

(v) The value of $\sin\theta + \cos\theta$ is always greater than 1.

25. What is the value of $\cos^2 67° - \sin^2 23°$? [CBSE 2018]

ANSWERS

2. 1	3. 1	4. 1	5. $\frac{1}{x}$	6. $\frac{1}{\alpha}$	7. 1	8. 1
9. -1	10. a^2b^2	11. 2	12. -9	13. -6	14. 1	15. 1
16. $\frac{625}{168}$	17. 18	18. 16	19. 1	20. 1	21. 1	22. $\frac{1}{5}$
23. $\frac{1}{2}$	24. (i) False	(ii) False	(iii) False	(iv) False	(v) False	25. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\sec\theta + \tan\theta = x$, then $\sec\theta =$

(a) $\frac{x^2+1}{x}$ (b) $\frac{x^2+1}{2x}$ (c) $\frac{x^2-1}{2x}$ (d) $\frac{x^2-1}{x}$

2. If $\sec\theta + \tan\theta = x$, then $\tan\theta =$

(a) $\frac{x^2+1}{x}$ (b) $\frac{x^2-1}{x}$ (c) $\frac{x^2+1}{2x}$ (d) $\frac{x^2-1}{2x}$

3. $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ is equal to

(a) $\sec\theta + \tan\theta$ (b) $\sec\theta - \tan\theta$ (c) $\sec^2\theta + \tan^2\theta$ (d) $\sec^2\theta - \tan^2\theta$

4. The value of $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$ is

(a) $\cot\theta - \operatorname{cosec}\theta$ (b) $\operatorname{cosec}\theta + \cot\theta$ (c) $\operatorname{cosec}^2\theta + \cot^2\theta$ (d) $(\cot\theta + \operatorname{cosec}\theta)^2$

5. $\sec^4 A - \sec^2 A$ is equal to

(a) $\tan^2 A - \tan^4 A$ (b) $\tan^4 A - \tan^2 A$ (c) $\tan^4 A + \tan^2 A$ (d) $\tan^2 A + \tan^4 A$

6. $\cos^4 A - \sin^4 A$ is equal to

(a) $2\cos^2 A + 1$ (b) $2\cos^2 A - 1$ (c) $2\sin^2 A - 1$ (d) $2\sin^2 A + 1$

7. $\dfrac{\sin\theta}{1+\cos\theta}$ is equal to

(a) $\dfrac{1+\cos\theta}{\sin\theta}$ (b) $\dfrac{1-\cos\theta}{\cos\theta}$ (c) $\dfrac{1-\cos\theta}{\sin\theta}$ (d) $\dfrac{1-\sin\theta}{\cos\theta}$

8. $\dfrac{\sin\theta}{1-\cot\theta}+\dfrac{\cos\theta}{1-\tan\theta}$ is equal to

(a) 0 (b) 1 (c) $\sin\theta+\cos\theta$ (d) $\sin\theta-\cos\theta$

9. The value of $(1+\cot\theta-\operatorname{cosec}\theta)(1+\tan\theta+\sec\theta)$ is

(a) 1 (b) 2 (c) 4 (d) 0

10. $\dfrac{\tan\theta}{\sec\theta-1}+\dfrac{\tan\theta}{\sec\theta+1}$ is equal to

(a) $2\tan\theta$ (b) $2\sec\theta$ (c) $2\operatorname{cosec}\theta$ (d) $2\tan\theta\sec\theta$

11. $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)$ is equal

(a) 0 (b) 1 (c) -1 (d) none of these

12. If $x=a\cos\theta$ and $y=b\sin\theta$, then $b^2x^2+a^2y^2=$

(a) a^2b^2 (b) ab (c) a^4b^4 (d) a^2+b^2

13. If $x=a\sec\theta$ and $y=b\tan\theta$, then $b^2x^2-a^2y^2=$

(a) ab (b) a^2-b^2 (c) a^2+b^2 (d) a^2b^2

14. $\dfrac{\cot\theta}{\cot\theta-\cot 3\theta}+\dfrac{\tan\theta}{\tan\theta-\tan 3\theta}$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

15. $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)$ is equal to

(a) 0 (b) 1 (c) -1 (d) none of these

16. If $a\cos\theta+b\sin\theta=4$ and $a\sin\theta-b\cos\theta=3$, then $a^2+b^2=$

(a) 7 (b) 12 (c) 25 (d) none of these

17. If $a\cot\theta+b\operatorname{cosec}\theta=p$ and $b\cot\theta+a\operatorname{cosec}\theta=q$, then $p^2-q^2=$

(a) a^2-b^2 (b) b^2-a^2 (c) a^2+b^2 (d) $b-a$

18. The value of $\sin^2 29^\circ+\sin^2 61^\circ$ is

(a) 1 (b) 0 (c) $2\sin^2 29^\circ$ (d) $2\cos^2 61$

19. If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$, then

(a) $x^2+y^2+z^2=r^2$ (b) $x^2+y^2-z^2=r^2$

(c) $x^2-y^2+z^2=r^2$ (d) $z^2+y^2-x^2=r^2$

20. If $\sin\theta + \sin^2\theta = 1$, then $\cos^2\theta + \cos^4\theta =$
(a) -1 (b) 1 (c) 0 (d) none of these

21. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, then $a^2 + b^2 =$
(a) $m^2 - n^2$ (b) m^2n^2 (c) $n^2 - m^2$ (d) $m^2 + n^2$

22. If $\cos A + \cos^2 A = 1$, then $\sin^2 A + \sin^4 A =$
(a) -1 (b) 0 (c) 1 (d) none of these

23. If $x = a\sec\theta\cos\phi$, $y = b\sec\theta\sin\phi$ and $z = c\tan\theta$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} =$
(a) $\frac{z^2}{c^2}$ (b) $1 - \frac{z^2}{c^2}$ (c) $\frac{z^2}{c^2} - 1$ (d) $1 + \frac{z^2}{c^2}$

24. If $a\cos\theta - b\sin\theta = c$, then $a\sin\theta + b\cos\theta =$
(a) $\pm\sqrt{a^2 + b^2 + c^2}$ (b) $\pm\sqrt{a^2 + b^2 - c^2}$ (c) $\pm\sqrt{c^2 - a^2 - b^2}$ (d) none of thes

25. $9\sec^2 A - 9\tan^2 A$ is equal to
(a) 1 (b) 9 (c) 8 (d) 0
[NCER

26. $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\,\theta) =$
(a) 0 (b) 1 (c) 1 (d) -1

27. $(\sec A + \tan A)(1 - \sin A) =$
(a) $\sec A$ (b) $\sin A$ (c) $\text{cosec}\,A$ (d) $\cos A$
[NCER

28. $\frac{1 + \tan^2 A}{1 + \cot^2 A}$ is equal to
(a) $\sec^2 A$ (b) -1 (c) $\cot^2 A$ (d) $\tan^2 A$
[NCER

29. If $\sin\theta - \cos\theta = 0$, then the value of $\sin^4\theta + \cos^4\theta$ is
(a) 1 (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

30. The value of $\sin(45° + \theta) - \cos(45° - \theta)$ is equal to
(a) $2\cos\theta$ (b) 0 (c) $2\sin\theta$ (d) 1

31. If ΔABC is right angled at C, then the value of $\cos(A + B)$ is
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}$

32. If $\cos 9\theta = \sin \theta$ and $9\theta < 90°$, then the value of $\tan 6\theta$ is

(a) $1/\sqrt{3}$ (b) $\sqrt{3}$ (c) 1 (d) 0

33. If $\cos(\alpha + \beta) = 0$, then $\sin(\alpha - \beta)$ can be reduced to

(a) $\cos \beta$ (b) $\cos 2\beta$ (c) $\sin \alpha$ (d) $\sin 2\alpha$

ANSWERS

1. (b)	2. (d)	3. (a)	4. (b)	5. (c)	6. (b)
7. (c)	8. (c)	9. (b)	10. (c)	11. (b)	12. (a)
13. (d)	14. (b)	15. (c)	16. (c)	17. (b)	18. (a)
19. (a)	20. (b)	21. (d)	22. (c)	23. (d)	24. (b)
25. (b)	26. (c)	27. (d)	28. (d)	29. (c)	30. (b)
31. (a)	32. (b)	33. (b)			

SUMMARY

1. An equation is called an identity if it is true for all values of the variable (s) involved.
2. An equation involving trigonometric ratios of an angle is called a trigonometric identity if it is true for all values of the angle.
3. Following are some trigonometric identities:

   (i) $\sin^2\theta + \cos^2\theta = 1$ or, $1 - \cos^2\theta = \sin^2\theta$ or, $1 - \sin^2\theta = \cos^2\theta$

   (ii) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec^2\theta - \tan^2\theta = 1$

   (iii) $1 + \cot^2\theta = \text{cosec}^2\theta$ or, $\text{cosec}^2\theta - \cot^2\theta = 1$.

CHAPTER 12

HEIGHT AND DISTANCES

12.1 INTRODUCTION

In this chapter, we shall be applying the trigonometric results to discuss problems regarding heights and distances. We begin by defining some terms which will be used in this chapter.

12.2 ANGLES OF ELEVATION AND DEPRESSION

Let O and P be two points such that the point P is at higher level. Let OA and PB be horizontal lines through O and P respectively.

If an observer is at O and the point P is the object under consideration, then the line OP is called the *line of sight* of the point P and the angle AOP, between the line of sight and the horizontal line OA, is known as the angle of elevation of point P as seen from O.

If an observer is at P and the object under consideration is at O, then the angle BPO is known as the angle of *depression* of O as seen from P.

Obviously, the angle of elevation of a point P as seen from a point O is equal to the angle of *depression* of O as seen from P.

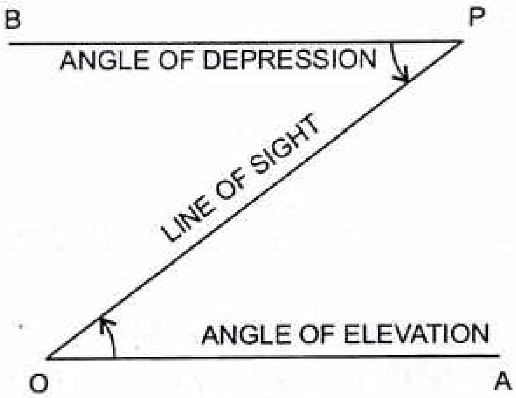

Fig. 12.1

LEVEL-1

EXAMPLE 1 *A tower is $100\sqrt{3}$ metres high. Find the angle of elevation if its top from a point 100 metres away from its foot.*

SOLUTION Let AB be the tower of height $100\sqrt{3}$ metres, and let C be a point at a distance of 100 metres from the foot of the tower.

Let θ be the angle of elevation of the top of the tower from point C.

Clearly, in ΔCAB the lengths of base AC and perpendicular AB are known. So, we will use the trigonometric ratio containing base and perpendicular. Such a ratio is tangent. Taking tangent of angle $\angle ACB$ in ΔCAB, we have

$$\tan\theta = \frac{AB}{AC}$$

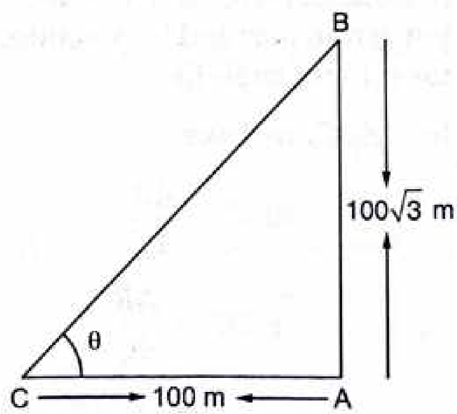

Fig. 12.2

$$\Rightarrow \qquad \tan\theta = \frac{100\sqrt{3}}{100} = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan 60^\circ$$

$$\Rightarrow \qquad \theta = 60^\circ$$

Hence, the angle of elevation of the top of the tower from a point 100 metres away from its foot is 60°.

EXAMPLE 2 *The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower is 30°. Find the height of the tower.* **[NCERT]**

SOLUTION Let AB be the tower of height h meters and C be a point on the ground such that the angle of elevation of the top A of tower AB is of 30°.

In ΔABC, we are given $\angle C = 30^\circ$ and Base $BC = 30$ m and we have to find perpendicular AB. So, we use that trigonometrical ratios which contains base and perpendicular. Clearly, such ratio is tangent. So, we take tangent of $\angle C$.

In ΔABC, taking tangent of $\angle C$, we have,

$$\tan C = \frac{AB}{BC}$$

$$\Rightarrow \qquad \tan 30^\circ = \frac{AB}{BC}$$

$$\Rightarrow \qquad \frac{1}{\sqrt{3}} = \frac{h}{30}$$

$$\Rightarrow \qquad h = \frac{30}{\sqrt{3}} \text{ metres} = 10\sqrt{3} \text{ metres}$$

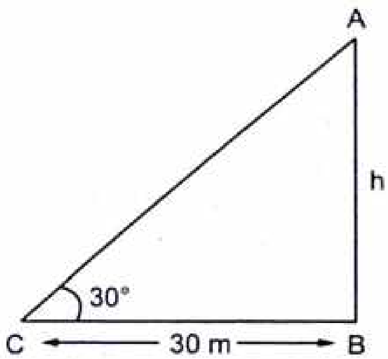

Fig. 12.3

Hence, the height of the tower is $10\sqrt{3}$ metres.

EXAMPLE 3 *A kite is flying at a height of 60 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string assuming that there is no slack in the string.* **[NCERT]**

SOLUTION Let A be the kite and CA be the string attached to the kite such that its one end is tied to a point C on the ground. The inclination of the string CA with the ground is 60°.

In ΔABC, we are given that $\angle C = 60^\circ$ and perpendicular $AB = 60$ m and we have to find hypotenuse AC. So, we use the trigonometric ratio involving perpendicular and hypotenuse. Clearly, such ratio is sine. So, we take sine of angle C.

In ΔABC, we have

$$\sin C = \frac{AB}{AC}$$

$$\Rightarrow \qquad \sin 60^\circ = \frac{AB}{AC}$$

$$\Rightarrow \qquad \frac{\sqrt{3}}{2} = \frac{60}{AC}$$

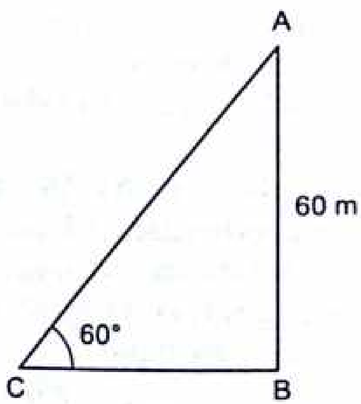

Fig. 12.4

$$\Rightarrow \qquad AC = \frac{120}{\sqrt{3}} = 40\sqrt{3} \text{ m}.$$

Hence, the length of the string is $40\sqrt{3}$ m

EXAMPLE 4 *The string of a kite is 100 metres long and it makes an angle of 60° with the horizontal. Find the height of the kite, assuming that there is no slack in the string.*

SOLUTION Let OA be the horizontal ground, and let K be the position of the kite at a height h above the ground. Then, $AK = h$.

It is given that $OK = 100$ metres and $\angle AOK = 60°$.

Thus, in $\Delta\, OAK$, we have hypotenuse $OK = 100$ m and $\angle\, AOK = 60°$ and we wish to find the perpendicular AK. So, we use the trigonometric ratio involving perpendicular and hypotenuse. Clearly, sine is such a ratio. So, we take the sine of $\angle\, AOK$ in ΔOAK.

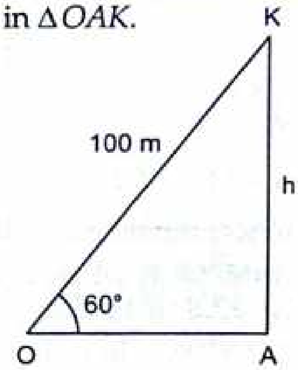

Fig. 12.5

In ΔAOK, we have

$$\sin 60° = \frac{AK}{OK}$$

$$\Rightarrow \qquad \sin 60° = \frac{h}{100}$$

$$\Rightarrow \qquad h = 100 \sin 60°$$

$$\Rightarrow \qquad h = 100\frac{\sqrt{3}}{2} = 50\sqrt{3} = 86.60 \text{ metres}.$$

Hence, the height of the kite is 86.60 metres.

EXAMPLE 5 *A circus artist is climbing from the ground along a rope stretched from the top of a vertical pole and tied at the ground. The height of the pole is 12 m and the angle made by the rope with ground level is 30°. Calculate the distance covered by the artist in climbing to the top of the pole.*

SOLUTION Clearly, distance covered by the artist is equal to the length of the rope AC. Let AB be the vertical pole of height 12 m.

It is given that $\angle ACB = 30°$.

Thus, in right-angled triangle ABC, we have

Perpendicular $AB = 12$ m, $\angle\, ACB = 30°$ and we wish to find hypotenuse AC.

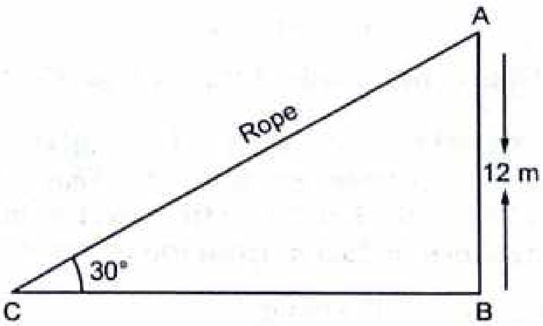

Fig. 12.6

$$\therefore \qquad \sin 30° = \frac{AB}{AC}$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{12}{AC}$$

$$\Rightarrow \qquad AC = 24 \text{ m}$$

Hence, the distance covered by the circus artist is 24 m.

EXAMPLE 6 *A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole if the angle made by the rope with the ground level is 30°.*

SOLUTION Let AB be the vertical pole and CA be the 20 m long rope such that its one end is tied from the top of the vertical pole AB and the other end C is tied to a point C on the ground.

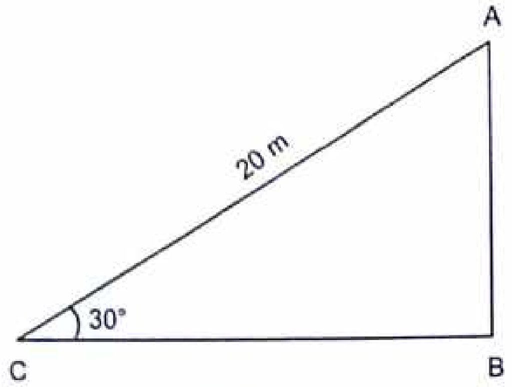

Fig. 12.7

In ΔABC, we have

$$\sin 30^\circ = \frac{AB}{AC}$$

$\Rightarrow$ $$\frac{1}{2} = \frac{AB}{AC}$$

$\Rightarrow$ $$AB = 10 \text{ m}.$$

Hence, the height of the pole is 10 m.

EXAMPLE 7 *A bridge across a river makes an angle of 45° with the river bank as shown in Fig. 12.8. If the length of the bridge across the river is 150 m, what is the width of the river?*

SOLUTION In right triangle ABC, we have

$$\sin 45^\circ = \frac{BC}{AC}$$

$\Rightarrow$ $$\frac{1}{\sqrt{2}} = \frac{BC}{150}$$

$\Rightarrow$ $$BC = \frac{150}{\sqrt{2}}$$

$\Rightarrow$ $$BC = 75\sqrt{2} \text{ m}$$

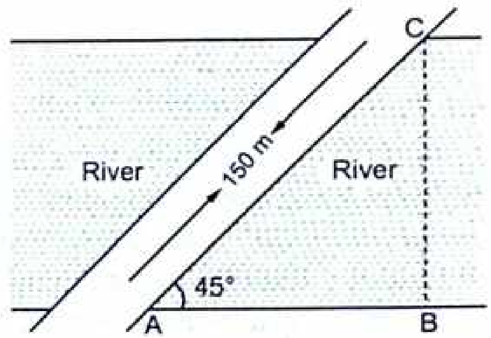

Fig. 12.8

Hence, the width of the river is $75\sqrt{2}$ metres.

EXAMPLE 8 *An observer 1.5 m tall is 28.5 m away from a tower. The angle of elevation of the top of the tower from her eyes is 45°. What is the height of the tower?* **[NCERT]**

SOLUTION Let AB be the tower of height h and CD be the observer of height 1.5 m at a distance of 28.5 m from the tower AB.

In ΔAED, we have

$$\tan 45^\circ = \frac{AE}{DE}$$

$\Rightarrow$ $$1 = \frac{AE}{28.5}$$

$\Rightarrow$ $$AE = 28.5 \text{ m}$$

$\therefore$ $$h = AE + BE = AE + DC$$
$$= (28.5 + 1.5) \text{ m} = 30 \text{ m}$$

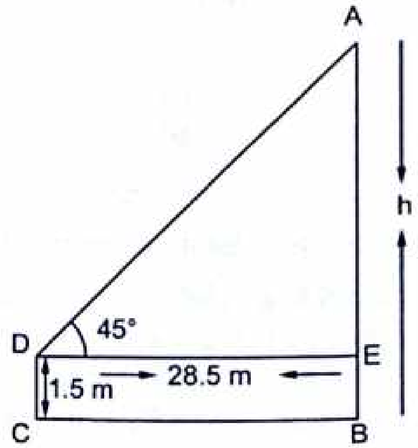

Fig. 12.9

Hence, the height of the tower is 30 m.

EXAMPLE 9 *An electrician has to repair an electric fault on a pole of height 4 m. He needs to reach a point 1.3 m below the top of the pole to undertake the repair work. What should be the length of the ladder that he should use which when inclined at an angle of 60° to the horizontal would enable him to reach the required position?* **[NCERT]**

SOLUTION Let AC be the electric pole of height 4 m. Let B be a point 1.3 m below the top A of the pole AC.

Then, $BC = AC - AB = (4 - 1.3)\text{ m} = 2.7\text{ m}$

Let BD be the ladder inclined at an angle of 60° to the horizontal.

In ΔBCD, we have

$$\sin 60° = \frac{BC}{BD}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{2.7}{BD} \qquad [\because BC = 2.7\text{ m}]$$

$$\Rightarrow \quad BD = \frac{2 \times 2.7}{\sqrt{3}}\text{ m} = \frac{5.4}{\sqrt{3}}\text{ m} = \frac{5.4 \times \sqrt{3}}{3}\text{ m}$$

$$\Rightarrow \quad BD = (1.8)\sqrt{3}\text{ m} = \frac{9}{5}\sqrt{3}\text{ m}$$

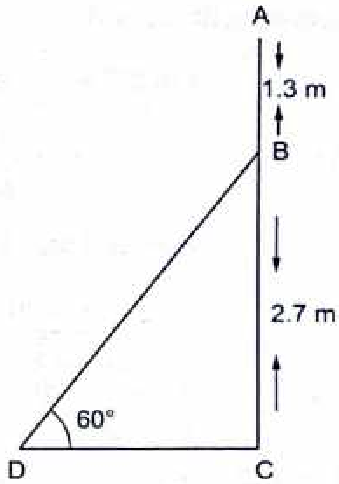

Fig. 12.10

Hence, the length of the ladder should be $\frac{9\sqrt{3}}{5}$ m.

EXAMPLE 10 *From a point on the ground 40 m away from the foot of a tower, the angle of elevation of the top of the tower is 30°. The angle of elevation of the top of a water tank (on the top of the tower) is 45°. Find the (i) height of the tower (ii) the depth of the tank.* **[NCERT]**

SOLUTION Let BC be the tower of height h metre and CD be the water tank of height h_1 metre. Let A be a point on the ground at a distance of 40 m away from the foot B of the tower.

In $\Delta\, ABD$, we have

$$\tan 45° = \frac{BD}{AB}$$

$$\Rightarrow \quad 1 = \frac{h + h_1}{40}$$

$$\Rightarrow \quad h + h_1 = 40\text{ m} \qquad \ldots\text{(i)}$$

In $\Delta\, ABC$, we have

$$\tan 30° = \frac{BC}{AB}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{40}$$

$$\Rightarrow \quad h = \frac{40}{\sqrt{3}}\text{ m} = \frac{40\sqrt{3}}{3}\text{ m} = 23.1\text{ m}$$

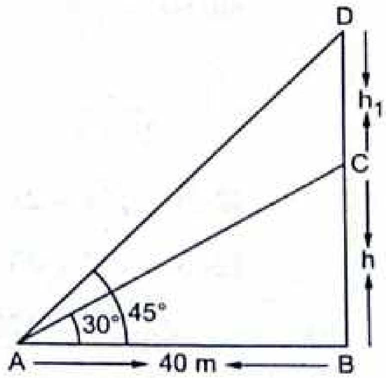

Fig. 12.11

Substituting the value of h in (i), we have

$$23.1 + h_1 = 40$$

$$\Rightarrow \quad h_1 = (40 - 23.1)\text{ m} = 16.9\text{ m}$$

Hence, the height of the tower is $h = 23.1$ m and the depth of the tank is $h_1 = 16.9$ m.

EXAMPLE 11 *A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is $60°$. When he retreates 20 m from the bank, he finds the angle to be $30°$. Find the height of the tree and the breadth of the river.*

SOLUTION Let AB be the width of the river and BC be the tree which makes an angle of $60°$ at a point A on the opposite bank. Let D be the position of the person after retreating 20 m from the bank. Let $AB = x$ metres and $BC = h$ metres.

From right angled triangles ABC and DBC, we have

$$\tan 60° = \frac{BC}{AB} \text{ and } \tan 30° = \frac{BC}{DB}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x} \text{ and } \frac{1}{\sqrt{3}} = \frac{h}{x+20}$$

$$\Rightarrow \quad h = x\sqrt{3} \text{ and } h = \frac{x+20}{\sqrt{3}}$$

$$\Rightarrow \quad x\sqrt{3} = \frac{x+20}{\sqrt{3}}$$

$$\Rightarrow \quad 3x = x + 20$$

$$\Rightarrow \quad x = 10 \text{ m}$$

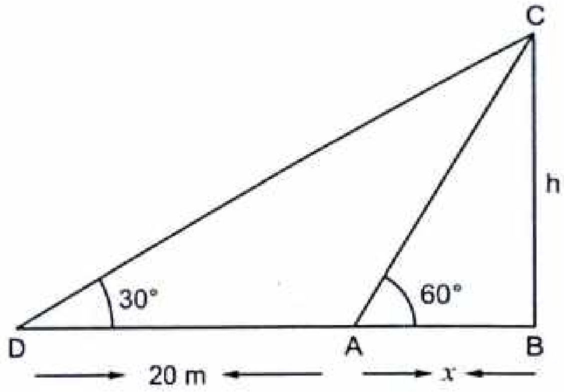

Fig. 12.12

Putting $x = 10$ in $h = \sqrt{3}x$, we get

$$h = 10\sqrt{3} = 17.32 \text{ m}$$

Hence, height of the tree is 17.32 m and the breadth of the river is 10 m.

EXAMPLE 12 *A tree 12 m high, is broken by the wind in such a way that its top touches the ground and makes an angle $60°$ with the ground. At what height from the bottom the tree is broken by the wind?*

SOLUTION Let AB be the tree of height 12 metres. Suppose the tree is broken by the wind at point C and the part CB assumes the position CO and meets the ground at O.

Let $AC = x$. Then, $CO = CB = 12 - x$. It is given that $\angle AOC = 60°$

In $\Delta\, OAC$, we have

$$\sin 60° = \frac{AC}{OC}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{x}{12-x}$$

$$\Rightarrow \quad 12\sqrt{3} - \sqrt{3}x = 2x$$

$$\Rightarrow \quad 12\sqrt{3} = x(2+\sqrt{3})$$

$$\Rightarrow \quad x = \frac{12\sqrt{3}}{2+\sqrt{3}} = \frac{12\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = 12\sqrt{3}(2-\sqrt{3})$$

$$\Rightarrow \quad x = 24\sqrt{3} - 36 = 5.569 \text{ metres}$$

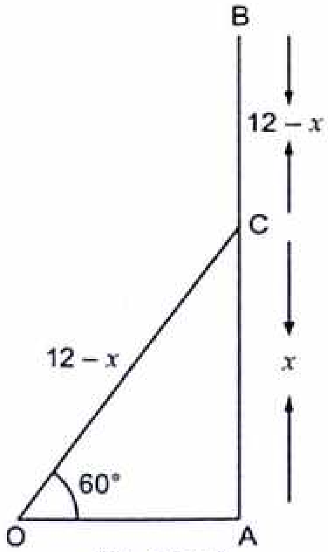

Fig. 12.13

Hence, the tree is broken at a height of 5.569 metres from the ground.

EXAMPLE 13 *A tree is broken by the wind. The top struck the ground at an angle of $30°$ and at a distance of 30 metres from the root. Find the whole height of the tree.*

SOLUTION Let AB be the tree broken at a point C such that the broken part CB takes the position CO and strikes the ground at O. It is given that $OA = 30$ metres and $\angle AOC = 30°$.

Let $AC = x$ and $CB = y$. Then, $CO = y$.

In ΔOAC, we have

$$\tan 30° = \frac{AC}{OA}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{x}{30}$$

$$\Rightarrow \quad x = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

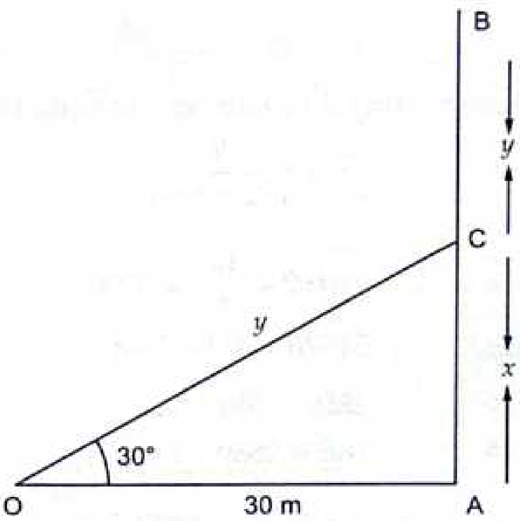

Fig. 12.14

Again in ΔOAC, we have

$$\cos 30° = \frac{OA}{OC}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{30}{y}$$

$$\Rightarrow \quad y = \frac{60}{\sqrt{3}} = 20\sqrt{3}$$

$\therefore$ Height of the tree $= (x + y)$ metres

$= (10\sqrt{3} + 20\sqrt{3})$ metres

$= 30\sqrt{3}$ metres $= 30 \times 1.732$ metres $= 51.96$ metres.

EXAMPLE 14 *At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent is 5/12. On walking 192 metres towards the tower, the tangent of the angle of elevation is 3/4. Find the height of the tower.*

SOLUTION Let AB be the tower and let the angle of elevation of its top at C be α. Let D be a point at a distance of 192 metres from C such that the angle of elevation of the top of the tower at D be β. Let h be the height of the tower and $AD = x$.

It is given that

$$\tan\alpha = \frac{5}{12} \text{ and } \tan\beta = \frac{3}{4}$$

In $\Delta\, CAB$, we have

$$\tan\alpha = \frac{AB}{AC}$$

$$\Rightarrow \quad \frac{5}{12} = \frac{h}{x + 192} \qquad \text{... (i)}$$

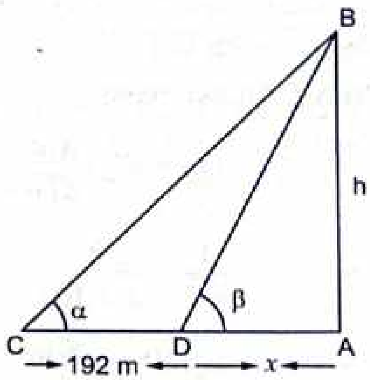

Fig. 12.15

In $\Delta\, DAB$, we have

$$\tan\beta = \frac{AB}{AD}$$

$$\Rightarrow \quad \tan\beta = \frac{h}{x}$$

$$\Rightarrow \quad \frac{3}{4} = \frac{h}{x} \qquad \text{... (ii)}$$

We have to find h. This means that we have to eliminate x from equations (i) and (ii). From equation (ii), we have

$$3x = 4h \Rightarrow x = \frac{4h}{3}$$

Substituting this value of x in equation (i), we get

$$\frac{5}{12} = \frac{h}{192 + 4h/3}$$

$\Rightarrow$ $$5\left(192 + \frac{4h}{3}\right) = 12h$$

$\Rightarrow$ $$5(576 + 4h) = 36h$$

$\Rightarrow$ $$2880 + 20h = 36h$$

$\Rightarrow$ $$16h = 2880$$

$\Rightarrow$ $$h = \frac{2880}{16} = 180$$

Hence, the height of the tower is 180 metres.

EXAMPLE 15 *The shadow of a vertical tower on level ground increases by 10 metres, when the altitude of the sun changes from angle of elevation 45° to 30°. Find the height of the tower, correct to one place of decimal. (Take $\sqrt{3} = 1.73$)*

SOLUTION Let AB be the tower and AC and AD be its shadows when the angles of elevation of the sun are 45° and 30° respectively. Then, $CD = 10$ metres. Let h be the height of the tower and let $AC = x$ metres.

In $\Delta\, CAB$, we have

$$\tan 45° = \frac{AB}{AC}$$

$\Rightarrow$ $$1 = \frac{h}{x}$$

$\Rightarrow$ $$x = h \qquad ...(i)$$

In $\Delta\, DAB$, we have

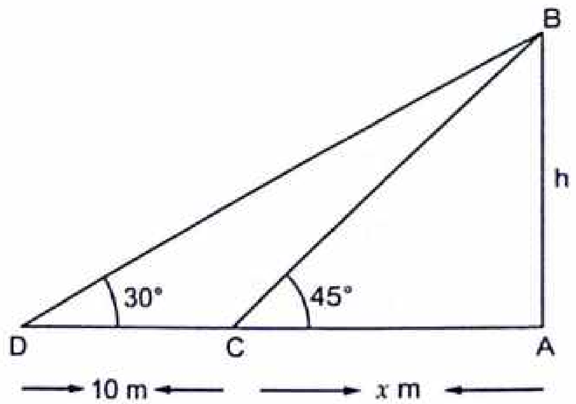

Fig. 12.16

$$\tan 30° = \frac{AB}{AD}$$

$\Rightarrow$ $$\frac{1}{\sqrt{3}} = \frac{h}{x + 10}$$

$\Rightarrow$ $$x + 10 = \sqrt{3}\, h \qquad ...(ii)$$

Substituting the value of x obtained from equation (i) and (ii), we get

$$h + 10 = \sqrt{3}\, h$$

$\Rightarrow$ $$h(\sqrt{3} - 1) = 10$$

$\Rightarrow$ $$h = \frac{10}{\sqrt{3} - 1} = \frac{10}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = 10\left(\frac{\sqrt{3} + 1}{2}\right) = 5(\sqrt{3} + 1)$$

$\Rightarrow$ $$h = 5(1.73 + 1) = 13.65 \text{ metres}$$

Hence, the height of the tower is 13.65 metres.

EXAMPLE 16 *From the top of a hill, the angles of depression of two consecutive kilometre stones due east are found to be 30° and 45°. Find the height of the hill.*

SOLUTION Let AB be the hill of height h km. Let C and D be two stones due east of the hill at a distance of 1 km from each other such that the angles of depression of C and D be $45°$ and $30°$ respectively. Let $AC = x$ km.

In $\Delta\, CAB$, we have

$$\tan 45° = \frac{AB}{AC}$$

$$\Rightarrow \quad 1 = \frac{h}{x} \Rightarrow h = x \qquad \text{...(i)}$$

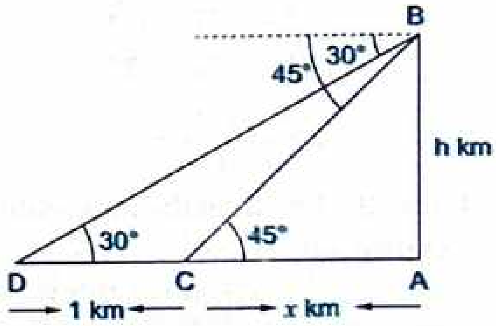

Fig. 12.17

In $\Delta\, DAB$, we have

$$\tan 30° = \frac{AB}{AD}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x+1}$$

$$\Rightarrow \quad \sqrt{3}h = x + 1 \qquad \text{...(ii)}$$

Substituting the value of x from equation (i) in equation (ii), we get

$$\sqrt{3}h = h + 1$$

$$\Rightarrow \quad h(\sqrt{3} - 1) = 1$$

$$\Rightarrow \quad h = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$\Rightarrow \quad h = \frac{\sqrt{3}+1}{2} = \frac{2.73}{2} = 1.365$$

Hence, the height of the hill is 1.365 km.

EXAMPLE 17 *Determine the height of a mountain if the elevation of its top at an unknown distance from the base is 30° and at a distance 10 km further off from the mountain, along the same line, the angle of elevation is 15°. (Use $\tan 15° = 0.27$)*

SOLUTION Let AB be the mountain of height h kilometres. Let C be a point at a distance of x km. from the base of the mountain such that the angle of elevation of the top at C is $30°$. Let D be a point at a distance of 10 km from C such that the angle of elevation at D is of $15°$.

In $\Delta\, CAB$, we have

$$\tan 30° = \frac{AB}{AC}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\Rightarrow \quad x = \sqrt{3}h \qquad \text{...(i)}$$

In $\Delta\, DAB$, we have

$$\tan 15° = \frac{AB}{AD}$$

$$\Rightarrow \quad 0.27 = \frac{h}{x+10}$$

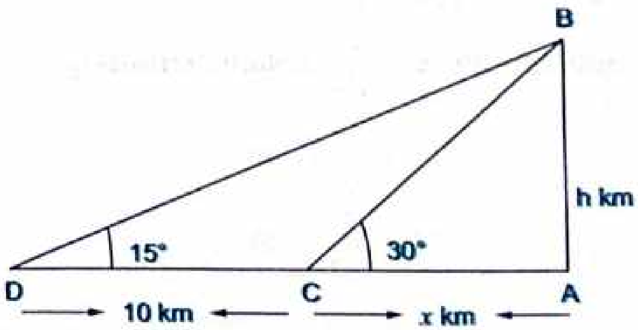

Fig. 12.18

$\Rightarrow \quad (0.27)(x + 10) = h$ ... (ii)

Substituting $x = \sqrt{3}\,h$ obtained from equation (i) in equation (ii), we get

$$0.27(\sqrt{3}\,h + 10) = h$$

$$\Rightarrow \quad 0.27 \times 10 = h - 0.27 \times \sqrt{3}\,h$$

$$\Rightarrow \quad h(1 - 0.27 \times \sqrt{3}) = 2.7$$

$$\Rightarrow \quad h(1 - 0.46) = 2.7$$

$$\Rightarrow \quad h = \frac{2.7}{0.54} = 5$$

Hence, the height of the mountain is 5 km.

EXAMPLE 18 *A person standing on the bank of a river observes that the angle of elevation of the top of a tree standing on the opposite bank is 60°. When he moves 40 metres away from the bank, he finds the angle of elevation to be 30°. Find the height of the tree and the width of the river.*

SOLUTION Let AB be the tree of height h metres standing on the bank of a river. Let C be the position of man standing on the opposite bank of the river such that $BC = x$ metres. Let D be the new position of the man. It is given that $CD = 40$ metres and the angles of elevation of the top of the tree at C and D are 60° and 30° respectively i.e., $\angle ACB = 60°$ and $\angle ADB = 30°$.

In $\Delta\, CBA$, we have

$$\tan 60° = \frac{AB}{BC}$$

$$\Rightarrow \quad \tan 60° = \frac{h}{x}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow \quad x = \frac{h}{\sqrt{3}} \quad \text{... (i)}$$

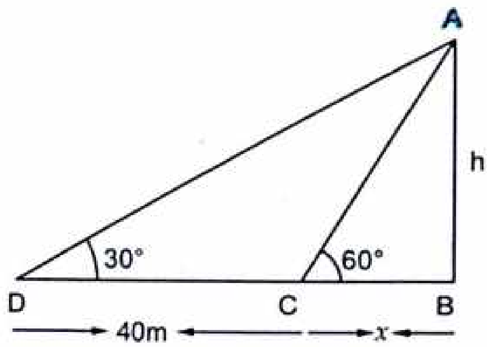

Fig. 12.19

In $\Delta\, DBA$, we have

$$\tan 30° = \frac{AB}{BD}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x + 40}$$

$$\Rightarrow \quad \sqrt{3}h = x + 40 \quad \text{...(ii)}$$

Substituting $x = \dfrac{h}{\sqrt{3}}$ obtained from equation (i) in equation (ii), we get

$$\sqrt{3}\,h = \frac{h}{\sqrt{3}} + 40$$

$$\Rightarrow \quad \sqrt{3}\,h - \frac{h}{\sqrt{3}} = 40$$

$$\Rightarrow \quad \frac{3h - h}{\sqrt{3}} = 40 \Rightarrow \frac{2h}{\sqrt{3}} = 40 \Rightarrow h = 20\sqrt{3} = 20 \times 1.73 = 34.64 \text{ metres}$$

Substituting h in equation (i), we get $x = \frac{20\sqrt{3}}{\sqrt{3}} = 20$ metres

Hence, the height of the tree is 34.64 metres and width of the river is 20 metres.

EXAMPLE 19 *An aeroplane at an altitude of 1200 metres finds that two ships are sailing towards it in the same direction. The angles of depression of the ships as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two ships.*

SOLUTION Let the aeroplane be at B and let the two ships be at C and D such that their angles of depression from B are 30° and 60° respectively.

We have, $AB = 1200$ metres. Let $AC = x$ and $CD = y$.

In ΔCAB, we have

$$\tan 60° = \frac{AB}{CA}$$

$$\Rightarrow \quad \sqrt{3} = \frac{1200}{x}$$

$$\Rightarrow \quad x = \frac{1200}{\sqrt{3}} = 400\sqrt{3}$$

In ΔBAD, we have

$$\tan 30° = \frac{AB}{AD}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{1200}{x+y}$$

$$\Rightarrow \quad x + y = 1200\sqrt{3}$$

$$\Rightarrow \quad y = 1200\sqrt{3} - x$$

$$\Rightarrow \quad y = 1200\sqrt{3} - 400\sqrt{3} = 800\sqrt{3} = 800 \times 1.732 = 1385.6$$

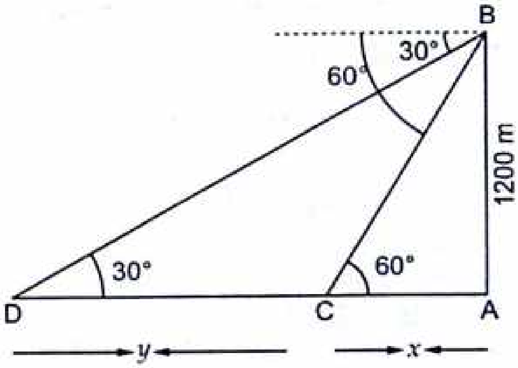

Fig. 12.20

Hence, the distance between the two ships is 1385.6 metres.

EXAMPLE 20 *The shadow of a flag-staff is three times as long as the shadow of the flag-staff when the sun rays meet the ground at an angle of 60°. Find the angle between the sun rays and the ground at the time of longer shadow.*

SOLUTION Let AB be the flag-staff and let $x = AC$ be the length of its shadow when the sun rays meet the ground at an angle of 60°. Let θ be the angle between the sun rays and the ground when the length of the shadow of the flag-staff is $AD = 3x$. Let h be the height of the flag-staff.

In ΔCAB, we have

$$\tan 60° = \frac{AB}{AC}$$

$$\Rightarrow \quad \tan 60° = \frac{h}{x}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x} \Rightarrow h = \sqrt{3}x$$

In ΔDAB, we have

$$\tan \theta = \frac{AB}{AD}$$

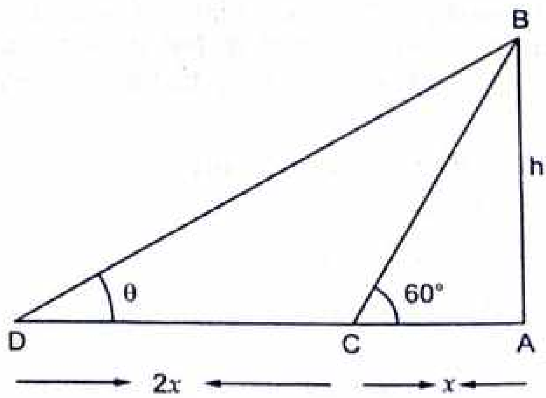

Fig. 12.21

$$\Rightarrow \quad \tan\theta = \frac{h}{3x}$$

$$\Rightarrow \quad \tan\theta = \frac{\sqrt{3}x}{3x} \qquad [\because h = \sqrt{3}\,x]$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \tan 30^\circ \Rightarrow \theta = 30^\circ$$

Thus, the angle between the sun rays and the ground is 30° at the time of longer shadow.

EXAMPLE 21 *An aeroplane at an altitude of 200 metres observes the angles of depression of opposite points on the two banks of a river to be 45° and 60°. Find the width of the river.*

SOLUTION Let P be the position of the aeroplane and let A and B be two points on the two banks of a river such that the angles of depression at A and B are 60° and 45° respectively. Let $AM = x$ metres and $BM = y$ metres. We have to find AB.

In $\Delta\, AMP$, we have

$$\tan 60^\circ = \frac{PM}{AM}$$

$$\Rightarrow \quad \sqrt{3} = \frac{200}{x}$$

$$\Rightarrow \quad 200 = \sqrt{3}\,x$$

$$\Rightarrow \quad x = \frac{200}{\sqrt{3}} \qquad \ldots \text{(i)}$$

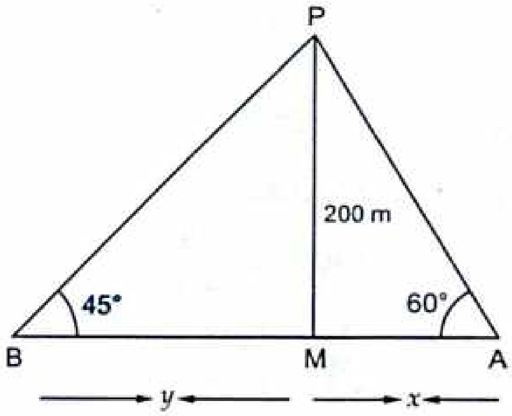

Fig. 12.22

In $\Delta\, BMP$, we have

$$\tan 45^\circ = \frac{PM}{BM}$$

$$\Rightarrow \quad 1 = \frac{200}{y}$$

$$\Rightarrow \quad y = 200 \qquad \ldots \text{(ii)}$$

From equation (i) and (ii), we get

$$AB = x + y = \frac{200}{\sqrt{3}} + 200 = 200\left(\frac{1}{\sqrt{3}} + 1\right) = 315.4 \text{ metres}.$$

Hence, the width of the river is 315.4 metres.

EXAMPLE 22 *Two pillars of equal height and on either side of a road, which is 100 m wide. The angles of elevation of the top of the pillars are 60° and 30° at a point on the road between the pillars. Find the position of the point between the pillars and the height of each pillar.*

[NCERT, CBSE 2005, 2013]

SOLUTION Let AB and CD be two pillars, each of height h metres. Let P be a point on the road such that $AP = x$ metres. Then, $CP = (100 - x)$ metres. It is given that $\angle APB = 60^\circ$ and $\angle CPD = 30^\circ$.

In $\Delta\, PAB$, we have

$$\tan 60^\circ = \frac{AB}{AP}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow \quad h = \sqrt{3}x \quad \text{...(i)}$$

In $\Delta\, PCD$, we have

$$\tan 30° = \frac{CD}{PC}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{100 - x}$$

$$\Rightarrow \quad h\sqrt{3} = 100 - x \quad \text{...(ii)}$$

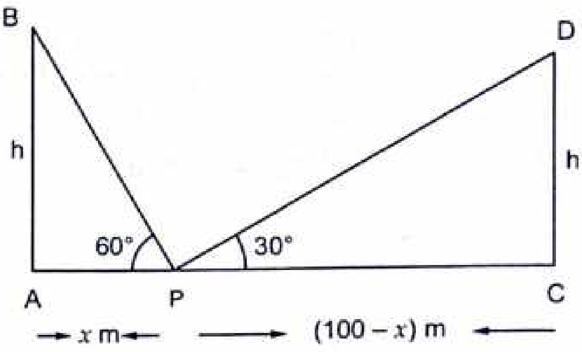

Fig. 12.23

Eliminating h between equation (i) and (ii), we get

$$3x = 100 - x \Rightarrow 4x = 100 \Rightarrow x = 25$$

Substituting $x = 25$ in equation (i), we get

$$h = 25\sqrt{3} = 25 \times 1.732 = 43.3$$

Thus, the required point is at a distance of 25 metres from the first pillar and 75 metres from the second pillar. The height of the pillars is 43.3 metres.

EXAMPLE 23 *As observed from the top of a light house, 100 m above sea level, the angle of depression of a ship, sailing directly towards it, changes from 30° to 45°. Determine the distance travelled by the ship during the period of observation.* **[CBSE 2004, 2018]**

SOLUTION Let A and B be the two positions of the ship. Let d be the distance travelled by the ship during the period of observation i.e. $AB = d$ metres.

Let the observer be at O, the top of the light house PO.

It is given that $PO = 100$ m and the angles of depression from O of A and B are 30° and 45° respectively.

$$\therefore \quad \angle OAP = 30° \text{ and } \angle OBP = 45°$$

In $\Delta\, OPB$, we have

$$\tan 45° = \frac{OP}{BP}$$

$$\Rightarrow \quad 1 = \frac{100}{BP}$$

$$\Rightarrow \quad BP = 100 \text{ m}$$

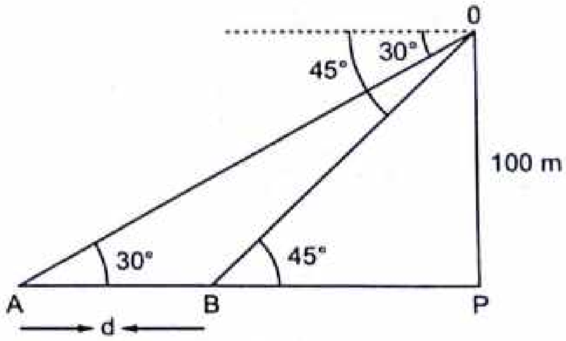

Fig. 12.24

In $\Delta\, OPA$, we have

$$\Rightarrow \quad \tan 30° = \frac{OP}{AP}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{100}{d + BP}$$

$$\Rightarrow \quad d + BP = 100\sqrt{3}$$

$$\Rightarrow \quad d + 100 = 100\sqrt{3} \quad [\because BP = 100 \text{ m}]$$

$$\Rightarrow \quad d = 100\sqrt{3} - 100$$

$$\Rightarrow \quad d = 100(\sqrt{3} - 1) = 100(1.732 - 1) = 73.2 \text{ m}$$

Hence, the distance travelled by the ship from A to B is 73.2 m.

EXAMPLE 24 *The angle of elevation of the top Q of a vertical tower PQ from a point X on the ground is 60°. At a point Y, 40 m vertically above X, the angle of elevation is 45°. Find the height of the tower PQ and the distance XQ.*

SOLUTION In $\Delta\, YRQ$, we have

$$\tan 45^\circ = \frac{QR}{YR}$$

$$\Rightarrow \quad 1 = \frac{x}{YR}$$

$$\Rightarrow \quad YR = x$$

$$\Rightarrow \quad XP = x \qquad [\because YR = XP]$$

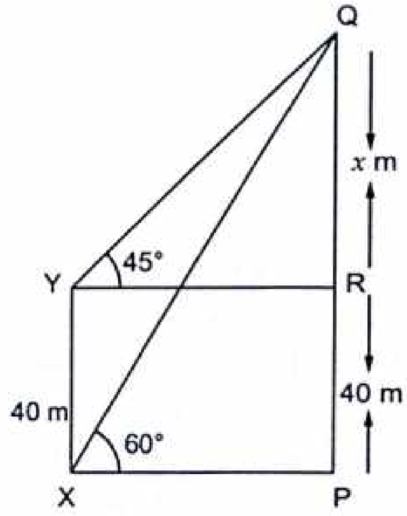

Fig. 12.25

In $\Delta\, XPQ$, we have

$$\tan 60^\circ = \frac{PQ}{PX}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x+40}{x} \qquad [\because PX = x]$$

$$\Rightarrow \quad \sqrt{3}x = x + 40$$

$$\Rightarrow \quad x(\sqrt{3} - 1) = 40$$

$$\Rightarrow \quad x = \frac{40}{\sqrt{3}-1}$$

$$\Rightarrow \quad x = \frac{40}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = 20(\sqrt{3}+1) = 54.64$$

So, height of the tower $PQ = x + 40 = 54.64 + 40 = 94.64$ metres

In $\Delta\, XPQ$, we have

$$\sin 60^\circ = \frac{PQ}{XQ}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{94.64}{XQ}$$

$$\Rightarrow \quad XQ = \frac{94.64 \times 2}{\sqrt{3}}$$

$$\Rightarrow \quad XQ = \frac{94.64 \times 2 \times \sqrt{3}}{3} = 109.3 \text{ metres.}$$

EXAMPLE 25 *From a window 15 metres high above the ground in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are 30° and 45° respectively show that the height of the opposite house is 23.66 metres* (*Take* $\sqrt{3} = 1.732$)

[CBSE 2006]

SOLUTION Let the window be at P at a height of 15 metres above the ground and CD be the house on the opposite side of the street such that the angles of devation of the top D of house CD as seen from P is of 30° and the angle of depression of the foot C of house CD as seen from P is of 45°.

Let h metres be the height of the house CD.

We have,

$$QD = CD - CQ = CD - AP = (h - 15) \text{ metres.}$$

In ΔPQC, we have

$$\tan 45° = \frac{QC}{PQ} \Rightarrow 1 = \frac{15}{PQ} \Rightarrow PQ = 15 \text{ metres.}$$

In ΔPQD, we have

$$\tan 30° = \frac{QD}{PQ}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h-15}{15} \Rightarrow h - 15 = \frac{15}{\sqrt{3}} \Rightarrow h - 15 = 5\sqrt{3}$$

$$\Rightarrow \quad h = 15 + 5 \times 1.732 = 23.66 \text{ metres,}$$

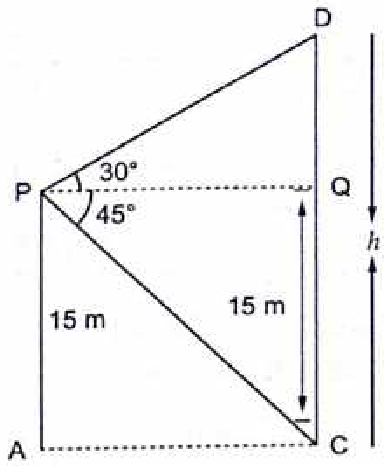

Fig. 12.26

Hence, the height of the opposite house is 23.66 metres

EXAMPLE 26 *From the top of a building 60 m high the angles of depression of the top and the bottom of a tower are observed to be 30° and 60°. Find the height of the tower.* **[CBSE 2005]**

SOLUTION Let AB be the building and CD be the tower. Let $CD = h$ metres. Let DE be horizontal from D. It is given that the angles of depression of the top D and the bottom C of the tower CD are 30° and 60° respectively.

$$\therefore \quad \angle EDB = 30° \text{ and } \angle ACB = 60°$$

Let $AC = DE = x$.

In ΔDEB, we have

$$\tan 30° = \frac{BE}{DE}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{60 - h}{x}$$

$$\Rightarrow \quad x = (60 - h)\sqrt{3} \quad \text{...(i)}$$

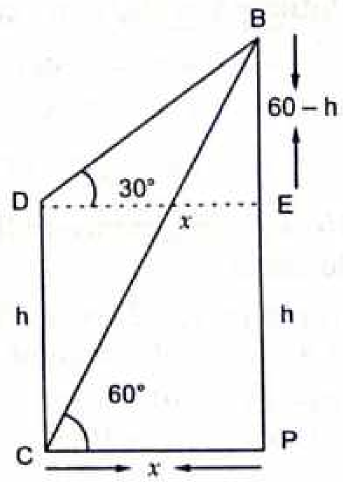

Fig. 12.27

In ΔCAB, we have

$$\tan 60° = \frac{AB}{CA}$$

$$\Rightarrow \quad \sqrt{3} = \frac{60}{x}$$

$$\Rightarrow \quad x = \frac{60}{\sqrt{3}} \quad \text{...(ii)}$$

From equations (i) and (ii), we have

$$(60 - h)\sqrt{3} = \frac{60}{\sqrt{3}}$$

$$\Rightarrow \quad 3(60 - h) = 60$$

$$\Rightarrow \quad 60 - h = 20$$

$$\Rightarrow \quad h = 40$$

Thus, the height of the tower is 40 metres.

EXAMPLE 27 *A man standing on the deck of a ship, which is 10 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill.*

[CBSE 2004, 2005, 2010]

SOLUTION Suppose the man is standing on the deck of a ship at point A and let CD be the hill. It is given that the angle of depression of the base C of the hill CD observed from A is 30° and the angle of elevation of the top D of the hill CD observed from A is 60°. Then, $\angle EAD = 60°$, $\angle BCA = 30°$.

Also, $AB = 10$ m

In ΔAED, we have

$$\tan 60° = \frac{DE}{EA}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x} \Rightarrow h = \sqrt{3}x \quad \text{...(i)}$$

In ΔABC, we have

$$\tan 30° = \frac{AB}{BC}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{10}{x} \Rightarrow x = 10\sqrt{3} \quad \text{...(ii)}$$

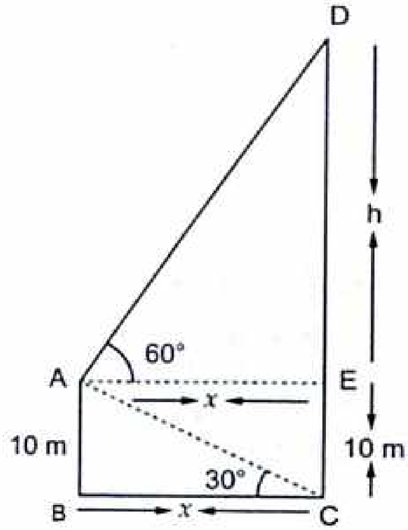

Fig. 12.28

Putting $x = 10\sqrt{3}$ in equation (i), we get

$$h = \sqrt{3} \times 10\sqrt{3} = 30$$

$$\Rightarrow \quad DE = 30 \text{ m}$$

$$\therefore \quad CD = CE + ED = 10 + 30 = 40 \text{ metres}$$

Hence, the distance of the hill from the ship is $10\sqrt{3}$ metres and the height of the hill is 40 metres.

EXAMPLE 28 *The angle of elevation of a jet plane from a point A on the ground is 60°. After a flight of 30 seconds, the angle of elevation changes to 30°. If the jet plane is flying at a constant height of $3600\sqrt{3}$ m, find the speed of the jet plane.* **[CBSE 2008, 2014]**

SOLUTION Let P and Q be the two positions of the plane and let A be the point of observation. Let ABC be the horizontal line through A. It is given that angles of elevation of the plane in two positions P and Q from a point A are 60° and 30° respectively.

$\therefore$ $\angle PAB = 60°$, $\angle QAB = 30°$. It is also given that $PB = 3600\sqrt{3}$ metres

In ΔABP, we have

$$\tan 60° = \frac{BP}{AB}$$

$$\Rightarrow \quad \sqrt{3} = \frac{3600\sqrt{3}}{AB}$$

$$\Rightarrow \quad AB = 3600 \text{ m}$$

In ΔACQ, we have

$$\tan 30° = \frac{CQ}{AC}$$

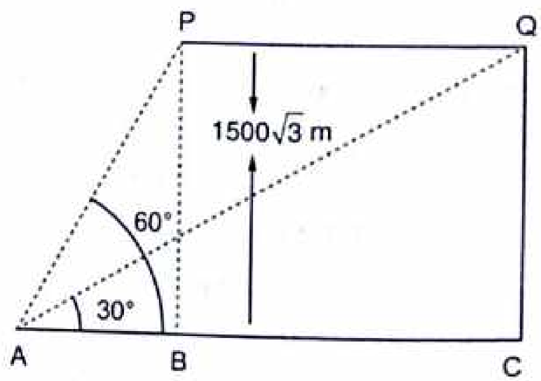

Fig. 12.29

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{3600\sqrt{3}}{AC}$$

$$\Rightarrow \quad AC = 3600 \times 3 = 10800 \text{ m}$$

$$\therefore \quad PQ = BC = AC - AB = 10800 - 3600 = 7200 \text{ m}$$

Thus, the plane travels 7200 m in 30 seconds.

Hence, Speed of plane $= \frac{7200}{30} = 240 \text{ m/sec} = \frac{240}{1000} \times 60 \times 60 = 864 \text{ km/hr}$

EXAMPLE 29 *There is a small island in the middle of a 100 m wide river and a tall tree stands on the island. P and Q are points directly opposite to each other on two banks and in line with the tree. If the angles of elevation of the top of the tree from P and Q are respectively 30° and 45°, find the height of the tree.*

SOLUTION Let OA be the tree of height h metre.

In triangle POA and QOA, we have

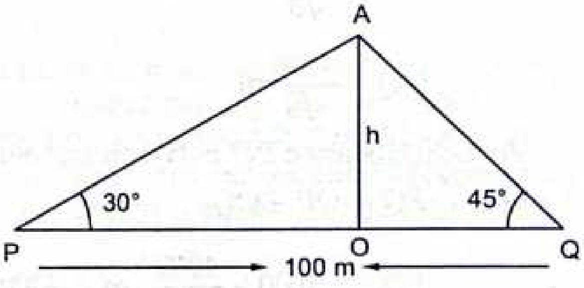

Fig. 12.30

$$\tan 30° = \frac{OA}{OP} \text{ and } \tan 45° = \frac{OA}{OQ}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{OP} \text{ and } 1 = \frac{h}{OQ}$$

$$\Rightarrow \quad OP = \sqrt{3}\,h \text{ and } OQ = h$$

$$\Rightarrow \quad OP + OQ = \sqrt{3}\,h + h$$

$$\Rightarrow \quad PQ = (\sqrt{3} + 1)\,h$$

$$\Rightarrow \quad 100 = (\sqrt{3} + 1)\,h \qquad [\because PQ = 100 \text{ m}]$$

$$\Rightarrow \quad h = \frac{100}{\sqrt{3}+1} \text{ m} = \frac{100(\sqrt{3}-1)}{2} \text{ m} = 50(1.732 - 1) \text{ m} = 36.6 \text{ m}$$

Hence, the height of the tree is 36.6 m.

EXAMPLE 30 *The horizontal distance between two towers is 140 m. The angle of elevation of the top of the first tower when seen from the top of the second tower is 30°. If the height of the second tower is 60 m, find the height of the first tower.*

SOLUTION Let AB and CD be two towers of height h metres and 60 metres respectively such that the distance AC between them is 140 m. The angle of elevation of top B of tower AB as seen from D (top of tower CD) is 30°.

In $\Delta\, DEB$, we have

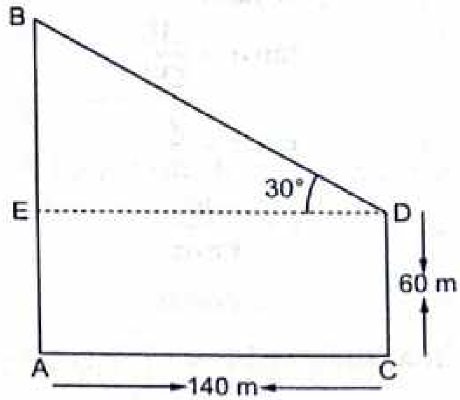

Fig. 12.31

$$\tan 30° = \frac{BE}{DE}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{BE}{140} \qquad [\because DE = AC = 140 \text{ m}]$$

$$\Rightarrow \quad BE = \frac{140}{\sqrt{3}} \text{ m} = \frac{140}{1.732} \text{ m} = 80.83 \text{ m}$$

$$\therefore \quad AB = AE + BE = CD + BE = 60 + 80.83 \text{ m} = 140.83 \text{ m}$$

Hence, the height of the second tower is 140.83 m.

EXAMPLE 31 *An aeroplane when flying at a height of 4000 m from the ground passes vertically above another aeroplane at an instant when the angles of the elevation of the two planes from the same point on the ground are 60° and 45° respectively. Find the vertical distance between the aeroplanes at that instant.* **[CBSE 2008, 2009, 2016]**

SOLUTION Let P and Q be the positions of two aeroplanes when Q is vertically below P and $OP = 4000$ m. Let the angles of elevation of P and Q at a point A on the ground be 60° and 45° respectively.

In triangles AOP and AOQ, we have

$$\tan 60° = \frac{OP}{OA} \text{ and } \tan 45° = \frac{OQ}{OA}$$

$$\Rightarrow \quad \sqrt{3} = \frac{4000}{OA} \text{ and } 1 = \frac{OQ}{OA}$$

$$\Rightarrow \quad OA = \frac{4000}{\sqrt{3}} \text{ and } OQ = OA$$

$$\Rightarrow \quad OQ = \frac{4000}{\sqrt{3}} \text{ m}$$

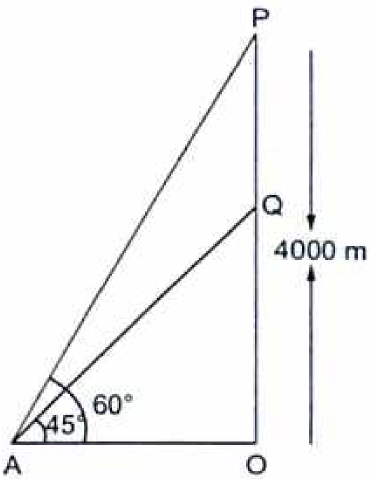

Fig. 12.32

$\therefore$ Vertical distance PQ between the aeroplanes is given by

$$PQ = OP - OQ$$

$$\Rightarrow \quad PQ = \left(4000 - \frac{4000}{\sqrt{3}}\right) \text{m} = 4000\frac{(\sqrt{3}-1)}{\sqrt{3}} \text{m} = 1690.53 \text{ m}$$

LEVEL-2

EXAMPLE 32 *A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff of height h. At a point on the plane, the angles of elevation of the bottom and the top of the flag-staff are α and β respectively. Prove that the height of the tower is* $\dfrac{h\tan\alpha}{\tan\beta - \tan\alpha}$.

[NCERT EXEMPLAR]

SOLUTION Let AB be the tower and BC be the flag-staff. Let O be a point on the plane containing the foot of the tower such that the angles of elevation of the bottom B and top C of the flag-staff at O are α and β respectively. Let $OA = x$ metres, $AB = y$ metres and $BC = h$ metres.

In ΔOAB, we have

$$\tan\alpha = \frac{AB}{OA}$$

$$\Rightarrow \quad \tan\alpha = \frac{y}{x}$$

$$\Rightarrow \quad x = \frac{y}{\tan\alpha} \qquad \text{...(i)}$$

$$\Rightarrow \quad x = y\cot\alpha$$

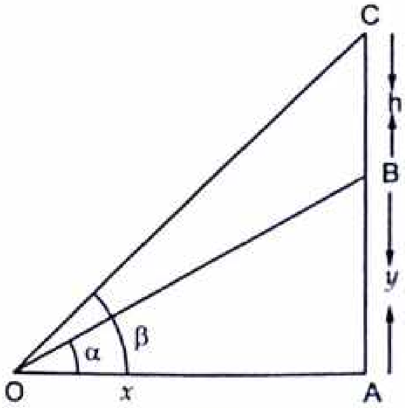

Fig. 12.33

In ΔOAC, we have

$$\tan\beta = \frac{y+h}{x}$$

$$\Rightarrow \quad x = \frac{y+h}{\tan\beta}$$

$$\Rightarrow \quad x = (y+h)\cot\beta \qquad \text{...(ii)}$$

On equating the values of x given in equations (i) and (ii), we get

$$\Rightarrow \quad y\cot\alpha = (y+h)\cot\beta$$

$$\Rightarrow \quad (y\cot\alpha - y\cot\beta) = h\cot\beta$$

$$\Rightarrow \quad y(\cot\alpha - \cot\beta) = h\cot\beta$$

$$\Rightarrow \quad y = \frac{h\cot\beta}{\cot\alpha - \cot\beta} = \frac{\frac{h}{\tan\beta}}{\frac{1}{\tan\alpha} - \frac{1}{\tan\beta}} = \frac{h\tan\alpha}{\tan\beta - \tan\alpha}$$

Hence, the height of the tower is $\dfrac{h\tan\alpha}{\tan\beta - \tan\alpha}$

EXAMPLE 33 *The angles of elevation of the top of a tower from two points at distances a and b metres from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ metres.* **[NCERT EXEMPLAR, CBSE 2002 C, 2004]**

SOLUTION Let AB be the tower. Let C and D be two points at distances a and b respectively from the base of the tower. Then, $AC = a$ and $AD = b$. Let $\angle ACB = \theta$ and $\angle ADB = 90° - \theta$. Let h be the height of the tower AB.

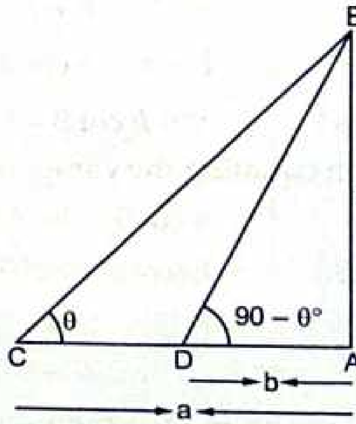

Fig. 12.34

In $\Delta\, CAB$, we have

$$\tan\theta = \frac{AB}{AC}$$

$$\Rightarrow \quad \tan\theta = \frac{h}{a} \qquad \text{...(i)}$$

In $\Delta\, DAB$, we have

$$\tan(90° - \theta) = \frac{AB}{AD}$$

$$\Rightarrow \quad \cot\theta = \frac{h}{b} \qquad \text{...(ii)}$$

From(i) and (ii), we have

$$\tan\theta \times \cot\theta = \frac{h^2}{ab}$$

$$\Rightarrow \quad 1 = \frac{h^2}{ab} \Rightarrow h^2 = ab \Rightarrow h = \sqrt{ab} \text{ metres.}$$

Hence, the height of the tower is $\sqrt{ab}$ metres

EXAMPLE 34 *Two stations due south of a leaning tower which leans towards the north are at distances a and b from its foot. If α, β be the elevations of the [illegible] of the tower from these stations, prove that its inclination θ to the horizontal is given* [illegible]

$$\cot\theta = \frac{b\cot\alpha - a\cot\beta}{b-a}$$

SOLUTION Let AB be the leaning tower and let C an[illegible] [illegible]en stations at distances a and b respectively from the foot A of the tower.

Let $AE = x$ and $BE = h$

In $\Delta\, AEB$, we have

$$\tan\theta = \frac{BE}{AE}$$

$$\Rightarrow \quad \tan\theta = \frac{h}{x}$$

$$\Rightarrow \quad x = h\cot\theta \qquad \text{...(i)}$$

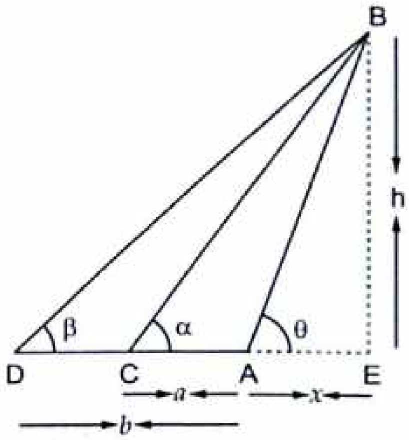

Fig. 12.35

In $\Delta\, CEB$, we have

$$\tan\alpha = \frac{BE}{CE}$$

$$\Rightarrow \quad \tan\alpha = \frac{h}{a+x}$$

$$\Rightarrow \quad a + x = h\cot\alpha$$

$$\Rightarrow \quad x = h\cot\alpha - a \qquad \text{... (ii)}$$

In $\Delta\, DEB$, we have

$$\tan\beta = \frac{BE}{DE}$$

$$\Rightarrow \quad \tan\beta = \frac{h}{b+x}$$

$$\Rightarrow \quad b + x = h\cot\beta$$

$$\Rightarrow \quad x = h\cot\beta - b \qquad \text{... (iii)}$$

On equating the values of x obtained from equations (i) and (ii), we have

$$h\cot\theta = h\cot\alpha - a$$

$$\Rightarrow \quad h(\cot\alpha - \cot\theta) = a$$

$$\Rightarrow \quad h = \frac{a}{\cot\alpha - \cot\theta} \qquad \text{... (iv)}$$

On equating the values of x obtained from equations (i) and (iii), we get

$$h\cot\theta = h\cot\beta - b$$

$$\Rightarrow \quad h(\cot\beta - \cot\theta) = b$$

$$\Rightarrow \quad h = \frac{b}{\cot\beta - \cot\theta} \qquad \text{...(v)}$$

Equating the values of h from equations (iv) and (v), we get

$$\frac{a}{\cot\alpha - \cot\theta} = \frac{b}{\cot\beta - \cot\theta}$$

$$\Rightarrow \quad a(\cot\beta - a\cot\theta) = b(\cot\alpha - \cot\theta)$$

$$\Rightarrow \quad (b-a)\cot\theta = b\cot\alpha - a\cot\beta$$

$$\Rightarrow \quad \cot\theta = \frac{b\cot\alpha - a\cot\beta}{b-a}$$

EXAMPLE 35 *If the angle of elevation of a cloud from a point h metres above a lake is α and the angle of depression of its reflection in the lake is β, prove that the height of the cloud is*

$$\frac{h(\tan\beta + \tan\alpha)}{\tan\beta - \tan\alpha}$$

[NCERT EXEMPLAR]

SOLUTION Let AB be the surface of the lake and let P be a point of observation such that $AP = h$ metres. Let C be the position of the cloud and C' be its reflection in the lake. Then, $CB = C'B$. Let PM be perpendicular from P on CB. Then, $\angle CPM = \alpha$ and $\angle MPC' = \beta$. Let $CM = x$. Then, $CB = CM + MB = CM + PA = x + h$.

In $\Delta\, CPM$, we have

$$\tan\alpha = \frac{CM}{PM}$$

$$\Rightarrow \quad \tan\alpha = \frac{x}{AB} \qquad [\because PM = AB]$$

$$\Rightarrow \quad AB = x\cot\alpha \qquad \text{...(i)}$$

In $\Delta\, PMC'$, we have

$$\tan\beta = \frac{C'M}{PM}$$

$$\Rightarrow \quad \tan\beta = \frac{x + 2h}{AB} \qquad [\because C'M = C'B + BM = x + h + h]$$

$$\Rightarrow \quad AB = (x + 2h)\cot\beta \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$x\cot\alpha = (x + 2h)\cot\beta$$

[On equating the values of AB]

$$\Rightarrow \quad x(\cot\alpha - \cot\beta) = 2h\cot\beta$$

$$\Rightarrow \quad x\left(\frac{1}{\tan\alpha} - \frac{1}{\tan\beta}\right) = \frac{2h}{\tan\beta}$$

$$\Rightarrow \quad x\left(\frac{\tan\beta - \tan\alpha}{\tan\alpha\tan\beta}\right) = \frac{2h}{\tan\beta}$$

$$\Rightarrow \quad x = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha}$$

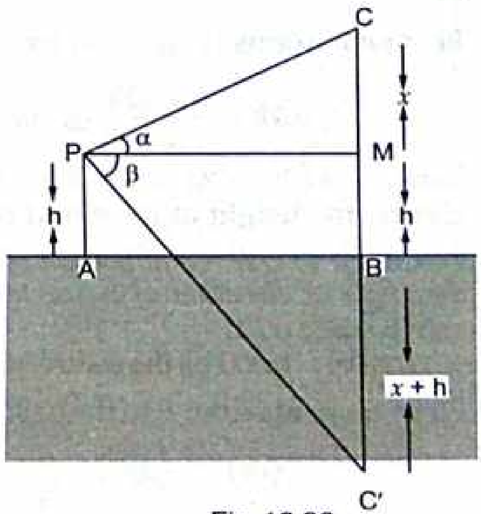

Fig. 12.36

Hence, the height CB of the cloud is given by

$$CB = x + h$$

$$\Rightarrow \quad CB = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha} + h$$

$$\Rightarrow \quad CB = \frac{2h\tan\alpha + h\tan\beta - h\tan\alpha}{\tan\beta - \tan\alpha} = \frac{h(\tan\alpha + \tan\beta)}{\tan\beta - \tan\alpha}$$

EXAMPLE 36 *The angle of elevation of a cloud from a point 60 m above a lake is 30° and the angle of depression of the reflection of cloud in the lake is 60°. Find the height of the cloud.*

[CBSE 2010, 2017]

SOLUTION Let AB be the surface of the lake and P be the point of observation such that $AP = 60$ metres. Let C be the position of the cloud and C' be its reflection in the lake. Then, $CB = C'B$. Let PM be perpendicular from P on CB. Then, $\angle CPM = 30°$ and $\angle C'PM = 60°$. Let $CM = h$. Then, $CB = h + 60$. Consequently, $C'B = h + 60$.

In ΔCMP, we have

$$\tan 30° = \frac{CM}{PM}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{PM}$$

$$\Rightarrow \quad PM = \sqrt{3}\,h \qquad \text{...(i)}$$

In $\Delta PMC'$, we have

$$\tan 60° = \frac{C'M}{PM}$$

$$\Rightarrow \quad \tan 60° = \frac{C'B + BM}{PM}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h + 60 + 60}{PM}$$

$$\Rightarrow \quad PM = \frac{h + 120}{\sqrt{3}} \qquad \text{...(ii)}$$

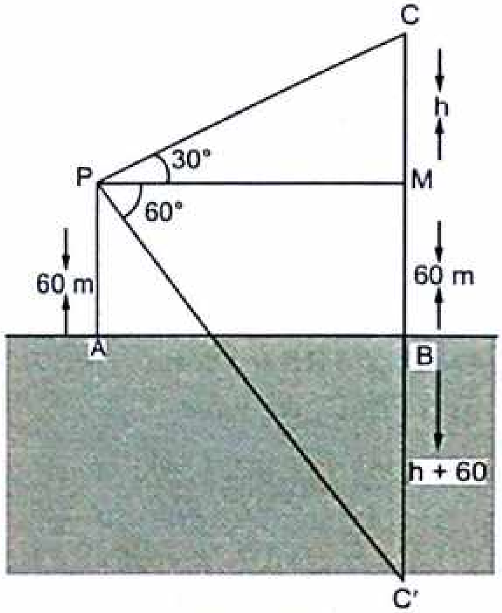

Fig. 12.37

From equations (i) and (ii), we get

$$\sqrt{3}\,h = \frac{h + 120}{\sqrt{3}} \Rightarrow 3h = h + 120 \Rightarrow 2h = 120 \Rightarrow h = 60$$

Now, $\quad CB = CM + MB = h + 60 = 60 + 60 = 120.$

Hence, the height of the cloud from the surface of the lake is 120 metres.

EXAMPLE 37 *A round balloon of radius r subtends an angle α at the eye of the observer while the angle of elevation of its centre is β. Prove that the height of the centre of the balloon is $r \sin \beta \operatorname{cosec} \alpha/2$.* **[NCERT EXEMPLAR]**

SOLUTION Let O be the centre of the balloon of radius r and P the eye of the observer. Let PA, PB be tangents from P to the balloon. Then, $\angle APB = \alpha$.

$$\therefore \quad \angle APO = \angle BPO = \frac{\alpha}{2}$$

Let OL be perpendicular from O on the horizontal PX. We are given that the angle of the elevation of the centre of the balloon is β i.e, $\angle OPL = \beta$.

In ΔOAP, we have

$$\sin \frac{\alpha}{2} = \frac{OA}{OP}$$

$$\Rightarrow \quad \sin \frac{\alpha}{2} = \frac{r}{OP}$$

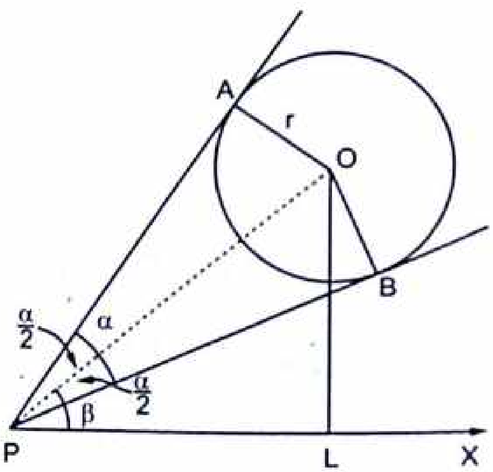

Fig. 12.38

$$\Rightarrow \quad OP = r \operatorname{cosec} \frac{\alpha}{2} \quad \text{...(i)}$$

In $\Delta\, OPL$, we have

$$\sin\beta = \frac{OL}{OP}$$

$$\Rightarrow \quad OL = OP \sin\beta = r \operatorname{cosec}\frac{\alpha}{2}\, \sin\beta \quad \text{[Using equation (i)]}$$

Hence, the height of the centre of the balloon is $r \sin\beta \operatorname{cosec}\frac{\alpha}{2}$

EXAMPLE 38 *The angle of elevation of a cliff from a fixed point is θ. After going up a distance of k metres towards the top of the cliff at an angle of ϕ, it is found that the angle of elevation is α. Show that the height of the cliff is*

$$\frac{k\,(\cos\phi - \sin\phi\cot\alpha)}{\cot\theta - \cot\alpha}\ \textit{metres}$$

SOLUTION Let AB be the cliff and O be the fixed point such that the angle of elevation of the cliff from O is θ i.e., $\angle AOB = \theta$. Let $\angle AOC = \phi$ and $OC = k$ metres. From C draw CD and CE perpendiculars on AB and OA respectively. Then, $\angle DCB = \alpha$. Let h be the height of the cliff AB.

In $\Delta\, OCE$, we have

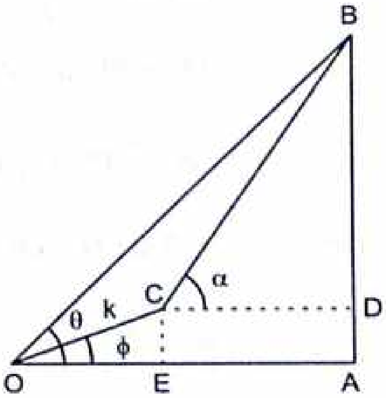

Fig. 12.39

$$\sin\phi = \frac{CE}{OC}$$

$$\Rightarrow \quad \sin\phi = \frac{CE}{k}$$

$$\Rightarrow \quad CE = k\sin\phi$$

$$\Rightarrow \quad AD = k\sin\phi \quad \text{...(i)} \quad [\because CE = AD]$$

and, $$\cos\phi = \frac{OE}{OC}$$

$$\Rightarrow \quad \cos\phi = \frac{OE}{k}$$

$$\Rightarrow \quad OE = k\cos\phi \quad \text{....(ii)}$$

In ΔOAB, we have

$$\tan\theta = \frac{AB}{OA}$$

$$\Rightarrow \quad \tan\theta = \frac{h}{OA}$$

$$\Rightarrow \quad OA = h\cot\theta \quad \text{...(iii)}$$

$$\therefore \quad CD = EA = OA - OE = h\cot\theta - k\cos\phi \quad \text{...(iv)} \quad \text{[Using (ii) and (iii)]}$$

and, $$BD = AB - AD = AB - CE = h - k\sin\phi \quad \text{...(v)} \quad \text{[Using (i)]}$$

In $\Delta\, BCD$, we have

$$\tan\alpha = \frac{BD}{CD}$$

$$\Rightarrow \quad \tan\alpha = \frac{h - k\sin\phi}{h\cot\theta - k\cos\phi} \qquad \text{[Using (iv) and (v)]}$$

$$\Rightarrow \quad \frac{1}{\cot\alpha} = \frac{h - k\sin\phi}{h\cot\theta - k\cos\phi}$$

$$\Rightarrow \quad h\cot\alpha - k\sin\phi\cot\alpha = h\cot\theta - k\cos\phi$$

$$\Rightarrow \quad h(\cot\theta - \cot\alpha) = k(\cos\phi - \sin\phi\cot\alpha)$$

$$\Rightarrow \quad h = \frac{k(\cos\phi - \sin\phi\cot\alpha)}{\cot\theta - \cot\alpha}$$

EXAMPLE 39 *At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.*

SOLUTION Let F be the foot and S be the summit of the mountain FOS. Then, $\angle OFS = 45°$ and therefore, $\angle OSF = 45°$. Consequently, $OF = OS = h$ km (say). Let $FP = 1000$ m $= 1$ km be the slope so that $\angle OFP = 30°$. Draw $PM \perp OS$ and $PL \perp OF$. Join PS. It is given that $\angle MPS = 60°$.

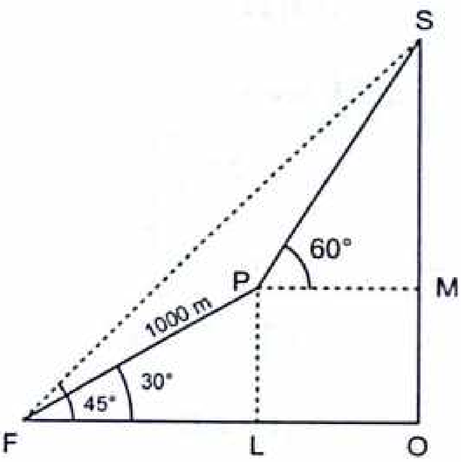

Fig. 12.40

In $\Delta\, FPL$, we have

$$\sin 30° = \frac{PL}{PF}$$

$$\Rightarrow \quad PL = PF\sin 30° = \left(1 + \frac{1}{2}\right)\text{km} = \frac{1}{2}\text{km}$$

$$\therefore \quad OM = PL = \frac{1}{2}\text{km}$$

$$\Rightarrow \quad MS = OS - OM = \left(h - \frac{1}{2}\right)\text{km} \qquad \text{...(i)}$$

Also, $$\cos 30° = \frac{FL}{PF}$$

$$\Rightarrow \quad FL = PF\cos 30° = \left(1 \times \frac{\sqrt{3}}{2}\right)\text{km} = \frac{\sqrt{3}}{2}\text{km}$$

Now, $$h = OS = OF = OL + LF$$

$$\Rightarrow \quad h = OL + \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad OL = \left(h - \frac{\sqrt{3}}{2}\right)\text{km}$$

$$\Rightarrow \quad PM = \left(h - \frac{\sqrt{3}}{2}\right)\text{km} \qquad \text{...(ii)}$$

In $\Delta\, SPM$, we have

$$\tan 60° = \frac{SM}{PM}$$

$$\Rightarrow \quad SM = PM \cdot \tan 60°$$

$$\Rightarrow \quad \left(h - \frac{1}{2}\right) = \left(h - \frac{\sqrt{3}}{2}\right)\sqrt{3} \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad h - \frac{1}{2} = h\sqrt{3} - \frac{3}{2}$$

$$\Rightarrow \quad \sqrt{3}h - h = \frac{3}{2} - \frac{1}{2}$$

$$\Rightarrow \quad h(\sqrt{3} - 1) = 1$$

$$\Rightarrow \quad h = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{\sqrt{3} + 1}{2} = \frac{2.732}{2} = 1.366 \text{ km}$$

Hence, the height of the mountain is 1.366 km.

EXAMPLE 40 *The angle of elevation of the top of a tower from a point A due south of the tower is α and from B due east of the tower is β. If AB = d, show that the height of the tower is* $\frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$.

SOLUTION Let OP be the tower and let A and B be two points due south and east respectively of the tower such that $\angle OAP = \alpha$ and $\angle OBP = \beta$. Let $OP = h$.

In $\Delta\, OAP$, we have

$$\tan \alpha = \frac{h}{OA}$$

$$\Rightarrow \quad OA = h \cot \alpha \qquad \text{...(i)}$$

In $\Delta\, OBP$, we have

$$\tan \beta = \frac{h}{OB}$$

$$\Rightarrow \quad OB = h \cot \beta \qquad \text{...(ii)}$$

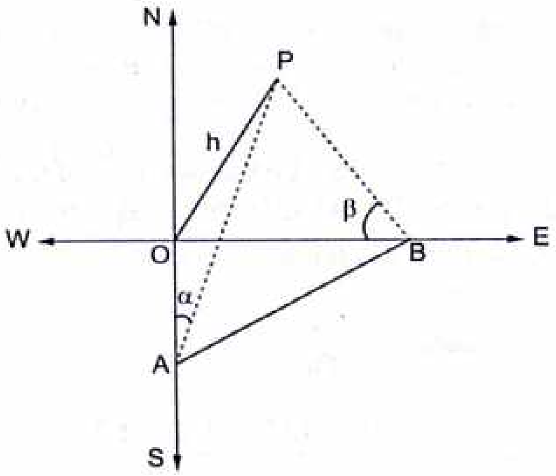

Fig. 12.41

Since OAB is a right-angled triangle.

$$\therefore \quad AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad d^2 = h^2 \cot^2 \alpha + h^2 \cot^2 \beta$$

$$\Rightarrow \quad h = \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}} \qquad \text{[Using (i) and (ii)]}$$

EXAMPLE 41 *The elevation of a tower at a station A due north of it is α and at a station B due west of A is β. Prove that the height of the tower is* $\frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$.

SOLUTION Let OP be the tower and let A be a point due north of the tower OP and let B be the point due west of A. Such that $\angle OAP = \alpha$ and $\angle OBP = \beta$. Let h be the height of the tower.
In right-angled triangle OAP and OBP, we have

$$\tan \alpha = \frac{h}{OA} \text{ and } \tan \beta = \frac{h}{OB}$$

$\Rightarrow \quad OA = h \cot \alpha$ and $OB = h \cot \beta$

In ΔOAB, we have

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad AB^2 = OB^2 - OA^2$$

$$\Rightarrow \quad AB^2 = h^2 \cot^2 \beta - h^2 \cot^2 \alpha$$

$$\Rightarrow \quad AB^2 = h^2 \left[\cot^2 \beta - \cot^2 \alpha \right]$$

$$\Rightarrow \quad AB^2 = h^2 \left[(\operatorname{cosec}^2 \beta - 1) - (\operatorname{cosec}^2 \alpha - 1) \right]$$

$$\Rightarrow \quad AB^2 = h^2 (\operatorname{cosec}^2 \beta - \operatorname{cosec}^2 \alpha)$$

$$\Rightarrow \quad AB^2 = h^2 \left(\frac{\sin^2 \alpha - \sin^2 \beta}{\sin^2 \alpha \sin^2 \beta} \right)$$

$$\Rightarrow \quad h = \frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$$

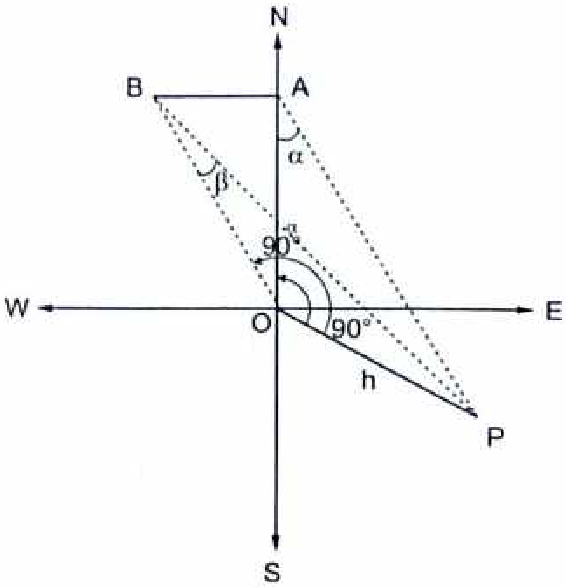

Fig. 12.42

EXAMPLE 42 *A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.* **[NCERT]**

SOLUTION Let P be the position of the balloon when its angle of elevation from the eyes of the girl is 60° and Q be the position when angle of elevation is 30°.

In ΔOLP, we have

$$\tan 60^\circ = \frac{PL}{OL}$$

$$\Rightarrow \quad \sqrt{3} = \frac{PL' - LL'}{OL} = \frac{88.2 - 1.2}{OL}$$

$$\Rightarrow \quad \sqrt{3} = \frac{87}{OL}$$

$$\Rightarrow \quad OL = \frac{87}{\sqrt{3}}$$

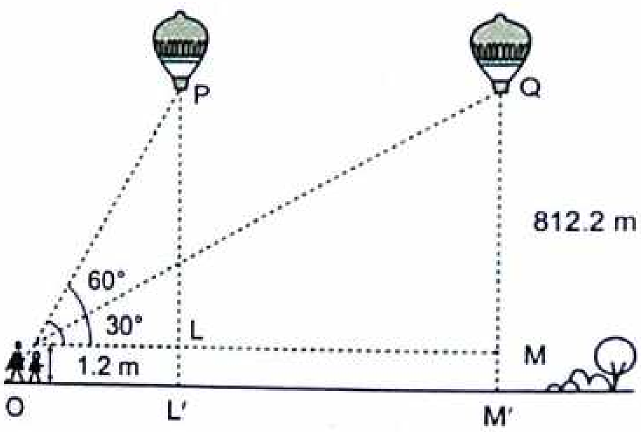

Fig. 12.43

In ΔOMQ, we have

$$\tan 30^\circ = \frac{QM}{OM} = \frac{QM' - MM'}{OM}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{88.2 - 1.2}{OM}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{87}{OM}$$

$$\Rightarrow \quad OM = 87 \times \sqrt{3}$$

$\therefore$ Distance travelled by the balloon $= PQ = LM = OM - OL$

$$= \left(87 \times \sqrt{3} - \frac{87}{\sqrt{3}}\right) \text{m}$$

$$= 87 \times \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) \text{m} = \frac{87 \times 2}{\sqrt{3}} \text{m} = \frac{174}{\sqrt{3}} \text{m}$$

$$= \frac{174}{3}\sqrt{3} \text{ m} = 58\sqrt{3} \text{ m}.$$

EXAMPLE 43 *A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at angle of depression of 30°, which is approaching to the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the further time taken by the car to reach the foot of the tower.* **[NCERT, CBSE 2008, 2009, 2017]**

SOLUTION Let P be the foot of the vertical tower PQ of height h metres. Let the speed of the car be v m/sec. At A the angle of depression of the car is 30° and six seconds later it reaches to B where the angle of depression is 60°.

Clearly, car travels distance AB in 6 seconds with speed v m/sec.

$$\therefore \quad AB = 6v \text{ metres}$$

Suppose car takes t seconds to reach to P from point B. Then, $BP = vt$ metres.

$$\therefore \quad AP = AB + BP = 6v + vt$$

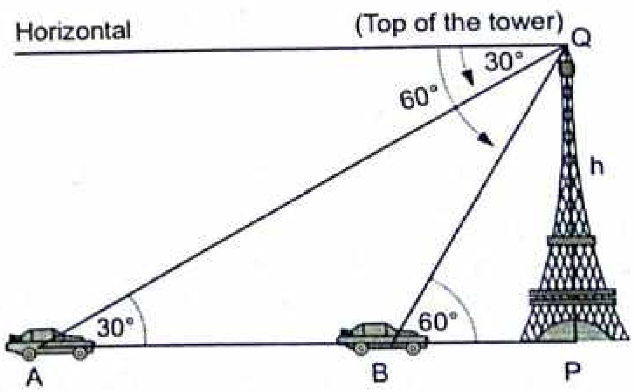

Fig. 12.44

In ΔAPQ, we have

$$\tan 30° = \frac{PQ}{AP}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{6v + vt}$$

$$\Rightarrow \quad \sqrt{3}h = 6v + vt \qquad \ldots(i)$$

In ΔBPQ, we have

$$\tan 60° = \frac{PQ}{BP}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{vt}$$

$$\Rightarrow \quad \sqrt{3}\, vt = h \qquad \ldots(ii)$$

From (i) and (ii), we have

$$\sqrt{3} \times \sqrt{3}\, vt = 6v + vt \Rightarrow 3vt = 6v + vt \Rightarrow 2vt = 6v \Rightarrow t = \frac{6v}{2v} = 3 \text{ seconds}$$

Hence, further time taken by the car to reach the foot of the tower is 3 seconds.

EXAMPLE 44 *A man on a cliff observes a boat at an angle of depression of 30° which is approaching the shore to the point immediately beneath the observer with a uniform speed. Six minutes later, the angle of depression of the boat is found to be 60°. Find the time taken by the boat to reach the shore.*

SOLUTION Let OA be the cliff and P be the initial position of the boat when the angle of depression is 30°. After 6 minutes the boat reaches to Q such that the angle of depression at Q is 60°. Let $PQ = x$ metres.

In Δ's POA and QOA, we have

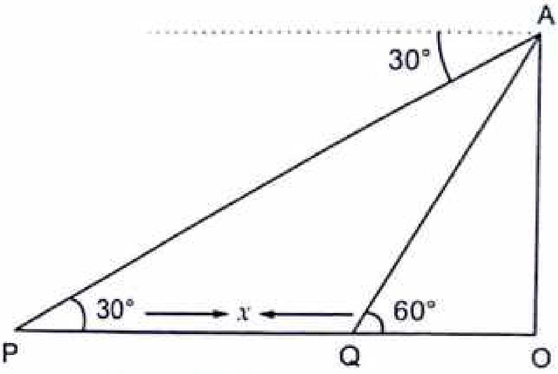

Fig. 12.45

$$\tan 30^\circ = \frac{OA}{OP} \text{ and } \tan 60^\circ = \frac{OA}{OQ}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{OA}{OP} \text{ and } \sqrt{3} = \frac{OA}{OQ}$$

$$\Rightarrow \quad OA = \frac{OP}{\sqrt{3}} \text{ and } OA = \sqrt{3}\, OQ$$

$$\Rightarrow \quad \frac{OP}{\sqrt{3}} = \sqrt{3}\, OQ$$

$$\Rightarrow \quad OP = 3\, OQ$$

$$\Rightarrow \quad PQ = OP - OQ = OP - \frac{OP}{3} = \frac{2}{3} OP \qquad \left[\because OQ = \frac{1}{3} OP\right]$$

Let the speed of the boat be v metre/minute. Then,

PQ = Distance travelled by the boat is 6 minutes

$$\Rightarrow \quad PQ = 6\, v$$

$$\Rightarrow \quad \frac{2}{3}(OP) = 6\, v \qquad \left[\because PQ = \frac{2}{3} OP\right]$$

$$\Rightarrow \quad OP = 9\, v$$

∴ Time taken by the boat to reach at the shore is given by

$$T = \frac{OP}{v} \qquad \left[\because \text{Time} = \frac{\text{Distance}}{\text{Speed}}\right]$$

$$\Rightarrow \quad T = \frac{9v}{v} \text{ minutes} = 9 \text{ minutes}.$$

EXAMPLE 45 *A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 12 minutes for the angle of depression to change from 30° to 45°, how soon after this, will the car reach the tower? Give your answer to the nearest second.* **[CBSE 2006C]**

SOLUTION Let AB be the tower of height h metres. Let C be the initial position of the car and let after 12 minutes the car be at D. It is given that the angles of depression at C and D are 30° and 45° respectively.

Let the speed of the car be v metre per minute. Then,

CD = Distance travelled by the car in 12 minutes.

$$\Rightarrow \quad CD = 12\, v \text{ metres} \qquad [\because \text{Distance} = \text{speed} \times \text{time}]$$

Suppose the car takes t minutes to reach the tower AB from D. Then, $DA = vt$ metres.

In Δ DAB, we have

$$\tan 45^\circ = \frac{AB}{AD}$$

$\Rightarrow \quad 1 = \dfrac{h}{vt}$

$\Rightarrow \quad h = vt \quad$ ...(i)

In $\Delta\, CAB$, we have

$\tan 30° = \dfrac{AB}{AC}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{h}{vt + 12v}$

$\sqrt{3}h = vt + 12v \quad$ ...(ii)

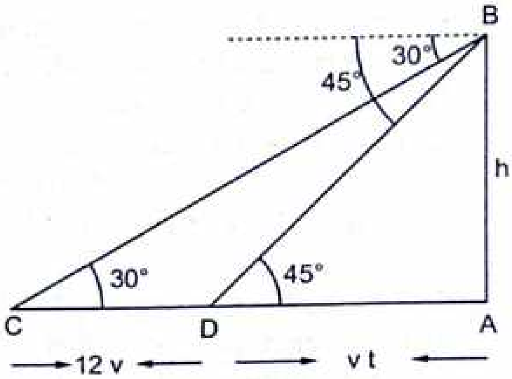

Fig. 12.46

Substituting the value of h from equation (i) in equation (ii), we get

$\sqrt{3}\, vt = vt + 12v$

$\Rightarrow \quad \sqrt{3}\, t = t + 12$

$\Rightarrow \quad t(\sqrt{3} - 1) = 12$

$\Rightarrow \quad t = \dfrac{12}{\sqrt{3} - 1} = \dfrac{12(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$

$\Rightarrow \quad t = 6(\sqrt{3} + 1) = 16.39$ minutes

$\Rightarrow \quad t = 16$ minutes 23 seconds $\quad [\because 0.39 \text{ minutes} = 0.39 \times 60 \text{ seconds}]$

Thus, the car will reach the tower from D in 16 minutes and 23 seconds.

EXERCISE 12.1

LEVEL-1

1. A tower stands vertically on the ground. From a point on the ground, 20 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°. What is the height of the tower?
2. The angle of elevation of a ladder leaning against a wall is 60° and the foot of the ladder is 9.5 m away from the wall. Find the length of the ladder.
3. A ladder is placed along a wall of a house such that its upper end is touching the top of the wall. The foot of the ladder is 2 m away from the wall and the ladder is making an angle of 60° with the level of the ground. Determine the height of the wall.
4. An electric pole is 10 m high. A steel wire tied to top of the pole is affixed at a point on the ground to keep the pole up right. If the wire makes an angle of 45° with the horizontal through the foot of the pole, find the length of the wire.
5. A kite is flying at a height of 75 metres from the ground level, attached to a string inclined at 60° to the horizontal. Find the length of the string to the nearest metre.
6. A ladder 15 metres long just reaches the top of a vertical wall. If the ladder makes an angle of 60° with the wall, find the height of the wall. **[NCERT EXEMPLAR]**
7. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff. At a point on the plane 70 metres away from the tower, an observer notices that the angles of elevation of the top and the bottom of the flag-staff are respectively 60° and 45°. Find the height of the flag-staff and that of the tower. **[CBSE 2014]**

8. A vertically straight tree, 15 m high, is broken by the wind in such a way that its top just touches the ground and makes an angle of $60°$ with the ground. At what height from the ground did the tree break?
9. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff of height 5 metres. At a point on the plane, the angles of elevation of the bottom and the top of the flag-staff are respectively $30°$ and $60°$. Find the height of the tower. **[CBSE 2015, 2016]**
10. A person observed the angle of elevation of the top of a tower as $30°$. He walked 50 m towards the foot of the tower along level ground and found the angle of elevation of the top of the tower as $60°$. Find the height of the tower.
11. The shadow of a tower, when the angle of elevation of the sun is $45°$, is found to be 10 m. longer than when it was $60°$. Find the height of the tower.
12. A parachutist is descending vertically and makes angles of elevation of $45°$ and $60°$ at two observing points 100 m apart from each other on the left side of himself. Find the maximum height from which he falls and the distance of the point where he falls on the ground from the just observation point.
13. On the same side of a tower, two objects are located. When observed from the top of the tower, their angles of depression are $45°$ and $60°$. If the height of the tower is 150 m, find the distance between the objects.
14. The angle of elevation of a tower from a point on the same level as the foot of the tower is $30°$. On advancing 150 metres towards the foot of the tower, the angle of elevation of the tower becomes $60°$. Show that the height of the tower is 129.9 metres (Use $\sqrt{3} = 1.732$). **[CBSE 2006]**
15. The angle of elevation of the top of a tower as observed from a point in a horizontal plane through the foot of the tower is $32°$. When the observer moves towards the tower a distance of 100 m, he finds the angle of elevation of the top to be $63°$. Find the height of the tower and the distance of the first position from the tower. [Take $\tan 32° = 0.6248$ and $\tan 63° = 1.9626$] **[CBSE 2001C]**
16. The angle of elevation of the top of a tower from a point A on the ground is $30°$. On moving a distance of 20 metres towards the foot of the tower to a point B the angle of elevation increases to $60°$. Find the height of the tower and the distance of the tower from the point A. **[CBSE 2002, 2015, 2017]**
17. From the top of a building 15 m high the angle of elevation of the top of a tower is found to be $30°$. From the bottom of the same building, the angle of elevation of the top of the tower is found to be $60°$. Find the height of the tower and the distance between the tower and building. **[CBSE 2002]**
18. On a horizontal plane there is a vertical tower with a flag pole on the top of the tower. At a point 9 metres away from the foot of the tower the angle of elevation of the top and bottom of the flag pole are $60°$ and $30°$ respectively. Find the height of the tower and the flag pole mounted on it. **[CBSE 2005]**
19. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of $30°$ with the ground. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the height of the tree.
20. From a point P on the ground the angle of elevation of a 10 m tall building is $30°$. A flag is hoisted at the top of the building and the angle of elevation of the top of the flag-staff from P is $45°$. Find the length of the flag-staff and the distance of the building from the point P. (Take $\sqrt{3} = 1.732$).

21. A 1.6 m tall girl stands at a distance of 3.2 m from a lamp-post and casts a shadow of 4.8 m on the ground. Find the height of the lamp-post by using (i) trigonometric ratios (ii) property of similar triangles.

22. A 1.5 m tall boy is standing at some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building. **[NCERT, CBSE 2014]**

23. The shadow of a tower standing on a level ground is found to be 40 m longer when Sun's altitude is 30° than when it was 60°. Find the height of the tower. **[NCERT EXEMPLAR]**

24. From a point on the ground the angles of elevation of the bottom and top of a transmission tower fixed at the top of 20 m high building are 45° and 60° respectively. Find the height of the transimission tower. **[NCERT]**

25. The angles of depression of the top and bottom of 8 m tall building from the top of a multistoried building are 30° and 45° respectively. Find the height of the multistoried building and the distance between the two buildings. **[NCERT, CBSE 2009]**

26. A statue 1.6 m tall stands on the top of pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal. **[NCERT, CBSE 2008,2014]**

27. A T.V. Tower stands vertically on a bank of a river. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From a point 20 m away this point on the same bank, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the river. **[NCERT]**

28. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower. **[NCERT, CBSE 2014, 2017]**

29. As observed from the top of a 75 m tall lighthouse, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. **[NCERT]**

30. The angle of elevation of the top of the building from the foot of the tower is 30° and the angle of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building. **[NCERT, CBSE 2012, 2015, 2017]**

31. From a point on a bridge across a river the angles of depression of the banks on opposite side of the river are 30° and 45° respectively. If bridge is at the height of 30 m from the banks, find the width of the river. **[NCERT]**

32. Two poles of equal heights are standing opposite to each other on either side of the road which is 80 m wide. From a point between them on the road the angles of elevation of the top of the poles are 60° and 30° respectively. Find the height of the poles and the distances of the point from the poles. **[NCERT]**

33. A man sitting at a height of 20 m on a tall tree on a small island in the middle of a river observes two poles directly opposite to each other on the two banks of the river and in line with the foot of tree. If the angles of depression of the feet of the poles from a point at which the man is sitting on the tree on either side of the river are 60° and 30° respectively. Find the width of the river.

34. A vertical tower stands on a horizontal plane and is surmounted by a flag-staff of height 7 m. From a point on the plane, the angle of elevation of the bottom of the flag-staff is 30° and that of the top of the flag-staff is 45°. Find the height of the tower. **[CBSE 2016]**

35. The length of the shadow of a tower standing on level plane is found to be $2x$ metres longer when the sun's altitude is 30° than when it was 45°. Prove that the height of tower is $x(\sqrt{3}+1)$ metres.

36. A tree breaks due to the storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with the ground. The distance from the foot of the tree to the point where the top touches the ground is 10 metres. Find the height of the tree.

37. A balloon is connected to a meteorological ground station by a cable of length 215 m inclined at 60° to the horizontal. Determine the height of the balloon from the ground. Assume that there is no slack in the cable.

38. Two men on either side of the cliff 80 m high observes the angles of elevation of the top of the cliff to be 30° and 60° respectively. Find the distance between the two men. **[CBSE 2016]**

39. Find the angle of elevation of the sun (sun's altitude) when the length of the shadow of a vertical pole is equal to its height.

40. An aeroplane is flying at a height of 210 m. Flying at this height at some instant the angles of depression of two points in a line in opposite directions on both the banks of the river are 45° and 60°. Find the width of the river. (Use $\sqrt{3} = 1.73$) **[CBSE 2015]**

41. The angle of elevation of the top of a chimney from the top of a tower is 60° and the angle of depression of the foot of the chimney from the top of the tower is 30°. If the height of the tower is 40 m, find the height of the chimney. According to pollution control norms, the minimum height of a smoke emitting chimney should be 100 m. State if the height of the above mentioned chimney meets the polution norms. What value is discussed in this question? **[CBSE 2014]**

42. Two ships are there in the sea on either side of a light house in such away that the ships and the light house are in the same straight line. The angles of depression of two ships are observed from the top of the light house are 60° and 45° respectively. If the height of the light house is 200 m, find the distance between the two ships. (Use $\sqrt{3} = 1.73$)

43. The horizontal distance between two poles is 15 m. The angle of depression of the top of the first pole as seen from the top of the second pole is 30°. If the height of the second pole is 24 m, find the height of the first pole. ($\sqrt{3} = 1.732$) **[CBSE 2013]**

44. The angles of depression of two ships from the top of a light house and on the same side of it are found to be 45° and 30° respectively. If the ships are 200 m apart, find the height of the light house. **[CBSE 2012]**

45. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6m. **[NCERT, CBSE 2016]**

46. From the top of a 50 m high tower, the angles of depression of the top and bottom of a pole are observed to be 45° and 60° respectively. Find the height of the pole. **[CBSE 2016]**

47. The horizontal distance between two trees of different heights is 60 m. The angle of depression of the top of the first tree when seen from the top of the second tree is 45°. If the height of the second tree is 80 m, find the height of the first tree.

48. A flag-staff stands on the top of a 5 m high tower. From a point on the ground, the angle of elevation of the top of the flag-staff is 60° and from the same point, the angle of elevation of the top of the tower is 45°. Find the height of the flag-staff. **[CBSE 2013]**

49. The angle of elevation of the top of a vertical tower PQ from a point X on the ground is $60°$. At a point Y, 40 m vertically above X, the angle of elevation of the top is $45°$. Calculate the height of the tower.

50. As observed from the top of a 150 m tall light house, the angles of depression of two ships approaching it are $30°$ and $45°$. If one ship is directly behind the other, find the distance between the two ships. **[CBSE 2013]**

51. The angles of elevation of the top of a rock from the top and foot of a 100 m high tower are respectively $30°$ and $45°$. Find the height of the rock.

52. A straight highway leads to the foot of a tower of height 50 m. From the top of the tower, the angles of depression of two cars standing on the highway are $30°$ and $60°$ respectively. What is the distance between the two cars and how far is each car from the tower?

53. From the top of a building AB, 60 m high, the angles of depression of the top and bottom of a vertical lamp post CD are observed to be $30°$ and $60°$ respectively. Find
    (i) the horizontal distance between AB and CD.
    (ii) the height of the lamp post.
    (iii) the difference between the heights of the building and the lamp post. **[CBSE 2009]**

54. Two boats approach a light house in mid-sea from opposite directions. The angles of elevation of the top of the light house from two boats are $30°$ and $45°$ respectively. If the distance between two boats is 100 m, find the height of the light house. **[CBSE 2014]**

55. The angle of elevation of the top of a hill at the foot of a tower is $60°$ and the angle of elevation of the top of the tower from the foot of the hill is $30°$. If the tower is 50 m high, what is the height of the hill? **[CBSE 2006C, 2013]**

56. A moving boat is observed from the top of a 150 m high cliff moving away from the cliff. The angle of depression of the boat changes from $60°$ to $45°$ in 2 minutes. Find the speed of the boat in m/h. **[CBSE 2017]**

57. From the top of a 120 m high tower, a man observes two cars on the opposite sides of the tower and in straight line with the base of tower with angles of depression as $60°$ and $45°$. Find the distance between the cars. (Take $\sqrt{3} = 1.732$) **[CBSE 2017]**

58. Two points A and B are on the same side of a tower and in the same straight line with its base. The angles of depression of these points from the top of the tower are $60°$ and $45°$ respectively. If the height of the tower is 15 m, then find the distance between these points. **[CBSE 2017]**

LEVEL-2

59. A fire in a building B is reported on telephone to two fire stations P and Q, 20 km apart from each other on a straight road. P observes that the fire is at an angle of $60°$ to the road and Q observes that it is at an angle of $45°$ to the road. Which station should send its team and how much will this team have to travel?

60. A man on the deck of a ship is 10 m above the water level. He observes that the angle of elevation of the top of a cliff is $45°$ and the angle of depression of the base is $30°$. Calculate the distance of the cliff from the ship and the height of the cliff.

61. A man standing on the deck of a ship, which is 8 m above water level. He observes the angle of elevation of the top of a hill as $60°$ and the angle of depression of the base of the hill as $30°$. Calculate the distance of the hill from the ship and the height of the hill.

62. There are two temples, one on each bank of a river, just opposite to each other. One temple is 50 m high. From the top of this temple, the angles of depression of the top and the foot of the other temple are 30° and 60° respectively. Find the width of the river and the height of the other temple.

63. The angle of elevation of an aeroplane from a point on the ground is 45°. After a flight of 15 seconds, the elevation changes to 30°. If the aeroplane is flying at a height of 3000 metres, find the speed of the aeroplane.

64. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60°. After 10 seconds, its elevation is observed to be 30°. Find the speed of the aeroplane in km/ hr.

65. A tree standing on a horizontal plane is leaning towards east. At two points situated at distances a and b exactly due west on it, the angles of elevation of the top are respectively α and β. Prove that the height of the top from the ground is

$$\frac{(b-a)\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}.$$

66. The angle of elevation of a stationery cloud from a point 2500 m above a lake is 15° and the angle of depression of its reflection in the lake is 45°. What is the height of the cloud above the lake level? (Use $\tan 15^\circ = 0.268$)

67. If the angle of elevation of a cloud from a point h metres above a lake is a and the angle of depression of its reflection in the lake be b, prove that the distance of the cloud from the point of observation is

$$\frac{2h\sec\alpha}{\tan\beta-\tan\alpha}$$ **[CBSE 2004]**

68. From an aeroplane vertically above a straight horizontal road, the angles of depression of two consecutive mile stones on opposite sides of the aeroplane are observed to be α and β. Show that the height in miles of aeroplane above the road is given by

$$\frac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$ **[CBSE 2004]**

69. PQ is a post of given height a, and AB is a tower at some distance. If α and β are the angles of elevation of B, the top of the tower, at P and Q respectively. Find the height of the tower and its distance from the post.

70. A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a, so that it slides a distance b down the wall making an angle β with the horizontal. Show that

$$\frac{a}{b}=\frac{\cos\alpha-\cos\beta}{\sin\beta-\sin\alpha}$$ **[NCERT EXEMPLAR]**

71. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a point b metres just above A is β. Prove that the height of the tower is $b\tan\alpha\cot\beta$.

72. An observer, 1.5 m tall, is 28.5 m away from a tower 30 m high. Determine the angle of elevation of the top of the tower from his eye. **[NCERT EXEMPLAR]**

73. A carpenter makes stools for electricians with a square top of side 0.5 m and at a height of 1.5 m above the ground. Also, each leg is inclined at an angle of 60° to the ground. Find the length of each leg and also the lengths of two steps to be put at equal distances.

74. A boy is standing on the ground and flying a kite with 100 m of string at an elevation of 30°. Another boy is standing on the roof of a 10 m high building and is flying his kite at an elevation of 45°. Both the boys are on opposite sides of both the kites. Find the length of the string that the second boy must have so that the two kites meet.

75. From the top of a light house, the angles of depression of two ships on the opposite sides of it are observed to be α and β. If the height of the light house be h metres and the line joining the ships passes through the foot of the light house, show that the distance between the ship is $\dfrac{h(\tan\alpha + \tan\beta)}{\tan\alpha \tan\beta}$ metres.

76. From the top of a tower h metre high, the angles of depression of two objects, which are in the line with the foot of the tower are α and β $(\beta > \alpha)$. Find the distance between the two objects. [NCERT EXEMPLAR]

77. A window of a house is h metre above the ground. From the window, the angles of elevation and depression of the top and bottom of another house situated on the opposite side of the lane are found to be α and β respectively. Prove that the height of the house is $h(1 + \tan\alpha \tan\beta)$ metres. [NCERT EXEMPLAR]

78. The lower window of a house is at a height of 2 m above the ground and its upper window is 4 m vertically above the lower window. At certain instant the angles of elevation of a balloon from these windows are observed to be 60° and 30° respectively. Find the height of the balloon above the ground. [NCERT EXEMPLAR]

ANSWERS

1. $20\sqrt{3}$ m 2. 19 m 3. $2\sqrt{3}$ m 4. 14.1 m
5. 87 m 6. 7.5 m 7. 51.24 m, 70 m 8. 6.9 m
9. 2.5 m 10. 43.25 m 11. 23.66 m
12. 236.6 m, 136.6 m 13. 63.4 m 15. 91.65 m, 146.7 m
16. Height = 17.3 m, Distance = 30 m 17. Height = 22.5 m, Distance = 12.975 m
18. $3\sqrt{3}$ m, $6\sqrt{3}$ m 19. $8\sqrt{3}$ m 20. 7.32 m, 17.32 m 21. $\frac{8}{3}$ m
22. $19\sqrt{3}$ m 23. $20\sqrt{3}$ m 24. $20(\sqrt{3} - 1)$ m
25. $4(3 + \sqrt{3})$ m, $4(3 + \sqrt{3})$ m 26. $\frac{4(\sqrt{3} + 1)}{5}$ m
27. $10\sqrt{3}$ m, 10 m 28. $7(\sqrt{3} + 1)$ m 29. $75(\sqrt{3} - 1)$ m 30. $\frac{50}{3}$ m
31. $30(\sqrt{3} + 1)$ m 32. $20\sqrt{3}$ m, 20 m, 60 m 33. $\frac{80}{\sqrt{3}}$ m
34. 9.56 m 36. 17.3 m 37. 186 m 38. 184.8 m
39. 45° 40. 331.38 m 41. 160 m, Yes polution control
42. 315.6 m 43. 15.34 m 44. 273.2 m 46. 21.13 m
47. 20 m 48. 3.65 m 49. 94.64 m 50. 109.5 m
51. 236.5 m 52. 57.67 m, 86.5 m, 28.83 m
53. (i) 34.64 m (ii) 40 m. (iii) 20 m. 54. $50(\sqrt{3} - 1)$ m 55. 150 m
56. 1902 m/hr 57. 189.28 m 58. 6.340 m
59. Station P, 14.64 km 60. Distance $= 10\sqrt{3}$ m, Height $= 27.32$ m
61. Distance $= 8\sqrt{3}$ m, Height $= 32$ m 62. 28.83 m, 33.33 m 63. 527.04 km/hr

64. 415.68 km/hr **65.** 20.87 m, 33.33 m **66.** $2500\sqrt{3}$ m

69. Distance $= \dfrac{a}{\tan\alpha - \tan\beta}$, Height $= \dfrac{a\tan\alpha}{\tan\alpha - \tan\beta}$ **72.** 45°

73. 1.732 m, 1.1077 m, 1.654 m **74.** $40\sqrt{2}$ m **76.** $h(\cot\alpha - \cot\beta)$ **78.** 8 m

HINT TO SELECTED PROBLEMS

22. In $\Delta\, ACB$, we have

$$\tan 30^\circ = \frac{QC}{AC}$$

$$\Rightarrow \quad \tan 30^\circ = \frac{30-1.5}{AC}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{28.5}{AC}$$

$$\Rightarrow \quad AC = 28.5 \times \sqrt{3}\ \text{m}$$

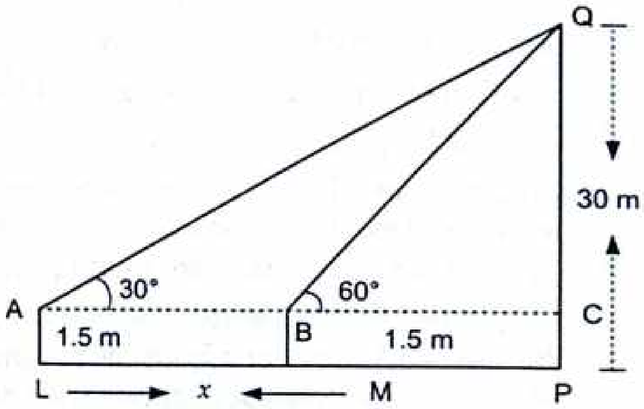

Fig. 12.47

In $\Delta\, BCQ$, we have

$$\tan 60^\circ = \frac{QC}{BC}$$

$$\Rightarrow \quad \sqrt{3} = \frac{30-1.5}{BC}$$

$$\Rightarrow \quad BC = \frac{28.5}{\sqrt{3}}\ \text{m}$$

$$\therefore \quad AB = AC - BC = 28.5 \times \sqrt{3} - \frac{28.5}{\sqrt{3}} = \frac{28.5 \times 2}{\sqrt{3}} = 19\sqrt{3}\ \text{m}.$$

24. Let PQ be the building of height 20 metre and QR be the transmission tower of height h metre.

Let the angles of elevation of the bottom and top of the tower at point O be 45° and 60° respectively.

Then, in triangles OPQ and OPR, we have

$$\tan 45^\circ = \frac{PQ}{OP} \text{ and } \tan 60^\circ = \frac{PR}{OP}$$

$$\Rightarrow \quad 1 = \frac{20}{OP} \text{ and } \sqrt{3} = \frac{20+h}{OP}$$

$$\Rightarrow \quad OP = 20\ \text{m and } \sqrt{3} \times OP = 20 + h$$

$$\Rightarrow \quad 20\sqrt{3} = 20 + h$$

$$\Rightarrow \quad h = (20\sqrt{3} - 20)\ \text{m} = 20(\sqrt{3} - 1)\ \text{m}.$$

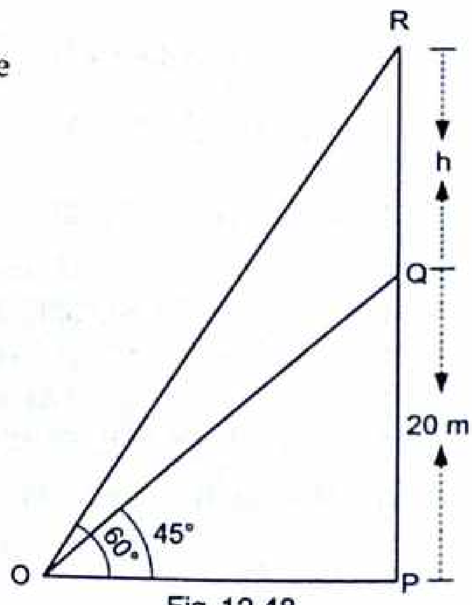

Fig. 12.48

26. Let OP be the pedestal and PQ be the statue of height 1.6 m

In Δ's AOP and AOQ, we have

$$\tan 45° = \frac{OP}{OA} \text{ and } \tan 60° = \frac{OQ}{OA}$$

$\Rightarrow$ $1 = \frac{OP}{OA}$ and $\sqrt{3} = \frac{OP + 1.6}{OA}$

$\Rightarrow$ $OA = OP$ and $\sqrt{3}\, OA = OP + 1.6$

$\Rightarrow$ $\sqrt{3}\, OP = OP + 1.6$

$\Rightarrow$ $(\sqrt{3} - 1)\, OP = 1.6$

$\Rightarrow$ $OP = \frac{1.6}{\sqrt{3} - 1} = 0.8\,(\sqrt{3} + 1)$ m

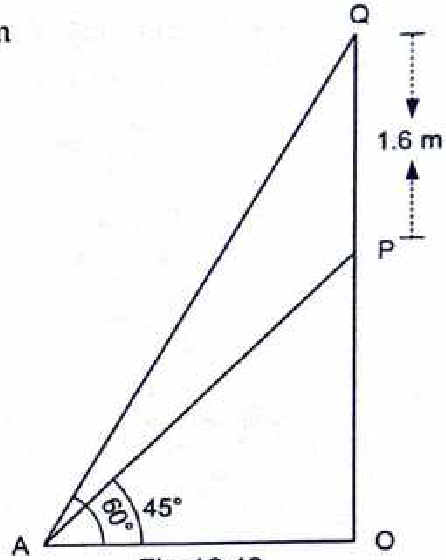

Fig. 12.49

27. Let AB be the tower of height h metre on a bank of the river and D be a point on the opposite bank or the river.

In Δ's DBA and CBA, we have

$\Rightarrow$ $\tan 30° = \frac{AB}{DB}$ and $\tan 60° = \frac{AB}{BC}$

$\Rightarrow$ $\frac{1}{\sqrt{3}} = \frac{h}{20 + BC}$ and $\sqrt{3} = \frac{h}{BC}$

$\Rightarrow$ $\sqrt{3}\, h = 20 + BC$ and $BC = \frac{h}{\sqrt{3}}$

$\Rightarrow$ $\sqrt{3}h = 20 + \frac{h}{\sqrt{3}}$ [On eliminating BC]

$\Rightarrow$ $\frac{2h}{\sqrt{3}} = 20$

$\Rightarrow$ $h = 10\sqrt{3}$ m

$\therefore$ $BC = \frac{h}{\sqrt{3}} = 10$ m

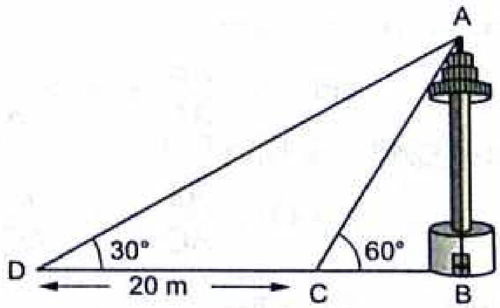

Fig. 12.50

29. Let OA be the light house of height 75 m and P and Q be the positions of two ships.

In Δ's AOQ and AOP, we have

$$\tan 45° = \frac{OA}{OQ} \text{ and } \tan 30° = \frac{OA}{OP}$$

$\Rightarrow$ $1 = \frac{75}{OQ}$ and $\frac{1}{\sqrt{3}} = \frac{75}{OP}$

$\Rightarrow$ $OQ = 75$ and $OP = 75\sqrt{3}$

$\therefore$ $PQ = (75\sqrt{3} - 75)$ m $= 75\,(\sqrt{3} - 1)$ m

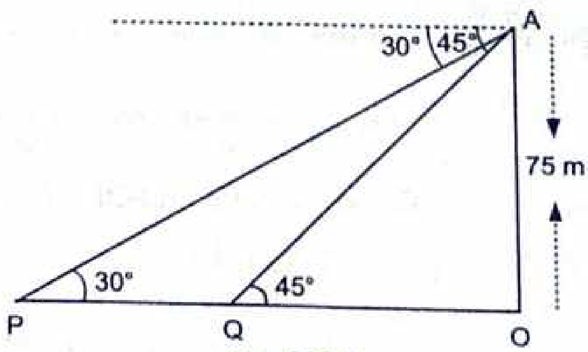

Fig. 12.51

30. Let AD be the building of height h metre.

In Δ's ABC and ABD, we have

$$\tan 60^\circ = \frac{BC}{AB} \text{ and } \tan 30^\circ = \frac{AD}{AB}$$

$$\Rightarrow \quad \sqrt{3} = \frac{50}{AB} \text{ and } \frac{1}{\sqrt{3}} = \frac{h}{AB}$$

$$\Rightarrow \quad AB = \frac{50}{\sqrt{3}} \text{ and } AB = \sqrt{3}h$$

$$\Rightarrow \quad \sqrt{3}h = \frac{50}{\sqrt{3}}$$

$$\Rightarrow \quad h = \frac{50}{3}$$

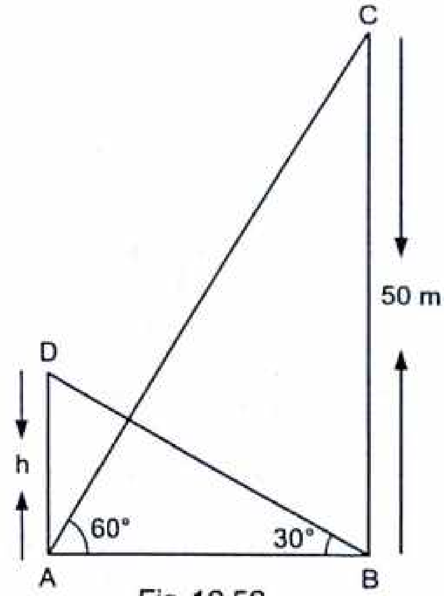

Fig. 12.52

59. Let AB be the building of height h.

Clearly, $\angle APB > \angle AQB$.

$$\Rightarrow \quad \angle ABP < \angle ABQ$$

$$\Rightarrow \quad AP < AQ$$

$\Rightarrow$ Station P is nearer to the building.

So, station P must send its team.

In ΔPAB, we have

$$\tan 60^\circ = \frac{AB}{AP} \Rightarrow \sqrt{3} = \frac{h}{AP} \Rightarrow AP = \frac{h}{\sqrt{3}}$$

In ΔQAB, we have

$$\tan 45^\circ = \frac{AB}{AQ} \Rightarrow 1 = \frac{h}{AQ} \Rightarrow AQ = h$$

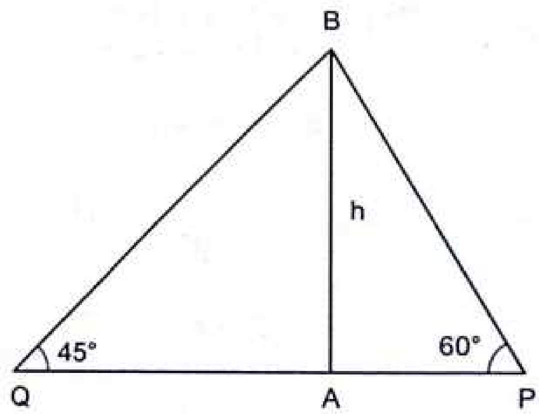

Fig. 12.53

Now, $\quad PQ = 20$ km

$$\Rightarrow \quad AP + AQ = 20$$

$$\Rightarrow \quad \frac{h}{\sqrt{3}} + h = 20 \Rightarrow h = \frac{20\sqrt{3}}{\sqrt{3}+1} = 10(3-\sqrt{3}) = 17.32 \text{ km}$$

65. Let OP be the tree and A, B be two points such that $OA = a$ and $OB = b$.

In Δ's ALP and BLP, we have

$$\tan\alpha = \frac{h}{OL+a} \text{ and } \tan\beta = \frac{h}{OL+b}$$

$$\Rightarrow \quad OL + a = h\cot\alpha \text{ and } OL + b = h\cot\beta$$

$$\Rightarrow \quad b - a = h\cot\beta - h\cot\alpha$$

$$\Rightarrow \quad h = \frac{(b-a)}{\cot\beta - \cot\alpha} = \frac{(b-a)\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$$

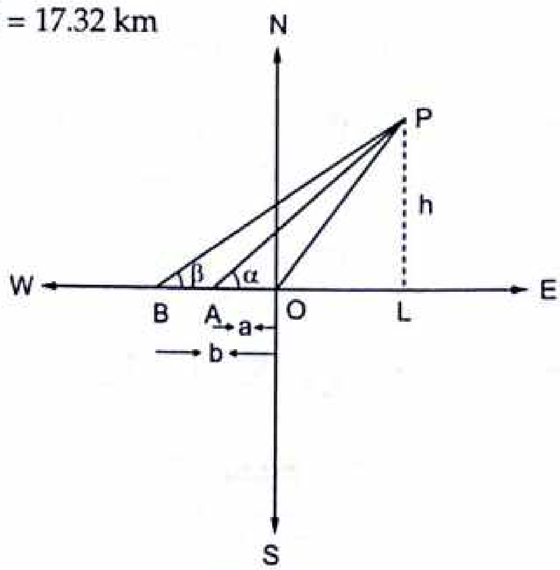

Fig. 12.54

67. Let C' be the image of clould C in the lake.

In Δ's PQC and PQC', we have

$$\tan\alpha = \frac{x}{PQ} \text{ and } \tan\beta = \frac{QC'}{PQ}$$

$$\Rightarrow \quad \tan\alpha = \frac{x}{PQ} \text{ and } \tan\beta = \frac{x+2h}{PQ}$$

$$\Rightarrow \quad \tan\beta - \tan\alpha = \frac{x+2h}{PQ} - \frac{x}{PQ}$$

$$\Rightarrow \quad \tan\beta - \tan\alpha = \frac{2h}{PQ}$$

$$\Rightarrow \quad PQ = \frac{2h}{\tan\beta - \tan\alpha}$$

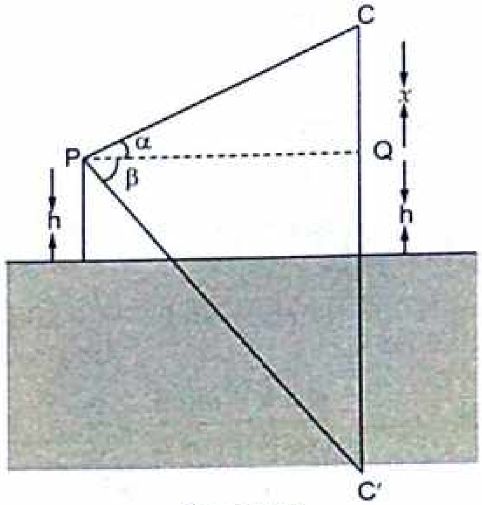

Fig. 12.55

Again, in $\Delta\, PQC$, we have

$$\cos\alpha = \frac{PQ}{CP}$$

$$\Rightarrow \quad CP = PQ\sec\alpha$$

$$\Rightarrow \quad CP = \frac{2h\sec\alpha}{\tan\beta - \tan\alpha}$$

68. Let h be the height of aeroplane P above the road and A and B be two consecutive milestones. Then,

$$AB = 1 \text{ mile}$$

In Δ's AQP and BQP, we have

$$\tan\alpha = \frac{h}{AQ} \text{ and } \tan\beta = \frac{h}{BQ}$$

$$\Rightarrow \quad AQ = h\cot\alpha \text{ and } BQ = h\cot\beta$$

$$\Rightarrow \quad AQ + BQ = h(\cot\alpha + \cot\beta)$$

$$\Rightarrow \quad AB = h\left(\frac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta}\right)$$

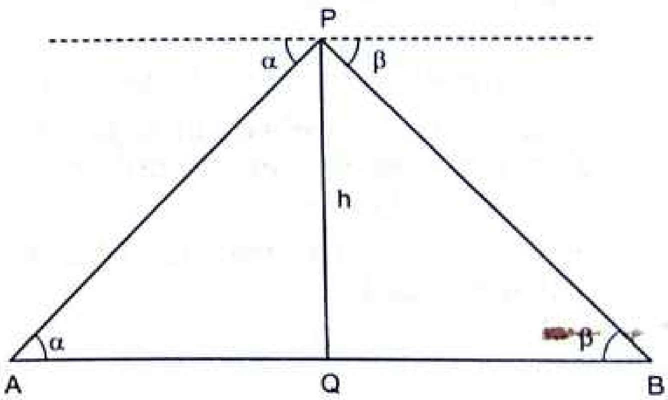

Fig. 12.56

$$\Rightarrow \quad h = \frac{\tan\alpha\tan\beta}{\tan\alpha + \tan\beta} \qquad [\because AB = 1]$$

69. Let PQ be the ladder such that its top Q is on the wall OQ and bottom P is on the ground. The ladder is pulled away from the wall through a distance a, so that its top Q slides and takes position Q'.

Clearly, $PQ = P'Q'$.

In $\Delta^{'s}$ POQ and $P'OQ'$, we have

$$\sin\alpha = \frac{OQ}{PQ}, \cos\alpha = \frac{OP}{PQ}, \sin\beta = \frac{OQ'}{P'Q'}, \cos\beta = \frac{OP'}{P'Q'}$$

$$\Rightarrow \quad \sin\alpha = \frac{b+y}{PQ}, \cos\alpha = \frac{x}{PQ}, \sin\beta = \frac{y}{PQ}, \cos\beta = \frac{a+x}{PQ}$$

$$\Rightarrow \quad \sin\alpha - \sin\beta = \frac{b+y}{PQ} - \frac{y}{PQ} \text{ and } \cos\beta - \cos\alpha = \frac{a+x}{PQ} - \frac{x}{PQ}$$

$$\Rightarrow \quad \sin\alpha - \sin\beta = \frac{b}{PQ} \text{ and } \cos\beta - \cos\alpha = \frac{a}{PQ}$$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \frac{b}{a}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\cos\alpha - \cos\beta}{\sin\beta - \sin\alpha}$$

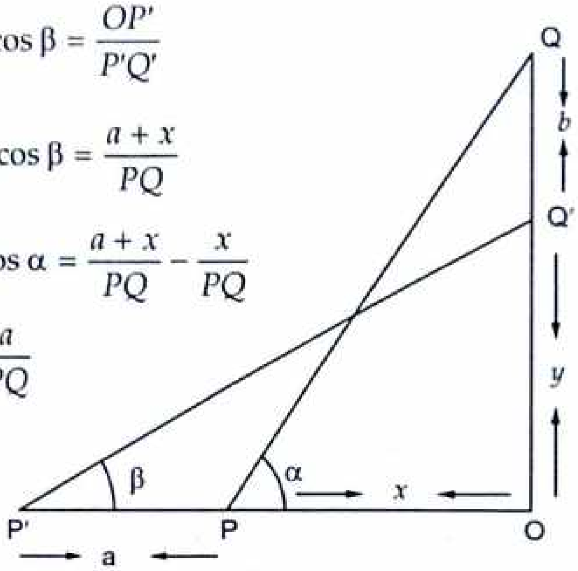

Fig. 12.57

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. The height of a tower is 10 m. What is the length of its shadow when Sun's altitude is 45°?
2. If the ratio of the height of a tower and the length of its shadow is $\sqrt{3} : 1$, what is the angle of elevation of the Sun? [CBSE 2017]
3. What is the angle of elevation of the Sun when the length of the shadow of a vertical pole is equal to its height?
4. From a point on the ground, 20 m away from the foot of a vertical tower, the angle of elevation of the top of the tower is 60°, what is the height of the tower?
5. If the angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary, find the height of the tower.
6. In Fig. 12.58, what are the angles of depression from the observing positions O_1 and O_2 of the object at A?

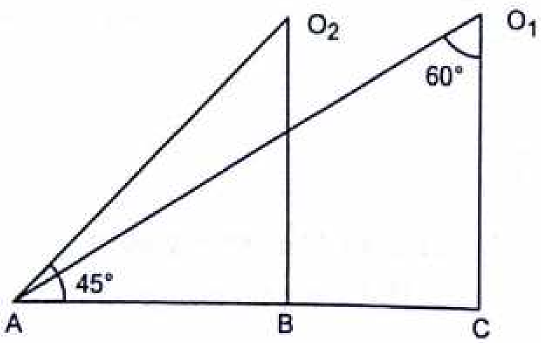

Fig. 12.58

7. The tops of two towers of height x and y, standing on level ground, subtend angles of 30° and 60° respectively at the centre of the line joining their feet, then find $x : y$. [CBSE 2015]

8. The angle of elevation of the top of a tower at a point on the ground is 30°. What will be the angle of elevation, if the height of the tower is tripled? [CBSE 2015]

9. AB is a pole of height 6 m standing at a point B and CD is a ladder inclined at angle of 60° to the horizontal and reaches upto a point D of pole. If $AD = 2.54$ m, find the length of the ladder. (Use $\sqrt{3} = 1.73$) [CBSE 2016]

10. An observer, 1.7 m tall, is $20\sqrt{3}$ m away from a tower. The angle of elevation from the eye of an observer to the top of tower is 30°. Find the height of the tower. [CBSE 2016]

11. An observer, 1.5 m tall, is 28.5 m away from a 30 m high tower. Determine the angle of elevation of the top of the tower from the eye of the observer. [CBSE 2017]

ANSWERS

1. 10 m	2. 60°	3. 45°	4. $20\sqrt{3}$ m	5. 6 m
6. 30°, 45°	7. 1 : 3	8. 60°	9. 4 m	10. 21.7 m
11. 45°				

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The ratio of the length of a rod and its shadow is $1 : \sqrt{3}$. The angle of elevation of the sum is
   (a) 30° (b) 45° (c) 60° (d) 90°

2. If the angle of elevation of a tower from a distance of 100 metres from its foot is 60°, then the height of the tower is
   (a) $100\sqrt{3}$ m (b) $\frac{100}{\sqrt{3}}$ m (c) $50\sqrt{3}$ m (d) $\frac{200}{\sqrt{3}}$ m

3. If the altitude of the sum is at 60°, then the height of the vertical tower that will cast a shadow of length 30 m is
   (a) $30\sqrt{3}$ m (b) 15 m (c) $\frac{30}{\sqrt{3}}$ m (d) $15\sqrt{2}$ m

4. If the angles of elevation of a tower from two points distant a and b $(a > b)$ from its foot and in the same straight line from it are 30° and 60°, then the height of the tower is
   (a) $\sqrt{a+b}$ (b) $\sqrt{ab}$ (c) $\sqrt{a-b}$ (d) $\sqrt{\frac{a}{b}}$

5. If the angles of elevation of the top of a tower from two points distant a and b from the base and in the same straight line with it are complementary, then the height of the tower is
   (a) ab (b) $\sqrt{ab}$ (c) $\frac{a}{b}$ (d) $\sqrt{\frac{a}{b}}$

6. From a light house the angles of depression of two ships on opposite sides of the light house are observed to be 30° and 45°. If the height of the light house is h metres, the distance between the ships is

(a) $(\sqrt{3}+1)\,h$ metres (b) $(\sqrt{3}-1)\,h$ metres

(c) $\sqrt{3}\,h$ metres (d) $1+\left(1+\frac{1}{\sqrt{3}}\right)h$ metres

7. The angle of elevation of the top of a tower standing on a horizontal plane from a point A is α. After walking a distance d towards the foot of the tower the angle of elevation is found to be β. The height of the tower is

(a) $\frac{d}{\cot\alpha+\cot\beta}$ (b) $\frac{d}{\cot\alpha-\cot\beta}$ (c) $\frac{d}{\tan\beta-\tan\alpha}$ (d) $\frac{d}{\tan\beta+\tan\alpha}$

8. The tops of two poles of height 20 m and 14 m are connected by a wire. If the wire makes an angle of 30° with horizontal, then the length of the wire is

(a) 12 m (b) 10 m (c) 8 m (d) 6 m

9. From the top of a cliff 25 m high the angle of elevation of a tower is found to be equal to the angle of depression of the foot of the tower. The height of the tower is

(a) 25 m (b) 50 m (c) 75 m (d) 100 m

10. The angles of depression of two ships from the top of a light house are 45° and 30° towards east. If the ships are 100 m apart, the height of the light house is

(a) $\frac{50}{\sqrt{3}+1}$ m (b) $\frac{50}{\sqrt{3}-1}$ m (c) $50(\sqrt{3}-1)$ m (d) $50(\sqrt{3}+1)$ m

11. If the angle of elevation of a cloud from a point 200 m above a lake is 30° and the angle of depression of its reflection in the lake is 60°, then the height of the cloud above the lake, is

(a) 200 m (b) 500 m (c) 30 m (d) 400 m

12. The height of a tower is 100 m. When the angle of elevation of the sun changes from 30° to 45°, the shadow of the tower becomes x metres less. The value of x is

(a) 100 m (b) $100\sqrt{3}$ m (c) $100(\sqrt{3}-1)$m (d) $\frac{100}{\sqrt{3}}$ m

13. Two persons are a metres apart and the height of one is double that of the other. If from the middle point of the line joining their feet, an observer finds the angular elevation of their tops to be complementary, then the height of the shorter post is

(a) $\frac{a}{4}$ (b) $\frac{a}{\sqrt{2}}$ (c) $a\sqrt{2}$ (d) $\frac{a}{2\sqrt{2}}$

14. The angle of elevation of a cloud from a point h metre above a lake is θ. The angle of depression of its reflection in the lake is 45°. The height of the cloud is

(a) $h\tan(45^\circ+\theta)$ (b) $h\cot(45^\circ-\theta)$ (c) $h\tan(45^\circ-\theta)$ (d) $h\cot(45^\circ+\theta)$

15. A tower subtends an angle of 30° at a point on the same level as its foot. At a second point h metres above the first, the depression of the foot of the tower is 60°. The height of the tower is

(a) $\frac{h}{2}$ m (b) $\sqrt{3}h$ m (c) $\frac{h}{3}$ m (d) $\frac{h}{\sqrt{3}}$ m

16. It is found that on walking x meters towards a chimney in a horizontal line through its base, the elevation of its top changes from 30° to 60°. The height of the chimney is

(a) $3\sqrt{2}x$ (b) $2\sqrt{3}x$ (c) $\frac{\sqrt{3}}{2}x$ (d) $\frac{2}{\sqrt{3}}x$

17. The length of the shadow of a tower standing on level ground is found to be $2x$ metres longer when the sun's elevation is 30° than when it was 45°. The height of the tower in metres is

(a) $(\sqrt{3}+1)\,x$ (b) $(\sqrt{3}-1)\,x$ (c) $2\sqrt{3}x$ (d) $3\sqrt{2}x$

18. Two poles are 'a' metres apart and the height of one is double of the other. If from the middle point of the line joining their feet an observer finds the angular elevations of their tops to be complementary, then the height of the smaller is

(a) $\sqrt{2}a$ metres (b) $\frac{a}{2\sqrt{2}}$ metres (c) $\frac{a}{\sqrt{2}}$ metres (d) $2a$ metres

19. The tops of two poles of height 16 m and 10 m are connected by a wire of length l metres. If the wire makes an angle of 30° with the horizontal, then $l =$

(a) 26 (b) 16 (c) 12 (d) 10

20. If a 1.5 m tall girl stands at a distance of 3 m from a lamp-post and casts a shadow of length 4.5 m on the ground, then the height of the lamp-post is

(a) 1.5 m (b) 2 m (c) 2.5 m (d) 2.8 m

21. The length of shadow of a tower on the plane ground is $\sqrt{3}$ times the height of the tower. The angle of elevation of sun is

(a) 45° (b) 30° (c) 60° (d) 90° [CBSE 2012]

22. The angle of depression of a car, standing on the ground, from the top of a 75 m tower, is 30°. The distance of the car from the base of the tower (in metres) is

(a) $25\sqrt{3}$ (b) $50\sqrt{3}$ (c) $75\sqrt{3}$ (d) 150 [CBSE 2013]

23. A ladder 15 m long just reaches the top of a vertical wall. If the ladder makes an angle of 60° with the wall, then the height of the wall is

(a) $15\sqrt{3}$ m (b) $\frac{15\sqrt{3}}{2}$ m (c) $\frac{15}{2}$ m (d) 15 m [CBSE 2013]

24. The angle of depression of a car parked on the road from the top of a 150 m high tower is 30°. The distance of the car from the tower (in metres) is

(a) $50\sqrt{3}$ (b) $150\sqrt{3}$ (c) $150\sqrt{2}$ (d) 75 [CBSE 2014]

25. If the height of a vertical pole is $\sqrt{3}$ times the length of its shadow on the ground, then the angle of elevation of the sun at that time is

(a) 30° (b) 60° (c) 45° (d) 75° [CBSE 2014]

26. The angle of elevation of the top of a tower at a point on the ground 50 m away from the foot of the tower is 45°. Then the height of the tower (in metres) is

(a) $50\sqrt{3}$ (b) 50 (c) $\frac{50}{\sqrt{2}}$ (d) $\frac{50}{\sqrt{3}}$ [CBSE 2014]

27. A ladder makes an angle of 60° with the ground when placed against a wall. If the foot of the ladder is 2 m away from the wall, then the length of the ladder (in metres) is

(a) $\frac{4}{\sqrt{3}}$ (b) $4\sqrt{3}$ (c) $2\sqrt{2}$ (d) 4 [CBSE 2014]

ANSWERS

1. (a)	2. (b)	3. (a)	4. (b)	5. (b)	6. (a)
7. (b)	8. (a)	9. (b)	10. (d)	11. (d)	12. (c)
13. (d)	14. (a)	15. (c)	16. (c)	17. (a)	18. (b)
19. (c)	20. (c)	21. (b)	22. (a)	23. (c)	24. (a)
25. (b)	26. (b)	27. (d)			

SUMMARY

1. The line drawn from the eye of an observer to a point in the object where the person is viewing is called the line of sight.
2. The angle formed by the line of sight with the horizontal when the object is above the horizontal level is called the angle of elevation.
3. The angle formed by the line of sight with the horizontal when the object is below the horizontal level is called the angle of depression.
4. The height of an object or the distance between distant objects can be determined with the help of trigonometric ratios.

CHAPTER 13

AREAS RELATED TO CIRCLES

13.1 INTRODUCTION

In earlier classes, we have studied methods of finding perimeters and areas of simple plane figures such as rectangles, squares, parallelograms, triangles and circles. In our daily life, we come across many objects which are related to circular shape in some form or the other. For example, cycle wheels, wheel arrow, drain cover, bangles, brooches, flower beds, circular paths etc. That is why the problem of finding perimeters and areas related to circular figures is of great practical importance. In this chapter, we shall discuss problems on finding the areas of the two special parts of a circular region known as sector and segment of a circle. We shall also discuss problems on finding the areas of some combinations of plane figures involving circles or parts of circles. Let us first recall the concepts related to the perimeter and area of a circle.

13.2 REVIEW OF PERIMETER AND AREA OF A CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point always remains same.

The fixed point is called the *centre* and the given constant distance is known as the *radius* of the circle.

CIRCUMFERENCE *The perimeter of a circle is generally known as its circumference.*

We know that circumference of a circle bears a constant ratio with its diameter. This constant ratio is denoted by the Greek letter π (read as 'pi').

Thus, if C denotes the circumference of a circle of radius r. Then,

$$\pi = \frac{\text{Circumference}}{\text{Diameter}} \Rightarrow \pi = \frac{C}{2r} \Rightarrow C = 2\pi r$$

Here, π stands for a particular irrational number whose approximate value upto two decimal place is 3.14 or $\frac{22}{7}$. The value of π upto four places of decimal is 3.1416 and up to eight decimal places its value is 3.14159265. For practical purposes, we generally take the value of π as $\frac{22}{7}$ or 3.14 approximately.

If r is the radius of a circle, then

(i) Circumference $= 2\pi r$

Also, Circumference $= \pi d$, where $d = 2r$ is the diameter of the circle.

(ii) Area $= \pi r^2$, Also Area $= \pi\left(\frac{d}{2}\right)^2 = \frac{1}{4}\pi d^2$

(iii) Area of semi-circle $= \frac{1}{2}\pi r^2$

(iv) Area of a quadrant of a circle $= \frac{1}{4}\pi r^2$

AREA ENCLOSED BY TWO CONCENTRIC CIRLES If R and r are radii of two concentric circles, then

Area enclosed by the two circles $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)(R - r)$

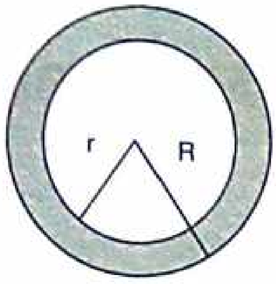

Fig. 13.1

Some useful results:

(i) If two circles touch internally, then the distance between their centres is equal to the difference of their radii.

(ii) If two circles touch externally, then the distance between their centres is equal to the sum of their radii.

(iii) Distance moved by a rotating wheel in one revolution is equal to the circumference of the wheel.

(iv) The number of revolutions completed by a rotating wheel in one minute

$$= \frac{\text{Distance moved in one minute}}{\text{Circumference}}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the circumference and area of a circle of radius 8.4 cm.*

SOLUTION We know that the circumference C and area A of a circle of radius r are given by $C = 2\pi r$ and $A = \pi r^2$ respectively.

Here, $r = 8.4$ cm.

$$\therefore \quad C = \text{Circumference} = 2\pi r = 2 \times \frac{22}{7} \times 8.4 \text{ cm} = 52.8 \text{ cm}$$

$$A = \text{Area} = \pi r^2 = \frac{22}{7} \times 8.4 \times 8.4 \text{ cm}^2 = 221.76 \text{ cm}^2$$

EXAMPLE 2 *Find the area of a circle whose circumference is 22 cm.*

SOLUTION Let r be the radius of the circle. It is given that the circumfernce of the circle is 22 cm.

Now, Circumference = 22 cm

$$\Rightarrow \quad 2\pi r = 22 \Rightarrow 2 \times \frac{22}{7} \times r = 22 \Rightarrow r = \frac{7}{2} \text{ cm}$$

$$\therefore \quad \text{Area of the circle} = \pi r^2 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \text{ cm}^2 = 38.5 \text{ cm}^2$$

EXAMPLE 3 *Find the area of a quadrant of a circle whose circumference is 22 cm.*

SOLUTION Let r be the radius of the circle. It is given that the circumference of the circle is 22 cm.

Now, Circumference = 22 cm

$\Rightarrow$ $2\pi r = 22 \Rightarrow 2 \times \frac{22}{7} \times r = 22 \Rightarrow r = \frac{7}{2}$ cm

$\therefore$ Area of a quadrant $= \frac{1}{4}\pi r^2 = \left\{\frac{1}{4} \times \frac{22}{7}\left(\frac{7}{2}\right)^2\right\}$ cm^2

$$= \left\{\frac{1}{4} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right\} \text{cm}^2 = \frac{77}{8} \text{cm}^2 = 9.625 \text{ cm}^2$$

EXAMPLE 4 *If the perimeter of a semi-circular protractor is 108 cm, find the diameter of the protractor* (Take $\pi = 22/7$).

SOLUTION Let the radius of the protractor be r cm. It is given that its perimeter is 108 cm.

Now, Perimeter = 108 cm

$\Rightarrow$ $\frac{1}{2}(2\pi r) + 2r = 108$ $\left[\therefore \text{ Perimeter of a semi-circle} = \frac{1}{2}(2\pi r)\right]$

$\Rightarrow$ $\pi r + 2r = 108 \Rightarrow \frac{22}{7} \times r + 2r = 108 \Rightarrow 36r = 108 \times 7 \Rightarrow r = 3 \times 7 = 21$

$\therefore$ Diameter of the protractor $= 2r = (2 \times 21)$ cm $= 42$ cm

EXAMPLE 5 *The circumference of a circle exceeds the diameter by 16.8 cm. Find the radius of the circle.*

SOLUTION Let the radius of the circle be r cm. Then,

Diameter $= 2r$ cm and, Circumference $= 2\pi r$ cm

It is given that the circumference exceeds the diameter by 16.8 cm. That is,

Circumference = Diameter + 16.8

$\Rightarrow$ $2\pi r = 2r + 16.8$

$\Rightarrow$ $2 \times \frac{22}{7} \times r = 2r + 16.8$ $\left[\because \pi = \frac{22}{7}\right]$

$\Rightarrow$ $44r = 14r + 16.8 \times 7$

$\Rightarrow$ $44r - 14r = 117.6 \Rightarrow 30r = 117.6 \Rightarrow r = \frac{117.6}{30} = 3.92$

Hence, radius of the circle is 3.92 cm.

EXAMPLE 6 *Find the diameter of the circle whose area is equal to the sum of the areas of two circles of diameters 20 cm and 48 cm.* **[NCERT EXEMPLAR]**

SOLUTION Let d be the diameter of the circle whose area is equal to the sum of the areas of two circles of diameters $d_1 = 20$ cm and $d_2 = 48$ cm. Then,

$$\pi\left(\frac{d}{2}\right)^2 = \pi\left(\frac{20}{2}\right)^2 + \pi\left(\frac{48}{2}\right)^2$$

$\Rightarrow$ $\frac{d^2}{4} = 10^2 + 24^2$

$\Rightarrow$ $\frac{d^2}{4} = 100 + 576 = 676$

$\Rightarrow$ $d^2 = 676 \times 4 = 26^2 \times 2^2$

$\Rightarrow$ $d = 26 \times 2 = 52$ cm.

EXAMPLE 7 *All the vertices of a rhombus lie on a circle. Find the area of the rhombus, if area of the circle is 1256 cm^2.* (Use $\pi = 3.14$) **[NCERT EXEMPLAR]**

SOLUTION Let $ABCD$ be a rhombus whose vertices A, B, C, D lie on a circle with centre O and radius r.

It is given that the area of the circle is 1256 cm^2.

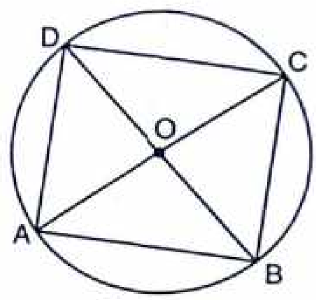

Fig. 13.2

$$\therefore \quad \pi r^2 = 1256$$

$$\Rightarrow \quad 3.14 r^2 = 1256$$

$$\Rightarrow \quad r^2 = 400 \quad \text{...(i)}$$

We know that the diagonals of a rhombus intersect at right angle. Therefore, AC and BD are perpendicular and so these two are diameters of the circle.

$$\therefore \quad AC = BD = 2r$$

Area of rhombus $ABCD = \frac{1}{2}(AC \times BD) = \frac{1}{2}(2r \times 2r) = 2r^2 = 2 \times 400 \text{ cm}^2 = 800 \text{ cm}^2$

EXAMPLE 8 *A copper wire, when bent in the form of a square, encloses an area of 484 cm^2. If the same wire is bent in the form of a circle, find the area enclosed by it. (Use $\pi = 22/7$).*

SOLUTION It is given that

Area of the square = 484 cm^2

$\therefore$ Side of the square $\sqrt{484}$ cm = 22cm $\quad [\because \text{Area} = (\text{Side})^2 \therefore \text{Side} = \sqrt{\text{Area}}]$

So, Perimeter of the square = 4 (Side) = (4 × 22) cm = 88 cm

Let r be the radius of the circle. Then,

Circumference of the circle = Perimeter of the square.

$$\Rightarrow \quad 2\pi r = 88$$

$$\Rightarrow \quad 2 \times \frac{22}{7} \times r = 88 \Rightarrow r = 14 \text{ cm}$$

$$\therefore \quad \text{Area of the circle} = \pi r^2 = \left\{\frac{22}{7} \times (14)^2\right\} \text{cm}^2 = 616 \text{ cm}^2$$

EXAMPLE 9 *A wire is looped in the form of a circle of radius 28 cm. It is re-bent into a square form. Determine the length of the side of the square.*

SOLUTION Let the side of the square be x cm. The wire is in the form of a circle of radius 28 cm.

$$\therefore \quad \text{Length of the wire} = \text{Circumference of the circle}$$

$$= \left\{2 \times \frac{22}{7} \times 28\right\} \text{cm} \quad [\text{Using } C = 2\pi r]$$

$$= 176 \text{ cm} \quad \text{...(i)}$$

The wire is re-bent in the form of a square of side x cm.

$\therefore$ Perimeter of the square = Length of the wire

$$\Rightarrow \quad 4x = 176 \quad [\text{Using (i)}]$$

$$\Rightarrow \quad x = 44 \text{ cm}$$

Hence, the length of the side of the square is 44 cm.

EXAMPLE 10 *A race track is in the form of a ring whose inner circumference is 352 m, and the outer circumference is 396 m. Find the width of the track.*

SOLUTION Let the outer and inner radii of the ring be R metres and r metres respectively.

It is given that the outer and inner circumferences of the ring are 396 m and 352 m respectively.

$\therefore \quad 2\pi R = 396$ and $2\pi r = 352$

$\Rightarrow \quad 2 \times \frac{22}{7} \times R = 396$ and $2 \times \frac{22}{7} \times r = 352$

$\Rightarrow \quad R = 396 \times \frac{7}{22} \times \frac{1}{2}$ and $r = 352 \times \frac{7}{22} \times \frac{1}{2}$

$\Rightarrow \quad R = 63$ m and $r = 56$ m

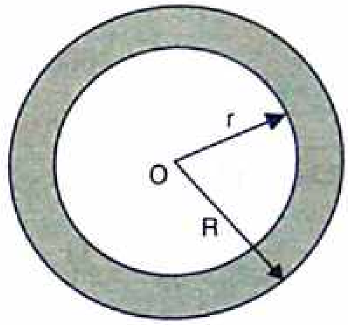

Fig. 13.3

Hence, width of the track $= (R - r)$ metres

$= (63 - 56)$ metres $= 7$ metres

EXAMPLE 11 *The inner circumference of a circular track is 220 m. The track is 7 m wide everywhere. Calculate the cost of putting up a fence along the outer circle at the rate of ₹2 per metre. (Use $\pi = 22/7$)*

SOLUTION Let the inner and outer radii of the circular track be r metres and R metres respectively. It is given that

Inner circumference = 220 metres

$\Rightarrow \quad 2\pi r = 220 \Rightarrow 2 \times \frac{22}{7} \times r = 220 \Rightarrow r = 35$ m

The track is 7 metre wide everywhere. Therefore, the outer radius R is given by

$R = r + 7 = (35 + 7)$ m $= 42$ m

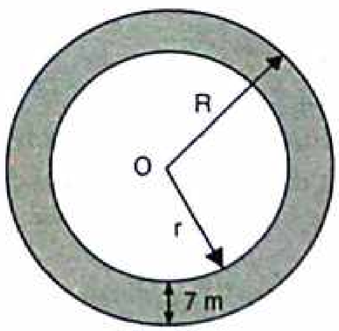

Fig. 13.4

$\therefore \quad$ Outer circumference $= 2\pi R = 2 \times \frac{22}{7} \times 42$ m $= 264$ m

Rate of fencing = ₹ 2 per metre

$\therefore \quad$ Total cost of fencing = (Circumference × Rate) = ₹ (264 × 2) = ₹ 528

EXAMPLE 12 *A bicycle wheel makes 5000 revolutions in moving 11 km. Find the diameter of the wheel.*

SOLUTION Let the radius of the wheel be r cm. We observe that the distance covered by the wheel in one revolution is equal to the circumference of the wheel.

$\therefore \quad$ Distance covered by the wheel in one revolution $= 2\pi r$ cm

$\Rightarrow \quad$ Distance covered by the wheel in 5000 revolutions $= 5000 \times 2\pi r$ cm

$$= 10000 \times \frac{22}{7} \times r \text{ cm}$$

$$= \frac{10000 \times \frac{22}{7} \times 4}{100} \text{ m}$$

$$= \frac{10000 \times \frac{22}{7} \times r}{100 \times 1000} \text{ km} = \frac{11}{35} r \text{ km}$$

It is given that the bicycle wheel covers 11 km distance in 5000 revolutions.

$\therefore \quad \frac{11}{35} r = 11 \Rightarrow r = 35$

$\therefore \quad \text{Diameter} = 2r \text{ cm} = (2 \times 35) \text{ cm} = 70 \text{ cm}$

Hence, the diameter of the wheel is 70 cm.

EXAMPLE 13 *A wheel has diameter 84 cm. Find how many complete revolutions must it take to cover 792 meters.*

SOLUTION Suppose the wheel makes n complete revolutions in covering 792 meters.

Let r be the radius of the wheel. It is given that the diameter of the wheel is 84 cm.

$\therefore \quad 2r = 84 \Rightarrow r = 42 \text{ cm}$

$\therefore \quad \text{Circumference of the wheel} = 2\pi r \text{ cm} = 2 \times \frac{22}{7} \times 42 \text{ cm} = 264 \text{ cm} = 2.64 \text{ m}$

Distance covered by the wheel in one revoluton = 2.64 cm

Distance covered by wheel in n revolutions = $(2.64)\, n$ metres

It is given that the wheel covers 792 metres in n revolutions.

$\therefore \quad (2.64)\, n = 792 \Rightarrow n = \frac{792}{2.64} = 300$

Hence, the wheel takes 300 revolutions in covering 792 meters.

EXAMPLE 14 *A boy is cycling such that the wheels of the cycle are making 140 revolutions per minute. If the diameter of the wheel is 60 cm, calculate the speed per hour with which the boy is cycling.*

SOLUTION The speed with which the boy is cycling per hour is the distance covered by the wheel in one hour. Clearly, distance covered by the wheel in one revolution is equal to its circumference. So, let us first compute the circumference.

We have,

$$\text{Radius of the wheel} = r = \frac{60}{2} \text{ cm} = 30 \text{ cm}.$$

$\therefore \quad \text{Circumference of the wheel} = 2\pi r = 2 \times \frac{22}{7} \times 30 \text{ cm} = \frac{1320}{7} \text{ cm}$

So, $\quad \text{Distance covered by the wheel in one revolution} = \frac{1320}{7} \text{ cm}$

$\Rightarrow \quad \text{Distance covered by the wheel in 140 revolutions} = \frac{1320}{7} \times 140 \text{ cm}$

$$= (1320 \times 20) \text{ cm} = 26400 \text{ cm}$$

$$= \frac{26400}{100} \text{ m} = 264 \text{ m} = \frac{264}{1000} \text{ km}$$

It is given that the wheels are making 140 revolutions per minute. So, distance covered by the wheels in one minute is equal to the distance covered by the wheels in 140 revolutions

$\Rightarrow$ Distance covered by the wheel in one minute $= \dfrac{264}{1000}$ km

$\Rightarrow$ Distance covered by the wheel in one hour i.e. in 60 minutes $= \dfrac{264}{1000} \times 60 \text{ km} = 15.84 \text{ km}$

Hence, the speed with which the boy is cycling is 15.84 km/hr.

EXAMPLE 15 *The diameter of the driving wheel of a bus is 140 cm. How many revolutions per minute must the wheel make in order to keep a speed of 66 km per hour?*

SOLUTION Suppose the driving wheel of the bus makes n revolutions per minute to keep a speed of 66 km/hr. Clearly, distance covered by the wheel in one revolution is equal to its circumference. It is given that r = Radius of the wheel = 70 cm.

$\therefore$ Circumference of the wheel $= 2\pi r = 2 \times \dfrac{22}{7} \times 70 \text{ cm} = 440 \text{ cm}$

So, distance covered by the wheel in one revolution = 440 cm

$\Rightarrow$ Distance covered by the wheel in n revolutions $= (440 \times n)$ cm

$\Rightarrow$ Distance covered by the wheel in one minute $= (440 \times n)$ cm

$\Rightarrow$ Distance covered by the wheel in one hour $= (440 \times n \times 60)$ cm

We are given that the speed of the bus is 66 km/hr. This means that the wheel covers 66 km in one hour.

So, the wheel covers $\dfrac{66 \times 1000 \times 100}{60} = 110000$ cm in one minute.

$\therefore \quad 440 \times n \times 60 = 110000 \Rightarrow n = \dfrac{110000}{440 \times 60} = 250$

Hence, the wheel makes 250 revolutions per minute.

EXAMPLE 16 *A car has wheels which are 80 cm in diameter. How many complete revolutions does each wheel make in 10 minutes when the car is travelling at a speed of 66 km per hour?* **[NCERT]**

SOLUTION Suppose each wheel of the car makes n complete revolutions in 10 minutes. This means that the distance covered by each wheel in n revolutions is same as the distance travelled by the car in 10 minutes.

It is given that :

Speed of the car = 66 km/hr

$\therefore$ Distance travelled by the car in 1 hour = 66 km

$\Rightarrow$ Distance travelled by the car in 10 min $= \left(\dfrac{66}{60} \times 10\right) \text{km} = 11 \text{ km} = 11 \times 1000 \times 100 \text{ cm}$ ...(i)

It is given that :

Radius of car wheels = 40 cm

$\therefore$ Circumference of the wheels $= 2 \times \dfrac{22}{7} \times 40$ cm

In a revolution each wheel covers the distance equal to its circumference.

$\therefore$ Distance covered by each wheel in one complete revolution $= 2\times\frac{22}{7}\times 40$ cm

$\Rightarrow$ Distance covered by each wheel in n revolutions $= \left(2\times\frac{22}{7}\times 40\times n\right)$ cm ...(ii)

But, distance covered by each wheel in completing n revoluitions is equal to the distance travelled by the car in 10 minutes.

$$\therefore \quad 2\times\frac{22}{7}\times 40\times n = 11\times 1000\times 100 \Rightarrow n = \frac{11\times 1000\times 100\times 7}{2\times 22\times 40} = 4375$$

Hence, each wheel makes 4375 revolutions in 10 minutes.

EXAMPLE 17 *Find the number of revolutions made by a circular wheel of area 1.54 m^2 in rolling a distance of 176 m.* **[NCERT EXEMPLAR]**

SOLUTION Let r be the radius of the circular wheel. It is given that its area is 1.54 m^2

$$\therefore \quad \pi r^2 = 1.54 \Rightarrow \frac{22}{7}r^2 = 1.54 \Rightarrow r^2 = 7\times 0.07 = 0.49 \Rightarrow r = 0.7$$

Suppose the wheel makes n revolutions in rolling a distance of 176 m.

$\therefore \quad n\times$ Distance rolled in one revolution $= 176$

$\Rightarrow \quad n\times 2\pi r = 176$ [$\because$ Distance rolled in one revolution = Circumference]

$$\Rightarrow \quad n\times 2\times\frac{22}{7}\times 0.7 = 176 \Rightarrow n = \frac{176\times 7}{2\times 22\times 0.7} = 40$$

Hence, the circular wheel makes 40 revolutions.

EXAMPLE 18 *The diameters of front and rear wheels of a tractor are 80 cm and 2 m respectively. Find the number of revolutions that rear wheel makes in covering a distance in which the front wheel makes 1400 revolutions.* **[NCERT EXEMPLAR]**

SOLUTION Let r_1 and r_2 be the radii of front and rear wheels of the tractor. It is given that $r_1 = 0.40$ m and $r_2 = 1$ m.

Distance covered by the front wheel in one revolution $= 2\pi r_1 = 2\pi\times 0.4$ m $= 0.8\,\pi$ m

$\therefore$ Distance covered by the front wheel in 1400 revolutions $= 1400\times 0.8\,\pi$ m $= 1120\,\pi$ m

Suppose the rear wheel makes n revolutions to cover this distance. Then,

(Distance covered by the rear wheel in one revolution) $\times n = 1120\,\pi$

$$\Rightarrow \quad 2\pi r_2\times n = 1120\pi \Rightarrow 2\pi\times 1\times n = 1120\,\pi \Rightarrow n = 560$$

Hence, the rear wheel makes 560 revolutions.

EXAMPLE 19 *The cost of fencing a circular field at the rate ₹ 24 per metre is ₹ 5280. The field is to be ploughed at the rate of ₹ 0.50 per m^2. Find the cost of ploughing the field.* (Take $\pi = 22/7$) **[NCERT]**

SOLUTION We have,

Rate of fencing = ₹ 24 per metre and, Total cost of fencing = ₹ 5280

$\therefore$ Length of the fence $= \dfrac{\text{Total cost}}{\text{Rate}} = \dfrac{5280}{24}$ metre $= 220$ metre

$\Rightarrow$ Circumference of the field $= 220$ metre

$\Rightarrow$ $2\pi r = 220$, where r is the radius of the field

$\Rightarrow$ $2 \times \dfrac{22}{7} \times r = 220$

$\Rightarrow$ $r = \dfrac{220 \times 7}{22 \times 2} = 35$ metres

$\therefore$ Area of the field $= \pi r^2 = \dfrac{22}{7} \times 35 \times 35 \text{ m}^2 = 22 \times 5 \times 35 \text{ m}^2$

It is given that the field is ploughed at the rate of ₹ 0.50 per m^2

$\therefore$ Cost of ploughing the field $=$ ₹ $(22 \times 5 \times 35 \times 0.50) =$ ₹ 1925

ALITER Let the radius of the circular field be r metres. Then,

Length of its circular fence $= 2\pi r$ metres.

It is given that the cost of fencing the field at the rate of ₹ 24 per metre is ₹ 5280.

Length of the fence $\times 24 = 5280$

$\Rightarrow$ $2\pi r \times 24 = 5280$

$\Rightarrow$ $2 \times \dfrac{22}{7} \times r \times 24 = 5280$

$\Rightarrow$ $r = \dfrac{5280 \times 7}{2 \times 22 \times 24} = 35$ metre

$\therefore$ Area of the circular field $= \pi r^2 = \dfrac{22}{7} \times 35 \times 35 \text{ m}^2$

So, the cost of ploughing the field at the rate of ₹ 0.50 per square metre is $=$ ₹ $\left(\dfrac{22}{7} \times 35 \times 35 \times 0.50\right)$

$=$ ₹ 1925

EXAMPLE 20 *The difference between the radii of the smaller circle and the larger circle is 7 cm and the difference between the areas of the two circles is 1078 sq. cm. Find the radius of the smaller circle.* **[CBSE 2017]**

SOLUTION Let the lengths of the radii of the smaller and larger circles be r cm and R cm respectively.

It is given that

$R - r = 7$ ...(i)

It is also given that the difference between the areas of two circles is 1078 cm^2

$\therefore$ $\pi R^2 - \pi r^2 = 1078$

$\Rightarrow$ $\pi(R^2 - r^2) = 1078$

$\Rightarrow$ $\dfrac{22}{7}(R + r)(R - r) = 1078$

$\Rightarrow$ $\dfrac{22}{7}(R + r) \times 7 = 1078$ [Using (i)]

$\Rightarrow$ $R + r = 49$ ...(ii)

Subtracting (i) from (ii), we get

$2r = 42 \Rightarrow r = 21$

Hence, the radius of the smaller circle is of length 21 cm.

LEVEL-2

EXAMPLE 21 *Two circles touch externally. The sum of their areas is 130 π sq. cm. and the distance between their centres is 14 cm. Find the radii of the circles.*

SOLUTION If two circles touch externally, then the distance between their centres is equal to the sum of their radii. Let the radii of the two circles be r_1 cm and r_2 cm respectively. Let C_1 and C_2 be the centres of the given circles. Then,

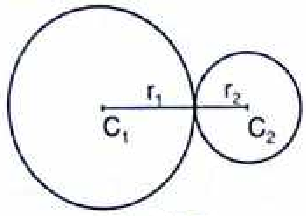

Fig. 13.5

$$C_1C_2 = r_1 + r_2$$

$\Rightarrow \quad 14 = r_1 + r_2 \qquad [\because C_1C_2 = 14 \text{ cm (given)}]$

$\Rightarrow \quad r_1 + r_2 = 14 \qquad \ldots(i)$

It is given that the sum of the areas of two circles is equal to 130 $\pi \text{ cm}^2$.

$\therefore \quad \pi r_1^2 + \pi r_2^2 = 130\pi$

$\Rightarrow \quad r_1^2 + r_2^2 = 130 \qquad \ldots(ii)$

Now, $\quad (r_1 + r_2)^2 = r_1^2 + r_2^2 + 2r_1r_2$

$\Rightarrow \quad 14^2 = 130 + 2r_1r_2 \qquad$ [Using (i) and (ii)]

$\Rightarrow \quad 196 - 130 = 2r_1r_2$

$\Rightarrow \quad r_1r_2 = 33 \qquad \ldots(iii)$

Now,

$(r_1 - r_2)^2 = r_1^2 + r_2^2 - 2r_1r_2$

$\Rightarrow \quad (r_1 - r_2)^2 = 130 - 2 \times 33 \qquad$ [Using (ii) and (iii)]

$\Rightarrow \quad (r_1 - r_2)^2 = 64$

$\Rightarrow \quad r_1 - r_2 = 8 \qquad \ldots(iv)$

Solving (i) and (iv), we get $r_1 = 11$ cm and $r_2 = 3$ cm.

Hence, the radii of the two circles are 11 cm and 3 cm.

EXAMPLE 22 *Two circles touch internally. The sum of their areas is 116 π cm^2 and distance between their centres is 6 cm. Find the radii of the circles.*

SOLUTION Let R and r be the radii of the circles having centres at O and O' respectively. It is given that the sum of the areas is $116\pi \text{ cm}^2$ and the distance between the centres is 6 cm.

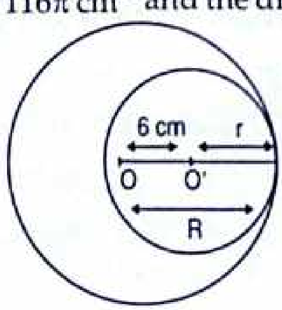

Fig. 13.6

Now, $\quad$ Sum of areas $= 116\pi \text{ cm}^2$

$\Rightarrow \quad \pi R^2 + \pi r^2 = 116\pi$

$\Rightarrow \quad R^2 + r^2 = 116 \qquad \ldots(i)$

Distance between the centres = 6 cm

$\Rightarrow$ $OO' = 6\text{ cm}$

$\Rightarrow$ $R - r = 6$ ...(ii)

Now, $(R+r)^2 + (R-r)^2 = 2(R^2 + r^2)$

$\Rightarrow$ $(R+r)^2 + 36 = 2 \times 116$ [Using (i) and (ii)]

$\Rightarrow$ $(R+r)^2 = (2 \times 116 - 36) = 196$

$\Rightarrow$ $R + r = 14$ ...(iii)

Solving (ii) and (iii), we get: $R = 10$ and $r = 4$.

Hence, radii of the given circles are 10 cm and 4 cm respectively.

EXAMPLE 23 *Figure 13.7, depicts an archery target marked with its five scoring areas from the centre outwards as Gold, Red, Blue Black and white. The diameter of the region representing Gold score is 21 cm and each of the other bands is 10.5 cm wide. Find the area of each of the five scoring regions.* [NCERT]

SOLUTION We have,

r = Radius of the region representing Gold score = 10.5 cm

$\therefore$ r_1 = Radius of the region representing Gold and Red scoring areas
$= (10.5 + 10.5)\text{ cm} = 21\text{ cm} = 2r\text{ cm}$

r_2 = Radius of the region representing Gold, Red and Blue scoring areas
$= (21 + 10.5)\text{ cm} = 31.5\text{ cm} = 3r\text{ cm}$

r_3 = Radius of the region represing Gold, Red, Blue and Black scoring areas
$= (31.5 + 10.5)\text{ cm} = 42\text{ cm} = 4r\text{ cm}$

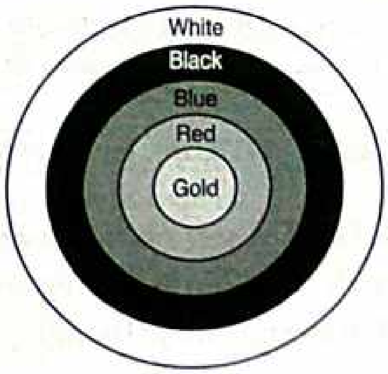

Fig. 13.7

r_4 = Radius of the region representing Gold, Red, Blue, Black and white scoring areas
$= (42 + 10.5)\text{ cm} = 52.5\text{ cm} = 5r\text{ cm}$

Now,

A_1 = Area of the region representing Gold scoring area $= \pi r^2 = \frac{22}{7} \times (10.5)^2 = \frac{22}{7} \times 10.5 \times 10.5$
$= 22 \times 1.5 \times 10.5 = 346.5\text{ cm}^2$

A_2 = Area of the region representing Red scoring area $= \pi(2r)^2 - \pi r^2 = 3\pi r^2 = 3A_1$
$= 3 \times 346.5\text{ cm}^2 = 1039.5\text{ cm}^2$

A_3 = Area of the region representing Blue scoring area $= \pi(3r)^2 - \pi(2r)^2 = 9\pi r^2 - 4\pi r^2$
$= 5\pi r^2 = 5A_1 = 5 \times 346.5\text{ cm}^2$
$= 1732.5\text{ cm}^2$

A_4 = Area of the region representing Black scoring area $= \pi(4r)^2 - \pi(3r)^2 = 7\pi r^2 = 7A_1$
$= 7 \times 346.5\text{ cm}^2 = 2425.5\text{ cm}^2$

A_5 = Area of the region representing White scoring area $= \pi(5r)^2 - \pi(4r)^2 = 9\pi r^2 = 9A_1$

$$= 9 \times 346.5 \text{ cm}^2 = 3118.5 \text{ cm}^2$$

EXERCISE 13.1

LEVEL-1

1. Find the circumference and area of a circle of radius 4.2 cm.
2. Find the circumference of a circle whose area is 301.84 cm^2.
3. Find the area of a circle whose circumference is 44 cm.
4. The circumference of a circle exceeds the diameter by 16.8 cm. Find the circum-ference of the circle.
5. A horse is tied to a pole with 28 m long string. Find the area where the horse can graze. (Take $\pi = 22/7$).
6. A steel wire when bent in the form of a square encloses an area of 121 cm^2. If the same wire is bent in the form of a circle, find the area of the circle.
7. The circumference of two circles are in the ratio 2 : 3. Find the ratio of their areas.
8. The sum of the radii of two circles is 140 cm and the difference of their circumferences is 88 cm. Find the diameters of the circles.
9. Find the radius of a circle whose circumference is equal to the sum of the circumferences of two circles of radii 15 cm and 18 cm. [NCERT EXEMPLAR]
10. The radii of two circles are 8 cm and 6 cm respectively. Find the radius of the circle having its area equal to the sum of the areas of the two circles. [NCERT]
11. The radii of two circles are 19 cm and 9 cm respectively. Find the radius and area of the circle which has its circumference equal to the sum of the circumferences of the two circles. [NCERT]
12. The area of a circular playground is 22176 m^2. Find the cost of fencing this ground at the rate of ₹ 50 per metre. [NCERT EXEMPLAR]
13. The side of a square is 10 cm. Find the area of circumscribed and inscribed circles.
14. If a square is inscribed in a circle, find the ratio of the areas of the circle and the square.
15. The area of a circle inscribed in an equilateral triangle is 154 cm^2. Find the perimeter of the triangle. [Use $\pi = 22/7$ and $\sqrt{3} = 1.73$]
16. A field is in the form of a circle. A fence is to be erected around the field. The cost of fencing would be ₹ 2640 at the rate of ₹ 12 per metre. Then, the field is to be thoroughly ploughed at the cost of ₹ 0.50 per m^2. What is the amount required to plough the field? [Take $\pi = 22/7$].
17. A park is in the form of a rectangle 120 m × 100 m. At the centre of the park there is a circular lawn. The area of park excluding lawn is 8700 m^2. Find the radius of the circular lawn. (Use $\pi = 22/7$).
18. A car travels 1 kilometre distance in which each wheel makes 450 complete revolutions. Find the radius of the its wheels.
19. The area enclosed between the concentric circles is 770 cm^2. If the radius of the outer circle is 21 cm, find the radius of the inner circle.
20. An archery target has three regions formed by thre concentric circles as shown in Fig. 13.8. If the diameters of the concentric circles are in the ratio 1 : 2 : 3, then find the ratio of the areas of three regions. [NCERT EXEMPLAR]

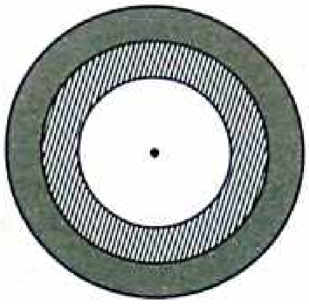

Fig. 13.8

21. The wheel of a motor cycle is of radius 35 cm. How many revolutions per minute must the wheel make so as to keep a speed of 66 km/hr? [NCERT EXEMPLAR]
22. A circular pond is 17.5 m in diameter. It is surrounded by a 2 m wide path. Find the cost of constructing the path at the rate of ₹ 25 per m^2. [NCERT EXEMPLAR]
23. A circular park is surrounded by a rod 21 m wide. If the radius of the park is 105 m, find the area of the road. [NCERT EXEMPLAR]
24. A square of diagonal 8 cm is inscribed in a circle. Find the area of the region lying outside the circle and inside the square. [NCERT EXEMPLAR]
25. A path of 4 m width runs round a semi-circular grassy plot whose circumference is 163 $\frac{3}{7}$ m Find:
    (i) the area of the path
    (ii) the cost of gravelling the path at the rate of ₹ 1.50 per square metre
    (iii) the cost of turfing the plot at the rate of 45 paise per m^2.
26. Find the area enclosed between two concentric circles of radii 3.5 cm and 7 cm. A third concentric circle is drawn outside the 7 cm circle, such that the area enclosed between it and the 7 cm circle is same as that between the two inner circles. Find the radius of the third circle correct to one decimal place.
27. A path of width 3.5 m runs around a semi-circular grassy plot whose perimeter is 72 m. Find the area of the path. (Use $\pi = 22/7$) [CBSE 2015]
28. A circular pond is of diameter 17.5 m. It is surrounded by a 2 m wide path. Find the cost of constructing the path at the rate of ₹ 25 per square metre (Use $\pi = 3.14$) [CBSE 2014]
29. The outer circumference of a circular race-track is 528 m. The track is everywhere 14 m wide. Calculate the cost of levelling the track at the rate of 50 paise per square metre (Use $\pi = 22/7$).
30. A road which is 7 m wide surrounds a circular park whose circumference is 352 m. Find the area of the road.
31. Prove that the area of a circular path of uniform width h surrounding a circular region of radius r is $\pi h (2r + h)$.

ANSWERS

1. 26.4 cm, 55.44 cm^2
2. 61.6 cm
3. 154 cm^2
4. 24.64 cm
5. 2464 m^2
6. 154 cm^2
7. 4 : 9
8. 154 cm, 126 cm
9. 33 cm
10. 10 cm
11. 28 cm, 2464 cm^2
12. ₹ 26400
13. 157 cm^2, 78.5 cm^2
14. $\pi : 2$
15. 72.7 cm
16. ₹ 1925
17. 32.40 m
18. 35.35 cm
19. 14 cm
20. 1 : 3 : 5
21. 500
22. ₹ 3061.50
23. 15246 cm^2
24. $(16\pi - 32)$ cm^2
25. (i) 352 m^2 (ii) ₹ 528 (iii) ₹ 478

26. 115.5 cm^2, 9.26 cm 27. 173.25 m^2 28. ₹ 3061.50
29. ₹ 3388 30. 2618 m^2

HINT TO SELECTED PROBLEMS

5. Length of the string = 28 m. Area over which the horse can graze is the area of a circle of radius 28 m. Hence, required area $= \pi(28)^2 = 2464\text{ m}^2$

6. Let r cm be the radius of the circle. Side of the square $= \sqrt{121}$ cm = 11 cm
   $\therefore$ Perimeter of the square = (4×11) cm = 44 cm
   So, length of the wire = 44 cm.
   Now, Circumference of the circle = Length of the wire $\Rightarrow$ $2\pi r = 44$ cm $\Rightarrow$ $r = 7$ cm
   Hence, Area of the circle $= \pi r^2 = \pi \times 7^2\text{ cm}^2 = 154\text{ cm}^2$

7. Let r_1 and r_2 be the radii of two given circles and C_1 and C_2 be their circumferences. Then,
$$C_1 : C_2 = 2:3$$
$$\Rightarrow \quad 2\pi r_1 : 2\pi r_2 = 2:3$$
$$\Rightarrow \quad r_1 : r_2 = 2:3 \Rightarrow r_1^2 : r_2^2 = 4:9 \Rightarrow \pi r_1^2 : \pi r_2^2 = 4\pi : 9\pi \Rightarrow \pi r_1^2 : \pi r_2^2 = 4:9$$

8. Let r_1 and r_2 be the radii of two given circles. Then,
$$r_1 + r_2 = 140 \text{ and } 2\pi r_1 - 2\pi r_2 = 88 \Rightarrow 2\pi(r_1 - r_2) = 88 \Rightarrow r_1 - r_2 = 14$$

10. Let r be the radius of the circle whose area is equal to the sum of the areas of circles of radii 8 cm and 6 cm. Then,
$$\pi r^2 = \pi \times 8^2 + \pi \times 6^2 \Rightarrow r^2 = 100 \Rightarrow r = 10\text{ cm}$$

11. Let the radius of the circle be r cm. Then,
$$2\pi r = 2\pi \times 19 + 2\pi \times 9 \Rightarrow r = 28\text{ cm}$$
$$\therefore \quad \text{Area of the circle } = \pi r^2 = \frac{22}{7} \times 28 \times 28\text{ cm}^2 = 2464\text{ cm}^2$$

13. We have, Diameter of the circumscribed circle = Diagonal of the square $= \sqrt{10^2 + 10^2}$
$$= 10\sqrt{2}\text{ cm}$$
    Diameter of the inscribed circle = Length of the side of the square.

15. Let r be the radius of the inscribed circle. Then,
    Area = $154\text{ cm}^2 \Rightarrow \pi r^2 = 154 \Rightarrow r = 7$ cm
    Let h be the height of the triangle. Then,
$$r = \frac{h}{3} \Rightarrow h = 3r = 21\text{ cm}$$
    If a is the side of the triangle. Then,
$$h = \frac{\sqrt{3}}{2}a \Rightarrow a = \frac{2h}{\sqrt{3}} = \frac{42}{\sqrt{3}} = 14\sqrt{3}\text{ cm}$$
    Hence, perimeter $= 3a = 3 \times 14\sqrt{3}$ cm $= 72.7$ cm

16. Length of the fence $= \dfrac{\text{Total Cost}}{\text{Rate per metre}} = \dfrac{2640}{12} = 220$ m

$\therefore\quad 2\pi r = 220 \text{ m} \Rightarrow r = 35 \text{ m}$

$\therefore\quad \text{Area} = \pi r^2 = \pi (35)^2$

Cost of ploughing the whole field $= ₹\,[\pi (35)^2 \times 0.50]$

17. Area of the park $= (120 \times 100)\ \text{m}^2 = 12000\ \text{m}^2$

Area of the park excluding the lawn $= 8700\ \text{m}^2$

$\therefore$ Area of the circular lawn $= (12000 - 8700)\ \text{m}^2$

$$\Rightarrow \quad \pi r^2 = 3300 \Rightarrow r^2 = 3300 \times \frac{7}{22} = 150 \times 7 \Rightarrow r = \sqrt{150 \times 7} = 32.40 \text{ m}$$

18. Let the radius of each wheel be r metres. Then,

Circumference of each wheel $= 2\pi r = 2 \times \frac{22}{7} \times r$ metres

$\Rightarrow$ Distance covered by the wheels in one revolution $= 2 \times \frac{22}{7} \times r$ metres

$\Rightarrow$ Distance covered by the wheels in 450 revolutions $= 2 \times \frac{22}{7} \times r \times 450$ metres

It is given that the car travels 1 kilometre i.e. 1000 metres distance when its each wheel makes 450 revolutions.

$$2 \times \frac{22}{7} \times r \times 450 = 1000 \Rightarrow r = \frac{7 \times 1000}{2 \times 22 \times 450} = 0.3535 \text{ metres} = 35.35 \text{ cm}$$

19. Let the radius of the inner circle be r cm. Then,

$$\pi \times 21 \times 21 - \pi \times r^2 = 770 \Rightarrow \pi(441 - r^2) = 770 \Rightarrow \frac{22}{7} \times (441 - r^2) = 770 \Rightarrow 441 - r^2 = 245$$

$$\Rightarrow r^2 = 196 \Rightarrow r = 14 \text{ cm}$$

13.3 SECTOR OF A CIRCLE AND ITS AREA

Consider a circle of radius r having its centre at the point O. Let A, B, and C be three points on the circle as shown in Fig. 13.9. The area enclosed by the circle is divided into two regions, namely, OBA and $OBCA$. These regions are called sectors of the circle. Each of these two sectors has an arc of the circle as a part of its boundary. The sector OBA has arc AB as a part of its boundary whereas the sector $OBCA$ has arc ACB as a part of its boundary. These sectors are known as minor and major sectors of the circle as defined below.

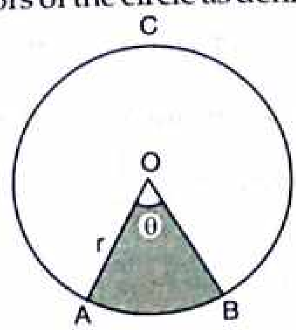

Fig. 13.9

MINOR SECTOR *A sector of a circle is called a minor sector if the minor arc of the circle is a part of its boundary*

In Fig. 13.9, sector OAB is the minor sector.

MAJOR SECTOR *A sector of a circle is called a major sector if the major arc of the circle is a part of its boundary.*

In Fig. 13.9, sector $OACB$ is the major sector.

Following are some important points to remember:

(i) *A minor sector has an angle* θ, *(say), subtended at the centre of the circle, whereas a major sector has no angle.*

(ii) *The sum of the arcs of major and minor sectors of a circle is equal to the circumference of the circle.*

(iii) *The sum of the areas of major and minor sectors of a circle is equal to the area of the circle.*

(iv) *The boundary of a sector consists of an arc of the circle and the two radii.*

13.3.1 AREA OF A SECTOR

Consider a circle of radius r having its centre at O. Let AOB be a sector of the circle such that $\angle AOB = \theta$. If $\theta < 180°$, then the arc AB is a minor arc of the circle. Now, if θ increases the length of the arc AB also increases and if θ becomes $180°$, then arc AB becomes the circumference of a semi-circle. Thus, if an arc subtends an angle of $180°$ at the centre, then its arc length is πr.

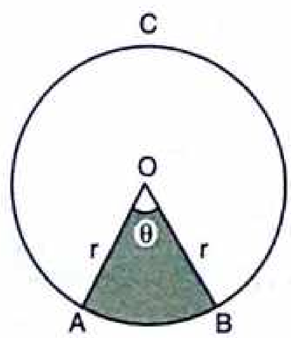

Fig. 13.10

$\therefore$ If the arc subtends an angle of θ at the centre, then its arc length $= \dfrac{\theta}{180} \times \pi r$

Hence, the arc length l of a sector of angle θ in a circle of radius r is given by

$$l = \frac{\theta}{180} \times \pi r \qquad \ldots(i)$$

$$\Rightarrow \quad l = \frac{\theta}{360} \times 2\pi r = \frac{\theta}{360} \times (\text{Circumference of the circle})$$

As discussed above, if the arc subtends an angle of $180°$ then the area of the corresponding sector is equal to the area of a semi-circle i.e $\dfrac{1}{2}\pi r^2$.

$\therefore$ If the arc subtends an angle θ, then area of the corresponding sector is $\dfrac{\theta}{180} \times \dfrac{1}{2}\pi r^2 = \dfrac{\pi r^2 \theta}{360}$

Thus, the area A of a sector of angle θ in a circle of radius r is given by

$$A = \frac{\theta}{360} \times \pi r^2 = \frac{\theta}{360} \times (\text{Area of the circle}) \qquad \ldots(ii)$$

Also, $A = \frac{\theta}{360} \times \pi r^2 \Rightarrow A = \frac{1}{2}\left(\frac{\theta}{180} \times \pi r\right) r \Rightarrow A = \frac{1}{2} lr$ [Using (i)]

REMARK *Area of major sector* $= \pi r^2$ *– Area of minor segment*

Some useful results to remember:

(i) Angle described by minute hand in 60 minutes $= 360°$

$\therefore$ Angle described by minute hand in one minute $= \left(\frac{360}{60}\right)^\circ = 6°$

Thus , minute hand rotates through an angle of $6°$ in one minute.

(ii) Angle described by hour hand in 12 hours $= 360°$

$\therefore$ Angle described by hour hand in one hour $= \left(\frac{360}{12}\right)^\circ = 30°$

$\Rightarrow$ Angle described by hour hand in one minute $= \left(\frac{30}{60}\right)^\circ = \frac{1}{2}^\circ$

Thus, hour hand rotates through $\left(\frac{1}{2}\right)^\circ$ in one minute.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the area of a sector of a circle of radius 28 cm and central angle 45°.* [NCERT EXEMPLAR]

SOLUTION We know that the area A of a sector of a circle of radius r and central angle θ (in degrees) is given by

$$A = \frac{\theta}{360} \times \pi r^2$$

Here, $r = 28$ cm and $\theta = 45$.

$$\therefore \quad A = \frac{45}{360} \times \pi \times (28)^2 = \frac{1}{8} \times \frac{22}{7} \times 28 \times 28 \text{cm}^2 = 308 \text{ cm}^2$$

EXAMPLE 2 *Find the difference of the areas of a sector of angle 120° and its corresponding major sector of a circle of radius 21 cm.* [NCERT EXEMPLAR]

SOLUTION Let A_1 and A_2 be the areas of the given sector and the corresponding major sector respectively. We have, $\theta = 120$ and $r = 21$ cm .

$$\therefore \quad A_1 = \frac{\theta}{360} \times \pi r^2 = \frac{120}{360} \times \pi \times (21)^2 = 147\,\pi \text{ cm}^2$$

and, $A_2 =$ Area of the circle $- A_1$

$$\Rightarrow \quad A_2 = \{\pi \times (21)^2 - 147\pi\} \text{ cm}^2 = \pi(441 - 147) \text{ cm}^2 = 294\,\pi \text{ cm}^2$$

Required differences $= A_2 - A_1 = (294\pi - 147\pi)\,\text{cm}^2 = 147\pi\,\text{cm}^2 = \left(147 \times \frac{22}{7}\right)\text{cm}^2 = 462\,\text{cm}^2$

EXAMPLE 3 *Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector. (Use $\pi = 3.14$).*

SOLUTION Here, $\theta = 30°$ and $r = 4$ cm.

$$\therefore \quad \text{Area of sector } OAPB = \frac{\theta}{360} \times \pi r^2$$

$$= \frac{30}{360} \times 3.14 \times 4 \times 4\,\text{cm}^2 = \frac{3.14 \times 4}{3}\,\text{cm}^2 = 4.153\,\text{cm}^2$$

Let A be the area of corresponding major sector. Then,

$$A = \text{Area of sector } OAQB$$

$$\Rightarrow \quad A = \text{Area of the circle} - \text{Area of the corresponding minor sector}$$

$$\Rightarrow \quad A = \pi r^2 - \frac{\theta}{360} \times \pi r^2$$

$$\Rightarrow \quad A = \pi r^2\left(1 - \frac{\theta}{360}\right)$$

$$\Rightarrow \quad A = 3.14 \times 4 \times 4\left(1 - \frac{30}{360}\right)\text{cm}^2$$

$$\Rightarrow \quad A = 3.14 \times 4 \times 4 \times \frac{11}{12}\,\text{cm}^2 = \frac{3.14 \times 44}{3}\,\text{cm}^2 = 46.05\,\text{cm}^2$$

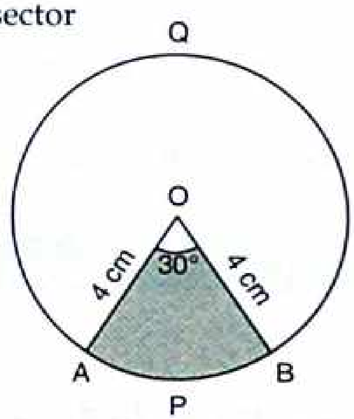

Fig. 13.11

EXAMPLE 4 *A sector is cut from a circle of radius 21 cm. The angle of the sector is 150°. Find the length of its arc and area.*

SOLUTION The arc length l and area A of a sector of angle θ in a circle of radius r are given by $l = \frac{\theta}{360} \times 2\pi r$ and $A = \frac{\theta}{360} \times \pi r^2$ respectively. Here, $r = 21$ cm and $\theta = 150$

$$\therefore \quad l = \left\{\frac{150}{360} \times 2 \times \frac{22}{7} \times 21\right\}\text{cm} = 55\,\text{cm}$$

$$\text{and,} \quad A = \left\{\frac{150}{360} \times \frac{22}{7} \times (21)^2\right\}\text{cm}^2 = \frac{1155}{2}\,\text{cm}^2 = 577.5\,\text{cm}^2$$

EXAMPLE 5 *The central angles of two sectors of circles of radii 7 cm and 21 cm are respectively 120° and 40°. Find the areas of the two sectors as well as the length of the corresponding arcs. What do you observe?* **[NCERT EXEMPLAR]**

SOLUTION	Sector -I	Sector -II
Radius:	$r_1 = 7$ cm	$r_2 = 21$ cm
Sector angle:	$\theta_1 = 120°$	$\theta_2 = 40°$
Sector areas:	$A_1 = \frac{\theta_1}{360} \times \pi r_1^2$	$A_2 = \frac{\theta_2}{360} \times \pi r_2^2$

Sector arc: $l_1 = \frac{\theta_1}{360} \times 2\pi r_1$ $l_2 = \frac{\theta_2}{360} \times 2\pi r_2$

We find that

$$A_1 = \frac{\theta_1}{360} \times \pi r_1^2 = \frac{120}{360} \times \frac{22}{7} \times 7^2 \text{ cm}^2 = \frac{154}{3} \text{cm}^2$$

$$A_2 = \frac{\theta_2}{360} \times \pi r_2^2 = \frac{40}{360} \times \frac{22}{7} \times 21^2 \text{ cm}^2 = 154 \text{ cm}^2$$

$$l_1 = \frac{\theta_1}{360} \times 2\pi r_1 = \frac{120}{360} \times 2 \times \frac{22}{7} \times 7 \text{ cm}^2 = \frac{44}{3} \text{ cm}$$

$$l_2 = \frac{\theta_2}{360} \times 2\pi r_2 = \frac{40}{360} \times 2 \times \frac{22}{7} \times 21 \text{ cm} = \frac{44}{3} \text{ cm}$$

We observe that the arc lengths of two circles of different radii may be same but areas need not be equal.

EXAMPLE 6 *A car has two wipers which do not overlap. Each wiper has a blade of length 25 cm sweeping through an angle of 115°. Find the total area cleaned at each sweep of the blades.* [NCERT]

SOLUTION Clearly, each wiper sweeps a sector of a circle of radius 25 cm and sector angle 115°. Therefore, total area A cleaned at each sweep is given by

$$\therefore \quad A = 2 \times \frac{\theta}{360} \times \pi r^2$$

$$\Rightarrow \quad A = 2 \times \frac{115}{360} \times \frac{22}{7} \times 25 \times 25 \text{ cm}^2 = 1254.96 \text{ cm}^2$$

EXAMPLE 7 *To warm ships for underwater rocks, a light house throws a red coloured light over a sector of 80° angle to a distance of 16.5 km. Find the area of the sea over which the ships area warmed.* ($Use\ \pi = 3.14$) [NCERT]

SOLUTION We have, $r = 16.5$ km and $\theta = 80$.

Let A be the area of the sea over which the ships are warmed. Then,

$$A = \frac{\theta}{360} \times \pi r^2 = \frac{80}{360} \times 3.14 \times 16.5 \times 16.5 \text{ km}^2 = 189.97 \text{ km}^2$$

EXAMPLE 8 *In Fig. 13.12, there are shown sectors of two concentric circles of radii 7 cm and 3.5 cm. Find the area of the shaded region.* (Use $\pi = 22/7$).

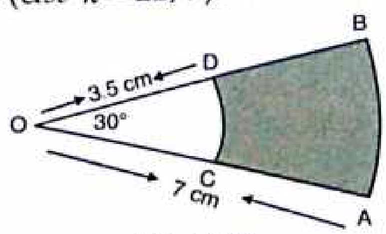

Fig. 13.12

SOLUTION Let A_1 and A_2 be the areas of sectors OAB and OCD respectively. Then,

A_1 = Area of a sector of angle 30° in a circle of radius 7 cm

$$\Rightarrow \quad A_1 = \left\{\frac{30}{360} \times \frac{22}{7} \times 7^2\right\} \text{cm}^2 = \frac{77}{6} \text{cm}^2 \qquad \left[\text{Using: } A = \frac{\theta}{360} \times \pi r^2\right]$$

A_2 = Area of a sector of angle 30° in a circle of radius 3.5 cm.

$$\Rightarrow \quad A_2 = \left\{\frac{30}{360} \times \frac{22}{7} \times (3.5)^2\right\} \text{cm}^2 = \left\{\frac{1}{12} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right\} \text{cm}^2 = \frac{77}{24} \text{cm}^2$$

$\therefore$ Area of the shaded region $= A_1 - A_2$

$$= \left(\frac{77}{6} - \frac{77}{24}\right) \text{cm}^2$$

$$= \frac{77}{24} \times (4-1) \text{cm}^2 = \frac{77}{8} \text{cm}^2 = 9.625 \text{ cm}^2$$

EXAMPLE 9 *A pendulum swings through an angle of 30° and describes an arc 8.8 cm in length. Find the length of the pendulum.* [*Use* $\pi = 22/7$].

SOLUTION Here, $\theta = 30°$, l = arc = 8.8 cm

$$\therefore \quad l = \frac{\theta}{360} \times 2\pi r \Rightarrow 8.8 = \frac{30}{360} \times 2 \times \frac{22}{7} \times r \Rightarrow r = \frac{8.8 \times 6 \times 7}{22} \text{ cm} = 16.8 \text{ cm}$$

EXAMPLE 10 *The perimeter of a sector of a circle of radius 5.2 cm is 16.4 cm. Find the area of the sector.*

SOLUTION Let OAB be the given sector. It is given that

Perimeter of sector OAB = 16.4 cm

$\Rightarrow \quad OA + OB + \text{arc } AB = 16.4$ cm

$\Rightarrow \quad 5.2 + 5.2 + \text{arc } AB = 16.4$

$\Rightarrow \quad \text{arc } AB = 6 \text{ cm} \Rightarrow l = 6$ cm

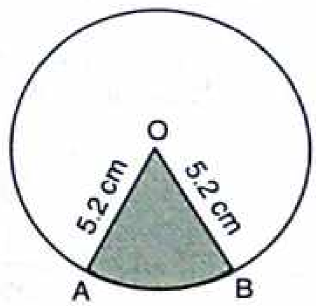

Fig. 13.13

$$\therefore \quad \text{Area of sector } OAB = \frac{1}{2} lr = \frac{1}{2} \times 6 \times 5.2 \text{ cm}^2 = 15.6 \text{ cm}^2$$

EXAMPLE 11 *An arc of a circle is of length* 5π *cm and the sector it bounds has an area of* $20\,\pi$ *cm*^{2}. *Find the radius of the circle.*

SOLUTION Let the radius of the circle be r cm and the arc AB of length 5π cm subtends angle θ at the centre O of the circle. Then,

Arc $AB = 5\pi$ cm and, Area of sector $OAB = 20\pi \text{ cm}^2$

$$\Rightarrow \quad \frac{\theta}{360} \times 2\pi r = 5\pi \text{ and, } \frac{\theta}{360} \times \pi r^2 = 20\pi$$

$$\Rightarrow \quad \frac{\frac{\theta}{360} \times \pi r^2}{\frac{\theta}{360} \times 2\pi r} = \frac{20\pi}{5\pi}$$

$$\Rightarrow \quad \frac{r}{2} = 4$$

$$\Rightarrow \quad r = 8 \text{ cm}$$

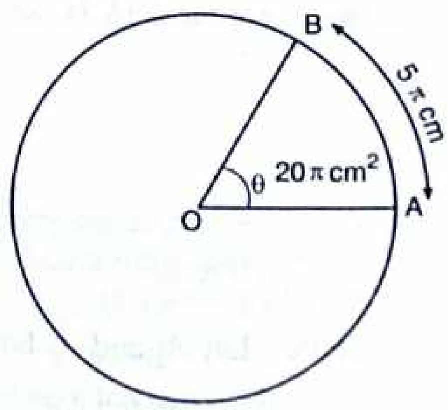

Fig. 13.14

ALITER We know that Area $= \frac{1}{2} lr \Rightarrow 20\pi = \frac{1}{2} \times 5\pi \times r \Rightarrow r = 8$ cm

EXAMPLE 12 *Area of a sector of a circle of radius 36 cm is $54\pi\ cm^2$. Find the length of the corresponding arc of the sector.*

SOLUTION Let A be the area of the sector of a circle of radius $r = 36$ cm and l be the length of the corresponding arc. Then,

$$A = \frac{1}{2} l r$$

$$\Rightarrow \quad 54\pi = \frac{1}{2} \times l \times 36 \qquad [\because A = 54\pi \text{ cm}^2 \text{ (given) and } r = 36 \text{ cm}]$$

$$\Rightarrow \quad l = 3\pi \text{ cm}$$

ALITER Let the central angle (in degrees) be θ. It is given that $r = 36$ cm and area of the sector is 54π cm².

$$\therefore \quad \frac{\theta}{360} \times \pi \times (36)^2 = 54\pi \qquad \left[\text{Using : Area} = \frac{\theta}{360} \times \pi r^2\right]$$

$$\Rightarrow \quad \theta = \frac{54\pi \times 360}{\pi (36)^2} = 15$$

Let l be the length of the corresponding arc. Then,

$$l = \frac{\theta}{360} \times 2\pi r \Rightarrow l = \frac{15}{360} \times 2\pi \times 36 \text{ cm} = 3\pi \text{ cm}$$

EXAMPLE 13 *A piece of wire 20 cm long is bent into the form of an arc of a circle subtending an angle of 60° at its centre. Find the radius of the circle.* **[NCERT EXEMPLAR]**

SOLUTION Let r be the radius of the circle. Here, $\theta = 60$ and $l = 20$ cm.

$$\therefore \quad l = \frac{\theta}{360} \times 2\pi r$$

$$\Rightarrow \quad 20 = \frac{60}{360} \times 2\pi r$$

$$\Rightarrow \quad r = \frac{60}{\pi} \text{ cm}$$

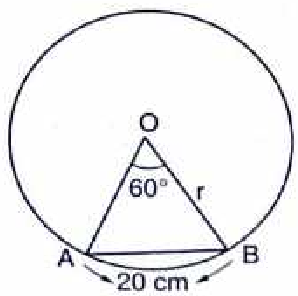

Fig. 13.15

Hence, the radius of the circle is $\frac{60}{\pi}$ cm.

EXAMPLE 14 *The length of minute hand of a clock is 14 cm. Find the area swept by the minute hand in one minute. (Use $\pi = 22/7$)*

SOLUTION Clearly, minute hand of a clock describes a circle of radius equal to its length i.e. 14 cm. Since the minute hand rotates through 6° in one minute. Therefore, area swept by the minute hand in one minute is the area of a sector of angle 6° in a circle of radius 14 cm. Hence, required area A is given by

$$A = \frac{\theta}{360} \times \pi r^2$$

$$\Rightarrow \quad A = \left\{\frac{6}{360} \times \frac{22}{7} \times (14)^2\right\} \text{cm}^2 = \left\{\frac{1}{60} \times \frac{22}{7} \times 14 \times 14\right\} \text{cm}^2 = \frac{154}{15} \text{cm}^2 = 10.26 \text{ cm}^2$$

EXAMPLE 15 *The minute hand of a clock is 10 cm long. Find the area of the face of the clock described by the minute hand between 9 A.M. and 9.35 A.M.*

SOLUTION We know that:

Angle described by the minute hand in one minute = 6°

$\therefore$ Angle described by the minute hand in 35 minutes $= (6 \times 35)^\circ = 210^\circ$

$\therefore$ Area swept by the minute hand in 35 minutes

= Area of a sector of angle 210° in a circle of radius 10 cm

$$= \left\{\frac{210}{360} \times \frac{22}{7} \times (10)^2\right\} \text{cm}^2 = 183.3 \text{ cm}^2 \qquad \left[\text{Using: } A = \frac{\theta}{360^\circ} \times \pi r^2\right]$$

EXAMPLE 16 *The short and long hands of a clock are 4 cm and 6 cm long respectively. Find the sum of distances travelled by their tips in 2 days. (Take $\pi = 22/7$)*

SOLUTION In 2 days, the short hand will complete 4 rounds.

$\therefore$ Distance moved by its tip = 4 (Circumference of a circle of radius 4 cm)

$$= 4 \times \left(2 \times \frac{22}{7} \times 4\right) \text{cm} = \frac{704}{7} \text{ cm}$$

In 2 days, the long hand will complete 48 rounds.

$\therefore$ Distance moved by its tip = 48 (Circumference of a circle of radius 6 cm)

$$= 48 \times \left(2 \times \frac{22}{7} \times 6\right) \text{cm} = \frac{12672}{7} \text{ cm}$$

$\therefore$ Sum of the distances moved by the tips of two hands of the clock $= \left(\frac{704}{7} + \frac{12672}{7}\right)$ cm

$= 1910.85$ cm

LEVEL-2

EXAMPLE 17 *In a circle with centre O and radius 5 cm, AB is a chord of length $5\sqrt{3}$ cm. Find the area of sector AOB.*

SOLUTION It is given that $AB = 5\sqrt{3}$ cm.

$$\Rightarrow \quad AL = BL = \frac{5\sqrt{3}}{2} \text{ cm}$$

Let $\angle AOB = 2\theta$. Then, $\angle AOL = \angle BOL = \theta$.

In ΔOLA, we have

$$\sin\theta = \frac{AL}{OA} = \frac{\frac{5\sqrt{3}}{2}}{5} = \frac{\sqrt{3}}{2}$$

$\Rightarrow \quad \theta = 60^\circ$

$\Rightarrow \quad \angle AOB = 120^\circ$

$$\therefore \quad \text{Area of sector } AOB = \frac{120}{360} \times \pi \times 5^2 \text{ cm}^2 = \frac{25\pi}{3} \text{ cm}^2$$

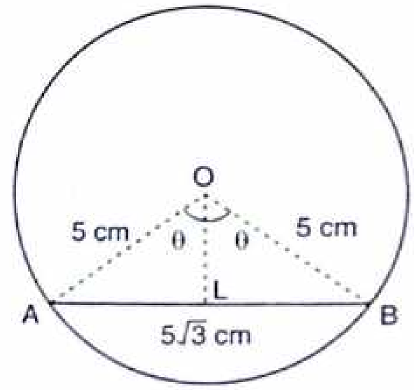

Fig. 13.16

EXAMPLE 18 *An umbrella has 8 ribs which are equally spaced. Assuming umbrella to be a flat circle of radius 45 cm. Find the area between the two consecutive ribs of the umbrella.*

[NCERT]

SOLUTION We know that the ribs of an umbrella are equally spaced.

$$\therefore \quad \text{Angle made by two consecutive ribs at the centre} = \frac{360^\circ}{8} = 45^\circ$$

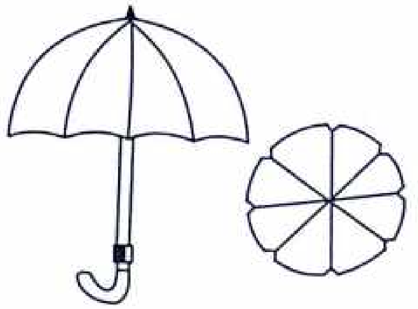

Fig. 13.17

Let A be the area between two consecutive ribs. Then,

$A =$ Area of a sector of a circle of radius 45 cm and sector angle 45°

$$\Rightarrow \quad A = \left\{\frac{45}{360} \times \frac{22}{7} \times 45 \times 45\right\} \text{cm}^2 \qquad \left[\text{Using: Area} = \frac{\theta}{360} \times \pi r^2\right]$$

$$\Rightarrow \quad A = \frac{1}{8} \times \frac{22}{7} \times 45 \times 45 \text{ cm}^2 = 795.53 \text{ cm}^2$$

EXAMPLE 19 *A brooch is made with silver wire in the form of a circle with diameter 35 mm. The wire also used in making 5 diameters which divide the circle into 10 equal sectors as shown in Fig. 13.18. Find: (i) the total length of the silver wire required (ii) the area of each sector of the brooch.* **[NCERT]**

SOLUTION (i) Let l be the total length of the silver wire. Then,

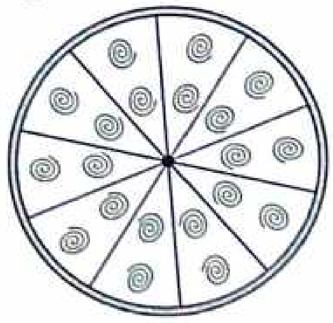

Fig. 13.18

$$l = \text{ Circumference of the circle of radius } \frac{35}{2} \text{ mm } + \text{Length of five diameters}$$

$$\Rightarrow \quad l = 2\pi \times \frac{35}{2} + 5 \times 35 \text{ mm} = \left(2 \times \frac{22}{7} \times \frac{35}{2} + 175\right) \text{mm} = 285 \text{ mm}$$

(ii) The circle is divided into 10 equal sectors. Therefore, area A of each sector of the brooch is given by

$$A = \frac{1}{10} (\text{Area of the circle}) = \frac{1}{10} \times \pi \times \left(\frac{35}{2}\right)^2 \text{mm}^2 = \frac{1}{10} \times \frac{22}{7} \times \frac{35}{2} \times \frac{35}{2} \text{mm}^2 = \frac{385}{4} \text{mm}^2$$

EXAMPLE 20 *An elastic belt is placed round the rim of a pulley of radius 5 cm. One point on the belt is pulled directly away from the centre O of the pulley until it is at P, 10 cm from O. Find the length of the belt that is in contact with the rim of the pulley. Also, find the shaded area.* **[CBSE 2016]**

SOLUTION In Figure 13.19, let $\angle AOP = \angle BOP = \theta$. Clearly, portion AB of the belt is not in contact with the rim of the pulley in right triangle OAP, we have

$$\cos\theta = \frac{OA}{OP} = \frac{5}{10} = \frac{1}{2} \Rightarrow \theta = 60° \Rightarrow \angle AOB = 2\theta = 120°$$

$\therefore$ Arc $AB = \frac{120° \times 2 \times \pi \times 5}{360}$ cm $= \frac{10\pi}{3}$ cm $\left[\text{Using: } l = \frac{\theta}{360} \times 2\pi r\right]$

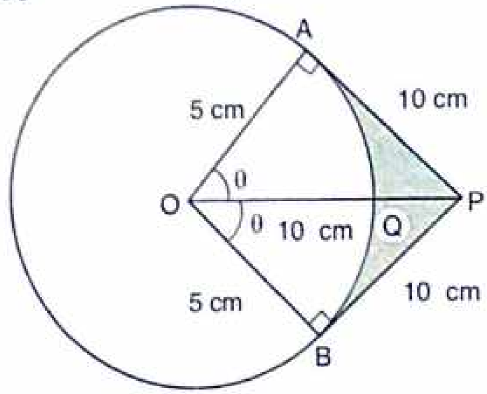

Fig. 13.19

Let l be the length of the belt that is in contact with the rim of the pulley. Then,

$$l = \text{Circumference of the rim} - \text{Length of arc } AB = 2\pi \times 5 \text{ cm} - \frac{10\pi}{3} \text{ cm} = \frac{20\pi}{3} \text{ cm}$$

Now,

$$\text{Area of sector } OAQB = \frac{120}{360} \times \pi \times 5^2 \text{ cm}^2 = \frac{25\pi}{3} \text{ cm}^2 \left[\text{Using: Area} = \frac{\theta}{360} \times \pi r^2\right]$$

Applying Pythagoras theorem in ΔOAP, we obtain

$$OP^2 = OA^2 + AP^2 \Rightarrow AP = \sqrt{OP^2 - OA^2} = \sqrt{100 - 25} = 5\sqrt{3} \text{ cm}.$$

$\therefore$ Area of quadrilateral $OAPB = 2(\text{Area of } \Delta OAP)$

$$= 2 \times \left(\frac{1}{2} \times OA \times AP\right) = 5 \times 5\sqrt{3} \text{ cm}^2 = 25\sqrt{3} \text{ cm}^2$$

Hence, Shaded area $=$ Area of quadrilateral $OAPB$ $-$ Area of sector $OAQB$.

$$= \left(25\sqrt{3} - \frac{25\pi}{3}\right) \text{cm}^2 = \frac{25}{3}(3\sqrt{3} - \pi) \text{ cm}^2$$

EXERCISE 13.2

LEVEL-1

1. Find, in terms of π, the length of the arc that subtends an angle of $30°$ at the centre of a circle of radius 4 cm.
2. Find the angle subtended at the centre of a circle of radius 5 cm by an arc of length $(5\pi/3)$ cm.
3. An arc of length 20π cm subtends an angle of $144°$ at the centre of a circle. Find the radius of the circle.
4. An arc of length 15 cm subtends an angle of $45°$ at the centre of a circle. Find in terms of π, the radius of the circle.
5. Find the angle subtended at the centre of a circle of radius 'a' by an arc of length $(a\pi/4)$ cm.
6. A sector of a circle of radius 4 cm contains an angle of $30°$. Find the area of the sector.
7. A sector of a circle of radius 8 cm contains an angle of $135°$. Find the area of the sector.

8. The area of a sector of a circle of radius 2 cm is π cm^2. Find the angle contained by the sector.
9. The area of a sector of a circle of radius 5 cm is 5 π cm^2. Find the angle contained by the sector.
10. Find the area of the sector of a circle of radius 5 cm, if the corresponding arc length is 3.5 cm. **[NCERT EXEMPLAR]**
11. In a circle of radius 35 cm, an arc subtends an angle of 72° at the centre. Find the length of the arc and area of the sector.
12. The perimeter of a sector of a circle of radius 5.7 m is 27.2 m. Find the area of the sector.
13. The perimeter of a certain sector of a circle of radius 5.6 m is 27.2 m. Find the area of the sector.
14. A sector is cut-off from a circle of radius 21 cm. The angle of the sector is 120°. Find the length of its arc and the area.
15. The minute hand of a clock is $\sqrt{21}$ cm long. Find the area described by the minute hand on the face of the clock between 7.00 AM and 7.05 AM.
16. The minute hand of a clock is 10 cm long. Find the area of the face of the clock described by the minute hand between 8 AM and 8.25 AM.
17. A sector of 56° cut out from a circle contains area 4.4 cm^2. Find the radius of the circle.
18. Area of a sector of central angle 200° of a circle is 770 cm^2. Find the length of the corresponding arc of this sector. **[NCERT EXEMPLAR]**
19. The length of minute hand of a clock is 5 cm. Find the area swept by the minute hand during the time period 6:05 amd and 6: 40 am. **[NCERT EXEMPLAR]**
20. The length of the minute hand of a clock is 14 cm. Find the area swept by the minute hand in 5 minutes. **[CBSE 2013]**
21. In a circle of radius 21 cm, an arc subtends an angle of 60° at the centre. Find (i) the length of the arc (ii) area of the sector formed by the arc. (Use $\pi = 22/7$) **[CBSE 2013, 2017]**
22. From a circular piece of carboard of radius 3 cm two sectors of 90° have been cut off. Find the perimeter of the remaining portion nearest hundredth centimeters (Take $\pi = 22/7$).
23. The area of a sector is one-twelfth that of the complete circle. Find the angle of the sector.

LEVEL-2

24. AB is a chord of a circle with centre O and radius 4 cm. AB is of length 4 cm. Find the area of the sector of the circle formed by chord AB.
25. In a circle of radius 6 cm, a chord of length 10 cm makes an angle of 110° at the centre of the circle. Find:
    (i) the circumference of the circle, (ii) the area of the circle,
    (iii) the length of the arc AB, (iv) the area of the sector OAB.
26. Figure 13.20, shows a sector of a circle, centre O, containing an angle $\theta°$. Prove that:

    (i) Perimeter of the shaded region is $r\left(\tan\theta + \sec\theta + \frac{\pi\theta}{180} - 1\right)$

    (ii) Area of the shaded region is $\frac{r^2}{2}\left(\tan\theta - \frac{\pi\theta}{180}\right)$

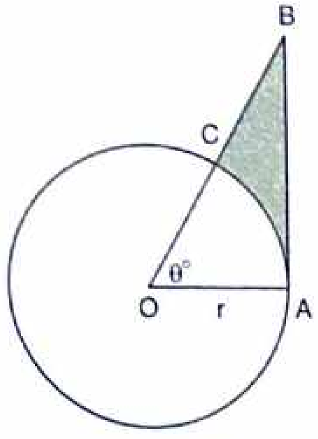

Fig. 13.20

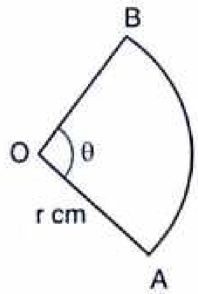

Fig. 13.21

27. Figure 13.21 shows a sector of a circle of radius r cm containing an angle $\theta°$. The area of the sector is $A\,\text{cm}^2$ and perimeter of the sector is 50 cm. Prove that

(i) $\theta = \dfrac{360}{\pi}\left(\dfrac{25}{r} - 1\right)$ (ii) $A = 25r - r^2$

ANSWERS

1. $\dfrac{2\pi}{3}$ cm 2. $60°$ 3. 25 cm 4. $\dfrac{60}{\pi}$ cm

5. $45°$ 6. $\dfrac{4\pi}{3}\,\text{cm}^2$ 7. $24\pi\,\text{cm}^2$ 8. $90°$

9. $72°$ 10. $8.7\,\text{cm}^2$ 11. 44 cm, $770\,\text{cm}^2$ 12. $45.03\,\text{m}^2$

13. $44.8\,\text{m}^2$ 14. 44 cm, $462\,\text{cm}^2$ 15. $5.5\,\text{cm}^2$

16. $130.95\,\text{cm}^2$ 17. 3 cm 18. $\dfrac{220}{3}$ cm 19. $45\dfrac{5}{6}\,\text{cm}^2$

20. $51.30\,\text{cm}^2$ 21. (i) 22 cm, (ii) $231\,\text{cm}^2$ 22. 9.428 cm

23. $30°$ 24. $\dfrac{8\pi}{3}\,\text{cm}^2$

25. (i) 37.68 cm (ii) $113.1\,\text{cm}^2$ (iii) 11.51 cm (iv) $34.5\,\text{cm}^2$

13.4 SEGMENT OF A CIRCLE AND ITS AREA

Consider a circle of radius r having centre at point O. Let PQ be a chord of the circle and let R and S be two points on it as shown in Fig. 13.22. The area enclosed by the circle is divided by the chord PQ into two segments, viz. PRQ and PSQ. Each of these two segments has an arc of the circle as a part of its boundary. Arc PRQ is the minor one and the arc PSQ is the major one.

SEGMENT OF A CIRCLE *The region enclosed by an arc and a chord is called the segment of the circle.*

In Fig. 13.22, the shaded region PRQ is a segment of the circle. The boundary of a segment consists of an arc of the circle and the chord determining the segment.

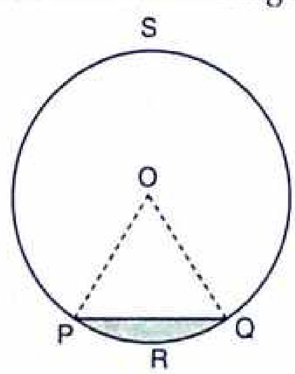

Fig. 13.22

MINOR SEGMENT *If the boundary of a segment is a minor arc of a circle, then the corresponding segment is called a minor segment.*

In Fig. 13.22, segment PQR is a minor segment.

MAJOR SEGMENT *A segment corresponding a major arc of a circle is known as the major segment.*

In Fig. 13.22, segment PQS is a major segment.

13.4.1 AREA OF A SEGMENT OF A CIRCLE

Draw a circle of radius r. Let O be the centre of the circle and PQ be a chord dividing the circle into two segments PRQ and PSQ as shown in Fig. 13.23. Suppose we wish to find the area of the minor segment PRQ (shaded region in Fig. 13.23). Let $\angle POQ = \theta$.

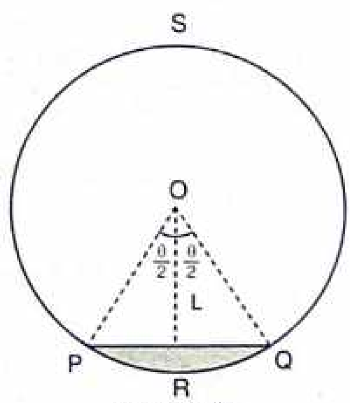

Fig. 13.23

It is evident from Fig. 13.23 that

Area of the sector $OPRQ$ = Area of the segment PRQ + Area of ΔOPQ

$\Rightarrow$ Area of the segment PRQ = Area of the sector $OPRQ$ − Area of ΔOPQ

Clearly, Area of the sector $OPRQ = \frac{\theta}{360} \times \pi r^2$

In ΔOLP, we have

$$\cos\frac{\theta}{2} = \frac{OL}{OP} \text{ and, } \sin\frac{\theta}{2} = \frac{PL}{OP}$$

$$\Rightarrow \quad OL = OP\cos\frac{\theta}{2} = r\cos\frac{\theta}{2} \text{ and, } PL = OP\sin\frac{\theta}{2} = r\sin\frac{\theta}{2}$$

$$\Rightarrow \quad OL = r\cos\frac{\theta}{2} \text{ and, } PQ = 2PL = 2r\sin\frac{\theta}{2}$$

$$\therefore \quad \Delta OPQ = \frac{1}{2}(PQ \times OL) = \frac{1}{2}\left(2r\sin\frac{\theta}{2} \times r\cos\frac{\theta}{2}\right) = r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Hence,

$$\text{Area of segment } PRQ = \frac{\theta}{360} \times \pi r^2 - r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left\{\frac{\pi}{360} \times \theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$$

REMARK *Area of the major segment $PSQ = \pi r^2$ − Area of minor segment PQR.*

NOTE *It should be noted that the area of the minor segment of a circle is always less than the area of its corresponding sector but the area of the major segment of a circle is greater than the area of its corresponding sector.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the area of the segment of a circle, given that the angle of the sector is 120° and the radius of the circle is 21 cm. (Take $\pi = 22/7$)*

SOLUTION The area A of a minor segment of a circle of radius r and the corresponding sector angle θ (in degrees) is given by

$$A = \left\{\frac{\pi}{360} \times \theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$$

Here, $r = 21$ cm and $\theta = 120°$.

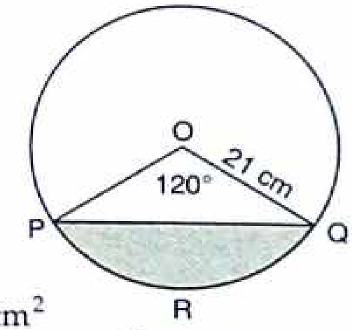

Fig. 13.24

$\therefore$ Area of the segment $= \left\{\frac{\pi}{360} \times \theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$

$$= \left\{\frac{22}{7} \times \frac{120}{360} - \sin 60° \cos 60°\right\} (21)^2 \text{ cm}^2$$

$$= \left\{\frac{22}{21} - \frac{1}{2} \times \frac{\sqrt{3}}{2}\right\} (21)^2 \text{ cm}^2$$

$$= \left\{\frac{22}{21} \times (21)^2 - (21)^2 \times \frac{\sqrt{3}}{4}\right\} \text{cm}^2$$

$$= \left(462 - \frac{441}{4}\sqrt{3}\right) \text{cm}^2 = \frac{21}{4}(88 - 21\sqrt{3}) \text{ cm}^2$$

EXAMPLE 2 *A chord AB of a circle of radius 10 cm makes a right angle at the centre of the circle. Find the area of the major and minor segments (Take $\pi = 3.14$)* **[CBSE 2016]**

SOLUTION We know that the area of a minor segment of angle θ (in degrees) in a circle of radius r is given by

$$A = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$$

Here, $r = 10$ and $\theta = 90°$

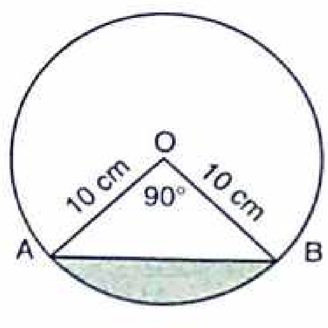

Fig. 13.25

$\therefore$ $A = \left\{\frac{3.14 \times 90}{360} - \sin 45° \cos 45°\right\} (10)^2 \text{ cm}^2$

$\Rightarrow$ $A = \left\{\frac{3.14}{4} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right\} (10)^2 \text{ cm}^2$

$\Rightarrow$ $A = \{3.14 \times 25 - 50\} \text{ cm}^2 = (78.5 - 50) \text{ cm}^2 = 28.5 \text{ cm}^2$

Area of the major segment = Area of the circle − Area of the minor segment

$$= (3.14 \times 10^2 - 28.5) \text{cm}^2 = (314 - 28.5) \text{cm}^2 = 285.5 \text{ cm}^2$$

EXAMPLE 3 *A chord AB of a circle of radius 15 cm makes an angle of 60° at the centre of the circle. Find the area of the major and minor segment. (Take $\pi = 3.14$, $\sqrt{3} = 1.73$)* **[CBSE 2017]**

SOLUTION We know that the area of a minor segment of angle θ (in degrees) in a circle of radius r is given by

$$A = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$$

$$\Rightarrow \quad A = \left\{\frac{3.14 \times 60}{360} - \sin 30^\circ \cos 30^\circ\right\}(15)^2 \text{ cm}^2$$

$$\Rightarrow \quad A = \left\{\frac{3.14}{6} - \frac{\sqrt{3}}{4}\right\} \times 225 \text{ cm}^2$$

$$\Rightarrow \quad A = (0.5233 - 0.4330)\, 225 \text{ cm}^2 = 225 \times 0.902 \text{ cm}^2 = 20.295 \text{ cm}^2$$

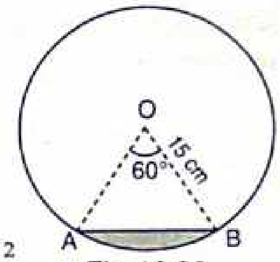

Fig. 13.26

Area of the major segment = Area of the circle − Area of the minor segment

$$= \{3.14 \times (15)^2 - 20.295\} \text{ cm}^2 = \{706.5 - 20.295\} \text{ cm}^2 = 686.205 \text{ cm}^2$$

EXAMPLE 4 *In a circle of radius 21 cm, an arc subtends an angle of 60° at the centre. Find:*
(i) length of the arc *(ii) area of the sector formed by the arc* **[NCERT]**
(iii) area of the segment formed by the corresponding chord of the arc.

SOLUTION Let O be the centre of the circle of radius 21 cm such that an arc APB subtends 60° angle at the centre O.

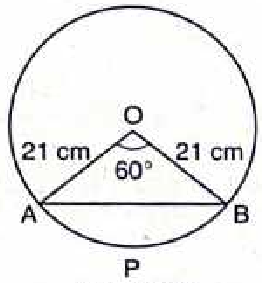

Fig. 13.27

(i) Length of the arc $APB = \frac{\theta}{360} \times 2\pi r = \frac{60}{360} \times 2 \times \frac{22}{7} \times 21 \text{ cm} = 22 \text{ cm}$

(ii) Area of sector $OAPB = \frac{\theta}{360} \times \pi r^2 = \frac{60}{360} \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 231 \text{ cm}^2$

(iii) Area of the segment $APB = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$

$$= \left\{\frac{22}{7} \times \frac{60}{360} - \sin 30^\circ \cos 30^\circ\right\} \times 21 \times 21 \text{ cm}^2$$

$$= \left\{\frac{11}{21} - \frac{1}{2} \times \frac{\sqrt{3}}{2}\right\} \times 21 \times 21 \text{ cm}^2$$

$$= \left\{11 \times 21 - \frac{\sqrt{3}}{4} \times 21 \times 21\right\} \text{ cm}^2$$

$$= \left\{231 - \frac{441\sqrt{3}}{4}\right\} \text{ cm}^2 = (231 - 190.95) \text{ cm}^2 = 40.05 \text{ cm}^2$$

EXAMPLE 5 *A chord of a circle of radius 10 cm subtends a right angle at the centre. Find:*
(i) area of the minor sector (ii) area of the minor segment
(iii) area of the major sector (iv) area of the major segment (Use $\pi = 3.14$) **[CBSE 2016]**

SOLUTION The area of the minor sector is a circle of radius r and sector angle θ (in degrees) is given by

$$A = \frac{\theta}{360} \times \pi r^2$$

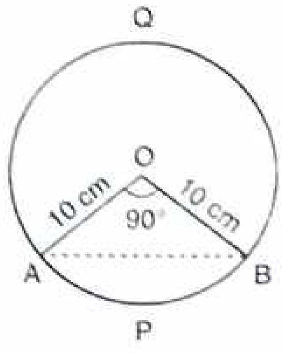

Fig. 13.28

The area of the corresponding minor segment is given by $\left\{\frac{\pi}{360} \times \theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$.

Here, $r = 10$ cm and $\theta = 90°$

(i) Area of the minor sector $OAPB = \frac{\theta}{360} \times \pi r^2 = \frac{90}{360} \times 3.14 \times 10 \times 10 \text{ cm}^2 = 78.5 \text{ cm}^2$

(ii) Let A be the area of the minor segment APB. Then,

$$A = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$$

$$\Rightarrow \quad A = \left\{3.14 \times \frac{90}{360} - \sin 45° \cos 45°\right\} \times 10 \times 10 \text{ cm}^2$$

$$\Rightarrow \quad A = \left\{\frac{3.14}{4} - \frac{1}{2}\right\} \times 100 \text{ cm}^2 = \left(\frac{314}{4} - 50\right) \text{cm}^2 = (78.5 - 50) \text{ cm}^2 = 28.5 \text{ cm}^2$$

(iii) Let A_1 be the area of the major sector $OAQB$. Then,

$A_1 =$ Area of the circle – Area of the minor sector $OA\ PB$.

$$\Rightarrow \quad A_1 = (3.14 \times 10 \times 10 - 78.5) \text{ cm}^2 = (314 - 78.5) \text{ cm}^2 = 235.5 \text{ cm}^2$$

(iv) Let A_2 be the area of the major segment AQB. Then,

$A_2 =$ Area of the circle – Area of the minor segment APB

$$\Rightarrow \quad A_2 = (3.14 \times 10 \times 10 - 28.5) \text{ cm}^2 = 285.5 \text{ cm}^2$$

LEVEL-2

EXAMPLE 6 *Figure 13.29 shows two arcs, A and B. Arc A is part of the circle with centre O and radius OP. Arc B is part of the circle with centre M and radius PM, where M is the mid-point of PQ. Show that the area enclosed by the two arcs is equal to* $25\left(\sqrt{3} - \frac{\pi}{6}\right) \text{cm}^2$. **[CBSE 2016]**

SOLUTION Let A_1 be the area enclosed by arc B and chord PQ. Then,

$$A_1 = \text{Area of semi-circle of radius 5 cm} = \frac{1}{2} \times \pi \times 5^2 \text{ cm}^2 = \frac{25\pi}{2} \text{ cm}^2$$

Let $\angle MOQ = \angle MOP = \theta$

In ΔOMP, we have

$$\sin\theta = \frac{PM}{OP} = \frac{5}{10} = \frac{1}{2}$$

$\Rightarrow \quad \theta = 30°$

$\Rightarrow \quad \angle POQ = 2\theta = 60°$

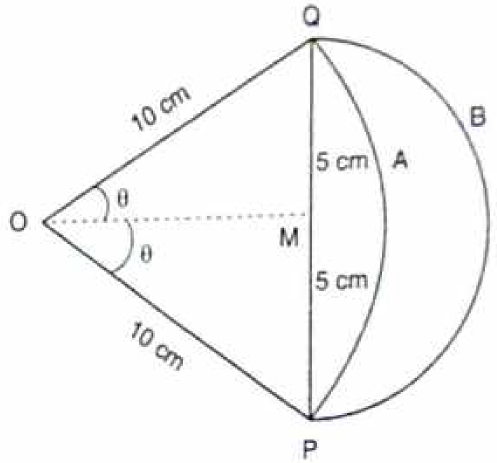

Fig. 13.29

Let A_2 be the area enclosed by arc A and chord PQ. Then,

A_2 = Area of segment of circle of radius 10 cm and sector containing angle 60°

$$\Rightarrow \quad A_2 = \left\{\frac{\pi\times 60}{360} - \sin 30°\times\cos 30°\right\}\times 10^2 \text{ cm}^2 \qquad \left[\because A = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}r^2\right]$$

$$\Rightarrow \quad A_2 = \left\{\frac{50\pi}{3} - 25\sqrt{3}\right\}\text{cm}^2$$

Clearly, Required area $= A_1 - A_2 = \left\{\frac{25\pi}{2} - \left(\frac{50\pi}{3} - 25\sqrt{3}\right)\right\}\text{cm}^2$

$$= \left\{25\sqrt{3} - \frac{25\pi}{6}\right\}\text{cm}^2 = 25\left\{\sqrt{3} - \frac{\pi}{6}\right\}\text{cm}^2$$

EXAMPLE 7 *Find the difference of the areas of two segments of a circle formed by a chord of length 5 cm subtending an angle of 90° at the centre.* **[NCERT EXEMPLAR]**

SOLUTION Let r be the radius of the circle. Using Pythagoras theorem in ΔAOB, we obtain

$$AB^2 = OA^2 + OB^2$$

$\Rightarrow \quad 5^2 = r^2 + r^2$

$\Rightarrow \quad 2r^2 = 25 \Rightarrow r^2 = \frac{25}{2} \Rightarrow r = \frac{5}{\sqrt{2}}$

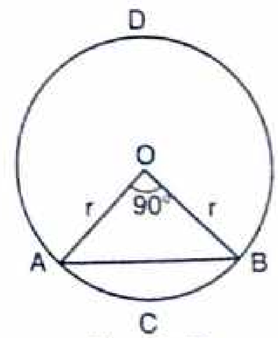

Fig. 13.30

Let A_1 and A_2 be the areas of minor segment ACB and major segment ADB respectively. Then,

$$A_1 = \left(\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)r^2$$

$$\Rightarrow \quad A_1 = \left(\frac{\pi}{360}\times 90 - \sin 45°\cos 45°\right)\times\left(\frac{5}{\sqrt{2}}\right) \qquad \left[\because \theta = 90° \text{ and } r = \frac{5}{\sqrt{2}}\right]$$

$\Rightarrow \qquad A_1 = \left(\frac{\pi}{4} - \frac{1}{2}\right) \times \frac{25}{2}\,\text{cm}^2 = \left(\frac{25\pi}{8} - \frac{25}{4}\right)\text{cm}^2$

and,

$A_2 = \text{Area of the circle} - A_1$

$\Rightarrow \qquad A_2 = \left\{\pi \times \left(\frac{5}{\sqrt{2}}\right)^2 - \left(\frac{25\pi}{8} - \frac{25}{4}\right)\right\}\text{cm}^2 = \left(\frac{25\pi}{2} - \frac{25\pi}{8} + \frac{25}{4}\right)\text{cm}^2 = \left(\frac{75\pi}{8} + \frac{25}{4}\right)\text{cm}^2$

$\therefore \qquad \text{Required difference} = A_2 - A_1 = \left\{\left(\frac{75\pi}{8} + \frac{25}{4}\right) - \left(\frac{25\pi}{8} - \frac{25}{4}\right)\right\}\text{cm}^2$

$$= \left(\frac{25\pi}{4} + \frac{25}{2}\right)\text{cm}^2 = \frac{25}{4}(\pi + 2)\,\text{cm}^2$$

EXERCISE 13.3

LEVEL-1

1. AB is a chord of a circle with centre O and radius 4 cm. AB is of length 4 cm and divides the circle into two segments. Find the area of the minor segment.
2. A chord PQ of length 12 cm subtends an angle of 120° at the centre of a circle. Find the area of the minor segment cut off by the chord PQ. **[NCERT]**
3. A chord of a circle of radius 14 cm makes a right angle at the centre. Find the areas of the minor and major segments of the circle.
4. A chord 10 cm long is drawn in a circle whose radius is $5\sqrt{2}$ cm. Find area of both the segments. (Take $\pi = 3.14$).
5. A chord AB of a circle, of radius 14 cm makes an angle of 60° at the centre of the circle. Find the area of the minor segment of the circle. (Use $\pi = 22/7$)
6. Find the area of the minor segment of a circle of radius 14 cm, when the angle of the corresponding sector is 60°. **[NCERT EXEMPLAR]**
7. A chord of a circle of radius 20 cm subtends an angle of 90° at the centre. Find the area of the corresponding major segment of the circle. (Use $\pi = 3.14$) **[NCERT EXEMPLAR]**
8. The radius of a circle with centre O is 5 cm (Fig. 13.31). Two radii OA and OB are drawn at right angles to each other. Find the areas of the segments made by the chord AB (Take $\pi = 3.14$).

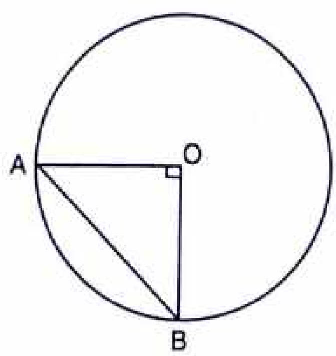

Fig. 13.31

LEVEL-2

9. AB is the diameter of a circle, centre O. C is a point on the circumference such that $\angle COB = \theta$. The area of the minor segment cut off by AC is equal to twice the area of the sector BOC. Prove that $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \pi\left(\frac{1}{2} - \frac{\theta}{120}\right)$.

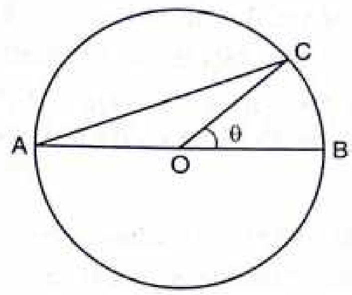

Fig. 13.32

10. A chord of a circle subtends an angle of θ at the centre of the circle. The area of the minor segment cut off by the chord is one eighth of the area of the circle. Prove that $8\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \pi = \frac{\pi\theta}{45}$.

ANSWERS

1. $\left(\frac{8\pi}{3} - 4\sqrt{3}\right)\text{cm}^2$
2. $4(4\pi - 3\sqrt{3})\ \text{cm}^2$
3. $56\ \text{cm}^2, 560\ \text{cm}^2$
4. $14.25\ \text{cm}^2, 142.75\ \text{cm}^2$
5. $17.80\ \text{cm}^2$
6. $\left(\frac{308}{3} - 49\sqrt{3}\right)\text{cm}^2$
7. $285.5\ \text{cm}^2$
8. $7.135\ \text{cm}^2, 71.425\ \text{cm}^2$

13.5 AREAS OF COMBINATIONS OF PLANE FIGURES

In our daily life we come across various plane figures which are combinations of two or more plane figures. For example, window designs, flower beds, drain covers, circular paths etc. In this section, we shall discuss problems on calculating areas of such figures by using the knowledge of computing areas of different plane figures studied in earlier classes.

Following examples will illustrate the process of computing areas of plane figures which are combinations of two or more plane figures.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In figure 13.33, find the area of the shaded region* [*Use* $\pi = 3.14$]

SOLUTION Let r be the radius of the circle. Clearly, Diameter of the circle = Diagonal BD of rectangle $ABCD$.

Applying Pythagoras theorem in ΔBCD, we obtain

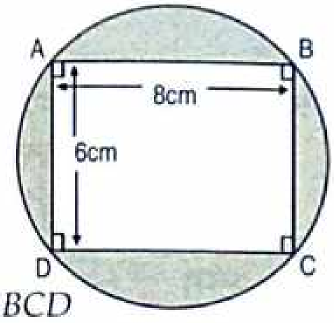

Fig. 13.33

$BD = \sqrt{BC^2 + CD^2} = \sqrt{6^2 + 8^2}$ cm $= 10$ cm

$\therefore\ 2r = BD \Rightarrow 2r = 10 \Rightarrow r = 5$

Area of rectangle $ABCD = AB \times BC = (8 \times 6)\text{ cm}^2 = 48\text{ cm}^2$

Area of the circle $= \pi r^2 = 3.14 \times (5)^2\text{ cm}^2 = 78.50\text{ cm}^2$

$\therefore$ Area of the shaded region = Area of the circle – Area of rectangle $ABCD$

$= (78.50 - 48)\text{ cm}^2 = 30.50\text{ cm}^2$

EXAMPLE 2 *A paper is in the form of a rectangle ABCD in which AB = 20 cm and BC = 14 cm. A semi-circular portion with BC as diameter is cut off. Find the area of a remaining part.*

SOLUTION We have,

Length of the rectangle $ABCD = AB = 20$ cm

Breadth of the rectangle $ABCD = BC = 14$ cm

$\therefore$ Area of rectangle $ABCD = (20 \times 14)\text{ cm}^2 = 280\text{ cm}^2$

Diameter of the semi-circle $= BC = 14$ cm

$\therefore$ Radius of the semi-circle $= 7$ cm

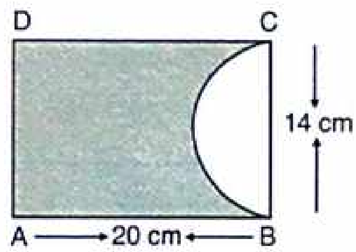

Fig. 13.34

Let A_1 be the area of the semi-circular portion cut off from the rectangle $ABCD$. Then,

$$A_1 = \frac{1}{2}(\pi r^2) = \left(\frac{1}{2} \times \frac{22}{7} \times 7^2\right)\text{cm}^2 = 77\text{ cm}^2$$

$\therefore$ Area of the remaining part = Area of rectangle $ABCD$ – Area of semi-circle

$$= (280 - 77)\text{ cm}^2 = 203\text{ cm}^2$$

EXAMPLE 3 *Find the area of the shaded region in Fig. 13.35, if ABCD is a square of side 14 cm and APD and BPC are semi-circles.* **[NCERT]**

SOLUTION Let A be the area of the shaded region. Then,

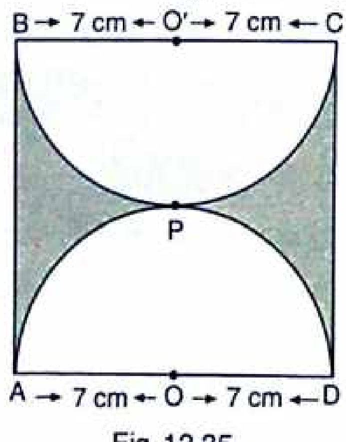

Fig. 13.35

$$A = \text{Area of square } ABCD - \text{Area of two semi-circles}$$

$$\Rightarrow \quad A = 14 \times 14 \text{ cm}^2 - 2\left(\frac{1}{2} \times \frac{22}{7} \times 7^2\right) \text{cm}^2 = 196 \text{ cm}^2 - 154 \text{ cm}^2 = 42 \text{ cm}^2$$

EXAMPLE 4 *A horse is placed for grazing inside a rectangular field 70 m by 52 m and is tethered to one corner by a rope 21 m long. On how much area can it graze?*

SOLUTION Shaded portion indicates the area which the horse can graze. Clearly, shaded area is the area of a quadrant of a circle of radius $r = 21$ m.

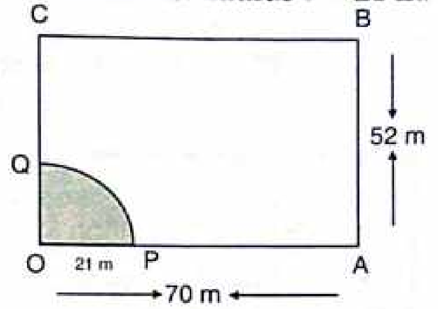

Fig. 13.36

$$\therefore \quad \text{Required area} = \frac{1}{4}\pi r^2$$

$$\Rightarrow \quad \text{Required area} = \left\{\frac{1}{4} \times \frac{22}{7} \times (21)^2\right\} \text{cm}^2 = \frac{693}{2} \text{cm}^2 = 346.5 \text{ cm}^2$$

EXAMPLE 5 *A square park has each side of 100 m. At each corner of the park, there is a flower bed in the form of a quadrant of radius 14 m as shown in Fig. 13.37. Find the area of the remaining part of the park (Use $\pi = 22/7$).*

SOLUTION Let A be the area of each quadrant of a circle of radius 14 m. Then,

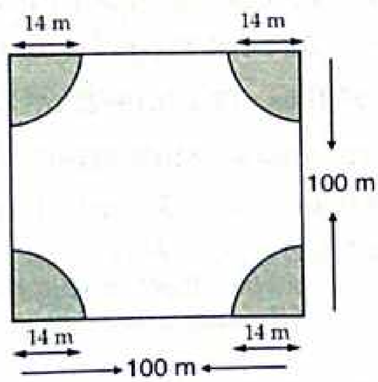

Fig. 13.37

$$A = \frac{1}{4}(\pi r^2) = \frac{1}{4} \times \frac{22}{7} \times 14 \times 14 = 154 \text{ m}^2$$

$$\therefore \quad \text{Area of 4 quadrants} = 4A = (4 \times 154) \text{ m}^2 = 616 \text{ m}^2$$

Area of square park having side 100 m long $= (100 \times 100) \text{ m}^2 = 10{,}000 \text{ m}^2$

Hence,

$$\text{Area of the remaining part of the park} = 10{,}000 - 616 = 9384 \text{ m}^2$$

EXAMPLE 6 *A drain cover is made from a square metal plate of side 40 cm having 441 holes of diameter 1 cm each drilled in it. Find the area of the remaining square plate.*

SOLUTION Area of square metal plate $= 40 \times 40\ \text{cm}^2 = 1600\ \text{cm}^2$

Area of each hole $= \pi r^2 = \frac{22}{7} \times \left(\frac{1}{2}\right)^2\ \text{cm}^2 = \frac{11}{14}\ \text{cm}^2$

$\therefore$ Area of 441 holes $= 441 \times \frac{11}{14}\ \text{cm}^2 = 346.5\ \text{cm}^2$

Hence, Area of the remaining square plate $= (1600 - 346.5)\ \text{cm}^2 = 1253.5\ \text{cm}^2$

EXAMPLE 7 *Floor of a room is a dimensions 5m × 4m and it is covered with circular tiles of diameter 50 cm each as shown in Fig. 13. 38. Find the area of the floor that remains uncovered with tiles (Use π 3.14).* **[NCERT EXEMPLAR]**

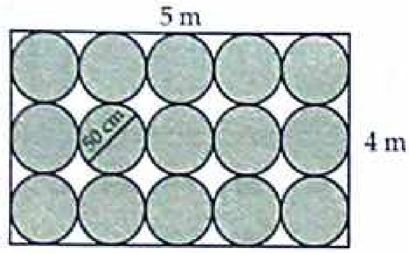

Fig. 13.38

SOLUTION Length of the room = 5 m, Breadth of the room = 4 m,

Diameter of each tile = 50 cm = 0.5 m

$\therefore$ Number of tiles along the length in a row $= \frac{5}{0.5} = 10$

Number of tiles along the breadth in a column $= \frac{4}{0.5} = 8$

$\therefore$ Total number of tiles used to cover the floor $= 10 \times 8 = 80$

Area of each tile $= \pi r^2 = 3.14 \times (0.25)^2\ \text{m}^2 = 0.19625\ \text{m}^2$

Total area covered by 80 tiles $= 80 \times 0.19625\ \text{m}^2 = 15.7\ \text{m}^2$

Area of the floor of the room $= 5 \times 4\ \text{m}^2 = 20\ \text{m}^2$

$\therefore$ Area of the uncovered floor $= (20 - 15.7)\ \text{m}^2 = 4.3\ \text{m}^2$

EXAMPLE 8 *On a square cardboard sheet of area 784 cm², four circular plates of maximum size are placed such that each circular plate touches the other two plates and each side of the square sheet is tangent to circular plates. Find the area of the square sheet not covered by the circular plates.* **[NCERT EXEMPLAR]**

SOLUTION Let the radius of each circular plate be r cm. Then,

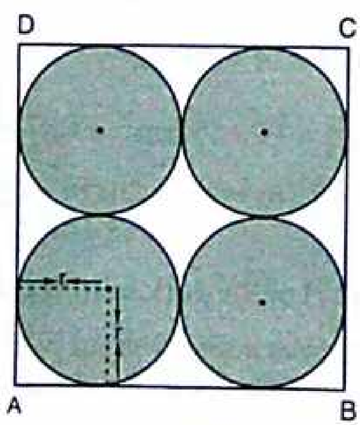

Fig. 13.39

Length of each side of the square sheet = $4r$ cm.

$\therefore$ Area of the square cardboard sheet = $(4r \times 4r)\ \text{cm}^2 = 16r^2\ \text{cm}^2$

But, the area of the cardboard sheet is given to be $784\ \text{cm}^2$

$\therefore$ $16r^2 = 784 \Rightarrow r^2 = 49 \Rightarrow r = 7$

Area of one circular plate $= \pi r^2 = \frac{22}{7} \times 7^2\ \text{cm}^2 = 154\ \text{cm}^2$

$\therefore$ Area of four circular plates $= 4 \times 154\ \text{cm}^2 = 616\ \text{cm}^2$

$\therefore$ Uncovered area of the square sheet $= (784 - 616)\ \text{cm}^2 = 168\ \text{cm}^2$

EXAMPLE 9 *On a square handkerchief, nine circular designs each of radius 7 cm are made. Find the area of the remaining portion of the handkerchief See (Fig. 13.40).* **[NCERT]**

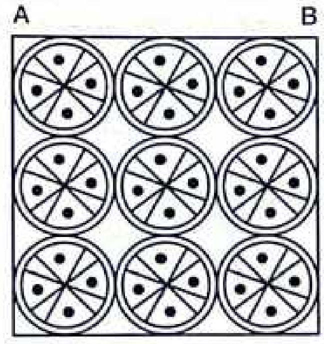

Fig. 13.40

SOLUTION Radius of each circle = 7 cm

$\therefore$ Diameter of each circle = 14 cm.

Length of each side of the square = 14 cm + 14 cm + 14 cm = 42 cm

So, area of the handkerchief = $42 \times 42\ \text{cm}^2 = 1764\ \text{cm}^2$

Area of 9 circles each of 7 cm radius $= (9 \times \pi \times 7^2)\ \text{cm}^2 = \left(9 \times \frac{22}{7} \times 7^2\right)\text{cm}^2$

$= 1386\ \text{cm}^2$

Hence, Area of the remaining portion of handkerchief $= 1764\ \text{cm}^2 - 1386\ \text{cm}^2 = 378\ \text{cm}^2$

EXAMPLE 10 *Four equal circles are described about the four corners of a square so that each touches two of the others as shown in Fig. 13.41. Find the area of the shaded region, each side of the square measuring 14 cm.* **[NCERT]**

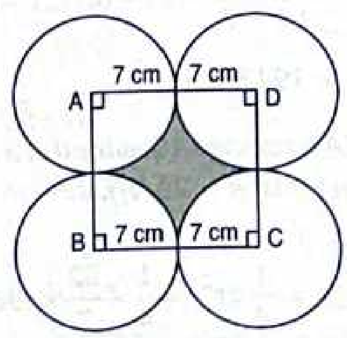

Fig. 13.41

SOLUTION Let $ABCD$ be the given square each side of which is 14 cm long. Clearly, the radius of each circle is 7 cm.

Area of the square of side 14 cm long $= (14 \times 14)\text{ cm}^2 = 196\text{ cm}^2$

Area of each quadrant of a circle of radius 7 cm $= \frac{1}{4}(\pi r^2)$

$$= \left\{\frac{1}{4} \times \frac{22}{7} \times (7)^2\right\}\text{cm}^2 = 38.5\text{ cm}^2$$

$\therefore$ Area of 4 quadrants $= 4 \times 38.5\text{ cm}^2 = 154\text{ cm}^2$

Hence,

Area of the shaded region = Area of the square $ABCD$ – Area of 4 quadrants

$$= (196 - 154)\text{ cm}^2 = 42\text{ cm}^2$$

EXAMPLE 11 *ABCD is a flower bed. If OA = 21 m and OC = 14 m, find the area of the bed. (Take $\pi = 22/7$).* **[NCERT]**

SOLUTION We have, $OA = R = 21$ m and $OC = r = 14$ m

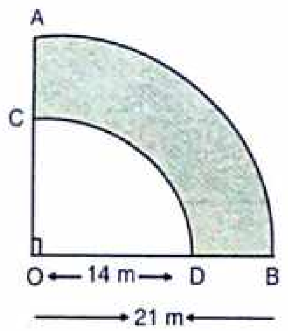

Fig. 13.42

$\therefore$ Area of the flower bed = Area of a quadrant of a circle of radius R – Area of a quadrant of a circle of radius r

$$= \frac{1}{4}\pi R^2 - \frac{1}{4}\pi r^2$$

$$= \frac{\pi}{4}(R^2 - r^2)$$

$$= \frac{1}{4} \times \frac{22}{7}(21^2 - 14^2)\text{ cm}^2 \qquad [\because R = 21\text{ m and } r = 14\text{ m}]$$

$$= \left\{\frac{1}{4} \times \frac{22}{7} \times (21+14)(21-14)\right\}\text{m}^2 = \left\{\frac{1}{4} \times \frac{22}{7} \times 35 \times 7\right\}\text{m}^2$$

$$= 192.5\text{ m}^2$$

EXAMPLE 12 *In Fig. 13.43, AOBCA represents a quadrant of a circle of radius 3.5 cm with centre O. Calculate the area of the shaded portion (Take $\pi = 22/7$).* **[CBSE 2014, 2017, NCERT]**

SOLUTION We find that :

Area of quadrant $AOBCA = \frac{1}{4}\pi r^2 = \frac{1}{4} \times \frac{22}{7} \times (3.5)^2$

$$= \frac{1}{4} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} = \frac{27}{8}\text{ cm}^2 = 9.625\text{ cm}^2$$

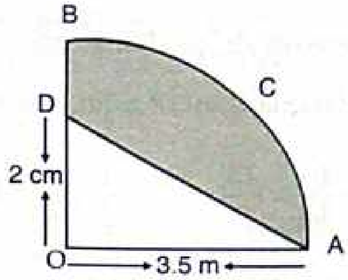

Fig. 13.43

Area of $\Delta AOD = \frac{1}{2} \times \text{Base} \times \text{Height} = \frac{1}{2}(OA \times OB) = \frac{1}{2}(3.5 \times 2)\ \text{cm}^2 = 3.5\ \text{cm}^2$

Hence,

Area of the shaded portion = Area of quadrant − Area of ΔAOD

$= (9.625 - 3.5)\ \text{cm}^2 = 6.125\ \text{cm}^2$

EXAMPLE 13 *A circular grassy plot of land, 42 m in diameter, has a path 3.5 m wide running round it on the outside. Find the cost of gravelling the path at ₹ 4 per square metre.*

SOLUTION Radius of the plot = 21 m.

Radius of the plot including the path = (21 + 3.5) m = 24.5 m

$\therefore$ Area of the path $= \left\{\pi(24.5)^2 - \pi(21)^2\right\}\ \text{m}^2$

$= \pi\left\{(24.5)^2 - (21)^2\right\}\ \text{m}^2$

$= \pi\left\{(24.5 + 21)(24.5 - 21)\right\}\ \text{m}^2$

$= \left\{\pi(45.5) \times (3.5)\right\}\ \text{m}^2$

$= \frac{22}{7} \times 45.5 \times 3.5\ \text{m}^2 = 500.5\ \text{m}^2$

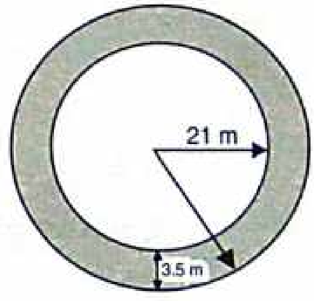

Fig. 13.44

Hence, cost of gravelling the path = ₹ (500.5 × 4) = ₹ 2002.

EXAMPLE 14 *ABCP is a quadrant of a circle of radius 14 cm. With AC as diameter, a semi-circle is drawn. Find the area of the shaded portion.* **[NCERT, CBSE 2008, 2014]**

SOLUTION Applying Pythagoras theorem in the right-angled triangle ABC, we obtain

$AC^2 = AB^2 + BC^2$

$\Rightarrow \quad AC^2 = 14^2 + 14^2 = 2 \times 14^2$

$\Rightarrow \quad AC = \sqrt{2 \times 14^2} = 14\sqrt{2}$ cm

$\Rightarrow \quad \frac{1}{2}AC = \frac{14\sqrt{2}}{2}\ \text{cm} = 7\sqrt{2}$ cm

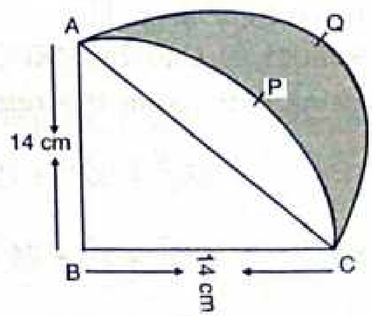

Fig. 13.45

Let A be the area of the shaded portion. Then,

A = Area $APCQA$

$\Rightarrow \quad A$ = Area $ACQA$ − Area $ACPA$

$\Rightarrow \quad A$ = Area $ACQA$ − (Area $ABCPA$ − Area of ΔABC)

$\Rightarrow \quad A =$ (Area of sem-circle with AC as diameter)

–[Area of a quadrant of a circle with AB as radius– Area of $\Delta\ ABC$]

$$\Rightarrow \quad A = \left[\frac{1}{2}\left\{\frac{22}{7}\times(7\sqrt{2})^2\right\} - \left\{\frac{1}{4}\times\frac{22}{7}\times 14^2 - \frac{1}{2}\times 14\times 14\right\}\right]$$

$$\Rightarrow \quad A = \left\{\frac{1}{2}\times\frac{22}{7}\times 49\times 2 - \frac{1}{4}\times\frac{22}{7}\times 14\times 14 + \frac{1}{2}\times 14\times 14\right\}\text{cm}^2$$

$$\Rightarrow \quad A = (154 - 154 + 98)\ \text{cm}^2 = 98\ \text{cm}^2$$

EXAMPLE 15 *The inner and outer diameters of ring I of a dartboard are 32 cm and 34 cm respectively and those of rings II are 19 cm and 21 cm respectively. What is the total area of these two rings?*

SOLUTION We find that:

$$A_1 = \text{Area of ring I} = (\pi\times 17^2 - \pi\times 16^2)\ \text{cm}^2$$

$$= \frac{22}{7}\times(17^2 - 16^2)\ \text{cm}^2$$

$$= \frac{22}{7}\times(17+16)(17-16)\ \text{cm}^2$$

$$= \frac{22}{7}\times 33\ \text{cm}^2$$

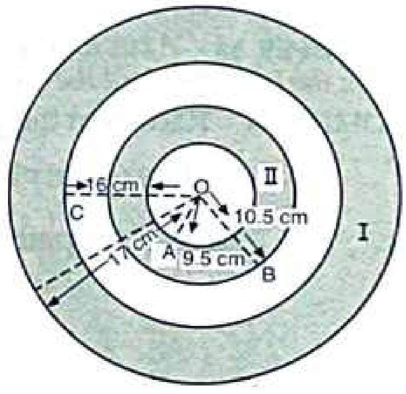

Fig. 13.46

$$A_2 = \text{Area of ring II} = (\pi\times 10.5^2 - \pi\times 9.5^2)\ \text{cm}^2$$

$$= \pi(10.5^2 - 9.5^2)\ \text{cm}^2 = \frac{22}{7}\times(10.5+9.5)(10.5-9.5)\ \text{cm}^2 = \frac{22}{7}\times 20\ \text{cm}^2$$

Hence,

$$\text{Total area of two rings} = A_1 + A_2 = \frac{22}{7}\times 33 + \frac{22}{7}\times 20\ \text{cm}^2$$

$$= \frac{22}{7}\times(33+20)\ \text{cm}^2 = 166.57\ \text{cm}^2$$

EXAMPLE 16 *Find the area of the shaded region in Fig. 13.47, if $PQ = 24$ cm $PR = 7$ cm and O is the centre of the circle.* **[NCERT, CBSE 2009]**

SOLUTION Clearly, $\angle RPQ$ is the angle in a semi-circle. Therefore, it is a right angle. Using Pythagoras theorem in $\Delta\ RPQ$, we obtain

$$RQ^2 = RP^2 + PQ^2$$

$$\Rightarrow \quad RQ^2 = 7^2 + 24^2 = 625$$

$$\Rightarrow \quad RQ = 25\ \text{cm}$$

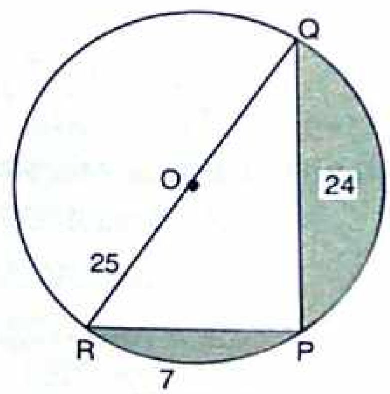

Fig. 13.47

$$\therefore \quad \text{Radius of the circle} = \frac{1}{2}RQ = \frac{25}{2}\text{cm}$$

Let A be the area of the shaded region. Then,

$$A = \text{Area of the semi-circle} - \text{Area of } \Delta RPQ.$$

$$\Rightarrow \quad A = \frac{1}{2}\pi r^2 - \frac{1}{2} \times PR \times PQ$$

$$\Rightarrow \quad A = \left\{\frac{1}{2} \times \frac{22}{7} \times \left(\frac{25}{2}\right)^2 - \frac{1}{2} \times 7 \times 24\right\} \text{cm}^2 = \left\{\frac{6875}{28} - 84\right\} \text{cm}^2 = \frac{4523}{28} \text{cm}^2$$

EXAMPLE 17 *Find the area of the shaded region in Fig. 13.48, where radii of the two concentric circles with centre O are 7 cm and 14 cm respectively and* $\angle AOC = 40°$.

[CBSE 2014, NCERT]

SOLUTION We have,

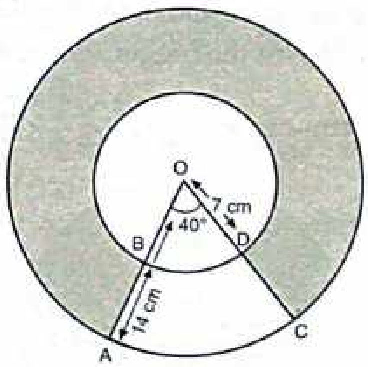

Fig. 13.48

$$\text{Area of the region } ABDC = \text{Area of sector } AOC - \text{Area of sector } BOD$$

$$= \left(\frac{40}{360} \times \frac{22}{7} \times 14 \times 14 - \frac{40}{360} \times \frac{22}{7} \times 7 \times 7\right) \text{cm}^2$$

$$= \left(\frac{1}{9} \times 22 \times 14 \times 2 - \frac{1}{9} \times 22 \times 7 \times 1\right) \text{cm}^2$$

$$= \frac{22}{9} \times (28 - 7) \text{ cm}^2 = \frac{154}{3} \text{cm}^2$$

$$\text{Area of the circular ring} = \left(\frac{22}{7} \times 14 \times 14 - \frac{22}{7} \times 7 \times 7\right) \text{cm}^2$$

$$= (22 \times 14 \times 2 - 22 \times 7 \times 1) \text{ cm}^2 = 22 \times 21 \text{ cm}^2 = 462 \text{ cm}^2$$

Hence, $\quad \text{Required shaded area} = \left(462 - \frac{154}{3}\right) \text{cm}^2 = \frac{1232}{3} \text{cm}^2 = 410.67 \text{ cm}^2$

EXAMPLE 18 *AB and CD are respectively arcs of two concentric circles of radii 21 cm and 7 cm and centre O. If* $\angle AOB = 30°$, *find the area of the shaded region.* **[NCERT, CBSE 2012]**

SOLUTION Let A be the area oif the shaded region. Then,

$$A = \text{Area of sector } OAB - \text{Area of sector } OCD$$

$$\Rightarrow \quad A = \left(\frac{30}{360} \times \frac{22}{7} \times 21 \times 21 - \frac{30}{360} \times \frac{22}{7} \times 7 \times 7\right) \text{cm}^2$$

$$\Rightarrow \quad A = \frac{30}{360} \times \frac{22}{7} \times (21 \times 21 - 7 \times 7) \text{ cm}^2$$

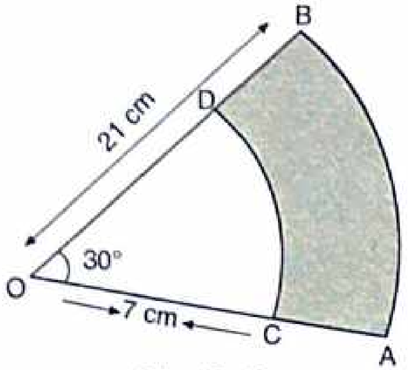

Fig. 13.49

$$\Rightarrow \quad A = \frac{11}{42} \times (21+7) \times (21-7)\ \text{cm}^2 = \frac{11}{42} \times 28 \times 14\ \text{cm}^2 = 102.67\ \text{cm}^2$$

EXAMPLE 19 *Find the areas of the shaded region in the Fig. 13.50.*

SOLUTION It is given that the radius of the bigger semi-circle is $r = 14$ cm

$$\therefore \quad A_1 = \text{area of the bigger semi-circle} = \frac{1}{2}\pi r^2 = \frac{1}{2} \times \frac{22}{7} \times (14)^2\ \text{cm}^2 = 308\ \text{cm}^2$$

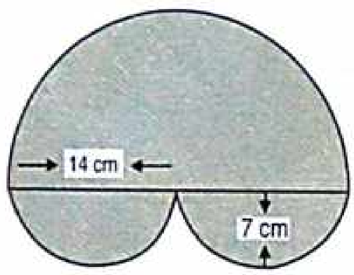

Fig. 13.50

Radius of each of the smaller circle is $r_1 = 7$ cm

$$\therefore \quad A_2 = \text{Area of 2 smaller semi-circles} = 2\left(\frac{1}{2}\pi r_1^2\right) = 2\left(\frac{1}{2} \times \frac{22}{7} \times 7^2\right)\text{cm}^2 = 154\ \text{cm}^2$$

Hence, required area $= A_1 + A_2 = (308 + 154)\ \text{cm}^2 = 462\ \text{cm}^2$

EXAMPLE 20 *In Fig. 13.51, ABCD is a square of side 10 cm. Semi-circles are drawn with each side of square as diameter. Find the area of (i) the unshaded region (ii) the shaded region*

[NCERT, CBSE 2016]

SOLUTION Let us mark the four unshaded regions as R_1, R_2, R_3 and R_4.

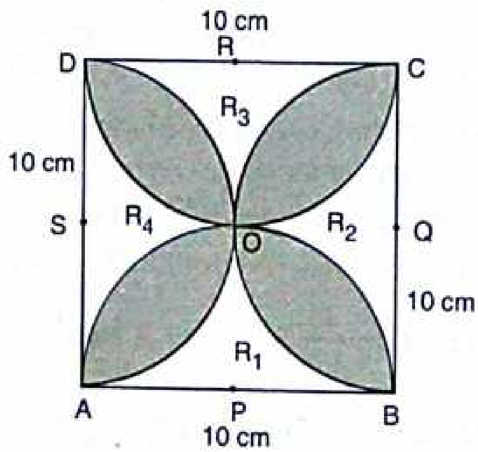

Fig. 13.51

Clealry,
Area of R_1 + Area of R_3

$$= \text{Area of square } ABCD - \text{Area of two semi-circles having centres at } Q \text{ and } S$$

$$= \left(10 \times 10 - 2 \times \frac{1}{2} \times 3.14 \times 5^2\right) \text{cm}^2 \qquad [\because \text{Radius} = AP = 5\,\text{cm}]$$

$$= (100 - 3.14 \times 25)\,\text{cm}^2 = (100 - 78.5)\,\text{cm}^2 = 21.5\,\text{cm}^2$$

Similarly, we have

Area of R_2 + Area of R_4 = 21.5 cm^2

(i) Area of the unshaded region = Area R_1 + Area R_2 + Area R_3 + Area R_4

$$= (\text{Area } R_1 + \text{Area } R_3) + (\text{Area } R_2 + \text{Area } R_4)$$

$$= 2(21.5)\,\text{cm}^2 = 43\,\text{cm}^2$$

(ii) Area of the shaded region

$$= \text{Area of square } ABCD - (\text{Area of } R_1 + \text{Area of } R_2 + \text{Area of } R_3 + \text{Area of } R_4)$$

$$= (100 - 2 \times 21.5)\,\text{cm}^2 = 57\,\text{cm}^2$$

EXAMPLE 21 *In Fig. 13.52, ABC is an equilateral triangle inscribed in a circle of radius 4 cm with centre O. Find the area of the shaded region.*

SOLUTION Let h be the height of ΔABC and R be the radius of the circumcircle. Then,

$$OA = \frac{2}{3} AD \text{ and } h = \frac{\sqrt{3}}{2} a$$

$$\Rightarrow \qquad R = \frac{2}{3} h \text{ and } h = \frac{\sqrt{3}a}{2} \Rightarrow R = \frac{a}{\sqrt{3}} \Rightarrow a = R\sqrt{3} \Rightarrow a = 4\sqrt{3}$$

Let the length of each side of equilateral ΔABC be a.

We find that : $\angle AOC = 2\angle ABC = 2 \times 60^\circ = 120^\circ$

$$\therefore \qquad \text{Required area} = \frac{1}{3}(\text{Area of the circle} - \text{Area of } \Delta ABC)$$

$$= \frac{1}{3}\left\{\pi R^2 - \frac{\sqrt{3}}{4} \times (\text{Side})^2\right\}$$

$$= \frac{1}{3}\left\{16\pi - \frac{\sqrt{3}}{4} \times (4\sqrt{3})^2\right\} \text{cm}^2$$

$$= \frac{1}{3}(16\pi - 12\sqrt{3})\,\text{cm}^2 = \frac{4}{3}(4\pi - 3\sqrt{3})\,\text{cm}^2$$

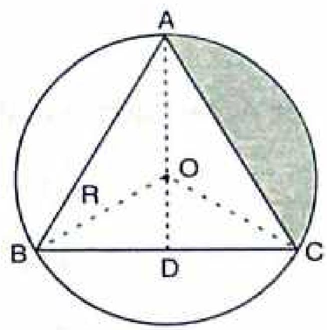

Fig. 13.52

EXAMPLE 22 *PQRS is a diameter of a circle of radius 6 cm. The lengths PQ, QR and RS are equal. Semi-circles are drawn on PQ and QS as diameters as shown in Fig. 13.53. Find the perimeter and area of the shaded region.*

SOLUTION PS = Diameter of a circle of radius 6 cm = 12 cm

$$\therefore \qquad PQ = QR = RS = \frac{12}{3} = 4\,\text{cm}, \quad QS = QR + RS = (4 + 4)\,\text{cm} = 8\,\text{cm}$$

Let P be the perimeter and A be the area of the shaded region.

P = Arc of semi-circle of radius 6 cm + Arc of semi-circle of radius 4 cm
+ Arc of semi-circle of radius 2 cm

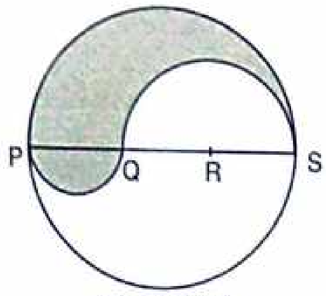

Fig. 13.53

$\Rightarrow \quad P = (\pi \times 6 + \pi \times 4 + \pi \times 2)\text{ cm} = 12\pi\text{ cm}$

and,

A = Area of semi-circle with PS as diameter + Area of semi-circle with PQ as diameter – Area of semi-circle with QS as diameter.

$$\Rightarrow \quad A = \frac{1}{2} \times \frac{22}{7} \times (6)^2 + \frac{1}{2} \times \frac{22}{7} \times 2^2 - \frac{1}{2} \times \frac{22}{7} \times (4)^2$$

$$\Rightarrow \quad A = \frac{1}{2} \times \frac{22}{7}(6^2 + 2^2 - 4^2) = \frac{1}{2} \times \frac{22}{7} \times 24 = \frac{264}{7}\text{ cm}^2 = 37.71\text{ cm}^2$$

EXAMPLE 23 *Find to the three places of decimals the radius of the circle whose area is the sum of the areas of two triangles whose sides are 35, 53, 66 and 33, 56, 65 measured in centimetres (Use $\pi = 22/7$).*

SOLUTION For the first triangle, we have $a = 35$, $b = 53$ and $c = 66$.

$$\therefore \qquad s = \frac{a+b+c}{2} = \frac{35+53+66}{2} = 77\text{ cm}$$

Let Δ_1 be the area of the first triangle. Then,

$$\Delta_1 = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \qquad \Delta_1 = \sqrt{77(77-35)(77-53)(77-66)} = \sqrt{77 \times 42 \times 24 \times 11}$$

$$\Rightarrow \qquad \Delta_1 = \sqrt{7 \times 11 \times 7 \times 6 \times 6 \times 4 \times 11} = \sqrt{7^2 \times 11^2 \times 6^2 \times 2^2} = 7 \times 11 \times 6 \times 2 = 924\text{ cm}^2 \qquad \text{...(i)}$$

For the second triangle, we have $a = 33$, $b = 56$, $c = 65$

$$\therefore \qquad s = \frac{a+b+c}{2} = \frac{33+56+65}{2} = 77\text{ cm}$$

Let Δ_2 be the area of the second triangle. Then,

$$\Delta_2 = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \qquad \Delta_2 = \sqrt{77(77-33)(77-56)(77-65)}$$

$$\Rightarrow \qquad \Delta_2 = \sqrt{77 \times 44 \times 21 \times 12} = \sqrt{7 \times 11 \times 4 \times 11 \times 3 \times 7 \times 3 \times 4} = \sqrt{7^2 \times 11^2 \times 4^2 \times 3^2}$$

$$\Rightarrow \qquad \Delta_2 = 7 \times 11 \times 4 \times 3 = 924\text{ cm}^2 \qquad \text{...(ii)}$$

Let r be the radius of the circle. Then,

Area of the circle = Sum of the areas of two triangles

$\Rightarrow \quad \pi r^2 = \Delta_1 + \Delta_2$

$\Rightarrow \quad \pi r^2 = 924 + 924$ [Using: (i) & (ii)]

$\Rightarrow \quad \frac{22}{7} \times r^2 = 1848$

$\Rightarrow \quad r^2 = 1848 \times \frac{7}{22} = 3 \times 4 \times 7 \times 7 \Rightarrow r = \sqrt{3 \times 2^2 \times 7^2} = 2 \times 7 \times \sqrt{3} = 14\sqrt{3}$ cm

EXAMPLE 24 *In an equilateral triangle of side 24 cm, a circle is inscribed touching its sides. Find the area of the remaining portion of the triangle* (*Take* $\sqrt{3} = 1.732$).

SOLUTION Let ABC be an equilateral triangle of side 24 cm, and let AD be perpendicular from A on BC. Since the triangle is equilateral, so D bisects BC.

$\therefore \quad BD = CD = 12$ cm

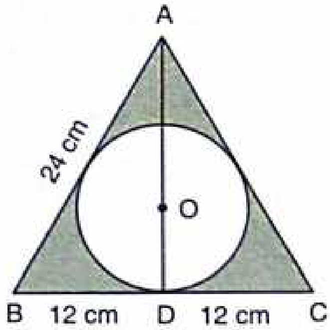

Fig. 13.54

The centre of the inscribed circle will coincide with the centroid of ΔABC.

$\therefore \quad OD = \frac{1}{3} AD$

In ΔABD, we have

$AB^2 = AD^2 + BD^2$ [Using Pythagoras Theorem]

$\Rightarrow \quad 24^2 = AD^2 + 12^2$

$\Rightarrow \quad AD = \sqrt{24^2 - 12^2} = \sqrt{(24-12)(24+12)} = \sqrt{36 \times 12} = 12\sqrt{3}$ cm

$\therefore \quad OD = \frac{1}{3} AD = \left(\frac{1}{3} \times 12\sqrt{3}\right)$ cm $= 4\sqrt{3}$ cm

Now, Area of the incircle $= \pi (OD)^2 = \left\{\frac{22}{7} \times (4\sqrt{3})^2\right\} \text{cm}^2 = \left\{\frac{22}{7} \times 48\right\} \text{cm}^2 = 150.85 \text{ cm}^2$

and, Area of the triangle $ABC = \frac{\sqrt{3}}{4} (\text{Side})^2 = \frac{\sqrt{3}}{4} (24)^2 = 249.4 \text{ cm}^2$

$\therefore$ Area of the remaining portion of the triangle $= (249.4 - 150.85) \text{ cm}^2 = 98.55 \text{ cm}^2$

EXAMPLE 25 *Find the area of the shaded region in Fig. 13.55, where a circular arc of radius 6 cm has been drawn with vertex O of an equilateral triangle OAB of side 12 cm as centre.* **[NCERT]**

SOLUTION Let A be the area of the shaded region. Then,

$A =$ Area of $\Delta OAB +$ Area of the circle $-$ Area of a sector of a circle of radius 6 cm and of angle $60°$

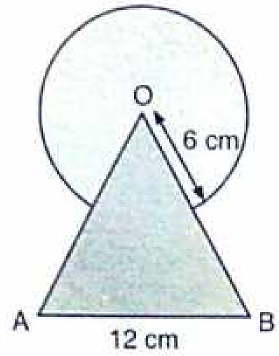

Fig. 13.55

$$\Rightarrow \quad A=\left\{\frac{\sqrt{3}}{4}\times 12^2+\pi\times 6^2-\frac{60}{360}\times\pi\times 6^2\right\}\text{cm}^2$$

$$\Rightarrow \quad A=\left(36\sqrt{3}+36\pi-6\pi\right)\text{cm}^2=\left(36\sqrt{3}+30\times\frac{22}{7}\right)\text{cm}^2=\left(\frac{660}{7}+36\sqrt{3}\right)\text{cm}^2$$

EXAMPLE 26 *The area of an equilateral triangle is $49\sqrt{3}\ cm^2$. Taking eqch angular point as centre, a circle is described with radius equal to half the length of the side of the triangle as shown in Fig. 13.56. Find the area of the triangle not included in the circle.* **[CBSE 2009]**

SOLUTION Let each side of the triangle be a cm. Then,

$$\text{Area of }\Delta\, ABC = 49\sqrt{3}\text{ cm}^2$$

$$\Rightarrow \quad \frac{\sqrt{3}}{4}a^2=49\sqrt{3} \qquad \left[\because \text{ Area}=\frac{\sqrt{3}}{4}(\text{Side})^2\right]$$

$$\Rightarrow \quad a^2=49\times 4 \Rightarrow a=14\text{ cm}$$

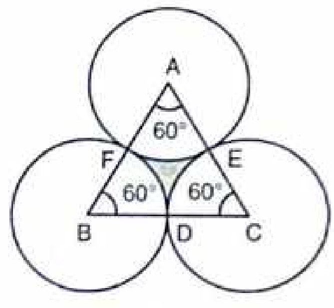

Fig. 13.56

Thus, the radius of each circle is $r=7$ cm

Let A be the required area. Then,

$$A=\text{Area of }\Delta ABC-3\times(\text{Area of a sector of angle }60^\circ\text{ in a circle of radius 7 cm})$$

$$\Rightarrow \quad A=\left\{49\sqrt{3}-3\left(\frac{60}{360}\times\frac{22}{7}\times 7^2\right)\right\}\text{cm}^2=\left(49\sqrt{3}-77\right)\text{cm}^2=(49\times 1.73-77)\text{cm}^2=7.77\text{ cm}^2$$

EXAMPLE 27 *The area of an equilateral triangle is 1732.05 cm². About each angular point as centre, a circle is described with radius equal to half the length of the side of the triangle. Find the area of the triangle not included in the circles. (Use $\pi=3.14$).* **[NCERT]**

SOLUTION Let each side of the equilateral triangle be a cm. It is given that its area is 1732.05 cm²

$$\therefore \quad \frac{\sqrt{3}}{4}a^2=1732.05$$

$$\Rightarrow \quad \frac{a^2}{4}=\frac{1732.05}{\sqrt{3}} \Rightarrow \left(\frac{a}{2}\right)^2=\frac{1732.05}{\sqrt{3}} \qquad \ldots(i)$$

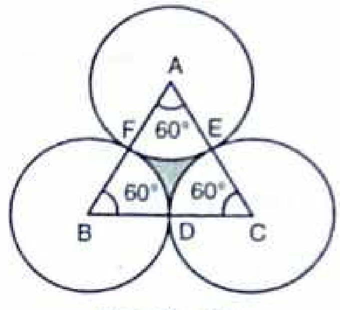

Fig. 13.57

Clearly, radius of each circle is $\frac{a}{2}$ cm.

Let A be the area of three sectors each of angle $60°$ in a circle of radius $\frac{a}{2}$ cm. Then,

$$A = 3\left\{\frac{60}{360} \times 3.14 \times \left(\frac{a}{2}\right)^2\right\} \text{cm}^2 = \frac{1}{2} \times 3.14 \times \left(\frac{a}{2}\right)^2 = \frac{1}{2} \times 3.14 \times \frac{1732.05}{\sqrt{3}} = 1570.05 \text{ cm}^2$$

Let A_1 be the required area. Then,

$$A = \text{Area of } \Delta ABC - A_1$$

$$\Rightarrow \quad A = (1732.05 - 1570.04) \text{ cm}^2 = 162.01 \text{ cm}^2$$

EXAMPLE 28 *An athletic track 14 m wide consists of two straight sections 120 m long joining semi-circular ends whose inner radius is 35 m. Calculate the area of the shaded region.*

SOLUTION Let A be the area of the shaded region.

We have, $OB = O'C = 35$ m and $AB = CD = 14$ m

$$\therefore \quad OA = O'D = (35 + 14) \text{ m} = 49 \text{ m}$$

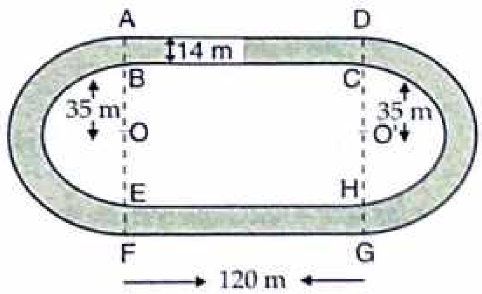

Fig. 13.58

The area A of the shaded region is given by

A = Area of rectangle $ABCD$ + Area of rectangle $EFGH$ + 2 {Area of the semi-circle with radius 49 m} − {Area of the semi-circle with radius 35 m}

$$A = (14 \times 120) + (14 \times 120) + 2\left\{\frac{1}{2} \times \frac{22}{7} \times (49)^2\right\} - 2\left\{\frac{1}{2} \times \frac{22}{7} \times (35)^2\right\}$$

$$\Rightarrow \quad A = \left\{1680 + 1680 + \frac{22}{7}(49^2 - 35^2)\right\} \text{m}^2 = \left\{3360 + \frac{22}{7}(49 + 35)(49 - 35)\right\} \text{m}^2$$

$$\Rightarrow \quad A = \left\{3360 + \frac{22}{7} \times 84 \times 14\right\} \text{m}^2 = \{3360 + 44 \times 84\} \text{ m}^2 = 7056 \text{ m}^2$$

Hence, the area of the shaded region is 7056 m^2

EXAMPLE 29 *It is proposed to add to a square lawn measuring 58 cm on a side, two circular ends. The centre of each circle being the point of intersection of the diagonals of the square. Find the area of the whole lawn.*

SOLUTION The length of each side of a square lawn is 58 cm.

$$\therefore \quad \text{Length of the diagonal of the square} = \sqrt{58^2 + 58^2} = 58\sqrt{2} \text{ cm}$$

So, radius of the circle having centre at the point of intersection of diagonals is $29\sqrt{2}$ cm.

Let A be the area of one of the circular ends. Then,

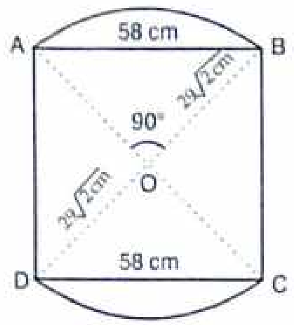

Fig. 13.59

A = Area of a segment of angle 90° in a circle of radius $29\sqrt{2}$ cm

$\Rightarrow \quad A = \left\{\frac{22}{7} \times \frac{90}{360} - \sin 45° \cos 45°\right\} \times (29\sqrt{2})^2 \text{ cm}^2 \quad \left[\because \text{Area} = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2} \cos\frac{\theta}{2}\right\} r^2\right]$

$\Rightarrow \quad A = \left(\frac{11}{14} - \frac{1}{2}\right) \times 29 \times 29 \times 2 \text{ cm}^2 = 29 \times 29 \times 2 \times \frac{4}{14} \text{ cm}^2 = \frac{3364}{7} \text{ cm}^2$

$\therefore$ Area of the whole lawn = Area of the square + 2 (Area of a circular end)

$$= \left\{58 \times 58 + 2 \times \frac{3364}{7}\right\} \text{cm}^2 = \left\{3364 + 2 \times \frac{3364}{7}\right\} \text{cm}^2$$

$$= 3364\left(1 + \frac{2}{7}\right) \text{cm}^2 = 3364 \times \frac{9}{7} \text{cm}^2 = 4325.14 \text{ cm}^2$$

EXAMPLE 30 *In Fig. 13.60, two circular flower beds have been shown on two sides of a square lawn ABCD of side 56 m. If the centre of each circular flower bed is the point of intersection of the diagonals of the square lawn, find the sum of the areas of the lawns and the flower beds.* **[CBSE 2014]**

SOLUTION Using Pythagoras theorem in ΔABD, we obtain

$$BD^2 = AB^2 + AD^2 = 56^2 + 56^2 = 2 \times (56)^2$$

$\Rightarrow \quad BD = 56\sqrt{2}$

$\therefore \quad AC = BD = 56\sqrt{2}$ m

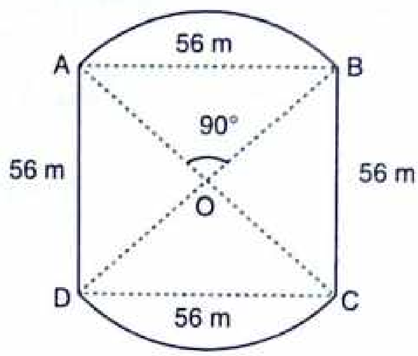

Fig. 13.60

$\therefore \quad OA = OB = \frac{1}{2}AC = 28\sqrt{2}$ m

So, the radius of the circle having centre at the point of intersection of diagonals is $28\sqrt{2}$ m. Let A be the area of one of the circular ends. Then,

A = Area of a segment of angle 90° in a circle of radius $28\sqrt{2}$ m.

$\Rightarrow \quad A = \left\{\frac{22}{7} \times \frac{90}{360} - \sin 45^\circ \cos 45^\circ\right\} \times (28\sqrt{2})^2 \text{ m}^2$ $\left[\begin{array}{l}\text{Substituting } r = 28\sqrt{2} \text{ and } \theta = 90^\circ \\ \text{in } A = \left(\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) r^2\end{array}\right]$

$\Rightarrow \quad A = \left\{\frac{11}{14} - \frac{1}{2}\right\} \times 28 \times 28 \times 2 \text{ m}^2 = 28 \times 28 \times 2 \times \frac{4}{14} \text{ cm}^2 = 448 \text{ m}^2$

$\therefore$ Area of two flower beds $2A = 2 \times 448 \text{ m}^2 = 896 \text{ m}^2$

Area of the square lawn $= 56 \times 56 \text{ m}^2 = 3136 \text{ m}^2$

Hence, Total area $= (3136 + 896) \text{ m}^2 = 4032 \text{ m}^2$

EXAMPLE 31 *A round table cover has six equal designs as shown in Fig. 13.61. If the radius of the cover is 28 cm, find the cost of making the designs at the rate of ₹3.50 per cm². (Use $\sqrt{3} = 1.7$)* **[NCERT]**

SOLUTION We observe that the designs form six segments of a circle of radius $r = 28$ cm and each of angle $\theta = 60^\circ$.

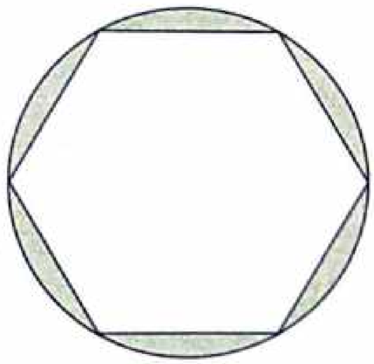

Fig. 13.61

Let A be the area of the six designs. Then,

$$A = 6\left\{\frac{\theta}{360} \times \pi r^2 - \sin\frac{\theta}{2}\cos\frac{\theta}{2} r^2\right\} \text{cm}^2$$

$$\Rightarrow \quad A = 6\left\{\frac{60}{360} \times \frac{22}{7} \times (28)^2 - \sin 30^\circ \cos 30^\circ \times (28)^2\right\} \text{cm}^2 \; [\because r = 28 \text{ cm and } \theta = 60^\circ]$$

$$\Rightarrow \quad A = 6\left\{\frac{1}{6} \times \frac{22}{7} \times 28 \times 28 - \frac{1}{2} \times \frac{\sqrt{3}}{2} \times 28 \times 28\right\} \text{cm}^2$$

$$\Rightarrow \quad A = (88 \times 28 - 6 \times \sqrt{3} \times 7 \times 28) \text{ cm}^2 = (2464 - 1999.2) \text{ cm}^2 = 464.8 \text{ cm}^2$$

Hence, Cost of making the designs at the rate of ₹ 3.50 per cm² = ₹ 464.8 × 3.50 = ₹ 1626.80

EXAMPLE 32 *In Figure 13.62, ABC is a right angled triangle at A. Find the area of the shaded region, if $AB = 6$ cm, $BC = 10$ cm and I is the centre of incircle of ΔABC.* **[CBSE 2009]**

SOLUTION Applying Pythagoras theorem in ΔABC, we obtain

$$BC^2 = AB^2 + AC^2$$

$$\Rightarrow \quad AC^2 = BC^2 - AB^2$$

$$\Rightarrow \quad AC^2 = 100 - 36 = 64$$

$$\Rightarrow \quad AC = 8\text{ cm}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2} \times AB \times AC = \frac{1}{2} \times 6 \times 8\text{ cm}^2 = 24\text{ cm}^2$$

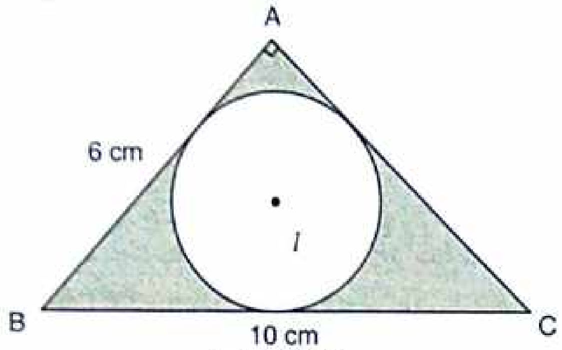

Fig. 13.62

Let r cm be the radius of the incircle. (circle inscribed in ΔABC). We observe that :

$$\text{Area of } \Delta ABC = \text{Area of } \Delta IBC + \text{Area of } \Delta ICA + \text{Area of } \Delta IAB$$

$$\Rightarrow \quad 24 = \frac{1}{2}(BC \times r) + \frac{1}{2}(CA \times r) + \frac{1}{2}(AB \times r)$$

$$\Rightarrow \quad 24 = \frac{1}{2} r (BC + CA + AB)$$

$$\Rightarrow \quad 24 = \frac{1}{2} \times r \times (10 + 8 + 6)$$

$$\Rightarrow \quad 24 = 12r$$

$$\Rightarrow \quad r = 2$$

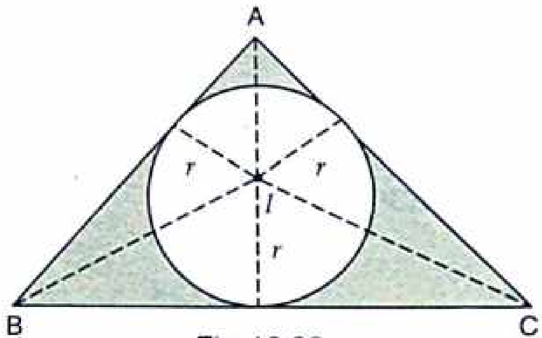

Fig. 13.63

Let A be the area of the shaded region. Then,

$$A = \text{Area of } \Delta ABC - \text{Area of the incircle}$$

$$\Rightarrow \quad A = 24 - \pi r^2 = \left(24 - \frac{22}{7} \times 4\right)\text{cm}^2 = \frac{80}{7}\text{cm}^2$$

LEVEL-2

EXAMPLE 33 *ABCD is a field in the shape of a trapezium. $AB \parallel DC$ and $\angle ABC = 90°$, $\angle DAB = 60°$. Four sectors are formed with centres A, B, C and D (See Fig. 13.64). The radius of each sector is 17.5 m. Find the*

(i) total area of the four sectors.

(ii) area of remaining portion given that $AB = 75$ m and $CD = 50$ m.

SOLUTION Since $AB \parallel CD$ and $\angle ABC = 90°$. Therefore $\angle BCD = 90°$. Also, $\angle BAD = 60°$.

$$\therefore \quad \angle CDA = 180° - 60° = 120° \qquad \text{[Co-interior angles]}$$

(i) Let A be the total area of the four sectors. Then,

$$A = \text{Area of sector at } A + \text{Area of sector at } B + \text{Area of sector at } C + \text{Area of sector at } D.$$

$$\Rightarrow \quad A = \frac{60}{360} \times \pi \times (17.5)^2 + \frac{90}{360} \times \pi \times (17.5)^2 + \frac{90}{360} \times \pi \times (17.5)^2 + \frac{120}{360} \times \pi \times (17.5)^2$$

$\Rightarrow \quad A = \left\{\left(\frac{1}{6}+\frac{1}{4}+\frac{1}{4}+\frac{1}{3}\right)\times \pi \times (17.5)^2\right\} m^2$

$\Rightarrow \quad A = \pi \times \left(\frac{35}{2}\right)^2 m^2 = \frac{22}{7}\times\frac{35}{2}\times\frac{35}{2} m^2 = 962.5 m^2$

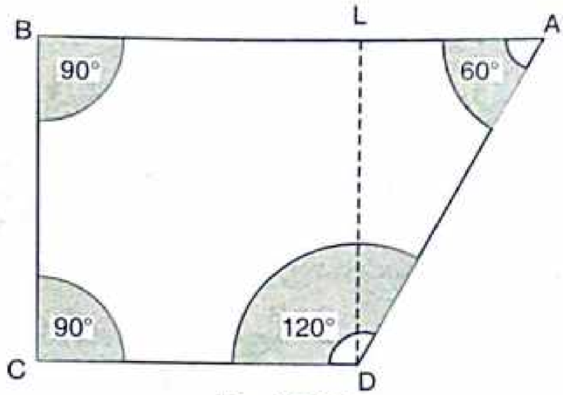

Fig. 13.64

(ii) Let DL be perpendicular drawn from D on AB. Then,

$$AL = AB - BL = AB - CD = (75 - 50) \text{ m} = 25 \text{ m}$$

In ΔALD, we have

$$\tan 60° = \frac{DL}{AL} \Rightarrow \sqrt{3} = \frac{DL}{25} \Rightarrow DL = 25\sqrt{3} \text{ m}$$

$\therefore$ Area of trapezium $ABCD = \frac{1}{2}(AB + CD)\times DL$

$$= \frac{1}{2}(75+50)\times 25\sqrt{3} \text{ m}^2 = 1562.5\times 1.732 \text{ m}^2 = 2706.25 \text{ m}^2$$

Let A be the area of the remaining portion. Then,

A = Area of trapezium $ABCD$ – Area of 4 sectors

$\Rightarrow \quad A = 2706.25 \text{ m}^2 - 962.5 \text{ m}^2 = 1743.75 \text{ m}^2$

EXAMPLE 34 *On a circular table cover of radius 32 cm, a design is formed leaving an equilateral triangle ABC in the middle as shown in Fig. 13.65. Find the area of the design (shaded region).* **[NCERT]**

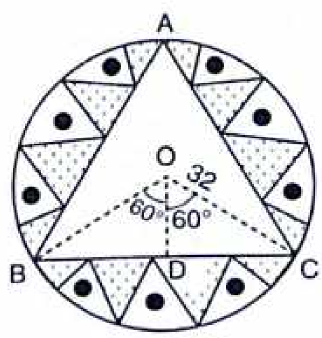

Fig. 13.65

SOLUTION In ΔOBD, we have

$$\cos 60° = \frac{OD}{OB} \text{ and } \sin 60° = \frac{BD}{OB}$$

$\Rightarrow \quad \frac{1}{2} = \frac{OD}{32} \text{ and } \frac{\sqrt{3}}{2} = \frac{BD}{32}$

$\Rightarrow \quad OD = 16$ and $BD = 16\sqrt{3}$

$\Rightarrow \quad BC = 2\,BD = 32\sqrt{3}$

Let A be the area of the shaded region. Then,

$$A = \text{Area of the circle} - \text{Area of } \Delta\, ABC = \left\{\pi \times 32^2 - \frac{\sqrt{3}}{4} \times (32\sqrt{3})^2\right\} \text{cm}^2$$

$$\Rightarrow \quad A = \left\{\frac{22}{7} \times 32 \times 32 - 768\sqrt{3}\right\} \text{cm}^2 = \left\{\frac{22528}{7} - 768\sqrt{3}\right\} \text{cm}^2$$

EXAMPLE 35 *In Fig. 13.66, AB and CD are two diameters of a circle (with centre O) perpendicular to each other and OD is the diameter of the smaller circle. If OA = 7 cm, find the area of the shaded region.* **[NCERT, CBSE 2010, 2013]**

SOLUTION Let A be the area of the shaded region. Then,

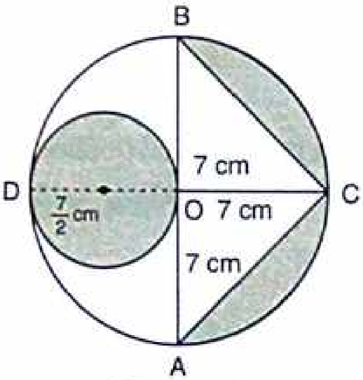

Fig. 13.66

$$A = (\text{Area of circle with } OD\ (=7\text{ cm}) \text{ as diameter}) + \text{Area of semi-circle with } AB \text{ as diameter} - \text{Area of } \Delta\, ABC$$

$$\Rightarrow \quad A = \pi \times \left(\frac{7}{2}\right)^2 + \frac{1}{2} \times \pi \times (7)^2 - \frac{1}{2} \times AB \times OC = \left\{\frac{\pi}{4} \times 49 + \frac{\pi}{2} \times 49 - \frac{1}{2} \times 14 \times 7\right\} \text{cm}^2$$

$$\Rightarrow \quad A = \left(\frac{3\pi}{4} \times 49 - 49\right) \text{cm}^2 = \left(\frac{3}{4} \times \frac{22}{7} \times 49 - 49\right) \text{cm}^2 = \frac{231 - 98}{2} \text{cm}^2 = 66.5 \text{ cm}^2$$

EXAMPLE 36 *Calculate the area of the designed region in Fig. 13.67 common between two quadrants of circles of radius 8 cm each.* **[NCERT]**

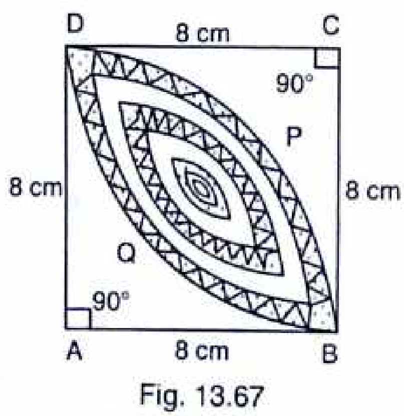

Fig. 13.67

SOLUTION Let A be the area of the shaded region. Then,

$$A = 2\left(\text{Area of quadrant } ABPD - \text{Area of } \Delta\, ABD\right)$$

$$\Rightarrow \quad A = 2\left\{\frac{\pi}{4} \times (8)^2 - \frac{1}{2} \times 8 \times 8\right\} \text{cm}^2$$

$\Rightarrow \quad A = 2\left\{\frac{22}{7} \times \frac{1}{4} \times 64 - 32\right\} cm^2$

$\Rightarrow \quad A = 2\left\{\frac{22 \times 16}{7} - 32\right\} cm^2 = 2\left(\frac{352 - 224}{7}\right) cm^2 = \frac{256}{7} cm^2$

EXAMPLE 37 *In Fig. 13.68, a crescent is formed by two circles which touch at A. C is the centre of the larger circle. The width of the crescent at BD is 9 cm and at EF it is 5 cm. Find (i) the radii of two circles (ii) the area of the shaded region.*

SOLUTION (i) Let the radii of the larger and smaller circles be R and r respectively. Then,

$BD = 9 \text{ cm} \Rightarrow 2R - 2r = 9 \Rightarrow R - r = 4.5$ ...(i)

Join AE and DE. Let $\angle CAE = \theta$ Then, $\angle AEC = 90° - \theta$.

Now, $\angle AED = 90° \Rightarrow \angle AEC + \angle DEC = 90° \Rightarrow \angle DEC = 90° - (90° - \theta) = \theta$.

Thus, in Δ's ACE and DCE, we have

$\angle CAE = \angle CED = \theta$ and $\angle ACE = \angle ECD = 90°$

So, by AA similarity criterion, we obtain

$\Delta ACE \sim \Delta ECD$

$\Rightarrow \quad \frac{AC}{EC} = \frac{CE}{CD}$

$\Rightarrow \quad \frac{AC}{CF - EF} = \frac{CF - EF}{BC - BD}$

$\Rightarrow \quad \frac{R}{R-5} = \frac{R-5}{R-9}$

$\Rightarrow \quad R(R-9) = (R-5)^2 \Rightarrow 0 = -R + 25 \Rightarrow R = 25 \text{ cm}$

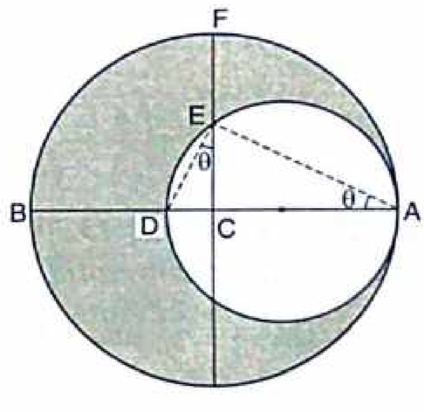

Fig. 13.68

Substituting the value of R in (i), we get

$25 - r = 4.5 \Rightarrow r = 20.5 \text{ cm}$

Hence, the radii of two circles are $r = 20.5$ cm and $R = 25$ cm.

(ii) Let A be the area of the shaded region. Then,

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)(R - r)$$

$$= 3.14(25 + 20.5)(25 - 20.5) \text{ cm}^2 = 3.14 \times 45.5 \times 4.5 \text{ cm}^2 = 642.915 \text{ cm}^2$$

EXAMPLE 38 *In Fig. 13.69, three circles of radius 2 cm touch one another externally. These circle are circumscribed by a circle of radius R cm. Find the value of R and the area of the shaded region in terms of π and $\sqrt{3}$.*

SOLUTION Clearly, ΔABC is an equilateral triangle of side 4 cm.

In ΔBDO, we have

$\cos \angle OBD = \frac{BD}{OB}$

$\Rightarrow \quad \cos 30° = \frac{2}{OB}$ $\quad [\because \angle OBD = 30°]$

$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{2}{OB}$

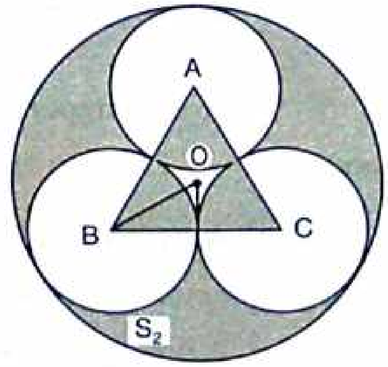

Fig. 13.69

$\Rightarrow \qquad OB = \dfrac{4}{\sqrt{3}}$

$\therefore \qquad OP = OB + BP \Rightarrow R = \left(\dfrac{4}{\sqrt{3}} + 2\right)\text{cm}$

Let A be the area of the shaded region. Then,

A = Area of the larger circle of radius $R - 3 \times$ Area of a smaller circle of radius 2 cm
+ 3 (Area of a sector of angle 60° in a circle of radius 2 cm)
– {Area of ΔABC – 3 (Area of sector of angle 60° in a circle of radius 2 cm)}

$\Rightarrow$ A = Area of the larger circle of radius $R - 3 \times$ Area of a smaller circle of radius 2 cm
+ 6 × Area of a sector of angle 60° in a circle of radius 2 cm – Area of ΔABC

$$\Rightarrow \qquad A = \left\{\pi\left(\frac{4}{\sqrt{3}} + 2\right)^2 - 3 \times \pi \times 2^2 + 6 \times \left(\frac{60}{360} \times \pi \times 2^2\right) - \frac{\sqrt{3}}{4} \times 4^2\right\}\text{cm}^2$$

$$\Rightarrow \qquad A = \left\{\pi\left(\frac{16}{3} + 4 + \frac{16}{\sqrt{3}}\right) - 12\pi + 4\pi - 4\sqrt{3}\right\}\text{cm}^2$$

$$\Rightarrow \qquad A = \left\{\pi\left(\frac{4}{3} + \frac{16}{\sqrt{3}}\right) - 4\sqrt{3}\right\}\text{cm}^2 = \left\{\frac{4\pi}{3}(4\sqrt{3} + 1) - 4\sqrt{3}\right\}\text{cm}^2$$

EXAMPLE 39 *In Fig. 13.70, ABCD is a trapezium with $AB \parallel DC$ and $\angle BCD = 60°$. If BFEC is a sector of a circle with centre C and $AB = BC = 7$ cm and $DE = 4$ cm, then find the area of the shaded region (Use $\pi = \frac{22}{7}$ and $\sqrt{3} = 1.732$).* **[CBSE 2010]**

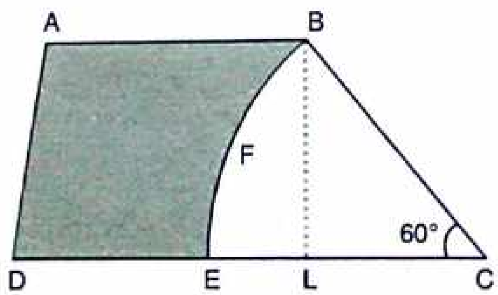

Fig. 13.70

SOLUTION Clearly, $CE = CB = 7$ cm.

$\therefore \qquad CD = CE + ED = (7 + 4)\text{ cm} = 11\text{ cm}$

In ΔCLB, we have

$$\sin 60° = \frac{BL}{BC} \Rightarrow \frac{\sqrt{3}}{2} = \frac{BL}{7} \Rightarrow BL = \frac{7\sqrt{3}}{2}\text{cm}$$

$$\therefore \qquad \text{Area of trapezium} = \frac{1}{2}(AB + CD) \times BL = \frac{1}{2}(7 + 11) \times \frac{7\sqrt{3}}{2}\text{cm}^2 = \frac{63\sqrt{3}}{2}\text{cm}^2$$

and,

$$\text{Area of sector } BFEC = \frac{60°}{360°} \times \frac{22}{7} \times 7^2\text{ cm}^2 = \frac{77}{3}\text{cm}^2$$

Let A be the area of the shaded region. Then,

$$A = \left(\frac{63\sqrt{3}}{2} - \frac{77}{3}\right)\text{cm}^2 = (54.558 - 25.666)\text{ cm}^2 = 28.89\text{ cm}^2$$

EXAMPLE 40 *With vertices A, B and C of a triangle ABC as centres, arcs are drawn with radii 5 cm each as shown in Fig. 13.71. If AB = 14 cm, BC = 48 cm and CA = 50 cm, then find the area of the shaded region. (Use $\pi = 3.14$).*

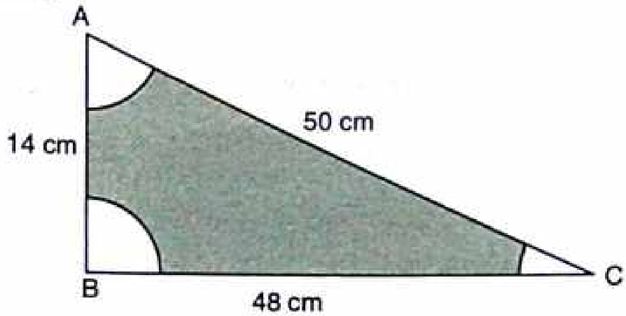

Fig. 13.71

SOLUTION In ΔABC, we have

$$a = BC = 48 \text{ cm}, b = CA = 50 \text{ cm and } c = AB = 14 \text{ cm}$$

Let s be the semi-perimeter of ΔABC. Then,

$$s = \frac{a+b+c}{2} = \frac{48+50+14}{2} = 56 \text{ cm}$$

Let Δ be the area of ΔABC. Then, by Heron's formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{56 \times 8 \times 6 \times 42} \text{ cm}^2 = 336 \text{ cm}^2$$

Let A_1, A_2 and A_3 be the areas of sectors with sector angles A, B and C respectively and sector radius $r = 5$ cm. Then,

$$A_1 = \frac{A}{360} \times \pi r^2 = \frac{A}{360} \times \pi \times 5^2 \text{ cm}^2 = \frac{A}{360} \times 25\pi \text{ cm}^2$$

$$A_2 = \frac{B}{360} \times \pi r^2 = \frac{B}{360} \times \pi \times 5^2 \text{ cm}^2 = \frac{B}{360} \times 25\pi \text{ cm}^2$$

$$A_3 = \frac{C}{360} \times \pi r^2 = \frac{C}{360} \times \pi \times 5^2 \text{ cm}^2 = \frac{C}{360} \times 25\pi \text{ cm}^2$$

$$\therefore \quad A_1 + A_2 + A_3 = \left(\frac{A}{360} \times 25\pi + \frac{B}{360} \times 25\pi + \frac{C}{360} \times 25\pi\right) \text{cm}^2$$

$$= (A+B+C) \times \frac{25\pi}{360} \text{cm}^2$$

$$= \frac{180}{360} \times 25\pi \text{ cm}^2 = \frac{25\pi}{2} \text{cm}^2 = \frac{25 \times 3.14}{2} \text{cm}^2 = 39.25 \text{ cm}^2$$

Let A be the area of the shaded region. Then,

$$A = \text{Area of } \Delta ABC - (A_1 + A_2 + A_3) = (336 - 39.25) \text{ cm}^2 = 296.75 \text{ cm}^2$$

REMARK *The above solution is the general solution. In this case, ΔABC is a right triangle right angled at B. So, its area can also be computed as follows:*

$$\Delta = \frac{1}{2}(BC \times AB) = \frac{1}{2} \times 48 \times 14 \text{ cm}^2 = 336 \text{ cm}^2$$

EXERCISE 13.4

LEVEL-1

1. A plot is in the form of a rectangle $ABCD$ having semi-circle on BC as shown in Fig. 13.72. If $AB = 60$ m and $BC = 28$ m, find the area of the plot.

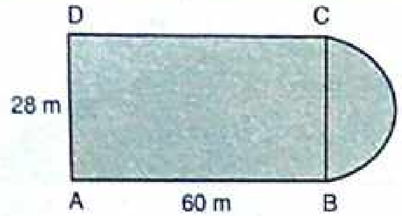

Fig. 13.72

2. A play ground has the shape of a rectangle, with two semi-circles on its smaller sides as diameters, added to its outside. If the sides of the rectangle are 36 m and 24.5 m, find the area of the playground. (Take $\pi = 22/7$).
3. Find the area of the circle in which a square of area 64 cm^2 is inscribed. [Use $\pi = 3.14$]
4. A rectangular piece is 20 m long and 15 m wide. From its four corners, quadrants of radii 3.5 m have been cut. Find the area of the remaining part.
5. In Fig. 13.73, $PQRS$ is a square of side 4 cm. Find the area of the shaded square. **[NCERT]**

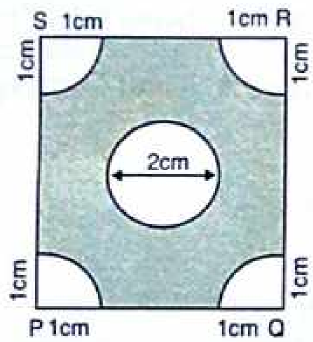

Fig. 13.73

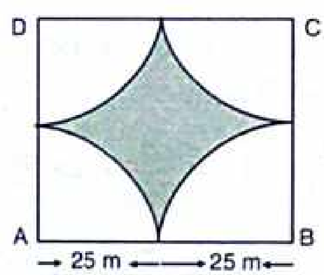

Fig. 13.74

6. Four cows are tethered at four corners of a square plot of side 50 m, so that they just cannot reach one another. What area will be left ungrazed? (Fig. 13.74) **[CBSE 2018]**
7. A cow is tied with a rope of length 14 m at the corner of a rectangular field of dimensions 20 m × 16 m, find the area of the field in which the cow can graze. **[NCERT EXEMPLAR]**
8. A calf is tied with a rope of length 6 m at the corner of a square grassy lawn of side 20 m. If the length of the rope is increased by 5.5 m, find the increase in area of the grassy lawn in which the calf can graze. **[NCERT EXEMPLAR]**
9. A square water tank has its side equal to 40 m. There are four semi-circular grassy plots all round it. Find the cost of turfing the plot at ₹ 1.25 per square metre (Take $\pi = 3.14$).
10. A rectangular park is 100 m by 50 m. It is surrounded by semi-circular flower beds all round. Find the cost of levelling the semi-circular flower beds at 60 paise per square metre (Use $\pi = 3.14$).
11. The inside perimeter of a running track (shown in Fig. 13.75) is 400 m. The length of each of the straight portion is 90 m and the ends are semi-circles. If the track is everywhere 14 m wide, find the area of the track. Also, find the length of the outer running track.

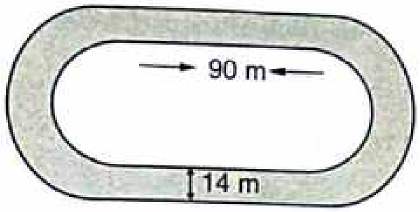

Fig. 13.75

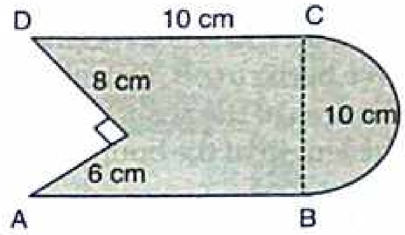

Fig. 13.76

12. Find the area of Fig. 13.76, in square cm, correct to one place of decimal. (Take $\pi = 22/7$).

13. In Fig. 13.77, from a rectangular region $ABCD$ with $AB = 20$ cm, a right triangle AED with $AE = 9$ cm and $DE = 12$ cm, is cut off. On the other end, taking BC as diameter, a semicircle is added on outside the region. Find the area of the shaded region. (Use $\pi = 22/7$). **[CBSE 2014]**

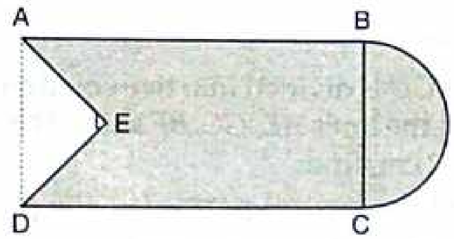

Fig. 13.77

14. From each of the two opposite corners of a square of side 8 cm, a quadrant of a circle of radius 1.4 cm is cut. Another circle of radius 4.2 cm is also cut from the centre as shown in Fig. 13.78. Find the area of the remaining (shaded) portion of the square. (Use $\pi = 22/7$). **[CBSE 2010]**

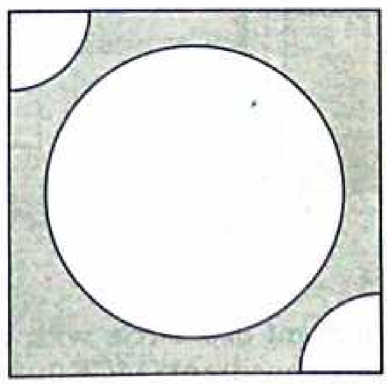

Fig. 13.78

15. In Fig. 13.79, $ABCD$ is a rectangle with $AB = 14$ cm and $BC = 7$ cm. Taking DC, BC and AD as diameters, three semi-circles are drawn as shown in the figure. Find the area of the shaded region.

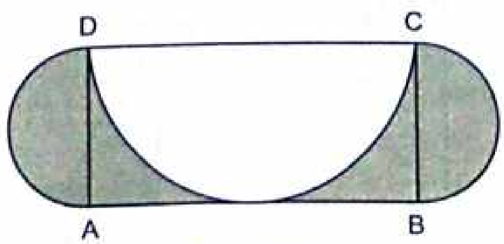

Fig. 13.79

16. In Fig. 13.80, $ABCD$ is a rectangle, having $AB = 20$ cm and $BC = 14$ cm. Two sectors of $180°$ have been cut off. Calculate:
   (i) the area of the shadded region.
   (ii) the length of the boundary of the shaded region.

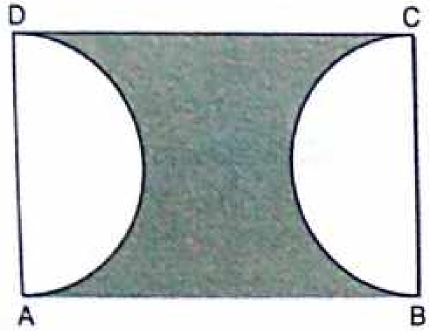

Fig. 13.80

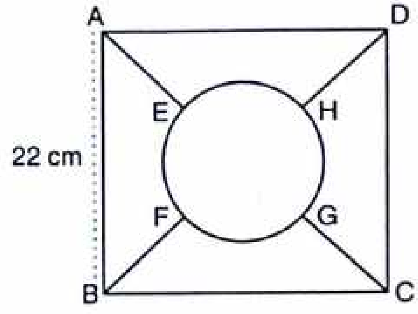

Fig. 13.81

17. In Fig. 13.81, the square $ABCD$ is divided into five equal parts, all having same area. The central part is circular and the lines AE, GC, BF and HD lie along the diagonals AC and BD of the square. If $AB = 22$ cm, find:
   (i) the circumference of the central part. (ii) the perimeter of the part $ABEF$.

18. In Fig. 13.82, find the area of the shaded region. (Use $\pi = 3.14$). **[CBSE 2015]**

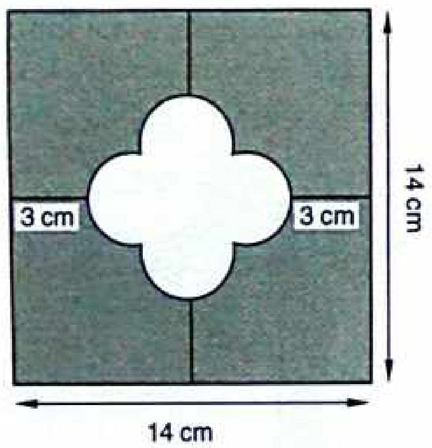

Fig. 13.82

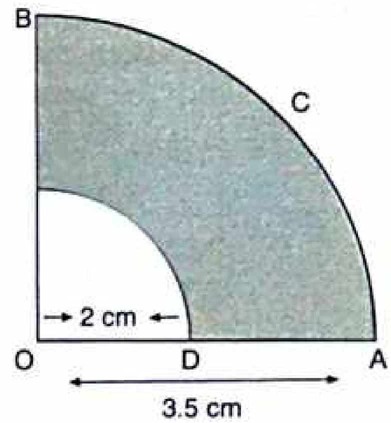

Fig. 13.83

19. In Fig. 13.83, $OACB$ is a quadrant of a circle with centre O and radius 3.5 cm. If $OD = 2$ cm, find the area of the (i) quadrant $OACB$ (ii) shaded region.

20. In Fig. 13.84, a square $OABC$ is inscribed in a quadrant $OPBQ$ of a circle. If $OA = 21$ cm, find the area of the shaded region. **[CBSE 2013, 2014]**

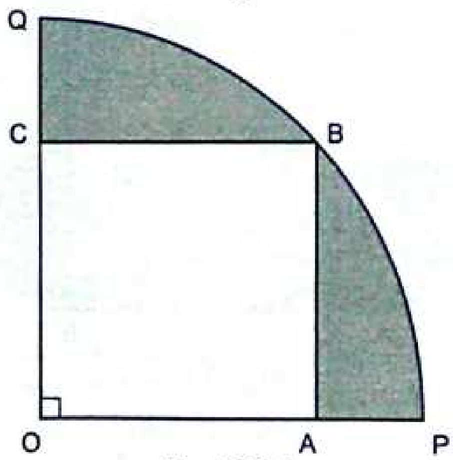

Fig. 13.84

21. In Fig. 13.85, $OABC$ is a square of side 7 cm. If $OAPC$ is a quadrant of a circle with centre O, then find the area of the shaded region. (Use $\pi = 22/7$) **[CBSE 2012]**

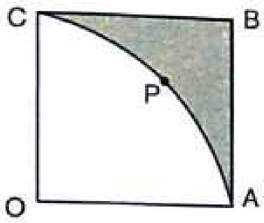

Fig. 13.85

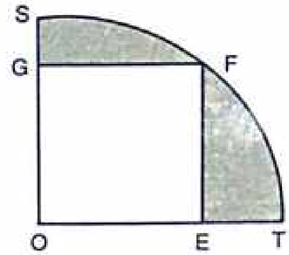

Fig. 13.86

22. In Fig. 13.86, $OE = 20$ cm. In sector $OSFT$, square $OEFG$ is inscribed. Find the area of the shaded region. **[CBSE 2013, 2014]**

23. Find the area of the shaded region in Fig. 13.87, if $AC = 24$ cm, $BC = 10$ cm and O is the centre of the circle. (Use $\pi = 3.14$) **[CBSE 2010]**

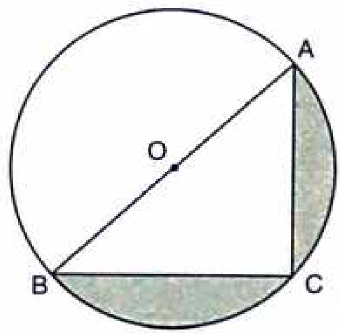

Fig. 13.87

24. A circle is inscribed in an equilateral triangle ABC is side 12 cm, touching its sides (Fig. 13.88). Find the radius of the inscribed circle and the area of the shaded part. **[CBSE 2014]**

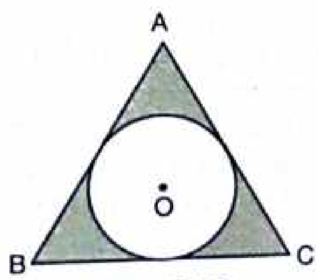

Fig. 13.88

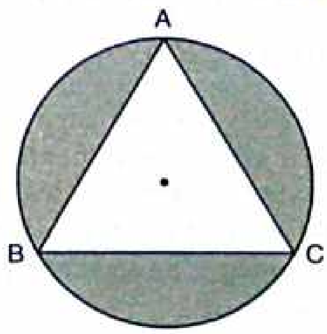

Fig. 13.89

25. In Fig. 13.89, an equilateral triangle ABC of side 6 cm has been inscribed in a circle. Find the area of the shaded region. (Take $\pi = 3.14$).

26. A circular field has a perimeter of 650 m. A square plot having its vertices on the circumference of the field is marked in the field. Calculate the area of the square plot.

27. Find the area of a shaded region in the Fig. 13.90, where a circular arc of radius 7 cm has been drawn with vertex A of an equilateral triangle ABC of side 14 cm as centre. (Use $\pi = 22/7$ and $\sqrt{3} = 1.73$) **[CBSE 2015, 2016]**

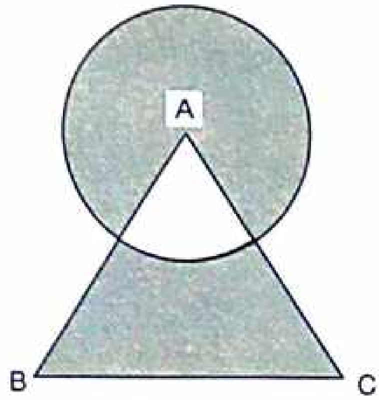

Fig. 13.90

28. A regular hexagon is inscribed in a circle. If the area of hexagon is $24\sqrt{3}$ cm^2, find the area of the circle. (Use $\pi = 3.14$) **[CBSE 2015]**

29. *ABCDEF* is a regular hexagon with centre *O* (Fig. 13.91). If the area of triangle *OAB* is 9 cm^2, find the area of: (i) the hexagon and (ii) the circle in which the haxagon is incribed.

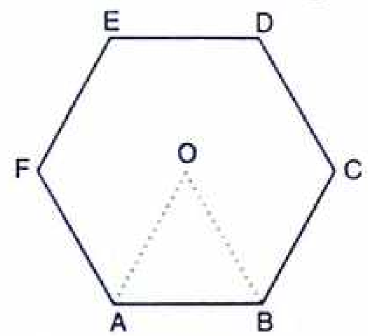

Fig. 13.91

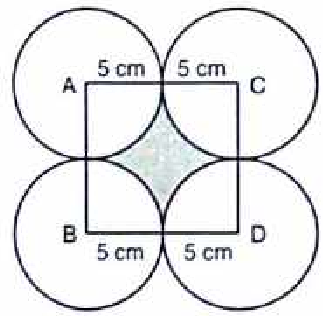

Fig. 13.92

30. Four equal circles, each of radius 5 cm, touch each other as shown in Fig. 13.92. Find the area included between them (Take $\pi = 3.14$).

31. Four equal circles, each of radius a, touch each other. Show that the area between them is $\frac{6}{7}a^2$. (Take $\pi = 22/7$).

32. A child makes a poster on a chart paper drawing a square *ABCD* of side 14 cm. She draws four circles with centre *A, B, C* and *D* in which she suggests different ways to save energy. The circles are drawn in such a way that each circle touches externally two of the three remaining circles (Fig. 13.93). In the shaded region she write a message 'Save Energy'. Find the perimeter and area of the shaded region. (Use $\pi = 22/7$) **[CBSE 2015]**

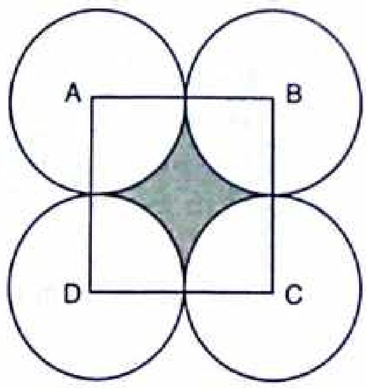

Fig. 13.93

33. The diameter of a coin is 1 cm (Fig. 13.94). If four such coins be placed on a table so that the rim of each touches that of the other two, find the area of the shaded region (Take $\pi = 3.1416$).

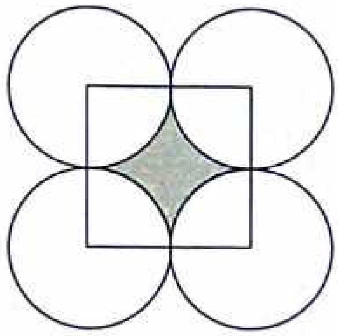

Fig. 13.94

34. Two circular pieces of equal radii and maximum area, touching each other are cut out from a rectangular card board of dimensions 14 cm × 7 cm. Find the area of the remaining card board. (Use $\pi = 22/7$) **[CBSE 2013]**

35. In Fig. 13.95, AB and CD are two diameters of a circle perpendicular to each other and OD is the diameter of the smaller circle. If $OA = 7$ cm, find the area of the shaded region.

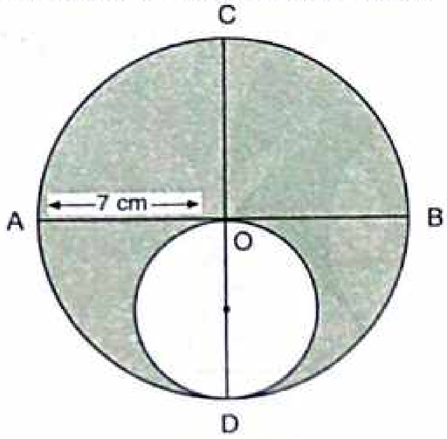

Fig. 13.95

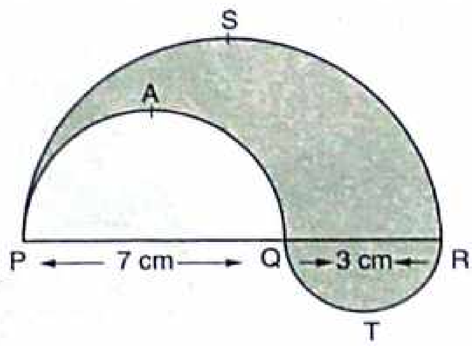

Fig. 13.96

36. In Fig. 13.96, PSR, RTQ and PAQ are three semi-circles of diameters 10 cm, 3 cm and 7 cm respectively. Find the perimeter of the shaded region. **[CBSE 2014]**

37. In Fig. 13.97, two circles with centres A and B touch each other at the point C. If $AC = 8$ cm and $AB = 3$ cm, find the area of the shaded region.

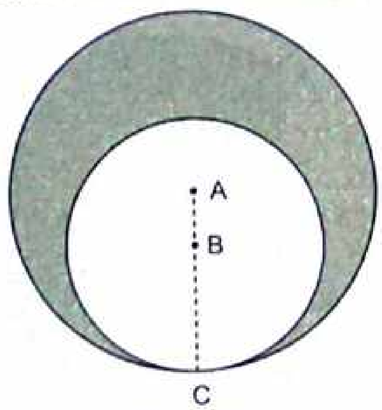

Fig. 13.97

38. In Fig. 13.98, $ABCD$ is a square of side $2a$. Find the ratio between
   (i) the circumferences
   (ii) the areas of the incircle and the circum-circle of the square.

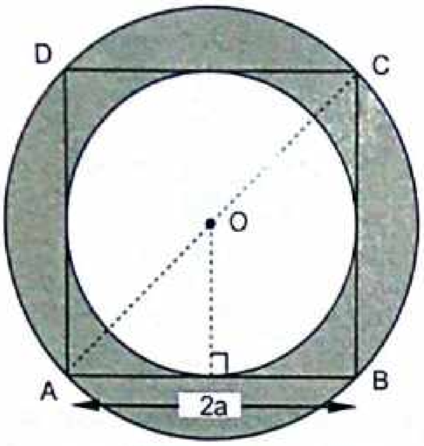

Fig. 13.98

39. In Fig. 13.99, there are three semicircles, A, B and C having diameter 3 cm each, and another semicircle E having a circle D with diameter 4.5 cm are shown. Calculate:
   (i) the area of the shaded region
   (ii) the cost of painting the shaded region at the rate of 25 paise per cm^2, to the nearest rupee.

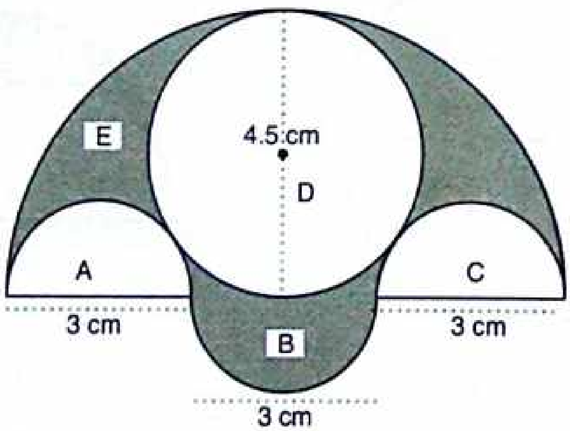

Fig. 13.99

40. In Fig. 13.100, ABC is a right-angled triangle, $\angle B = 90°$, $AB = 28$ cm and $BC = 21$ cm. With AC as diameter a semicircle is drawn and with BC as radius a quarter circle is drawn. Find the area of the shaded region correct to two decimal places.

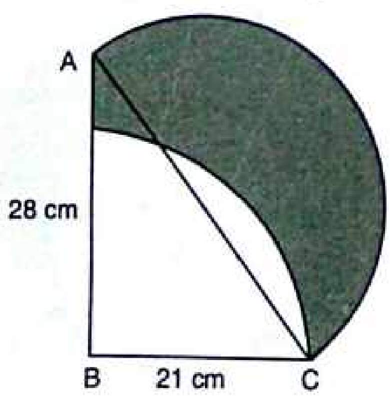

Fig. 13.100

41. In Fig. 13.101, O is the centre of a circular arc and AOB is a straight line. Find the perimeter and the area of the shaded region correct to one decimal place. (Take $\pi = 3.142$)

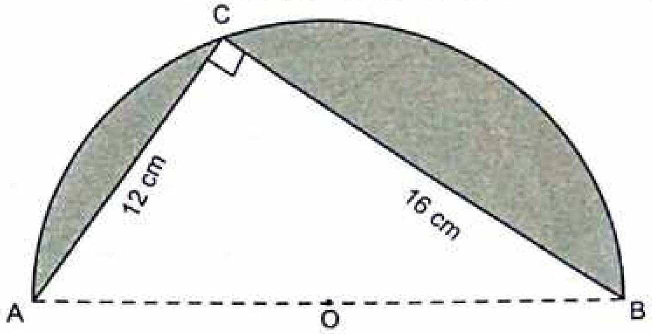

Fig. 13.101

42. In Fig. 13.102, the boundary of the shaded region consists of four semi-circular arcs, the smallest two being equal. If the diameter of the largest is 14 cm and of the smallest is 3.5 cm, find

(i) the length of the boundary. (ii) the area of the shaded region.

[CBSE 2010, 2016]

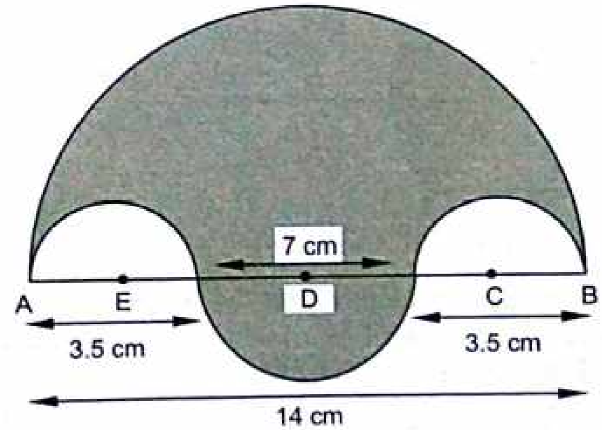

Fig. 13.102

43. In Fig. 13.103, $AB = 36$ cm and M is mid-point of AB. Semi-circles are drawn on AB, AM and MB as diameters. A circle with centre C touches all the three circles. Find the area of the shaded region.

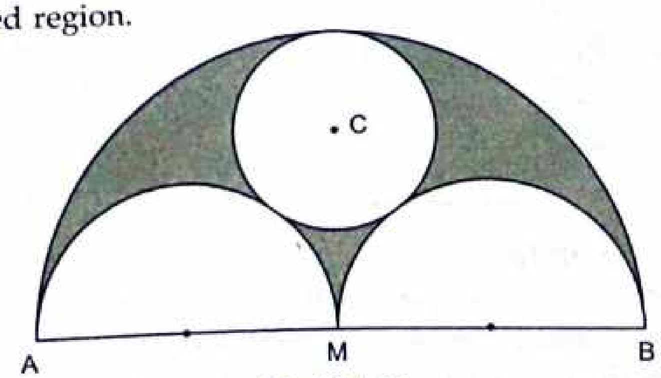

Fig. 13.103

44. In Fig. 13.104, ABC is a right angled triangle in which $\angle A = 90°$, $AB = 21$ cm and $AC = 28$ cm. Semi-circles are described on AB, BC and AC as diameters. Find the area of the shaded region.

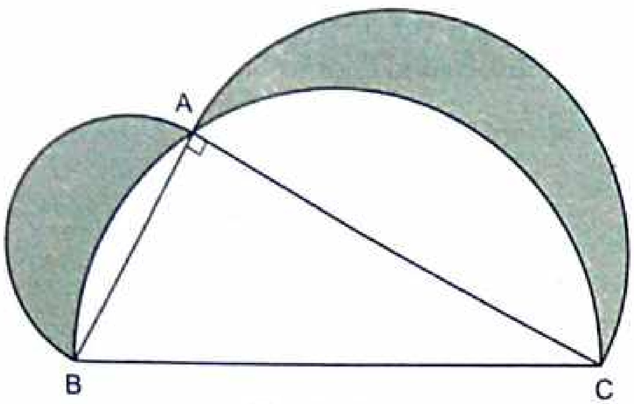

Fig. 13.104

45. Figure 13.105, shows the cross-section of railway tunnel. The radius OA of the circular part is 2 m. If $\angle AOB = 90°$, calculate:

(i) the height of the tunnel

(ii) the perimeter of the cross-section

(iii) the area of the cross-section.

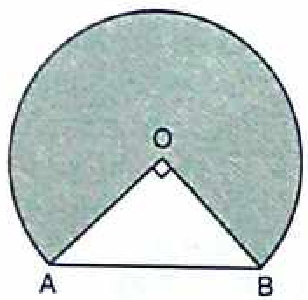

Fig. 13.105

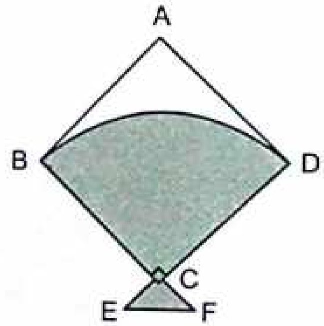

Fig. 13.106

46. Figure 13.106., shows a kite in which BCD is the shape of a quadrant of a circle of radius 42 cm. $ABCD$ is a square and $\Delta\, CEF$ is an isosceles right angled triangle whose equal sides are 6 cm long. Find the area of the shaded region.

47. In Fig. 13.107, $ABCD$ is a trapezium of area 24.5 cm^2. In it, $AD \parallel BC$, $\angle DAB = 90°$, $AD = 10$ cm and $BC = 4$ cm. If ABE is a quadrant of a circle, find the area of the shaded region. (Take $\pi = 22/7$). **[CBSE 2014]**

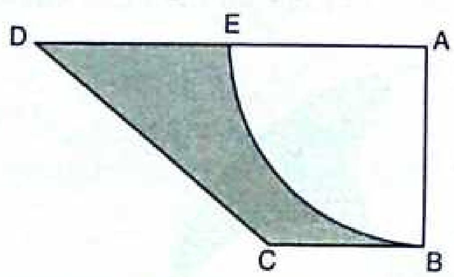

Fig. 13.107

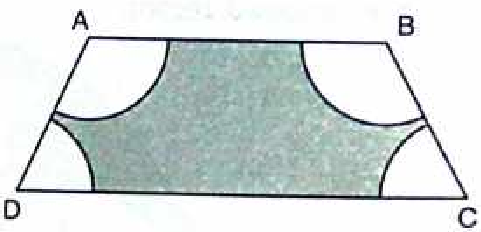

Fig. 13.108

48. In Fig. 13.108, $ABCD$ is a trapezium with $AB \parallel DC$, $AB = 18$ cm, $DC = 32$ cm and the distance between AB and DC is 14 cm. Circles of equal radii 7 cm with centres A, B, C and D have been drawn. Then, find the area of the shaded region of the figure. (Use $\pi = 22/7$). **[CBSE 2014]**

49. From a thin metallic piece, in the shape of a trapezium $ABCD$, in which $AB \parallel CD$ and $\angle BCD = 90°$, a quarter circle $BEFC$ is removed (see Fig. 13.109). Given $AB = BC = 3.5$ cm and $DE = 2$ cm, calculate the area of the remaining piece of the metal sheet.

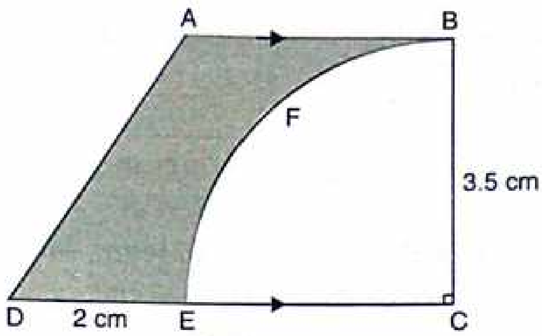

Fig. 13.109

50. In Fig. 13.110, ABC is an equilateral triangle of side 8 cm. A, B and C are the centres of circular arcs of radius 4 cm. Find the area of the shaded region correct upto 2 decimal places. (Take $\pi = 3.142$ and $\sqrt{3} = 1.732$).

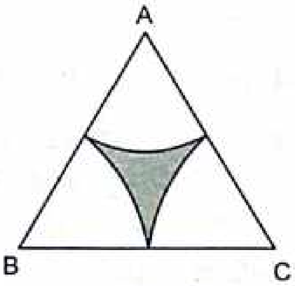

Fig. 13.110

51. Sides of a triangular field are 15 m, 16 m and 17 m. With the three corners of the field a cow, a buffalo and a horse are tied separately with ropes of length 7 m each to graze in the field. Find the area of the field which cannot be grazed by three animals.

[NCERT EXEMPLAR]

52. In the given Fig. 13.111, the side of square is 28 cm, and radius of each circle is half of the length of the side of the square where O and O' are centres of the circles. Find the area of shaded region. **[CBSE 2017]**

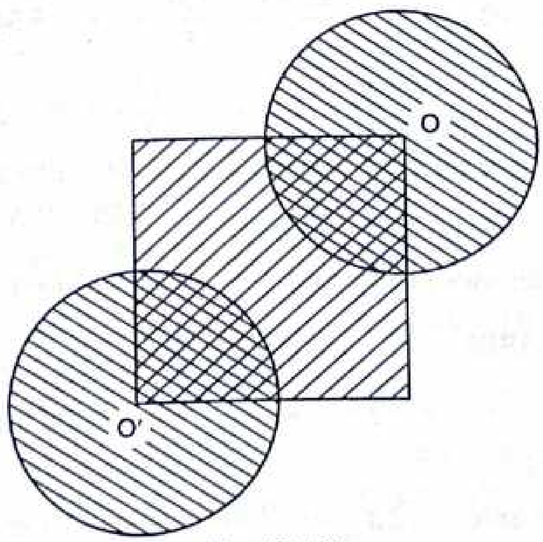

Fig. 13.111

53. In a hospital used water is collected in a cylindrical tank of diameter 2 m and height 5 m. After recycling, this water is used to irrigate a park of hospital whose length is 25 m and

breadth is 20 m. If tank is filled completely then what will be the height of standing water used for irrigating the park? **[CBSE 2017]**

ANSWERS

1. $1988\ m^2$
2. $1353.625\ m^2$
3. $100.48\ cm^2$
4. $261.5\ m^2$
5. $(16 - 2\pi)\ cm^2$
6. $535.71\ m^2$
7. $154\ m^2$
8. $75.625\ m^2$
9. ₹ 3140
10. ₹ 5887.50
11. $6216\ m^2$, 488 m
12. $115.3\ cm^2$
13. $334.3125\ cm^2$
14. $5.48\ cm^2$
15. $59.5\ cm^2$
16. (i) $126\ cm^2$ (ii) 84 cm
17. (i) 34.88 cm (ii) 50.64 cm
18. $154.88\ cm^2$
19. (i) $9.625\ cm^2$ (ii) $6.482\ cm^2$
20. $252\ cm^2$
21. $10.5\ cm^2$
22. $228\ cm^2$
23. $145.33\ cm^2$
24. $2\sqrt{3}$ cm, $24.638\ cm^2$
25. $22.126\ cm^2$
26. $21387\ m^2$
27. $187.44\ cm^2$
28. $50.24\ cm^2$
29. (i) 54 cm (ii) $65.23\ cm^2$
30. $21.5\ cm^2$
31. $\frac{6}{7}a^2$
32. Area = $42\ cm^2$, Perimeter = 44 cm
33. $0.2146\ cm^2$
34. $21\ cm^2$
35. $115.5\ cm^2$
36. 31.4 cm
37. $122.57\ cm^2$
38. (i) $1 : \sqrt{2}$ (ii) $1 : 2$
39. (i) $12.375\ cm^2$ (ii) ₹ 3
40. $428.75\ cm^2$
41. 59.4 cm, $61.1\ cm^2$
42. (i) 44 cm (ii) $86.625\ cm^2$
43. $45\pi\ cm^2$
44. $294\ cm^2$
45. (i) $(2 + \sqrt{2})$ m (ii) $(3\pi + 2\sqrt{2})$ m (iii) $(3\pi + 2)\ m^2$
46. $1404\ cm^2$
47. $14.875\ cm^2$
48. $196\ cm^2$
49. $6.125\ cm^2$
50. $2.576\ cm^2$
51. $(24\sqrt{21} - 77)\ m^2$
52. $3688\ cm^2$
53. π cm

HINTS TO SELECTED PROBLEMS

17. Let the radius of the central part be r cm. Then,

Area of the central part $= \frac{1}{5} \times$ Area of the square

$$\Rightarrow \frac{22}{7} \times r^2 = \frac{1}{5} \times 22 \times 22 \Rightarrow r^2 = \frac{22 \times 7}{5} = \frac{154}{5} \Rightarrow r = 5.549 \simeq 5.55 \text{ cm}$$

(i) Circumference of central part $= 2\pi r = 2 \times \frac{22}{7} \times 5.55 = 34.88$ cm

(ii) Let O be the centre of the central part. Clearly, O is also the centre of the square.

$$AE = BF = OA - OE = 11\sqrt{2} - 5.55 = 15.51 - 5.55 = 9.96 \text{ cm}$$

$$EF = \frac{1}{4}(\text{Circumference of the circle}) = \frac{1}{4}(2\pi r) = \frac{1}{2}(\pi r) = \frac{1}{2} \times \frac{22}{7} \times 5.55 = 8.72 \text{ cm}$$

$\therefore$ Perimeter of part $ABEF = AB + AE + EF + BF = 22 + 2 \times 9.96 + 8.72$ cm $= 50.64$ cm

37. Required area $= (\pi \times 8^2 - \pi \times 5^2) = 39\pi\ cm^2 = 122.57\ cm^2$

38. $AC = \sqrt{2} \times 2a = 2\sqrt{2}\,a$

$\therefore$ Radius of larger circle $= \sqrt{2}\,a$ and, Radius of smaller circle $= a$

(i) Ratio of circumferences $= 2\pi a : 2\pi\sqrt{2}\,a = 1 : \sqrt{2}$

(ii) Ratio of area's $= \pi a^2 : \pi(\sqrt{2}a)^2 = 1 : 2$

41. Area of the shaded region = Area of semi-cricle with AB as diameter − Area of ΔABC

$$= \left(\frac{1}{2} \times \pi \times 10^2 - \frac{1}{2} \times 12 \times 16\right) \text{cm}^2 = 61.1 \text{ cm}^2$$

Perimeter of the shaded region $= (\pi \times 10 + 12 + 16)$ cm $= 59.4$ cm

42. (i) Length of the boundary $= \left\{\pi \times 7 + \pi \times \frac{7}{2} + \pi\left(\frac{7}{4}\right) + \pi\left(\frac{7}{4}\right)\right\}$ cm $= 14\pi$ cm $= 44$ cm

(ii) Area of the shaded region $= \frac{\pi}{2} \times 7^2 + \frac{\pi}{2} \times \left(\frac{7}{2}\right)^2 - \frac{\pi}{2} \times \left(\frac{7}{4}\right)^2 - \frac{\pi}{2} \times \left(\frac{7}{4}\right)^2$

$$= \frac{\pi}{2} \times 7^2\left(1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{16}\right) = 86.625 \text{ cm}^2$$

43. Radius of circle with C as centre $= \frac{1}{6}AB = 6$ cm

$\therefore$ Area of the shaded region $= \frac{1}{2}\pi \times 18^2 - 2\left(\frac{1}{2} \times \pi \times 9^2\right) - \pi \times 6^2 = 45\pi \text{ cm}^2$

44. Area of the shaded region = Area of semi-circle with AB as diamater

+ Area of semi-circle with AC as diameter + Area of ΔABC

− Area of semi-circle with BC as diameter

$$= \frac{\pi}{2}\left\{\left(\frac{21}{2}\right)^2 + \left(\frac{28}{2}\right)^2\right\} + \frac{1}{2} \times 21 \times 28 - \frac{\pi}{2} \times \left(\frac{35}{2}\right)^2 = 294 \text{ cm}^2$$

46. Area of shaded region = Area of quadrant BCD + Area of ΔEFC

$$= \frac{1}{4} \times \frac{22}{7} \times 42^2 + \frac{1}{2} \times 6 \times 6 \text{ cm}^2 = 1404 \text{ cm}^2$$

50. Area of shaded region = Area of ΔABC − Area of 3 sectors of sector angle $60°$

= Area of ΔABC − Area of semi-circle of radius 4 cm

$$= \left(\frac{\sqrt{3}}{4} \times 8^2 - \frac{1}{2} \times 3.142 \times 4^2\right) \text{cm}^2 = (27.712 - 25.136) \text{ cm}^2 = 2.576 \text{ cm}^2$$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions either in one word or one sentence or as per requirement of the questions:

1. What is the ratio of the areas of a circle and an equilateral triangle whose diameter and a side are respectively equal?
2. If the circumference of two circles are in the ratio 2 : 3, what is the ratio of their areas?
3. Write the area of the sector of a circle whose radius is r and length of the arc is l.
4. What is the length (in terms of π) of the arc that subtends an angle of $36°$ at the centre of a circle of radius 5 cm?
5. What is the angle subtended at the centre of a circle of radius 6 cm by an arc of length 3 π cm?

6. What is the area of a sector of a circle of radius 5 cm formed by an arc of length 3.5 cm?
7. In a circle of radius 10 cm, an arc subtends an angle of 108° at the centre. What is the area of the sector in terms of π?
8. If a square is inscribed in a circle, what is the ratio of the areas of the circle and the square?
9. Write the formula for the area of a sector of angle θ (in degrees) of a circle of radius r.
10. Write the formula for the area of a segment in a circle a circle of radius r given that the sector angle is θ (in degrees).
11. If the adjoining figure is a sector of a circle of radius 10.5 cm, what is the perimeter of the sector? (Take $\pi = 22/7$)

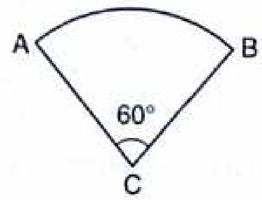

Fig. 13.112

12. If the diameter of a semi-circular protractor is 14 cm, then find its perimeter. **[CBSE 2009]**
13. An arc subtends an angle of 90° at the centre of the circle of radius 14 cm. Write the area of minor sector thus formed in terms of π.
14. Find the area of the largest triangle that can be inscribed in a semi-circle of radius r units. **[CBSE 2015]**
15. Find the area of a sector of circle of radius 21 cm and central angle 120°.
16. What is the area of a square inscribed in a circle of diameter p cm?
17. Is it true to say that area of a segment of a circle is less than the area of its corresponding sector? Why?
18. If the numerical value of the area of a circle is equal to the numerical value of its circumference, find its radius.
19. How many revolutions a circular wheel of radius r metres makes in covering a distance of s metres?
20. Find the ratio of the area of the circle circumscribing a square to the area of the circle inscribed in the square.

ANSWERS

1. $4\pi : \sqrt{3}$ 2. $4:9$ 3. $\frac{1}{2}lr$ 4. π cm 5. 90° 6. 8.75 cm^2 7. 30π cm^2

8. $\pi : 2$ 9. $\frac{\theta}{360} \times \pi r^2$ 10. $\left(\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) r^2$ 11. 32 cm 12. 36 cm

13. 49π cm^2 14. r^2 15. 462 cm^2 16. $\frac{p^2}{2}$ cm^2

17. No; it is only true for minor segment. 18. 2 units 19. $\frac{s}{2\pi r}$

20. 2 : 1

MULTIPLE CHOICE QUESTIONS (MCQs)

1. If the circumference and the area of a circle are numerically equal, then diameter of the circle is

(a) $\frac{\pi}{2}$ (b) 2π (c) 2 (d) 4

2. If the difference between the circumference and radius of a circle is 37 cm., then using $\pi = \frac{22}{7}$, the circumference (in cm) of the circle is

(a) 154 (b) 44 (c) 14 (d) 7 [CBSE 2013]

3. A wire can be bent in the form of a circle of radius 56 cm. If it is bent in the form of a square, then its area will be

(a) 3520 cm^2 (b) 6400 cm^2 (c) 7744 cm^2 (d) 8800 cm^2

4. If a wire is bent into the shape of a square, then the area of the square is 81 cm^2. When wire is bent into a semi-circular shape, then the area of the semi-circle will be

(a) 22 cm^2 (b) 44 cm^2 (c) 77 cm^2 (d) 154 cm^2

5. A circular park has a path of uniform width around it. The difference between the outer and inner circumferences of the circular path is 132 m. Its width is

(a) 20 m (b) 21 m (c) 22 m (d) 24 m

6. The radius of a wheel is 0.25 m. The number of revolutions it will make to travel a distance of 11 km will be

(a) 2800 (b) 4000 (c) 5500 (d) 7000

7. The ratio of the outer and inner perimeters of a circular path is 23 : 22. If the path is 5 metres wide, the diameter of the inner circle is

(a) 55 m (b) 110 m (c) 220 m (d) 230 m

8. The circumference of a circle is 100 cm. The side of a square inscribed in the circle is

(a) $50\sqrt{2}$ cm (b) $\frac{100}{\pi}$ cm (c) $\frac{50\sqrt{2}}{\pi}$ cm (d) $\frac{100\sqrt{2}}{\pi}$ cm

9. The area of the incircle of an equilateral triangle of side 42 cm is

(a) $22\sqrt{3}$ cm^2 (b) 231 cm^2 (c) 462 cm^2 (d) 924 cm^2

10. The area of incircle of an equilateral triangle is 154 cm^2. The perimeter of the triangle is

(a) 71.5 cm (b) 71.7 cm (c) 72.3 cm (d) 72.7 cm

11. The area of the largest triangle that can be inscribed in a semi-circle of radius r, is

(a) r^2 (b) $2r^2$ (c) r^3 (d) $2r^3$

12. The perimeter of a triangle is 30 cm and the circumference of its incircle is 88 cm. The area of the triangle is

(a) 70 cm^2 (b) 140 cm^2 (c) 210 cm^2 (d) 420 cm^2

13. The area of a circle is 220 cm^2. The area of a square inscribed in it is

(a) 49 cm^2 (b) 70 cm^2 (c) 140 cm^2 (d) 150 cm^2

14. If the circumference of a circle increases from 4π to 8π, then its area is
(a) halved (b) doubled (c) tripled (d) quadrupled

15. If the radius of a circle is diminished by 10%, then its area is diminished by
(a) 10% (b) 19% (c) 20% (d) 36%

16. If the area of a square is same as the area of a circle, then the ratio of their perimeters, in terms of π, is
(a) $\pi : \sqrt{3}$ (b) $2 : \sqrt{\pi}$ (c) $3 : \pi$ (d) $\pi : \sqrt{2}$

17. The area of the largest triangle that can be inscribed in a semi-circle of radius r is
(a) $2r$ (b) r^2 (c) r (d) $\sqrt{r}$

18. The ratio of the areas of a circle and an equilateral triangle whose diameter and a side are respectively equal, is
(a) $\pi : \sqrt{2}$ (b) $\pi : \sqrt{3}$ (c) $\sqrt{3} : \pi$ (d) $\sqrt{2} : \pi$

19. If the sum of the areas of two circles with radii r_1 and r_2 is equal to the area of a circle of radius r, then $r_1^2 + r_2^2$
(a) $>r^2$ (b) $=r^2$ (c) $<r^2$ (d) None of these

20. If the perimeter of a semi-circular protractor is 36 cm, then its diameter is
(a) 10 cm (b) 12 cm (c) 14 cm (d) 16 cm

21. The perimeter of the sector OAB shown in Fig. 13.113, is
(a) $\frac{64}{3}$ cm (b) 26 cm (c) $\frac{64}{5}$ cm (d) 19 cm

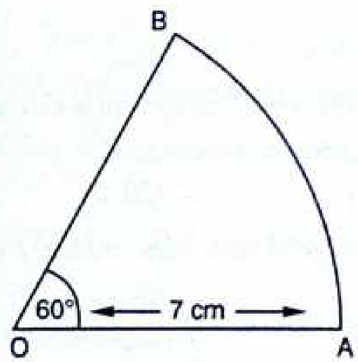

Fig. 13.113

22. If the perimeter of a sector of a circle of radius 6.5 cm is 29 cm, then its area is
(a) 58 cm^2 (b) 52 cm^2 (c) 25 cm^2 (d) 56 cm^2

23. If the area of a sector of a circle bounded by an arc of length 5π cm is equal to 20π cm^2, then its radius is
(a) 12 cm (b) 16 cm (c) 8 cm (d) 10 cm

24. The area of the circle that can be inscribed in a square of side 10 cm is
(a) 40π cm^2 (b) 30π cm^2 (c) 100π cm^2 (d) 25π cm^2

25. If the difference between the circumference and radius of a circle is 37 cm, then its area is
(a) 154 cm^2 (b) 160 cm^2 (c) 200 cm^2 (d) 150 cm^2

26. The area of a circular path of uniform width h surrounding a circular region of radius r is

(a) $\pi(2r+h)r$ (b) $\pi(2r+h)h$ (c) $\pi(h+r)r$ (d) $\pi(h+r)h$

27. If AB is a chord of length $5\sqrt{3}$ cm of a circle with centre O and radius 5 cm, then area of sector OAB is

(a) $\frac{3\pi}{8}\text{cm}^2$ (b) $\frac{8\pi}{3}\text{cm}^2$ (c) $25\pi\text{ cm}^2$ (d) $\frac{25\pi}{3}\text{cm}^2$

28. The area of a circle whose area and circumference are numerically equal, is

(a) 2π sq. units (b) 4π sq. units (c) 6π sq. units (d) 8π sq. units

29. If diameter of a circle is increased by 40%, then its area increases by

(a) 96% (b) 40% (c) 80% (d) 48%

30. In Fig. 13.114, the shaded area is

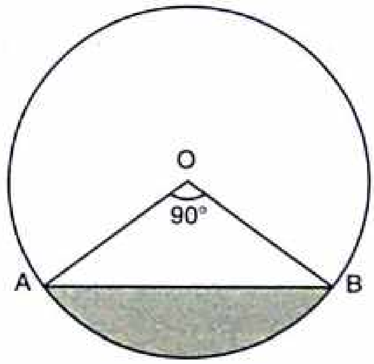

Fig. 13.114

(a) $50(\pi-2)\text{ cm}^2$ (b) $25(\pi-2)\text{ cm}^2$ (c) $25(\pi+2)\text{ cm}^2$ (d) $5(\pi-2)\text{ cm}^2$

31. In Fig. 13.115, the area of the segment PAQ is

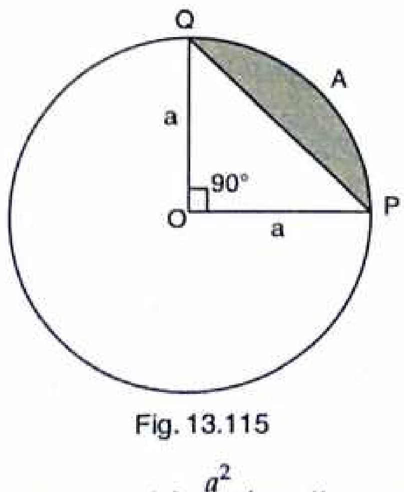

Fig. 13.115

(a) $\frac{a^2}{4}(\pi+2)$ (b) $\frac{a^2}{4}(\pi-2)$ (c) $\frac{a^2}{4}(\pi-1)$ (d) $\frac{a^2}{4}(\pi+1)$

32. In Fig. 13.116, the area of segment ACB is

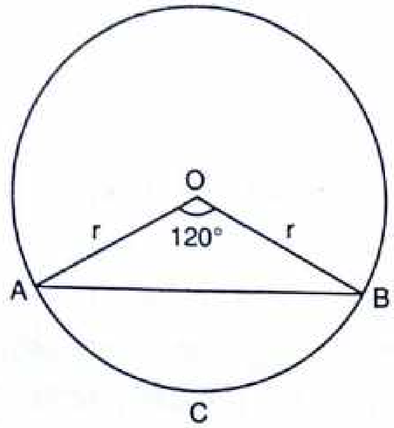

Fig. 13.116

(a) $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right) r^2$ (b) $\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right) r^2$ (c) $\left(\frac{\pi}{3}-\frac{2}{\sqrt{3}}\right) r^2$ (d) None of these

33. If the area of a sector of a circle bounded by an arc of length 5π cm is equal to $20\pi\ \text{cm}^2$, then the radius of the circle is
(a) 12 cm (b) 16 cm (c) 8 cm (d) 10 cm

34. In Fig. 13.117, the ratio of the areas of two sectors S_1 and S_2 is

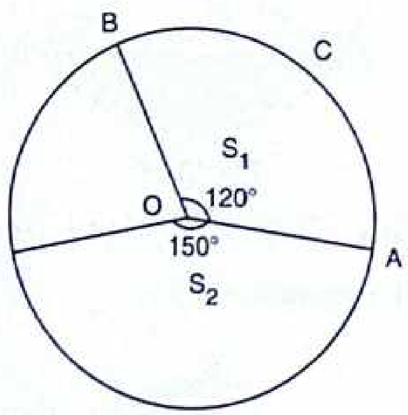

Fig. 13.117

(a) 5 : 2 (b) 3 : 5 (c) 5 : 3 (d) 4 : 5

35. If the area of a sector of a circle is $\frac{5}{18}$ of the area of the circle, then the sector angle is equal to
(a) 60° (b) 90° (c) 100° (d) 120°

36. If the area of a sector of a circle is $\frac{7}{20}$ of the area of the circle, then the sector angle is equal to
(a) 110° (b) 130° (c) 100° (d) 126°

37. In Fig. 13.118, if ABC is an equilateral triangle, then shaded area is equal to

(a) $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) r^2$ (b) $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right) r^2$ (c) $\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) r^2$ (d) $\left(\frac{\pi}{3}+\sqrt{3}\right) r^2$

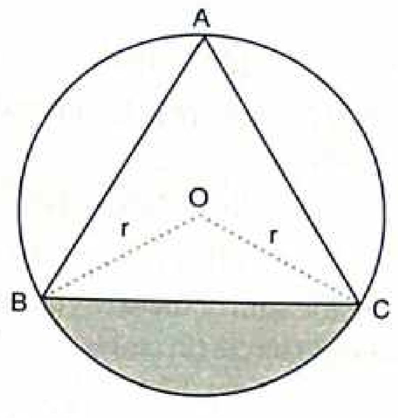

Fig. 13.118

38. In Fig. 13.119, the area of the shaded region is

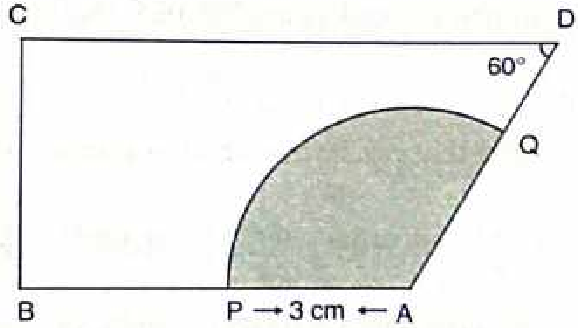

Fig. 13.119

(a) $3\pi\ cm^2$ (b) $6\pi\ cm^2$ (c) $9\pi\ cm^2$ (d) $7\pi\ cm^2$

39. If the perimeter of a circle is equal to that of a square, then the ratio of their areas is
(a) 13 : 22 (b) 14 : 11 (c) 22 : 13 (d) 11: 14

40. The radius of a circle is 20 cm. It is divided into four parts of equal area by drawing three concentric circles inside it. Then, the radius of the largest of three concentric circles drawn is
(a) $10\sqrt{5}$ cm (b) $10\sqrt{3}$ cm (c) 10 cm (d) $10\sqrt{2}$ cm

41. The area of a sector whose perimeter is four times its radius r units, is

(a) $\frac{r^2}{4}$ sq. units (b) $2r^2$ sq. units (c) r^2 sq. units (d) $\frac{r^2}{2}$ sq. units

42. If a chord of a circle of radius 28 cm makes an angle of $90°$ at the centre, then the area of the major segment is
(a) $392\ cm^2$ (b) $1456\ cm^2$ (c) $1848\ cm^2$ (d) $2240\ cm^2$

43. If area of a circle inscribed in an equilateral triangle is 48π square units, then perimeter of the triangle is
(a) $17\sqrt{3}$ units (b) 36 units (c) 72 units (d) $48\sqrt{3}$ units

44. The hour hand of a clock is 6 cm long. The area swept by it between 11.20 am and 11.55 am is

(a) 2.75 cm^2 (b) 5.5 cm^2 (c) 11 cm^2 (d) 10 cm^2

45. $ABCD$ is a square of side 4 cm. If E is a point in the interior of the square such that ΔCED is equilateral, then area of ΔACE is

(a) $2(\sqrt{3}-1)\ cm^2$ (b) $4(\sqrt{3}-1)\ cm^2$

(c) $6(\sqrt{3}-1)\ cm^2$ (d) $8(\sqrt{3}-1)\ cm^2$

46. If the area of a circle is equal to the sum of the areas of two circles of diameters 10 cm and 24 cm, then diameter of the larger circle (in cm) is

(a) 34 (b) 26 (c) 17 (d) 14 **[CBSE 2012]**

47. If π is taken as 22/7, the distance (in metres) covered by a wheel of diameter 35 cm, in one revolution, is

(a) 2.2 (b) 1.1 (c) 9.625 (d) 96.25 **[CBSE 2013]**

48. $ABCD$ is a rectangle whose three vertices are $B(4,0)$, $C(4,3)$ and $D(0,3)$. The length of one of its diagonals is

(a) 5 (b) 4 (c) 3 (d) 25 **[CBSE 2014]**

49. Area of the largest triangle that can be inscribed in a semi-circle of radius r units is

(a) r^2 sq. units (b) $\frac{1}{2}r^2$ sq. units (c) $2r^2$ sq. units (d) $\sqrt{2}r^2$ sq. units

50. If the sum of the areas of two circles with radii r_1 and r_2 is equal to the area of a circle of radius r, then

(a) $r = r_1 + r_2$ (b) $r_1^2 + r_2^2 = r^2$ (c) $r_1 + r_2 < r$ (d) $r_1^2 + r_2^2 < r^2$

51. If the sum of the circumferences of two circles with radii r_1 and r_2 is equal to the circumference of a circle of radius r, then

(a) $r = r_1 + r_2$ (b) $r_1 + r_2 > r$ (c) $r_1 + r_2 < r$ (d) None of these

52. If the circumference of a circle and the perimeter of a square are equal, then

(a) Area of the circle = Area of the square (b) Area of the circle < Area of the square

(c) Area of the circle > Area of the square (d) Nothing definite can be said

53. If the perimeter of a circle is equal to that of a square, then the ratio of their areas is

(a) 22 : 7 (b) 14 : 11 (c) 7 : 22 (d) 11 : 14

ANSWERS

1. (d)	2. (b)	3. (c)	4. (c)	5. (b)
6. (d)	7. (c)	8. (c)	9. (c)	10. (d)
11. (a)	12. (c)	13. (c)	14. (d)	15. (b)
16. (d)	17. (b)	18. (c)	19. (b)	20. (c)
21. (a)	22. (b)	23. (c)	24. (d)	25. (a)
26. (b)	27. (d)	28. (b)	29. (a)	30. (b)

31. (b)	32. (b)	33. (c)	34. (d)	35. (c)
36. (d)	37. (a)	38. (a)	39. (d)	40. (b)
41. (c)	42. (a)	43. (d)	44. (b)	45. (b)
46. (b)	47. (b)	48. (a)	49. (a)	50. (b)
51. (a)	52. (c)	53. (b)		

SUMMARY

1. For a circle of a radius r, we have
   (i) Circumference $= 2\pi r$ (ii) Area $= \pi r^2$
   (iii) Area of semi-circle $= \dfrac{\pi r^2}{2}$ (iv) Area of a quadrant $= \dfrac{\pi r^2}{4}$
2. If R and r are the radii of two concentric circles such that $R > r$ then.
   Area enclosed by the two circles $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$
3. If a sector of a circle of radius r contains an angle of $\theta°$. Then,
   (i) Length of the arc of the sector $= \dfrac{\theta}{360} \times 2\pi r$
   $= \dfrac{\theta}{360} \times$ (Circumference of the circle)
   (ii) Perimeter of the sector $= 2r + \dfrac{\theta}{360} \times 2\pi r$
   (iii) Area of the sector $= \dfrac{\theta}{360} \times \pi r^2 = \dfrac{\theta}{360} \times$ (Area of the circle)
   (iv) Area of the segment
   = Area of the corresponding sector − Area of the corresponding triangle
   $$= \frac{\theta}{360} \times \pi r^2 - r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left\{\frac{\pi\theta}{360} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\} r^2$$

CHAPTER 14

SURFACE AREAS AND VOLUMES

14.1 INTRODUCTION

What had been learnt in previous classes regarding surface areas and volumes of solids like cuboid, cube, right circular cylinder, right circular cone and sphere has been reviewed in the previous chapter. In this chapter, we shall discuss problems on conversion of one of these solids in another.

In our day-to-day life we come across various solids which are combinations of two or more such solids. For example, a conical circus tent with cylindrical base is a combination of a right circular cylinder and a right circular cone, also an ice-cream cone is a combination of a cone and a hemisphere. We shall discuss problems on finding surface areas and volumes of such solids. We also come across solids which are a part of a cone. For example, a bucket, a glass tumbler, a friction clutch etc. These solids are known as frustums of a cone. In the end of the chapter, we shall discuss problems on surface area and volume of frustum of a cone.

14.2 SOME USEFUL FORMULAE

CUBOID *Let l, b and h denote respectively the length, breadth and height of a cuboid. Then,*

(i) *Total surface area of the cuboid* $= 2(lb + bh + lh)$ *square units*

(ii) *Volume of the cuboid* = *Area of the base* × *Height* = *Length* × *Breadth* × *Height*

$= lbh$ *cubic units*

(iii) *Diagonal of the cuboid* $= \sqrt{l^2 + b^2 + h^2}$ *units.*

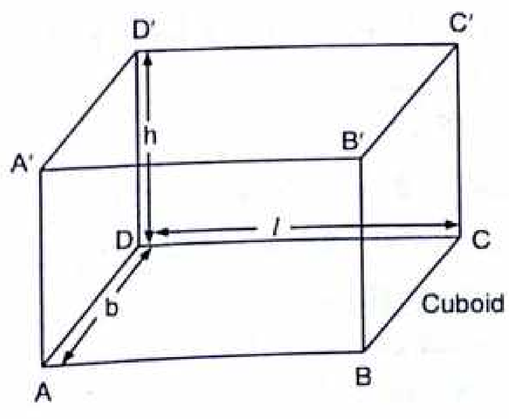

Fig. 14.1

(iv) *Area of four walls of a room* $= lh + lh + bh + bh = 2(l + b)h$ *square units.*

CUBE *If the length of each edge of a cube is 'a' units, then*

(i) *Total surface area of the cube* $= 6a^2$ *square units*

(ii) *Volume of the cube* $= a^3$ *cubic units*

(iii) *Diagonal of the cube* $= \sqrt{3}\,a$ *units*

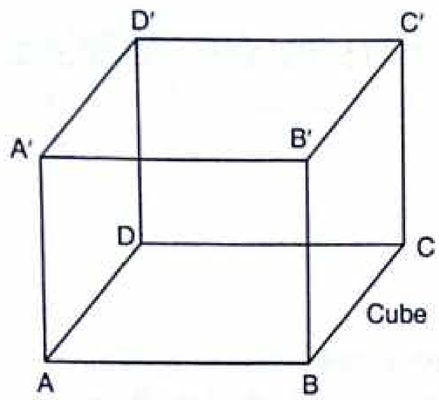

Fig. 14.2

RIGHT CIRCULAR CYLINDER *For a right circular cylinder of base radius r and height (or length) h, we have*

(i) *Area of each end* $=$ *Area of base* $= \pi r^2$

(ii) *Curved surface area* $= 2\pi r h$

$= 2\pi r \times h$

$=$ *Perimeter of the base* $\times$ *Height*

(iii) *Total surface area* $=$ *Curved surface area* $+$ *Area of circular ends*

$= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$

(iv) *Volume* $= \pi r^2 h$

$=$ *Area of the base* $\times$ *Height*

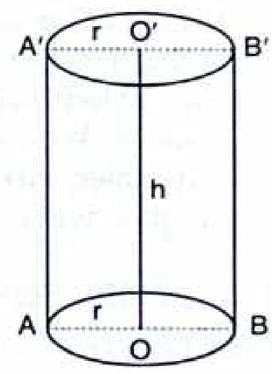

Fig. 14.3

RIGHT CIRCULAR HOLLOW CYLINDER *Let R and r be the external and internal radii of a hollow cylinder of height h. Then,*

(i) *Area of each end* $= \pi(R^2 - r^2)$

(ii) *Curved surface area of hollow cylinder*

$=$ *External surface area* $+$ *Internal surface area*

$= 2\pi Rh + 2\pi rh$

$= 2\pi h(R + r)$

(iii) *Total surface area* $= 2\pi Rh + 2\pi rh + 2(\pi R^2 - \pi r^2)$

$= 2\pi h(R + r) + 2\pi(R + r)(R - r)$

$= 2\pi(R + r)(R + h - r)$

(iv) *Volume of material* $=$ *External volume* $-$ *Internal volume*

$= \pi R^2 h - \pi r^2 h$

$= \pi h(R^2 - r^2)$

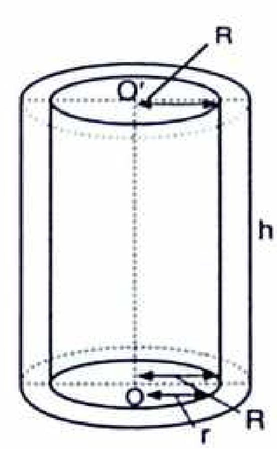

Fig. 14.4

RIGHT CIRCULAR CONE *For a right circular cone of height h, slant height l and radius of base r, we have*

(i) $l^2 = r^2 + h^2$

(ii) *Curved surface area* $= \pi r l$ *sq. units*

(iii) *Total surface area* $=$ *Curved surface area* $+$ *Area of the base*

$$= \pi r l + \pi r^2$$

$$= \pi r (l + r) \text{ sq. units}$$

(iv) *Volume* $= \frac{1}{3} \pi r^2 h$

$$= \frac{1}{3} (\textit{Area of the base}) \times \textit{Height}$$

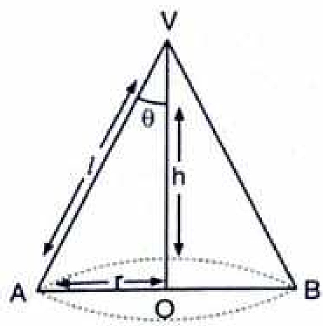

Fig. 14.5

SPHERE *For a sphere of radius r, we have*

(i) *Surface area* $= 4 \pi r^2$

(ii) *Volume* $= \frac{4}{3} \pi r^3$

For a hemisphere of radius r, we have

(i) *Surface area* $= 2\pi r^2$

(ii) *Total surface area* $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

(iii) *Volume* $= \frac{2}{3} \pi r^3$

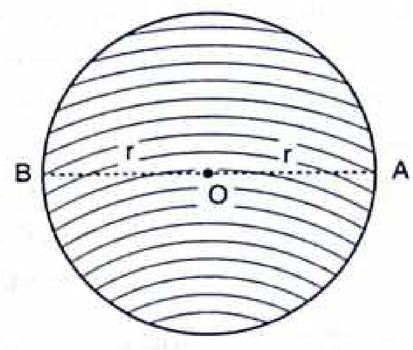

Fig. 14.6

SPHERICAL SHELL *If R and r are respectively the outer and inner radii of a spherical shell, then*

(i) *Outer surface area* $= 4 \pi R^2$

(ii) *Volume of material* $= \frac{4}{3} \pi (R^3 - r^3)$

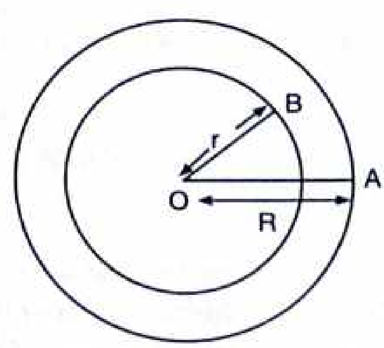

Fig. 14.7

14.3 CONVERSION OF SOLIDS

In this section, we shall discuss problems pertaining to conversion of a solid (discussed in the previous classes) into another solid of different shape. For example, a metallic sphere is melted and recast into a cylindrical wire, the earth taken out by digging a well and spreading it uniformly around the well to form an embankment in the form of a cylindrical shell from its original shape of right circular cylinder, etc. The computation of surface areas and volumes in such cases are illustrated below.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Two cubes each of 10 cm edge are joined end to end. Find the surface area of the resulting cuboid.*

SOLUTION If two cubes are joined end to end, we get a cuboid such that

l = Length of the resulting cuboid = 10 cm + 10 cm = 20 cm

b = Breadth of the resulting cuboid = 10 cm

h = Height of the resulting cuboid = 10 cm

$\therefore$ Surface area of the cuboid $= 2(lb + bh + lh)$

$\Rightarrow$ Surface area of the cuboid $= 2(20 \times 10 + 10 \times 10 + 20 \times 10)\text{ cm}^2 = 1000\text{ cm}^2$

EXAMPLE 2 *Three cubes whose edges measure 3 cm, 4 cm and 5 cm respectively to form a single cube. Find its edge. Also, find the surface area of the new cube.*

SOLUTION Let x cm be the edge of the new cube. Then,

Volume of the new cube = Sum of the volumes of three cubes.

$\Rightarrow$ $x^3 = 3^3 + 4^3 + 5^3 = 27 + 64 + 125$

$\Rightarrow$ $x^3 = 216$

$\Rightarrow$ $x^3 = 6^3 \Rightarrow x = 6\text{ cm}$

$\therefore$ Edge of the new cube is 6 cm long.

Surface area of the new cube $= 6x^2 = 6 \times (6)^2\text{ cm}^2 = 216\text{ cm}^2$

EXAMPLE 3 *Three cubes each of side 5 cm are joined end to end. Find the surface area of the resulting cuboid.*

SOLUTION The dimensions of the cuboid so formed are as under:

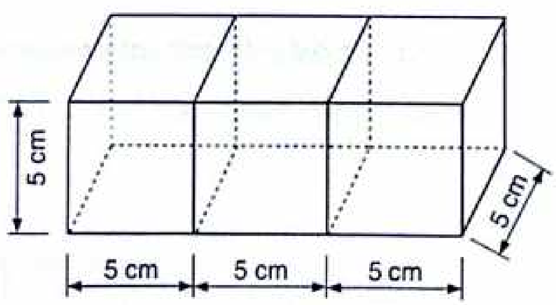

Fig. 14.8

l = Length = 15 cm, b = Breadth = 5 cm, and h = Height = 5 cm

$\therefore$ Surface area of the cuboid $= 2(15 \times 5 + 5 \times 5 + 15 \times 5)\text{ cm}^2$

$\Rightarrow$ Surface area of the cuboid $= 2(75 + 25 + 75)\text{ cm}^2 = 350\text{ cm}^2$

EXAMPLE 4 *Two cubes each of volume 64 cm³ are joined end to end. Find the surface area and volume of the resulting cuboid.* **[NCERT, NCERT EXEMPLAR]**

SOLUTION Let the length of each edge of the cube of volume 64 cm³ be x cm. Then,

Volume $= 64\text{ cm}^3$

$\Rightarrow$ $x^3 = 64$

$\Rightarrow$ $x^3 = 4^3$

$\Rightarrow$ $x = 4\text{ cm}$

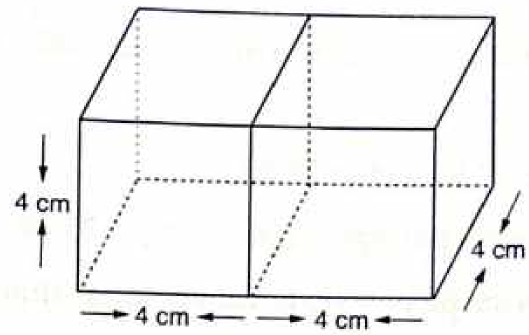

Fig. 14.9

The dimensions of the cuboid so formed are:

L = Length = $(4 + 4)$ cm = 8 cm, b = Breadth = 4 cm and, h = Height = 4 cm

$\therefore$ Surface area of the cuboid $= 2(lb + bh + lh)$

$= 2(8 \times 4 + 4 \times 4 + 8 \times 4)\text{ cm}^2 = 160\text{ cm}^2$

Volume of the cuboid $= lbh = 8 \times 4 \times 4\text{ cm}^3 = 128\text{ cm}^3$

EXAMPLE 5 *The dimensions of a metallic cuboid are: 100 cm × 80 cm × 64 cm. It is melted and recast into a cube. Find the surface area of the cube.*

SOLUTION Let the length of each edge of the recasted cube be a cm.

Volume of the metallic cuboid $= 100 \times 80 \times 64\text{ cm}^3 = 512000\text{ cm}^3$

The metallic cuboid is melted and is recasted into a cube.

$\therefore$ Volume of the cube = Volume of the metallic cuboid

$\Rightarrow \quad a^3 = 512000$

$\Rightarrow \quad a^3 = 8^3 \times 10^3 = (8 \times 10)^3$

$\Rightarrow \quad a = 8 \times 10\text{ cm} = 80\text{ cm}$

$\therefore$ Surface area of the cube $= 6a^2\text{ cm}^2 = 6 \times (80)^2\text{ cm}^2 = 38400\text{ cm}^2$

EXAMPLE 6 *Three metallic solid cubes whose edges are 3 cm, 4 cm and 5 cm, are melted and formed into a single cube. Find the edge of the cube so formed.* **[NCERT EXEMPLAR]**

SOLUTION Let the length of the edge of the new cube formed be x cm. Then,

Volume of the new cube = Sum of the volumes of three metallic cubes

$\Rightarrow \quad x^3 = 3^2 + 4^3 + 5^3$

$\Rightarrow \quad x^3 = 27 + 64 + 125$

$\Rightarrow \quad x^3 = 216$

$\Rightarrow \quad x^3 = 6^3$

$\Rightarrow \quad x = 6$

Hence, the length of an edge of the new cube is 6 cm.

EXAMPLE 7 *A solid iron rectangular block of dimensions 4.4 m, 2.6 m and 1 m is cast into a hollow cylindrical pipe of internal radius 30 cm and thickness 5 cm. Find the length of the pipe.*

[NCERT EXEMPLAR]

SOLUTION Let the length of the pipe be h cm. Then, volume of iron in the pipe is equal to the volume of iron in the block.

$$\text{Volume of the block} = (4.4 \times 2.6 \times 1)\ m^3 = (440 \times 260 \times 100)\ m^3$$

We have,

$$r = \text{Internal radius of the pipe} = 30\ cm$$

$$R = \text{External radius of the pipe} = (30 + 5)\ cm = 35\ cm$$

$$\therefore \quad \text{Volume of iron in the pipe} = (\text{External Volume}) - (\text{Internal Volume})$$

$$= \pi R^2 h - \pi r^2 h = \pi(R^2 - r^2)h$$

$$= \pi(R + r)(R - r)\, h$$

$$= \pi \times (35 + 30) \times (35 - 30) \times h\ cm^3$$

$$= \pi \times 65 \times 5 \times h\ cm^3$$

Now, Volume of iron in the pipe = Volume of iron in the block

$$\Rightarrow \quad \pi \times 65 \times 5 \times h = 440 \times 260 \times 100$$

$$\Rightarrow \quad \frac{22}{7} \times 65 \times 5 \times h = 440 \times 260 \times 100$$

$$\Rightarrow \quad h = \left(440 \times 260 \times 100 \times \frac{7}{22} \times \frac{1}{65} \times \frac{1}{5}\right) cm = 11200\ cm = 112\ m.$$

Hence, the length of the pipe is 112 m.

EXAMPLE 8 *The radii of the bases of two right circular solid cones of same height are r_1 and r_2 respectively. The cones are melted and recast into a solid sphere of radius R. Show that the height of each cone is given by* $h = \dfrac{4R^3}{r_1^2 + r_2^2}$

SOLUTION Let h be the height of each cone. Then,

Sum of the volumes of two cones = Volume of the sphere

$$\Rightarrow \quad \frac{1}{3}\pi r_1^2 h + \frac{1}{3}\pi r_2^2 h = \frac{4}{3}\pi R^3$$

$$\Rightarrow \quad (r_1^2 + r_2^2)h = 4R^3$$

$$\Rightarrow \quad h = \frac{4R^3}{r_1^2 + r_2^2}$$

EXAMPLE 9 *Two solid right circular cones have the same height. The radii of their bases are r_1 and r_2. They are melted and recast into a cylinder of same height. Show that the radius of the base of the cylinder is* $\sqrt{\dfrac{r_1^2 + r_2^2}{3}}$

SOLUTION Let h be the height of two given cones of base radii r_1 and r_2 respectively. Further, let R be the radius of the cylinder. It is given that the cylinder is also of height h.

$\therefore$ Volume of the cylinder = Sum of the Volumes of two cones

$$\Rightarrow \quad \pi R^2 h = \frac{1}{3}\pi r_1^2 h + \frac{1}{3}\pi r_2^2 h$$

$$\Rightarrow \quad \pi R^2 h = \frac{1}{3}\pi h (r_1^2 + r_2^2)$$

$$\Rightarrow \quad R^2 = \frac{1}{3}\left(r_1^2 + r_2^2\right)$$

$$\Rightarrow \quad R = \sqrt{\frac{r_1^2 + r_2^2}{3}}$$

EXAMPLE 10 *The diameter of a metallic sphere is 6 cm. It is melted and drawn into a wire having diameter of the cross-section as 0.2 cm. Find the length of the wire.*

SOLUTION We have,

Diameter of metallic sphere = 6 cm

$\therefore$ Radius of metallic sphere = 3 cm

Also, we have

Diameter of cross-section of cylindrical wire = 0.2 cm

$\therefore$ Radius of cross-section of cylindrical wire = 0.1 cm

Let the length of the wire be h cm. Since metallic sphere is converted into a cylindrical shaped wire of length h cm.

$\therefore$ Volume of the metal used in wire = Volume of the sphere

$$\Rightarrow \quad \pi \times (0.1)^2 \times h = \frac{4}{3} \times \pi \times 3^3$$

$$\Rightarrow \quad \pi \times \left(\frac{1}{10}\right)^2 \times h = \frac{4}{3} \times \pi \times 27$$

$$\Rightarrow \quad \pi \times \frac{1}{100} \times h = 36\pi$$

$$\Rightarrow \quad h = \frac{36\pi \times 100}{\pi} \text{ cm} = 3600 \text{ cm} = 36 \text{ metres}$$

EXAMPLE 11 *The diameter of a metallic sphere is 6 cm. The sphere is melted and drawn into a wire of uniform cross-section. If the length of the wire is 36 m, find its radius.* **[CBSE 2013]**

SOLUTION We have,

Diameter of the sphere = 6 cm

$\therefore$ Radius of the sphere $= \frac{6}{2}$ cm = 3 cm

$\Rightarrow$ Volume of the sphere $= \frac{4}{3} \times \pi \times 3^3 \text{ cm}^3 = 36\pi \text{ cm}^3$ $\left[\text{Using } V = \frac{4}{3}\pi r^3\right]$

Let the radius of cross-section of wire be r cm. It is given that the length of the cylindrical shaped wire is 36 m.

$\therefore$ Volume of the wire $= (\pi r^2 \times 3600)\ \text{cm}^3$ $\left[\text{Using } V = \pi r^2 h\right]$

Since metallic sphere is converted into cylindrical shaped wire. Therefore,

Volume of the wire = Volume of the sphere

$$\Rightarrow \quad \pi r^2 \times 3600 = 36\pi$$

$$\Rightarrow \quad r^2 = \frac{36\pi}{3600\pi} = \frac{1}{100}$$

$$\Rightarrow \quad r = \frac{1}{10}\ \text{cm} = 1\ \text{mm}$$

EXAMPLE 12 *How many shots each having diameter 3 cm can be made from a cuboidal lead solid of dimensions 9 cm × 11 cm × 12 cm?* **[NCERT EXEMPLAR]**

SOLUTION Volume of the lead in cubical solid $= (9 \times 11 \times 12)\ \text{cm}^3 = 1188\ \text{cm}^3$

Suppose x shots can be made from the cubical solid. Then

Volume of lead in x spherical shots = Volume of the solid

$$\Rightarrow \quad \left\{\frac{4}{3}\pi \times \left(\frac{3}{2}\right)^3\right\} x = 1188$$

$$\Rightarrow \quad \left(\frac{4}{3} \times \frac{22}{7} \times \frac{27}{8}\right) x = 1188$$

$$\Rightarrow \quad x = \frac{1188 \times 3 \times 7 \times 8}{4 \times 22 \times 27} = 84$$

Hence, 84 shots can be made from the cubical solid.

EXAMPLE 13 *A right circular cone of radius 3 cm had a curved surface area of 47.1 cm^2. Find the volume of the cone. (Use $\pi = 3.14$)* **[CBSE 2016]**

SOLUTION Let the height and the slant height of the cone be h cm and l cm respectively. It is given that the radius of the base is $r = 3$ cm. It is also given that the curved surface area of the cone is 47.1 cm^2

$$\therefore \quad \pi r l = 47.1$$

$$\Rightarrow \quad 3.14 \times 3 \times l = 47.1$$

$$\Rightarrow \quad l = \frac{47.1}{9.42}\ \text{cm} = 5\ \text{cm}$$

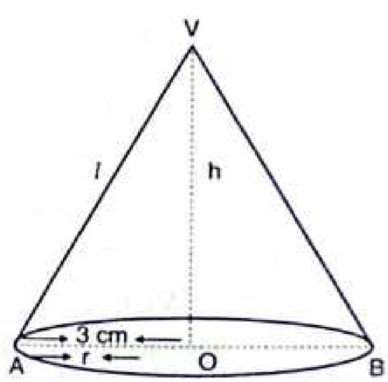

Fig. 14.10

Thus, we obtain $l = 5$ cm and $r = 3$ cm

$$\therefore \quad l^2 = r^2 + h^2$$

$$\Rightarrow \quad 25 = 9 + h^2$$

$$\Rightarrow \quad h^2 = 16$$

$$\Rightarrow \quad h = 4\ \text{cm}$$

Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3} \times 3.14 \times 3^2 \times 4 \text{ cm}^3 = 37.68 \text{ cm}^3$$

Hence, the volume of the cone is 37.68 cm^3.

EXAMPLE 14 *A right circular cone is of height 8.4 cm and the radius of its base is 2.1 cm. It is melted and recast into a sphere. Find the radius of the sphere.*

SOLUTION We have,

$$r = \text{Radius of the base of the cone} = 2.1 \text{ cm}, \quad h = \text{Height of the cone} = 8.4 \text{ cm}$$

$$\therefore \quad \text{Volume of the cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times (2.1)^2 \times 8.4 \text{ cm}^3$$

Let R cm be the radius of the sphere obtained by recasting the melted cone. Then,

$$\text{Volume of the sphere} = \frac{4}{3}\pi R^3$$

Since the volume of the material in the form of cone and sphere remains the same.

$$\therefore \quad \frac{4}{3}\pi R^3 = \frac{1}{3} \times \pi \times (2.1)^2 \times (8.4)$$

$$\Rightarrow \quad R^3 = \frac{(2.1)^2 \times 8.4}{4} = (2.1)^3$$

$$\Rightarrow \quad R = 2.1$$

Hence, the radius of the sphere is 2.1 cm.

EXAMPLE 15 *A metallic sphere of radius 4.2 cm is melted and recast into the shape of a cylinder of radius 6 cm. Find the height of the cylinder.* **[NCERT]**

SOLUTION Let the height of the cylinder be h cm. Then,

Volume of the cylinder = Volume of the sphere

$$\Rightarrow \quad \pi \times 6^2 \times h = \frac{4}{3} \times \pi \times (4.2)^3$$

$$\Rightarrow \quad h = \frac{4 \times 4.2 \times 4.2 \times 4.2}{3 \times 6 \times 6}$$

$$\Rightarrow \quad h = 4 \times 0.7 \times 0.7 \times 1.4 \text{ cm}$$

EXAMPLE 16 *Metallic spheres of radii 6 cm, 8 cm and 10 cm respectively, are melted to form a single solid sphere. Find the radius of the resulting sphere.* **[NCERT]**

SOLUTION Let the radius of the resulting sphere be r cm. Then,

Volume of the resulting sphere = Sum of the volumes of three spheres of radii 6 cm, 8 cm and 10 cm

$$\Rightarrow \quad \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 6^3 + \frac{4}{3}\pi \times 8^3 + \frac{4}{3}\pi \times 10^3$$

$$\Rightarrow \quad r^3 = 216 + 512 + 1000$$

$$\Rightarrow \quad r^3 = 1728$$

$$\Rightarrow \quad r^3 = 12^3$$

$$\Rightarrow \quad r = 12 \text{ cm}.$$

EXAMPLE 17 *A solid sphere of radius 3 cm is melted and then cast into small spherical balls each of diameter 0.6 cm. Find the number of balls thus obtained.*

SOLUTION Let the total number of balls be x.

$$\text{Volume of the solid sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 3^3 \text{ cm}^3 = 36\pi \text{ cm}^3$$

$$\text{Radius of spherical ball} = \frac{0.6}{2} \text{ cm} = 0.3 \text{ cm}$$

$$\text{Volume of a spherical ball} = \frac{4}{3}\pi \times (0.3)^3 \text{ cm}^3 = \frac{4}{3}\pi \times \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} \text{ cm}^3 = \frac{36\pi}{1000} \text{ cm}^3$$

$\therefore$ $$\text{Volume of } x \text{ spherical balls} = \frac{36\pi}{1000} x \text{ cm}^3$$

Clearly, Volume of the solid sphere = Volume of x spherical balls.

$\Rightarrow$ $$36\pi = \frac{36\pi}{1000} x \Rightarrow x = 1000$$

Hence, 1000 spherical balls are obtained by melting the given solid sphere.

EXAMPLE 18 *Find the number of coins, 1.5 cm in diameter and 0.2 cm thick, to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm.*

SOLUTION Let the total number of coins be x. Clearly, each coin is a cylinder of radius $r = 0.75$ cm and height $h = 0.2$ cm.

$$\text{Volume of a coin} = [\pi \times (0.75)^2 \times 0.2] \text{ cm}^3 \qquad [\text{Using: } V = \pi r^2 h]$$

$\therefore$ $$\text{Volume of } x \text{ coins} = [\pi \times (0.75)^2 \times 0.2]\, x \text{ cm}^3$$

$$\text{Volume of the cylinder} = [\pi \times (2.25)^2 \times 10] \text{cm}^3 \quad \left[\because \text{Radius} = \frac{4.5}{2} \text{cm} = 2.25 \text{ cm}\right]$$

Clearly, Volume of metal in x coins = Volume of the cylinder

$\Rightarrow$ $$\{\pi \times (0.75)^2 \times 0.2\}\, x = [\pi \times (2.25)^2 \times 10] \text{ cm}^3$$

$\Rightarrow$ $$x = \frac{\pi(2.25 \times 2.25 \times 10)}{\pi(0.75 \times 0.75 \times 0.2)} = 3 \times 3 \times 50 = 450$$

EXAMPLE 19 *How many spherical bullets can be made out of a solid cube of lead whose edge measures 44 cm, each bullet being 4 cm in diameter.*

SOLUTION Let the total number of bullets be x.

$$\text{Radius of a spherical bullet} = \frac{4}{2} \text{ cm} = 2 \text{ cm}$$

Now, $$\text{Volume of a spherical bullet} = \frac{4}{3}\pi \times (2)^3 \text{ cm}^3 = \left(\frac{4}{3} \times \frac{22}{7} \times 8\right) \text{cm}^3$$

$\therefore$ $$\text{Volume of } x \text{ spherical bullets} = \left(\frac{4}{3} \times \frac{22}{7} \times 8 \times x\right) \text{cm}^3$$

$$\text{Volume of the solid cube} = (44)^3 \text{ cm}^3$$

Clearly, Volume of x spherical bullets = Volume of cube

$\Rightarrow \quad \frac{4}{3} \times \frac{22}{7} \times 8 \times x = (44)^3$

$\Rightarrow \quad \frac{4}{3} \times \frac{22}{7} \times 8 \times x = 44 \times 44 \times 44$

$\Rightarrow \quad x = \frac{44 \times 44 \times 44 \times 3 \times 7}{4 \times 22 \times 8} = 2541$

Hence, total number of spherical bullets = 2541

EXAMPLE 20 *How many spherical lead shots each 4.2 cm in diameter can be obtained from a rectangular solid of lead with dimensions 66 cm, 42 cm, 21 cm. (Use $\pi = 22/7$).*

SOLUTION Let the number of lead shots be x

Volume of lead in the rectangular solid $= (66 \times 42 \times 21)\ \text{cm}^3$

Radius of a lead shot $= \frac{4.2}{2}\ \text{cm} = 2.1\ \text{cm}$

Volume of a spherical lead shot $= \frac{4}{3} \times \frac{22}{7} \times (2.1)^3\ \text{cm}^3$

$\therefore$ Volume of x spherical lead shots $= \left\{\frac{4}{3} \times \frac{22}{7} \times (2.1)^3 \times x\right\} \text{cm}^3$

$\because$ Volume of x spherical lead shots = Volume of lead in rectangular solid

$\therefore \quad \left\{\frac{4}{3} \times \frac{22}{7} \times (2.1)^3 \times x\right\} = 66 \times 42 \times 21$

$\Rightarrow \quad x = \frac{66 \times 42 \times 21 \times 3 \times 7}{4 \times 22 \times (2.1)^3} = \frac{66 \times 42 \times 21 \times 21 \times 1000}{4 \times 22 \times 21 \times 21 \times 21} = 1500$

Hence, the number of spherical lead shots is 1500.

EXAMPLE 21 *The radii of the internal and external surfaces of a metallic spherical shell are 3 cm and 5 cm respectively. It is melted and recast into a solid right circular cylinder of height $10\frac{2}{3}$ cm. Find the diameter of the base of the cylinder.*

SOLUTION Let the radius of the base of the cylinder be r cm. Then,

Volume of the metallic solid cylinder of height $10\frac{2}{3}$ cm

= Volume of the metal in the spherical shell

$\Rightarrow \quad \pi \times r^2 \times \frac{32}{3} = \frac{4}{3}\pi(5^3 - 3^3)$

$\Rightarrow \quad \frac{32}{3} r^2 = \frac{4}{3}(125 - 27)$

$\Rightarrow \quad r^2 = \frac{3}{32} \times \frac{4}{3} \times 98$

$\Rightarrow \quad r^2 = \frac{49}{4}$

$\Rightarrow \qquad r = \frac{7}{2}$ cm

Hence, diameter of the base of the cylinder = 7 cm

EXAMPLE 22 *A hemispherical bowl of internal diameter 36 cm contains a liquid. This liquid is to be filled in cylindrical bottles of radius 3 cm and height 6 cm. How many bottles are required to empty the bowl?*

SOLUTION We have,

Radius of hemispherical bowl = 18 cm

$\therefore$ Volume of hemispherical bowl $= \frac{2}{3}\pi \times (18)^3 \text{ cm}^3$ $\qquad \left[\because V = \frac{2}{3}\pi r^3\right]$

and, Radius of a cylindrical bottle = 3 cm

Height of a cylindrical bottle = 6 cm

$\therefore$ Volume of a cylindrical bottle $= (\pi \times 3^2 \times 6) \text{ cm}^3$ $\qquad \left[\because V = \pi r^2 h\right]$

Suppose x bottles are required to empty the bowl.

Volume of x cylindrical bottles $= (\pi \times 9 \times 6 \times x) \text{ cm}^3$

Clearly, Volume of liquid in x bottles = Volume of bowl

$$\Rightarrow \qquad \pi \times 9 \times 6 \times x = \frac{2\pi}{3} \times (18)^3$$

$$\Rightarrow \qquad x = \frac{2\pi \times 18^3}{3 \times \pi \times 9 \times 6} = 72$$

Hence, 72 bottles are required to empty the bowl.

EXAMPLE 23 *A right circular cone is 3.6 cm high and radius of its base is 1.6 cm. It is melted and recast into a right circular cone with radius of its base as 1.2 cm. Find its height.*

SOLUTION We have,

	First cone	*Second cone*
Radii	$r_1 = 1.6$ cm	$r_2 = 1.2$ cm
Heights	$h_1 = 3.6$ cm	$h_2 = ?$
Volumes	V_1	V_2

Clearly, two cones have the same volume.

$$\therefore \qquad V_1 = V_2$$

$$\Rightarrow \qquad \frac{1}{3}\pi r_1^2 h_1 = \frac{1}{3}\pi r_2^2 h_2$$

$$\Rightarrow \qquad r_1^2 h_1 = r_2^2 h_2$$

$$\Rightarrow \qquad h_2 = \frac{r_1^2 h_1}{r_2^2}$$

$$\Rightarrow \qquad h_2 = \frac{1.6 \times 1.6 \times 3.6}{1.2 \times 1.2} \text{ cm} = \frac{16 \times 16 \times 36}{12 \times 12 \times 10} = 6.4 \text{ cm}$$

Hence, the height of new cone is 6.4 cm.

EXAMPLE 24 *Solid cylinder of brass 8 m high and 4 m diameter is melted and recast into a cone of diameter 3 m. Find the height of the cone.*

SOLUTION We have,

	Cylinder	*Cone*
Radii	$r_1 = 2\text{m}$	$r_2 = 1.5\text{ m}$
Heights	$h_1 = 8\text{ m}$	$h_2 = ?$
Volumes	V_1	V_2

Clearly, Volume of the cone = Volume of the cylinder

i.e., $V_1 = V_2$

$$\Rightarrow \quad \frac{1}{3}\pi r_2^2 h_2 = \pi r_1^2 h_1$$

$$\Rightarrow \quad r_2^2 h_2 = 3 r_1^2 h_1$$

$$\Rightarrow \quad h_2 = \frac{3r_1^2 h_1}{r_2^2} \Rightarrow h_2 = \frac{3 \times 2^2 \times 8}{(1.5)^2}\text{ m} \Rightarrow h_2 = \frac{96}{2.25}\text{ m} = 42.66\text{ m}$$

Hence, the height of the cone is 42.66 m.

EXAMPLE 25 *A sphere of diameter 6 cm is dropped in a right circular cylindrical vessel partly filled with water. The diameter of the cylindrical vessel is 12 cm. If the sphere is completely submerged in water, by how much will the level of water rise in the cylindrical vessel?*

SOLUTION We have, Radius of the sphere = 3 cm

$\therefore$ Volume of the sphere $= \frac{4}{3}\pi \times (3)^3\text{ cm}^3 = 36\pi\text{cm}^3$ $\left[\because V = \frac{4}{3}\pi r^3\right]$

Radius of the cylindrical vessel = 6 cm

Suppose water level rises by h cm in the cylindrical vessel. Then,

Volume of the cylinder of height h cm and radius 6 cm

$$= (\pi \times 6^2 \times h)\text{ cm}^3 = 36\pi h\text{ cm}^3 \quad [\because V = \pi r^2 h]$$

This is the volume of water displaced by the sphere. Clearly, volume of water displaced by the sphere is equal to the volume of the sphere.

$\therefore$ $36\pi h = 36\pi \Rightarrow h = 1\text{ cm}$

Hence, water level rises by 1 cm.

EXAMPLE 26 *A conical vessel whose internal radius is 5 cm and height 24 cm is full of water. The water is emptied into a cylindrical vessel with internal radius 10 cms. Find the height to which the water rises.*

SOLUTION We have,

r_1 = radius of the conical vessel = 5 cm, h_1 = height of the conical vessel = 24 cm

and, r_2 = radius of the cylindrical vessel = 10 cm

Suppose water rises upto the height of h_2 cm in the cylindrical vessel.

Clearly, Volume of water in conical vessel = Volume of water in cylindrical vessel

$$\Rightarrow \quad \frac{1}{3}\pi r_1^2 h_1 = \pi r_2^2 h_2$$

$$\Rightarrow \quad r_1^2 h_1 = 3 r_2^2 h_2$$

$\Rightarrow \quad 5 \times 5 \times 24 = 3 \times 10 \times 10 \times h_2$

$\Rightarrow \quad h_2 = \dfrac{5 \times 5 \times 24}{3 \times 10 \times 10} = 2 \text{ cm}$

Hence, the height of water in the cylindrical vessel is 2 cm.

EXAMPLE 27 *A glass cylinder with diameter 20 cm has water to a height of 9 cm. A metal cube of 8 cm edge is immersed in it completely. Calculate the height by which water will rise in the cylinder. (Take $\pi = 3.142$)*

SOLUTION Suppose the water rises by h cm. Clearly, water in the cylinder forms a cylinder of height h cm and radius 10 cm.

$\therefore$ Volume of the water displaced = Volume of the cube of edge 8 cm

$\Rightarrow \quad \pi r^2 h = 8^3$

$\Rightarrow \quad 3.142 \times 10^2 \times h = 8 \times 8 \times 8 \qquad [\because r = 10 \text{ cm}]$

$\Rightarrow \quad h = \dfrac{8 \times 8 \times 8}{3.142 \times 10 \times 10} \text{ cm} = 1.6 \text{ cm}$

EXAMPLE 28 *500 persons are taking a dip into a cuboidal pond which is 80 m long and 50 m broad. What is the rise of water level in the pond, if the average displacement of the water by a person is 0.04 m^3?*

[NCERT EXEMPLAR]

SOLUTION Suppose level of water rises by h metres in the pond. Then, clearly, water risen in the pond forms a cuboidal of dimensions $80 \text{ m} \times 80 \text{ m} \times h \text{ m}$.

$\therefore \quad 80 \times 50 \times h$ = Volume of water displaced by 500 persons

$\Rightarrow \quad 80 \times 50 \times h = 500 \times 0.04$

$\Rightarrow \quad 4000h = 20$

$\Rightarrow \quad h = \dfrac{1}{200} \text{ m} = 0.5 \text{ cm}$

EXAMPLE 29 *The barrel of a fountain-pen, cylindrical in shape, is 7 cm long and 5 mm in diameter. A full barrel of ink in the pen will be used up on writing 330 words on an average. How many words would use up a bottle of ink containing one fifth of a litre ?*

SOLUTION We have,

Volume of a barrel $= \left(\dfrac{22}{7} \times 0.25 \times 0.25 \times 7\right) \text{cm}^3 = 1.375 \text{ cm}^3$

Volume of ink in the bottle $= \dfrac{1}{5}$ litre $= \dfrac{1000}{5} \text{ cm}^3 = 200 \text{ cm}^3$

$\therefore$ Total number of barrels that can be filled from the given volume of ink $= \dfrac{200}{1.375}$

So, required number of words $= \dfrac{200}{1.375} \times 330 = 48000$

EXAMPLE 30 *The cost of painting the total outside surface of a closed cylindrical oil tank at 60 paise per sq. dm is ₹237.60. The height of the tank is 6 times the radius of the base of the tank. Find its volume correct to two decimal places.*

SOLUTION Let r dm be the radius of the base and h dm be the height of the cylindrical tank. Then, $h = 6r$ (given)

Total surface area $= 2\pi r(r+h) = 2\pi r(r+6r) = 14\pi r^2$

$\Rightarrow$ Cost of painting $= ₹\,(14\pi r^2) \times \frac{60}{100} = ₹\,\frac{42}{5}\pi r^2$

It is given that the cost of painting is ₹ 237.60

$\therefore \quad \frac{42}{5}\pi r^2 = 237.60$

$\Rightarrow \quad \frac{42}{5} \times \frac{22}{7} \times r^2 = 237.60$

$\Rightarrow \quad r^2 = 237.60 \times \frac{5}{42} \times \frac{7}{22} = 9 \Rightarrow r = 3 \text{ dm}$

$\therefore \quad h = 6r = 18 \text{ dm}$

Hence, Volume of the cylinder $= \pi r^2 h = (\pi \times 3 \times 3 \times 18)\,\text{dm}^3 = \left(\frac{22}{7} \times 9 \times 18\right)\text{dm}^3 = 509.14\,\text{dm}^3$

EXAMPLE 31 *A well with 10 m inside diameter is dug 14 m deep. Earth taken out of it is spread all a round to a width of 5 m to form an embankment. Find the height of embankment.*

SOLUTION We have,

Volume of the earth dugout $= (\pi r^2 h)\ \text{m}^3$

$\Rightarrow$ Volume of the earth dugout $= \frac{22}{7} \times 5 \times 5 \times 14\ \text{m}^3 = 1100\ \text{m}^3$

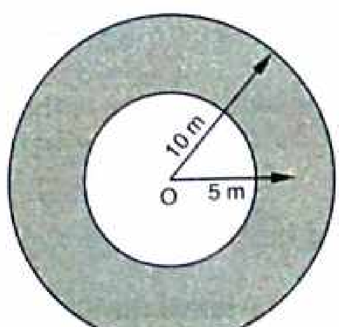

Fig. 14.11

Area of the embankment (shaded region) $= \pi(R^2 - r^2) = \pi(10^2 - 5^2)\,\text{m}^2 = \frac{22}{7} \times 75\,\text{m}^2$

$\therefore$ Height of the embankment $= \frac{\text{Volume of the earth dugout}}{\text{Area of the embankment}} = \frac{1100}{\frac{22}{7} \times 75} = \frac{7 \times 1100}{22 \times 75} = 4.66\,\text{m}.$

EXAMPLE 32 *A 20 m deep well with diameter 7 m is dug and the earth from digging is evenly spread out to form a platform 22 m by 14 m. Find the height of the platform.*

[NCERT, CBSE 2015]

SOLUTION We have,

Volume of the earth taken out of the well

= Volume of a cylinder of radius $\frac{7}{2}$ m and height 20 m

$= \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 20 \text{ m}^3 = 770 \text{ m}^3$

Let the height raised of 22 m × 14 m platform be equal to h metres. Then,

Volume of the earth in platform = Volume of the earth taken out of the well

$\Rightarrow \quad 22 \times 14 \times h = 770$

$\Rightarrow \quad h = \frac{770}{22 \times 14} \text{ m} \Rightarrow h = \frac{5}{2} \text{ m} = 2.5 \text{ m}.$

EXAMPLE 33 *An agriculture field is in the form of a rectangle of length 20 m width 14 m. A 10 m deep well of diameter 7m is dug in a corner of the field and the earth taken out of the well is spread evenly over the remaining part of the field. Find the rise in its level.*

SOLUTION We have,

Radius of the well $= \frac{7}{2}$, Depth of the well = 10 m

$\therefore$ Volume of the earth dug $= \pi\left(\frac{7}{2}\right)^2 \times 10 \text{ m}^3 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 10 \text{ m}^3 = 358 \text{ m}^3$

Also, we have

Length of the field = 14 m, Breadth of the field = 14 m

$\therefore$ Area of the field $= 20 \times 14 \text{ m}^2 = 280 \text{ m}^2$

Area of the base of the well $= \pi \times \left(\frac{7}{2}\right)^2 \text{ m}^2 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \text{ m}^2 = \frac{77}{2} \text{ m}^2$

$\therefore$ Area of the remaining part of the field = Area of the field – Area of the base of the field

$= \left(280 - \frac{77}{2}\right) \text{m}^2 = \left(\frac{560 - 77}{2}\right) \text{m}^2 = \frac{483}{2} \text{ m}^3$

Let the rise in the level of the field be h metres.

$\therefore$ Volume of the raised field = Area of the base × Height $= \left(\frac{483}{2} \times h\right) \text{m}^3$

But, Volume of the raised field = Volume of the earth dugout

$\therefore \quad \frac{483}{2} \times h = 385$

$\Rightarrow \quad h = \frac{2 \times 385}{483} = \frac{770}{483} = 1.594 \text{ m}$

Hence, rise in the level of the field = 1.594 m.

EXAMPLE 34 *A cylindrical pipe has inner diameter of 7 cm and water flows through it at 192.5 litres per minute. Find the rate of flow in kilometres per hour.* **[CBSE 2013]**

SOLUTION We have,

Volume of water that flows per hour = (192.50 × 60) litres

$= (192.50 \times 60 \times 1000) \text{ cm}^3$ ...(i)

Inner diameter of the pipe $= 7$ cm

$\Rightarrow$ Inner radius of the pipe $\frac{7}{2}$ cm $= 3.5$ cm

Let h cm be the length of the column of water that flows in one hour.

Clearly, water column forms a cylinder of radius 3.5 cm and length h cm.

$\therefore$ Volume of water that flows in one hour = Volume of the cylinder of radius 3.5 cm and length h cm

$$= \left(\frac{22}{7} \times (3.5)^2 \times h\right) \text{cm}^3 \quad \ldots \text{(ii)}$$

From (i) and (ii), we have

$$\frac{22}{7} \times 3.5 \times 3.5 \times h = 192.50 \times 60 \times 1000$$

$$\Rightarrow \quad h = \frac{192.50 \times 60 \times 1000 \times 7}{22 \times 3.5 \times 3.5} \text{ cm} = 300000 \text{ cm} = 3 \text{ km}$$

Hence, the rate of flow of water is 3 km per hour.

EXAMPLE 35 *Water is being pumped out through a circular pipe whose internal diameter is 7 cm. If the flow of water is 72 cm per second, how many litres of water are being pumped out in one hour?*

SOLUTION We have, Radius of the circular pipe $= \frac{7}{2}$ cm

Clearly, water column forms a cylinder of radius $\frac{7}{2}$ cm. It is given that the water flows out at the rate of 72 cm/sec.

$\therefore$ Length of the water column flowing out in one second = 72 cm.

Volume of the water flowing out per second

= Volume of the cylinder of radius $\frac{7}{2}$ cm and length 72 cm.

$$= \pi \times \left(\frac{7}{2}\right)^2 \times 72 \text{ cm}^3 = \pi \times \frac{7}{2} \times \frac{7}{2} \times 72 \text{ cm}^3 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 72 \text{ cm}^3 = 2772 \text{ cm}^3$$

$\therefore$ Volume of the water flowing out in one hour $= (2772 \times 3600)$ cm^3 [$\because$ 1 hr = 3600 sec.]

$$= 9979200 \text{ cm}^3$$

$$= \frac{9979200}{1000} \text{ litres} = 9979.2 \text{ litres}$$

Hence, 9979.2 litres of water flows out per hour.

EXAMPLE 36 *Water is flowing at the rate of 3 km/hr through a circular pipe of 20 cm internal diameter into a circular cistern of diameter 10 m and depth 2 m. In how much time will the cistern be filled?* **[CBSE 2008]**

SOLUTION Suppose the cistern is filled in x hours. Since water is flowing at the rate of 3 km/hr. Therefore,

Length of the water column in x hours $= 3x$ km $= 3000x$ metres.

Clearly, the water column forms a cylinder of radius

$$r = \frac{20}{2} \text{ cm} = 10 \text{ cm} = \frac{1}{10} \text{ m and } h = \text{height (length)} = 3000x \text{ metres}$$

$\therefore$ Volume of the water that flows in the cistern in x hours

$$= \pi r^2 h = \left(\frac{22}{7} \times \frac{1}{10} \times \frac{1}{10} \times 3000x\right) \text{m}^3$$

Also, Volume of the cistern $= \left(\frac{22}{7} \times 5 \times 5 \times 2\right) \text{m}^3$ $[\because r = 5 \text{ m}, h = 2 \text{ m}]$

Since the cistern is filled in x hours.

$\therefore$ Volume of the water that flows in the cistern in x hours = Volume of the cistern.

$$\Rightarrow \quad \frac{22}{7} \times \frac{1}{10} \times \frac{1}{10} \times 3000x = \frac{22}{7} \times 5 \times 5 \times 2$$

$$\Rightarrow \quad x = \left(\frac{5 \times 5 \times 2 \times 10 \times 10}{3000}\right) \text{hrs} = \frac{5}{3} \text{ hours} = 1 \text{ hour } 40 \text{ minutes.}$$

EXAMPLE 37 *Water is flowing at the rate of 7 metres per second through a circular pipe whose internal diameter is 2 cm into a cylindrical tank the radius of whose base is 40 cm. Determine the increase in the water level in 1/2 hour.* **[CBSE 2006C, 2013]**

SOLUTION We have,

Rate of flow of water = 7 m/sec = 700 cm/sec.

Length of the water column in $\frac{1}{2}$ hours $= (700 \times 30 \times 60)$ cm

Internal radius of circular pipe = 1 cm.

Clearly, water column forms a cylinder of radius 1 cm and length $= (700 \times 30 \times 60)$ cm.

$\therefore$ Volume of the water that flows in the tank in $\frac{1}{2}$ hr

$$= \left(\frac{22}{7} \times 1 \times 1 \times 700 \times 30 \times 60\right) \text{cm}^3 \qquad [\text{Using: } V = \pi r^2 h] \qquad \ldots\text{(i)}$$

Let h cm be the rise in the level of water in the tank. Then,

$$\text{Volume of the water in the tank} = \frac{22}{7} \times 40 \times 40 \times h \text{ cm}^3 \qquad \ldots\text{(ii)}$$

From (i) and (ii), we obtain

$$\frac{22}{7} \times 40 \times 40 \times h = \frac{22}{7} \times 1 \times 1 \times 700 \times 30 \times 60$$

$$\Rightarrow \quad h = \frac{700 \times 30 \times 60}{40 \times 40} \text{ cm} = 787.5 \text{ cm}$$

Hence, the rise in the level of water in the tank in $\frac{1}{2}$ hr is 787.5 cm.

EXAMPLE 38 *Water is flowing at the rate of 5 km/hr through a pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. Determine the time in which the level of the water in the tank will rise by 7 cm.* **[CBSE 2017]**

SOLUTION Suppose the level of the water in the tank will rise by 7 cm in x hours.

Since the water is flowing at the rate of 5 km/hr. Therefore,

Length of the water column in x hours $= 5x$ km $= 5000\,x$ metres

Clearly, the water column forms a cylinder whose radius $r = \frac{14}{2} \text{ cm} = \frac{7}{100} \text{ m}$

and, Length $= h = 5000\,x$ metres

$\therefore$ Volume of the water flowing through the cylindrical pipe in x hours

$$= \pi r^2 h = \frac{22}{7} \times \left(\frac{7}{100}\right)^2 \times 5000x \text{ m}^3 = \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 5000x \text{ m}^3 = 77x \text{ m}^3$$

Also,

Volume of the water that falls into the tank in x hours $= 50 \times 44 \times \frac{7}{100} \text{ m}^3 = 154 \text{ m}^3$

But, Volume of the water flowing through the cylindrical pipe in x hours

= Volume of the water that falls in the tank in x hours

$\Rightarrow \quad 77x = 154$

$\Rightarrow \quad x = \frac{154}{77} = 2$

Hence, the level of the water in the tank will rise by 7 cm in 2 hours.

EXAMPLE 39 *The rain water from a roof of 22 m × 20 m drains into a cylindrical vessel having diameter of base 2 m and height 3.5 m. If the vessel is just full, find the rain fall in cm.*

[CBSE 2010]

SOLUTION We have,

$r =$ Radius of cylindrical vessel = 1 m, $h =$ Height of cylindrical vessel = 3.5 m

$\therefore$ Volume of cylindrical vessel $= \pi r^2 h = \frac{22}{7} \times 1^2 \times 3.5 \text{ m}^3 = 11 \text{ m}^3$

Let the rain fall be x m. Then,

Volume of the water = Volume of a cuboid of base 22 m × 20 m and height x metres

$= (22 \times 20 \times x) \text{ m}^3$

Since the vessel is just full of the water that drains out of the roof into the vessel.

$\therefore$ Volume of the water = Volume of the cylindrical vessel

$\Rightarrow \quad 22 \times 20 \times x = 11$

$\Rightarrow \quad x = \frac{11}{22 \times 20} = \frac{1}{40} \text{ m} = \frac{100}{40} \text{ cm} = 2.5 \text{ cm}$

EXAMPLE 40 *Water in a canal, 30 dm wide and 12 dm deep is flowing with velocity of 10 km/hr. How much area will it irrigate in 30 minutes, if 8 cm of standing water is required for irrigation?*

[CBSE 2014]

SOLUTION We have,

Width of the canal = 30 dm = 300 cm = 3 m

Depth of the canal = 12 dm = 120 cm = 1.2 m.

It is given that the water is flowing with velocity 10 km/hr. Therefore,

Length of the water column formed in $\frac{1}{2}$ hour = 5 km = 5000 m

$\therefore$ Volume of the water flowing in $\frac{1}{2}$ hour = Volume of the cuboid of length 5000 m,

width 3 m and depth 1.2 m

$\Rightarrow$ Volume of the water following in ½ hour $= 5000 \times 3 \times 1.2 \text{ m}^3 = 18000 \text{ m}^3$

Suppose x m^2 area is irrigated in $\frac{1}{2}$ hour. Then,

$$x \times \frac{8}{100} = 18000 \Rightarrow x = \frac{1800000}{8} \text{ m}^2 \Rightarrow x = 225000 \text{ m}^2$$

Hence, the canal irrigates 225000 m^2 area in $\frac{1}{2}$ hour.

EXAMPLE 41 *Water flows at the rate of 10 metre per minute through a cylindrical pipe having its diameter as 5 mm. How much time will it take to fill a conical vessel whose diameter of base is 40 cm and depth 24 cm?*

SOLUTION We have,

Volume of the water that flows out in one minute

= Volume of the cylinder of diameter 5 mm and length 10 metre

= Volume of the cylinder of radius $\frac{5}{2}$ mm $\left(= \frac{1}{4}\right)$ cm and length 1000 cm

$$= \frac{22}{7} \times \frac{1}{4} \times \frac{1}{4} \times 1000 \text{ cm}^3$$

Volume of a conical vessel of base radius 20 cm and depth 24 cm $= \frac{1}{3} \times \frac{22}{7} \times (20)^2 \times 24 \text{ cm}^3$

Suppose the conical vessel is filled in x minutes.

$\therefore$ Volume of the water that flows out in x minutes = Volume of the conical vessel

$$\Rightarrow \quad \frac{22}{7} \times \frac{1}{4} \times \frac{1}{4} \times 1000 \times x = \frac{1}{3} \times \frac{22}{7} \times 20^2 \times 24$$

$$\Rightarrow \quad x = \frac{1}{3} \times \frac{400 \times 24 \times 4 \times 4}{1000} = \frac{512}{10} \text{ minutes}$$

$$\Rightarrow \quad x = 51 \text{ minutes } 12 \text{ seconds.}$$

EXAMPLE 42 *A hemispherical tank of radius 1.75 m is full of water. It is connected with a pipe which empties it at the rate of 7 litres per second. How much time will it take to empty the tank completely?*

SOLUTION Suppose the pipe takes x seconds to empty the tank. Then,

Volume of the water that flows out of the tank in x seconds

= Volume of the hemispherical tank.

$\Rightarrow$ Volume of the water that flows out of the tank x in seconds

= Volume of the hemispherical shell of radius 175 cm

$$\Rightarrow \quad 7000x = \frac{2}{3} \times \frac{22}{7} \times 175 \times 175 \times 175$$

$$\Rightarrow \quad x = \frac{2}{3} \times \frac{22}{7} \times \frac{175 \times 175 \times 175}{7000} = 1604.16 \text{ seconds}$$

$$\Rightarrow \quad x = \frac{1604.16}{60} \text{ minutes} = 26.73 \text{ minutes}$$

EXAMPLE 43 *A hemispherical tank full of water is emptied by a pipe at the rate of $3\frac{4}{7}$ litres per second. How much time will it take to make the tank half-empty, if the tank is 3 m in diameter?*

[CBSE 2016]

SOLUTION We have, Radius of hemispherical tank $= \frac{3}{2}$ m

$\therefore$ Volume of the tank $= \frac{2}{3} \times \frac{22}{7} \times \left(\frac{3}{2}\right)^3 m^3 = \frac{99}{14} m^3$

Volume of the water to be emptied $= \frac{1}{2} \times \frac{99}{14} m^3 = \frac{99}{28} m^3 = \frac{99}{28} \times 1000$ litres $= \frac{99000}{28}$ litres

Since $\frac{25}{7}$ litres of water is emptied in one second. Therefore,

Total time taken to empty half the tank i.e $\frac{99000}{28}$ litres of water $= \frac{99000}{28} \div \frac{25}{7}$ seconds

$$= \frac{99000}{28} \times \frac{7}{25} \text{ seconds}$$

$$= \frac{99000}{28} \times \frac{7}{25} \times \frac{1}{60} \text{ minutes}$$

$$= 16.5 \text{ minutes}$$

EXAMPLE 44 *The largest sphere is carved out of a cube of a side 7 cm. Find the volume of the sphere.*

SOLUTION The diameter of the largest sphere which can be carved out of a cube of side 7 cm is 7 cm.

$\therefore$ Radius of the sphere $= r = \frac{7}{2}$ cm

Hence,

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3} \times \frac{22}{7} \times \left(\frac{7}{2}\right)^3 cm^3 = \frac{4}{3} \times \frac{22}{7} \times \frac{343}{8} cm^3 = 179.66 \ cm^3$$

EXAMPLE 45 *Determine the ratio of the volume of a cube to that of a sphere which will exactly fit inside the cube.*

SOLUTION Let the radius of the sphere which fits exactly into a cube be r units. Then, Length of each edge of the cube $= 2r$ units

Let V_1 and V_2 be the volumes of the cube and sphere respectively. Then,

$$V_1 = (2r)^3 \text{ and } V_2 = \frac{4}{3}\pi r^3$$

$$\therefore \quad \frac{V_1}{V_2} = \frac{8r^3}{\frac{4}{3}\pi r^3} = \frac{6}{\pi} \Rightarrow V_1 : V_2 = 6 : \pi$$

EXAMPLE 46 *Find the maximum volume of a cone that can be carved out of a solid hemisphere of radius r.*

SOLUTION Clearly,

Radius of the base of cone = Radius of the hemisphere = r

and, Height of the cone = Radius of the hemisphere = r

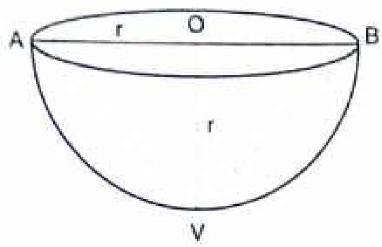

Fig. 14.12

$\therefore$ Volume of the cone $= \frac{1}{3}\pi r^2 \times r = \frac{1}{3}\pi r^3$ cubic units

LEVEL-2

EXAMPLE 47 *The radius of a solid iron sphere is 8 cm. Eight rings of iron plate of external radius $6\frac{2}{3}$ cm and thickness 3 cm are made by melting this sphere. Find the internal diameter of each ring.*

SOLUTION We have,

$$\text{Volume of solid iron sphere} = \frac{4}{3}\pi \times 8^3 \text{ cm}^3 = \frac{2048}{3}\pi \text{ cm}^3$$

$$\text{External radius of each iron ring} = 6\frac{2}{3} \text{ cm} = \frac{20}{3} \text{ cm}$$

Let the internal radius of each ring be r cm. Since each ring forms a hollow cylindrical shell of external and internal radii $\frac{20}{3}$ cm and r cm respectively and height 3 cm.

$$\therefore \quad \text{Volume of each ring} = \pi\left\{\left(\frac{20}{3}\right)^2 - r^2\right\} \times 3 \text{ cm}^3$$

$$\text{Volume of 8 such rings} = 8\pi\left(\frac{400}{9} - r^2\right) \times 3 \text{ cm}^3 = 24\pi\left(\frac{400}{9} - r^2\right) \text{cm}^3$$

Clearly, Volume of 8 rings = Volume of the sphere

$$\Rightarrow \quad 24\pi\left(\frac{400}{9} - r^2\right) = \frac{2048}{3}\pi$$

$$\Rightarrow \quad \frac{400}{9} - r^2 = \frac{2048}{3}\pi \times \frac{1}{24\pi}$$

$$\Rightarrow \quad r^2 = \frac{400}{9} - \frac{256}{9} = \frac{144}{9} = 16$$

$$\Rightarrow \quad r = 4 \text{ cm}$$

Hence, internal radius of each ring is 4 cm.

EXAMPLE 48 *A conical vessel of radius 6 cm and height 8 cm is completely filled with water. A sphere is lowered into the water and its size is such that when it touches the sides, it is just immersed as shown in Fig.14.13. What fraction of water over flows?*

SOLUTION Let the radius of the sphere be r cm.

In $\Delta VO'A$, we have

$$\tan\theta = \frac{6}{8} = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5}$$

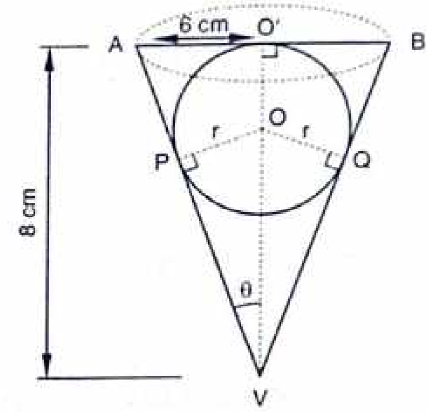

Fig. 14.13

In ΔVPO, We have

$$\sin\theta = \frac{r}{VO}$$

$$\Rightarrow \quad \frac{3}{5} = \frac{r}{8-r}$$

$$\Rightarrow \quad 24 - 3r = 5r$$

$$\Rightarrow \quad 8r = 24$$

$$\Rightarrow \quad r = 3 \text{ cm}$$

$$\therefore \quad V_1 = \text{Volume of the sphere } = \frac{4}{3}\pi \times 3^3 \text{ cm}^3 = 36\pi \text{ cm}^3$$

$$V_2 = \text{Volume of the water} = \text{Volume of the cone } = \frac{1}{3}\pi \times 6^2 \times 8 \text{ cm}^3 = 96\pi \text{ cm}^3$$

Clearly, volume of the water that flows out of the cone is same as the volume of the sphere i.e., V_1.

$\therefore$ Fraction of the water that flows out $= V_1 : V_2 = 36\pi : 96\pi = 3 : 8$

ALITER In $\Delta VO'A$, we have

$$VA^2 = VO'^2 + O'A^2$$

$$\Rightarrow \quad VA^2 = 8^2 + 6^2 = 100$$

$$\Rightarrow \quad VA = 10 \text{ cm}.$$

Now, $\quad AO' = AP$ $\quad$ [$\because$ Tangents drawn from A to the circle are equal]

$\Rightarrow \quad AP = 6$ $\quad$ [$\because AO' = 6$ cm]

$\therefore \quad VP = VA - AP = (10-6)$ cm $= 4$ cm

Now, $\quad VO = VO' - OO' = (8 - r)$ cm

In ΔVPO, we have

$$VO^2 = VP^2 + OP^2$$

$$\Rightarrow \quad (8-r)^2 = 16 + r^2$$

$$\Rightarrow \quad 64 - 16r + r^2 = 16 + r^2 \Rightarrow 16r = 48 \Rightarrow r = 3 \text{ cm}.$$

Now, proceed as in the previous solution.

EXAMPLE 49 *Selvi's house has an overhead tank in the shape of a cylinder. This is filled by pumping water from a sump (underground tank). Which is in the shape of a cuboid. The sump has dimensions 1.57 m × 1.44 m × 0.95 m. The overhead tank has its radius of 60 cm and its height is 95 cm. Find the height of the water, left in the sump after the overhead tank has been completely filled with water from a sump which had been full. Compare the capacity of the tank with that of the sump. (Use $\pi = 3.14$).* **[NCERT]**

SOLUTION Clearly, the volume of the water in the overhead tank is equal to the volume of the water removed from the sump.

Now,

$$\text{Volume of water in the overhead tank} = 3.14 \times 0.6 \times 0.6 \times 0.95 \text{ m}^3 \qquad \left[\text{Using: } V = \pi r^2 h\right]$$

$$= 3.14 \times 0.36 \times 0.95 \text{ m}^3$$

$$\text{Volume of water in the sump when it is full of water} = 1.57 \times 1.44 \times 0.95 \text{ m}^3$$

$$= 1.57 \times 4 \times 0.36 \times 0.95 \text{ m}^3$$

$$= 2 \times 3.14 \times 0.36 \times 0.95 \text{ m}^3$$

$\therefore$ Volume of water left in the sump after filling the tank

$$= (2 \times 3.14 \times 0.36 \times 0.95 - 3.14 \times 0.36 \times 0.95) \text{ m}^3$$

$$= 3.14 \times 0.36 \times 0.95 (2-1) \text{ m}^3 = 3.14 \times 0.36 \times 0.95 \text{ m}^3$$

$$\text{Area of the base of the sump} = 1.57 \times 1.44 \text{ m}^2 = 1.57 \times 4 \times 0.36 \text{ m}^2 = 2 \times 3.14 \times 0.36 \text{ m}^2$$

$$\therefore \quad \text{Height of water in the sump} = \frac{3.14 \times 0.36 \times 0.95}{2 \times 3.14 \times 0.36} \text{ m} = \frac{0.95}{2} = 0.475 \text{ m} = 47.5 \text{ cm}$$

$$\frac{\text{Capacity of tank}}{\text{Capacity of sump}} = \frac{3.14 \times 0.36 \times 0.95}{2 \times 3.14 \times 0.36 \times 0.95} = \frac{1}{2}$$

Hence, the capacity of the tank is half the capacity of the sump.

EXAMPLE 50 *If the diameter of cross-section of a wire is decreased by 5% how much percent will the length be increased so that the volume remains the same ?*

SOLUTION Let r be the radius of cross-section of wire and h be its length. Then,

$$\text{Volume} = \pi r^2 h \qquad \text{...(i)}$$

$$5\% \text{ of diameter of cross-section} = \frac{5}{100} \times 2r = \frac{r}{10}$$

$$\text{New diameter} = 2r - \frac{r}{10} = \frac{19r}{10}$$

$$\Rightarrow \quad \text{New radius} = \frac{19r}{20}$$

Let the new length be h_1. Then,

$$\text{Volume} = \pi\left(\frac{19r}{20}\right)^2 h_1 \quad ..(ii)$$

From (i) and (ii), we obtain

$$\pi r^2 h = \pi\left(\frac{19r}{20}\right)^2 h_1 \Rightarrow h = \frac{361}{400}h_1 \Rightarrow h_1 = \frac{400}{361}h$$

$$\therefore \quad \text{Increase in length} = h_1 - h = \frac{400h}{361} - h = \frac{39h}{361}$$

$$\Rightarrow \quad \text{Percentage increase in length} = \frac{h_1 - h}{h} \times 100 = \frac{39h}{h} \times 100 = \frac{3900}{361} = 10.8\%$$

Hence, the length of the wire increases by 10.8%

EXAMPLE 51 *A well, whose diameter is 7m, has been dug 22.5 m deep and the earth dugout is used to form an embankment around it. If the height of the embankment is 1.5 m, find the width of the embankment.*

SOLUTION We have, Radius of the well $= \frac{7}{2}$ m $= 3.5$ m and, Depth of the well $= 22.5$ m

$$\therefore \quad \text{Volume of the earth dugout} = \pi \times (3.5)^2 \times 22.5 \text{ m}^3 = \pi \times \frac{7}{2} \times \frac{7}{2} \times \frac{45}{2} \text{ m}^3$$

Let the width of the embankment be r metres. Clearly, embankment forms a cylindrical shell whose inner and outer radii are 3.5 m and $(r + 3.5)$ m respectively and height 1.5 m.

$$\therefore \quad \text{Volume of the embankment} = \pi\left\{(r+3.5)^2 - (3.5)^2\right\} \times 1.5 \text{ m}^3 = \pi(r+7)r \times \frac{3}{2} \text{ m}^3$$

But, Volume of the embankment $=$ Volume of the earth dugout

$$\Rightarrow \quad \pi r(r+7) \times \frac{3}{2} = \pi \times \frac{7}{2} \times \frac{7}{2} \times \frac{45}{2}$$

$$\Rightarrow \quad r(r+7) = \frac{49}{4} \times 15$$

$$\Rightarrow \quad 4r^2 + 28r = 735$$

$$\Rightarrow \quad 4r^2 + 28r - 735 = 0$$

$$\Rightarrow \quad r = \frac{-28 \pm \sqrt{784 + 11760}}{8}$$

$$\Rightarrow \quad r = \frac{-28 \pm \sqrt{12544}}{8} = \frac{-28 \pm 112}{8} = \frac{84}{8} = 10.5 \quad [\because r > 0]$$

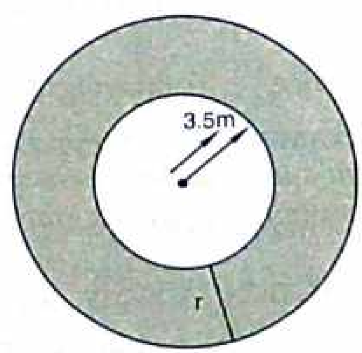

Fig. 14.14

Hence, the width of the embankment is 10.5 m.

EXAMPLE 52 *A lead pencil consists of a cylinder of wood with a solid cylinder of graphite filled into it. The diameter of the pencil is 7 mm, the diameter of the graphite is 1 mm and the length of the pencil is 10 cm. Calculate the weight of the whole pencil, if the specific gravity of the wood is 0.7 gm/cm³ and that of the graphite is 2.1 gm/cm³.*

SOLUTION We have, Diameter of the graphite cylinder $= 1 \text{ mm} = \frac{1}{10}$ cm

$\therefore$ Radius $= \frac{1}{20}$ cm

Length of the graphite cylinder $= 10$ cm

Volume of the graphite cylinder $= \left(\frac{22}{7} \times \frac{1}{20} \times \frac{1}{20} \times 10\right) \text{cm}^3$

Weight of graphite = Volume × Specific gravity

$$= \left(\frac{22}{7} \times \frac{1}{20} \times \frac{1}{20} \times 10 \times 2.1\right) \text{gm} = 0.165 \text{ gm}$$

Diameter of pencil $= 7 \text{ mm} = \frac{7}{10}$ cm

$\therefore$ Radius of pencil $= \frac{7}{20}$ cm and, Length of pencil $= 10$ cm

$\therefore$ Volume of pencil $= \frac{22}{7} \times \frac{7}{20} \times \frac{7}{20} \times 10 \text{ cm}^3$

Volume of wood $= \left(\frac{22}{7} \times \frac{7}{20} \times \frac{7}{20} \times 10 - \frac{22}{7} \times \frac{1}{20} \times \frac{1}{20} \times 10\right) \text{cm}^3$

$$= \frac{22}{7} \times \frac{1}{20} \times \frac{1}{20} \times 10(7 \times 7 - 1) \text{ cm}^3 = \frac{11}{7} \times \frac{1}{20} \times 48 \text{ cm}^3$$

$\therefore$ Weight of wood $= \left(\frac{11}{7} \times \frac{1}{20} \times 48 \times 0.7\right) \text{gm} = \left(\frac{11}{7} \times \frac{1}{20} \times 48 \times \frac{7}{10}\right) \text{gm} = 2.64 \text{ gm}$

Hence, Total weight $= (2.64 + 0.165) \text{ gm} = 2.805$ gm.

EXAMPLE 53 *A copper wire 4 mm in diameter is evenly wound about a cylinder whose length is 24 cm and diameter 20 cm so as to cover the whole surface. Find the length and weight of the wire assuming the specific gravity to be 8.88 gm/cm^3.*

SOLUTION Clearly, one round of wire covers 4 mm $\left(= \frac{4}{10} \text{ cm}\right)$ in thickness of the surface of the cylinder and length of the cylinder is 24 cm.

$\therefore$ Number of rounds to cover 24 cm $= \frac{24}{4/10} = \frac{24 \times 10}{4} = 60$

Diameter of the cylinder = 20 cm

Radius of the cylinder = 10 cm

Length of the wire in completing one round $= 2\pi r = 2\pi \times 10 \text{ cm} = 20\pi$ cm.

$\therefore$ Length of the wire in covering the whole surface

= Length of the wire in completing 60 rounds

$= (20\pi \times 60) \text{ cm} = 1200\pi$ cm

Radius of copper wire $= 2 \text{ mm} = \frac{2}{10}$ cm

$\therefore$ Volume of wire $= \left(\pi \times \frac{2}{10} \times \frac{2}{10} \times 1200\pi\right) \text{cm}^3 = 48\pi^2 \text{ cm}^3$

So, Weight of wire $= (48\pi^2 \times 8.88)$ gm $= 426.24\,\pi^2$gm

EXAMPLE 54 *A copper wire 3 mm in diameter is wound about a cylinder whose length is 1.2 m, and diameter 10 cm, so as to cover the curved surface of the cylinder. Find the length and mass of the wire, assuming the density of the copper wire to be 8.88 gram per cm.* [NCERT]

SOLUTION We have, $d =$ Diameter of copper wire $= 3 \text{ mm} = \frac{3}{10}$ cm.

and, $h =$ Height (length) of the cylinder $= 1.2$ m $= 120$ cm.

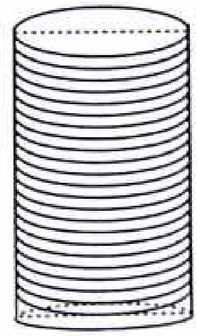

Fig. 14.15

$\therefore$ Number of rounds taken by the wire to cover the curved surface of the cylinder $= \frac{h}{d}$

$$= \frac{120}{\frac{3}{10}} = 400$$

Length of wire used in taking one round $= 2\pi r = (2 \times 3.14 \times 5)$ cm $= 31.4$ cm

$\therefore$ Total length of wire used in covering the curved surface of the cylinder $= (31.4 \times 400)$ cm

$= 12560$ cm $= 125.6$ m

Mass of the wire $=$ Length $\times$ Density $= 12560 \times 8.88$ gram $= 111532.8$ gm $\cong 111.533$ kg

EXERCISE 14.1

LEVEL-1

1. How many balls, each of radius 1 cm, can be made from a solid sphere of lead of radius 8 cm?
2. How many spherical bullets each of 5 cm in diameter can be cast from a rectangular block of metal 11 dm × 1 m × 5 dm?
3. A spherical ball of radius 3 cm is melted and recast into three spherical balls. The radii of the two of the balls are 1.5 cm and 2 cm respectively. Determine the diameter of the third ball.
4. 2.2 cubic dm of brass is to be drawn into a cylindrical wire 0.25 cm in diameter. Find the length of the wire.

5. What length of a solid cylinder 2 cm in diameter must be taken to recast into a hollow cylinder of length 16 cm, external diameter 20 cm and thickness 2.5 mm?

6. A cylindrical vessel having diameter equal to its height is full of water which is poured into two identical cylindrical vessels with diameter 42 cm and height 21 cm which are filled completely. Find the diameter of the cylindrical vessel.

7. 50 circular plates each of diameter 14 cm and thickness 0.5 cm are placed one above the other to form a right circular cylinder. Find its total surface area.

8. 25 circular plates, each of radius 10.5 cm and thickness 1.6 cm, are placed one above the other to form a solid circular cylinder. Find the curved surface area and the volume of the cylinder so formed.

9. Find the number of metallic circular discs with 1.5 cm base diameter and of height 0.2 cm to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm. **[NCERT EXEMPLAR]**

10. How many spherical lead shots each of diameter 4.2 cm can be obtained from a solid rectangular lead piece with dimensions $6 \text{ cm} \times 42 \text{ cm} \times 21 \text{ cm}$. **[NCERT EXEMPLAR]**

11. How many spherical lead shots of diameter 4 cm can be made out of a solid cube of lead whose edge measures 44 cm. **[NCERT EXEMPLAR]**

12. Three cubes of a metal whose edges are in the ratio $3:4:5$ are melted and converted into a single cube whose diagonal is $12\sqrt{3}$ cm. Find the edges of the three cubes. **[NCERT EXEMPLAR]**

13. A solid metallic sphere of radius 10.5 cm is melted and recast into a number of smaller cones, each of radius 3.5 cm and height 3 cm. Find the number of cones so formed. **[CBSE 2017, NCERT EXEMPLAR]**

14. The diameter of a metallic sphere is equal to 9 cm. It is melted and drawn into a long wire of diameter 2 mm having uniform cross-section. Find the length of the wire.

15. An iron spherical ball has been melted and recast into smaller balls of equal size. If the radius of each of the smaller balls is 1/4 of the radius of the original ball, how many such balls are made? Compare the surface area, of all the smaller balls combined together with that of the original ball.

16. A copper sphere of radius 3 cm is melted and recast into a right circular cone of height 3 cm. Find the radius of the base of the cone.

17. A copper rod of diameter 1 cm and length 8 cm is drawn into a wire of length 18 m of uniform thickness. Find the thickness of the wire. **[NCERT]**

18. The diameters of internal and external surfaces of a hollow spherical shell are 10 cm and 6 cm respectively. If it is melted and recast into a solid cylinder of length of $2\frac{2}{3}$ cm, find the diameter of the cylinder.

19. How many coins 1.75 cm in diameter and 2 mm thick must be melted to form a cuboid $11 \text{ cm} \times 10 \text{ cm} \times 7 \text{ cm}$? **[NCERT]**

20. The surface area of a solid metallic sphere is 616 cm^2. It is melted and recast into a cone of height 28 cm. Find the diameter of the base of the cone so formed. (Use $\pi = 22/7$). **[CBSE 2010]**

21. A cylindrical bucket, 32 cm high and with radius of base 18 cm, is filled with sand. This bucket is emptied out on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, find the radius and slant height of the heap. **[CBSE 2012, 2014]**

22. A solid metallic sphere of radius 5.6 cm is melted and solid cones each of radius 2.8 cm and height 3.2 cm are made. Find the number of such cones formed. **[CBSE 2014, 2017]**

23. A solid cuboid of iron with dimensions 53 cm × 40 cm × 15 cm is melted and recast into a cylindrical pipe. The outer and inner diameters of pipe are 8 cm and 7 cm respectively. Find the length of pipe. **[CBSE 2015]**

24. The diameters of the internal and external surfaces of a hollow spherical shell are 6 cm and 10 cm respectively. If it is melted and recast into a solid cylinder of diameter 14 cm, find the height of the cylinder. **[CBSE 2001 C]**

25. A hollow sphere of internal and external diameters 4 cm and 8 cm respectively is melted into a cone of base diameter 8 cm. Calculate the height of the cone.

26. A hollow sphere of internal and external radii 2 cm and 4 cm respectively is melted into a cone of base radius 4 cm. Find the height and slant height of the cone.

27. A spherical ball of radius 3 cm is melted and recast into three spherical balls. The radii of two of the balls are 1.5 cm and 2 cm. Find the diameter of the third ball.

28. A path 2 m wide surrounds a circular pond of diameter 40 m. How many cubic metres of gravel are required to grave the path to a depth of 20 cm?

29. A 16 m deep well with diameter 3.5 m is dug up and the earth from it is spread evenly to form a platform 27.5 m by 7 m. Find the height of the platform.

30. A well of diameter 2 m is dug 14 m deep. The earth taken out of it is spread evenly all around it to form an embankment of height 40 cm. Find the width of the embankment. **[CBSE 2015]**

31. A well with inner radius 4 m is dug 14 m deep. Earth taken out of it has been spread evenly all around a width of 3 m it to form an embankment. Find the height of the embankment. **[CBSE 2016]**

32. A well of diameter 3 m is dug 14 m deep. The earth taken out of it has been spread evenly all around it to a width of 4 m to form an embankment. Find the height of the embankment. **[CBSE 2016]**

33. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 9 cm.

34. A cylindrical bucket, 32 cm high and 18 cm of radius of the base, is filled with sand. This bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, find the radius and slant height of the heap.

35. Rain water, which falls on a flat rectangular surface of length 6 m and breadth 4 m is transferred into a cylindrical vessel of internal radius 20 cm. What will be the height of water in the cylindrical vessel if a rainfall of 1 cm has fallen ? [Use $\pi = 22/7$]

36. The rain water from a roof of dimensions 22 m × 20 m drains into a cylindrical vessel having diameter of base 2 m and height 3.5 m. If the rain water collected from the roof just fills the cylindrical vessel, then find the rain fall in cm. **[NCERT EXEMPLAR]**

37. A conical flask is full of water. The flask has base-radius r and height h. The water is poured into a cylindrical flask of base-radius mr. Find the height of water in the cylindrical flask.

38. A rectangular tank 15 m long and 11 m broad is required to receive entire liquid contents from a full cylindrical tank of internal diameter 21 m and length 5 m. Find the least height of the tank that will serve the purpose.

39. A hemispherical bowl of internal radius 9 cm is full of liquid. This liquid is to be filled into cylindrical shaped small bottles each of diameter 3 cm and height 4 cm. How many bottles are necessary to empty the bowl?

40. A cylindrical tub of radius 12 cm contains water to a depth of 20 cm. A spherical ball is dropped into the tub and the level of the water is raised by 6.75 cm. Find the radius of the ball.

41. 500 persons have to dip in a rectangular tank which is 80 m long and 50 m broad. What is the rise in the level of water in the tank, if the average displacement of water by a person is $0.04\ m^3$? **[NCERT EXEMPLAR]**

42. A cylindrical jar of radius 6 cm contains oil. Iron spheres each of radius 1.5 cm are immersed in the oil. How many spheres are necessary to raise the level of the oil by two centimetres?

43. A cylindrical tub of radius 12 cm contains water to a depth of 20 cm. A spherical form ball of radius 9 cm is dropped into the tub and thus the level of water is raised by h cm. What is the value of h?

44. Metal spheres, each of radius 2 cm, are packed into a rectangular box of internal dimension 16 cm × 8 cm × 8 cm when 16 spheres are packed the box is filled with preservative liquid. Find the volume of this liquid. [Use $\pi = 669/213$]

45. A vessel in the shape of a cuboid contains some water. If three indentical spheres are immersed in the water, the level of water is increased by 2 cm. If the area of the base of the cuboid is $160\ cm^2$ and its height 12 cm, determine the radius of any of the spheres.

46. 150 spherical marbles, each of diameter 1.4 cm are dropped in a cylindrical vessel of diameter 7 cm containing some water, which are completely immersed in water. Find the rise in the level of water in the vessel. **[CBSE 2014]**

47. Sushant has a vessel, of the form of an inverted cone, open at the top, of height 11 cm and radius of top as 2.5 cm and is full of water. Metallic spherical balls each of diameter 0.5 cm are put in the vessel due to which $\left(\frac{2}{5}\right)^{th}$ of the water in the vessel flows out. Find how many balls were put in the vessel. Sushant made the arrangement so that the water that flows out irrigates the flower beds. What value has been shown by Sushant? **[CBSE 2014]**

48. 16 glass spheres each of radius 2 cm are packed into a cuboidal box of internal dimensions 16 cm × 8 cm × 8 cm and then the box is filled with water. Find the volume of water filled in the box. **[NCERT EXEMPLAR]**

49. Water flows through a cylindrical pipe, whose inner radius is 1 cm, at the rate of 80 cm/sec in an empty cylindrical tank, the radius of whose base is 40 cm. What is the rise of water level in tank in half an hour? **[NCERT EXEMPLAR]**

50. Water in a canal 1.5 m wide and 6 m deep is flowing with a speed of 10 km/hr. How much area will it irrigate in 30 minutes if 8 cm of standing water is desired? **[NCERT]**

51. A farmer runs a pipe of internal diameter 20 cm from the canal into a cylindrical tank in his field which is 10 m in diameter and 2 m deep. If water flows through the pipe at the rate of 3 km/h, in how much time will the tank be filled? **[CBSE 2014, NCERT]**

52. A cylindrical tank full of water is emptied by a pipe at the rate of 225 litres per minute. How much time will it take to empty half the tank, if the diameter of its base is 3 m and its height is 3.5 m? [Use $\pi = 22/7$] [CBSE 2014]

53. Water is flowing at the rate of 2.52 km/h through a cylinderical pipe into a cylindrical tank, the radius of the base is 40 cm. If the increase in the level of water in the tank, in half an hour is 3.15 m, find the internal diameter of the pipe. [CBSE 2015]

54. Water flows at the rate of 15 km/hr through a pipe of diameter 14 cm into a cuboidal pond which is 50 m long and 44 m wide. In what time will the level of water in the pond rise by 21 cm? [NCERT EXEMPLAR]

55. A canal is 300 cm wide and 120 cm deep. The water in the canal is flowing with a speed of 20 km/h. How much area will it irrigate in 20 minutes if 8 cm of standing water is desired? [NCERT EXEMPLAR]

56. The sum of the radius of base and height of a solid right circular cylinder is 37 cm. If the total surface area of the solid cylinder is 1628 cm^2, find the volume of cylinder. (Use $\pi = 22/7$) [CBSE 2016]

57. A tent of height 77 dm is in the form a right circular cylinder of diameter 36 m and height 44 dm surmounted by a right circular cone. Find the cost of the canvas at ₹ 3.50 per m^2. [Use $\pi = 22/7$]

58. The largest sphere is to be curved out of a right circular cylinder of radius 7 cm. and height 14 cm. Find the volume of the sphere.

59. A right angled triangle whose sides are 3 cm, 4 cm and 5 cm is revolved about the sides containing the right angle in two ways. Find the difference in volumes of the two cones so formed. Also, find their curved surfaces.

60. A 5 m wide cloth is used to make a conical tent of base diameter 14 m and height 24 m. Find the cost of cloth used at the rate of ₹ 25 per metre. [Use $\pi = 22/7$] [CBSE 2014]

61. The volume of a hemi-sphere is $2425\frac{1}{2}\text{cm}^3$. Find its curved surface area. (Use $\pi = 22/7$) [CBSE 2012]

62. The difference between the outer and inner curved surface areas of a hollow right circular cylinder 14 cm long is 88 cm^2. If the volume of metal used in making the cylinder is 176 cm^3, find the outer and inner diameters of the cylinder. (Use $\pi = 22/7$) [CBSE 2010]

63. The internal and external diameters of a hollow hemispherical vessel are 21 cm and 25.2 cm respectively. The cost of painting 1 cm^2 of the surface is 10 paise. Find the total cost to paint the vessel all over.

64. Prove that the surface area of a sphere is equal to the curved surface area of the circumscribed cylinder.

65. If the total surface area of a solid hemisphere is 462 cm^2, find its volume (Take $\pi = 22/7$) [CBSE 2014]

66. Water flows at the rate of 10 m/minute through a cylindrical pipe 5 mm in diameter. How long would it take to fill a conical vessel whose diameter at the base is 40 cm and depth 24 cm? [NCERT EXEMPLAR]

67. A solid right circular cone of height 120 cm and radius 60 cm is placed in a right circular cylinder full of water of height 180 cm such that it touches the bottom. Find the volume of water left in the cylinder, if the radius of the cylinder is equal to the radius of the cone. [NCERT EXEMPLAR]

68. A heap of rice in the form of a cone of diameter 9 m and height 3.5 m. Find the volume of rice. How much canvas cloth is required to cover the heap?

[NCERT EXEMPLAR, CBSE 2018]

69. A cylindrical bucket of height 32 cm and base radius 18 cm is filled with sand. This bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, find the radius and slant height of the heap.

[NCERT EXEMPLAR]

70. A hemispherical bowl of internal radius 9 cm is full of liquid. The liquid is to be filled into cylindrical shaped bottles each of radius 1.5 cm and height 4 cm. How many bottles are needed to empty the bowl? **[NCERT EXEMPLAR]**

71. A factory manufactures 120,000 pencils daily The pencils are cylindrical in shape each of length 25 cm and circumference of base as 1.5 cm. Determine the cost of colouring the curved surfaces of the pencils manufactured in one day at ₹ 0.05 per dm^2.

[NCERT EXEMPLAR]

72. The $\frac{3}{4}$th part of a conical vessel of internal radius 5 cm and height 24 cm is full of water. The water is emptied into a cylindrical vessel with internal radius 10 cm. Find the height of water in cylindrical vessel. **[CBSE 2017]**

ANSWERS

1. 512
2. 8400
3. 5 cm
4. 448 metres
5. 79 cm
6. 42 cm
7. 1408 cm^2
8. 2640 cm^2, 13860 cm^3
9. 450
10. 1500
11. 2541
12. 6 cm, 8 cm, 10 cm
13. 126
14. 12150 cm
15. 64 balls, 4 : 1
16. 6 cm
17. 0.67 mm
18. 14 cm
19. 400
20. 14 cm
21. 36 cm, $12\sqrt{13}$ cm
22. 28
23. 2698.18 cm
24. $\frac{8}{3}$ cm
25. 14 cm
26. 14 cm, 14.56 cm
27. 2.5 cm
28. 52.8 m^3
29. 80 cm
30. 5 m
31. 6.78 m
32. 9/8
33. 190.93 cm^3
34. 36 cm, 43.27cm
35. 191 cm
36. 2.5 cm
37. $\frac{h}{3m^2}$
38. 10.5 m
39. 54
40. 9 cm
41. 0.5 cm
42. 16
43. 6.75 cm
44. 448 cm^3
45. 2.94 cm
46. 5.6 cm
47. 440
48. 487.6 cm^3
49. 90 cm
50. 562500 m^2
51. 1 hour 40 minutes
52. 55 minutes
53. 4 cm
54. 2 hour
55. 30 hectares
56. 4620 cm^3
57. ₹ 5365. 80
58. 1437
59. 4π cm^3, 20π cm^2, 15π cm^2
60. ₹ 2750
61. 693 cm^2
62. 5 cm, 3 cm
63. ₹ 184.34
65. 7 cm
66. 51 minutes 12 sec
67. 1.584 m^3
68. 74.25 m^3, 80.61 m^2
69. 36 cm, 43.27 cm
70. 54
71. ₹ 2250
72. 1.5 cm.

HINT TO SELECTED PROBLEMS

1. Number of balls $= \dfrac{\text{Volume of sphere of radius 8 cm}}{\text{Volume of sphere of radius 1 cm}} = \dfrac{\frac{4}{3}\pi \times 8^3}{\frac{4}{3}\pi \times 1^3} = 512$

19. Total number of coins $= \dfrac{11 \times 10 \times 7}{\dfrac{22}{7} \times \left(\dfrac{1.75}{2}\right)^2 \times \left(\dfrac{2}{10}\right)} = \dfrac{11 \times 10 \times 7}{\dfrac{22}{7} \times \left(\dfrac{7}{4}\right)^2 \times \dfrac{2}{10}} = 400$

27. Let r be the radius of the third ball. Then,

$$\frac{4}{3}\pi \times (3)^3 = \frac{4}{3}\pi \times (1.5)^3 + \frac{4}{3}\pi \times (2)^3 + \frac{4}{3}\pi r^3 \Rightarrow 27 = \frac{27}{8} + 8 + r^3 \Rightarrow r^3 = \frac{125}{8} \Rightarrow r = \frac{5}{2}$$

31. Height of embankment $= \dfrac{\text{Volume of the earth dugout}}{\text{Area of the embankment}} = \dfrac{\dfrac{22}{7} \times 4^2 \times 14}{\dfrac{22}{7} \times (7^2 - 4^2)} = 6.78$ m

32. Height of the embankment $= \dfrac{\pi \times \dfrac{3}{2} \times \dfrac{3}{2} \times 14}{\pi\left(\dfrac{11}{2} \times \dfrac{11}{2} - \dfrac{3}{2} \times \dfrac{3}{2}\right)}$ m $= \dfrac{9}{8}$ m

34. Let r be the radius and 1 the slant height. Then,

Volume of the bucket = Volume of the heap $\Rightarrow \pi \times 18^2 \times 32 = \dfrac{1}{3}\pi \times r^2 \times 24 \Rightarrow r = 36$

38. Let the least height be h metre. Then, $15 \times 11 \times h = \dfrac{1}{3}\pi \times \left(\dfrac{21}{2}\right)^2 \times 5.$

42. Suppose x iron spheres are required to raise the level of the oil by two cm. Then,

Volume of x iron spheres = Volume of oil raised

$$\Rightarrow x \times \frac{4}{3}\pi \times (1.5)^3 = \pi \times 6^2 \times 2 \Rightarrow x \times \frac{4}{3}\pi\left(\frac{3}{2}\right)^2 = 72\pi \Rightarrow x = 16$$

43. Let r cm be the radius of the ball. Then,

Volume of ball = Volume of water raised $\Rightarrow \dfrac{4}{3}\pi r^2 = \pi \times (12)^2 \times 6.75 \Rightarrow r = 9$ cm.

50. Required area $= \dfrac{1.5 \times 6 \times 10000 \times \dfrac{1}{2}}{\left(\dfrac{8}{100}\right)}\text{ m}^2 = \dfrac{9 \times 5000 \times 25}{2}\text{ m}^2 = 562500\text{ m}^2$

51. Volume of cylindrical tank $= \pi \times 5^2 \times 2\text{ m}^3 = 50\pi\text{ m}^3$

Volume of the water that flows through the pipe in t hours

= Volume of a cylinder of radius of 10 cm and length = $3t$ km = $3000\,t$ m

$$= \left\{\pi \times \left(\frac{1}{10}\right)^2 \times 3000\,t\right\}\text{m}^3 = 30\pi t\text{ m}^3$$

$\therefore \quad 30\pi t = 50\pi \Rightarrow t = \dfrac{5}{3}$ hours = 1 hour 40 minutes

64. Height of the circumscribing cylinder = $2r$,

Radius of the circumscribing cylinder = r, where r is the radius of the sphere.

14.4 SURFACE AREAS AND VOLUMES OF COMBINATIONS OF SOLIDS

Uptill now we have learnt about various applications of the formulas for finding the surface areas and volumes of basic solids, namely, a right circular cone, a right circular cylinder, a sphere, a hemisphere etc. In this section, we shall apply them to find the surface areas and volumes of solids which are combinations of the basic solids. For example, a circus tent consisting of a cylindrical base surmounted by a conical roof, a toy in the form of a cone mounted on a hemisphere etc. are combinations of two or more basic solids. Following examples will illustrate the method of finding surface areas and volumes of such combinations of solids.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *An iron pillar has some part in the form of a right circular cylinder and remaining in the form of a right circular cone. The radius of the base of each of cone and cylinder is 8 cm. The cylindrical part is 240 cm high and the conical part is 36 cm high. Find the weight of the pillar if one cubic cm of iron weighs 7.8 grams.*

SOLUTION Let r_1 cm and r_2 cm denote the radii of the base of the cylinder and cone respectively. Then,

$$r_1 = r_2 = 8 \text{ cm}$$

Let h_1 and h_2 cm be the heights of the cylinder and the cone respectively. Then,

$$h_1 = 240 \text{ cm and } h_2 = 36 \text{ cm}$$

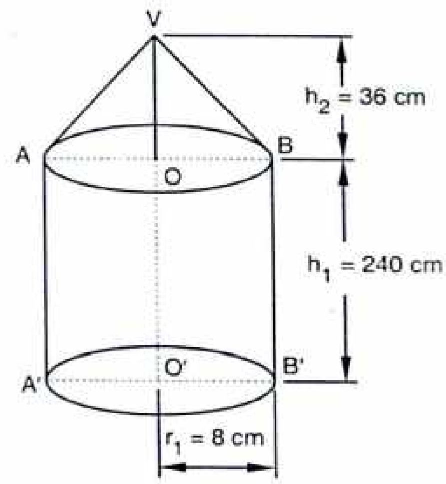

Fig. 14.16

$\therefore$ Volume of the cylinder $= \pi r_1^2 h_1 \text{ cm}^3$

$$= \pi \times 8 \times 8 \times 240) \text{ cm}^3$$

$$= (\pi \times 64 \times 240) \text{ cm}^3$$

Volume of the cone $= \frac{1}{3}\pi r_2^2 h_2 \text{ cm}^3$

$$= \left(\frac{1}{3}\pi \times 8 \times 8 \times 36\right) \text{cm}^3$$

$$= \left(\frac{1}{3}\pi \times 64 \times 36\right) \text{cm}^3$$

$\therefore$ Total volume of the iron = Volume of the cylinder + Volume of the cone

$$= \left(\pi \times 64 \times 240 + \frac{1}{3}\pi \times 64 \times 36\right) \text{cm}^3$$

$$= \pi \times 64 \times (240 + 12) \text{ cm}^3$$

$$= \frac{22}{7} \times 64 \times 252 \text{ cm}^3 = 22 \times 64 \times 36 \text{ cm}^3$$

Hence, Total weight of the pillar = Volume $\times$ Weight per cm^3

$$= (22 \times 64 \times 36) \times 7.8 \text{ gms}$$

$$= 395366.4 \text{ gms} = 395.3664 \text{ kg}$$

EXAMPLE 2 *The interior of a building is in the form of a right circular cylinder of diameter 4.2 m and height 4 m surmounted by a cone. The vertical height of cone is 2.1 m. Find the outer surface area and volume of the building.* (Use $\pi = 22/7$)

SOLUTION Let r_1 be the radius of base of the cylinder and h_1 m be its height. It is given that $r_1 = 2.1$ m and $h_2 = 4$ m. Let r_2 m be the radius of the base of the cone, h_2 m be its height and l_2 m and be its slant height. It is also given that $r_2 = 2.1$ m, $h_2 = 2.1$ m.

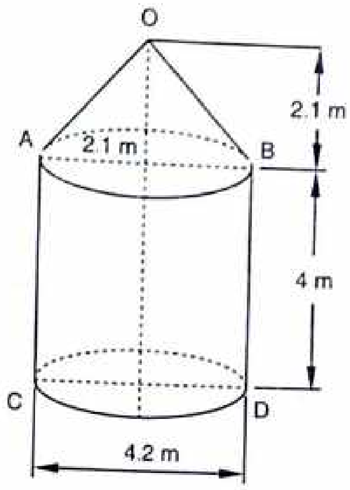

Fig. 14.17

$\therefore \quad l_2^2 = r_2^2 + h_2^2$

$\Rightarrow \quad l_2 = \sqrt{r_2^2 + h_2^2} = \sqrt{(2.1)^2 + (2.1)^2} = \sqrt{(2.1)^2 \times 2} = 2.1 \times \sqrt{2}$ m

Let S be the outer surface area and V be the volume of the building. Then,

$$S = \text{Curved surface area of cylinder} + \text{Curved surface area of cone}$$
$$= (2\pi r_1 h_1 + \pi r_2 l_2)\ \text{m}^2$$
$$= \pi(2r_1 h_1 + r_2 l_2)\ \text{m}^2$$
$$= \frac{22}{7}(2 \times 2.1 \times 4 + 2.1 \times 2.1 \times \sqrt{2})\ \text{m}^2$$
$$= \frac{22}{7} \times 2.1 \times (8 + 2.1 \times \sqrt{2})\ \text{m}^2$$
$$= \frac{22}{7} \times 2.1 \times (8 + 2.1 \times 1.414)\ \text{m}^2$$
$$= \frac{22}{7} \times 2.1 \times (8 + 2.9694)\ \text{m}^2$$
$$= \frac{22}{7} \times 2.1 \times 10.9694\ \text{m}^2 = 22 \times 0.3 \times 10.9694\ \text{m}^2 = 72.3980\ \text{m}^2 = 72.40\ \text{m}^2$$

and, $\quad V = \text{Volume of the cylinder} + \text{Volume of the cone}$

$$= \left(\pi r_1^2 h_1 + \frac{1}{3}\pi r_2^2 h_2\right) \text{m}^3$$
$$= \left(\pi r_1^2 h_1 + \frac{1}{3}\pi r_1^2 h_2\right) \text{m}^3 \qquad [\because r_2 = r_1]$$
$$= \pi r_1^2 \left(h_1 + \frac{1}{3}h_2\right) \text{m}^3$$
$$= \frac{22}{7} \times 2.1 \times 2.1 \times \left(4 + \frac{1}{3} \times 2.1\right) \text{m}^3$$
$$= \frac{22}{7} \times 2.1 \times 2.1 \times (4 + 0.7)\ \text{m}^3 = 22 \times 0.3 \times 2.1 \times 4.7\ \text{m}^3 = 65.142\ \text{m}^3$$

EXAMPLE 3 *A circus tent is cylindrical upto a height of 3 m and conical above it. If the diameter of the base is 105 m and the slant height of the conical part is 53 m, find the total canvas used in making the tent.* **[CBSE 2004]**

SOLUTION The total canvas used is equal to the outer surface area of the tent.

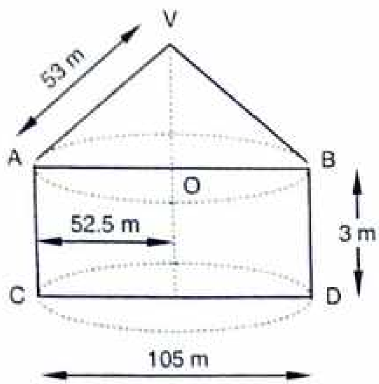

Fig. 14.18

Total canvas used = Curved surface area of cylinder + Curved surface area of cone

$$= \left(2 \times \frac{22}{7} \times 52.5 \times 3 + \frac{22}{7} \times 52.5 \times 53\right) \text{m}^2 \qquad [\because S = 2\pi rh + \pi rl]$$

$$= \frac{22}{7} \times 52.5\,(6 + 53)\ \text{m}^2 = 9735\ \text{m}^2$$

EXAMPLE 4 *A right circular cylinder having diameter 12 cm and height 15 cm is full of ice-cream. The ice-cream is to be filled in cones of height 12 cm and diameter 6 cm having a hemispherical shape on the top. Find the number of such cones which can be filled with ice-cream.* **[CBSE 2006C]**

SOLUTION We have, r = Radius of the cylinder = 6 cm, h = Height of the cylinder = 15 cm

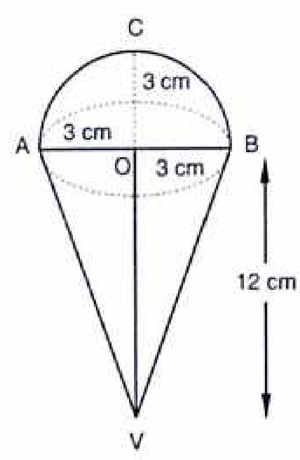

Fig. 14.19

$\therefore$ Volume of the cylinder $= \pi r^2 h = \pi \times 6^2 \times 15\ \text{cm}^3 = 540\,\pi\ \text{cm}^3$

It is given that

r_1 = Radius of the ice-cream cone = 3 cm and, h_1 = Height of the ice-cream cone = 12 cm

$\therefore$ Volume of the conical part of ice-cream cone $= \frac{1}{3}\pi r_1^2 h_1 = \frac{1}{3} \times \pi \times 3^2 \times 12\ \text{cm}^3 = 36\pi\ \text{cm}^3$

Volume of the hemispherical top of the ice-cream cone $= \frac{2}{3}\pi r_1^3 = \frac{2}{3} \times \pi \times 3^3 = 18\pi \text{ cm}^3$

Total volume of the ice-cream cone $= (36\pi + 18\pi) \text{ cm}^3 = 54\pi \text{ cm}^3$

$\therefore$ Number of ice-cream cones $= \dfrac{\text{Volume of the cylinder}}{\text{Total volume of ice-cream cone}} = \dfrac{540\pi}{54\pi} = 10$

EXAMPLE 5 *A tent is of the shape of a right circular cylinder upto a height of 3 metres and then becomes a right circular cone with a maximum height of 13.5 metres above the ground. Calculate the cost of painting the inner side of the tent at the rate of ₹2 per square metre, if the radius of the base is 14 metres.* **[CBSE 2005]**

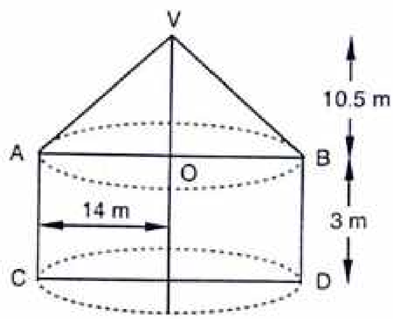

Fig. 14.20

SOLUTION Let r m be the radius of the base of the cylinder and h metres be its height. It is given that $r = 14$ m and $h = 3$ m.

Curved surface area of the cylinder $= 2\pi r h \text{ m}^2 = \left(2 \times \frac{22}{7} \times 14 \times 3\right) \text{m}^2 = 264 \text{ m}^2$

Let r_1 m be the radius of the base, h_1 m be the height and l_1 m be the slant height of the cone. It is given that $r_1 = 14$ m, $h_1 = (13.5 - 3)$ m $= 10.5$ m.

$$\therefore \quad l_1 = \sqrt{r_1^2 + h_1^2} = \sqrt{14^2 + (10.5)^2} = \sqrt{196 + 110.25} = \sqrt{306.25} = 17.5$$

$$\therefore \quad \text{Curved surface area of the cone} = \pi r_1 l_1 = \left(\frac{22}{7} \times 14 \times 17.5\right) \text{m}^2 = 770 \text{ m}^2$$

Let S be the total area which is to be painted. Then,

$S =$ Curved surface area of the cylinder + Curved surface area of the cone

$\Rightarrow \quad S = (264 + 770) \text{ m}^2 = 1034 \text{ m}^2$

Hence, Cost of painting $= S \times$ Rate $=$ ₹ $(1034 \times 2) =$ ₹ 2068

EXAMPLE 6 *A solid wooden toy is in the shape of a right circular cone mounted on a hemisphere. If the radius of the hemisphere is 4.2 cm and the total height of the toy is 10.2 cm, find the volume of the wooden toy.* **[CBSE 2012]**

SOLUTION We have, $VO' = 10.2$ cm, $OA = OO' = 4.2$ cm

Let r be the radius of the hemisphere and h be the height of the conical part of the toy. Then,
$r = OA = 4.2$ cm, $h = VO = VO' - OO' = (10.2 - 4.2)$ cm $= 6$ cm

Also, radius of the base of the cone $= OA = r = 4.2$ cm

Let V be the volume of the wooden toy. Then,

V = Volume of the conical part + Volume of the hemispherical part

$$\Rightarrow \quad V = \left(\frac{1}{3}\pi r^2 h + \frac{2\pi}{3} r^3\right) \text{cm}^3$$

$$\Rightarrow \quad V = \frac{\pi r^2}{3}(h + 2r) \text{ cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times 4.2 \times 4.2 \times (6 + 2 \times 4.2) \text{ cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times 4.2 \times 4.2 \times 14.4 \text{ cm}^3 = 266.11 \text{ cm}^3$$

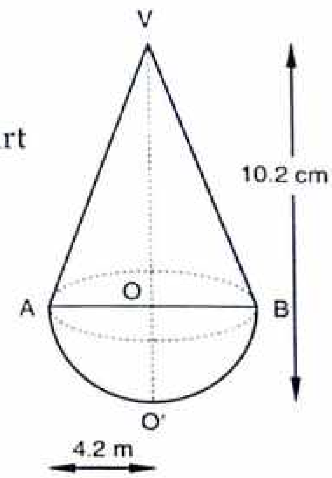

Fig. 14.21

EXAMPLE 7 *A solid is in the form of a right circular cone mounted on a hemisphere. The radius of the hemisphere is 3.5 cm and the height of the cone is 4 cm. The solid is placed in a cylindrical tub, full of water, in such a way that the whole solid is submerged in water. If the radius of the cylinder is 5 cm and its height is 10.5 cm, find the volume of water left in the cylindrical tub.* (*Use* $\pi = 22/7$)

SOLUTION We have, $VO = 4$ cm, $OA = OB = OO' = 3.5$ cm.

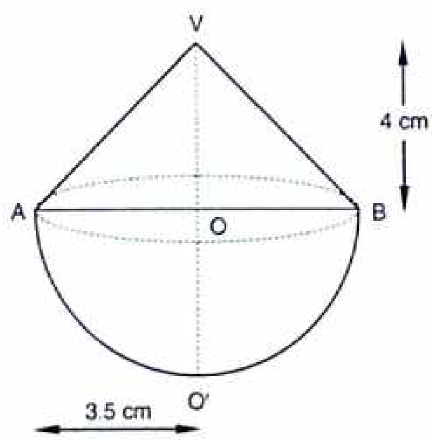

Fig. 14.22

$\therefore$ Volume of the solid = Volume of its conical part + Volume of its hemispherical part

$$= \left\{\frac{1}{3} \times \frac{22}{7} \times (3.5)^2 \times 4 + \frac{2}{3} \times \frac{22}{7} \times (3.5)^3\right\} \text{cm}^3$$

$$= \frac{1}{3} \times \frac{22}{7} \times (3.5)^2 \{4 + 2 \times 3.5\} \text{ cm}^3 = \left\{\frac{1}{3} \times \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 11\right\} \text{cm}^3$$

Clearly, when the solid is submerged in the cylindrical tub the volume of water that flows out of the cylinder is equal to the volume of the solid.

Hence,

Volume of water left in the cylinder = Volume of cylinder – Volume of the solid

$$= \left\{\frac{22}{7} \times (5)^2 \times 10.5 - \frac{1}{3} \times \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 11\right\} \text{cm}^3$$

$$= \left\{\frac{22}{7} \times 25 \times \frac{21}{2} - \frac{1}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 11\right\} \text{cm}^3$$

$$= \left(11 \times 25 \times 3 - \frac{1}{3} \times 11 \times \frac{7}{2} \times 11\right) \text{cm}^3$$

$$= (825 - 141.16)\ \text{cm}^3 = 683.83\ \text{cm}^3$$

EXAMPLE 8 *A cylindrical container of radius 6 cm and height 15 cm is filled with ice-cream. The whole ice-cream has to be distributed to 10 children in equal cones with hemispherical tops. If the height of the conical portion is four times the radius of its base, find the radius of the ice-cream cone.*

[CBSE 2016]

SOLUTION Let the radius of the base of the conical portion be r cm. Then, height of the conical portion $= 4r$ cm.

Let V be the volume of cone with hemispherical top. Then,

$$V = \text{Volume of the cone} + \text{Volume of the hemispherical top}$$

$$= \left(\frac{1}{3}\pi r^2 \times 4r + \frac{2}{3}\pi r^3\right) \text{cm}^3 = \left(\frac{6}{3}\pi r^3\right) \text{cm}^3 = (2\pi r^3)\ \text{cm}^3$$

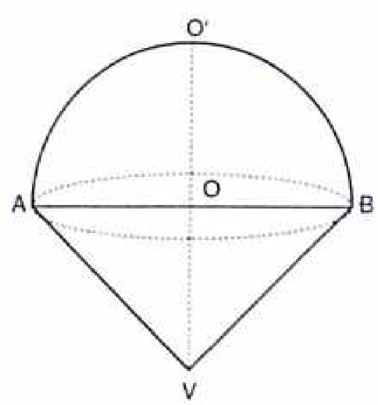

Fig. 14.23

Volume of 10 cones with hemispherical tops $= 10\ V = (10 \times 2\pi r^3)\ \text{cm}^3 = 20\pi r^3\ \text{cm}^3$

Volume of the cylindrical container $= (\pi \times 6^2 \times 15)\ \text{cm}^3 = 540\pi\ \text{cm}^3$

Clearly,

Volume of 10 cones with hemispherical tops $=$ Volume of the cylindrical container

$\Rightarrow \qquad 20\pi r^3 = 540\pi$

$\Rightarrow \qquad r^3 = 27$

$\Rightarrow \qquad r = 3$ cm

Hence, radius of the ice-cream cone is 3 cm.

EXAMPLE 9 *A solid is in the form of a cylinder with hemispherical ends. The total height of the solid is 19 cm and the diameter of the cylinder is 7 cm. Find the volume and total surface area of the solid (Use $\pi = 22/7$)*

SOLUTION Let r cm be the radius and h cm the height of the cylinder. Then,

$$r = \frac{7}{2}\ \text{cm} \ \text{and,}\ h = \left(19 - 2 \times \frac{7}{2}\right) \text{cm} = 12\ \text{cm}$$

Also, radius of hemisphere $= \frac{7}{2}$ cm $= r$ cm

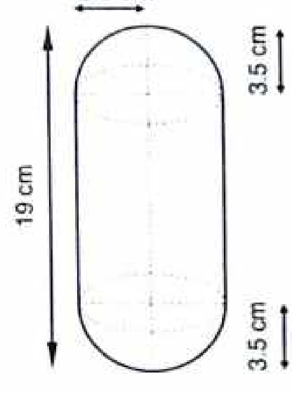

Fig. 14.24

Let V be the volume and S be the surface area of the solid. Then,

$V =$ Volume of the cylinder + Volume of two hemispheres

$$\Rightarrow \quad V = \left\{\pi r^2 h + 2\left(\frac{2}{3}\pi r^3\right)\right\} \text{cm}^3$$

$$\Rightarrow \quad V = \pi r^2\left(h + \frac{4r}{3}\right) \text{cm}^3$$

$$\Rightarrow \quad V = \left\{\frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times \left(12 + \frac{4}{3} \times \frac{7}{2}\right)\right\} \text{cm}^3 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times \frac{50}{3} \text{cm}^3 = 641.66 \text{ cm}^3$$

and,

$S =$ Curved surface area of cylinder + Surface area of two hemispheres

$$\Rightarrow \quad S = (2\pi r h + 2 \times 2\pi r^2) \text{cm}^2$$

$$\Rightarrow \quad S = 2\pi r(h + 2r) \text{cm}^2$$

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times \frac{7}{2} \times \left(12 + 2 \times \frac{7}{2}\right) \text{cm}^2 = \left(2 \times \frac{22}{7} \times \frac{7}{2} \times 19\right) \text{cm}^2 = 418 \text{ cm}^2$$

EXAMPLE 10 *A solid is composed of a cylinder with hemispherical ends. If the whole length of the solid is 108 cm and the diameter of the hemispherical ends is 36 cm, find the cost of polishing the surface of the solid at the rate of 7 paise per sq. cm.* (*Use* $\pi = 22/7$).

SOLUTION We have,

$r =$ radius of the cylinder $=$ radius of hemispherical ends $= 18$ cm

$h =$ height of the cylinder $= 72$ cm

Let S be the total surface area of the solid. Then,

$S =$ Curved surface area of the cylinder + Surface areas of hemispherical ends

$$\Rightarrow \quad S = (2\pi r h + 2 \times 2\pi r^2) \text{cm}^2$$

$$\Rightarrow \quad S = (2\pi r h + 4\pi r^2) \text{cm}^2$$

$$\Rightarrow \quad S = 2\pi r(h + 2r) \text{cm}^2$$

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 18 \times (72 + 36) \text{cm}^2 \qquad [\because r = 18 \text{ cm}, h = 72 \text{ cm}]$$

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 18 \times 108 \text{ cm}^2 = 12219.42 \text{ cm}^2$$

Rate of polishing = 7 paise per sq. cm.

$$\therefore \quad \text{Cost of polishing} = ₹\left(12219.42 \times \frac{7}{100}\right) = ₹\,855.36$$

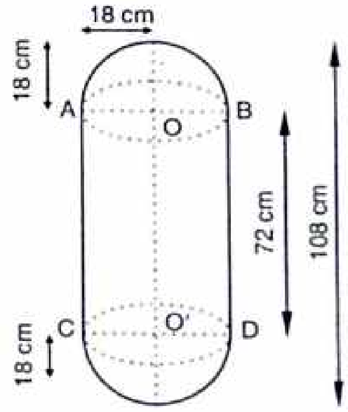

Fig. 14.25

EXAMPLE 11 *A toy is in the shape of a right circular cylinder with a hemisphere on one end and a cone on the other. The height and radius of the cylindrical part are 13 cm and 5 cm respectively. The radii of the hemispherical and conical parts are the same as that of the cylindrical part. Calculate the surface area of the toy if height of the conical part is 12 cm.*

SOLUTION Let r cm be the radius and h cm the height of the cylindrical part. It is given that $r = 5$ cm and $h = 13$ cm. Clearly, radii of the spherical part and base of the conical part are also r cm. Let h_1 cm be the height, l cm be the slant height of the conical part. Then,

$$l^2 = r^2 + h_1^2$$

$$\Rightarrow \quad l = \sqrt{r^2 + h_1^2}$$

$$\Rightarrow \quad l = \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \text{ cm} \quad [\because h_1 = 12 \text{ cm}, r = 5 \text{ cm}]$$

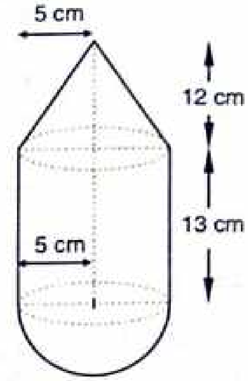

Fig. 14.26

Let S be the surface area of the toy. Then,

$S =$ Curved surface area of the cylindrical part
+ Curved surface area of hemispherical part
+ Curved surface area of conical part

$$\Rightarrow \quad S = (2\pi rh + 2\pi r^2 + \pi r l) \text{ cm}^2$$

$$\Rightarrow \quad S = \pi r(2h + 2r + l) \text{ cm}^2$$

$$\Rightarrow \quad S = \left\{\frac{22}{7} \times 5 \times (2 \times 13 + 2 \times 5 + 13)\right\} \text{cm}^2 = \left(\frac{22}{7} \times 5 \times 49\right) \text{cm}^2 = 770 \text{ cm}^2$$

EXAMPLE 12 *A vessel is in the form of a hemispherical bowl mounted by a hollow cylinder. The diameter of the sphere is 14 cm and the total height of the vessel is 13 cm. Find its capacity. (Take $\pi = 22/7$)* **[NCERT, CBSE 2006C]**

SOLUTION Let r be the radius of the hemispherical bowl and h be the height of the cylinder. It is given that $r = 7$ cm and $h = 6$ cm. Let V be the total capacity of the bowl. Then,

$V =$ Volume of the cylinder + Volume of the hemisphere

$$\Rightarrow \quad V = \left(\pi r^2 h + \frac{2}{3}\pi r^3\right) \text{cm}^3$$

$$\Rightarrow \quad V = \pi r^2 \left(h + \frac{2}{3} r\right) \text{cm}^3$$

$$\Rightarrow \quad V = \frac{22}{7} \times 7^2 \times \left(6 + \frac{2}{3} \times 7\right) \text{cm}^3$$

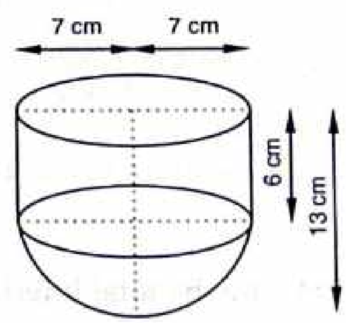

Fig. 14.27

$$\Rightarrow \quad V = 22 \times 7 \times \frac{32}{3} \text{ cm}^3 = \frac{4928}{3} \text{ cm}^3 = 1642.66 \text{ cm}^3$$

EXAMPLE 13 *A solid toy is in the form of a right circular cylinder with a hemispherical shape at one end and a cone at the other end. Their common diameter is 4.2 cm and the height of the cylindrical and conical portions are 12 cm and 7 cm respectively. Find the volume of the solid toy. (Use $\pi = 22/7$)*

[NCERT, CBSE 2002C]

SOLUTION Let V be the volume of the solid toy. Then,

$V =$ Volume of the conical portion + Volume of the cylindrical portion
+ Volume of the hemispherical portion.

$$\Rightarrow \quad V = \frac{1}{3}\pi \times (2.1)^2 \times 7 + \pi \times (2.1)^2 \times 12 + \frac{2}{3} \times \pi \times (2.1)^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \pi \times (2.1)^2\ (7 + 3 \times 12 + 2 \times 2.1)\ \text{cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \pi \times (2.1)^2 \times 47.2\ \text{cm}^3$$

$$\Rightarrow \quad V = \pi \times 0.7 \times 2.1 \times 47.2\ \text{cm}^3$$

$$\Rightarrow \quad V = \frac{22}{7} \times 0.7 \times 2.1 \times 47.2\ \text{cm}^3 = 218.064\ \text{cm}^3$$

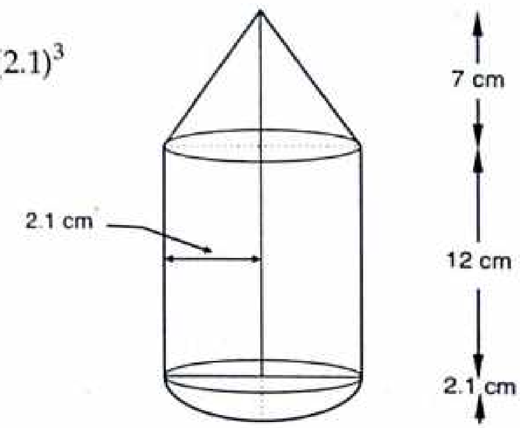

Fig. 14.28

EXAMPLE 14 *A godown building is in the form as shown in Fig. 14.29. The vertical cross-section parallel to the width side of the building is a rectangle 7 m × 3 m, mounted by a semi-circle of radius 3.5 m. The inner measurements of the cuboidal portion of the building are 10 m × 7 m × 3 m. Find the volume of the godown and the total interior surface area excluding the floor (base). (Take $\pi = 22/7$)*

SOLUTION Since the top of the building is in the form of half of the cylinder of radius 3.5 m, and length 10 m, split along the diameter. Let V be the volume of the godown. Then,

V = Volume of the cuboid $+\frac{1}{2}$ (Volume of the cylinder of radius 3.5 m and length 10 m)

$$\Rightarrow V = \left\{10 \times 7 \times 3 + \frac{1}{2}\left(\frac{22}{7} \times 3.5 \times 3.5 \times 10\right)\right\}\text{m}^3 = (210 + 192.5)\ \text{m}^3 = 402.5\ \text{m}^3$$

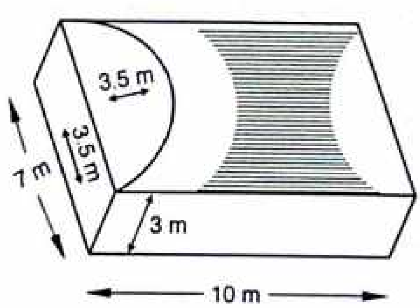

Fig. 14.29

Let S be the total interior surface area excluding the base floor. Then,

$$S = \text{Area of four walls} + \frac{1}{2}(\text{Curved surface area of the cylinder}) + 2\,(\text{Area of the semi-circles})$$

$$\Rightarrow \quad S = \left[2(10+7) \times 3 \times \frac{1}{2}\left(2 \times \frac{22}{7} \times 3.5 \times 10\right) + 2\left(\frac{1}{2} \times \frac{22}{7} \times (3.5)^2\right)\right]\text{m}^2$$

$$\Rightarrow \quad S = (102 + 110 + 38.5)\ \text{m}^2 = 250.5\ \text{m}^2$$

EXAMPLE 15 *A solid toy is in the form of a hemisphere surmounted by a right circular cone. Height of the cone is 2 cm and the diameter of the base is 4 cm. If a right circular cylinder circumscribes the solid. Find how much more space it will cover.* **[NCERT]**

SOLUTION Let BPC be the hemisphere and ABC be the cone mounted on the base of the hemisphere. Let $EFGH$ be the right circular cylinder circumscribing the given toy.

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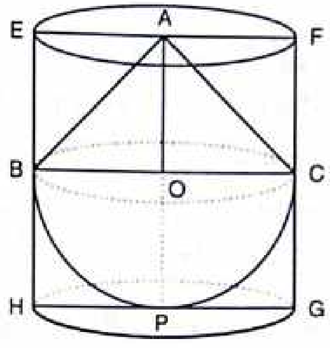

Fig. 14.30

We have,

$$OA = \text{Height of the cone} = 2 \text{ cm}$$

and, $$BC = \text{Diameter of the base of the cone} = 4 \text{ cm}$$

$\therefore$ $$BO = \text{Radius of the hemisphere} = \frac{1}{2} BC = \frac{4}{2} \text{ cm} = 2 \text{ cm}$$

$\Rightarrow$ $$OP = 2 \text{ cm} \qquad [\because OP = OB]$$

$\therefore$ $$AP = OP + OA = (2+2) \text{ cm} = 4 \text{ cm}$$

Now, Volume of the right circular cylinder $= \pi \times 2^2 \times 4 \text{ cm}^3 = 16\pi \text{ cm}^3$

Volume of the solid toy $= \left\{\frac{2}{3}\pi \times 2^3 + \frac{1}{3}\pi \times 2^2 \times 2\right\} \text{cm}^3 = 8\pi \text{ cm}^3$

$\therefore$ Required space = Volume of the right circular cylinder – Volume of the toy

$$= 16\pi \text{cm}^3 - 8\pi \text{cm}^3 = 8\pi \text{cm}^3.$$

Hence, the right circular cylinder covers $8\pi \text{cm}^3$ more space than the solid toy.

EXAMPLE 16 *From a solid circular cylinder with height 10 cm and radius of the base 6 cm, a right circular cone of the same height and same base is removed. Find the volume of the remaining solid. Also, find the whole surface area.* **[CBSE 2009]**

SOLUTION Let V be the volume of the remaining solid and S be the whole surface area. Then,

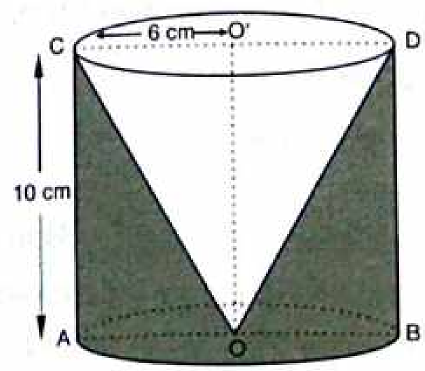

Fig. 14.31

$$V = \text{Volume of the cylinder} - \text{Volume of the cone.}$$

$\Rightarrow$ $$V = \left\{\pi \times 6^2 \times 10 - \frac{1}{3} \times \pi \times 6^2 \times 10\right\} \text{cm}^3 = (360\pi - 120\pi) \text{ cm}^3 = 240\pi \text{cm}^3$$

Slant height of the cone $= OC = \sqrt{OO'^2 + O'C^2} = \sqrt{10^2 + 6^2} = \sqrt{136}$ cm $= 2\sqrt{34}$ cm

and,

S = Curved surface area of the cylinder
+ Area of the base of the cylinder + Curved surface area of cone

$\Rightarrow \quad S = \{2\pi \times 6 \times 10 + \pi \times 6^2 + \pi \times 6 \times 2\sqrt{34}\}\ \text{cm}^2 = (156 + 12\sqrt{34})\ \pi\ \text{cm}^2$

EXAMPLE 17 *A wooden article was made by scooping out a hemisphere from each end of a solid cylinder, as shown in Fig. 14.32. If the height of the cylinder is 10 cm, and its base is of radius 3.5 cm, find the total surface area of the article.* **[CBSE 2014, 2018, NCERT]**

SOLUTION Let r be the radius of the base of the cylinder and h be its height. Let S be the total surface area of the article. Then,

S = Curved surface area of the cylinder + 2 (Surface area of a hemisphere)

$\Rightarrow \quad S = 2\pi rh + 2(2\pi r^2)$

$\Rightarrow \quad S = 2\pi r(h + 2r)$

$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 3.5(10 + 2 \times 3.5)\ \text{cm}^2$

$\Rightarrow \quad S = 22 \times 17\ \text{cm}^2 = 374\ \text{cm}^2$

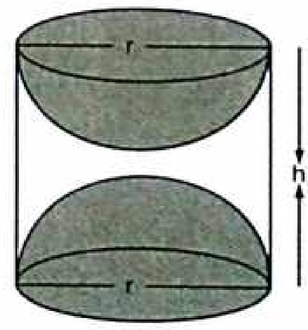

Fig. 14.32

EXAMPLE 18 *A tent is in the shape of a cylinder surmounted by a conical top. If the height and diameter of the cylindrical part are 2.1 m and 4 m, and slant height of the top is 2.8 m, find the area of the canvas used for making the tent. Also, find the cost of canvas of the tent at the rate of ₹ 500 per m^2.* **[NCERT]**

SOLUTION For conical portion, $r = 2$ m and $l = 2.8$ m. Let S_1 the curved surface area of conical portion. Then,

$$S_1 = \pi rl = \pi \times 2 \times 2.8\ \text{m}^2 = 5.6\pi\ \text{m}^2$$

For cylindrical portion, we have $r = 2$ m, $h = 2.1$ m

Let S_2 be the curved surface area of cylindrical portion. Then,

$$S_2 = 2\pi rh = 2\pi \times 2 \times 2.1\ \text{m} = 8.4\pi\ \text{m}^2$$

Let S be the area of the canvas used. Then,

$$S = S_1 + S_2 = (5.6\pi + 8.4\pi)\ \text{m}^2 = 14 \times \frac{22}{7}\ \text{m}^2 = 44\ \text{m}^2$$

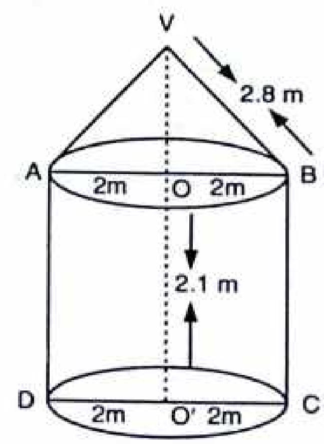

Fig. 14.33

Total cost of the canvas at the rate of ₹ 500 per m^2 = ₹ (500 × 44) = ₹ 22000

EXAMPLE 19 *From a solid cylinder whose height is 2.4 cm and diameter 1.4 cm, a conical cavity of the same height and same diameter is hollowed out. Find the total surface area of the remaining solid to the nearest cm^2.* **[NCERT, CBSE 2012, 2017]**

SOLUTION Let S be the total surface area of the remaining solid. Then,

S = Curved surface area of the cylinder + Area of the base of the cylinder
+ Curved surface area of the cone

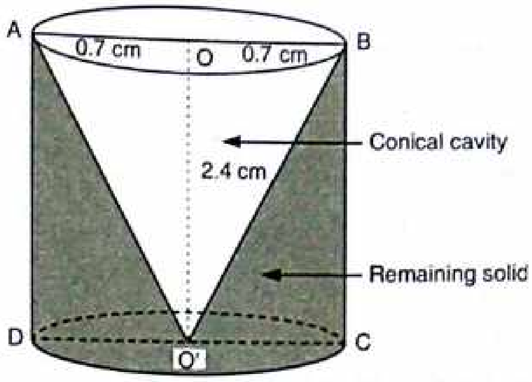

Fig. 14.34

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 0.7 \times 2.4 + \frac{22}{7} \times (0.7)^2 + \frac{22}{7} \times 0.7 \times \sqrt{(2.4)^2 + (0.7)^2}$$

$$\left[\because\ l = \sqrt{r^2 + h^2} = \sqrt{(0.7)^2 + (2.4)^2}\right]$$

$$\Rightarrow \quad S = 44 \times 0.24 + 22 \times 0.1 \times 0.7 + 22 \times 0.1 \times 2.5 = 10.56 + 1.54 + 5.5 \text{ cm}^2 = 17.6 \text{ cm}^2$$

Hence, the total surface area to the nearest cm^2 is 18 cm^2

EXAMPLE 20 *A solid is in the shape of a cone standing on a hemisphere with both their radii being equal to 1 cm and the height of the cone is equal to its radius. Find the volume of the solid in terms of π.* **[NCERT]**

SOLUTION We have,

r_1 = Radius of the cone = 1 cm, h_1 = Height of the cone = 1 cm

and, r_2 = Radius of the hemisphere = 1 cm

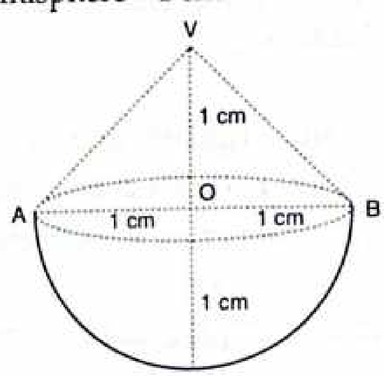

Fig. 14.35

Let V be the volume of the solid. Then,

V = Volume of the cone + Volume of the hemisphere

$$\Rightarrow \quad V = \frac{1}{3}\pi r_1^2 h_1^2 + \frac{2}{3}\pi r_2^3 = \left(\frac{1}{3}\pi \times 1^2 \times 1 + \frac{2}{3}\pi \times 1^3\right) \text{cm}^3 = \pi \text{ cm}^3$$

EXAMPLE 21 *Rachel, an engineering student was asked to make a model in her workshop, which was shaped like a cylinder with two cones attached to its two ends, using thin aluminium sheet. The diameter of the model is 3 cm and its length is 12 cm. If each cone has a height of 2 cm, find the volume of air contained in the model that Rachel made.* **[NCERT]**

SOLUTION We have,

r_1 = Radius of two conical parts = 1.5 cm, h_1= Height of two conical parts = 2 cm

r_2 = Radius of the cylindrical part = 1.5 cm, h_2= Height of the cylindrical part = 8 cm

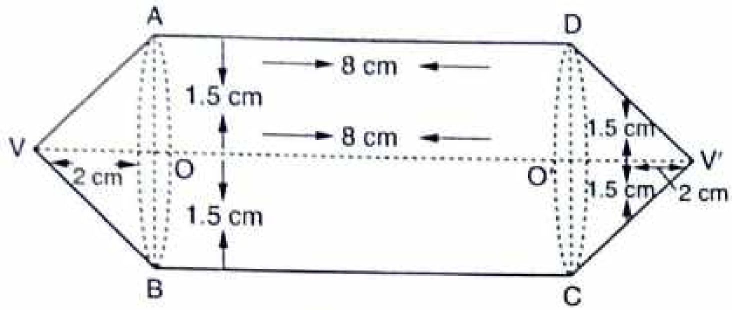

Fig. 14.36

Let V be the volume of the air contained in the model. Then,

V = Volume of the cylindrical part + Volumes of two conical parts

$$\Rightarrow \quad V = \pi r_2^2 h_2 + 2 \times \left(\frac{1}{3}\pi r_1^2 h_1\right)$$

$$\Rightarrow \quad V = \left\{\pi \times (1.5)^2 \times 8 + 2 \times \frac{1}{3} \times \pi \times (1.5)^2 \times 2\right\} \text{cm}^3$$

$$\Rightarrow \quad V = \left\{\pi \times \left(\frac{3}{2}\right)^2 \times 8 + \frac{2}{3} \times \pi \times \left(\frac{3}{2}\right)^2 \times 2\right\} \text{cm}^3$$

$$\Rightarrow \quad V = \{18\pi + 3\pi\} \text{ cm}^3 = 21\pi \text{ cm}^3 = 21 \times \frac{22}{7} \text{ cm}^3 = 66 \text{ cm}^3.$$

EXAMPLE 22 *A gulabjamun when completely ready for eating contains sugar syrup up to about 30% of its volume. Find approximately how much syrup would be found in 45 gulabjamuns shaped like a cylinder with two hemispherical ends, if the complete length of each of the gulabjamun is 5 cm and its diameter is 2.8 cm.* **[NCERT]**

SOLUTION We have,

h = Length of the cylindrical part of a gulabjamun = $(5 - 1.4 - 1.4)$ cm = 2.2 cm

r = Radius of the cylindrical part of a gulabjamun = 1.4 cm

Also, r = Radius of a hemispherical part of a gulabjamun = 1.4 cm

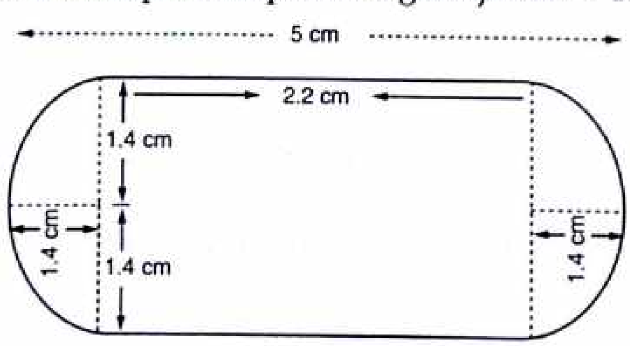

Fig. 14.37

Let V be the volume of a gulabjamun. Then,

V = Volume of two hemispherical part + Volume of cylindrical part

$$\Rightarrow \quad V = 2\left(\frac{2}{3}\pi r^3\right) + \pi r^2 h$$

$$\Rightarrow \quad V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2\left(\frac{4}{3}r + h\right)$$

$$\Rightarrow \quad V = \frac{22}{7} \times 1.4 \times 1.4 \times \left(\frac{4}{3} \times 1.4 + 2.2\right) \text{cm}^3 = 22 \times 0.2 \times 1.4 \times \frac{12.2}{3} \text{cm}^3 = \frac{75.152}{3} \text{cm}^3$$

$$\text{Volume of 45 gulabjamuns} = \frac{75.152}{3} \times 45 \text{ cm}^3 = 1127.28 \text{ cm}^3$$

$$\therefore \quad \text{Volume of syrup} = 30\% \text{ of } 1127.28 \text{ cm}^3 = \frac{30}{100} \times 1127.28 \text{ cm}^3 = 338.184 \text{ cm}^3$$

EXAMPLE 23 *A vessel in the form of inverted cone. Its height is 8 cm and radius of its top, which is open, is 5 cm. It is filled with water upto the brim. When lead shots, each of which is a sphere of radius 0.5 cm are dropped in the vessel, one fourth of the water flows out. Find the number of lead shots dropped in the vessel.* **[NCERT]**

SOLUTION We have,

r = Radius of the base of the cone = 5 cm, h = Height of the cone = 8 cm

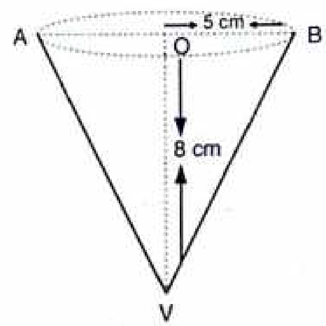

Fig. 14.38

$$\therefore \quad \text{Volume of water in the cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 5^2 \times 8 \text{ cm}^3 = \frac{200}{3}\pi \text{ cm}^3$$

$$\text{Volume of the water that flows out of the cone} = \frac{1}{4} \times \frac{200\pi}{3} \text{cm}^3 = \frac{50\pi}{3} \text{cm}^3$$

$$\text{Volume of a spherical lead shot of radius 0.5 cm} = \frac{4}{3}\pi \times (0.5)^3 \text{ cm}^3 = \frac{4}{3}\pi \times \frac{1}{8} \text{cm}^3 = \frac{\pi}{6} \text{cm}^3$$

Suppose n spherical lead shots are dropped in the vessel so that $\frac{1}{4}$th of the water contained in the vessel flows out of the vessel.

$\therefore$ Volume of n spherical lead-shots = Volume of the water that flows out

$$\Rightarrow \quad n \times \frac{\pi}{6} = \frac{50}{3}\pi$$

$$\Rightarrow \quad n = 100$$

Hence, 100 lead shots are dropped in the vessel.

EXAMPLE 24 *A solid consisting of a right cone standing on a hemisphere is placed upright in a right circular cylinder full of water and touches the bottom. Find the volume of water left in the cylinder, if the radius of the cylinder is 60 cm and its height is 180 cm, the radius of the hemisphere is 60 cm and height of the cone is 120 cm, assuming that the hemisphere and the cone have common base.* **[NCERT]**

SOLUTION For the cylinder: r = Radius of the base = 60 cm, h = Height = 180 cm

$\therefore$ V = Volume of water that the cylinder contains = $\pi r^2 h = \left\{\pi \times (60)^2 \times 180\right\} \text{cm}^3$

For conical part: r = Radius of the base = 60 cm, h_1 = Height = 120 cm.

Let V_1 be the volume of the conical part. Then,

$$V_1 = \frac{1}{3}\pi r^2 h_1$$

$$\Rightarrow \quad V_1 = \frac{1}{3} \times \pi \times 60^2 \times 120 \text{ cm}^3 = \left\{\pi \times 60^2 \times 40\right\} \text{cm}^3$$

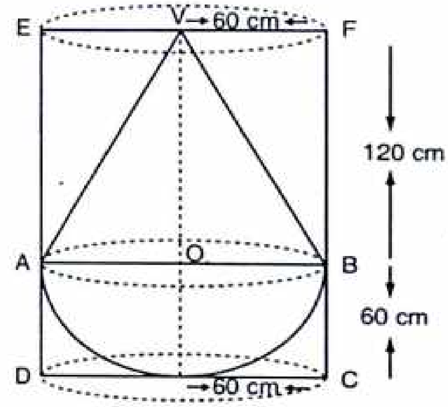

Fig. 14.39

For hemispherical part: r = Radius = 60 cm

Let V_2 be the volume of the hemisphere. Then,

$$V_2 = \left\{\frac{2}{3}\pi \times 60^3\right\} \text{cm}^3$$

$$\Rightarrow \quad V_2 = \left\{2\pi \times 20 \times 60^2\right\} \text{cm}^3 = \left\{40\pi \times 60^2\right\} \text{cm}^3$$

Let V_3 the the volume of the water left-out in the cylinder. Then,

$$V_3 = V - V_1 - V_2$$

$$\Rightarrow \quad V_3 = \left\{\pi \times 60^2 \times 180 - \pi \times 60^2 \times 40 - 40\pi \times 60^2\right\} \text{cm}^3$$

$$\Rightarrow \quad V_3 = \pi \times 60^2 \times \{180 - 40 - 40\} \text{cm}^3$$

$$\Rightarrow \quad V_3 = \frac{22}{7} \times 3600 \times 100 \text{ cm}^3$$

$$\Rightarrow \quad V_3 = \frac{22 \times 360000}{7} \text{cm}^3 = \frac{22 \times 360000}{7 \times (100)^3} \text{m}^3 = \frac{22 \times 36}{700} \text{m}^3 = 1.1314 \text{ m}^3.$$

EXAMPLE 25 *A solid iron pole consists of a cylinder of height 220 cm and base diameter 24 cm, which is surmounted by another cylinder of height 60 cm and radius 8 cm. Find the mass of the pole, given that 1 cm*^{3} *of iron has approximately 8 g mass (Use π = 3.14).* **[NCERT]**

SOLUTION Let V be the total volume of iron in two cylinders. Then,

$$V = \left\{\pi \times 12^2 \times 220 + \pi \times 8^2 \times 60\right\} \text{cm}^3$$

$$\Rightarrow \quad V = \{\pi \times 144 \times 220 + \pi \times 64 \times 60\} \text{cm}^3$$

$$\Rightarrow \quad V = 35520\pi \text{ cm}^3 = 35520 \times 3.14 \text{ cm}^3 = 111532.8 \text{ cm}^3$$

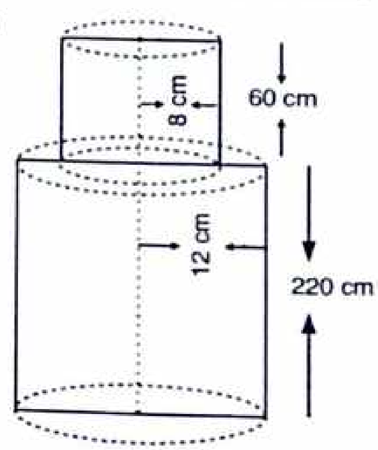

Fig. 14.40

Let M be the total mass of the iron pole. Then,

$$M = 111532.8 \times 8 \text{ grams} = \frac{111532.8 \times 8}{1000} \text{kg} = 892.2624 \text{ kg}$$

EXAMPLE 26 *A spherical glass vessel has a cylindrical neck 8 cm long, 2 cm in diameter: the diameter of the spherical part is 8.5 cm. By measuring the amount of water it holds, a child finds its volume to be 345 cm^3. Check whether she is correct, taking the above as the inside measurements and $\pi = 3.16$.* **[NCERT]**

SOLUTION We have,

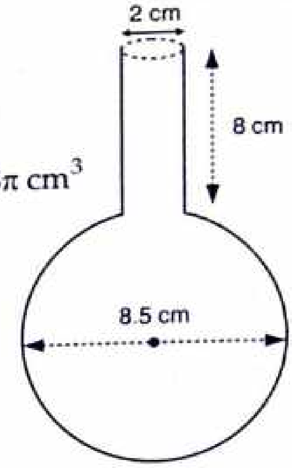

Fig. 14.41

h = Length of the cylindrical neck = 8 cm

r = Radius of the cylindrical neck = 1 cm

$\therefore$ Volume of the cylindrical neck $= \pi r^2 h = \pi \times 1^2 \times 8 \text{ cm}^3 = 8\pi \text{ cm}^3$

Volume of the spherical part $= \frac{4}{3}\pi \times \left(\frac{8.5}{2}\right)^3 \text{ cm}^3$

$$= \frac{4\pi}{3} \times (4.25)^3 \text{ cm}^3$$

$\therefore$ Amount of water in the vessel $= \left\{8\pi + \frac{4\pi}{3} \times (4.25)^3\right\} \text{cm}^3$

$$= \pi\left\{8 + \frac{4}{3} \times (4.25)^3\right\} \text{cm}^3$$

$$= 3.14 \times \left\{8 + \frac{4}{3} \times 4.25 \times 4.25 \times 4.25\right\} \text{cm}^3$$

$$= 3.14 \times (8 + 102.354) \text{ cm}^3$$

$$= 346.511 \text{ cm}^3 \cong 346.5 \text{ cm}^3$$

Hence, the volume found by the child is not correct.

EXAMPLE 27 *Rasheed got a playing top (lattu) as his birthday present, which surprisingly had no colour on it. He wanted to colour it with his crayons. The top is shaped like a cone surmounted by a hemisphere. The entire top is 5 cm in height and the diameter of the top is 3.5 cm. Find the area he has to colour.* (Take $\pi = 22/7$). **[NCERT]**

SOLUTION We have,

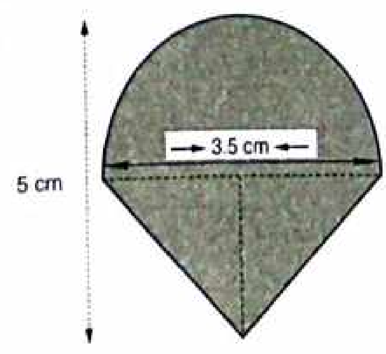

Fig. 14.42

r = Radius of hemispherical portion of the lattu $= \frac{3.5}{2} \text{ cm} = \frac{7}{4} \text{cm}$

r = Radius of the conical portion $= \frac{3.5}{2} = \frac{7}{4} \text{cm}$

h = Height of the conical portion $= \left(5 - \frac{3.5}{2}\right) \text{cm} = \frac{13}{4} \text{cm}$

Let l be the slant height of the conical part. Then,

$$l = \sqrt{r^2 + h^2} = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{13}{4}\right)^2} = \sqrt{\frac{49+169}{4}} \text{ cm} = \sqrt{\frac{218}{4}} \text{ cm} = 3.69 \text{ cm} = 3.7 \text{ cm}$$

Let S be the total surface area of the top. Then,

$$S = 2\pi r^2 + \pi r l$$

$$\Rightarrow \quad S = \pi r(2r + l)$$

$$\Rightarrow \quad S = \frac{22}{7} \times \frac{7}{4}\left(2 \times \frac{7}{4} + 3.7\right) \text{cm}^2$$

$$\Rightarrow \quad S = \frac{11}{2}(3.5 + 3.7) \text{ cm}^2 = \frac{11}{2} \times 7.2 \text{ cm}^2 = 11 \times 3.6 \text{ cm}^2 = 39.6 \text{ cm}^2$$

EXAMPLE 28 *A decorative block shown in Fig. 14.43 is made of two solids — a cube and a hemi-sphere. The base of the block is a cube with edge 5 cm, and the hemisphere fixed on the top has a diameter 4.2 cm. Find the total surface area of the block (Take $\pi = 22/7$).*

[NCERT, CBSE 2009, 2016]

SOLUTION Let S be the total surface area of the decorative block. Then,

S = Total surface area of the cube – Base area of hemisphere + Curved surface area of hemisphere

$$\Rightarrow \quad S = (6 \times 5 \times 5 - \pi r^2 + 2\pi r^2) \text{ cm}^2$$

$$\Rightarrow \quad S = (150 + \pi r^2) \text{ cm}^2$$

$$\Rightarrow \quad S = \left\{150 + \frac{22}{7} \times (2.1)^2\right\} \text{cm}^2$$

$$\Rightarrow \quad S = \{150 + 22 \times 0.3 \times 2.1\} \text{ cm}^2$$

$$\Rightarrow \quad S = (150 + 13.86) \text{ cm}^2 = 163.86 \text{ cm}^2$$

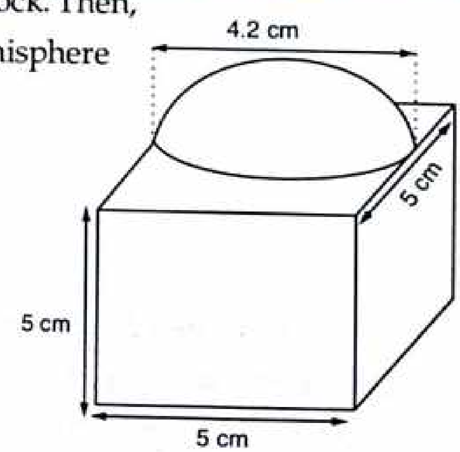

Fig. 14.43

EXAMPLE 29 *A cubical block of side 7 cm is surmounded by a hemisphere. What is the greatest diameter of the hemisphere can have? Find the total surface area of the solid.* **[NCERT]**

SOLUTION Clearly, greatest diameter of the hemisphere is equal to the length of an edge of the cube i.e. 7 cm.

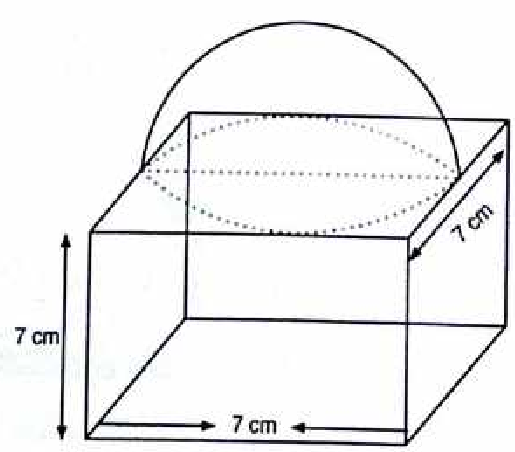

Fig. 14.44

$\therefore$ Radius of the hemisphere $= \dfrac{7}{2}$ cm

Let S be the total surface area of the solid. Then,

S = Surface area of the cube + Curved surface area of hemisphere − Area of the base of the hemisphere

$$\Rightarrow \quad S = \left\{6 \times 7^2 + 2 \times \frac{22}{7} \times \left(\frac{7}{2}\right)^2 - \frac{22}{7} \times \left(\frac{7}{2}\right)^2\right\} \text{cm}^2$$

$$\Rightarrow \quad S = \left\{294 + 77 - \frac{77}{2}\right\} \text{cm}^2 = \left(294 + \frac{77}{2}\right) \text{cm}^2 = 332.5 \text{ cm}^2$$

EXAMPLE 30 *A hemispherical depression is cut-out from one face of the cubical wooden block such that the diameter l of the hemisphere is equal to the edge of the cube. Determine the surface area of the remaining solid.* **[NCERT, CBSE 2010, 2014]**

SOLUTION It is given that a hemisphere of radius $\frac{l}{2}$ is cut-out from the top face of the cuboidal wooden box. Let S be the surface area of the remaining solid. Then,

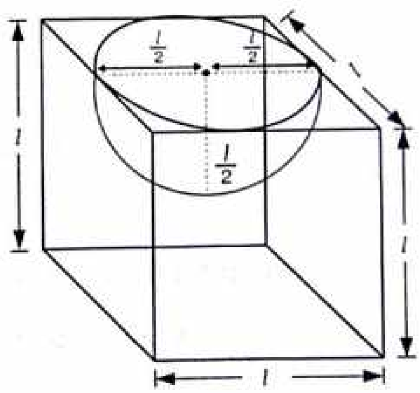

Fig. 14.45

S = Surface area of the cuboidal box whose each edge is of length l − Area of the top of the hemispherical part + Curved surface area of the hemispherical part

$$\Rightarrow \quad S = 6l^2 - \pi\left(\frac{l}{2}\right)^2 + 2\pi\left(\frac{l}{2}\right)^2 = 6l^2 - \frac{\pi l^2}{4} + \frac{\pi l^2}{2} = 6l^2 + \frac{\pi l^2}{4} = \frac{l^2}{4}(24 + \pi) \text{ sq. units}$$

EXAMPLE 31 *A medicine capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends as shown in Fig. 14.46. The length of the entire capsule is 14 mm and the diameter of the capsule is 5 mm. Find its surface area.* **[NCERT]**

SOLUTION For cylindrical part, we have

$$r = \text{Radius} = \frac{5}{2} \text{ mm} = 2.5 \text{ mm}, \quad h = \text{length} = \{14 - (2.5 + 2.5)\} \text{ mm} = 9 \text{ mm}$$

Let S_1 be the curved surface area of the cylindrical part. Then,

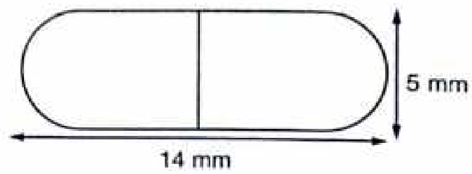

Fig. 16.46

$\Rightarrow \quad S_1 = 2\pi rh = 2 \times \frac{22}{7} \times 2.5 \times 9 \text{ mm}^2$

For two hemispherical parts, we have

$r = \text{Radius} = 2.5 \text{ mm}$

Let S_2 be the curved surface area of two hemispherical parts. Then,

$$S_2 = 2 \times 2\pi r^2 = 4\pi r^2 = 4 \times \frac{22}{7} \times (2.5)^2 \text{ mm}^2$$

Let S be the surface area of the capsule. Then,

$$S = S_1 + S_2 = \left\{2 \times \frac{22}{7} \times 2.5 \times 9 + 4 \times \frac{22}{7} \times (2.5)^2\right\} \text{mm}^2$$

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 2.5\,(9 + 2 \times 2.5) \text{ mm}^2$$

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 2.5 \times 14 \text{ mm}^2 = 2 \times 22 \times 2.5 \times 2 \text{ mm}^2 = 220 \text{ mm}^2$$

EXAMPLE 32 *Shanta runs an industry in a shed which was in the shape of a cuboid surmounted by half-cylinder. If the base of the shed is 7 m by 15 m, and height of the cuboidal portion is 8 m, find the volume of the air that the shed can hold. If the industry requires machinery which would occupy a total space of 300 m^3 and there are 20 workers each of whom would occupy 0.08 m^3 space on an average, how much air would be in the shed when it is working? (Take $\pi = 22/7$).* **[NCERT]**

SOLUTION Clearly, volume of air inside the shed (when there is no people or machinery) is equal to the volume of air inside the cuboid and inside the half-cylinder taken together.
For cuboidal part, we have

length = 15 m, breadth = 7 m and height = 8 m

$\therefore$ Volume of cuboidal part $= 15 \times 7 \times 8 \text{ m}^2 = 840 \text{ m}^3$

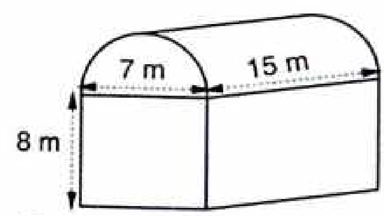

Fig. 14.47

Clearly,

$r = \text{Radius of half-cylinder} = \frac{1}{2}\text{(Width of the cuboid)} = \frac{7}{2} \text{ m}$

and,

$h = \text{Height (length) of half-cylinder} = \text{Length of cuboid} = 15 \text{ m}$

$\therefore$ Volume of half-cylinder $= \frac{1}{2}\pi r^2 h = \frac{1}{2} \times \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 15 \text{ m}^3 = \frac{1155}{4} \text{ m}^3 = 288.75 \text{ m}^3$

Volume of air inside the shed when there is no people or machinery $= (840 + 288.75) \text{ m}^3$

$= 1128.75 \text{ m}^3$

Now,

Total space occupied by 20 workers $= 20 \times 0.08 \text{ m}^3 = 1.6 \text{ m}^3$

Total space occupied by the machinery $= 300 \text{ m}^3$

$\therefore$ Volume of the air inside the shed when there are machine and workers in side it

$= (1128.75 - 1.6 - 300) \text{ m}^3 = 827.15 \text{ m}^3$

EXAMPLE 33 *A juice seller was serving his customers using glasses. The inner diameter of the cylindrical glass was 5 cm, but the bottom of the glass had a hemispherical raised portion which reduced the capacity of the glass. If the height of the glass was 10 cm, find what the apparent capacity of the glass was and what the actual capacity was (Use $\pi = 3.14$).* **[NCERT, CBSE 2009]**

SOLUTION We have, Inner diameter of the glass = 5 cm, Height of the glass = 10 cm

$\therefore$ Apparent capacity of the glass $= 3.14 \times \left(\frac{5}{2}\right)^2 \times 10 \text{ cm}^3$

$= 3.14 \times \frac{25}{4} \times 10 \text{ cm}^3 = 196.25 \text{ cm}^3$

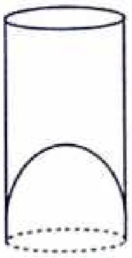

Fig. 14.48

Volume of hemispherical part $= \frac{2}{3} \times 3.14 \times \left(\frac{5}{2}\right)^3 \text{ cm}^3$

$= \frac{2}{3} \times 3.14 \times 2.5 \times 2.5 \times 2.5 \text{ cm}^3 = 32.71 \text{ cm}^3$

$\therefore$ Actual capacity of glass = Apparent capacity of glass – Volume of hemispherical part

$= (196.25 - 32.71) \text{ cm}^3 = 163.54 \text{ cm}^3$

EXAMPLE 34 *A container shaped like a right circular cylinder having diameter 12 cm and height 15 cm is full of ice-cream. The ice-cream is to be filled into cones of height 12 cm and diameter 6 cm, having a hemispherical shape on the top. Find the number of such cones which can be filled with ice-cream.*

SOLUTION Let V_1 be the volume of ice-cream in the container shaped like a right circular cylinder having radius 6 cm and height 15 cm. Then,

$$V_1 = \pi \times 6^2 \times 15 \text{ cm}^3$$

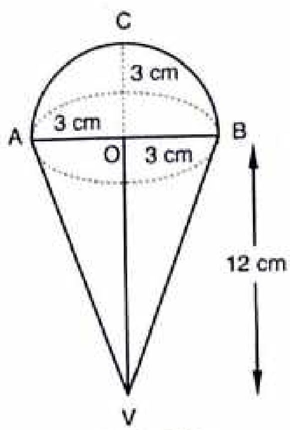

Fig. 14.49

Let V_2 be the volume of one ice-cream cone shown in Fig. 14.48. Then,

$$V_2 = \left\{\frac{2}{3}\pi \times 3^3 + \frac{1}{3}\pi \times 3^2 \times 12\right\} \text{cm}^3 = (18\pi + 36\pi) \text{ cm}^3 = 54\pi \text{ cm}^3$$

Let the total number of cones that can be filled with the ice-cream given in the container be n. Then,

Volume of ice-cream in n cones = Volume of ice-cream in the container

$\Rightarrow \quad n V_2 = V_1$

$\Rightarrow \quad 54\pi \times n = \pi \times 36 \times 15 \Rightarrow n = \frac{\pi \times 36 \times 15}{54\pi} = 10.$

LEVEL-2

EXAMPLE 35 *The interior of a building is in the form of cylinder of diameter 4.3 m and height 3.8 m, surmounted by a cone whose vertical angle is a right angle. Find the area of the surface and the volume of the building. (Take $\pi = 3.14$).*

SOLUTION We have,

$$r_1 = \text{Radius of the base of the cylinder} = \frac{4.3}{2}\text{ m} = 2.15 \text{ m}$$

$\therefore$ r_2 = Radius of the base of the cone = 2.15 m, h_1 = Height of the cylinder = 3.8 m

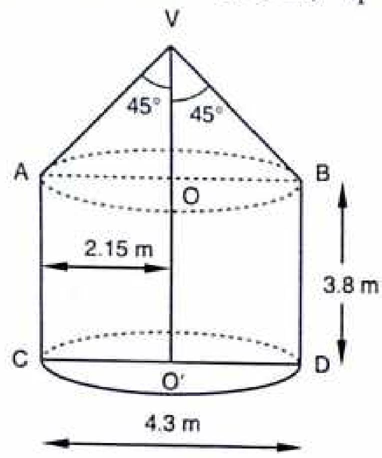

Fig. 14.50

In ΔVOA, we have

$$\sin 45° = \frac{OA}{VA} \Rightarrow \frac{1}{\sqrt{2}} = \frac{2.15}{VA} \Rightarrow VA = (\sqrt{2} \times 2.15)\text{ m} = (1.414 \times 2.15)\text{ m} = 3.04 \text{ m}$$

Clearly, ΔVOA is an isosceles triangle. Therefore, $VO = OA = 2.15$ m

Thus, we have

h_2 = Height of the cone = VO = 2.15 m , l_2 = Slant height of the cone = VA = 3.04 m

Let S be the Surface area of the building. Then,

S = Surface area of the cylinder + Surface area of cone

$\Rightarrow$ $S = (2\pi r_1 h_1 + \pi r_2 l_2)\text{ m}^2$

$\Rightarrow$ $S = (2\pi r_1 h_1 + \pi r_1 l_2)\text{ m}^2$ $[\because r_1 = r_2 = 2.15 \text{ m}]$

$\Rightarrow$ $S = \pi r_1 (2h_1 + l_2)\text{ m}^2$

$\Rightarrow$ $S = 3.14 \times 2.15 \times (2 \times 3.8 + 3.04)\text{ m}^2 = 3.14 \times 2.15 \times 10.64 \text{ m}^2 = 71.83 \text{ m}^2$

Let V be the volume of the building. Then,

V = Volume of the cylinder + Volume of the cone

$$\Rightarrow \quad V = \left(\pi r_1^2 h_1 + \frac{1}{3}\pi r_2^2 h_2\right)\text{m}^3$$

$$\Rightarrow \quad V = \left(\pi r_1^2 h_1 + \frac{1}{3}\pi r_1^2 h_2\right)\text{m}^3 \quad [\because r_2 = r_1]$$

$$\Rightarrow \quad V = \pi r_1^2 \left(h_1 + \frac{1}{3}h_2\right)\text{m}^3$$

$\Rightarrow \quad V = 3.14 \times 2.15 \times 2.15 \times \left(3.8 + \frac{2.15}{3}\right) m^3$

$\Rightarrow \quad V = [3.14 \times 2.15 \times 2.15 \times (3.8 + 0.7166)]\, m^3$

$\Rightarrow \quad V = (3.14 \times 2.15 \times 2.15 \times 4.5166)\, m^3 = 65.55\, m^3$

EXAMPLE 36 *Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 9 cm.*

SOLUTION The base of the largest right circular cone will be the circle inscribed in a face of the cube and its height will be equal to an edge of the cube.

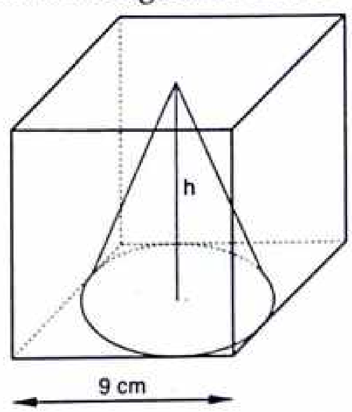

Fig. 14.51

$\therefore \quad r = \text{Radius of the base of the cone} = \frac{9}{2}$ cm $\qquad [\because \text{ edge} = 9 \text{ cm}]$

and, $\quad h = \text{Height of cone} = 9$ cm

$\therefore \quad \text{Volume of the cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times \frac{9}{2} \times \frac{9}{2} \times 9\, cm^3 = \frac{2673}{14}\, cm^3 = 190.93\, cm^3$

EXAMPLE 37 *A right triangle, whose sides are 15 cm and 20 cm, is made to revolve about its hypotenuse. Find the volume and surface area of the double cone so formed.* (Use $\pi = 3.14$)

SOLUTION Let ABC be the right angled triangle such that $AB = 15$ cm and $AC = 20$ cm. Using Pythagoras theorem, we have

$BC^2 = AB^2 + AC^2$

$\Rightarrow \quad BC^2 = 15^2 + 20^2$

$\Rightarrow \quad BC^2 = 225 + 400 = 625$

$\Rightarrow \quad BC = 25$ cm

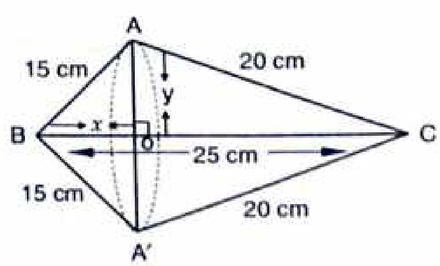

Fig. 14.52

Let $OB = x$ and $OA = y$.

Applying Pythagoras theorems in triangles OAB and OAC, we have

$AB^2 = OB^2 + OA^2$ and $AC^2 = OA^2 + OC^2$

$\Rightarrow \quad 15^2 = x^2 + y^2$ and $20^2 = y^2 + (25 - x)^2$

$\Rightarrow \quad x^2 + y^2 = 225$ and $(25 - x)^2 + y^2 = 400$

$\Rightarrow \quad \left\{(25 - x)^2 + y^2\right\} - \left\{x^2 + y^2\right\} = 400 - 225$

$\Rightarrow \quad (25-x)^2 - x^2 = 175$

$\Rightarrow \quad (25-x-x)(25-x+x) = 175$

$\Rightarrow \quad (25-2x) \times 25 = 175 \Rightarrow 25-2x = 7 \Rightarrow 2x = 18 \Rightarrow x = 9$

Putting $x = 9$ in $x^2 + y^2 = 225$, we get

$$81 + y^2 = 225 \Rightarrow y^2 = 144 \Rightarrow y = 12.$$

Thus, we have $OA = 12$ cm and $OB = 9$ cm.

Let V be the volume and S be the surface area of the double cone. Then,

$$V = \text{Vol. of cone } CAA' + \text{Vol. of cone } BAA'$$

$\Rightarrow \quad V = \frac{1}{3}\pi(OA^2) \times OC + \frac{1}{3}\pi(OA^2) \times OB$

$\Rightarrow \quad V = \frac{1}{3}\pi \times 12^2 \times 16 + \frac{1}{3}\pi \times 12^2 \times 9$

$\Rightarrow \quad V = \frac{1}{3}\pi \times 144(16+9)$

$\Rightarrow \quad V = \frac{1}{3} \times 3.14 \times 144 \times 25 \text{ cm}^3 = 3768 \text{ cm}^3$

and, $\quad S =$ Curved surface area of cone CAA' + Curved surface area of cone BAA'

$\Rightarrow \quad S = \pi \times OA \times AC + \pi \times OA \times AB$

$\Rightarrow \quad S = (\pi \times 12 \times 20 + \pi \times 12 \times 15) \text{ cm}^2$

$\Rightarrow \quad S = 420\,\pi \text{cm}^2 = 420 \times 3.14 \text{ cm}^2 = 1318.8 \text{ cm}^2$

EXAMPLE 38 *A cone made of paper has height 3h and vertical angle 2α. It contains two other cones of height 2h and h and vertical angles 4α and 6α respectively. Find the ratio of the two volumes in between the cones.*

SOLUTION Let U, V, and W be the volumes of cones VAB, $V_1A_1B_1$ and $V_2A_2B_2$ respectively.

For cone VAB, we have

$$VO = 3h \text{ and } OA = 3h \tan\alpha$$

$\therefore \quad U = \frac{1}{3}\pi(3h\tan\alpha)^2 \times 3h = \frac{27\pi}{3}h^2\tan^2\alpha$

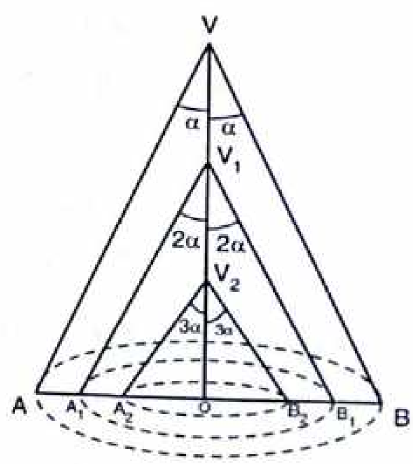

Fig. 14.53

From cone $V_1\,A_1\,B_1$, we have

$$V_1O = 2h \text{ and } OA_1 = 2h\tan 2\alpha$$

$\therefore \quad V = \frac{1}{3}\pi(2h\tan 2\alpha)^2 \times 2h = \frac{8}{3}\pi h^3\tan^2 2\alpha$

For cone $V_2\,A_2\,B_2$, we have

$$V_2O = h \text{ and } OA_2 = h\tan 3\alpha$$

$\therefore \quad W = \frac{1}{3}\pi(h\tan 3\alpha)^2 \times h = \frac{1}{3}\pi h^3\tan^2 3\alpha$

Now, $U - V = \frac{27\pi}{3} h^3 \tan^2 \alpha - \frac{8\pi}{3} h^3 \tan^2 2\alpha = \frac{\pi h^3}{3}(27 \tan^2 \alpha - 8 \tan^2 2\alpha)$

and, $V - W = \frac{8}{3}\pi h^3 \tan^2 2\alpha - \frac{\pi}{3} h^3 \tan^2 3\alpha = \frac{\pi h^3}{3}(8 \tan^2 2\alpha - \tan^2 3a)$

$\therefore$ Required ratio $= (U - V) : (V - W)$

$$= (27 \tan^2 \alpha - 8 \tan^2 2\alpha) : (8 \tan^2 2\alpha - \tan^2 3\alpha).$$

EXAMPLE 39 *A golf ball has diameter equal to 4.1 cm. Its surface has 150 dimples each of radius 2 mm. Calculate total surface area which is exposed to the surroundings assuming that the dimples are hemispherical.*

SOLUTION We observe that:

$$\text{Surface area of the ball} = 4\pi \times \left(\frac{4.1}{2}\right)^2 \text{cm}^2 = 16.81\, \pi \text{cm}^2$$

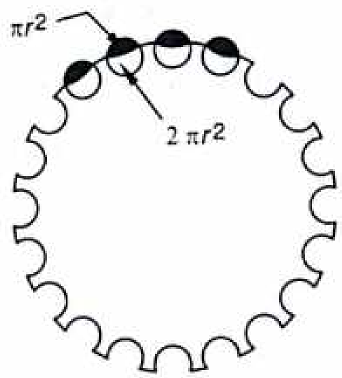

Fig. 14.54

In case of each dimple, surface area equal to πr^2 (r is the radius of each dimple) is removed from the surface of the ball where as the surface area of hemisphere i.e. $2\pi r^2$ is exposed to the surroundings. Let S be the total surface area exposed to the surroundings. Then,

$$S = \text{Surface area of the ball } -150 \times \pi r^2 + 150 \times 2\pi r^2$$

$\Rightarrow$ $S = 16.81\pi + 150\pi r^2$

$\Rightarrow$ $S = \left\{16.81\pi + 150\pi \times \left(\frac{2}{10}\right)^2\right\} \text{cm}^2$

$\Rightarrow$ $S = (16.81\pi + 6\pi) \text{cm}^2 = 22.81\pi \text{ cm}^2 = 22.81 \times \frac{22}{7} \text{cm}^2 = 71.68 \text{ cm}^2$

EXAMPLE 40 *A wooden toy rocket is in the shape of a cone mounted on a cylinder as shown in Fig. 14.55. The height of the entire rocket is 26 cm, while the height of the conical part is 6 cm. The base of the conical portion has a diameter of 5 cm, while the base diameter of the cylindrical portion is 3 cm. If the conical portion is to be painted orange and the cylindrical portion yellow, find the area of the rocket painted with each of these colours. (Take $\pi = 3.14$)*

[NCERT]

SOLUTION Let r be the radius of the base of the cone and its slant height be l. Further, let r_1 be the radius of the cylinder and h_1 be its height. It is given that

$$r = 2.5 \text{ cm}, \; h = 6 \text{ cm}, \; r_1 = 1.5 \text{ cm and } h_1 = 20 \text{ cm}$$

$\therefore \quad l = \sqrt{r^2 + h^2} \quad = \sqrt{(2.5)^2 + 6^2} = 6.5 \text{ cm} = \sqrt{4.25 + 36} = \sqrt{40.25}$

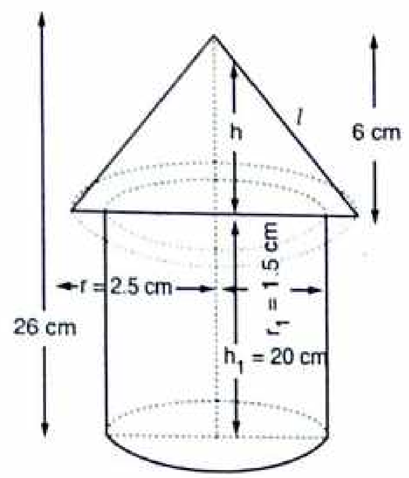

Fig. 14.55

Let S_1 and S_2 be the areas to be painted orange and yellow respectively.

S_1 = Curved surface area of the cone + Base area of the cone – Base area of the cylinder

$$\Rightarrow \quad S_1 = \pi r l + \pi r^2 - \pi r_1^2$$

$$\Rightarrow \quad S_1 = \pi \left\{ rl + r^2 - r_1^2 \right\}$$

$$\Rightarrow \quad S_1 = \pi \left\{ 2.5 \times 6.5 + (2.5)^2 - (1.5)^2 \right\} \text{cm}^2$$

$$\Rightarrow \quad S_1 = 3.14\,(16.25 + 6.25 - 2.25) \text{ cm}^2 = 3.14 \times 20.25 \text{ cm}^2 = 63.585 \text{ cm}^2$$

and,

S_2 = Curved surface area of the cylinder + Area of the base of the cylinder

$$\Rightarrow \quad S_2 = 2\pi r_1 h_1 + \pi r_1^2$$

$$\Rightarrow \quad S_2 = \pi r_1 \left(2h_1 + r_1 \right)$$

$$\Rightarrow \quad S_2 = 3.14 \times 1.5\,(2 \times 20 + 1.5) \text{ cm}^2$$

$$\Rightarrow \quad S_2 = 3.14 \times 1.5 \times 41.5 \text{ cm}^2 = 4.71 \times 41.5 \text{ cm}^2 = 195.465 \text{ cm}^2$$

EXAMPLE 41 *Mayank made a bird-bath for his garden in the shape of a cylinder with a hemispherical dipression at one end as shown in Fig. 14.56. The height of the hollow cylinder is 1.45 m and its radius is 30 cm. Find the total surface area of the bird-bath.* (*Take* $\pi = 22/7$)

[NCERT]

SOLUTION Let r be the common radius of the cylinder and hemisphere and h be the height of the hollow cylinder. Then, $r = 30$ cm and $h = 1.45$ m $= 145$ cm. Let S be the total surface area of the bird-bath. Then,

S = Curved surface area of the cylinder + Curved surface area of the hemisphere

$$\Rightarrow \quad S = 2\pi r h + 2\pi r^2 = 2\pi r\,(h + r)$$

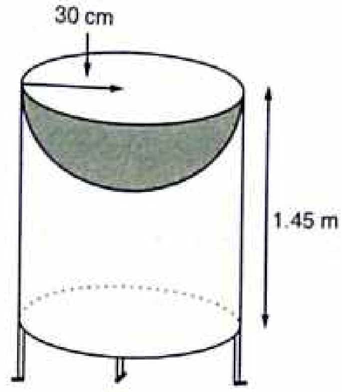

Fig. 14.56

$$\Rightarrow \quad S = 2 \times \frac{22}{7} \times 30\,(145 + 30)\ \text{cm}^2 = 33000\ \text{cm}^2 = 3.3\ \text{m}^2$$

EXAMPLE 42 *A pen stand made of wood is in the shape of a cuboid with four conical depressions to hold pens. The dimensions of the cuboid are 15 cm by 10 cm by 3.5 cm. The diameter of each of the depression is 1 cm and the depth is 1.4 cm. Find the volume of the wood in the entire stand.*

[NCERT, CBSE 2017]

SOLUTION It is given that the dimensions of the cuboidal part are 15 cm by 10 cm by 3.5 cm

$\therefore$ Volume of the cuboid $= (15 \times 10 \times 3.5)\ \text{cm}^3 = 525\ \text{cm}^3$

It is given that there are four conical depressions such that the radius of each depression is 0.5 cm and the depth is 1.4 cm. Let V be the volume of wood taken out to make four cavities. Then,

$$V = 4 \times \text{Volume of a cone of base radius 0.5 cm and height 1.4 cm}$$

$$\Rightarrow \quad V = 4 \times \frac{1}{3} \times \frac{22}{7} \times 0.5 \times 0.5 \times 1.4\ \text{cm}^3$$

$$\Rightarrow \quad V = 4 \times \frac{1}{3} \times \frac{22}{7} \times \frac{1}{2} \times \frac{1}{2} \times \frac{14}{10}\ \text{cm}^3$$

$$\Rightarrow \quad V = \frac{22}{15}\ \text{cm}^3 = 1.47\ \text{cm}^3\ \text{(approximately)}$$

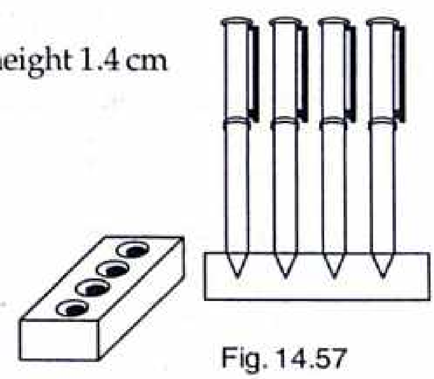
Fig. 14.57

Hence, Volume of the wood in the entire stand $= (525 - 1.47)\ \text{cm}^3 = 523.53\ \text{cm}^3$

EXAMPLE 43 *A cistern, internally measuring 150 cm × 120 cm × 110 cm has 129600 cm³ of water in it. Porous bricks are placed in the water until the cistern is full to the brim. Each brick absorbs one seventeenth of its own volume of water. How many bricks can be put in without the water overflowing, each brick being 22.5 cm × 7.5 cm × 6.5 cm?* **[NCERT]**

SOLUTION We have,

$$\text{Volume of cistern} = 150 \times 120 \times 110\ \text{cm}^3 = 1980000\ \text{cm}^3$$

$$\text{Volume of water in cistern} = 129600\ \text{cm}^3$$

$$\text{Volume of one brick} = 22.5 \times 7.5 \times 6.5\ \text{cm}^3 = 1096.875\ \text{cm}^3$$

$$\text{Volume of water absorbed by one brick} = \frac{1}{17} \times 1096.875\ \text{cm}^3$$

Let n be the total number of bricks which can be put in the cistern without water overflowing. Then, Volume of water absorbed by n bricks $= n \times \frac{1}{17} \times 1096.875 \text{ cm}^3$

$\therefore$ Volume of water left in the cistern $= \left(129600 - \frac{n}{17} \times 1096.875\right) \text{cm}^3$

Since the cistern is filled upto the brim. Therefore,

Volume of water left in the cistern + Volume of bricks = Volume of the cistern

$$\Rightarrow \quad 129600 - \frac{n}{17} \times 1096.875 + n \times 1096.875 = 1980000$$

$$\Rightarrow \quad n \times 1096.875 - \frac{n}{17} \times 1096.875 = 1980000 - 129600$$

$$\Rightarrow \quad 1096.875 \times \left(n - \frac{n}{17}\right) = 1850400$$

$$\Rightarrow \quad 1096.875 \times \frac{16n}{17} = 1850400$$

$$\Rightarrow \quad 17550 \times \frac{n}{17} = 1850400 \Rightarrow n = \frac{1850400 \times 17}{17550} = 1792.41 \cong 1792$$

EXERCISE 14.2

LEVEL-1

1. A tent is in the form of a right circular cylinder surmounted by a cone. The diameter of cylinder is 24 m. The height of the cylindrical portion is 11 m while the vertex of the cone is 16 m above the ground. Find the area of canvas required for the tent.
2. A rocket is in the form of a circular cylinder closed at the lower end with a cone of the same radius attached to the top. The cylinder is of radius 2.5 m and height 21 m and the cone has the slant height 8 m. Calculate the total surface area and the volume of the rocket.
3. A tent of height 77 dm is in the form of a right circular cylinder of diameter 36 m and height 44 dm surmounted by a right circular cone. Find the cost of the canvas at ₹ 3.50 per m^2 (Use $\pi = 22/7$).
4. A toy is in the form of a cone surmounted on a hemisphere. The diameter of the base and the height of the cone are 6 cm and 4 cm, respectively. Determine the surface area of the toy. (Use $\pi = 3.14$)
5. A solid is in the form of a right circular cylinder, with a hemisphere at one end and a cone at the other end. The radius of the common base is 3.5 cm and the heights of the cylindrical and conical portions are 10 cm. and 6 cm, respectively. Find the total surface area of the solid. (Use $\pi = 22/7$)
6. A toy is in the shape of a right circular cylinder with a hemisphere on one end and a cone on the other. The radius and height of the cylindrical part are 5 cm and 13 cm respectively. The radii of the hemispherical and conical parts are the same as that of the cylindrical part. Find the surface area of the toy if the total height of the toy is 30 cm. **[CBSE 2002]**
7. A cylindrical tub of radius 5 cm and length 9.8 cm is full of water. A solid in the form of a right circular cone mounted on a hemisphere is immersed in the tub. If the radius of the hemisphere is immersed in the tub. If the radius of the hemisphere is 3.5 cm and height of the cone outside the hemisphere is 5 cm, find the volume of the water left in the tub. (Take $\pi = 22/7$) **[CBSE 2000C]**

8. A circus tent has cylindrical shape surmounted by a conical roof. The radius of the cylindrical base is 20 m. The heights of the cylindrical and conical portions are 4.2 m and 2.1 m respectively. Find the volume of the tent.
9. A petrol tank is a cylinder of base diameter 21 cm and length 18 cm fitted with conical ends each of axis length 9 cm. Determine the capacity of the tank.
10. A conical hole is drilled in a circular cylinder of height 12 cm and base radius 5 cm. The height and the base radius of the cone are also the same. Find the whole surface and volume of the remaining cylinder.
11. A tent is in the form of a cylinder of diameter 20 m and height 2.5 m, surmounted by a cone of equal base and height 7.5 m. Find the capacity of the tent and the cost of the canvas at ₹ 100 per square metre.
12. A boiler is in the form of a cylinder 2 m long with hemispherical ends each of 2 metre diameter. Find the volume of the boiler.
13. A vessel is a hollow cylinder fitted with a hemispherical bottom of the same base. The depth of the cylinder is $\frac{14}{3}$ m and the diameter of hemisphere is 3.5 m. Calculate the volume and the internal surface area of the solid.
14. A solid is composed of a cylinder with hemispherical ends. If the whole length of the solid is 104 cm and the radius of each of the hemispherical ends is 7 cm, find the cost of polishing its surface at the rate of ₹ 10 per dm^2. **[CBSE 2006C]**
15. A cylindrical vessel of diameter 14 cm and height 42 cm is fixed symmetrically inside a similar vessel of diameter 16 cm and height 42 cm. The total space between the two vessels is filled with cork dust for heat insulation purposes. How many cubic centimeters of cork dust will be required?
16. A cylindrical road roller made of iron is 1 m long. Its internal diameter is 54 cm and the thickness of the iron sheet used in making the roller is 9 cm. Find the mass of the roller, if 1 cm^3 of iron has 7.8 gm mass. (Use $\pi = 3.14$)
17. A vessel in the form of a hollow hemisphere mounted by a hollow cylinder. The diameter of the hemisphere is 14 cm and the total height of the vessel is 13 cm. Find the inner surface area of the vessel. **[CBSE 2013]**
18. A toy is in the form of a cone of radius 3.5 cm mounted on a hemisphere of same radius. The total height of the toy is 15.5 cm. Find the total surface area of the toy. **[CBSE 2013]**
19. The difference between outside and inside surface areas of cylindrical metallic pipe 14 cm long is 44 m^2. If the pipe is made of 99 cm^3 of metal, find the outer and inner radii of the pipe.
20. A right circular cylinder having diameter 12 cm and height 15 cm is full ice-cream. The ice-cream is to be filled in cones of height 12 cm and diameter 6 cm having a hemispherical shape on the top. Find the number of such cones which can be filled with ice-cream.
21. A solid iron pole having cylindrical portion 110 cm high and of base diameter 12 cm is surmounted by a cone 9 cm high Find the mass of the pole, given that the mass of 1 cm^3 of iron is 8 gm.
22. A solid toy is in the form of a hemisphere surmounted by a right circular cone. Height of the cone is 2 cm and the diameter of the base is 4 cm. If a right circular cylinder circumscribes the toy, find how much more space it will cover.
23. A solid consisting of a right circular cone of height 120 cm and radius 60 cm standing on a hemisphere of radius 60 cm is placed upright in a right circular cylinder full of water such that it touches the bottoms. Find the volume of water left in the cylinder, if the radius of the cylinder is 60 cm and its height is 180 cm.

24. A cylindrical vessel with internal diameter 10 cm and height 10.5 cm is full of water. A solid cone of base diameter 7 cm and height 6 cm is completely immersed in water. Find the value of water (i) displaced out of the cylinder. (ii) left in the cylinder. (Take $\pi = 22/7$) **[CBSE 2009]**
25. A hemispherical depression is cut out from one face of a cubical wooden block of edge 21 cm, such that the diameter of the hemisphere is equal to the edge of the cube. Determine the volume and total surface area of the remaining block. **[CBSE 2010]**
26. A toy is in the form of a hemisphere surmounted by a right circular cone of the same base radius as that of the hemisphere. If the radius of the base of the cone is 21 cm and its volume is 2/3 of the volume of the hemisphere, calculate the height of the cone and the sufrace area of the toy. (Use $\pi = 22/7$). **[CBSE 2010]**
27. A solid is in the shape of a cone surmounted on a hemisphere, the radius of each of them is being 3.5 cm and the total height of solid is 9.5 cm. Find the volume of the solid. (Use $\pi = 22/7$). **[CBSE 2012]**
28. An wooden toy is made by scooping out a hemisphere of same radius from each end of a solid cylinder. If the height of the cylinder is 10 cm, and its base is of radius 3.5 cm, find the volume of wood in the toy. (Use $\pi = 22/7$). **[CBSE 2013]**
29. The largest possible sphere is carved out of a wooden solid cube of side 7 cm. Find the volume of the wood left. (Use $\pi = 22/7$). **[CBSE 2014]**
30. From a solid cylinder of height 2.8 cm and diameter 4.2 cm, a conical cavity of the same height and same diameter is hollowed out. Find the total surface area of the remaining solid. (Take $\pi = 22/7$). **[CBSE 2014]**
31. The largest cone is curved out from one face of solid cube of side 21 cm. Find the volume of the remaining solid. **[CBSE 2015]**
32. A solid wooden toy is in the form of a hemisphere surmounted by a cone of same radius. The radius of hemisphere is 3.5 cm and the total wood used in the making of toy is $166\frac{5}{6}\text{ cm}^3$. Find the height of the toy. Also, find the cost of painting the hemispherical part of the toy at the rate of ₹ 10 per cm^2. (Take $\pi = 22/7$). **[CBSE 2015]**
33. In Fig. 14.58, from a cuboidal solid metalic block, of dimensions 15 cm × 10 cm × 5 cm, a cylindrical hole of diameter 7 cm is drilled out. Find the surface area of the remaining block. (Take $\pi = 22/7$). **[CBSE 2015]**

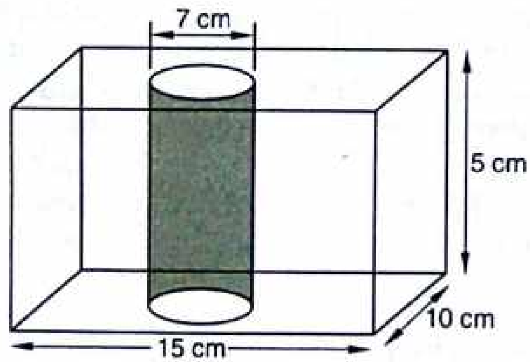

Fig. 14.58

34. A building is in the form of a cylinder surmounted by a hemispherical vaulted dome and contains $41\frac{19}{21}\text{ m}^3$ of air. If the internal diameter of dome is equal to its total height above the floor, find the height of the building? **[NCERT EXEMPLAR]**
35. A pen stand made of wood is in the shape of a cuboid with four conical depressions and a cubical depression to hold the pens and pins, respectively. The dimension of the cuboid are 10 cm × 5 cm × 4 cm. The radius of each of the conical depression is 0.5 cm and the depth is 2.1 cm. The edge of the cubical depression is 3 cm. Find the volume of the wood in the entire stand. **[NCERT EXEMPLAR]**

36. A building is in the form of a cylinder surmounted by a hemispherical dome. The base diameter of the dome is equal to $\frac{2}{3}$ of the total height of the building. Find the height of the building, if it contains $67\frac{1}{21}$ m^3 of air. [NCERT EXEMPLAR]

37. A solid toy is in the form of a hemisphere surmounted by a right circular cone. The height of cone is 4 cm and the diameter of the base is 8 cm. Determine the volume of the toy. If a cube circumscribes the toy, then find the difference of the volumes of cube and the toy. Also, find the total surface area of the toy. [NCERT EXEMPLAR]

38. A circus tent is in the shape of a cylinder surmounted by a conical top of same diameter. If their common diameter is 56 m, the height of the cylindrical part is 6 m and the total height of the tent above the ground is 27 m, find the area oif the canvas used in making the tent. [CBSE 2017]

ANSWERS

1. 1320 m^2
2. 412.5 m^2, 461.77 cm^3
3. ₹ 5365.80
4. 103. 62 cm^2
5. 372. 56 cm^2
6. 770 cm^2
7. 616 cm^3
8. 6160 m^3
9. 8316 cm^3
10. Volume = 200π cm^3, Surface area = 210π cm^2
11. 500π m^3, ₹ 55000
12. $\frac{220}{21}$ m^3
13. 56.15 m^3, $70\frac{7}{12}$ m^2
14. ₹ 457.60
15. 1980 cm^3
16. 1388.7 kg
17. 572 cm^2
18. 214.5 cm^2
19. $\frac{5}{2}$ cm and 2 cm
20. 10
21. 102.188 kg
22. 8π cm^3
23. 1.131 m^3
24. (i) 77 cm^3 (ii) 748 cm^3
25. 2992.50 cm^2, 9030 cm^3
26. 28 cm, 5082 cm^2
27. 166.83 cm^3
28. 205.33 cm^3
29. 163.33 cm^3
30. 73.92 cm^2
31. 4410 cm^3
32. 9.5 cm, ₹ 770
33. 583 cm^2
34. 4 m
35. 170.8 cm^2
36. 6 m
37. 310.86 cm^3, 171.68 cm^2
38. 4136 m^2

HINTS TO SELECTED PROBLEMS

19. Let outer and inner radii of the pipe be R and r respectively. Then,

$$2\times\frac{22}{7}\times(R-r)\times 14 = 44 \text{ and, } \frac{22}{7}\times(R^2-r^2)\times 14 = 99$$

$$\Rightarrow R-r=\frac{1}{2} \text{ and } R^2-r^2=\frac{9}{4} \Rightarrow R-r=\frac{1}{2} \text{ and, } (R+r)\times\frac{1}{2}=\frac{9}{4}$$

$$\Rightarrow R-r=\frac{1}{2} \text{ and } R+r=\frac{9}{2} \Rightarrow R=\frac{5}{2} \text{ and } r=2$$

20. Number of ice-cream cones $= \dfrac{\text{Volume of the circular cylinder}}{\text{Volume of one ice-cream cone}}$

$$= \frac{\pi\times 6\times 6\times 15}{\frac{1}{3}\times\pi\times 3\times 3\times 12+\frac{2\pi}{3}\times 3^3} = \frac{\pi\times 36\times 15}{\frac{\pi}{3}\times(108+54)} = 10$$

21. Mass of the pole $= \left\{\pi\times 6^2\times 110+\frac{\pi}{3}\times 6^2\times 9\right\}\times\frac{8}{1000}$ kg

22. Required space = Volume of the cylinder – Volume of the toy

$$= \left[\pi \times 2^2 \times 4 - \left\{\frac{2}{3} \times \pi \times 2^3 + \frac{1}{3} \times \pi \times 2^2 \times 2\right\}\right] = 8\pi \text{ cm}^3$$

14.5 FRUSTUM OF A RIGHT CIRCULAR CONE

In the previous section, we have learnt about surface areas and volumes of combina-tions of two or more basic solids like right circular cone, right circular cylinder, sphere etc. In this section, we shall learn about a solid which is a part of a right circular cone when it is cut by a plane parallel to the base of the cone. Such a solid is called a frustum as defined below.

FRUSTUM *If a right circular cone is cut off by a plane parallel to its base, then the portion of the cone between the cutting plane and the base of the cone is called a frustum of the cone.*

In Fig. 14.59 (i) right circular cone VAB is cut by a plane parallel to its circular base with centre O and diameter AB. The portion containing the vertex V is removed (Fig. 14.59 (ii)). The left out portion $ABB'A'$ shown in Fig. 14.59 (iii) is the frustum of the cone VAB. The circular faces AOB and $A'O'B'$ are called the circular ends of the frustum.

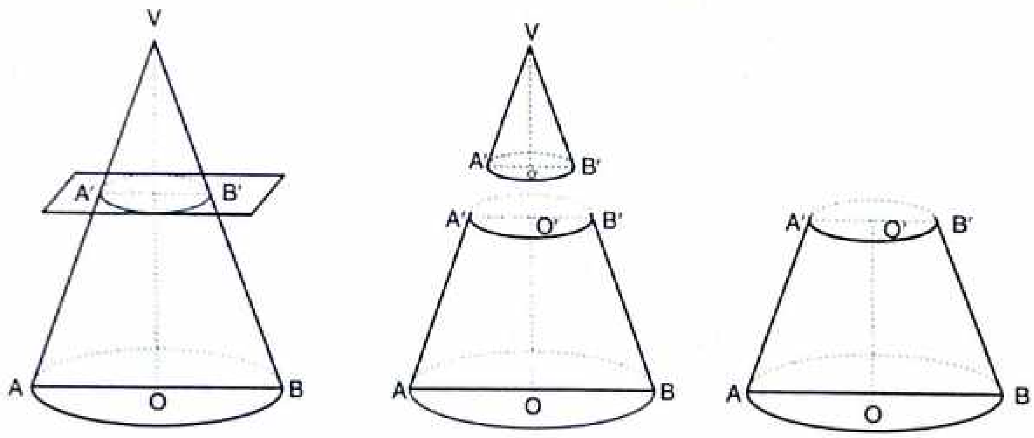

Fig. 14.59 (i), (ii), (iii)

Clearly, a frustum of a right circular cone has two unequal flat circular bases and a curved surface.

Let us now define some other terms like height, lateral (slant) height etc. related to a frustum.

HEIGHT *The height or thickness of a frustum is the perpendicular distance between its two circular bases.*

Clearly, the line segment OO' joining the centres of two circular bases (Fig. 14.59 (iii)) is perpendicular to them. So, OO' is the height of the frustum.

Also, $OO' = VO - VO'$

Thus, the height of the frustum $ABB'A'$ is equal to the difference between the heights of the cones VAB and $VA'B'$.

SLANT HEIGHT *The slant height of a frustum of a right circular cone is the length of the line segment joining the extremities of two parallel radii, drawn in the same direction, of the two circular bases.*

In Fig. 14.59, slant height of the frustum $ABA'B' = AA' = BB'$.

Clearly, $AA' = VA - VA'$ and $BB' = VB - VB'$

Thus, the slant height of the frustum equals the difference between the slant heights of the cones VAB and $VA'B'$.

14.6 VOLUME AND SURFACE AREA OF A FRUSTUM OF A RIGHT CIRCULAR CONE

Let h be the height, l the slant height and r_1 and r_2 the radii of the circular bases of the frustum $ABB'A'$ shown in Fig. 14.60 such that $r_1 > r_2$. Clearly, this frustum can be viewed as the difference of the two right circular cones VAB and $VA'B'$.

Let the height of the cone VAB be h_1 and its slant height be l_1 i.e., $VO = h_1$ and $VA = VB = l_1$

$\therefore \quad VA' = VA - AA' = l_1 - l$ and $VO' = VO - OO' = h_1 - h$

Clearly, right triangles VOA and $VO'A'$ are similar.

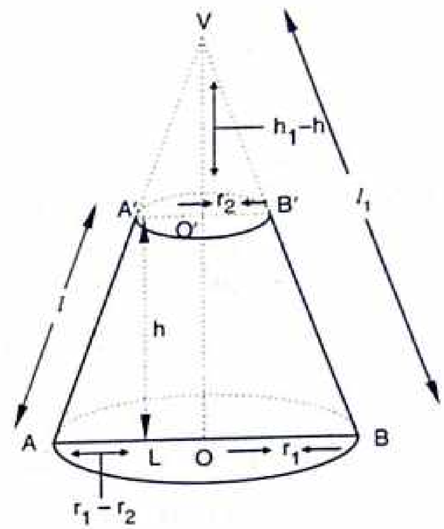

Fig. 14.60

$$\therefore \quad \frac{VO}{VO'} = \frac{OA}{O'A'} = \frac{VA}{VA'}$$

$$\Rightarrow \quad \frac{h_1}{h_1 - h} = \frac{r_1}{r_2} = \frac{l_1}{l_1 - l}$$

$$\Rightarrow \quad \frac{h_1 - h}{h_1} = \frac{r_2}{r_1} = \frac{l_1 - l}{l_1}$$

$$\Rightarrow \quad 1 - \frac{h}{h_1} = \frac{r_2}{r_1} = 1 - \frac{l}{l_1}$$

$$\Rightarrow \quad \frac{h}{h_1} = 1 - \frac{r_2}{r_1} \text{ and } \frac{l}{l_1} = 1 - \frac{r_2}{r_1}$$

$$\Rightarrow \quad \frac{h}{h_1} = \frac{r_1 - r_2}{r_1} \text{ and } \frac{l}{l_1} = \frac{r_1 - r_2}{r_1}$$

$$\Rightarrow \quad h_1 = \frac{hr_1}{r_1 - r_2} \text{ and } l_1 = \frac{lr_1}{r_1 - r_2} \qquad \text{...(i)}$$

Now,

$$\text{Height of the cone } VA'B' = h_1 - h = \frac{hr_1}{r_1 - r_2} - h = \frac{hr_2}{r_1 - r_2} \qquad \text{...(ii)}$$

$$\text{Slant height of the cone } VA'B' = l_1 - l = \frac{lr_1}{r_1 - r_2} - l = \frac{lr_2}{r_1 - r_2} \qquad \text{...(iii)}$$

Let V be the volume of the frustum of cone. Then,

V = Volume of cone VAB – Volume of cone $VA'B'$

$$\Rightarrow \quad V = \frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 (h_1 - h)$$

$$\Rightarrow \quad V = \frac{\pi}{3}\left\{h_1 r_1^2 - (h_1 - h) r_2^2\right\}$$

$$\Rightarrow \quad V = \frac{\pi}{3}\left\{\left(\frac{hr_1^3}{r_1 - r_2}\right) - \left(\frac{hr_2^3}{r_1 - r_2}\right)\right\}$$

$$\Rightarrow \quad V = \frac{\pi}{3}\left\{\frac{h}{r_1 - r_2}(r_1^3 - r_2^3)\right\}$$

$$\Rightarrow \quad V = \frac{\pi}{3}\left\{\frac{h}{r_1 - r_2}(r_1 - r_2)(r_1^2 + r_1 r_2 + r_2^2)\right\}$$

$$\Rightarrow \quad V = \frac{\pi}{3} h \left(r_1^2 + r_1 r_2 + r_2^2\right)$$

Thus, the volume V of the frustum of the cone is given by $V = \frac{1}{3}\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)h$.

Let S denote the curved surface area of the frustum of cone. Then,

S = Lateral (curved) surface area of cone VAB – Curved surface area of cone $VA'B'$

$\Rightarrow \quad S = \pi r_1 l_1 - \pi r_2 (l_1 - l)$

$\Rightarrow \quad S = \pi r_1 \cdot \dfrac{l r_1}{r_1 - r_2} - \pi r_1 \cdot \dfrac{l r_2}{r_1 - r_2}$ [Using (i) and (iii)]

$\Rightarrow \quad S = \pi \left(\dfrac{r_1^2 - r_2^2}{r_1 - r_2} \right) l$

$\Rightarrow \quad S = \pi (r_1 + r_2)$

Thus, Curved surface area of a the frustum $= \pi (r_1 + r_2) l$

Total surface area of the frustum

= Lateral (curved) surface area + Surface area of circular bases

$= \pi (r_1 + r_2) l + \pi r_1^2 + \pi r_2^2$

$= \pi \{(r_1 + r_2) l + r_1^2 + r_2^2\}$

REMARK 1 *If A_1 and A_2 denote the surface areas of circular bases, with centres O and O' respectively, of the frustum. Then, $A_1 = \pi r_1^2$ and $A_2 = \pi r_2^2$*

$\therefore$ Volume of the frustum of cone $= \dfrac{\pi}{3}(r_1^2 + r_1 r_2 + r_2^2) h$

$\Rightarrow$ Volume of the frustum of cone $= \dfrac{h}{3}\left\{\pi r_1^2 + \pi r_2^2 + \sqrt{\pi r_1^2 \times \pi r_2^2}\right\} l$

$\Rightarrow$ Volume of the frustum of cone $= \dfrac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2})$

REMARK 2 *In right triangle ALA' (Fig. 14.60), we have*

$$AA'^2 = AL^2 + A'L^2 \Rightarrow l^2 = (r_1 - r_2)^2 + h^2 \Rightarrow l = \sqrt{(r_1 - r_2)^2 + h^2}$$

$\therefore$ Slant height of the frustum of cone $= \sqrt{(r_1 - r_2)^2 + h^2}$

Height of the cone of which the frustum is a part $= \dfrac{h r_1}{r_1 - r_2}$

Slant height of the cone of which the frustum is a part $= \dfrac{l r_1}{r_1 - r_2}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If the radii of the circular ends of a conical bucket which is 45 cm high, are 28 cm and 7 cm, find the capacity of the bucket* (Use $\pi = 22/7$). [CBSE 2004, 2005]

SOLUTION Clearly, bucket forms a frustum of a cone such that the radii of its circular ends are $r_1 = 28$ cm, $r_2 = 7$ cm and height $h = 45$ cm. Let V be the capacity of the bucket. Then,

V = Volume of the frustum

$\Rightarrow \quad V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$

$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times 45(28^2 + 7^2 + 28 \times 7)$

$\Rightarrow \quad V = 22 \times 15 \times (28 \times 4 + 7 + 28) = 330 \times 147 \text{ cm}^3 = 48510 \text{ cm}^3$

EXAMPLE 2 *The radii of the circular ends of a frustum of height 6 cm are 14 cm and 6 cm respectively. Find the lateral surface area and total surface area of the frustum.*

SOLUTION We have, $r_1 = 14$ cm, $r_2 = 6$ cm and $h = 6$ cm. Let l be the slant height of the frustum. Then,

$$l = \sqrt{h^2 + (r_1 - r_2)^2} \Rightarrow l = \sqrt{36 + (14-6)^2} = \sqrt{36 + 64} = 10 \text{ cm}$$

Let LSA and TSA respectively be the lateral surface area and total surface area of the frustum. Then,

$\therefore \quad LSA = \pi(r_1 + r_2)l = \frac{22}{7} \times (14 + 6) \times 10 \text{ cm}^2 = \frac{22}{7} \times 200 \text{ cm}^2 = 628.57 \text{ cm}^2$

$$TSA = \pi\{r_1^2 + r_2^2 + (r_1 + r_2)l\} = \frac{22}{7} \times (196 + 36 + 20 \times 10) \text{ cm}^2 = \frac{22}{7} \times 432 \text{ cm}^2 = 1357.71 \text{ cm}^2$$

EXAMPLE 3 *The perimeters of the ends of a frustum are 48 cm and 36 cm. If the height of the frustum be 11 cm, find its volume.*

SOLUTION Let r_1 and r_2 be the radii of the circular ends of the frustum and h be its height. Then,

$$2\pi r_1 = 48, 2\pi r_2 = 36 \text{ and } h = 11 \text{ cm} \Rightarrow r_1 = \frac{24}{\pi}, r_2 = \frac{18}{\pi} \text{ and } h = 11 \text{ cm}$$

Let V be the volume of the frustum. Then,

$V = \frac{1}{3}\pi(r_1^2 + r_2^2 + r_1 r_2)h$

$\Rightarrow \quad V = \frac{1}{3} \times \pi \times 11 \times \left\{\left(\frac{24}{\pi}\right)^2 + \left(\frac{18}{\pi}\right)^2 + \frac{24}{\pi} \times \frac{18}{\pi}\right\}$

$\Rightarrow \quad V = \frac{1}{3} \times \pi \times 11 \times \frac{(576 + 324 + 432)}{\pi^2} \text{ cm}^3$

$\Rightarrow \quad V = \frac{11}{3} \times \frac{1332}{\pi} \text{cm}^3 = \frac{11}{3} \times \frac{1332}{22} \times 7 \text{ cm}^3 = 1554 \text{ cm}^3$

EXAMPLE 4 *The slant height of the frustum of a cone is 4 cm, and the perimeter of its circular bases are 18 cm and 6 cm respectively. Find the curved surface area of the frustum.*

SOLUTION Let r_1 and r_2 be the radii of the circular bases of the frust, l be the slant height and h be the height. It is given that

$$l = 4 \text{ cm}, 2\pi r_1 = 18 \text{ and } 2\pi r_2 = 6 \Rightarrow l = 4 \text{ cm}, r_1 = \frac{9}{\pi} \text{ and } r_2 = \frac{3}{\pi}$$

$\therefore \quad \text{Curved surface area} = \pi(r_1 + r_2)l = \pi\left(\frac{9}{\pi} + \frac{3}{\pi}\right) \times 4 \text{ cm}^2 = 48 \text{ cm}^2$

EXAMPLE 5 *A bucket is in the form of a frustum of a cone and holds 28.490 litres of water. The radii of the top and bottom are 28 cm and 21 cm respectively. Find the height of the bucket.*

[CBSE 2012, 2014]

SOLUTION Let the height of the bucket be h cm. We have, $r_1 = 28$ cm, $r_2 = 21$ cm
and, V = Volume of the bucket = 28.490 litres = 28.490×1000 cm^3 = 28490 cm^3
Now,

$$V = 28490 \text{ cm}^3$$

$$\Rightarrow \quad \frac{1}{3}\pi h\,(r_1^2 + r_1 r_2 + r_2^2) = 28490$$

$$\Rightarrow \quad \frac{1}{3} \times \frac{22}{7} \times h \times (28^2 + 28 \times 21 + 21^2) = 28490$$

$$\Rightarrow \quad \frac{22}{21} \times h \times (784 + 588 + 441) = 28490$$

$$\Rightarrow \quad \frac{22}{21} \times h \times 1813 = 28490 \Rightarrow h = \frac{28490 \times 21}{22 \times 1813} \text{ cm} \Rightarrow h = 15 \text{ cm}$$

Thus, height of the bucket = 15 cm.

EXAMPLE 6 *A friction clutch is in the form of a frustum of a cone, the diameter of the ends being 32 cm and 20 cm and length 8 cm. Find its bearing surface and volume.*

SOLUTION Let $ABB'A'$ be the friction clutch of slant height l cm.
We have,

$$r_1 = 16 \text{ cm}, r_2 = 10 \text{ cm and } h = 8 \text{ cm}$$

$$\therefore \quad l^2 = h^2 + (r_1 - r_2)^2$$

$$\Rightarrow \quad l^2 = 64 + 36 \Rightarrow l = 10 \text{ cm}$$

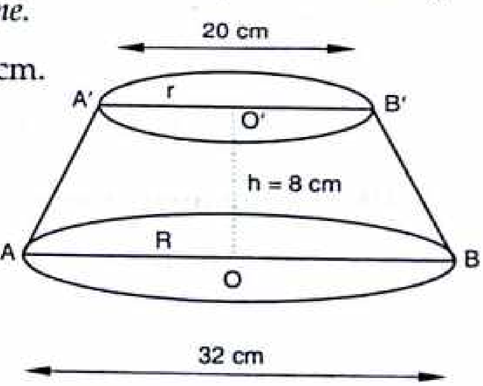

Fig. 14.61

Let S be the bearing surface and V be the volume of the clutch. Then,

$$S = \text{Lateral surface of the frustum}$$

$$\Rightarrow \quad S = \pi(r_1 + r_2)l = \frac{22}{7} \times (16 + 10) \times 10 \text{ cm}^2 = 817.14 \text{ cm}^2$$

and,

$$V = \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times 8 \times (16^2 + 16 \times 10 + 10^2) \text{ cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times 8 \times (256 + 160 + 100) \text{ cm}^3 = \frac{176}{21} \times 516 \text{ cm}^3 = 4324.57 \text{ cm}^3$$

EXAMPLE 7 *The height of a cone is 30 cm. A small cone is cut off at the top by a plane parallel to the base. If its volume be $\frac{1}{27}$ of the volume of the given cone, at what height above the base is the section made?* **[CBSE 2016, 2017]**

SOLUTION Let VAB be a cone of height 30 cm and base radius r cm. Suppose it is cut off by a plane parallel to the base at a height h from the base of the cone.

Clearly, $\Delta VOA \sim \Delta VO'A'$

$$\therefore \quad \frac{VO}{VO'} = \frac{OA}{O'A'} \Rightarrow \frac{30}{h_1} = \frac{r}{r_1} \qquad \ldots(i)$$

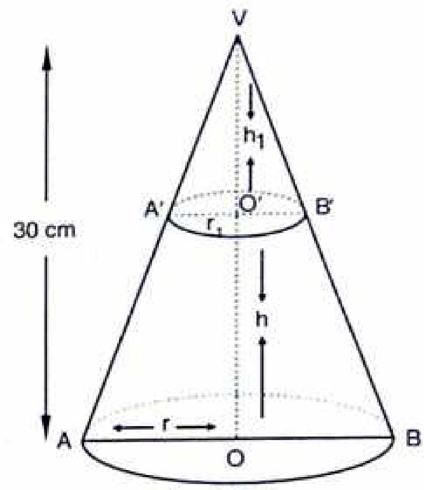

Fig. 14.62

It is given that

$$\text{Volume of cone } VA'B' = \frac{1}{27} \text{ Volume of cone } VAB$$

$$\Rightarrow \quad \frac{1}{3}\pi r_1^2 h_1 = \frac{1}{27} \times \frac{1}{3}\pi r^2 \times 30$$

$$\Rightarrow \quad \left(\frac{r_1}{r}\right)^2 h_1 = \frac{10}{9}$$

$$\Rightarrow \quad \left(\frac{h_1}{30}\right)^2 h_1 = \frac{10}{9} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad h_1^3 = 1000$$

$$\Rightarrow \quad h_1 = 10 \text{ cm}$$

$$\therefore \quad h = 30 - h_1 = (30 - 10) \text{ cm} = 20 \text{ cm}$$

Hence, the section is made at a height of 20 cm from the base of the cone.

EXAMPLE 8 *A container, open from the top, made up of a metal sheet is in the form of a frustum of a cone of height 16 cm with radii of its lower and upper ends as 8 cm and 20 cm respectively. Find the cost of milk which can completely fill the container at the rate of ₹15 per litre and the cost of metal sheet used, if the costs ₹5 per 100 cm². (Use $\pi = 3.14$)* **[CBSE 2008, 2014, 2016]**

SOLUTION Let l be the slant height of the frustum. It is given that $r_1 = 20$ cm, $r_2 = 8$ cm and $h = 16$ cm.

$$\therefore \quad l = \sqrt{(r_1 - r_2)^2 + h^2} \Rightarrow l = \sqrt{(20-8)^2 + 16^2} = \sqrt{144 + 256} = \sqrt{400} = 20 \text{ cm}$$

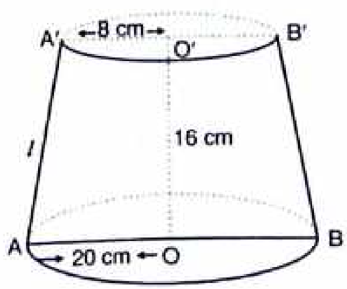

Fig. 14.63

Let V be the volume of the container. Then,

$$\therefore \quad V = \frac{\pi}{3}\left\{r_1^2 + r_2^2 + r_1 r_2\right\}h$$

$\Rightarrow \quad V = \frac{\pi}{3}\{20^2 + 8^2 + 20 \times 8\} \times 16 \text{ cm}^3$

$\Rightarrow \quad V = \frac{3.14 \times 624 \times 16}{3} \text{cm}^3 = 10449.92 \text{ cm}^3 = \frac{10449.92}{1000} \text{ litres} = 10.45 \text{ litres approx.}$

$\therefore$ Cost of milk at the rate of ₹ 15 per litre = ₹ (10.45 × 15) = ₹ 156.75

Let S be the surface area of the frustum. Then,

$S = \pi(r_1 + r_2)l + \pi r_2^2$ [$\because$ Top is open]

$\Rightarrow \quad S = \{3.14(20 + 8) \times 20 + 3.14 \times 8^2\} \text{ cm}^2$

$\Rightarrow \quad S = 3.14 \times (560 + 64) \text{ cm}^2 = 3.14 \times 624 \text{ cm}^2 = 1959.36 \text{ cm}^2$

$\therefore$ Cost of metal used = ₹ $\left(\frac{1959.36 \times 5}{100}\right)$ = ₹ 97.96 (Approx)

EXAMPLE 9 *A tent is made in the form of a conic frustum surmounted by a cone. The diameters of the base and the top of the frustum are 20 m and 6 m respectively and the height is 24 m. If the height of the tent is 28 m, find the quantity of canvas required.*

SOLUTION Let h be the height of the frustum and r_1 and r_2 be the radii of its circular bases. It is given that $h = 24$ m, $r_1 = 10$ m and $r_2 = 3$ m. Let l be the slant height of the frustum. Then,

$$l = \sqrt{(r_1 - r_2)^2 + h^2} = \sqrt{(10-3)^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} \text{ m} = 25 \text{ m}$$

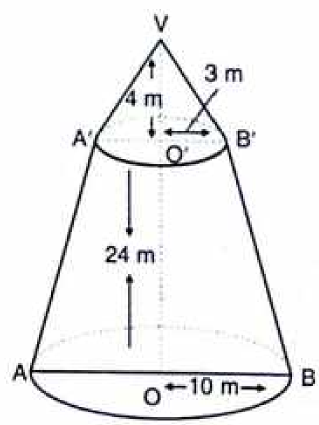

Fig. 14.64

Let l_2 be the slant height of the cone $VA'B'$. Then,

$$l_2 = \sqrt{O'B'^2 + VO'^2} = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

Let S be the quantity of canvas required. Then,

S = Lateral surface area of frustum + Lateral surface area of cone $VA'B'$

$\Rightarrow \quad S = \pi(r_1 + r_2)l + \pi r_2 l_2$

$\Rightarrow \quad S = \{\pi(10 + 3) \times 25 + \pi \times 3 \times 5\} \text{ m}^2 = (325\pi + 15\pi) \text{ m}^2 = 340\pi \text{ m}^2$

EXAMPLE 10 *An oil funnel of tin sheet consists of a cylindrical portion 10 cm long attached to a frustum of a cone. If the total height be 22 cm, diameter of the cylindrical portion be 8 cm and the diameter of the top of the funnel be 18 cm, find the area of the tin required to make the funnel.*

[NCERT]

SOLUTION Let l be the slant height of the frustum part of the funnel. Then,

$$l = \sqrt{(9-4)^2 + 12^2} = \sqrt{25+144} \text{ cm} = 13 \text{ cm}$$

Let S be the area of the tin required to make the funnel. Then,

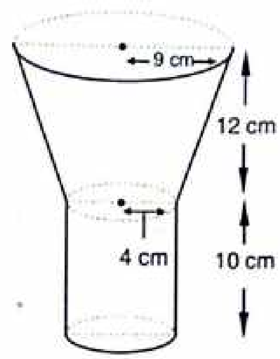

Fig. 14.65

S = Curved surface area of cylindrical portion + Curved surface area of frustum portion

$$\Rightarrow \quad S = 2\pi r_1 h + \pi(r_1 + r_2)l$$

$$\Rightarrow \quad S = \{2\pi \times 4 \times 10 + \pi(4+9) \times 13\} \text{ cm}^2 = (80\pi + 169\pi) \text{ cm}^2 = 249\pi \text{ cm}^2$$

EXAMPLE 11 *A solid metallic right circular cone 20 cm high with vertical angle 60° is cut into two parts at the middle point of its height by a plane parallel to the base. If the frustum, so obtained, be drawn into a wire of diameter $\frac{1}{16}$ cm, find the length of the wire.* **[CBSE 2014]**

SOLUTION Let VAB be the solid metallic right circular cone of height 20 cm. Suppose this cone is cut by a plane parallel to its base at a point O' such that $VO' = O'O$ i.e. O' is the mid-point of VO. Let r_1 and r_2 be the radii of circular ends of the frustum $ABB'A'$.

In triangles VOA and $VO'A'$, we have

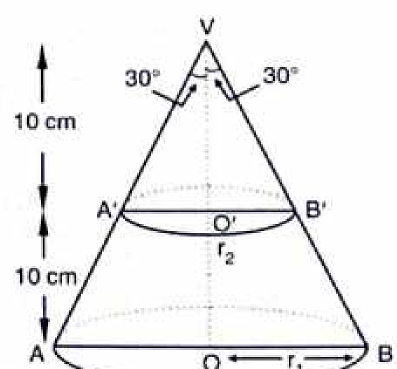

Fig. 14.66

$$\tan 30° = \frac{OA}{VO} \text{ and } \tan 30° = \frac{O'A'}{VO'}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{r_1}{20} \text{ and } \frac{1}{\sqrt{3}} = \frac{r_2}{10}$$

$$\Rightarrow \quad r_1 = \frac{20}{\sqrt{3}} \text{ cm and } r_2 = \frac{10}{\sqrt{3}} \text{ cm}$$

Let V be the volume of the frustum. Then,

$$V = \frac{1}{3}\pi\left(r_1^2 + r_2^2 + r_1 r_2\right)h$$

$$\Rightarrow \quad V = \frac{\pi}{3}\left(\frac{400}{3} + \frac{100}{3} + \frac{200}{3}\right) \times 10 \text{ cm}^2 = \frac{7000}{9}\pi \text{ cm}^2$$

Let the length of the wire of $\frac{1}{16}$ cm diameter be l cm and V_1 be the volume of the metal used in the wire.

$$V_1 = \pi \times \left(\frac{1}{32}\right)^2 \times l \text{ cm}^2 \qquad \left[\because \text{ radius} = \frac{1}{32} \text{ cm}\right]$$

$\Rightarrow \quad V_1 = \dfrac{\pi l}{1024}\text{ cm}^2$

The frustum is recast into a wire of length l cm and diameter $\dfrac{1}{16}$ cm.

$\therefore$ Volume of the metal used in wire = Volume of the frustum

$\Rightarrow \quad V_1 = V$

$\Rightarrow \quad \dfrac{\pi l}{1024} = \dfrac{7000\pi}{9}$

$\Rightarrow \quad l = \dfrac{7000\pi}{9} \times \dfrac{1024}{\pi}\text{ cm} = \dfrac{7000}{9} \times 1024\text{ cm} = 7964.4\text{ m}$

EXAMPLE 12 *A bucket of height 8 cm and made up of copper sheet is in the form of frustum of a right circular cone with radii of its lower and upper ends as 3 cm and 9 cm respectively. Calculate.*

(i) the height of the cone of which the bucket is a part.

(ii) the volume of water which can be filled in the bucket.

(iii) the area of copper sheet required to make the bucket. **[CBSE 2014]**

SOLUTION Let h be the height, l the slant height and r_1 and r_2 the radii of the circular bases of a frustum of a cone. It is given that $h = 8$ cm, $r_1 = 9$ cm and $r_2 = 3$ cm

(i) Let h_1 be the height of the cone of which the bucket is a part. Then,

$$h_1 = \frac{hr_1}{r_1 - r_2} \Rightarrow h_1 = \left(\frac{8 \times 9}{9-3}\right)\text{ cm} = 12\text{ cm}$$

(ii) Let V be the volume of the water which can be filled in the bucket. Then,

V = Volume of the frustum

$$\Rightarrow \quad V = \frac{\pi}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right)h = \frac{\pi}{3}\left(9^2 + 9 \times 3 + 3^2\right) \times 8\text{ cm}^3 = 312\,\pi\text{ cm}^3$$

(iii) Let S be the area of the copper sheet required to make the bucket. Then,

$S = \pi(r_1 + r_2)\,l + \pi r_2^2$, where l is the slant height of the frustum

$$\Rightarrow \quad S = \pi(9+3) \times \sqrt{(9-3)^2 + 8^2} + \pi \times 3^2 \qquad \left[\because l = \sqrt{(r_1 - r_2)^2 + h^2}\right]$$

$$\Rightarrow \quad S = 129\pi\text{ cm}^2$$

EXAMPLE 13 *An open metallic bucket is in the shape of a frustum of a cone mounted on hollow cylindrical base made of metallic sheet. If the diameters of the two circular ends of the bucket are 45 cm and 25 cm, the total vertical height of the bucket is 30 cm and that of the cylindrical portion is 6 cm, find the area of the metallic sheet used to make the bucket. Also, find the volume of the water it can hold. (Take $\pi = 22/7$).* **[NCERT]**

SOLUTION h = Height of the frustum of the cone $= (30 - 6)$ cm $= 24$ cm

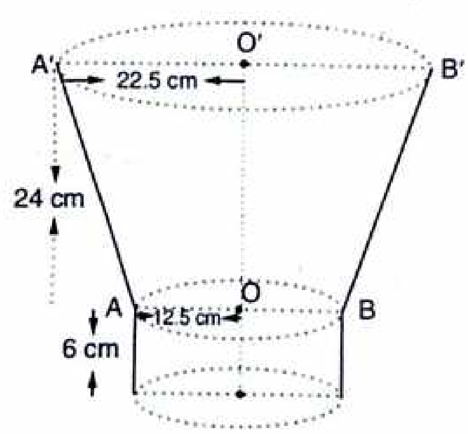

Fig. 14.67

Radii of the circular ends are $r_1 = 22.5$ cm and $r_2 = 12.5$ cm.
Let l be the slant height of the frustum. Then,

$$l = \sqrt{h^2 + (r_1 - r_2)^2} = \sqrt{24^2 + (22.5 - 12.5)^2} = \sqrt{576 + 100} = 26 \text{ cm}$$

Let A be the area of metallic sheet used. Then,

A = Curved surface area of the frustum of cone + Area of circular base + Curved surface area of cylinder.

$\Rightarrow \quad A = \pi(r_1 + r_2)l + \pi r_2^2 + 2\pi r_2 h_2$, where h_2 = height of the base = 6 cm

$$\Rightarrow \quad A = \pi\left[(22.5 + 12.5) \times 26 + 12.5^2 + 2 \times 12.5 \times 6\right] \text{cm}^2$$

$$\Rightarrow \quad A = \pi \times (1216.25) \text{ cm}^2 = \frac{22}{7} \times 1216.25 \text{ cm}^2 = 3822.5 \text{ cm}^2$$

Let V be the volume of water that the bucket can hold. Then,

$$V = \frac{1}{3} \times \pi \times \left(r_1^2 + r_2^2 + r_1 r_2\right) \times h$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times \left\{22.5^2 + 12.5^2 + 22.5 \times 12.5\right\} \times 24 \text{ cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times \left\{(9 \times 2.5)^2 + (5 \times 2.5)^2 + (9 \times 2.5) \times (5 \times 2.5)\right\} \times 24 \text{ cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times (2.5)^2\left(9^2 + 5^2 + 9 \times 5\right) \times 24 \text{ cm}^3$$

$$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times (2.5)^2 \times (151) \times 24 \text{ cm}^3 = 23728.57 \text{ cm}^3 = 23.728 \text{ litres}$$

So, the bucket can hold 23.728 litres of water.

EXAMPLE 14 *A shuttle cock used for playing badminton has the shape of a frustum of a cone mounted on a hemisphere as shown in Fig. 14.68. The external diameters of the frustum are 5 cm and 2 cm, the height of the entire shuttle cock is 7 cm. Find its external surface area.*

SOLUTION We have,

r_1 = Radius of the lower end of the frustum = 1 cm

r_2 = Radius of the upper end of the frustum = 2.5 cm

h = Height of the frustum = 6 cm.

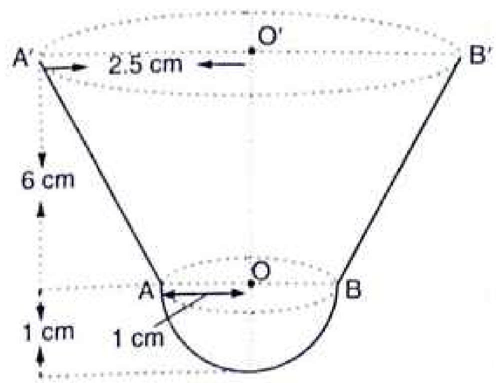

Fig. 14.68

Let l be the slant height of the frustum. Then,

$$l = \sqrt{h^2 + (r_2 - r_1)^2} = \sqrt{36 + (2.5 - 1)^2} = \sqrt{38.25} \text{ cm} = 6.18 \text{ cm}$$

Let S be the external surface area of shuttle cock. Then,

$$S = \text{Curved surface area of the frustum} + \text{Surface area of hemisphere}$$

$\Rightarrow \quad S = \pi(r_1 + r_2)l + 2\pi r_1^2$

$\Rightarrow \quad S = \left\{\pi(1 + 2.5) \times 6.18 + 2 \times \pi \times 1^2\right\} \text{cm}^2$

$\Rightarrow \quad S = \left\{\frac{22}{7} \times 3.5 \times 6.18 + 2 \times \frac{22}{7}\right\} \text{cm}^2 = (67.98 + 6.28) \text{ cm}^2 = 74.26 \text{ cm}^2$

EXAMPLE 15 *Hanumappa and his wife Gangavva are busy making Jaggery out of sugar-cane. They have processed the sugarcane juice to make the molasses which is poured into moulds of the shape shown in Fig. 14.69. It will be cooled to solidify in this shape to be sent to the market. Each mould is in the shape of a frustum of a cone having the diameters of its two circular ends as 30 cm and 35 cm and the height of the mould is 14 cm. If each cm^3 of molasses weighs about 1.2 gm, find the weight of molasses that can be poured into each mould (take $\pi = 22/7$).* **[NCERT]**

SOLUTION Clearly, the mould is in the shape of a frustum of a cone with radii of two circular ends as $r_1 = \frac{30}{2}$ cm $= 15$ cm, $r_2 = \frac{35}{2}$ cm $= 17.5$ cm and height $h = 14$ cm.

Let V be the volume of molasses that can be poured into the mould. Then,

$V = $ Volume of the mould

$\Rightarrow \quad V = \frac{\pi}{3} h \left(r_1^2 + r_2^2 + r_1 r_2\right)$

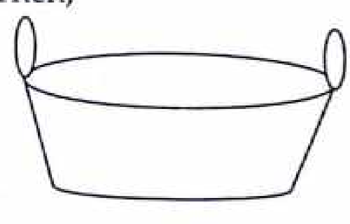

Fig. 14.69

$\Rightarrow \quad V = \frac{1}{3} \times \frac{22}{7} \times 14 \left(15^2 + 17.5^2 + 15 \times 17.5\right) \text{cm}^3$

$\Rightarrow \quad V = \frac{44}{3}(225 + 306.25 + 262.5) \text{ cm}^3 = \frac{44}{3} \times 793.75 \text{ cm}^3 = \frac{34925}{3} \text{ cm}^3$

Let M be the mass of molasses that can be poured into each mould.
It is given that 1 cm^3 of molasses has mass 1.2 gm.

$\therefore \quad M = V \times 1.2 = \frac{34925}{3} \times 1.2 \text{ gm} = \frac{34925 \times 0.4}{1000} \text{ kg} = \frac{13970}{1000} \text{ kg} = 13.97 \text{ kg}$

EXAMPLE 16 *A fez, the headgear cap used by the trucks is shaped like the frustum of a cone. If its radius on the open side is 10 cm, radius at the upper base is 4 cm and its slant height is 15 cm, find the area of material used for making it.* **[NCERT]**

SOLUTION Clearly, the fez is in the shape of a frustum of a cone with radii of two bases as $r_1 = 10$ cm, $r_2 = 4$ cm and slant height $l = 15$ cm. Let A be the area of the material used. Then,

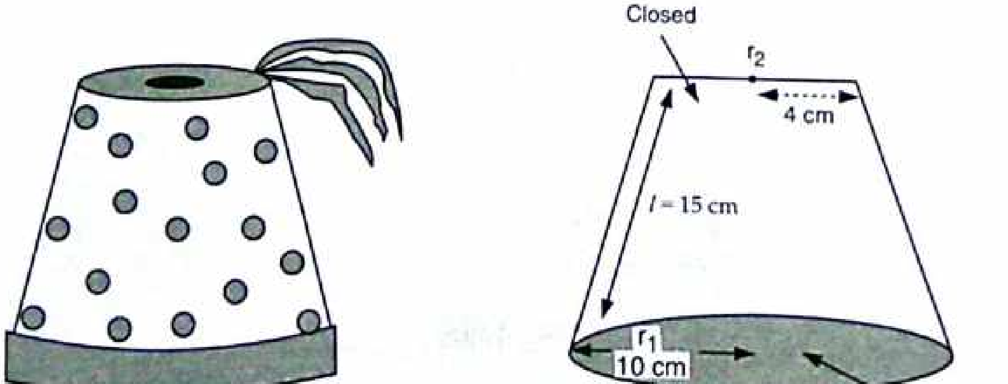

Fig. 14.70

$A =$ Curved surface area + Area of the closed base

$\Rightarrow \quad A = \pi(r_1 + r_2)l + \pi r_2^2$

$\Rightarrow \quad A = \left\{\frac{22}{7} \times (10+4) \times 15 + \frac{22}{7} \times 4^2\right\} \text{cm}^2$

$\Rightarrow \quad A = \left(352 \times 2 \times 15 + \frac{352}{7}\right) \text{cm}^2 = \left(660 + \frac{352}{7}\right) \text{cm}^2 = 660 + 50.28 \text{ cm}^2 = 710.28 \text{ cm}^2.$

LEVEL-2

EXAMPLE 17 *A hollow cone is cut by a plane parallel to the base and the upper portion is removed. If the curved surface of the remainder is $\frac{8}{9}$ of the curved surface of the whole cone, find the ratio of the line-segment into which the cone's altitude is divided by the plane.* **[CBSE 2004]**

SOLUTION Let VAB be a hollow cone of height H, slant height L and base radius R. Suppose this cone is cut by a plane parallel to the base such that O' is the centre of the circular section of the cone. Let h be the height, l be the slant height and r be the base radius of the smaller cone $VA'B'$.

Clearly, $\Delta VO'A' \sim \Delta VOA$

$\therefore \quad \frac{VO'}{VO} = \frac{O'A'}{OA} = \frac{VA'}{VA} \Rightarrow \frac{h}{H} = \frac{r}{R} = \frac{l}{L}$ ...(i)

It is given that

Curved surface area of the frustum $ABB'A' = \frac{8}{9} \times$ Curved surface area of the cone

$\Rightarrow \quad \pi(R+r)(L-l) = \frac{8}{9}\pi RL$

$\Rightarrow \quad (R+r)(L-l) = \frac{8}{9}RL$

$\Rightarrow \quad \left(\frac{R+r}{R}\right)\left(\frac{L-l}{L}\right) = \frac{8}{9}$

$\Rightarrow \quad \left(1 + \frac{r}{R}\right)\left(1 - \frac{l}{L}\right) = \frac{8}{9}$

$\Rightarrow \quad \left(1 + \frac{h}{H}\right)\left(1 - \frac{h}{H}\right) = \frac{8}{9}$ [Using (i)]

$\Rightarrow \quad 1 - \frac{h^2}{H^2} = \frac{8}{9}$

$\Rightarrow \quad \frac{h^2}{H^2} = 1 - \frac{8}{9}$

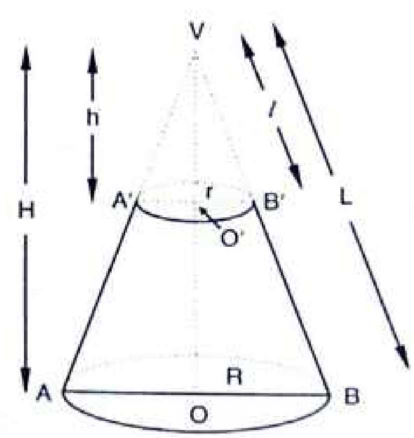

Fig. 14.71

$\Rightarrow \qquad \frac{h^2}{H^2} = \frac{1}{9} \Rightarrow \frac{h}{H} = \frac{1}{3} \Rightarrow h = \frac{H}{3}$

Hence, required ratio $= \frac{h}{H-h} = \frac{\frac{H}{3}}{H - \frac{H}{3}} = \frac{1}{2}$

EXAMPLE 18 *The height of a right circular cone is trisected by two planes drawn parallel to the base. Show that the volumes of the three portions starting from the top are in the ratio 1 : 7 : 19.*

SOLUTION Let VAB be a right circular cone of height $3h$ and base radius r. This cone is cut by planes parallel to its base at points O' and L such that $VL = LO' = h$.

Since triangles VOA and $VO'A'$ are similar.

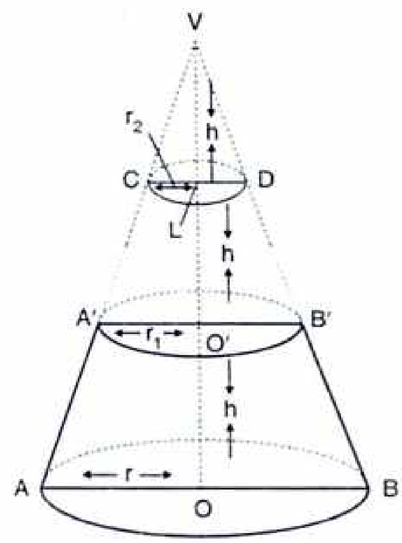

Fig. 14.72

$\therefore \qquad \frac{VO}{VO'} = \frac{OA}{O'A'} \Rightarrow \frac{r}{r_1} = \frac{3h}{2h} \Rightarrow r_1 = \frac{2r}{3}$

Also, $\qquad \Delta VOA \sim \Delta VLC$

$\therefore \qquad \frac{VO}{VL} = \frac{OA}{LC} \Rightarrow \frac{3h}{h} = \frac{r}{r_2} \Rightarrow r_2 = \frac{r}{3}$

Let V_1 be the volume of cone VCD. Then,

$$V_1 = \frac{1}{3}\pi r_2^2 h = \frac{1}{3}\pi\left(\frac{r}{3}\right)^2 h = \frac{1}{27}\pi r^2 h.$$

Let V_2 be the volume of the frustum $A'B'DC$. Then,

$$V_2 = \frac{1}{3}\pi(r_1^2 + r_2^2 + r_1 r_2)h$$

$\Rightarrow \qquad V_2 = \frac{1}{3}\pi\left(\frac{4r^2}{9} + \frac{r^2}{9} + \frac{2r^2}{9}\right)h \qquad \left[\because r_1 = \frac{2r}{3} \text{ and } r_2 = \frac{r}{3}\right]$

$\Rightarrow \qquad V_2 = \frac{7}{27}\pi r^2 h$

Let V_3 be the volume of the frustum $ABB'A'$. Then,

$\Rightarrow \qquad V_3 = \frac{1}{3}\pi\left(r^2 + r_1^2 + r_1 r\right)h$

$\Rightarrow \qquad V_3 = \frac{1}{3}\pi\left(r^2 + \frac{4r^2}{9} + \frac{2r^2}{3}\right)h = \frac{19\pi}{27}r^2 h$

$\therefore \qquad$ Required ratio $= V_1 : V_2 : V_3 = \frac{1}{27}\pi r^2 h : \frac{7}{27}\pi r^2 h : \frac{19\pi}{27}r^2 h = 1 : 7 : 19.$

EXAMPLE 19 *The radius of the base of a right circular cone is r. It is cut by a plane parallel to the base at a height h from the base. The distance of the boundary of the upper surface from the centre of the base of the frustum is* $\sqrt{h^2 + \frac{r^2}{9}}$. *Show that the volume of the frustum is* $\frac{13}{27}\pi r^2 h$.

SOLUTION We have, $OA = r, OO' = h$ and $OB' = \sqrt{h^2 + \frac{r^2}{9}}$

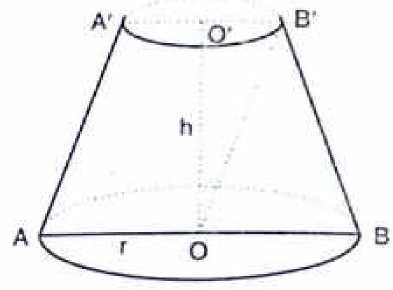

Fig. 14.73

Using Pythagoras theorem in $\Delta OO'B'$, we obtain

$$OB'^2 = OO'^2 + O'B'^2$$

$$\Rightarrow \quad h^2 + \frac{r^2}{9} = h^2 + O'B'^2$$

$$\Rightarrow \quad O'B' = \frac{r}{3}$$

Let V be the volume of the frustum. Then,

$$\therefore \quad V = \frac{1}{3}\pi\left\{r^2 + \left(\frac{r}{3}\right)^2 + r \times \frac{r}{3}\right\}h = \frac{1}{3}\pi\left\{r^2 + \frac{r^2}{9} + \frac{r^2}{3}\right\}h = \frac{13}{27}\pi r^2 h.$$

EXAMPLE 20 *A right circular cone is divided by a plane parallel to its base in two equal volumes. In what ratio will the plane divide the axis of the cone?*

SOLUTION Let VAB be a cone of height h and base radius r. Suppose it is cut by a plane parallel to the base of the cone at point O'. Let $O'A' = r_1$ and $VO' = h_1$.

Clearly, $\Delta VO'A' \sim \Delta VOA$

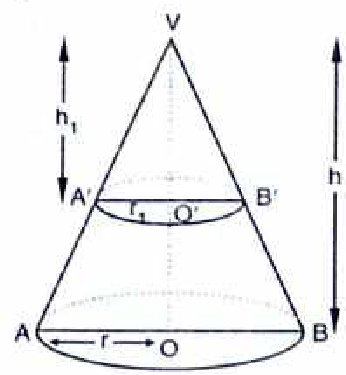

Fig. 14.74

$$\therefore \quad \frac{VO}{VO'} = \frac{OA}{O'A'} \Rightarrow \frac{h}{h_1} = \frac{r}{r_1}$$

It is given that

Volume of cone $VA'B'$ = Volume of the frustum $ABB'A'$

$$\Rightarrow \quad \frac{1}{3}\pi r_1^2 h_1 = \frac{1}{3}\pi\left(r^2 + r_1^2 + rr_1\right)(h - h_1)$$

$$\Rightarrow \quad r_1^2 h_1 = (r^2 + r_1^2 + rr_1)(h - h_1)$$

$$\Rightarrow \quad \frac{r_1^2 h_1}{r_1^2 h_1} = \left(\frac{r^2 + r_1^2 + rr_1}{r_1^2}\right)\left(\frac{h - h_1}{h_1}\right) \quad \text{[Dividing both sides by } r_1^2 h_1\text{]}$$

$$\Rightarrow \quad 1 = \left\{\left(\frac{r}{r_1}\right)^2 + 1 + \left(\frac{r}{r_1}\right)\right\}\left(\frac{h}{h_1} - 1\right)$$

$$\Rightarrow \quad 1 = \left\{\left(\frac{h}{h_1}\right)^2 + 1 + \left(\frac{h}{h_1}\right)\right\}\left(\frac{h}{h_1} - 1\right) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 1 = \left\{\left(\frac{h}{h_1}\right)^2 + \left(\frac{h}{h_1}\right) + 1\right\}\left(\frac{h}{h_1} - 1\right)$$

$$\Rightarrow \quad 1 = \left(\frac{h}{h_1}\right)^3 - 1^3 \quad [\because (a^2 + a + 1)(a - 1) = a^3 - 1]$$

$$\Rightarrow \quad \left(\frac{h}{h_1}\right)^3 = 2 \Rightarrow \frac{h}{h_1} = 2^{1/3}$$

Hence, required ratio $= \dfrac{h_1}{h - h_1} = \dfrac{1}{\left(\dfrac{h}{h_1} - 1\right)} = \dfrac{1}{2^{1/3} - 1}$

ALITER We have,

Volume of cone $VA'B'$ = Volume of frustum $ABB'A'$

$\Rightarrow$ Volume of cone $VA'B' = \dfrac{1}{2}$ (Volume of cone VAB)

$$\Rightarrow \quad \frac{1}{3}\pi r_1^2 h_1 = \frac{1}{2} \times \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad r_1^2 h_1 = \frac{1}{2} r^2 h^2$$

$$\Rightarrow \quad \left(\frac{r_1}{r}\right)^2 \left(\frac{h_1}{h}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \left(\frac{h_1}{h}\right)^2 \left(\frac{h_1}{h}\right) = \frac{1}{2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{h_1}{h} = \left(\frac{1}{2}\right)^{1/3} \Rightarrow \frac{h}{h_1} = 2^{1/3}$$

Hence, required ratio $= \dfrac{h_1}{h - h_1} = \dfrac{1}{\dfrac{h}{h_1} - 1} = \dfrac{1}{2^{1/3} - 1}$

EXERCISE 14.3

LEVEL-1

1. A bucket has top and bottom diameters of 40 cm and 20 cm respectively. Find the volume of the bucket if its depth is 12 cm. Also, find the cost of tin sheet used for making the bucket at the rate of ₹ 1.20 per dm^2. (Use $\pi = 3.14$)
2. A frustum of a right circular cone has a diameter of base 20 cm, of top 12 cm, and height 3 cm. Find the area of its whole surface and volume.
3. The slant height of the frustum of a cone is 4 cm and the perimeters of its circular ends are 18 cm and 6 cm. Find the curved surface of the frustum.
4. The perimeters of the ends of a frustum of a right circular cone are 44 cm and 33 cm. If the height of the frustum be 16 cm, find its volume, the slant surface and the total surface.
5. If the radii of the circular ends of a conical bucket which is 45 cm high be 28 cm and 7 cm, find the capacity of the bucket. (Use $\pi = 22/7$). [CBSE 2000]
6. The height of a cone is 20 cm. A small cone is cut off from the top by a plane parallel to the base. If its volume be $1/125$ of the volume of the original cone, determine at what height above the base the section is made.
7. If the radii of the circular ends of a bucket 24 cm high are 5 cm and 15 cm respectively, find the surface area of the bucket.

8. The radii of the circular bases of a frustum of a right circular cone are 12 cm and 3 cm and the height is 12 cm. Find the total surface area and the volume of the frustum.

9. A tent consists of a frustum of a cone capped by a cone. If the radii of the ends of the frustum be 13 m and 7 m, the height of the frustum be 8 m and the slant height of the conical cap be 12 m, find the canvas required for the tent. (Take : $\pi = 22/7$)

10. A milk container of height 16 cm is made of metal sheet in the form of a frustum of a cone with radii of its lower and upper ends as 8 cm and 20 cm respectively. Find the cost of milk at the rate of ₹ 44 per litre which the container can hold. [NCERT EXEMPLAR]

11. A bucket is in the form of a frustum of a cone of height 30 cm with radii of its lower and upper ends as 10 cm and 20 cm respectively. Find the capacity and surface area of the bucket. Also, find the cost of milk which can completely fill the cointainer, at the rate of ₹ 25 per litre. (Use $\pi = 3.14$) [NCERT EXEMPLAR]

12. A bucket is in the form of a frustum of a cone with a capacity of 12308.8 cm^3 of water. The radii of the top and bottom circular ends are 20 cm and 12 cm respectively. Find the height of the bucket and the area of the metal sheet used in its making. (Use $\pi = 3.14$).

[CBSE 2006C, 2016]

13. A bucket made of aluminium sheet is of height 20 cm and its upper and lower ends are of radius 25 cm and 10 cm respectively. Find the cost of making the bucket if the aluminium sheet costs ₹ 70 per 100 cm^2. (Use $\pi = 3.14$). [CBSE 2006C]

14. The radii of the circular ends of a solid frustum of a cone are 33 cm and 27 cm and its slant height is 10 cm. Find its total surface area. [CBSE 2005]

15. A bucket made up of a metal sheet is in the form of a frustum of a cone of height 16 cm with diameters of its lower and upper ends as 16 cm and 40 cm respectively. Find the volume of the bucket. Also, find the cost of the bucket if the cost of metal sheet used is ₹ 20 per 100 cm^2. (Use $\pi = 3.14$) [CBSE 2008, 2013]

16. A solid is in the shape of a frustum of a cone. The diameters of the two circular ends are 60 cm and 36 cm and the leight is 9 cm. Find the area of its whole surface and the volume.

[CBSE 2010]

17. A milk container is made of metal sheet in the shape of frustum of a cone whose volume is $10459\frac{3}{7}$ cm^3 The radii of its lower and upper circular ends are 8 cm and 20 cm respectively. Find the cost of metal sheet used in making the container at the rate of ₹ 1.40 per cm^2. (Use $\pi = 22.7$) [CBSE 2010]

18. A solid cone of base radius 10 cm is cut into two parts through the mid-point of its height, by a plane parallel to its base. Find the ratio in the volumes of two parts of the cone. [CBSE 2013, 2017]

19. A bucket open at the top, and made up of a metal sheet is in the form of a frustum of a cone. The depth of the bucket is 24 cm and the diameters of its upper and lower circular ends are 30 cm and 10 cm respectively. Find the cost of metal sheet used in it at the rate of ₹ 10 per 100 cm^2. (Use $\pi = 3.14$). [CBSE 2013]

20. In Fig. 14.75, from the top of a solid cone of height 12 cm and base radius 6 cm, a cone of height 4 cm is removed by a plane parallel to the base. Find the total surface area of the remaining solid. (Use $\pi = 22/7$ and $\sqrt{5} = 2.236$). [CBSE 2015]

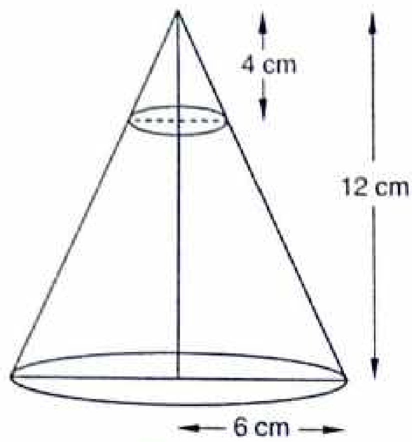

Fig. 14.75

21. The height of a cone is 10 cm. The cone is divided into two parts using a plane parallel to its base at the middle of its height. Find the ratio of the volumes of two parts.

[CBSE 2017]

22. A bucket, made of metal sheet, is in the form of a cone whose height is 35 cm and radii of circular ends are 30 cm and 12 cm. How many litres of milk it contains if it is full to the brim? If the milk is sold at ₹ 40 per litre, find the amount received by the person.

[CBSE 2017]

23. The diameters of the lower and upper ends of a bucket in the form of a frustum of a cone are 10 cm and 30 cm respectively. If its height is 24 cm,
   (i) Find the area of the metal sheet used to make the bucket.
   (ii) Why we should avoid the bucket made by ordinary plastic? (use $\pi = 3.14$)

[CBSE 2018]

LEVEL-2

24. A reservoir in the form of the frustum of a right circular cone contains 44×10^7 litres of water which fills it completely. The radii of the bottom and top of the reservoir are 50 metres and 100 metres respectively. Find the depth of water and the lateral surface area of the reservoir. (Take: $\pi = 22/7$)

ANSWERS

1. $8800\ cm^3$, ₹ 21.40
2. $678.85\ cm^2$, $616\ cm^3$
3. $48\ cm^2$
4. $1900\ cm^3$, $619.65\ cm^2$, $860.275\ cm^2$
5. $48510\ cm^3$
6. 16 cm
7. $545\pi\ cm^2$
8. $378\pi\ cm^2$, $756\pi\ cm^3$
9. $892.57\ m^2$
10. ₹ 460.24
11. Capacity = 21.980 litres, Surface area = $3292.6\ cm^2$, Cost of milk = ₹ 549.50
12. Height = 15 cm, Area = $2160.32\ cm^2$
13. ₹ 2143.05
14. $7599.42\ cm^2$
15. $10449.92\ cm^2$, ₹ 2089.98
16. $1944\pi\ cm^2$, $5292\pi\ cm^3$
17. ₹ 4224
18. 1 : 7
19. 171.13
20. $350.59\ cm^2$
21. 1 : 7
22. 51.48 litres, ₹ 2059.20
23. (i) $1711.30\ cm^3$
24. 24 m, $26145.9\ m^2$

REVISION EXERCISE

1. A metallic sphere 1 dm in diameter is beaten into a circular sheet of uniform thickness equal to 1 mm. Find the radius of the sheet.
2. Three solid spheres of radii 3, 4 and 5 cm respectively are melted and converted into a single solid sphere. Find the radius of this sphere.

3. A spherical shell of lead, whose external diameter is 18 cm, is melted and recast into a right circular cylinder, whose height is 8 cm and diameter 12 cm. Determine the internal diameter of the shell.
4. A well with 10 m inside diameter is dug 8.4 m deep. Earth taken out of it is spread all around it to a width of 7.5 m to form an embankment. Find the height of the embankment.
5. In the middle of a rectangular field measuring $30 \text{ m} \times 20 \text{ m}$, a well of 7 m diameter and 10 m depth is dug. The earth so removed is evenly spread over the remaining part of the field. Find the height through which the level of the field is raised.
6. The inner and outer radii of a hollow cylinder are 15 cm and 20 cm, respectively. The cylinder is melted and recast into a solid cylinder of the same height. Find the radius of the base of new cylinder.
7. Two cylindrical vessels are filled with oil. Their radii are 15 cm, 12 cm and heights 20 cm, 16 cm respectively. Find the radius of a cylindrical vessel 21 cm in height, which will just contain the oil of the two given vessels.
8. A cylindrical bucket 28 cm in diameter and 72 cm high is full of water. The water is emptied into a rectangular tank 66 cm long and 28 cm wide. Find the height of the water level in the tank.
9. A cubic cm of gold is drawn into a wire 0.1 mm in diameter, find the length of the wire.
10. A well of diameter 3 m is dug 14 m deep. The earth taken out of it is spread evenly all around it to a width of 4 m to form an embankment. Find the height of the embankment.
11. A conical vessel whose internal radius is 10 cm and height 48 cm is full of water. Find the volume of water. If this water is poured into a cylindrical vessel with internal radius 20 cm, find the height to which the water level rises in it.
12. The vertical height of a conical tent is 42 dm and the diameter of its base is 5.4 m. Find the number of persons it can accommodate if each person is to be allowed 29.16 cubic dm.
13. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, determine the ratio of the radius of the base to the height of either of them.
14. A sphere of diameter 5 cm is dropped into a cylindrical vessel partly filled with water. The diameter of the base of the vessel is 10 cm. If the sphere is completely submerged, by how much will the level of water rise?
15. A spherical ball of iron has been melted and made into smaller balls. If the radius of each smaller ball is one-fourth of the radius of the original one, how many such balls can be made?
16. Find the depth of a cylindrical tank of radius 28 m, if its capacity is equal to that of a rectangular tank of size $28 \text{ m} \times 16 \text{ m} \times 11 \text{ m}$.
17. A hemispherical bowl of internal radius 15 cm contains a liquid. The liquid is to be filled into cylindrical-shaped bottles of diameter 5 cm and height 6 cm. How many bottles are necessary to empty the bowl? [CBSE 2001 C]
18. In a cylindrical vessel of diameter 24 cm, filled up with sufficient quantity of water, a solid spherical ball of radius 6 cm is completely immersed. Find the increase in height of water level.
19. A hemisphere of lead of radius 7 cm is cast into a right circular cone of height 49 cm. Find the radius of the base.

20. A solid metallic sphere of diameter 28 cm is melted and recast into a number of smaller cones, each of diameter $4\frac{2}{3}$ cm and height 3 cm. Find the number of cones so formed. CBSE 2004]
21. The diameter of a copper sphere is 18 cm. The sphere is melted and is drawn into a long wire of uniform circular cross-section. If the length of the wire is 108 m, find its diameter.
22. A hemisphere of lead of radius 7 cm is cast into a right circular cone of height 49 cm. Find the radius of the base.
23. A metallic sphere of radius 10.5 cm is melted and thus recast into small cones, each of radius 3.5 cm and height 3 cm. Find how many cones are obtained. [CBSE 2004]
24. A cone, a hemisphere and a cylinder stand on equal bases and have the same height. Show that their volumes are in the ratio 1 : 2 : 3.
25. A hollow sphere of internal and external diameters 4 and 8 cm respectively is melted into a cone of base diameter 8 cm. Find the height of the cone.
26. The largest sphere is carved out of a cube of side 10.5 cm. Find the volume of the sphere.
27. Find the weight of a hollow sphere of metal having internal and external diameters as 20 cm and 22 cm, respectively if 1 m^3 of metal weighs 21 g.
28. A solid sphere of radius 'r' is melted and recast into a hollow cylinder of uniform thickness. If the external radius of the base of the cylinder is 4 cm, its height 24 cm and thickness 2 cm, find the value of 'r'.
29. Lead spheres of diameter 6 cm are dropped into a cylindrical beaker containing some water and are fully submerged. If the diameter of the beaker is 18 cm and water rises by 40 cm. find the number of lead spheres dropped in the water.
30. The height of a solid cylinder is 15 cm and the diameter of its base is 7 cm. Two equal conical holes each of radius 3 cm and height 4 cm are cut off. Find the volume of the remaining solid.
31. A solid is composed of a cylinder with hemispherical ends. If the length of the whole solid is 108 cm. and the diameter of the cylinder is 36 cm, find the cost of polishing the surface at the rate of 7 paise per cm^2. [Use $\pi = 3.1416$]
32. The surface area of a sphere is the same as the curved surface area of a cone having the radius of the base as 120 cm and height 160 cm. Find the radius of the sphere.
33. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, determine the ratio of the radius of the base to the height of either of them.
34. A rectangular vessel of dimensions 20 cm × 16 cm × 11 cm. is full of water. This water is poured into a conical vessel. The top of the conical vessel has its radius 10 cm. If the conical vessel is filled completely, determine its height. [Use $\pi = 22/7$]
35. If r_1 and r_2 be the radii of two solid metallic spheres and if they are melted into one solid sphere, prove that the radius of the new sphere is $(r_1^3 + r_2^3)^{1/3}$.
36. A solid metal sphere of 6 cm diameter is melted and a circular sheet of thickness 1 cm is prepared Determine the diameter of the sheet.
37. A hemispherical tank full of water is emptied by a pipe at the rate of $\frac{25}{7}$ litres per second. How much time will it take to half-empty the tank, if the tank is 3 metres in diameter?

38. Find the number of coins, 1.5 cm is diameter and 0.2 cm thick, to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm.

39. The radius of the base of a right circular cone of semi-vertical angle α is r. Show that its volume is $\frac{1}{3}\pi r^3 \cot \alpha$ and curved surface area is $\pi r^2 \operatorname{cosec} \alpha$.

40. An iron pillar consists of a cylindrical portion 2.8 m high and 20 cm in diameter and a cone 42 cm high is surmounting it. Find the weight of the pillar, given that 1 cubic cm of iron weighs 7.5 gm.

41. A circus tent is cylindrical to a height of 3 metres and conical above it. If its diameter is 105 m and the slant height of the conical portion is 53 m, calculate the length of the canvas 5 m wide to make the required tent.

42. Height of a solid cylinder is 10 cm and diameter 8 cm. Two equal conical hole have been made from its both ends. If the diameter of the holes is 6 cm and height 4 cm, find (i) volume of the cylinder, (ii) volume of one conical hole, (iii) volume of the remaining solid.

43. The height of a solid cylinder is 15 cm. and the diameter of its base is 7 cm. Two equal conical holes each of radius 3 cm, and height 4 cm are cut off. Find the volume of the remaining solid.

44. A solid is composed of a cylinder with hemispherical ends. If the length of the whole solid is 108 cm and the diameter of the cylinder is 36 cm, find the cost of polishing the surface at the rate of 7 paise per cm^2. (Use $\pi = 3.1416$)

45. The largest sphere is to be curved out of a right circular cylinder of radius 7 cm. and height 14 cm. Find the volume of the sphere.

46. A tent is in the form of a right circular cylinder surmounted by a cone. The diameter of the base of the cylinder or the cone is 24 m. The height of the cylinder is 11 m. If the vertex of the cone is 16 m above the ground, find the area of the canvas required for making the tent. (Use $\pi = 22/7$)

47. A toy is in the form of a cone mounted on a hemisphere of radius 3.5 cm. The total height of the toy is 15.5 cm find the total surface area and volume of the toy.
[CBSE 2000, 2002]

48. A cylindrical container is filled with ice-cream, whose diameter is 12 cm and height is 15 cm. The whole ice-cream is distributed to 10 children in equal cones having hemispherical tops. If the height of the conical portion is twice the diameter of its base, find the diameter of the ice-cream.

49. Find the volume of a solid in the form of a right circular cylinder with hemispherical ends whose total length is 2.7 m and the diameter of each hemispherical end is 0.7 m.

50. A tent of height 8.25 m is in the form of a right circular cylinder with diameter of base 30 m and height 5.5 m, surmounted by a right circular cone of the same base. Find the cost of the canvas of the tent at the rate of ₹ 45 per m^2.

51. An iron pole consisting of a cylindrical portion 110 cm high and of base diameter 12 cm is surmounted by a cone 9 cm high. Find the mass of the pole, given that 1 cm^3 of iron has 8 gram mass approximately. (Use: $\pi = 355/115$)

52. The interior of a building is in the form of a cylinder of base radius 12 m and height 3.5 m, surmounted by a cone of equal base and slant height 12.5 m. Find the internal curved surface area and the capacity of the building.

53. A right angled triangle with sides 3 cm and 4 cm is revolved around its hypotenuse. Find the volume of the double cone thus generated.

54. A toy is in the form of a cone mounted on a hemisphere with the same radius. The diameter of the base of the conical portion is 6 cm and its height is 4 cm. Determine the surface area of the toy. (Use $\pi = 3.14$)

55. Find the mass of a 3.5 m long lead pipe, if the external diameter of the pipe is 2.4 cm, thickness of the metal is 2 mm and the mass of 1 cm^3 of lead is 11.4 grams.

56. A solid is in the form of a cylinder with hemispherical ends. Total height of the solid is 19 cm and the diameter of the cylinder is 7 cm. Find the volume and total surface area of the solid.

57. A golf ball has diameter equal to 4.2 cm. Its surface has 200 dimples each of radius 2 mm. Calculate the total surface area which is exposed to the surroundings assuming that the dimples are hemispherical.

58. The radii of the ends of a bucket of height 24 cm are 15 cm and 5 cm. Find its capacity. (Take $\pi = 22/7$).

59. The radii of the ends of a bucket 30 cm high are 21 cm and 7 cm. Find its capacity in litres and the amount of sheet required to make this bucket.

60. The radii of the ends of a frustum of a right circular cone are 5 metres and 8 metres and its lateral height is 5 metres. Find the lateral surface and volume of the frustum.

61. A frustum of a cone is 9 cm thick and the diameters of its circular ends are 28 cm and 4 cm. Find the volume and lateral surface area of the frustum. (Take $\pi = 22/7$).

62. A bucket is in the form of a frustum of a cone and holds 15.25 litres of water. The diameters of the top and bottom are 25 cm and 20 cm respectively. Find its height and area of tin used in its construction.

63. If a cone of radius 10 cm is divided into two parts by drawing a plane through the mid-point of its axis, parallel to its base. Compare the volumes of the two parts. **[CBSE 2000 C]**

64. A tent is of the shape of a right circular cylinder upto a height of 3 metres and then becomes a right circular cone with a maximum height of 13.5 metres above the ground. Calculate the cost of painting the inner side of the tent at the rate of ₹ 2 per square metre, if the radius of the base is 14 metres.

65. An oil funnel of tin sheet consists of a cylindrical portion 10 cm long attached to a frustum of a cone. If the total height be 22 cm, the diameter of the cylindrical portion 8 cm and the diameter of the top of the funnel 18 cm, find the area of the tin required. (Use : $\pi = 22/7$). **[NCERT]**

66. A solid cylinder of diameter 12 cm and height 15 cm is melted and recast into toys with the shape of a right circular cone mounted on a hemisphere of radius 3 cm. If the height of the toy is 12 cm, find the number of toys so formed. **[CBSE 2006 C]**

67. A container open at the top, is in the form of a frustum of a cone of height 24 cm with radii of its lower and upper circular ends as 8 cm and 20 cm respectively. Find the cost of milk which can completely fill the container at the rate of ₹ 21 per litre. (Use $\pi = 22/7$)

[CBSE 2014]

68. A cone of maximum size is carved out from a cube of edge 14 cm. Find the surface area of the cone and of the remaining solid left out after the cone carved out.

[NCERT EXEMPLAR]

69. A cone of radius 4 cm is divided into two parts by drawing a plane through the mid point of its axis and parallel to its base. Compare the volumes of two parts.

[NCERT EXEMPLAR]

70. A wall 24 m, 0.4 m thick and 6 m high is constructed with the bricks each of dimensions 25 cm × 16 cm × 10 cm. If the mortar occupies $\frac{1}{10}$th of the volume of the wall, then find the number of bricks used in constructing the wall.

[NCERT EXEMPLAR]

71. A bucket is in the form of a frustum of a cone and holds 28.490 litres of water. The radii of the top and bottom are 28 cm and 21 cm respectively. Find the height of the bucket.

[NCERT EXEMPLAR]

72. Marbles of diameter 1.4 cm are dropped into a cylindrical beaker of diameter 7 cm containing some water. Find the number of marbles that should be dropped into the beaker so that the water level rises by 5.6 cm. **[NCERT EXEMPLAR]**

73. Two cones with same base radius 8 cm and height 15 cm are joined together along their bases. Find the surface area of the shape formed. **[NCERT EXEMPLAR]**

74. From a solid cube of side 7 cm, a conical cavity of height 7 cm and radius 3 cm is hollowed out. Find the volume of the remaining solid. **[NCERT EXEMPLAR]**

75. Two solid cones A and B are placed in a cylindrical tube as shown in Fig. 14.76. The ratio of their capacities are 2 : 1. Find the heights and capacities of the cones. Also, find the volume of the remaining portion of the cylinder. **[NCERT EXEMPLAR]**

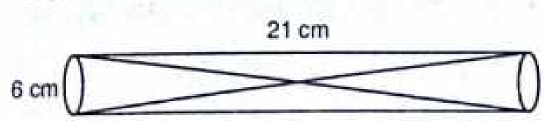

Fig. 14.76

76. An icecream cone full of icecream having radius 5 cm and height 10 cm as shown in Fig. 14.77. Calculate the volume of icecream, provided that its 1/6 part is left unfilled with icecream. **[NCERT EXEMPLAR]**

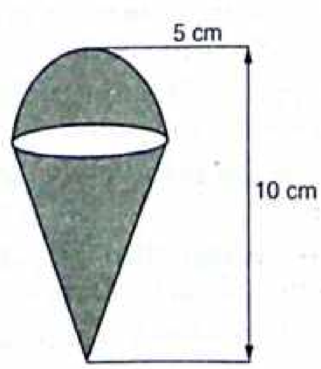

Fig. 14.77

ANSWERS

1. 4.08 cm	2. 6 cm	3. $6(19)^{1/3}$	4. 1.6 cm
5. 68.6 cm	6. 13.2 cm	7. 18 cm	8. 24 cm
9. 127.3 m	10. 1.125 m	11. 5024 cm^3, 4 cm	12. 11
13. 3 : 4	14. 5/6 cm	15. 64	16. 2 m
17. 60	18. 2 cm	19. 3.74 cm	20. 126
21. 0.6 cm	22. 3.74 cm	23. 126	25. 14 cm
26. 606.375	27. 29.13 kg	28. 6 cm	29. 90
30. 502.1 cm^3	31. ₹ 855. 02	32. 77.46 cm	33. 3 : 4
34. 33.6 cm	36. 12 cm	37. 16.5 minutes	38. 450
40. 693 kg	41. 1947 m	42. $160\pi\ cm^3$, $12\pi\ cm^3$, $136\pi\ cm^3$	
43. 502.1 cm^3	44. ₹ 855. 02	45. 1437	46. 1320 m^2
47. 214.5 cm^2, 243.83 cm^3		48. 6 cm	49. 0.95 m (appr.)
50. ₹ 55687.50	51. 102.24 kg	52. 735.43 m^2, 2112 m^3	
53. $30\frac{6}{35}$ cm^3	54. 103.62 cm^2	55. 5.518 kg	
56. 641.67 cm^3, 418 cm^2		57. 80.58 cm^2	58. 8171.42 cm^2
59. 20.02 litres, 3069 cm^2		60. 204.28 m^2, 540.56 cm^3	
61. $684\pi\ cm^3$, $240\pi\ cm^2$		62. 38.18 cm, 3017 cm^2	
63. 1 : 7	64. ₹ 2068	65. $249\pi\ cm^2$	66. 12
67. 329.47	68. $154(\sqrt{5}+1)\ cm^2$, $(1022+154\sqrt{5})\ cm^2$		69. 1 : 7
70. 12960	71. 15 cm	72. 150	73. 855 cm^2 (Approx.)
74. 277 cm^3	75. 14 cm, 7 cm, 132 cm^3, 66 cm^3, 396 cm^3		76. 327.4 cm^2

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

1. The radii of the bases of a cylinder and a cone are in the ratio 3 : 4 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?
2. If the heights of two right circular cones are in the ratio 1 : 2 and the perimeters of their bases are in the ratio 3 : 4, what is the ratio of their volumes?
3. If a cone and a sphere have equal radii and equal volumes. What is the ratio of the diameter of the sphere to the height of the cone?
4. A cone, a hemisphere and a cylinder stand on equal bases and have the same height. What is the ratio of their volumes?
5. The radii of two cylinders are in the ratio 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their curved surface areas?
6. Two cubes have their volumes in the ratio 1 : 27. What is the ratio of their surface areas?
7. Two right circular cylinders of equal volumes have their heights in the ratio 1 : 2. What is the ratio of their radii?

8. If the volumes of two cones are in the ratio 1 : 4 and their diameters are in the ratio 4 : 5, then write the ratio of their weights.
9. A sphere and a cube have equal surface areas. What is the ratio of the volume of the sphere to that of the cube?
10. What is the ratio of the volume of a cube to that of a sphere which will fit inside it?
11. What is the ratio of the volumes of a cylinder, a cone and a sphere, if each has the same diameter and same height?
12. A sphere of maximum volume is cut-out from a solid hemisphere of radius r. What is the ratio of the volume of the hemisphere to that of the cut-out sphere?
13. A metallic hemisphere is melted and recast in the shape of a cone with the same base radius R as that of the hemisphere. If H is the height of the cone, then write the value of H/R.
14. A right circular cone and a right circular cylinder have equal base and equal height. If the radius of the base and height are in the ratio 5 : 12, write the ratio of the total surface area of the cylinder to that of the cone.
15. A cylinder, a cone and a hemisphere are of equal base and have the same height. What is the ratio of their volumes?
16. The radii of two cones are in the ratio 2 : 1 and their volumes are equal. What is the ratio of their heights?
17. Two cones have their heights in the ratio 1 : 3 and radii 3 : 1. What is the ratio of their volumes?
18. A hemisphere and a cone have equal bases. If their heights are also equal, then what is the ratio of their curved surfaces?
19. If r_1 and r_2 denote the radii of the circular bases of the frustum of a cone such that $r_1 > r_2$, then write the ratio of the height of the cone of which the frustum is a part to the height of the frustum.
20. If the slant height of the frustum of a cone is 6 cm and the perimeters of its circular bases are 24 cm and 12 cm respectively. What is the curved surface area of the frustum?
21. If the areas of circular bases of a frustum of a cone are 4 cm^2 and 9 cm^2 respectively and the height of the frustum is 12 cm. What is the volume of the frustum?
22. The surface area of a sphere is 616 cm^2. Find its radius. **[CBSE 2008]**
23. A cylinder and a cone are of the same base radius and of same height. Find the ratio of the value of the cylinder to that of the cone **[CBSE 2009]**
24. The slant height of the frustum of a cone is 5 cm. If the difference between the radii of its two circular ends is 4 cm, write the height of the frustum. **[CBSE 2010]**
25. Volume and surface area of a solid hemisphere are numerically equal. What is the diameter of hemisphere? **[CBSE 2017]**

ANSWERS

1. 9 : 8	2. 9 : 32	3. 1 : 2	4. 1 : 2 : 3	5. 2 : 5	6. 1 : 9
7. $\sqrt{2} : 1$	8. 25 : 64	9. $\sqrt{\frac{6}{\pi}}$	10. $6 : \pi$	11. 3 : 1 : 2	12. 4 : 1
13. 2	14. 17 : 9	15. 3 : 1 : 2	16. 1 : 4	17. 3 : 1	18. $\sqrt{2} : 1$
19. $\frac{r_1}{r_1 - r_2}$	20. 108 cm^2	21. 44 cm^2	22. 7 cm	23. 3 : 1	24. 3 cm
25. 9 units					

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The diameter of a sphere is 6 cm. It is melted and drawn into a wire of diameter 2 mm. The length of the wire is
(a) 12 m (b) 18 m (c) 36 m (d) 66 m
2. A metallic sphere of radius 10.5 cm is melted and then recast into small cones, each of radius 3.5 cm and height 3 cm. The number of such cones is
(a) 63 (b) 126 (c) 21 (d) 130
3. A solid is hemispherical at the bottom and conical above. If the surface areas of the two parts are equal, then the ratio of its radius and the height of its conical part is
(a) $1:3$ (b) $1:\sqrt{3}$ (c) $1:1$ (d) $\sqrt{3}:1$
4. A solid sphere of radius r is melted and cast into the shape of a solid cone of height r, the radius of the base of the cone is
(a) $2r$ (b) $3r$ (c) r (d) $4r$
5. The material of a cone is converted into the shape of a cylinder of equal radius. If height of the cylinder is 5 cm, then height of the cone is
(a) 10 cm (b) 15 cm (c) 18 cm (d) 24 cm
6. A circus tent is cylindrical to a height of 4 m and conical above it. If its diameter is 105 m and its slant height is 40 m, the total area of the canvas required in m^2 is
(a) 1760 (b) 2640 (c) 3960 (d) 7920
7. The number of solid spheres, each of diameter 6 cm that could be moulded to form a solid metal cylinder of height 45 cm and diameter 4 cm, is
(a) 3 (b) 4 (c) 5 (d) 6
8. A sphere of radius 6 cm is dropped into a cylindrical vessel partly filled with water. The radius of the vessel is 8 cm. If the sphere is submerged completely, then the surface of the water rises by
(a) 4.5 cm (b) 3 cm (c) 4 cm (d) 2 cm
9. If the radii of the circular ends of a bucket of height 40 cm are of lengths 35 cm and 14 cm, then the volume of the bucket in cubic centimeters, is
(a) 60060 (b) 80080 (c) 70040 (d) 80160
10. If a cone is cut into two parts by a horizontal plane passing through the mid-point of its axis, the ratio of the volumes of the upper part and the cone is
(a) $1:2$ (b) $1:4$ (c) $1:6$ (d) $1:8$
11. The height of a cone is 30 cm. A small cone is cut off at the top by a plane parallel to the base. If its volume be $\frac{1}{27}$ of the volume of the given cone, then the height above the base at which the section has been made, is
(a) 10 cm (b) 15 cm (c) 20 cm (d) 25 cm
12. A solid consists of a circular cylinder with an exact fitting right circular cone placed at the top. The height of the cone is h. If the total volume of the solid is 3 times the volume of the cone, then the height of the circular cylinder is
(a) $2h$ (b) $\frac{2h}{3}$ (c) $\frac{3h}{2}$ (d) $4h$

13. A reservoir is in the shape of a frustum of a right circular cone. It is 8 m across at the top and 4 m across at the bottom. If it is 6 m deep, then its capacity is
(a) $176\ m^3$ (b) $196\ m^3$ (c) $200\ m^3$ (d) $110\ m^3$

14. Water flows at the rate of 10 metre per minute from a cylindrical pipe 5 mm in diameter. How long will it take to fill up a conical vessel whose diameter at the base is 40 cm and depth 24 cm?
(a) 48 minutes 15 sec (b) 51 minutes 12 sec
(c) 52 minutes 1 sec (d) 55 minutes

15. A cylindrical vessel 32 cm high and 18 cm as the radius of the base, is filled with sand. This bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, the radius of its base is
(a) 12 cm (b) 24 cm (c) 36 cm (d) 48 cm

16. The curved surface area of a right circular cone of height 15 cm and base diameter 16 cm is
(a) $60\pi\ cm^2$ (b) $68\pi\ cm^2$ (c) $120\pi\ cm^2$ (d) $136\pi\ cm^2$

17. A right triangle with sides 3 cm, 4 cm and 5 cm is rotated about the side of 3 cm to form a cone. The volume of the cone so formed is
(a) $12\pi\ cm^3$ (b) $15\pi\ cm^3$ (c) $16\pi\ cm^3$ (d) $20\pi\ cm^3$

18. The curved surface area of a cylinder is $264\ m^2$ and its volume is $924\ m^3$. The ratio of its diameter to its height is
(a) 3 : 7 (b) 7 : 3 (c) 6 : 7 (d) 7 : 6

19. A cylinder with base radius of 8 cm and height of 2 cm is melted to form a cone of height 6 cm. The radius of the cone is
(a) 4 cm (b) 5 cm (c) 6 cm (d) 8 cm

20. The volumes of two spheres are in the ratio 64 : 27. The ratio of their surface areas is
(a) 1 : 2 (b) 2 : 3 (c) 9 : 16 (d) 16 : 9

21. If three metallic spheres of radii 6 cm, 8 cm and 10 cm are melted to form a single sphere, the diameter of the sphere is
(a) 12 cm (b) 24 cm (c) 30 cm (d) 36 cm

22. The surface area of a sphere is same as the curved surface area of a right circular cylinder whose height and diameter are 12 cm each. The radius of the sphere is
(a) 3 cm (b) 4 cm (c) 6 cm (d) 12 cm

23. The volume of the greatest sphere that can be cut off from a cylindrical log of wood of base radius 1 cm and height 5 cm is
(a) $\frac{4}{3}\pi$ (b) $\frac{10}{3}\pi$ (c) 5π (d) $\frac{20}{3}\pi$

24. A cylindrical vessel of radius 4 cm contains water. A solid sphere of radius 3 cm is lowered into the water until it is completely immersed. The water level in the vessel will rise by
(a) $\frac{2}{9}$ cm (b) $\frac{4}{9}$ cm (c) $\frac{9}{4}$ cm (d) $\frac{9}{2}$ cm

25. 12 spheres of the same size are made from melting a solid cylinder of 16 cm diameter and 2 cm height. The diameter of each sphere is

(a) $\sqrt{3}$ cm (b) 2 cm (c) 3 cm (d) 4 cm

26. A solid metallic spherical ball of diameter 6 cm is melted and recast into a cone with diameter of the base as 12 cm. The height of the cone is

(a) 2 cm (b) 3 cm (c) 4 cm (d) 6 cm

27. A hollow sphere of internal and external diameters 4 cm and 8 cm respectively is melted into a cone of base diameter 8 cm. The height of the cone is

(a) 12 cm (b) 14 cm (c) 15 cm (d) 18 cm

28. A solid piece of iron of dimensions $49 \times 33 \times 24$ cm is moulded into a sphere. The radius of the sphere is

(a) 21 cm (b) 28 cm (c) 35 cm (d) None of these

29. The ratio of lateral surface area to the total surface area of a cylinder with base diameter 1.6 m and height 20 cm is

(a) 1 : 7 (b) 1 : 5 (c) 7 : 1 (d) 5 : 1

30. A solid consists of a circular cylinder surmounted by a right circular cone. The height of the cone is h. If the total height of the solid is 3 times the volume of the cone, then the height of the cylinder is

(a) $2h$ (b) $\frac{3h}{2}$ (c) $\frac{h}{2}$ (d) $\frac{2h}{3}$

31. The maximum volume of a cone that can be carved out of a solid hemisphere of radius r is

(a) $3\pi r^2$ (b) $\frac{\pi r^3}{3}$ (c) $\frac{\pi r^2}{3}$ (d) $3\pi r^3$

32. The radii of two cylinders are in the ratio 3 : 5. If their heights are in the ratio 2 : 3, then the ratio of their curved surface areas is

(a) 2 : 5 (b) 5 : 2 (c) 2 : 3 (d) 3 : 5

33. A right circular cylinder of radius r and height h ($h = 2r$) just encloses a sphere of diameter

(a) h (b) r (c) $2r$ (d) $2h$

34. The radii of the circular ends of a frustum are 6 cm and 14 cm. If its slant height is 10 cm, then its vertical height is

(a) 6 cm (b) 8 cm (c) 4 cm (d) 7 cm

35. The height and radius of the cone of which the frustum is a part are h_1 and r_1 respectively. If h_2 and r_2 are the heights and radius of the smaller base of the frustum respectively and $h_2 : h_1 = 1 : 2$, then $r_2 : r_1$ is equql to

(a) 1 : 3 (b) 1 : 2 (c) 2 : 1 (d) 3 : 1

36. The diameters of the ends of a frustum of a cone are 32 cm and 20 cm. If its slant height is 10 cm, then its lateral surface area is

(a) 321π cm^2 (b) 300π cm^2 (c) 260π cm^2 (d) 250π cm^2

37. A solid frustum is of height 8 cm. If the radii of its lower and upper ends are 3 cm and 9 cm respectively, then its slant height is

(a) 15 cm (b) 12 cm (c) 10 cm (d) 17 cm

38. The radii of the ends of a bucket 16 cm high are 20 cm and 8 cm. The curved surface area of the bucket is
(a) 1760 cm^2 (b) 2240 cm^2 (c) 880 cm^2 (d) 3120 cm^2

39. The diameters of the top and the bottom portions of a bucket are 42 cm and 28 cm respectively. If the height of the bucket is 24 cm, then the cost of painting its outer surface at the rate of 50 paise/cm^2 is
(a) ₹ 1582.50 (b) ₹ 1724.50 (c) ₹ 1683 (d) ₹ 1642

40. If four times the sum of the areas of two circular faces of a cylinder of height 8 cm is equal to twice the curve surface area, then diameter of the cylinder is
(a) 4 cm (b) 8 cm (c) 2 cm (d) 6 cm

41. If the radius of the base of a right circular cylinder is halved, keeping the height the same, then the ratio of the volume of the cylinder thus obtained to the volume of original cylinder is
(a) 1 : 2 (b) 2 : 1 (c) 1 : 4 (d) 4 : 1 [CBSE 2012

42. A metalic solid cone is melted to form a solid cylinder of equal radius. If the height of the cylinder is 6 cm, then the height of the cone was
(a) 10 cm (b) 12 cm (c) 18 cm (d) 24 cm [CBSE 2014]

43. A rectangular sheet of paper 40 cm × 22 cm, is rolled to form a hollow cylinder of height 40 cm. The radius of the cylinder (in cm) is
(a) 3.5 (b) 7 (c) 80/7 (d) 5 [CBSE 2014]

44. The number of solid spheres, each of diameter 6 cm that can be made by melting a solid metal cylinder of height 45 cm and diameter 4 cm is
(a) 3 (b) 5 (c) 4 (d) 6 [CBSE 2014]

45. Volumes of two spheres are in the ratio 64 : 27. The ratio of their surface areas is
(a) 3 : 4 (b) 4 : 3 (c) 9 : 16 (d) 16 : 9

46. A right circular cylinder of radius r and height h ($h > 2r$) just encloses a sphere of diameter
(a) r (b) $2r$ (c) h (d) $2h$

47. In a right circular cone, the cross-section made by a plane parallel to the base is a
(a) circle (b) frustyum of a cone (c) sphere (d) hemisphare

48. If two solid-hemispheres of same base radius r are joined together along their bases, then curved surface area of this new solid is
(a) $4\pi r^2$ (b) $6\pi r^2$ (c) $3\pi r^2$ (d) $8\pi r^2$

49. The diameters of two circular ends of the bucket are 44 cm and 24 cm. The height of the bucket is 35 cm. The capacity of the bucket is
(a) 32.7 litres (b) 33.7 litres (c) 34.7 litres (d) 31.7 litres

50. A spherical ball of radius r is melted to make 8 new identical balls each of radius r_1. Then $r : r_1 =$
(a) 2 : 1 (b) 1 : 2 (c) 4 : 1 (d) 1 : 4

ANSWERS

1. (c)	2. (b)	3. (b)	4. (a)	5. (b)	6. (d)
7. (c)	8. (a)	9. (b)	10. (d)	11. (c)	12. (b)
13. (a)	14. (b)	15. (c)	16. (d)	17. (a)	18. (b)

(i) Volume of the frustum $= \frac{\pi}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right) h$

(ii) Lateral surface area $= \pi(r_1 + r_2) l$

(iii) Total surface area $= \pi\{(r_1 + r_2) l + r_1^2 + r_2^2\}$

(iv) Slant height of the frustum $= \sqrt{h^2 + (r_1 - r_2)^2}$

(v) Height of the cone of which the frustum is a part $= \frac{h r_1}{r_1 - r_2}$

(vi) Slant height of the cone of which the frustum is a part $= \frac{l r_1}{r_1 - r_2}$

(vii) Volume of the frustum $= \frac{h}{3}\left\{A_1 + A_2 + \sqrt{A_1 A_2}\right\}$, where A_1 and A_2 denote the areas of circular bases of the frustum.

19. (d)	20. (d)	21. (b)	22. (c)	23. (a)	24. (c)
25. (d)	26. (b)	27. (b)	28. (a)	29. (b)	30. (d)
31. (b)	32. (a)	33. (c)	34. (a)	35. (b)	36. (c)
37. (c)	38. (a)	39. (c)	40. (b)	41. (d)	42. (c)
43. (a)	44. (b)	45. (d)	46. (b)	47. (a)	48. (a)
49. (a)	50. (a)				

SUMMARY

1. If l, b and h denote respectively the length, breadth and height of a cuboid, then
   (i) Total surface area of the cuboid $= 2(lb + bh + lh)$ square units.
   (ii) Volume of the cuboid $=$ Area of the base $\times$ height $= lbh$ cubic units.
   (iii) Diagonal of the cuboid $= \sqrt{l^2 + b^2 + h^2}$ units.
   (iv) Area of four walls of a room $= 2(l + b)\,h$ sq. units.
2. If the length of each edge of a cube is 'a' units, then
   (i) Total surface area of the cube $= 6a^2$ sq. units
   (ii) Volume of the cube $= a^3$ cubic units
   (iii) Diagonal of the cube $= \sqrt{3}\,a$ units.
3. If r and h denote respectively the radius of the base and height of a right circular cylinder, then
   (i) Area of each end $= \pi r^2$
   (ii) Curved surface area $= 2\pi rh$
   (iii) Total surface area $= 2\pi r(h + r)$ sq. units
   (iv) Volume $= \pi r^2 h =$ Area of the base $\times$ height
4. If R and r denote respectively the external and internal radii of a hollow right circular cylinder, then
   (i) Area of each end $= \pi(R^2 - r^2)$
   (ii) Curved surface area of hollow cylinder $= 2\pi(R + r)\,h$
   (iii) Total surface area $= 2\pi(R + r)(R + h - r)$
   (iv) Volume of material $= \pi h(R^2 - r^2)$
5. If r, h and l denote respectively the radius of base, height and slant height of a right circular cone, then
   (i) $l^2 = r^2 + h^2$
   (ii) Curved surface area $= \pi r l$
   (iii) Total surface area $= \pi r^2 + \pi r l$
   (iv) Volume $= \frac{1}{3}\pi r^2 h$
6. For a sphere of radius r, we have
   (i) Surface area $= 4\pi r^2$
   (ii) Volume $= \frac{4}{3}\pi r^3$
7. If h is the height, l the slant height and r_1 and r_2 the radii of the circular bases of a frustum of a cone, then

CHAPTER 15

STATISTICS

15.1 INTRODUCTION

In class IX, we have learnt about representation of statistical data in the form of histograms and frequency polygons. We have also learnt about mean, median and mode of ungrouped data. In this chapter, we will study about the techniques for finding mean, median and mode of grouped data. We will also learn about cumulative frequency graph of a frequency distribution.

15.2 MEAN OF GROUPED DATA

If $x_1, x_2, x_3, \dots, x_n$ are n values of a variable X, then the arithmetic mean or simply mean of these values is denoted by $\overline{X}$ and is defined as

$$\overline{X} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \text{ or, } \overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Here, the symbol $\sum_{i=1}^{n} x_i$ denotes the sum $x_1 + x_2 + x_3 + \dots + x_n$.

In other words, we can say that the arithmetic mean of a set of observations is equal to their sum divided by the total number of observations.

In this section, we will study about the arithmetic mean of grouped data or a discrete frequency distribution. In a discrete frequency distribution the arithmetic mean may be computed by any one of the following methods:

(i) Direct method,

(ii) Short-cut method,

(iii) Step-Deviation method.

Let us now learn about these methods one by one.

15.2.1 DIRECT METHOD

If a variate X takes values $x_1, x_2, \dots, x_n$ with corresponding frequencies $f_1, f_2, f_3, \dots, f_n$ respectively, then arithmetic mean of these values is given by

$$\overline{X} = \frac{f_1x_1 + f_2x_2 + \dots + f_nx_n}{f_1 + f_2 + \dots + f_n}$$

or, $$\overline{X} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \text{ where } N = \sum_{i=1}^{n} f_i = f_1 + f_2 \cdots + f_n$$

The following algorithm may be used to compute arithmetic mean by direct method.

ALGORITHM

STEP I *Prepare the frequency table in such a way that its first column consists of the values of the variate and the second column the corresponding frequencies.*

STEP II *Multiply the frequency of each row with the corresponding values of variable to obtain third column containing $f_i x_i$.*

STEP III *Find the sum of all entries in column III to obtain $\Sigma f_i x_i$*

STEP IV *Find the sum of all the frequencies in column II to obtain $\Sigma f_i = N$*

STEP V *Use the formula :* $\overline{X} = \frac{\Sigma f_i x_i}{N}$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the mean of the following distribution:*

x:	4	6	9	10	15
f:	5	10	10	7	8

SOLUTION *Calculation of Arithmetic Mean*

x_i	f_i	$f_i x_i$
4	5	20
6	10	60
9	10	90
10	7	70
15	8	120
	$N = \Sigma f_i = 40$	$\Sigma f_i x_i = 360$

$\therefore$ $$\text{Mean} = \overline{X} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{360}{40} = 9$$

EXAMPLE 2 *Following table shows the weight of 12 students:*

Weight (in kgs):	67	70	72	73	75
Number of students:	4	3	2	2	1

Find the mean weight of the students.

SOLUTION *Calculation of Arithmetic Mean*

Weight (in kgs) x_i	*Frequency* f_i	$f_i x_i$
67	4	268
70	3	210
72	2	144
73	2	146
75	1	75
	$N = \Sigma f_i = 12$	$\Sigma f_i x_i = 843$

$$\therefore \quad \text{Mean} = \overline{X} = \frac{\Sigma f_i x_i}{N} = \frac{843}{12} = 70.25 \text{ kg}$$

EXAMPLE 3 *Find the mean of the following distribution:*

x:	10	30	50	70	89
f:	7	8	10	15	10

SOLUTION *Calculation of Mean*

x_i	f_i	$f_i x_i$
10	7	70
30	8	240
50	10	500
70	15	1050
89	10	890
	$N = \Sigma f_i = 50$	$\Sigma f_i x_i = 2750$

$$\therefore \quad \text{Mean} = \frac{\Sigma f_i x_i}{N} = \frac{2750}{50} = 55$$

EXAMPLE 4 *If the mean of the following distribution is 6, find the value of p.*

x:	2	4	6	10	$p+5$
f:	3	2	3	1	2

SOLUTION *Calculation of Mean*

x_i	f_i	$f_i x_i$
2	3	6
4	2	8
6	3	18
10	1	10
$p+5$	2	$2p+10$
	$N = \Sigma f_i = 11$	$\Sigma f_i x_i = 2p+52$

We have, $N = \Sigma f_i = 11, \Sigma f_i x_i = 2p + 52$

$$\therefore \quad \text{Mean} = \frac{\Sigma f_i x_i}{N}$$

$$\Rightarrow \quad 6 = \frac{2p + 52}{11} \Rightarrow 66 = 2p + 52 \Rightarrow 2p = 14 \Rightarrow p = 7$$

EXAMPLE 5 *Find the value of p, if the mean of the following distribution is 7.5.*

x:	3	5	7	9	11	13
f:	6	8	15	p	8	4

SOLUTION *Calculation of Mean*

x_i	f_i	$f_i x_i$
3	6	18
5	8	40
7	15	105
9	p	$9p$
11	8	88
13	4	52
	$N = \Sigma f_i = 41 + p$	$\Sigma f_i x_i = 303 + 9p$

We have, $\Sigma f_i = 41 + p, \Sigma f_i x_i = 303 + 9p$

$\therefore$ $$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$\Rightarrow$ $$7.5 = \frac{303 + 9p}{41 + p}$$

$\Rightarrow$ $$7.5 \times (41 + p) = 303 + 9p$$

$\Rightarrow$ $$307.5 + 7.5p = 303 + 9p \Rightarrow 9p - 7.5p = 307.5 - 303 \Rightarrow 1.5p = 4.5 \Rightarrow p = 3$$

LEVEL-2

EXAMPLE 6 *Find the missing frequencies in the following frequency distribution if it is known that the mean of the distribution is 1.46.*

Number of accidents (x):	0	1	2	3	4	5	*Total*
Frequency (f):	46	?	?	25	10	5	200

SOLUTION Let the missing frequencies be f_1 and f_2.

Computation of Arithmetic Mean

x_i	f_i	$f_i x_i$
0	46	0
1	f_1	f_1
2	f_2	$2f_2$
3	25	75
4	10	40
5	5	25
	$N = 86 + f_1 + f_2$	$\Sigma f_i x_i = 140 + f_1 + 2f_2$

We have,

$$N = 200 \Rightarrow 200 = 86 + f_1 + f_2 \Rightarrow f_1 + f_2 = 114 \quad \ldots(i)$$

Also,

$$\text{Mean} = 1.46$$

$$\Rightarrow \quad 1.46 = \frac{\Sigma f_i x_i}{N}$$

$$\Rightarrow \quad 1.46 = \frac{140 + f_1 + 2f_2}{200}$$

$$\Rightarrow \quad 292 = 140 + f_1 + 2f_2 \Rightarrow f_1 + 2f_2 = 152 \quad \ldots(ii)$$

Solving equation (i) and (ii), we get

$$f_1 = 76 \text{ and } f_2 = 38$$

EXERCISE 15.1

LEVEL-1

1. Calculate the mean for the following distribution:

x:	5	6	7	8	9
f:	4	8	14	11	3

2. Find the mean of the following data:

x:	19	21	23	25	27	29	31
f:	13	15	16	18	16	15	13

3. If the mean of the following data is 20.6. Find the value of p.

x:	10	15	p	25	35
f:	3	10	25	7	5

4. If the mean of the following data is 15, find p.

x:	5	10	15	20	25
f:	6	p	6	10	5

5. Find the value of p for the following distribution whose mean is 16.6.

x:	8	12	15	p	20	25	30
f:	12	16	20	24	16	8	4

6. Find the missing value of p for the following distribution whose mean is 12.58.

x:	5	8	10	12	p	20	25
f:	2	5	8	22	7	4	2

7. Find the missing frequency (p) for the following distribution whose mean is 7.68.

x:	3	5	7	9	11	13
f:	6	8	15	p	8	4

8. The following table gives the number of boys of a particular age in a class of 40 students. Calculate the mean age of the students

Age (in years):	15	16	17	18	19	20
No. of students:	3	8	10	10	5	4

9. Candidates of four schools appear in a mathematics test. The data were as follows:

Schools	*No. of Candidates*	*Average Score*
I	60	75
II	48	80
III	Not available	55
IV	40	50

If the average score of the candidates of all the four schools is 66, find the number of candidates that appeared from school III.

10. Five coins were simultaneously tossed 1000 times and at each toss the number of heads were observed. The number of tosses during which 0, 1, 2, 3, 4 and 5 heads were obtained are shown in the table below. Find the mean number of heads per toss.

No. of heads per toss	*No. of tosses*
0	38
1	144
2	342
3	287
4	164
5	25
Total	1000

11. The arithmetic mean of the following data is 14. Find the value of k.

x_i:	5	10	15	20	25
f_i:	7	k	8	4	5.

[CBSE 2002C]

12. The arithmetic mean of the following data is 25, find the value of k.

x_i :	5	15	25	35	45
f_i:	3	k	3	6	2

[CBSE 2001]

13. If the mean of the following data is 18.75. Find the value of p.

x_i :	10	15	p	25	30
f_i :	5	10	7	8	2

[CBSE 2005]

LEVEL-2

14. Find the value of p, if the mean of the following distribution is 20.

x:	15	17	19	$20 + p$	23
f:	2	3	4	$5p$	6

15. Find the missing frequencies in the following frequency distribution if it is known that the mean of the distribution is 50.

x:	10	30	50	70	90	
f:	17	f_1	32	f_2	19	Total 120.

ANSWERS

1. 7.025 2. 25 3. $p = 20$ 4. 8
5. 18 6. 15 7. 9 8. 17.45 years
9. 52 10. 2.47 11. 6 12. 4
13. $p = 20$ 14. $p = 1$ 15. $f_1 = 28, f_2 = 24$

15.2.2 SHORT-CUT METHOD

If the values of x or (and) f are large, the calculation of AM by the direct method is quite tedious and time consuming, because calculations involved are lengthy. In such a case to minimize the time involved in calculation, we take deviations from an arbitrary point as discussed below.

Let $x_1, x_2 \cdots, x_n$ be values of a variable X with corresponding frequencies $f_1, f_2, f_3, \cdots, f_n$ respectively. Taking deviations about an arbitrary point 'A', we have

$$d_1 = x_i - A, i = 1, 2, 3, \cdots, n$$

$$\Rightarrow \quad f_i d_i = f_i(x_i - A); i = 1, 2, 3, \cdots, n$$

$$\Rightarrow \quad \sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n} f_i(x_i - A)$$

$$\Rightarrow \quad \sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n} f_i x_i - A \sum_{i=1}^{n} f_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n} f_i x_i - AN \qquad \left[\because N = \sum_{i=1}^{n} f_i\right]$$

$$\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{n} f_i d_i = \frac{1}{N}\sum_{i=1}^{n} f_i x_i - \frac{AN}{N}$$

$$\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{n} f_i d_i = \overline{X} - A \qquad \left[\because \overline{X} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i\right]$$

$$\Rightarrow \quad \overline{X} = A + \frac{1}{N}\sum_{i=1}^{n} f_i d_i$$

Finding AM by using the above formula is known as the short-cut method.

NOTE *The number 'A' is generally known as the assumed mean and is generally chosen in such a way that the deviations are small.*

Following algorithm may be used to find arithmetic mean by the short-cut method.

ALGORITHM

STEP I *Prepare the frequency table in such a way that its first column consists of the values of the variable and the second column consists of the corresponding frequencies.*

STEP II *Choose a number 'A' (preferable among the values in first column) and take deviations* $d_i = x_i - A$ *of the values* x_i *of variable X about A. Write these deviations against the corresponding frequencies in the third column.*

STEP III *Multiply the frequencies in column II with the corresponding deviations* d_i *in column III to prepare column IV consisting of* $f_i\ d_i$.

STEP IV *Find the sum of all entries in column III to obtain* $\sum_{i=1}^{n} f_i d_i$ *and the sum of all frequencies in column II to obtain* $\sum_{i=1}^{n} f_i = N$.

STEP V *Use the formula :* $\overline{X} = A + \frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i\right)$.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The following table shows the weights of 12 students:*

Weight (in kg):	67	70	72	73	75
Number of students:	4	3	2	2	1

Find the mean weight by using short-cut method.

SOLUTION Let the assumed mean be $A = 72$

Calculation of Mean

weight in Kg x_i	*No. of students* f_i	$d_i = x_i - A = x_i - 72$	$f_i d_i$
67	4	–5	–20
70	3	–2	–6
72	2	0	2
73	2	1	2
75	1	3	3
	$N = \Sigma f_i = 12$		$\Sigma f_i d_i = -21$

We have,

$$N = 12, \Sigma f_i d_i = -21, \text{ and } A = 72$$

$$\therefore \quad \text{Mean} = A + \frac{1}{N}(\Sigma f_i d_i)$$

$$\Rightarrow \quad \text{Mean} = 72 + \frac{(-21)}{12} = 72 - \frac{7}{4}$$

$$\Rightarrow \quad \text{Mean} = \frac{288-7}{4} = \frac{281}{4} = 70.25 \text{ kg}$$

Hence, mean weight = 70.25 kg.

EXAMPLE 2 *Find the mean wage from the data given below:*

Wage (in ₹):	800	820	860	900	920	980	1000
No. of wokers:	7	14	19	25	20	10	5

SOLUTION Let the assumed mean be $A = 900$.

Calculation of Mean

Wage (in ₹) x_i	No. of workers f_i	$d_i = x_i - A = x_i - 900$	$f_i d_i$
800	7	– 100	– 700
820	14	– 80	– 1120
860	19	– 40	– 760
900	25	0	0
920	20	20	400
980	10	80	800
1000	5	100	500
	$N = \Sigma f_i = 100$		$\Sigma f_i d_i = -880$

We have,

$$N = 100, \Sigma f_i d_i = -880 \text{ and } A = 900$$

$$\therefore \quad \text{Mean } \overline{X} = A + \frac{\Sigma f_i d_i}{N}$$

$$\Rightarrow \quad \text{Mean } \overline{X} = 900 + \frac{-880}{100} = 900 - 8.8 = 891.2$$

Hence, mean wage = ₹ 891.2.

15.2.3 STEP-DEVIATION METHOD

Sometimes, during the application of the short-cut method for finding AM, the deviations d_i are divisible by a common number h (say). In such a case the arithmetic is reduced to a great extent by taking

$$u_i = \frac{x_i - A}{h}; \; i = 1, 2, 3, \cdots, n$$

$$\Rightarrow \quad x_i = A + hu_i \,, \; i = 1, 2, 3, \cdots, n$$

$$\Rightarrow \quad f_i x_i = Af_i + hf_i u_i \,, \; i = 1, 2, 3, \cdots, n$$

$$\Rightarrow \quad \sum_{i=1}^{n} f_i x_i = A \sum_{i=1}^{n} f_i + h \sum_{i=1}^{n} f_i u_i$$

$$\Rightarrow \quad \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i\right) = \frac{AN}{N} + h\left\{\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right\}$$

$$\Rightarrow \quad \overline{X} = A + h\left\{\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right\}$$

Finding AM by using this formula is known as the step-deviation method.

Following algorithm may be used to find the arithmetic mean by step deviation method:

ALGORITHM

STEP I *Obtain the frequency distribution and prepare the frequency table in such a way that its first column consists of the values of the variable and the second column corresponding frequencies.*

STEP II *Choose a number 'A' (generally known as the assumed mean) and take deviations $d_i = x_i - A$ about A. Write these deviations against the corresponding frequencies in the third column.*

STEP III *Choose a number h, generally common factor of all d_i's in III column, divide deviations d_i by h to get u_i. Write these u_i's against the corresponding d_i's in the IV column.*

STEP IV *Multiply the frequencies in II column with the corresponding u_i's in IV column to prepare V column of $f_i u_i$.*

STEP V *Find the sum of all entries in V column to obtain $\sum_{i=1}^{n} f_i u_i$ and the sum of all frequencies in II column to obtain $N = \left(\sum_{i=1}^{n} f_i\right)$*

STEP VI *Use the formula:* $\overline{X} = A + h\left\{\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right\}$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the mean wage from the following data:*

Wage (in ₹):	800	820	860	900	920	980	1000
No. of workers:	7	14	19	25	20	10	5

SOLUTION Let the assumed mean be $A = 900$ and $h = 20$.

Calculation of Mean

Wage (in ₹) x_i	*No. of workers* f_i	$d_i = x_i - A = x_i - 900$	$u_i = \frac{x_i - 900}{20}$	$f_i u_i$
800	7	– 100	– 5	– 35
820	14	– 80	– 4	– 56
860	19	– 40	– 2	– 38
900	25	0	0	0
920	20	20	1	20
980	10	80	4	40
1000	5	100	5	25
	$N = \Sigma f_i = 100$			$\Sigma f_i u_i = -44$

We have,

$$N = 100, \Sigma f_i u_i = -44, A = 900 \text{ and } h = 20.$$

$\therefore$ $$\text{Mean} = \overline{X} = A + h\left(\frac{1}{N}\Sigma f_i u_i\right)$$

$\Rightarrow$ $$\overline{X} = 900 + 20 \times \frac{-44}{100} = 900 - 8.8 = 891.2$$

Hence, mean wage = ₹ 891.2.

EXAMPLE 2 *Apply step-deviation method to find the AM of the following frequency distribution*

Variate (x):	5	10	15	20	25	30	35	40	45	50
Frequency (f):	20	43	75	67	72	45	39	9	8	6

SOLUTION Let the assumed mean be $A = 25$ and $h = 5$.

Calculation of Mean

Variate x_i	*Frequency* f_i	*Deviations* $d_i = x_i - 25$	$u_i = \frac{x_i - 25}{5}$	$f_i u_i$
5	20	− 20	− 4	− 80
10	43	− 15	− 3	− 129
15	75	− 10	− 2	− 150
20	67	− 5	− 1	− 67
25	72	0	0	0
30	45	5	1	45
35	39	10	2	78
40	9	15	3	27
45	8	20	4	32
50	6	25	5	30
$N = \Sigma f_i = 384$				$\Sigma f_i u_i = -214$

We have,

$$N = 384, A = 25, h = 5 \text{ and } \Sigma f_i u_i = -214$$

$\therefore$ $$\text{Mean} = \overline{X} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$\Rightarrow$ $$\text{Mean} = 25 + 5 \times \left(\frac{-214}{384}\right) = 25 - 2.786 = 22.214$$

EXAMPLE 3 *The weights in kilograms of 60 workers in a factory are given in the following frequency table. Find the mean weight of a worker.*

Weight (in kg) x:	60	61	62	63	64	65
No. of workers f:	5	8	14	16	10	7

SOLUTION Let the assumed mean be $A = 63$

Calculation of Mean

Weight (in kg) x	No. of workers f	$d_i = x_i - 63$	$f_i d_i$
60	5	−3	−15
61	8	−2	−16
62	14	−1	−14
63	16	0	0
64	10	1	10
65	7	2	14
	$N = \Sigma f_i = 60$		$\Sigma f_i d_i = -21$

We have, $N = 60, A = 63$ and $\Sigma f_i d_i = -21$

$$\therefore \quad \text{Mean} = A + \left\{ \frac{1}{N} \Sigma f_i d_i \right\}$$

$$\Rightarrow \quad \text{Mean} = 63 + \left(\frac{-21}{60} \right) = 63 - \frac{7}{20} = 63 - 0.35 = 62.65$$

Hence, mean weight of a worker = 62.65 kg

EXAMPLE 4 *The table below gives the distribution of villages under different heights from sea level in a certain region. Compute the mean height of the region:*

Height (in metres):	*200*	*600*	*1000*	*1400*	*1800*	*2200*
No. of village:	*142*	*265*	*560*	*271*	*89*	*16*

SOLUTION Let the assumed mean be $A = 1400$ and $h = 400$

Calculation of Mean

Height (in metres) x_i	No. of villages f_i	$d_i = x_i - 1400$	$u_i = \dfrac{x_i - 1400}{400}$	$f_i u_i$
200	142	−1200	−3	−426
600	265	−800	−2	−530
1000	560	−400	−1	−560
1400	271	0	0	0
1800	89	400	1	89
2200	16	800	2	32
	$N = \Sigma f_i = 1343$			$\Sigma f_i u_i = -1395$

We have, $A = 1400, h = 400, \Sigma f_i u_i = -1395$ and $N = 1343$

$$\therefore \quad \text{Mean} = A + h \left\{ \frac{1}{N} \Sigma f_i u_i \right\}$$

$$\Rightarrow \quad \text{Mean} = 1400 + 400 \times \frac{-1395}{1343} = 1400 - 415.49 = 984.51$$

LEVEL-1

[illegible]

Compute the mean number of calls per interval.

[illegible]

[CBSE 2006 C]

$f_i u_i$
–30
–18
0
29
34
$\Sigma f_i u_i = 15$

[illegible] ure (in rupees) of

450-500

7

9. Find the mean from the following frequency distribution of marks at a test in statistics :

Marks (x):	5	10	15	20	25	30	35	40	45	50
No. of students (f):	15	50	80	76	72	45	39	9	8	6

ANSWERS

1. 3.54 2. 2.47 3. 3.62 (approx) 4. 2.35
5. 26.08 6. 3.53 (approx) 7. 0.73 8. 0.83
9. 22.075

15.2.4 ARITHMETIC MEAN OF A CONTINUOUS FREQUENCY DISTRIBUTION

Uptill now we have been discussing about various methods for computing arithmetic mean of a discrete frequency distribution. In case of a continuous frequency distribution or a frequency distribution with class intervals arithmetic mean may be computed by applying any of the methods discussed so far. The values of $x_1, x_2, x_3 \cdots, x_n$ are taken as the mid-points or class-marks of the various classes. It should be noted that the mid-value or class-marks of a class interval is equal to $\frac{1}{2}$ (lower limit + upper limit).

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the mean of the following frequency distribution:*

Class-interval:	*0 – 10*	*10 – 20*	*20 – 30*	*30 – 40*	*40 – 50*
No. of workers f:	*7*	*10*	*15*	*8*	*10*

SOLUTION *Calculation of Mean*

Class-interval	*Mid-values* (x_i)	*Frequency* f_i	$d_i = x_i - 25$	$u_i = \frac{x_i - 25}{10}$	$f_i u_i$
0 - 10	5	7	– 20	– 2	– 14
10 - 20	15	10	– 10	– 1	– 10
20 - 30	25	15	0	0	0
30 - 40	35	8	10	1	8
40 - 50	45	10	20	2	20
		$N = \Sigma f_i = 50$			$\Sigma f_i u_i = 4$

We have,

$$A = 25, h = 10, N = 50 \text{ and } \Sigma f_i u_i = 4.$$

$$\therefore \quad \text{Mean} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad \text{Mean} = 25 + 10 \times \frac{4}{50} = 25.8$$

EXAMPLE 2 *Find the mean of the following frequency distribution:*

Classes:	*0-20*	*20-40*	*40-60*	*60-80*	*80-100*
Frequency:	*15*	*18*	*21*	*29*	*17*

[CBSE 2006 C]

SOLUTION *Calculation of Mean*

Classes	*Mid-values* (x_i)	*Frequency* (f_i)	$d_i = x_i - 50$	$u_i = \frac{x_i - 50}{20}$	$f_i u_i$
0 - 20	10	15	−40	−2	−30
20 - 40	30	18	−20	−1	−18
40 - 60	50	21	0	0	0
60 - 80	70	29	20	1	29
80 - 100	90	17	40	2	34
		$\Sigma f_i = 100$			$\Sigma f_i u_i = 15$

We have,

$$A = 50, h = 20, N = 100 \text{ and } \Sigma f_i u_i = 15$$

$$\therefore \quad \text{Mean} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad \text{Mean} = 50 + 20 \times \frac{15}{100}$$

$$\Rightarrow \quad \text{Mean} = 50 + 3 = 53$$

EXAMPLE 3 *The following table gives the distribution of total household expenditure (in rupees) of manual workers in a city.*

Expenditure: (in ₹)	100-150	150-200	200-250	250-300	300-350	350-400	400-450	450-500
Frequency:	24	40	33	28	30	22	16	7

Find the average expenditure (in ₹) per household.

SOLUTION Let the assumed mean be $A = 325$ and $h = 50$.

Calculation of Mean

Expenditure (in ₹) x_i	Frequency f_i	Mid-values x_i	$d_i = x_i - A$ $= x_i - 325$	$u_i = \frac{x - A}{h}$ $= \frac{x_i - 325}{50}$	$f_i u_i$
100 - 150	24	125	– 200	– 4	– 96
150 - 200	40	175	– 150	– 3	– 120
200 - 250	33	225	– 100	– 2	– 66
250 - 300	28	275	– 50	– 1	– 28
300 - 350	30	325	0	0	0
350 - 400	22	375	50	1	22
400 - 450	16	425	100	2	32
450 - 500	7	475	150	3	21
	$N = \Sigma f_i = 200$				$\Sigma f_i u_i = -235$

We have,

$$N = 200, A = 325, h = 50, \text{ and } \Sigma f_i u_i = -235$$

$$\therefore \quad \overline{X} = A + h \frac{1}{N} \Sigma f_i u_i$$

$$\Rightarrow \quad \overline{X} = 325 + 50 \times \left\{ \frac{-235}{200} \right\}$$

$$\Rightarrow \quad \overline{X} = 325 - \frac{235}{4} = 325 - 58.75 = 266.25$$

Hence, the average expenditure is ₹ 266.25.

EXAMPLE 4 *A frequency distribution of the life times of 400 T.V. picture tubes tested in a tube company is given below. Find the average life of tube.*

Life time (in hrs)	*Frequency*	*Life time (in hrs)*	*Frequency*
300-399	14	800-899	62
400-499	46	900-999	48
500-599	58	1000-1099	22
600-699	76	1100-1199	6
700-799	68		

SOLUTION Here, the class-intervals are formed by exclusive method. If we make the series an inclusive one the mid-values remain same. So, there is no need to convert the series into an inclusive form.

Let the assumed mean be $A = 749.5$ and $h = 100$.

Calculation of Mean

Life time (in hrs):	Frequency f_i	Mid-Values x_i	$d_i = x_i - A$ $= x_i - 749.5$	$u_i = \frac{x_i - A}{h}$ $u_i = \frac{x_i - 749.5}{100}$	$f_i u_i$
300 - 399	14	349.5	− 400	−4	− 56
400 - 499	46	449.5	− 300	−3	− 138
500 - 599	58	549.5	− 200	−2	− 116
600 - 699	76	649.5	− 100	−1	− 76
700 - 799	68	749.5	0	0	0
800 - 899	62	849.5	100	1	62
900 - 999	48	949.5	200	2	96
1000 - 1099	22	1049.5	300	3	66
1100 - 1199	6	1149.5	400	4	24
	$N = \Sigma f_i = 400$				$\Sigma f_i u_i = -138$

We have, $N = 400, A = 749.5, h = 100$ and $\Sigma f_i u_i = -138$

$$\therefore \quad \overline{X} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad \overline{X} = 749.5 + 100 \times \left(\frac{-138}{400}\right) = 749.5 - \frac{138}{4} = 749.5 - 34.5 = 715$$

Hence, the average life time of a tube is 715 hours.

EXAMPLE 5 *If the mean of the following distributions 54, find the value of p:*

Class:	*0 - 20*	*20 - 40*	*40 - 60*	*60 - 80*	*80 - 100*
Frequency:	*7*	*p*	*10*	*9*	*13*

[CBSE 2006C]

SOLUTION *Computation of Arithmetic Mean*

Class	Mid-values (x_i)	Frequency (f_i)	$d_i = x_i - 50$	$u_i = \frac{x_i - 50}{20}$	$f_i u_i$
0 - 20	10	7	− 40	−2	− 14
20 -40	30	p	− 20	−1	$-p$
40 -60	50	10	0	0	0
60 - 80	70	9	20	1	9
80 - 100	90	13	40	2	26
	$\Sigma f_i = 39 + p$				$\Sigma f_i u_i = 21 - p$

We have,

$$A = 50, N = 39 + p, h = 20, \Sigma f_i u_i = 21 - p \text{ and } \overline{X} = 54$$

$$\therefore \quad \text{Mean} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad 54 = 50 + 20 \times \left\{\frac{21-p}{39+p}\right\}$$

$$\Rightarrow \quad 4 = 20 \times \left\{\frac{21-p}{39+p}\right\}$$

$$\Rightarrow \quad 1 = 5\left(\frac{21-p}{39+p}\right) \Rightarrow 39 + p = 105 - 5p \Rightarrow 6p = 66 \Rightarrow p = 11$$

LEVEL-2

EXAMPLE 6 *The following table gives weekly wages in rupees of workers in a certain commercial organization. The frequency of class 49-52 is missing. It is known that the mean of the frequency distribution is 47.2. Find the missing frequency.*

Weekly wages (₹):	*40 - 43*	*43 - 46*	*46 - 49*	*49 - 52*	*52 - 55*
Number of workers:	*31*	*58*	*60*	*?*	*27*

SOLUTION Let the missing frequency be f, the assumed mean be $A = 47$ and $h = 3$.

Calculation of Mean

Class-Intervals	*mid-values* x_i	f_i	$d_i = x_i - 47.5$	$u_i = \frac{x_i - 47.5}{3}$	$f_i u_i$
40 - 43	41.5	31	− 6	− 2	− 62
43 - 46	44.5	58	− 3	− 1	− 58
46 - 49	47.5	60	0	0	0
49 - 52	50.5	f	3	1	f
52 - 55	53.5	27	6	2	54
	$N = \Sigma f_i = 176 + f$				$\Sigma f_i u_i = f - 66$

We have,

$$\overline{X} = 47.2,\ A = 47.5 \text{ and } h = 3$$

$$\therefore \quad \overline{X} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad 47.2 = 47.5 + 3 \times \left\{\frac{f-66}{176+f}\right\}$$

$$\Rightarrow \quad -0.3 = 3 \times \left\{\frac{f-66}{176+f}\right\}$$

$$\Rightarrow \quad \frac{-1}{10} = \frac{f-66}{176+f} \Rightarrow -176 - f = 10f - 660 \Rightarrow 11f = 484 \Rightarrow f = 44$$

Hence, the missing frequency is 44.

EXAMPLE 7 *The mean of the following frequency table 50. But the frequencies f_1 and f_2 in class 20 - 40 and 60 - 80 are missing. Find the missing frequencies.*

Class:	*0 - 20*	*20 - 40*	*40 - 60*	*60 - 80*	*80 - 100*	*Total*
Frequency:	*17*	f_1	*32*	f_2	*19*	*120*

[CBSE 2006C]

SOLUTION Let the assumed mean be $A = 50$ and $h = 20$.

Calculation of Mean

Class	Frequency f_i	Mid-values x_i	$u_i = \dfrac{x_i - A}{h}$	$f_i u_i$
0 - 20	17	10	− 2	− 34
20 - 40	f_1	30	− 1	$-f_1$
40 - 60	32	50	0	0
60 - 80	f_2	70	1	f_2
80 - 100	19	90	2	38
	$N = \Sigma f_i = 68 + f_1 + f_2$			$\Sigma f_i u_i = 4 - f_1 + f_2$

We have,

$$N = \Sigma f_i = 120 \qquad \text{[Given]}$$

$$\Rightarrow \quad 68 + f_1 + f_2 = 120$$

$$\Rightarrow \quad f_1 + f_2 = 52 \qquad \text{...(i)}$$

Now,

$$\text{Mean} = 50$$

$$\Rightarrow \quad A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\} = 50$$

$$\Rightarrow \quad 50 + 20 \times \left\{\frac{4 - f_1 + f_2}{120}\right\} = 50$$

$$\Rightarrow \quad 50 + \frac{4 - f_1 + f_2}{6} = 50$$

$$\Rightarrow \quad \frac{4 - f_1 + f_2}{6} = 0$$

$$\Rightarrow \quad 4 - f_1 + f_2 = 0$$

$$\Rightarrow \quad f_1 - f_2 = 4 \qquad \text{...(ii)}$$

Solving equations (i) and (ii), we get $f_1 = 28$ and $f_2 = 24$.

EXAMPLE 8 *Find the mean marks of students from the following cumulative frequency distribution:*

Marks	Number of students	Marks	Number of students
0 and above	80	60 and above	28
10 and above	77	70 and above	16
20 and above	72	80 and above	10
30 and above	65	90 and above	8
40 and above	55	100 and above.	0
50 and above	43		

SOLUTION Here we have, the cumulative frequency distribution. So, first we convert it into an ordinary frequency distribution. We observe that there are 80 students getting marks greater than or equal to 0 and 77 students have secured 10 and more marks. Therefore, the number of students getting marks between 0 and 10 is $80-77=3$.
Similarly, the number of students getting marks between 10 and 20 is $77-72=5$ and so on. Thus, we obtain the following frequency distribution.

Marks	Number of students	Marks	Number of students
0 – 10	3	50 - 60	15
10 - 20	5	60 - 70	12
20 - 30	7	70 - 80	6
30 - 40	10	80 - 90	2
40 - 50	12	90 - 100	8

Now, we compute arithmetic mean by taking 55 as the assumed mean.

Computation of Mean

Marks (x_i)	Mid-value (f_i)	Frequency	$u_i = \frac{x_i - 55}{10}$	$f_i u_i$
0 - 10	5	3	−5	−15
10 - 20	15	5	−4	−20
20 - 30	25	7	−3	−21
30 - 40	35	10	−2	−20
40 - 50	45	12	−1	−12
50 - 60	55	15	0	0
60 - 70	65	12	1	12
70 - 80	75	6	2	12
80 - 90	85	2	3	6
90 - 100	95	8	4	32
Total		$\Sigma f_i = 80$		$\Sigma f_i u_i = -26$

We have,

$$N = \Sigma f_i = 80, \Sigma f_i u_i = -26, A = 55 \text{ and } h = 10$$

$$\therefore \quad \overline{X} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad \overline{X} = 55 + 10 \times \frac{-26}{80} = 55 - 3.25 = 51.75 \text{ Marks.}$$

EXAMPLE 9 *Find the mean marks of the students from the following cumulative frequency distribution:*

Marks:	*Below 10*	*Below 20*	*Below 30*	*Below 40*	*Below 50*	*Below 60*	*Below 70*	*Below 80*	*Below 90*	*Below 100*
Number of students:	5	9	17	29	45	60	70	78	83	85

SOLUTION Here we have, cumulative frequency distribution less than type. First we convert it into an ordinary frequency distribution. We observe that the number of students getting marks less than 10 is 5 and 9 students have secured marks less than 20. Therefore, number of students getting marks between 10 and 20 (inclusive 0 and exclusive 10) is $9 - 5 = 4$. Similarly, the number of students getting marks between 20 and 30 is $17 - 9 = 8$ and so on. Thus, we have the following frequency distribution:

Marks:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80	80 - 90	90 - 100
Number of students:	5	4	8	12	16	15	10	8	5	2

Let us now compute arithmetic mean by taking 55 and the assumed mean.

Computation of Mean

Marks	*Mid-value*	*Frequency*	$u_i = \dfrac{x_i - 55}{10}$	$f_i u_i$
0 - 10	5	5	− 5	− 25
10 - 20	10	4	− 4	− 16
20 - 30	25	8	− 3	− 24
30 - 40	35	12	− 2	− 24
40 - 50	45	16	− 1	− 16
50 - 60	55	15	0	0
60 - 70	65	10	1	10
70 - 80	75	8	2	16
80 - 90	85	5	3	15
90 - 100	95	2	4	8
Total		$N = \Sigma f_i = 85$		$\Sigma f_i u_i = -56$

We have,

$$N = \Sigma f_i = 85, \Sigma f_i u_i = -56, h = 10 \text{ and } A = 55$$

$$\therefore \quad \overline{X} = A + h\left\{\frac{1}{N}\Sigma f_i u_i\right\}$$

$$\Rightarrow \quad \overline{X} = 55 + 10 \times \frac{-56}{85} = 55 - 6.59 = 48.41 \text{ Marks}$$

Hence, mean marks scored by the students = 48.41.

EXERCISE 15.3

LEVEL-1

1. The following table gives the distribution of total household expenditure (in rupees) of manual workers in a city.

Expenditure (in rupees) (x_i)	*Frequency (f_i)*	*Expenditure (in rupees) (x_i)*	*Frequency (f_i)*
100-150	24	300-350	30
150-200	40	350-400	22
200-250	33	400-450	16
250-300	28	450-500	7

Find the average expenditure (in rupees) per household.

2. A survey was conducted by a group of students as a part of their environment awareness programme, in which they collected the following data regarding the number of plants in 20 houses in a locality. Find the mean number of plants per house.

Number os plants:	0 - 2	2 - 4	4 - 6	6 - 8	8 - 10	10 - 12	12 - 14
Number of houses:	1	2	1	5	6	2	3

[NCERT]

Which method did you use for finding the mean, and why?

3. Consider the following distribution of daily wages of 50 workers of a factory.

Daily wages (in ₹).	100 - 120	120 - 140	140 - 160	160 - 180	180 - 200
Number of workers:	12	14	8	6	10

Find the mean daily wages of the workers of the factory by using an appropriate method.

4. Thirty women were examined in a hospital by a doctor and the number of heart beats per minute recorded and summarised as follows. Find the mean heart beats per minute for these women, choosing a suitable method.

Number of heat beats per minute:	65 - 68	68 - 71	71 - 74	74 - 77	77 - 80	80 - 83	83 - 86
Number of women:	2	4	3	8	7	4	2

[NCERT]

Find the mean of each of the following frequency distributions: (5 – 14)

5.

Class interval:	0 - 6	6 - 12	12 - 18	18 - 24	24 - 30
Frequency:	6	8	10	9	7

6.

Class interval:	50 - 70	70 - 90	90 - 110	110 - 130	130 - 150	150 - 170
Frequency:	18	12	13	27	8	22

7.

Class interval:	0 - 8	8 - 16	16 - 24	24 - 32	32 - 40
Frequency:	6	7	10	8	9

8.

Class interval:	0 - 6	6 - 12	12 - 18	18 - 24	24 - 30
Frequency:	7	5	10	12	6

9.

Class interval:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency:	9	12	15	10	14

10.

Class interval:	0 - 8	8 - 16	16 - 24	24 - 32	32 - 40
Frequency:	5	9	10	8	8

11.

Class interval:	0 - 8	8 - 16	16 - 24	24 - 32	32 - 40
Frequency:	5	6	4	3	2

12.

Class interval:	10 - 30	30 - 50	50 - 70	70 - 90	90 - 110	110 - 130
Frequency:	5	8	12	20	3	2

13.

Class interval:	25 - 35	35 - 45	45 - 55	55 - 65	65 - 75
Frequency:	6	10	8	12	4

14.

Classes:	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54	55 - 59
Frequency:	14	22	16	6	5	3	4

[CBSE 2006C]

15. For the following distribution, calculate mean using all suitable methods:

Size of item:	1 - 4	4 - 9	9 - 16	16 - 27
Frequency:	6	12	26	20

16. The weekly observations on cost of living index in a certain city for the year 2004 - 2005 are given below. Compute the weekly cost of living index.

Cost of living Index	*Number of Students*	*Cost of living Index*	*Number of Students*
1400 - 1500	5	1700 - 1800	9
1500 - 1600	10	1800 - 1900	6
1600 - 1700	20	1900 - 2000	2

17. The following table shows the marks scored by 140 students in an examination of a certain paper:

Marks:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Number of students:	20	24	40	36	20

Calculate the average marks by using all the three methods: direct method, assumed mean deviation and shortcut method.

18. The mean of the following frequency distribution is 62.8 and the sum of all the frequencies is 50. Compute the missing frequency f_1 and f_2.

Class:	0 - 20	20 - 40	40 - 60	60 - 80	80 - 100	100 - 120
Frequency:	5	f_1	10	f_2	7	8

[CBSE 2004]

19. The following distribution shows the daily pocket allowance given to the children of a multistorey building. The average pocket allowance is ₹ 18.00. Find out the missing frequency.

Class interval:	11-13	13-15	15-17	17-19	19-21	21-23	23-25
Frequency:	7	6	9	13	–	5	4

[NCERT]

20. If the mean of the following distribution is 27, find the value of p.

Class:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency:	8	p	12	13	10

[CBSE 2006C]

21. In a retail market, fruit vendors were selling mangoes kept in packing boxes. These boxes contained varying number of mangoes. The following was the distribution of mangoes according to the number of boxes.

Number of mangoes:	50 - 52	53 - 55	56 - 58	59 - 61	62 -64
Number of boxes:	15	110	135	115	25

Find the mean number of mangoes kept in a packing box. Which method of finding the mean did you choose? [NCERT]

22. The table below shows the daily expenditure on food of 25 households in a locality

Daily expenditure (in ₹):	100 - 150	150 - 200	200 - 250	250 - 300	300 -350
Number of households:	4	5	12	2	2

Find the mean daily expenditure on food by a suitable method. [NCERT]

23. To find out the concentration of SO_2 in the air (in parts per million, i.e., ppm), the data was collected for 30 localities in a certain city and is presented below:

Concentration of SO_2 (in ppm)	*Frequency*
0.00-0.04	4
0.04-0.08	9
0.08-0.12	9
0.12-0.16	2
0.16-0.20	4
0.20-0.24	2

Find the mean concentration of SO_2 in the air. [NCERT]

24. A class teacher has the following absentee record of 40 students of a class for the whole term. Find the mean number of days a student was absent.

Number of days:	0 - 6	6 - 10	10 - 14	14 - 20	20 -28	28 - 38	38 - 40
Number of students:	11	10	7	4	4	3	1

[NCERT]

25. The following table gives the literacy rate (in percentage) of 35 cities. Find the mean literacy rate.

Literacy rate (in %):	45 - 55	55 - 65	65 - 75	75 - 85	85 -95
Number of cities:	3	10	11	8	3

[NCERT]

26. The following is the cummulative frequency distribution (of less than type) of 1000 persons each of age 20 years and above. Determine the mean age.

Age below (in years):	30	40	50	60	70	80
Number of persons:	100	220	350	750	950	1000

[NCERT EXEMPLAR]

27. If the mean of the following frequency distribution is 18, find the missing frequency.

Class interval:	11-13	13-15	15-17	17-19	19-21	21-23	23-25
Frequency:	3	6	9	13	f	5	4

[NCERT EXEMPLAR, CBSE 2018]

28. Find the missing frequencies in the following distribution, if the sum of the frequencies is 120 and the mean is 50.

Class:	0-20	20-40	40-60	60-80	80-100
Frequency:	17	f_1	32	f_2	19

[NCERT EXEMPLAR]

29. The daily income of a sample of 50 employees are tabulated as follows:

Income (in ₹):	1-200	201-400	401-600	601-800
No. of employees:	14	15	14	7

Find the mean daily income of employees. **[NCERT EXEMPLAR]**

ANSWERS

1. 266.25	**2.** 8.1 plants	**3.** 145.20	**4.** 75.9
5. 15.45	**6.** 112.20	**7.** 21.4	**8.** 15.75
9. 26.333	**10.** 21	**11.** 16.4	**12.** 65.6
13. 49.5	**14.** 36.357	**16.** 1663.3	**17.** 25.857
18. $f_1 = 8, f_2 = 12.$	**19.** 20	**20.** $p = 7$	**21.** 57.19
22. ₹ 211	**23.** 0.099 ppm	**24.** 12.475 days	**25.** 69.43%
26. 51.3 years	**27.** 8	**28.** $f_1 = 28, f_2 = 24$	**29.** ₹ 356.5

15.3 MEDIAN

The median is the middle value of a distribution i.e., median of a distribution is the value of the variable which divides it into two equal parts. It is the value of the variable such that the number of observations above it is equal to the number of observations below it.

In class IX, we have studied about the method for finding median of individual observations. If $x_1, x_2, x_3, \cdots, x_n$ are n values of a variable X, then to find the median we use the following algorithm.

ALGORITHM

STEP I *Arrange the observations $x_1, x_2 \cdots, x_n$ in ascending or descending order of magnitude.*

STEP II *Determine the total number of observations, say, n*

STEP III *If n is odd, then median is the value of $\left(\frac{n+1}{2}\right)^{th}$ observation.*

If n is even, then median is the AM of the values of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations.

ILLUSTRATION 1 *(i) The following are the marks of 9 students in a class. Find the median*

34, 32, 48, 38, 24, 30, 27, 21, 35

created by a car driven for 18,000 kilometres.

(ii) Find the median of the daily wages of ten workers from the following data:

₹20, 25, 17, 18, 8, 15, 22, 11, 9, 14.

SOLUTION (i) Arranging the data in ascending order of magnitude, we have

21, 24, 27, 30, 32, 34, 35, 38, 48

Since there are 9 i.e., an odd number of items. Therefore, median is the value of $\left(\frac{9+1}{2}\right)^{th}$ observation i.e., 32.

(ii) Arranging the wages in ascending order of magnitude, we have

8, 9, 11, 14, 15, 17, 18, 20, 22, 25

Since there are 10 observations Therefore, median is the arithmetic mean of

$$\left(\frac{10}{2}\right)^{th} \text{ and} \left(\frac{10}{2}+1\right)^{th} \text{ observations.}$$

Hence, Median $= \dfrac{15+17}{2} = 16$

15.3.1 MEDIAN OF DISCRETE FREQUENCY DISTRIBUTION

In case of a discrete frequency distribution $x_i / f_i; i = 1, 2, \cdots, n$ we calculate the median by using the following algorithm.

ALGORITHM

STEP I *Find the cumulative frequencies (c.f.)*

STEP II *Find* $\dfrac{N}{2}$, *where* $N = \sum_{i=1}^{n} f_i$

STEP III *See the cumulative frequency (c.f.) just greater than* $\dfrac{N}{2}$ *and determine the corresponding value of the variable.*

STEP IV *The value obtained in step III is the median.*

ILLUSTRATION 2 *Obtain the median for the following frequency distribution:*

x:	1	2	3	4	5	6	7	8	9
f:	8	10	11	16	20	25	15	9	6

SOLUTION

Calculation of Median

x	f	cf
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120
	$N = 120$	

Here, $N = 120 \Rightarrow \frac{N}{2} = 60$

We find that the cumulative frequency just greater than $\frac{N}{2}$ i.e., 60 is 65 and the value of x corresponding to 65 is 5. Therefore, Median = 5.

15.3.2 MEDIAN OF A GROUPED OR CONTINUOUS FREQUENCY DISTRIBUTION

In order to calculate the median of a grouped or continuous frequency distribution, we use the following algorithm.

ALGORITHM

STEP I *Obtain the frequency distribution.*

STEP II *Prepare the cumulative frequency column and obtain* $N = \Sigma f_i$.

STEP III *Find N/2.*

STEP IV *See the cumulative frequency just greater than N/2 and determine the corresponding class. This class is known as the median class.*

STEP V *Use the following formula:*

$$\text{Median} = l + \left\{ \frac{\frac{N}{2} - F}{f} \right\} \times h$$

where, l = lower limit of the median class

f = frequency of the median class

h = with (size) of the median class

F = Cumulative frequency of the class preceding the median class $N = \Sigma f_i$

ILLUSTRATION 3 *Calculate the median from the following distribution:*

Class:	5 - 10	10 - 15	15 - 20	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45
Frequency:	5	6	15	10	5	4	2	2

SOLUTION First we prepare the following cumulative table to compute the median

Class	*Frequency*	*Cumulative frequency*
5 - 10	5	5
10 - 15	6	11
15 - 20	15	26
20 - 25	10	36
25 - 30	5	41
30 - 35	4	45
35 - 40	2	47
40 - 45	2	49
	$N = 49$	

We have, $N = 49$

$$\therefore \quad \frac{N}{2} = \frac{49}{2} = 24.5$$

The cumulative frequency just greater than $N/2$ is 26 and the corresponding class is 15 - 20. Thus, 15 - 20 is the median class such that

$l = 15, f = 15, F = 11$ and $h = 5$

$$\therefore \quad \text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h = 15 + \frac{24.5 - 11}{15} \times 5 = 15 + \frac{13.5}{3} = 19.5$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The number of students absent in a school was recorded every day for 147 days and the raw data was presented in the form of the following frequency table.*

No. of students absent:	5	6	7	8	9	10	11	12	13	15	18	20
No. of days:	1	5	11	14	16	13	10	70	4	1	1	1

Obtain the median and describe what information it conveys.

SOLUTION

Calculation of median

x_i	f_i	cf
5	1	1
6	5	6
7	11	17
8	14	31
9	16	47
10	13	60
11	10	70
12	70	140
13	4	144
15	1	145
18	1	146
20	1	147
	$N = \Sigma f_i = 147$	

We have,

$$N = 147 \Rightarrow \frac{N}{2} = \frac{147}{2} = 73.5$$

The cumulative frequency just greater than $N/2$ is 140 and the corresponding value of variable x is 12.

Hence, median = 12. This means that for about half the number of days, more than 12 students were absent.

EXAMPLE 2 *Calculate the median from the following data:*

Marks:	*0-10*	*10-30*	*30-60*	*60-80*	*80-90*
No. of students:	5	15	30	8	2

SOLUTION Here, the class intervals are of unequal width. If the class intervals are of unequal width the frequencies need not be adjusted to make the class intervals equal.

Calculation of Median

Marks	*No. of students (Frequency)*	*Cumulative frequency*
0 - 10	5	5
10 - 30	15	20
30 - 60	30	50
60 - 80	8	58
80 - 90	2	60
		$N = \Sigma f_i = 60$

Here, $N = 60 \therefore N/2 = 30$

The cumulative frequency just greater than $N/2 = 30$ is 50 and the corresponding class is 30 - 60. Hence, 30 - 60 is the median class.

$\therefore \quad l = 30, f = 30, F = 20, h = 30$

Now, $$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h$$

$$\Rightarrow \quad \text{Median} = 30 + \frac{30 - 20}{30} \times 30 = 40$$

EXAMPLE 3 *If the median of the following frequency distribution is 46, find the missing frequencies.*

Variable:	*10 - 20*	*20 - 30*	*30 - 40*	*40 - 50*	*50 - 60*	*60 - 70*	*70 - 80*	*Total*
Frequency:	*12*	*30*	*?*	*65*	*?*	*25*	*18*	*229*

SOLUTION Let the frequency of the class 30 - 40 be f_1 and that of the class 50 - 60 be f_2. The total frequency is 229.

$$12 + 30 + f_1 + 65 + f_2 + 25 + 18 = 229 \Rightarrow f_1 + f_2 = 79$$

It is given that the median is 46.

Clearly, 46 lies in the class 40 - 50. So, 40 - 50 is the median class.

$\therefore \quad l = 40, h = 10, f = 65$ and $F = 12 + 30 + f_1 = 42 + f_1, N = 229$

Now,

$$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h$$

$$\Rightarrow \quad 46 = 40 + \frac{\frac{229}{2} - (42 + f_1)}{65} \times 10$$

$$\Rightarrow \quad 46 = 40 + \frac{145 - 2f_1}{13}$$

$$\Rightarrow \quad 6 = \frac{145 - 2f_1}{13} \Rightarrow 2f_1 = 67 \Rightarrow f_1 = 33.5 \text{ or } 34 \text{ (say)}$$

Since $f_1 + f_2 = 79$. Therefore, $f_2 = 45$.

Hence, $f_1 = 34$ and $f_2 = 45$.

LEVEL-2

EXAMPLE 4 *Find the median of the following frequency distribution:*

Weekly wages (in ₹):	*60-69*	*70-79*	*80-89*	*90-99*	*100-109*	*110-119*
No. of days:	*5*	*15*	*20*	*30*	*20*	*8*

SOLUTION Here, the frequency table is given in inclusive form. So, we first transform it into exclusive form by subtracting and adding $h/2$ to the lower and upper limits respectively of each class, where h denotes the difference of lower limit of a class and the upper limit of the previous class.

Transforming the above table into exclusive form and preparing the cumulative frequency table, we get

Weekly wages (in ₹)	*No. of workers*	*Cumulative frequency*
59.5 – 69.5	5	5
69.5 – 79.5	15	20
79.5 – 89.5	20	40
89.5 – 99.5	30	70
99.5 – 109.5	20	90
109.5 – 119.5	8	98
		$N = \Sigma f_i = 98$

We have, $N = 98$. $\therefore$ $N/2 = 49$

The cumulative frequency just greater than $N/2$ is 70 and the corresponding class is 89.5 – 99.5. So, 89.5 – 99.5 is the median class.

$\therefore$ $l = 89.5, h = 10, f = 30$ and $F = 40$

Now, $$\text{Median} = l + \frac{\frac{N}{2} - f}{f} \times h$$

$\Rightarrow$ $$\text{Median} = 89.5 + \frac{49 - 40}{30} \times 10 = 92.5$$

EXAMPLE 5 *Compute the median form the following data:*

Mid-value:	*115*	*125*	*135*	*145*	*155*	*165*	*175*	*185*	*195*
Frequency:	*6*	*25*	*48*	*72*	*116*	*60*	*38*	*22*	*3*

SOLUTION Here, we are given the mid-values. So, should first find the upper and lower limits of the various classes. The difference between two consecutive values is $h = 125 - 115 = 10$.

$\therefore$ Lower limit of a class = Mid-value $- h/2$, Upper limit = Mid-value + $h/2$.

Calculation of Median

Mid-value	*Class groups*	*Frequency*	*Cumulative Frequency*
115	110 - 120	6	6
125	120 - 130	25	31
135	130 - 140	48	79
145	140 - 150	72	151
155	150 - 160	116	267
165	160 - 170	60	327
175	170 - 180	38	365
185	180 - 190	22	387
195	190 - 200	3	390
			$N = \Sigma f_i = 390$

We have,

$$N = 390 \quad \therefore \quad \frac{N}{2} = \frac{390}{2} = 195$$

The cumulative frequency just greater than $N/2$ i.e., 195 is 267 and the corresponding class is 150 - 160. So, 150 - 160 is the median class.

$\therefore \qquad l = 150,\ f = 116,\ h = 10,\ F = 151$

Now,

$$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h$$

$$\Rightarrow \quad \text{Median} = 150 + \frac{195 - 151}{116} \times 10 = 153.80$$

EXAMPLE 6 *Compute the median for the following cumulative frequency distribution:*

Less than 20	*Less than 30*	*Less than 40*	*Less than 50*	*Less than 60*	*Less than 70*	*Less than 80*	*Less than 90*	*Less than 100*
0	*4*	*16*	*30*	*46*	*66*	*82*	*92*	*100*

SOLUTION We are given the cumulative frequency distribution. So, we first construct a frequency table from the given cumulative frequency distribution and then we will make necessary computations to compute median.

Class intervals	*Frequency (f)*	*Cumulative frequency (c.f.)*
20 - 30	4	4
30 - 40	12	16
40 - 50	14	30
50 - 60	16	46
60 - 70	20	66
70 - 80	16	82
80 - 90	10	92
90 - 100	8	100
		$N = \Sigma f_i = 100$

Here, $\qquad N = \Sigma f_i = 100 \quad \therefore \quad \frac{N}{2} = 50$

We observe that the cumulative frequency just greater than $\frac{N}{2} = 50$ is 66 and the corresponding class is 60 - 70.

So, 60 - 70 is the median class.

$\therefore \quad l = 60,\ f = 20,\ F = 46$ and $h = 10$

Now, $$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h$$

$$\Rightarrow \quad \text{Median} = 60 + \frac{50 - 46}{20} \times 10 = 62$$

EXAMPLE 7 *The median of the following data is 525. Find the values of x and y, if the total frequency is 100* **[NCERT]**

Class interval	*Frequency*
0 - 100	*2*
100 - 200	*5*
200 - 300	*x*
300 - 400	*12*
400 - 500	*17*
500 - 600	*20*
600 - 700	*y*
700 - 800	*9*
800 - 900	*7*
900 - 1000	*4*

SOLUTION

Computation of Median

Class intervals	*Frequency (f)*	*Cumulative frequency (cf)*
0 - 100	2	2
100 - 200	5	7
200 - 300	x	$7 + x$
300 - 400	12	$19 + x$
400 - 500	17	$36 + x$
500 - 600	20	$56 + x$
600 - 700	y	$56 + x + y$
700 - 800	9	$65 + x + y$
800 - 900	7	$72 + x + y$
900 - 1000	4	$76 + x + y$
		Total = 100

We have,

$$N = \Sigma f_i = 100$$

$$\Rightarrow \quad 76 + x + y = 100 \Rightarrow x + y = 24$$

It is given that the median is 525. Clearly, it lies in the class 500 - 600

$\therefore \quad l = 500,\ h = 100,\ f = 20,\ F = 36 + x$ and $N = 100$

Now,

$$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h$$

$\Rightarrow \quad 525 = 500 + \frac{50 - (36 + x)}{20} \times 100$

$\Rightarrow \quad 525 - 500 = (14 - x) \times 5$

$\Rightarrow \quad 25 = 70 - 5x \Rightarrow 5x = 45 \Rightarrow x = 9$

Putting $x = 9$ in $x + y = 24$, we get $y = 15$.

Hence, $x = 9$ and $y = 15$.

EXAMPLE 8 *If the median of the distribution given below is 28.5, find the value of x and y.*

Class interval:	*0 - 10*	*10 - 20*	*20 - 30*	*30 - 40*	*40 - 50*	*50 - 60*
No. of students:	*5*	*x*	*20*	*15*	*y*	*5*

Total 60

[NCERT]

SOLUTION

Computation of Median

Class intervals	*Frequency (f)*	*Cumulative frequency (cf)*
0 - 10	5	5
10 - 20	x	$5 + x$
20 - 30	20	$25 + x$
30 - 40	15	$40 + x$
40 - 50	y	$40 + x + y$
50 - 60	5	$45 + x + y$
	$\Sigma f_1 = 60$	

We have,

Median = 28.5

Clealry, it lies in the class interval 20 - 30. So, 20 - 30 is the median class.

$\therefore \quad l = 20, h = 10, f = 20, F = 5 + x$ and $N = 60$

Now,

$$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h$$

$\Rightarrow \quad 28.5 = 20 + \frac{30 - (5 + x)}{20} \times 10$

$\Rightarrow \quad 28.5 = 20 + \frac{25 - x}{2}$

$\Rightarrow \quad 8.5 = \frac{25 - x}{2} \Rightarrow 25 - x = 17 \Rightarrow x = 8$

We have,

$N = 60$

$\therefore \quad 45 + x + y = 60 \Rightarrow x + y = 15$

Putting $x = 8$ in $x + y = 15$, we get $y = 7$

Hence, $x = 8$ and $y = 7$.

EXERCISE 15.4

LEVEL-1

1. Following are the lives in hours of 15 pieces of the components of aircraft engine. Find the median:

   715, 724, 725, 710, 729, 745, 694, 699, 696, 712, 734, 728, 716, 705, 719.

2. The following is the distribution of height of students of a certain class in a certain city:

Height (in cms):	160 - 162	163 - 165	166 - 168	169 - 171	172 -174
No. of students:	15	118	142	127	18

   Find the median height.

3. Following is the distribution of I.Q. of 100 students. Find the median I.Q.

I.Q.:	55-64	65-74	75-84	85-94	95-104	105-114	115-124	125-134	135-144
No of Students:	1	2	9	22	33	22	8	2	1

4. Calculate the median salary of the following data giving salaries of 280 persons:

Salary (in thousands):	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45	45-50
No. of persons:	49	133	63	15	6	7	4	2	1

   [CBSE 2018]

5. Calculate the median from the following data:

Marks below:	10	20	30	40	50	60	70	80
No. of students:	15	35	60	84	96	127	198	250

6. Calculate the missing frequency from the following distribution, it being given that the median of the distribution is 24.

Age in years:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of persons:	5	25	?	18	7

7. The following table gives the frequency distribution of married women by age at marriage:

Age (in years)	*Frequency*	*Age (in years)*	*Frequency*
15-19	53	40-44	9
20-24	140	45-49	5
25-29	98	50-54	3
30-34	32	55-59	3
35-39	12	60 and above	2

   Calculate the median and interpret the results.

8. The following table gives the distribution of the life time of 400 neon lamps:

Lite time: (in hours)	*Number of lamps*
1500-2000	14
2000-2500	56
2500-3000	60
3000-3500	86
3500-4000	74
4000-4500	62
4500-5000	48

   Find the median life. **[NCERT]**

9. The distribution below gives the weight of 30 students in a class. Find the median weight of students:

Weight (in kg):	40-45	45-50	50-55	55-60	60-65	65-70	70-75
No. of students:	2	3	8	6	6	3	2

[NCERT]

LEVEL-2

10. Find the missing frequencies and the median for the following distribution if the mean is 1.46.

No. of accidents:	0	1	2	3	4	5	Total
Frequency (No. of days):	46	?	?	25	10	5	200

11. An incomplete distribution is given below:

Variable:	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency:	12	30	–	65	–	25	18

You are given that the median value is 46 and the total number of items is 230.

(i) Using the median formula fill up missing frequencies.

(ii) Calculate the *AM* of the completed distribution.

12. If the median of the following frequency distribution is 28.5 find the missing frequencies:

Class interval:	0-10	10-20	20-30	30-40	40-50	50-60	Total
Frequency:	5	f_1	20	15	f_2	5	60

13. The median of the following data is 525. Find the missing frequency, if it is given that there are 100 observations in the data:

Class interval	*Frequency*	*Class interval*	*Frequency*
0 - 100	2	500 - 600	20
100 - 200	5	600 - 700	f_2
200 - 300	f_1	700 - 800	9
300 - 400	12	800 - 900	7
400 - 500	17	900 - 1000	4

14. If the median of the following data is 32.5, find the missing frequencies.

Class interval:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	Total
Frequency:	f_1	5	9	12	f_2	3	2	40

15. Compute the median for each of the following data:

(i) Marks	*No. of students*	*(ii) Marks*	*No. of students*
Less than 10	0	More than 150	0
Less than 30	10	More than 140	12
Less than 50	25	More than 130	27
Less than 70	43	More than 120	60
Less than 90	65	More than 110	105
Less than 110	87	More than 100	124
Less than 130	96	More than 90	141
Less than 150	100	More than 80	150

created by a car driven for 18,000 kilometres.

16. A survey regarding the height (in cm) of 51 girls of class X of a school was conducted and the following data was obtained:

Height in cm	*Number of Girls*
Less than 140	4
Less than 145	11
Less than 150	29
Less than 155	40
Less than 160	46
Less than 165	51

Find the median height. **[NCERT]**

17. A life insurance agent found the following data for distribution of ages of 100 policy holders. Calculate the median age, if policies are only given to persons having age 18 years onwards but less than 60 years.

Age in years	*Number of policy holders*
Below 20	2
Below 25	6
Below 30	24
Below 35	45
Below 40	78
Below 45	89
Below 50	92
Below 55	98
Below 60	100

[NCERT]

18. The lengths of 40 leaves of a plant are measured correct to the nearest millimetre, and the data obtained is represented in the following table:

Length (in mm):	118-126	127-135	136-144	145-153	154-162	163-171	172-180
No. of leaves:	3	5	9	12	5	4	2

Find the mean length of leaf. **[NCERT]**

19. An incomplete distribution is given as follows:

Variable:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
Frequency:	10	20	?	40	?	25	15

You are given that the median value is 35 and the sum of all the frequencies is 170. Using the median formula, fill up the missing frequencies.

20. The median of the distribution given below is 14.4. Find the values of x and y, if the total frequency is 20.

Class interval:	0-6	6-12	12-18	18-24	24-30
Frequency:	4	x	5	y	1

[NCERT EXEMPLAR]

21. The median of the following data is 50. Find the values of p and q, if the sum of all the frequencies is 90.

Marks:	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Frequency:	p	15	25	20	q	8	10

[NCERT EXEMPLAR]

ANSWERS

1. 716 2. 167.13 cms 3. 99.35 4. ₹13421 5. 59.35 6. 25
7. Median = 24.5 years.
   Nearly half the women were married between the ages 15 and 24.5 years.
8. Median life = 3406.98 hours 9. 56.67 kg
10. Missing Frequencies 76 and 38, Median = 1.
11. Missing frequencies 34 and 46, Mean = 45.87.
12. $f_1 = 8, f_2 = 7$ 13. $f_1 = 9, f_2 = 15$ 14. $f_1 = 3, f_2 = 6$
15. (i) 76.36 (ii) 116.67 16. 149.03 cm 17. 35.76 years
18. 146.75 mm 19. Class 20 - 30 40 - 50
    Frequency 35 25
20. $x = 4, y = 6$ 21. $p = 5, q = 7$

15.4 MERITS AND DEMERITS OF MEDIAN

The following are some merits and demerits of median:

MERITS

(i) It is easy to compute and understand.
(ii) It is well defined an ideal average should be.
(iii) It can also be computed in case of frequency distribution with open ended classes.
(iv) It is not affected by extreme values.
(v) It can be determined graphically.
(vi) It is proper average for qualitative data where items are not measured but are scored.

DEMERITS

(i) For computing median data needs to be arranged in ascending or descending order.
(ii) It is not based on all the observations of the data.
(iii) It cannot be given further algebraic treatment.
(iv) It is affected by fluctuations of sampling.
(v) It is not accurate when the data is not large.
(vi) In some cases median is determined approximately as the mid-point of two observations whereas for mean this does not happen.

15.5 MODE

In earlier classes, we have studied about the computation of mode of raw data. In this section, we shall learn about the computation of mode of a discrete frequency distribution and frequency distribution with class intervals. But, let us first recall the definition of mode.

MODE *The mode or modal value of a distribution is that value of the variable for which the frequency is maximum.*

Thus, the mode of a distribution is that value of the variable around which the values of the variable are clustered densely.

15.5.1 COMPUTATION OF MODE OF A SERIES OF INDIVIDUAL OBSERVATIONS

In order to compute the mode of a series of individual observations, we first convert it into a discrete series frequency distribution by preparing a frequency table. From the frequency table, we identify the value having maximum frequency. The value of variable so obtained is the mode or modal value.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the mode of the following data:*

120, 110, 130, 110, 120, 140, 130, 120, 140, 120

SOLUTION Let us first form the frequency table for the given data as given below:

Value x_i:	110	120	130	140
Frequency f_i:	2	4	2	2

We observe that the value 120 has the maximum frequency.

Hence, the mode or modal value is 120.

EXAMPLE 2 *Find the mode of the following data:*

25, 16, 19, 48, 19, 20, 34, 15, 19, 20, 21, 24, 19, 16, 22, 16, 18, 20, 16, 19

SOLUTION The frequency table of the given data is as given below:

Value (x_i):	15	16	18	19	20	21	22	24	25	34	48
Frequency (f_i):	1	4	1	5	3	1	1	1	1	1	1

We observe that the value 19 has the maximum frequency i.e. it occurs maximum number of times. Therefore, mode of the given data is 19.

EXAMPLE 3 *Find the value of x, if the mode of the following data is 25:*

15, 20, 25, 18, 14, 15, 25, 15, 18, 16, 20, 25, 20, x, 18

SOLUTION The frequency table of the given data is as given below:

Value (x_i):	14	15	16	18	20	25	x
Frequency (f_i):	1	3	1	3	3	3	1

It is given that the mode of the given data is 25. So, it must have the maximum frequency. That is possible only when $x = 25$.

Hence, $x = 25$.

15.5.2 COMPUTATION OF MODE BY GROUPING

Sometimes there are two or more values having the same frequency. In such cases one cannot say which is modal value and hence mode is said to be ill-defined. Such a frequency distribution is also known as bimodal or multimodal distribution. For such frequency distribution mode is computed by grouping method.

Consider the following frequency distribution:

x:	5	6	7	8	9	10	11	12	13
f:	8	12	13	14	13	11	7	4	3

From the above frequency distribution, we can clearly say that modal value is 8, because the value 8 of variable x has occurred the maximum number of times i.e. 14. But, we find that the difference between the maximum frequency and the frequencies of the values of the variable on both sides of 8 which are very close to 8 is very small. This means that the values of variable are heavily concentrated on either side of 8. Therefore, if we find mode just by inspection, an error is possible. In such cases, we prepare a grouping table and an analysis table to find the mode. These tables help us in determining the correct value of mode. The grouping table consists of six columns which are constructed by using the following algorithm.

ALGORITHM

STEP I *Obtain the discrete frequency distribution.*

STEP II *Take the column of frequencies as column I and encircle the maximum frequency in it.*

STEP III *Construct column II, containing the sum of the frequencies taken two at a time and encircle the maximum frequency in it.*

STEP IV *Leave the first frequency and construct column III, containing the sum of the frequencies taken two at a time. Encircle the maximum frequency in column III.*

STEP V *Construct column IV, containing the sum of three frequencies at a time and encircle the maximum frequency in it.*

STEP VI *Exclude the first frequency and compute the sum of the frequencies taken three at a time to construct column V. Encircle the maximum frequency in this column.*

STEP VII *Exclude the first two frequencies and compute the sum of the frequencies taken three at a time to construct column VI. Encircle the maximum frequency in this column.*

After preparing the grouping table, we prepare an analysis table by using the following algorithm.

ALGORITHM

STEP I *Prepare a table in which in the top most row write all values of the variable and in the left most column write column numbers from I to VI.*

STEP II *See the maximum frequency in the first column of the grouping table and obtain the corresponding value of the variable. Now, mark a bar (|) in the first row of the analysis table against the value of the variable having the maximum frequency. Continue the same procedure for the remaining five columns.*

STEP III *Find the total number of bars corresponding to each value of the variable. That value of the variable which has the maximum number of bars is the mode of the frequency distribution.*

Following illustration will illustrate the grouping and analysis tables.

ILLUSTRATION 4 *Compute the modal value for the following frequency distribution:*

x:	95	105	115	125	135	145	155	165	175
y:	4	2	18	22	21	19	10	3	2

SOLUTION It is clear from the frequency distribution that the difference between the maximum frequency and frequency succeeding it is very small and values of the variable x are closely concentrated on its either side. So, we compute the modal value by grouping method.

Grouping Table

x	f Col. I	Col. II	Col. III	Col. IV	Col. V	Col. VI
95	4					
		6				
105	2			24		
			20			
115	18					
		40			45	
125	22					
			43			61
135	21			62		
		40				
145	19					
			29		50	
155	10					
		13				32
165	3			15		
			5			
175	2					

Analysis Table

Col. No.	95	105	115	125	135	145	155	165	175
I				|					
II			|	|					
III				|	|				
IV				|	|	|			
V					|	|	|		
VI			|	|	|				
Total			2	5	4	2	1		

From the analysis table, it is clear that the value 125 has the maximum number of bars. So, Modal value is 125.

15.5.3 COMPUTATION OF MODE FOR A CONTINUOUS FREQUENCY DISTRIBUTION

In case of a grouped or continuous frequency distribution with equal class intervals, we use the following algorithm to compute the mode.

ALGORITHM

STEP I *Obtain the continuous frequency distribution.*

STEP II *Determine the class of maximum frequency either by inspection or by grouping method. This class is called the modal class.*

STEP III *Obtain the values of the following from the frequency distribution:*

l = lower limit of the modal class, f = frequency of the modal class

h = *width of the modal class,*

f_1 = *frequency of the class preceding the modal class,*

f_2 = *frequency of the class following the modal class.*

STEP IV *Substitute the values obtained in step III in the following formula:*

$$\text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Compute the mode for the following frequency distribution:*

Size of items:	0-4	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40
Frequency:	5	7	9	17	12	10	6	3	1	0

SOLUTION Here, the maximum frequency is 17 and the corresponding class is 12-16. So, 12-16 is the modal class such that $l = 12, h = 4, f = 17, f_1 = 9$ and $f_2 = 12$.

$$\therefore \quad \text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

$$\Rightarrow \quad \text{Mode} = 12 + \frac{17 - 9}{34 - 9 - 12} \times 4 = 12 + \frac{8}{13} \times 4 = 12 + \frac{32}{13} = 12 + 2.46 = 14.46$$

EXAMPLE 2 *For the following grouped frequency distribution find the mode:*

Class:	3 - 6	6 - 9	9 - 12	12 - 15	15 - 18	18 - 21	21 - 24
Frequency:	2	5	10	23	21	12	3

SOLUTION We observe that the class 12-15 has maximum frequency. Therefore, this is the modal class such that $l = 12, h = 3, f = 23, f_1 = 10$ and $f_2 = 21$.

$$\therefore \quad \text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

$$\Rightarrow \quad \text{Mode} = 12 + \frac{23 - 10}{46 - 10 - 21} \times 3 = 12 + \frac{13}{15} \times 3 = 12 + \frac{13}{5} = 14.6$$

EXAMPLE 3 *Compute the value of mode for the following frequency distribution.*

Class:	100-110	110-120	120-130	130-140	140-150	150-160	160-170
Frequency:	4	6	20	32	33	8	2

SOLUTION Clearly, the difference between the maximum frequency and the frequency preceding is very small. So, we shall determine the modal class by grouping method.

Grouping Table

Class	Frequency					
	Col. I	Col. II	Col. III	Col. IV	Col. V	Col. VI
100-110	4					
		10				
110-120	6			30		
			26			
120-130	20				58	
		52				
130-140	32					
			65			85
140-150	33			73		
		41				
150-160	8				43	
			10			
160-170	2					

Analysis Table

Col. No.	100-110	110-120	120-130	130-140	140-150	150-160	160-170
I					|		
II			|	|			
III				|	|		
IV				|	|	|	
V		|	|	|			
VI			|	|	|		
Total		1	3	5	4	1	

Clearly, class 130-140 has maximum number of bars. So, 130-140 is the modal class.

$\therefore \quad l = 130, h = 10, f = 32, f_1 = 20, f_2 = 33$

Now, $\quad \text{Mode} = l + \dfrac{f - f_1}{2f - f_1 - f_2} \times h$

$\Rightarrow \quad \text{Mode} = 130 + \dfrac{32 - 20}{64 - 20 - 33} \times 10$

$\Rightarrow \quad \text{Mode} = 130 + \dfrac{12}{9} \times 10 = 140.9$

LEVEL-2

EXAMPLE 4 *The following data gives the distribution of total household expenditure (in rupees) of manual workers in a city:*

Expenditure (in ₹)	*Frequency*	*Expenditure (in ₹)*	*Frequency*
1000-1500	24	3000-3500	30
1500-2000	40	3500-4000	22
2000-2500	33	4000-4500	16
2500-3000	28	4500-5000	7

Find the average expenditure which is being done by the maximum number of manual workers.

[NCERT]

SOLUTION We know that the mode is the value of the variable which occurs maximum number of times in a frequency distribution. So, the average expenditure done by the maximum number of workers is the modal value. We observe that the class 1500-2000 has the maximum frequency 40. So, it is the modal class such that $l = 1500, h = 500, f = 40,$ $f_1 = 24$ and $f_2 = 33$.

$$\therefore \qquad \text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

$$\Rightarrow \qquad \text{Mode} = 1500 + \frac{40 - 24}{80 - 24 - 33} \times 500 = 1500 + \frac{16}{23} \times 500 = 1847.826$$

EXAMPLE 5 *Calculate the value of mode for the following frequency distribution:*

Class:	1-4	5-8	9-12	13-16	17-20	21-24	25-28	29-32	33-26	37-40
Frequency:	2	5	8	9	12	14	14	15	11	13

SOLUTION Here, the classes are not in the inclusive form. So, we first convert them in inclusive form by subtracting $h/2$ from the lower limit and adding $h/2$ to the upper limit of each class, where h is the difference between the lower limit of a class and the upper limit of the preceding class.

Class	*Frequency*	*Class*	*Frequency*
0.5-4.5	24	20.5-24.5	30
4.5-8.5	40	24.5-28.5	22
8.5-12.5	40	28.5-32.5	22
12.5-16.5	33	32.5-36.5	16
16.5-20.5	28	36.5-36.5	7

To find the modal class, we use the grouping method:

Grouping Table

Class:	*Frequency:*					
	Col I	Col II	Col III	Col IV	Col V	Col VI
0.5-4.5	2	7				
4.5-8.5	5		13	15		
8.5-12.5	8	17			22	29
12.5-16.5	9		21			
16.5-20.5	12	26		35		
20.5-24.5	14		28		40	
24.5-28.5	14	29				43
28.5-32.5	15		26	40	39	
32.5-36.5	11	24				
36.5-40.5	13					

Analysis Table

Col. No.	0.5-4.5	4.5-8.5	8.5-12.5	12.5-16.5	16.5-20.5	20.5-24.5	24.5-28.5	28.5-32.5	32.5-36.5	36.5-40.5
I								|		
II							|	|		
III						|	|			
IV							|	|	|	
V					|	|	|			
VI						|	|	|		
Total					1	3	5	4	1	

Since 24.5-28.5 has the maximum number of bars. So, 24.5-28.5 is the modal class.

$\therefore \quad l = 24.5, h = 4, f = 14, f_1 = 14, f_2 = 15$

$$\therefore \quad \text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

$$\Rightarrow \quad \text{Mode} = 24.5 + \frac{14 - 14}{28 - 14 - 15} \times 4$$

$$\Rightarrow \quad \text{Mode} = 24.5 + 0 = 24.5$$

EXAMPLE 6 *The following table shows the age distribution of cases of a certain disease admitted during a year in a particular hospital.*

Age (in years):	*5 - 14*	*15 - 24*	*25 -34*	*35 - 44*	*45 - 54*	*55 - 64*
No. of cases:	*6*	*11*	*21*	*23*	*14*	*5*

Find the average age for which maximum cases occurred.

SOLUTION Here, class intervals are not in inclusive form. So, we first convert them in inclusive form by subtracting $h/2$ from the lower limit and adding $h/2$ to the upper limit of each class, where h is the difference between the lower limit of a class and the upper limit of the preceding class. The given frequency distribution in inclusive form is as follows.

Age (in years):	4.5 -14.5	14.5 - 24.5	24.5 - 34.5	34.5 - 44.5	44.5 - 54.5	54.5 - 64.5
No. of cases:	6	11	21	23	14	5

We observe that the class 34.5 - 44.5 has the maximum frequency. So, it is the modal class such that

$$l = 34.5, h = 10, f = 23, f_1 = 21 \text{ and } f_2 = 14$$

$$\therefore \qquad \text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

$$\Rightarrow \qquad \text{Mode} = 34.5 + \frac{23 - 21}{46 - 21 - 14} \times 10 = 34.5 + \frac{2}{11} \times 10 = 36.31$$

EXERCISE 15.5

LEVEL-1

1. Find the mode of the following data:
   (i) 3, 5, 7, 4, 5, 3, 5, 6, 8, 9, 5, 3, 5, 3, 6, 9, 7, 4
   (ii) 3, 3, 7, 4, 5, 3, 5, 6, 8, 9, 5, 3, 5, 3, 6, 9, 7, 4
   (iii) 15, 8, 26, 25, 24, 15, 18, 20, 24, 15, 19, 15

2. The shirt sizes worn by a group of 200 persons, who bought the shirt from a store, are as follows:

Shirt size:	37	38	39	40	41	42	43	44
Number of persons:	15	25	39	41	36	17	15	12

   Find the modal shirt size worn by the group.

3. Find the mode of the following distribution.

   (i)

Class-interval:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency:	5	8	7	12	28	20	10	10

   (ii)

Class-interval:	10-15	15-20	20-25	25-30	30-35	35-40
Frequency:	30	45	75	35	25	15

   (iii)

Class-interval:	25-30	30-35	35-40	40-45	45-50	50-60
Frequency:	25	34	50	42	38	14

4. Compare the modal ages of two groups of students appearing for an entrance test:

Age (in years):	16-18	18-20	20-22	22-24	24-26
Group A:	50	78	46	28	23
Group B:	54	89	40	25	17

5. The marks in science of 80 students of class X are given below: Find the mode of the marks obtained by the students in science.

Marks:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency:	3	5	16	12	13	20	5	4	1	1

6. The following is the distribution of height of students of a certain class in a certain city:

Height (in cms):	160-162	163-165	166-168	169-171	172-174
No. of students:	15	118	142	127	18

Find the average height of maximum number of students.

7. The following table shows the ages of the patients admitted in a hospital during a year:

Age (in years):	5-15	15-25	25-35	35-45	45-55	55-65
No. of students:	6	11	21	23	14	5

Find the mode and the mean of the data given above. Compare and interpret the two measures of central tendency. **[NCERT]**

8. The following data gives the information on the observed lifetimes (in hours) of 225 electrical components:

Lifetimes (in hours):	0-20	20-40	40-60	60-80	80-100	100-120
No. of components:	10	35	52	61	38	29

Determine the modal lifetimes of the components. **[NCERT]**

9. The following table gives the daily income of 50 workers of a factory:

Daily income (in ₹)	100 - 120	120 - 140	140 - 160	160 -180	180 - 200
Number of workers:	12	14	8	6	10

Find the mean, mode and median of the above data. **[CBSE 2009]**

10. The following distribution gives the state-wise teacher-student ratio in higher secondary schools of India. Find the mode and mean of this data. Interpret, the two measures: **[NCERT]**

Number of students per Teacher	*Number of States/U.T.*	*Number of students per Teacher*	*Number of States/U.T.*
15 – 20	3	35 – 40	3
20 – 25	8	40 – 45	0
25 – 30	9	45 – 50	0
30 – 35	10	50 – 55	2

[NCERT]

11. Find the mean, median and mode of the following data:

Classes:	0-50	50-100	100-150	150-200	200-250	250-300	300-350
Frequency:	2	3	5	6	5	3	1

[CBSE 2008]

12. A student noted the number of cars passing through a spot on a road for 100 periods each of 3 minutes and summarised it in the table given below. Find the mode of the data:

Number of cars:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency:	7	14	13	12	20	11	15	8

[NCERT]

13. The following frequency distribution gives the monthly consumption of electricity of 68 consumers of a locality. Find the median, mean and mode of the data and compare them.

Monthly consumption: (in units)	65-85	85-105	105-125	125-145	145-165	165-185	185-205
No. of consumers:	4	5	13	20	14	8	4

14. 100 surnames were randomly picked up from a local telephone directly and the frequency distribution of the number of letters in the English alphabets in the surnames was obtained as follows:

Number of letters:	1-4	4-7	7-10	10-13	13-16	16-19
Number surnames:	6	30	40	16	4	4

Determine the median number of letters in the surnames. Find the mean number of letters in the surnames. Also, find the modal size of the surnames.

15. Find the mean, median and mode of the following data:

Classes:	0-20	20-40	40-60	60-80	80-100	100-120	120-140
Frequency:	6	8	10	12	6	5	3

[CBSE 2008]

LEVEL-2

16. The following data gives the distribution of total monthly houshold expenditure of 200 families of a village. Find the modal monthly expenditure of the families. Also, find the mean monthly expenditure: **[NCERT]**

Expenditure (in ₹)	*Frequency*	*Expenditure* (*in* ₹)	*Frequency*
1000-1500	24	3000-3500	30
1500-2000	40	3500-4000	22
2000-2500	33	4000-4500	16
2500-3000	28	4500-5000	7

17. The given distribution shows the number of runs scored by some top batsmen of the world in one-day international cricket matches.

Runs scored	*Number of bastsman*	*Runs scored*	*Number of bastsman*
3000-4000	4	7000-8000	6
4000-5000	18	8000-9000	3
5000-6000	9	9000-10000	1
6000-7000	7	10000-11000	1

Find the mode of the data.

18. The frequency distribution table of agriculture holdings in a village is given below:

Area of land (in hectares):	1-3	3-5	5-7	7-9	9-11	11-13
Number of families:	20	45	80	55	40	12

Find the modal agriculture holdings of the village. **[NCERT EXEMPLAR]**

19. The monthly income of 100 families are given as below:

Income in (in ₹)	Number of families
0-5000	8
5000-10000	26
10000-15000	41
15000-20000	16
20000-25000	3
25000-30000	3
30000-35000	2
35000-40000	1

Calculate the modal income.

[NCERT EXEMPLAR]

ANSWERS

1. (i) 5 (ii) 3 (iii) 15 2. 40
3. (i) 46.67 (ii) 22.14 (iii) 38.33
4. Group A: 18.93 years, Group B: 18.83 years 5. 53.17 6. 167.35
7. Mode = 36.8 years, Mean = 35.37 years 8. 65.625 hours
9. Mean = 145.20, Median = 138.57, Mode = 125
10. Mode = 30.6, Mean = 29.2 11. Median = 170.83, Mean = 169, Mode = 175
12. Mode = 44.7 cars
13. Median = 137 units, Mean = 137.05 units, Mode = 135.76 units
14. Median = 8.05, Mean = 8.32, Modal size = 7.88
15. Median = 61.66, Mean = 62.4, Mode = 65
16. ₹ 1847.83, ₹ 2662.50 17. Mode = 4608.7 runs
18. 6.2 hectares 19. ₹ 11875

15.6 MERITS, DEMERITS AND USES OF MODE

The following are some merits and demerits of mode :

MERITS

(i) It is readily comprehensible and easy to compute. In some case it can be computed merly by inspection.

(ii) It is not affected by extreme values. It can be obtained even if the extreme values are not known.

(iii) Mode can be determined in distributions with open classes.

(iv) Mode can be located on graph also.

DEMERITS

(i) It is ill-defined. It is not always possible to find a clearly defined mode. In some cases, we may come across distributions with two modes. Such distributions are called bimodal. If a distribution has more than two modes, it is said to be multimodal.

(ii) It is not based upon all the observation.

(iii) Mode can be calculated by various formulae as such the value may differ from one to other. Therefore, it is not rigidly defined.

(iv) It is affected to a greater extent by fluctuations of sampling.

USES OF MODE

Mode is used by the manufacturers of readymade garments, shoes and accessories in common use etc. The readymade garment manufacturers made those sizes more which are used by most of the persons than other sizes. Similarly, the makers of shoes will make that size maximum which the majority people use and others in less quantity.

15.7 RELATIONSHIP AMONG MEAN, MEDIAN AND MODE

We have learnt about three measures of central value, namely, arithmetic mean, median and mode. These three measures are closely connected by the following relations.

$$\text{Mode} = 3\ \text{Median} - 2\ \text{Mean}$$

or, $$\text{Median} = \text{Mode} + \frac{2}{3}(\text{Mean} - \text{Mode})$$

or, $$\text{Mean} = \text{Mode} + \frac{3}{2}(\text{Median} - \text{Mode})$$

15.8 CUMULATIVE FREQUENCY POLYGON CURVE (AN OGIVE)

In class IX, we have learnt about graphical representation of frequency distributions by using bargraphs, histograms and frequency polygons. In this section, we will learn about the construction of cumulative frequency polygon and cumulative frequency curves or ogives. The technique of drawing the cumulative frequency polygons and cumulative frequency curves or ogives is more or less the same. The only difference is that in case of simple frequency curves and polygons the frequencies are plotted against class marks of the class intervals where as in case of a cumulative frequency polygon or curves the cumulative frequencies are plotted against the lower or upper limits of the class intervals depending upon the manner in which the series has been cumulated. There are two methods of constructing a frequency polygon and an ogive. Let us now discuss the two methods.

(i) Less than method (ii) More than method.

15.8.1 LESS THAN METHOD

To construct a cumulative frequency polygon and an ogive by less than method, we use the following algorithm

ALGORITHM

STEP I *Start with the upper limits of class intervals and add class frequencies to obtain the cumulative frequency distribution.*

STEP II *Mark upper class limits along X-axis on a suitable scale.*

STEP III *Mark cumulative frequencies along Y-axis on a suitable scale.*

STEP IV *Plot the points (x_i, f_i), where x_i is the upper limit of a class and f_i is corresponding cumulative frequency.*

STEP V *Join the points obtained in step IV by a free hand smooth curve to get the ogive and to get the cumulative frequency polygon join the points obtained in step IV by line segments.*

15.8.2 MORE THAN METHOD

To construct a cumulative frequency polygon and an ogive by more than method, we use the following algorithm.

ALGORITHM

STEP I Start with the lower limits of the class intervals and from the total frequency subtract the frequency of each class to obtain the cumulative frequency distribution.

STEP II Mark the lower class limits along X-axis on a suitable scale.

STEP III Mark the cumulative frequencies along Y-axis on a suitable scale.

STEP IV Plot the points (x_i, f_i), where x_i is the lower limit of a class and f_i is the corresponding cumulative frequency.

STEP V Join the points obtained in step IV by a free hand smooth curve to get the ogive and to get the cumulative frequency polygon join these points by line segments.

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Draw an ogive and the cumulative frequency polygon for the following frequency distribution by less than method.*

Marks:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
Number of students:	7	10	23	51	6	3

SOLUTION We first prepare the cumulative frequency distribution table by less than method as given below:

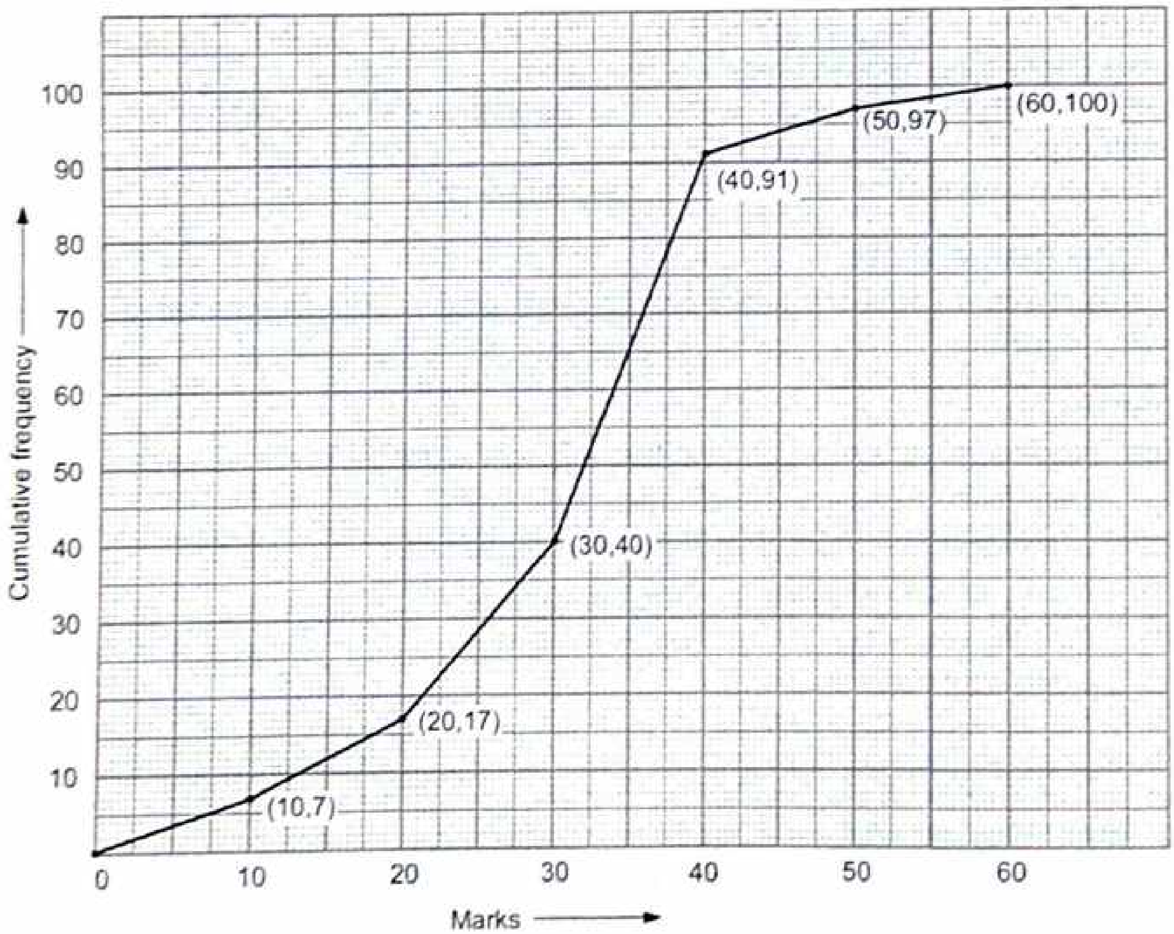

Fig. 15.1 Cumulative frequency curve or ogive

Marks	*No. of Students*	*Marks less than*	*Cumulative Frequency*
0 - 10	7	10	7
10 - 20	10	20	17
20 - 30	23	30	40
30 - 40	51	40	91
40 - 50	6	50	97
50 - 60	2	60	100

Other than the given class intervals, we assume a class – 10 - 0 before the first class interval 0 - 10 with zero frequency.

Now, we mark the upper class limits (including the imagined class) along X-axis on a suitable scale and the cumulative frequencies along Y-axis on a suitable scale.

Thus, we plot the points (0, 0), (10, 7), (20, 17), (30, 40), (40, 91), (50, 97) and (60, 100).

Now, we join the plotted points by a free hand curve to obtain the required ogive as shown in Fig. 15.1. In order to obtain the cumulative frequency polygon, we join the plotted points by line segments as shown in Fig. 15.2.

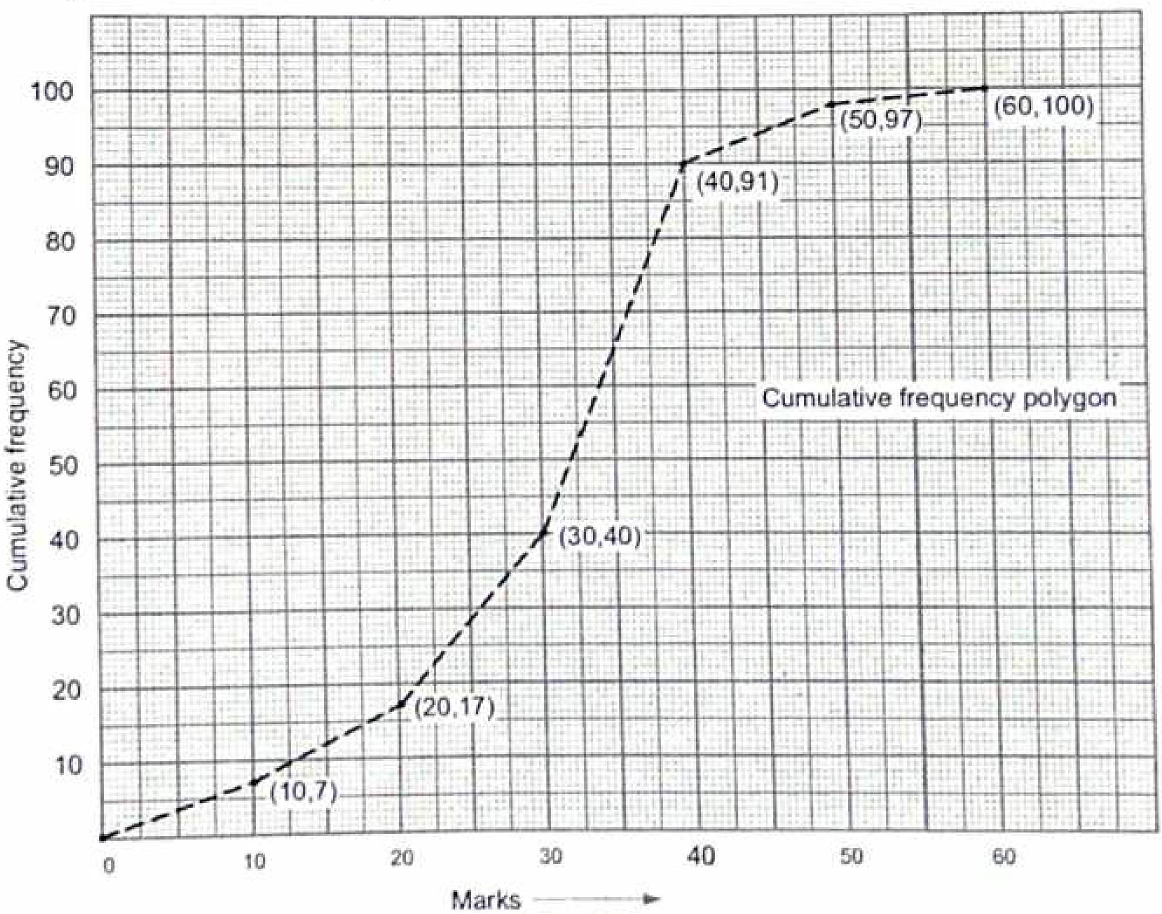

Fig. 15.2

EXAMPLE 2 *Draw a cumulative frequency curve and cumulative frequency polygon for the following frequency distribution by less than method.*

Age (in years):	*0 - 9*	*10 - 19*	*20 - 29*	*30 - 39*	*40 - 49*	*50 - 59*	*60 - 69*
No. of Persons:	*5*	*15*	*20*	*23*	*17*	*11*	*9*

SOLUTION The given frequency distribution is not continuous. So, we first make it continuous and prepare the cumulative frequency distribution as under:

Age (in years)	*Frequency*	*Age less than*	*Cumulative Frequency*
−0.5 - 9.5	5	9.5	5
9.5 - 19.5	15	19.5	20
19.5 - 29.5	20	29.5	40
29.5 - 39.5	23	39.5	63
39.5 - 49.5	17	49.5	80
49.5 - 59.5	11	59.5	91
59.5 - 69.5	9	69.5	100

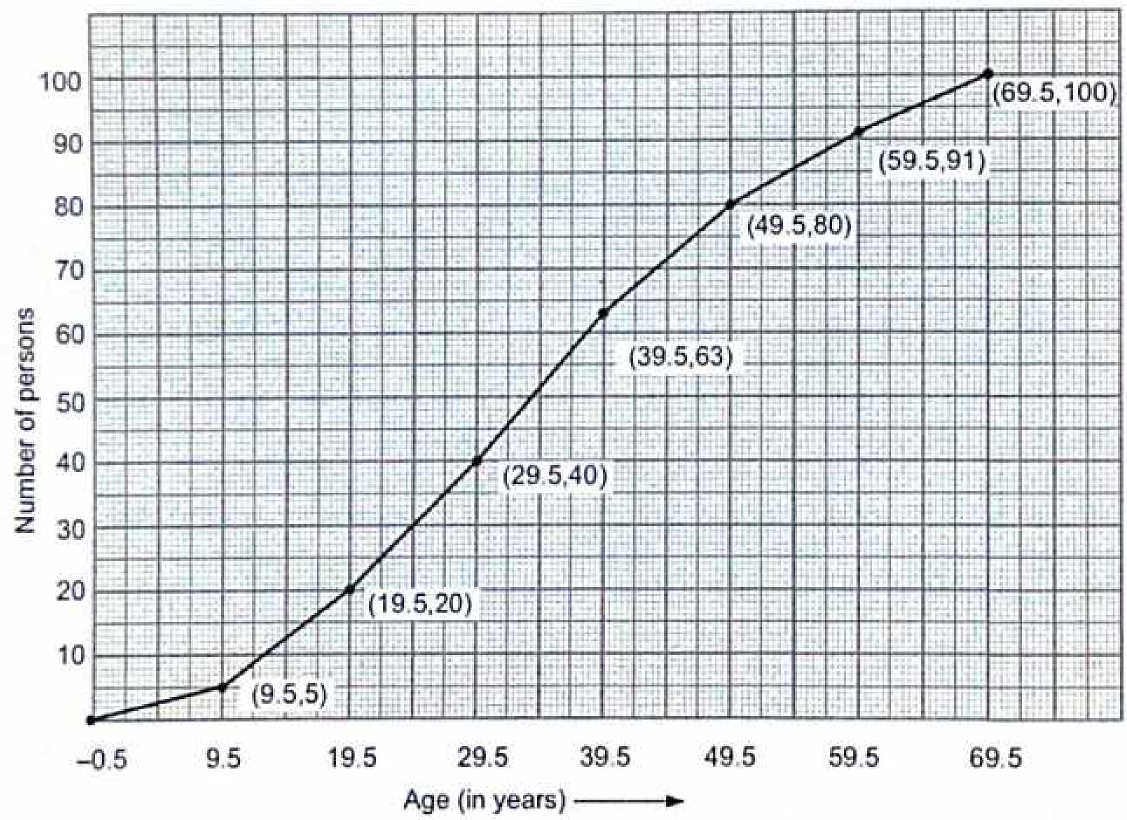

Fig. 15.3 Cumulative frequency curve

Now, we plot points (9.5, 5), (19.5, 20), (29.5, 40), (39.5, 63, (49.5, 80), (59.5, 91) and (69.5, 100) and join them by a free hand smooth curve to obtain the required ogive as shown in Fig. 15.3. The cumulative frequency polygon is obtained by joining these points by line segments as shown in Fig. 15.4.

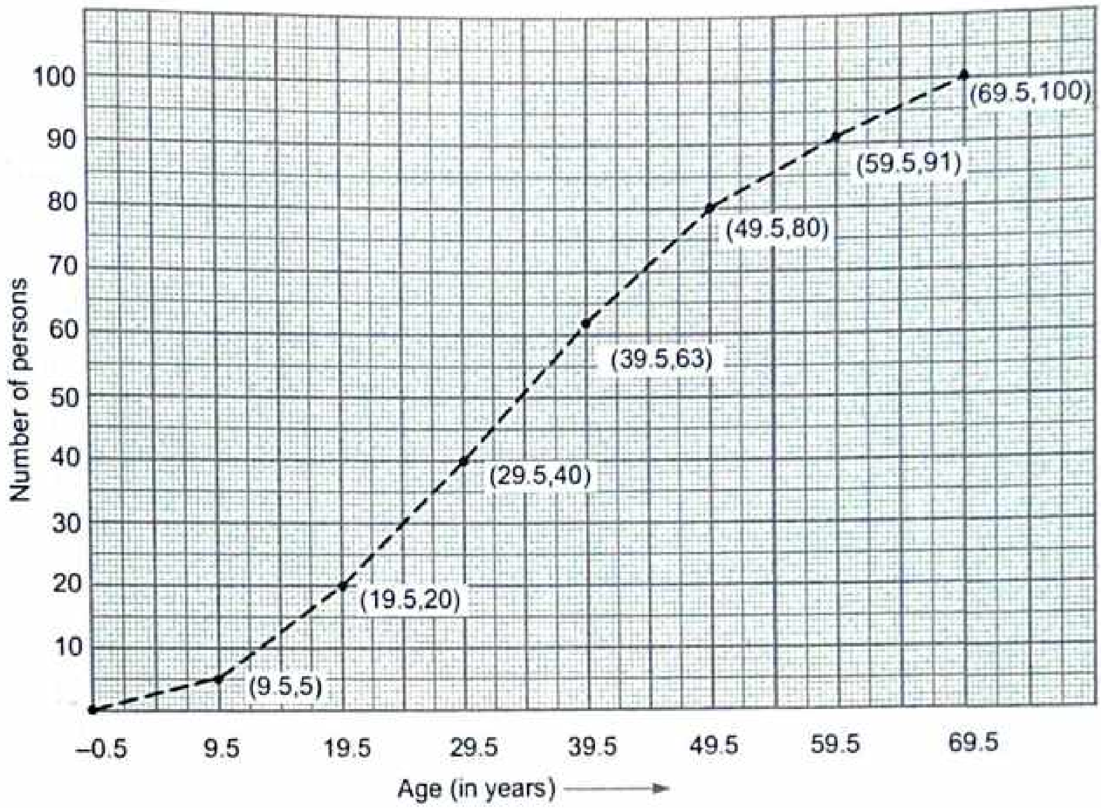

Fig. 15.4 Cumulative frequency polygon

LEVEL-2

EXAMPLE 3 *The frequency distribution of scores obtained by 230 candidates in a medical entrance test is as follows.*

Scores:	*400-450*	*450-500*	*500-550*	*550-600*	*600-650*	*650-700*	*700-750*	*750-800*
Number of Candidates:	*20*	*35*	*40*	*32*	*24*	*27*	*18*	*24*

Draw cumulative frequency curves by less than and more than method on the same axes. Also, draw the two types of cumulative frequency polygons.

SOLUTION *Less than method:* We first prepare the cumulative frquency table by less than method as given below:

Scores	*Number of Candidates*	*Scores less than*	*Cumulative Frequency*
400 - 450	20	450	20
450 - 500	35	500	55
500 - 550	40	550	95
550 - 600	32	600	127
600 - 650	24	650	151
650 - 700	27	700	178
700 - 750	18	750	196
750 - 800	34	800	230

Other than the given class intervals, we assume a class interval 350 - 400 prior the first class interval 400 - 500 with zero frequency.

SOLUTION The given frequency distribution is not continuous. So, we first make it continuous and prepare the cumulative frequency distribution as under:

Age (in years)	*Frequency*	*Age less than*	*Cumulative Frequency*
−0.5 - 9.5	5	9.5	5
9.5 - 19.5	15	19.5	20
19.5 - 29.5	20	29.5	40
29.5 - 39.5	23	39.5	63
39.5 - 49.5	17	49.5	80
49.5 - 59.5	11	59.5	91
59.5 - 69.5	9	69.5	100

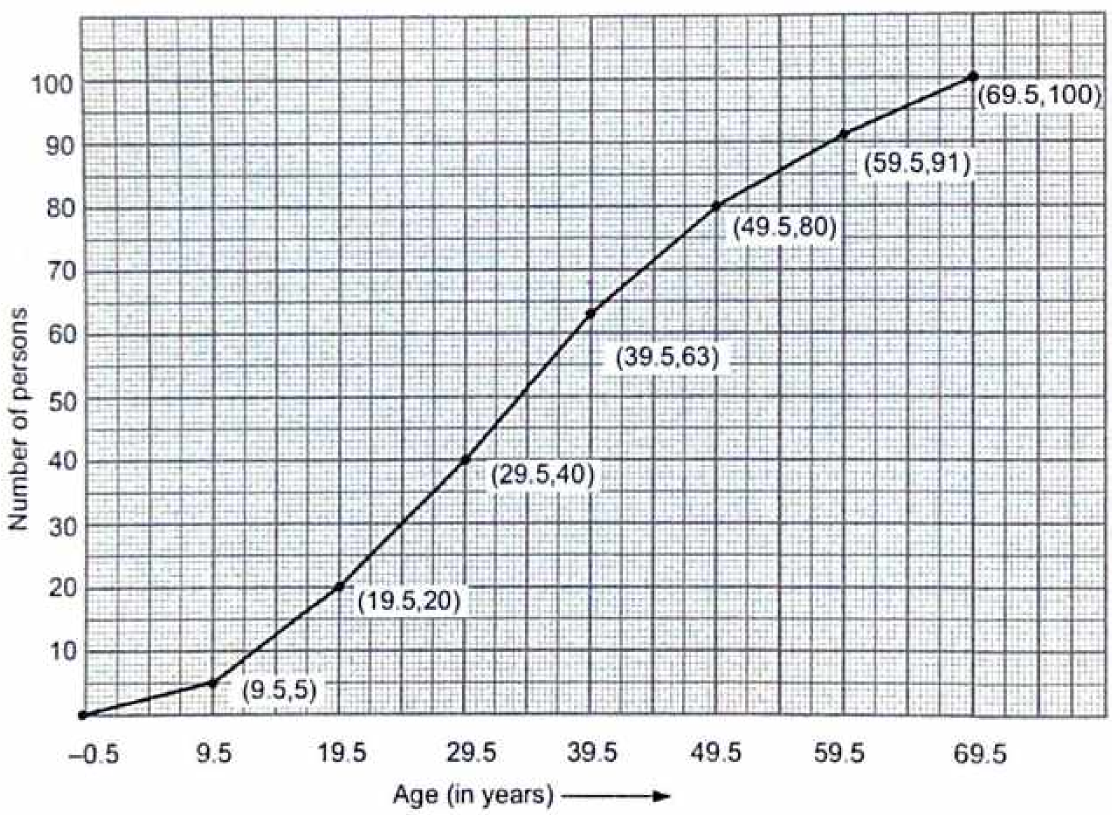

Fig. 15.3 Cumulative frequency curve

Now, we plot points (9.5, 5), (19.5, 20), (29.5, 40), (39.5, 63, (49.5, 80), (59.5, 91) and (69.5, 100) and join them by a free hand smooth curve to obtain the required ogive as shown in Fig. 15.3. The cumulative frequency polygon is obtained by joining these points by line segments as shown in Fig. 15.4.

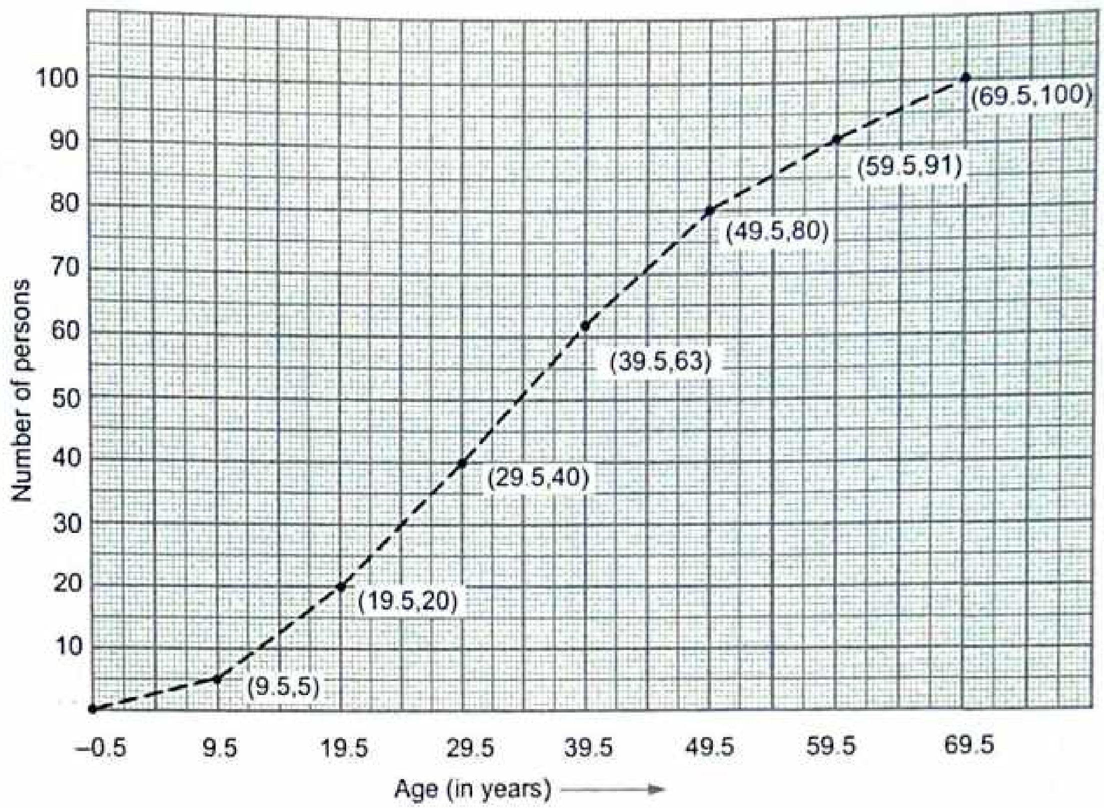

Fig. 15.4 Cumulative frequency polygon

LEVEL-2

EXAMPLE 3 *The frequency distribution of scores obtained by 230 candidates in a medical entrance test is as follows.*

Scores:	*400-450*	*450-500*	*500-550*	*550-600*	*600-650*	*650-700*	*700-750*	*750-800*
Number of Candidates:	*20*	*35*	*40*	*32*	*24*	*27*	*18*	*24*

Draw cumulative frequency curves by less than and more than method on the same axes. Also, draw the two types of cumulative frequency polygons.

SOLUTION *Less than method:* We first prepare the cumulative frquency table by less than method as given below:

Scores	*Number of Candidates*	*Scores less than*	*Cumulative Frequency*
400 - 450	20	450	20
450 - 500	35	500	55
500 - 550	40	550	95
550 - 600	32	600	127
600 - 650	24	650	151
650 - 700	27	700	178
700 - 750	18	750	196
750 - 800	34	800	230

Other than the given class intervals, we assume a class interval 350 - 400 prior the first class interval 400 - 500 with zero frequency.

Now, we mark the upper class limits on X-axis and the cumulative frequencies along Y-axis on suitable scales.

Thus, we plot the points (400,0), (450, 20), (500, 55), (550, 95), (600, 127), (650, 151), (700, 178), (750, 196) and (800, 230).

Join these points by a free hand smooth curve to obtain an ogive by less than method as shown in Fig. 15.5.

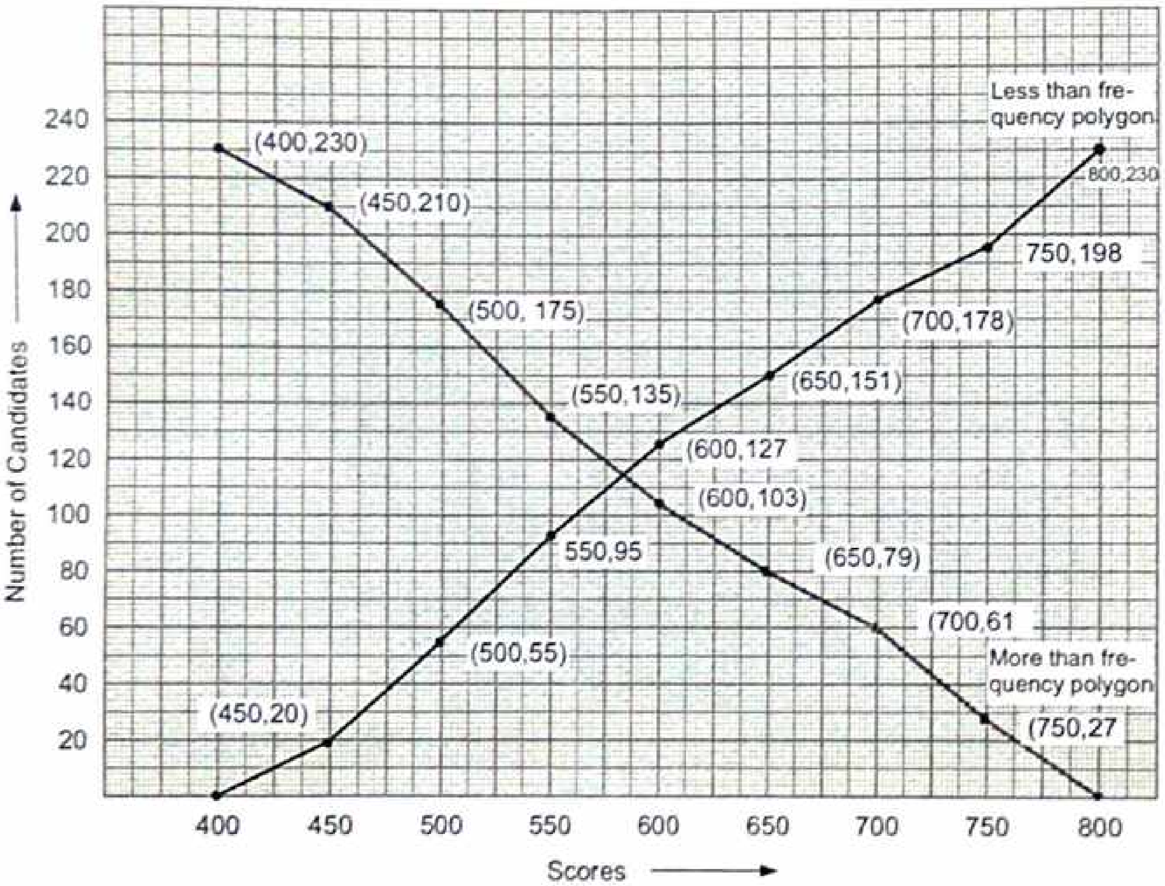

Fig. 15.5

In order to obtain the cumulative frequency polygon by less than method join these points by line segments as given in Fig. 15.6.

Other than the given class intervals, we assume class interval 800 - 850 after the last class interval 750 - 800 with zero frequency.

Now, we mark the lower class limits on X-axis and the cumulative frequencies along Y-axis on suitable scales.

Thus, we plot the points (400, 230), (450, 210), (500, 175), (550, 175), (600, 103), (650, 79), (700, 61), (750, 27) and (800, 0).

By joining these points by a free hand smooth curve, we obtain an ogive by more than method as shown in Fig. 15.3.

The cumulative frequency polygon by more than method is obtained by joining these points by line segments as shown in Fig. 15.4.

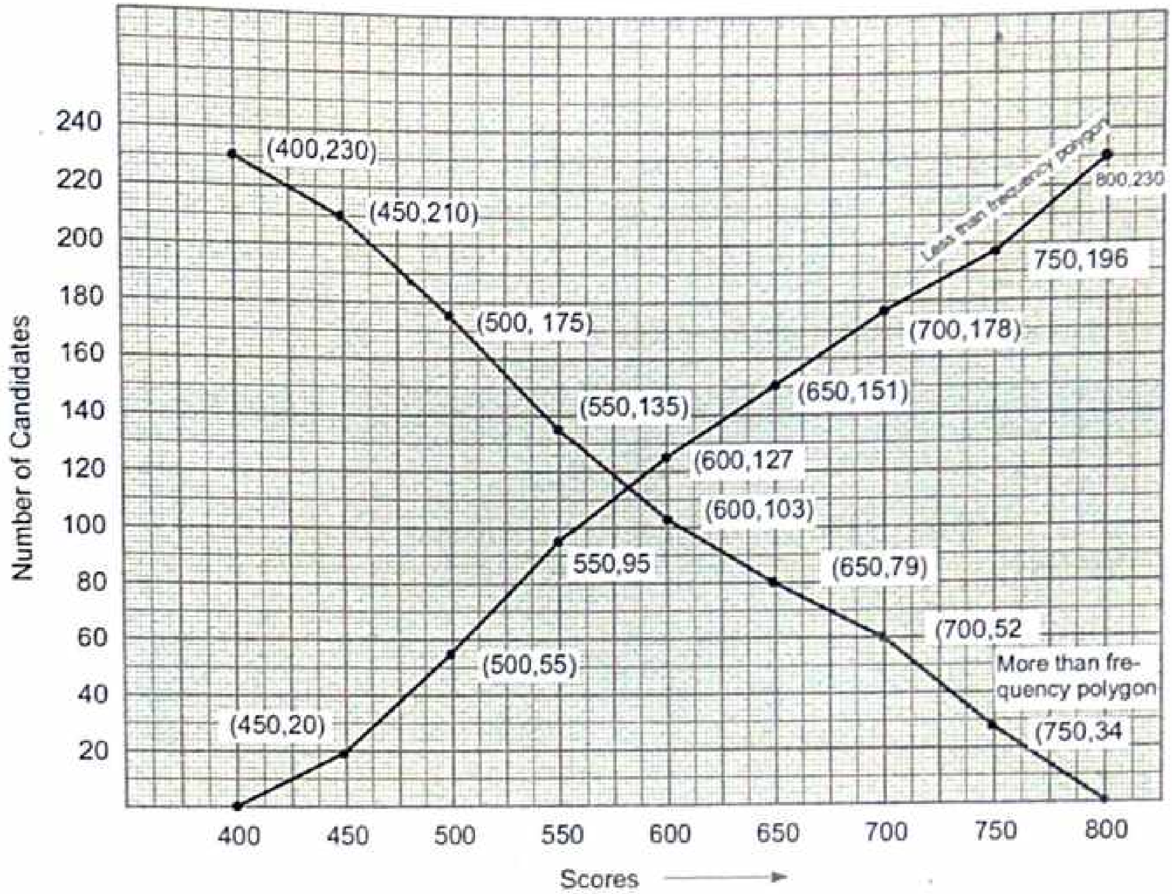

Fig. 15.6

More than method: Let us first prepare the cumulative frequency table by more than method as given below:

Scores	*Number of Candidates*	*Scores more than*	*Cumulative Frequency*
400 - 450	20	400	230
450 - 500	35	450	210
500 - 550	40	500	175
550 - 600	32	550	135
600 - 650	24	600	103
650 - 700	27	650	79
700 - 750	18	700	52
750 - 800	34	750	34

15.8.3 SOME APPLICATIONS OF OGIVES

Ogives can be used to find the median of a frequency distribution. In order to determine the same, we may use the following algorithms:

ALGORITHM 1

STEP I *Draw any one of the two types of frequency curves on the graph paper.*

STEP II *Compute* $\frac{N}{2}(N = \Sigma f_i)$ *and mark the corresponding point on y-axis.*

STEP III *Draw a line parallel to x-axis, from the point marked in step II, cutting the cumulative frequency curve at a point P (say).*

STEP IV *Draw perpendicular PM from P on the X-axis. The x-coordinate of point M gives the median.*

ALGORITHM 2

STEP I *Draw less than type and greater than type cumulative frequency curves on the graph paper.*

STEP II *Mark the point of intersection of the two curves drawn in step I. Let this point be P*

STEP III *Draw perpendicular PM from P on the X-axis. The x-coordinate of point M gives the Median.*

Following examples will illustrate these algorithms.

ILLUSTRATIVE EXAMPLES

LEVEL-2

EXAMPLE 1 *Following is the age distribution of a group of students. Draw the cumulative frequency polygon, cumulative frequency curve (less than type) and hence obtain the median value.*

Age	*Frequency*	*Age*	*Frequency*
5 - 6	40	11 - 12	92
6 - 7	56	12 - 13	80
7 - 8	60	13 - 14	64
8 - 9	66	14 - 15	44
9 - 10	84	15 - 16	20
10 - 11	96	16 - 17	8

SOLUTION We first prepare the cumulative frequency table by less then method as given below.

Age	*Frequency*	*Age less than*	*Cumulative frequency*
5-6	40	6	40
6-7	56	7	96
7-8	60	8	156
8-9	66	9	222
9-10	84	10	306
10-11	96	11	402
11-12	92	12	494
12-13	80	13	574
13-14	64	14	638
14-15	44	15	682
15-16	20	16	702
16-17	8	17	710

Other than the given class intervals, we assume a class 4-5 before the first class-interval 5-6 with zero frequency.

Now, we mark the upper class limits (including the imagined class) along X-axis on a suitable scale and the cumulative frequencies along Y-axis on a suitable scale.

Thus, we plot the points (5, 0), (6, 40), (7, 96), (8, 156), (9, 222), (10, 306), (11, 402), (12, 494), (13, 574), (14, 638), (15, 682), (16, 702) and (17, 710). These points are marked and joined by line segments to obtain the cumulative frequency polygon shown in Fig. 15.7.

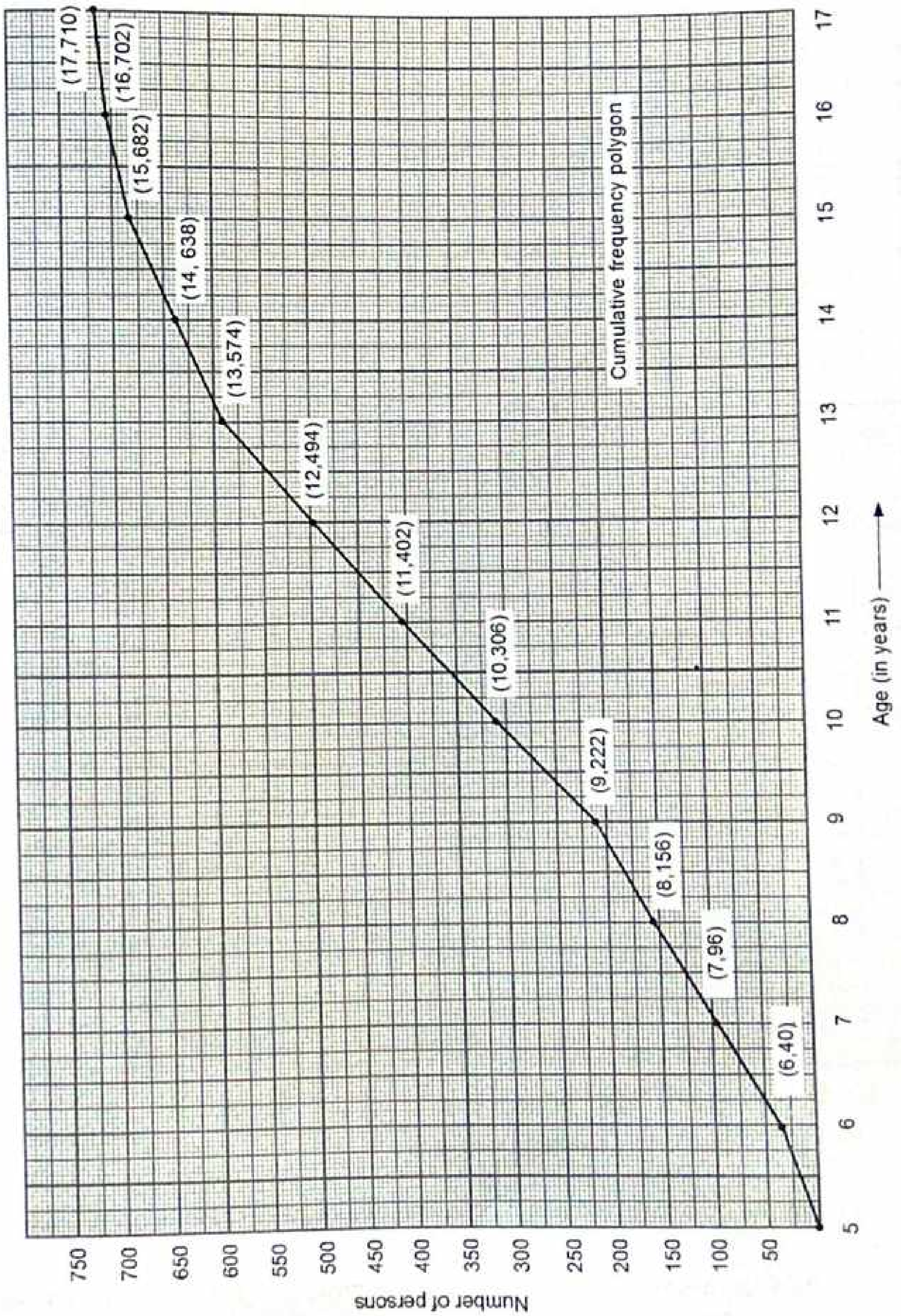

Fig. 15.7

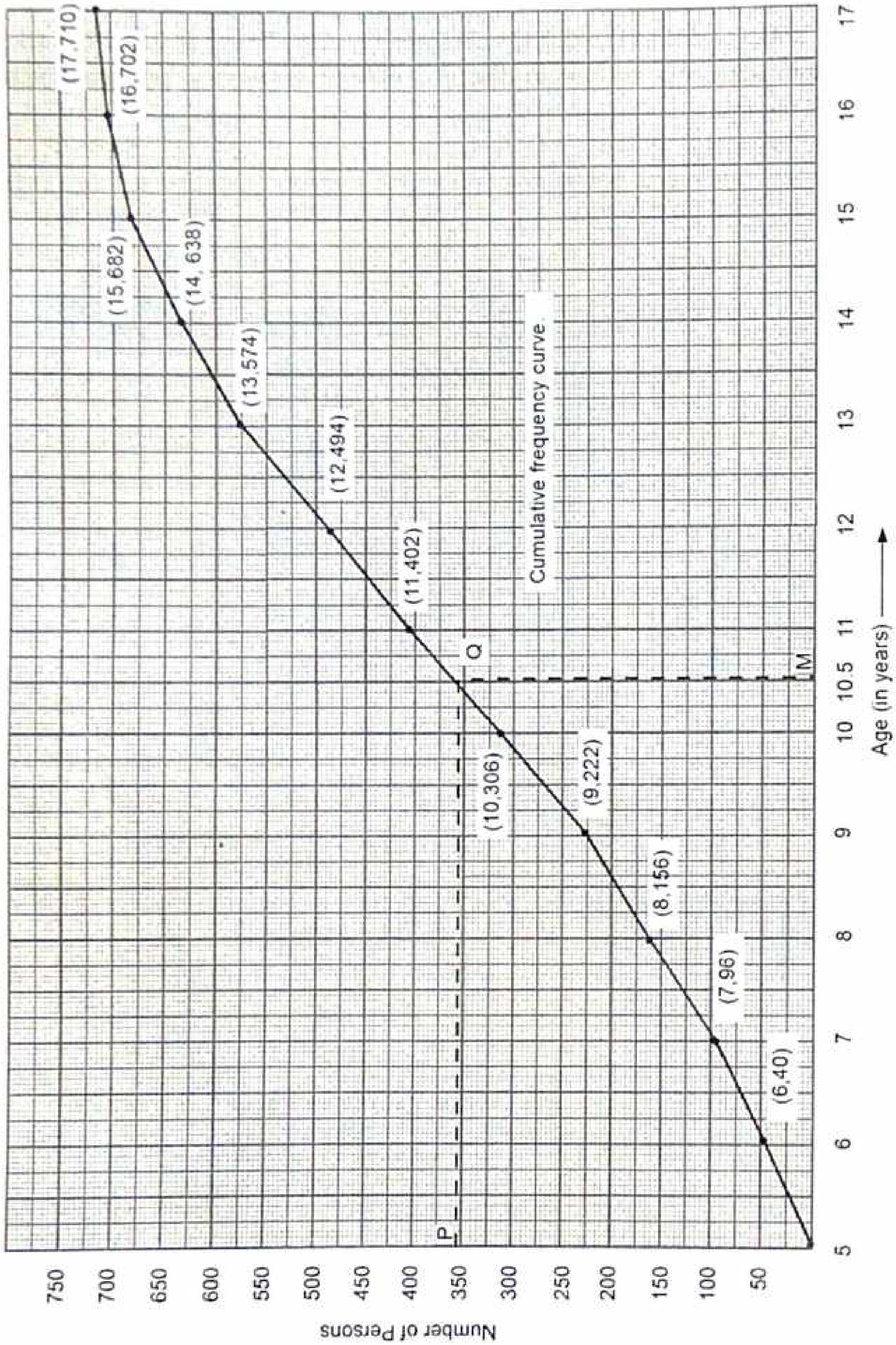

Fig. 15.8

In order to obtain the cumulative frequency curve, we draw a smooth curve passing through the points discussed above.

The graph (Fig. 15.8) shows the total number of students as 710. The median is the age corresponding to $\frac{N}{2} = \frac{710}{2} = 355$ students. In order to find the median, we first locate the point corresponding to 355th student on Y-axis. Let the point be P. From this point draw a line parallel to the X-axis cutting the curve at Q. From this point Q draw a line parallel to Y-axis and meeting X-axis at the point M. The x-coordinate of M is 10.5 (See Fig. 15.8). Hence, median is 10.5.

EXAMPLE 2 *The following observations relate to the height of a group of persons. Draw the two types of cumulative frequency polygons and cumulative frequency curves and determine the median.*

Height in cms	*Frequency*
140-143	3
143 -146	9
146 -149	26
149 -152	31
152 -155	45
155 -158	64
158 -161	78
161 -164	85
164 -167	96
167 -170	72
170 -173	60
173 -176	43
176 -179	20
179 -182	6

SOLUTION *Less than method:* We first prepare the cumulative frequency table by less than method as given below:

Height in cms	*Frequency*	*Height less than*	*Cumulative frequency*
140-143	3	143	3
143-146	9	146	12
146-149	26	149	38
149-152	31	152	69
152-155	45	155	114
155-158	64	158	178
158-161	78	161	256
161-164	85	164	341
164-167	96	167	437
167-170	72	170	509
170-173	60	173	569
173-176	43	176	612
176-179	20	179	632
179-182	06	182	638

Other than the given class intervals, we assume a class interval 137-140 prior to the first class interval 140-143 with zero frequency.

Now, we mark the upper class limits on X-axis and cumulative frequencies along Y-axis on a suitable scale. We plot the points (140, 0), (143, 3), (146, 12), (149, 38), (152, 69), (155, 114), (158, 178), (161, 256), (164, 341), (167, 437), (170, 509), (173, 569), (176, 612), (179, 632) and (182, 638).

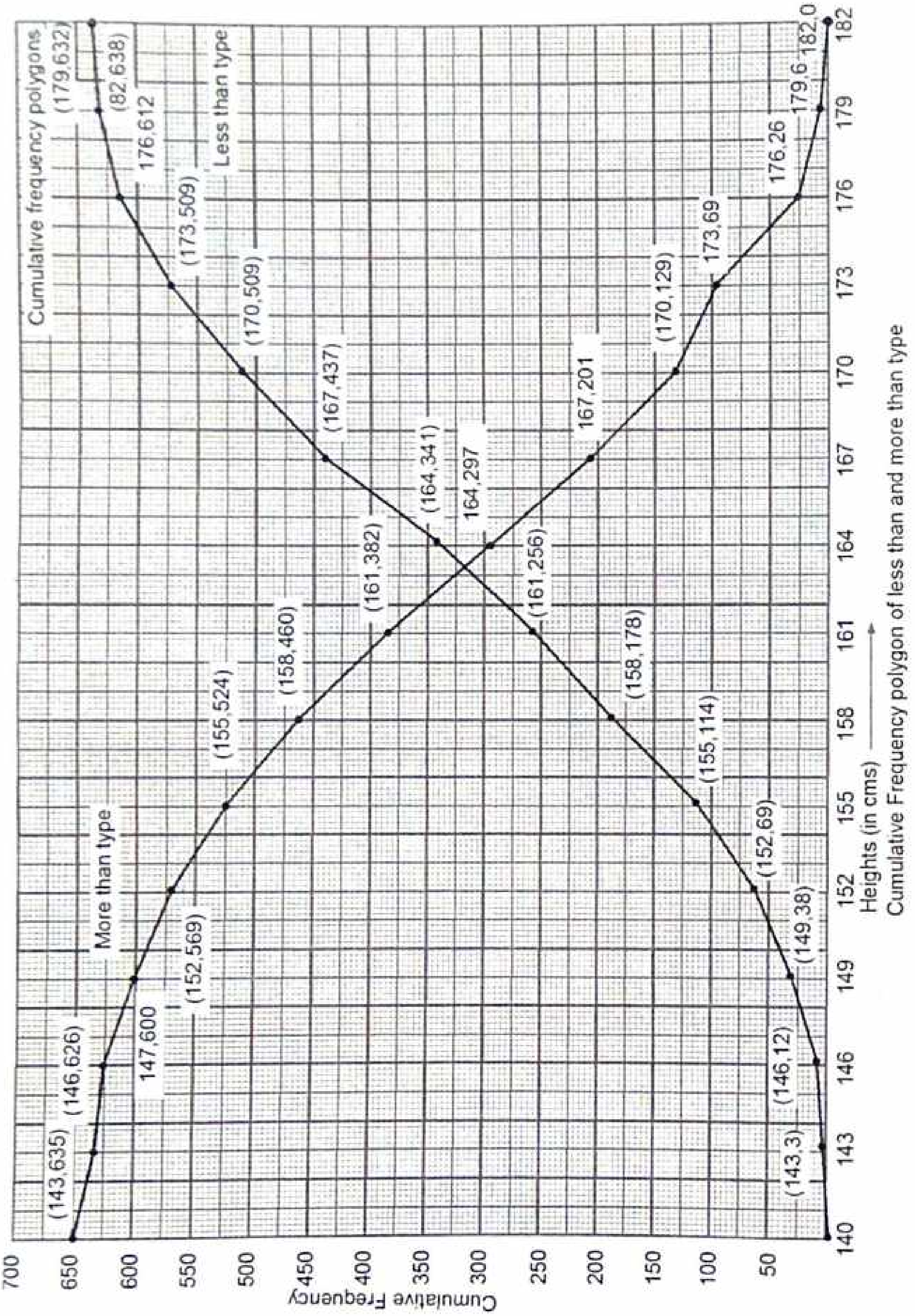

Fig. 15.9

These points are joined by line segments to obtain the cumulative frequency polygon as shown in Fig. 15.9 and by a free hand smooth curve to obtain an ogive by less than method as shown in fig. 15.10.

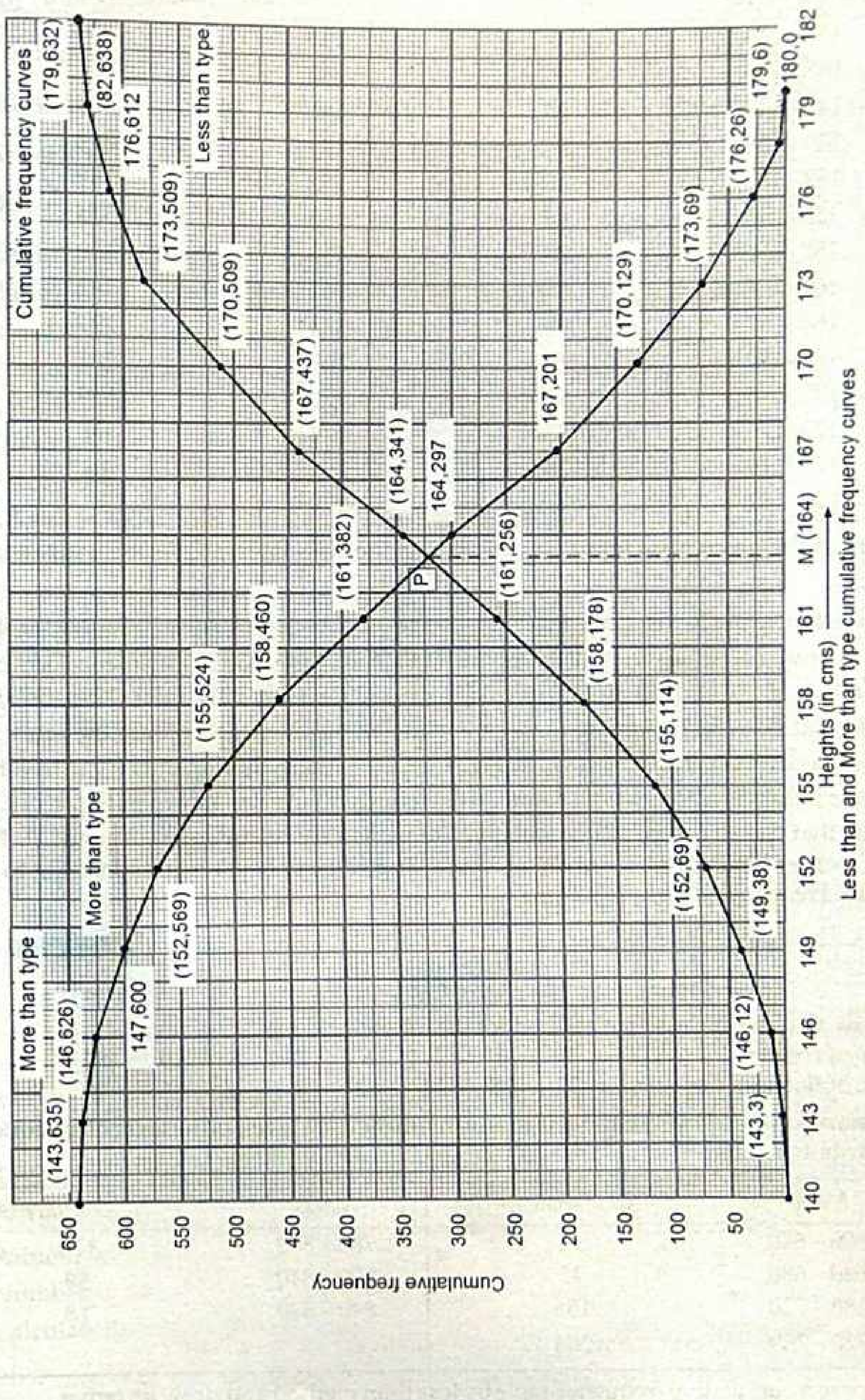
Cumulative frequency curves
More than type
More than type
Less than type
(143,635)
(146,626)
147,600
(152,569)
(155,524)
(158,460)
(161,382)
(164,341)
(167,437)
(170,509)
(173,509)
176,612
(82,638)
(179,632)
(143,3)
(146,12)
(149,38)
(152,69)
(155,114)
(158,178)
(161,256)
164,297
167,201
(170,129)
(173,69)
(176,26)
179,6)
180,0
P
Cumulative frequency
650
600
550
500
450
400
350
300
250
200
150
100
50
140
143
146
149
152
155
158
161
M (164)
167
170
173
176
179
182
Heights (in cms)
Less than and More than type cumulative frequency curves

Fig. 15.10

More than method: We prepare the cmulative frequency table by more than method as given below:

Height in cms	*Frequency*	*Height more than*	*Cumulative frequency*
140 - 143	3	140	638
143 - 146	9	143	635
146 - 149	26	146	626
149 - 152	31	149	600
152 - 155	45	152	569
155 -158	64	155	524
158 -161	78	158	460
161 -164	85	161	382
164 -167	96	164	297
167 -170	72	167	201
170 -173	60	170	129
173 -176	43	173	69
176 -179	20	176	26
179 -182	6	179	6

Other than the given class intervals, we assume the class interval 182-185 with zero frequency.

Now, we mark the lower class limits on X-axis and the cumulative frequencies along Y-axis on suitable scales to plot the points (140, 638), (143, 635), (146, 626), (149, 600), (152, 569), (155, 524), (158, 460), (161, 382), (164, 297), (167, 201), (170, 129), (173, 69), (176, 26) and (179, 6). By joining these points by line segments, we obtain the more than type frequency polygon as shown in Fig. 15.9. By joining these points by a free hand curve, we obtain more than type cumulative frequency curve as shown in Fig. 15.10.

We find that the two types of cumulative frequency curves intersect at point P. From point P perpendicular PM is drawn on X-axis. The value of height corresponding to M is 163.2 cm. Hence, median is 163.2 cm.

EXERCISE 15.6

LEVEL-1

1. Draw an ogive by less than method for the following data:

No. of rooms:	1	2	3	4	5	6	7	8	9	10
No. of houses:	4	9	22	28	24	12	8	6	5	2

2. The marks scored by 750 students in an examination are given in the form of a frequency distribution table:

Marks	*No. of students*	*Marks*	*No. of students*
600 - 640	16	760 - 800	172
640 - 680	45	800 - 840	59
680 - 720	156	840 - 880	18
720 - 760	284		

Prepare a cumulative frequency table by less than method and draw an ogive.

3. Draw an ogive to represent the following frequency distribution:

Class-interval:	0 - 4	5 - 9	10 - 14	15 - 19	20 - 24
No. of students:	2	6	10	5	3

4. The monthly profits (in ₹) of 100 shops are distributed as follows:

Profits per shop:	0 - 50	50 - 100	100 - 150	150 - 200	200 - 250	250 - 300
No. of shops:	12	18	27	20	17	6

Draw the frequency polygon for it.

5. The following distribution gives the daily income of 50 workers of a factory:

Daily income (in ₹):	100 - 120	120 - 140	140 - 160	160 - 180	180 - 200
Number of workers:	12	14	8	6	10

Convert the above distribution to a less than type cumulative frequency distribution and draw its ogive. **[CBSE 2018]**

6. The following table gives production yield per hectare of wheat of 100 farms of a village:

Production yield in kg per hectare:	50 - 55	55 - 60	60 -65	65 - 70	70 - 75	75 - 80
Number of farms:	2	8	12	24	38	16

Draw 'less than' ogive and 'more than' ogive.

7. During the medical check-up of 35 students of a class, their weights were recorded as follows:

Weight (in kg)	*Number of students*
Less than 38	0
Less than 40	3
Less than 42	5
Less than 44	9
Less than 46	14
Less than 48	28
Less than 50	32
Less than 52	35

Draw a less than type ogive for the given data. Hence, obtain the median weight from the graph and verify the result by using the formula. **[CBSE 2009]**

8. The annual rainfall record of a city for 66 days is given in the following table:

Rainfall (in cm):	0 - 10	10 - 20	20 -30	30 - 40	40 - 50	50 - 60
Number of days:	22	10	8	15	5	6

Calculate the median rainfall using ogives of more than type and less than type.

[NCERT EXEMPLAR]

10. In the graphical representation of a frequency distribution, if the distance between mode and mean is k times the distance between median and mean, then write the value of k.
11. Find the class marks of classes 10-25 and 35-55. **[CBSE 2008]**
12. Write the median class of the following distribution:

Classes:	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
Frequency:	4	4	8	10	12	8	4

[CBSE 2009]

ANSWERS

2. zero 3. Median 4. 4 5. Mode = 3 Median − 2 Mean 6. Median
7. 20 - 25 8. 50 9. 40 - 50 10. 3 11. 17.5, 45 12. 30 - 40

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Which of the following is not a measure of central tendency?
   (a) Mean (b) Median (c) Mode (d) Standard deviation
2. The algebraic sum of the deviations of a frequency distribution from its mean is
   (a) always positive (b) always negative (c) 0 (d) a non-zero number
3. The arithmetic mean of 1, 2, 3, ..., n is
   (a) $\frac{n+1}{2}$ (b) $\frac{n-1}{2}$ (c) $\frac{n}{2}$ (d) $\frac{n}{2}+1$
4. For a frequency distribution, mean, median and mode are connected by the relation
   (a) Mode = 3 Mean − 2 Median (b) Mode = 2 Median − 3 Mean
   (c) Mode = 3 Median − 2 Mean (d) Mode = 3 Median + 2 Mean
5. Which of the following cannot be determined graphically?
   (a) Mean (b) Median (c) Mode (d) None of these
6. The median of a given frequency distribution is found graphically with the help of
   (a) Histogram (b) Frequency curve (c) Frequency polygon (d) Ogive
7. The mode of a frequency distribution can be determined graphically from
   (a) Histogram (b) Frequency polygon
   (c) Ogive (d) Frequency curve
8. Mode is
   (a) least frequent value (b) middle most value
   (c) most frequent value (d) None of these
9. The mean of n observations is $\overline{X}$. If the first item is increased by 1, second by 2 and so on, then the new mean is
   (a) $\overline{X}+n$ (b) $\overline{X}+\frac{n}{2}$ (c) $\overline{X}+\frac{n+1}{2}$ (d) None of these
10. One of the methods of determining mode is
    (a) Mode = 2 Median − 3 Mean (b) Mode = 2 Median + 3 Mean
    (c) Mode = 3 Median − 2 Mean (d) Mode = 3 Median + 2 Mean

11. If the mean of the following distribution is 2.6, then the value of y is

Variable (x):	1	2	3	4	5
Frequency:	4	5	y	1	2

(a) 3 (b) 8 (c) 13 (d) 24

12. The relationship between mean, median and mode for a moderately skewed distribution is

(a) Mode = 2 Median − 3 Mean (b) Mode = Median − 2 Mean

(c) Mode = 2 Median − Mean (d) Mode = 3 Median − 2 mean

13. The mean of a discrete frequency distribution $x_i / f_i; i = 1, 2, \ldots, n$ is given by

(a) $\dfrac{\sum f_i x_i}{\sum f_i}$ (b) $\dfrac{1}{n}\sum_{i=1}^{n} f_i x_i$ (c) $\dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} x_i}$ (d) $\dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} i}$.

14. If the arithmetic mean of $x, x+3, x+6, x+9$, and $x+12$ is 10, the $x =$
(a) 1 (b) 2 (c) 6 (d) 4

15. If the median of the data: 24, 25, 26, $x+2$, $x+3$, 30, 31, 34 is 27.5, then $x =$
(a) 27 (b) 25 (c) 28 (d) 30

16. If the median of the data: 6, 7, $x-2$, x, 17, 20, written in ascending order, is 16. Then $x =$
(a) 15 (b) 16 (c) 17 (d) 18

17. The median of first 10 prime numbers is
(a) 11 (b) 12 (c) 13 (d) 14

18. If the mode of the data: 64, 60, 48, x, 43, 48, 43, 34 is 43, then $x + 3 =$
(a) 44 (b) 45 (c) 46 (d) 48

19. If the mode of the data: 16, 15, 17, 16, 15, x, 19, 17, 14 is 15, then $x =$
(a) 15 (b) 16 (c) 17 (d) 19

20. The mean of 1, 3, 4, 5, 7, 4 is m. The numbers 3, 2, 2, 4, 3, 3, p have mean $m - 1$ and median q. Then, $p + q =$
(a) 4 (b) 5 (c) 6 (d) 7

21. If the mean of a frequency distribution is 8.1 and $\Sigma f_i x_i = 132 + 5k$, $\Sigma f_i = 20$, then $k =$
(a) 3 (b) 4 (c) 5 (d) 6

22. If the mean of 6, 7, x, 8, y, 14 is 9, then
(a) $x + y = 21$ (b) $x + y = 19$ (c) $x - y = 19$ (d) $x - y = 21$

23. The mean of n observations is $\overline{x}$. If the first observation is increased by 1, the second by 2, the third by 3, and so on, then the new mean is

(a) $\overline{x} + (2n+1)$ (b) $\overline{x} + \dfrac{n+1}{2}$ (c) $\overline{x} + (n+1)$ (d) $\overline{x} - \dfrac{n+1}{2}$

24. If the mean of first n natural numbers is $\dfrac{5n}{9}$, then $n =$

(a) 5 (b) 4 (c) 9 (d) 10

25. The arithmetic mean and mode of a data are 24 and 12 respectively, then its median is
(a) 25 (b) 18 (c) 20 (d) 22

26. The mean of first n odd natural number is
(a) $\frac{n+1}{2}$ (b) $\frac{n}{2}$ (c) n (d) n^2

27. The mean of first n odd natural numbers is $\frac{n^2}{81}$, then $n =$
(a) 9 (b) 81 (c) 27 (d) 18

28. If the difference of mode and median of a data is 24, then the difference of median and mean is
(a) 12 (b) 24 (c) 8 (d) 36

29. If the arithmetic mean of 7, 8, x, 11, 14 is x, then $x =$
(a) 9 (b) 9.5 (c) 10 (d) 10.5

30. If mode of a series exceeds its mean by 12, then mode exceeds the median by
(a) 4 (b) 8 (c) 6 (d) 10

31. If the mean of first n natural number is 15, then $n =$
(a) 15 (b) 30 (c) 14 (d) 29

32. If the mean of observations $x_1, x_2, ..., x_n$ is $\overline{x}$, then the mean of $x_1 + a, x_2 + a, ..., x_n + a$ is
(a) $a\overline{x}$ (b) $\overline{x} - a$ (c) $\overline{x} + a$ (d) $\frac{\overline{x}}{a}$

33. Mean of a certain number of observations is $\overline{x}$. If each observation is divided by $m\,(m \neq 0)$ and increased by n, then the mean of new observation is
(a) $\frac{\overline{x}}{m} + n$ (b) $\frac{\overline{x}}{n} + m$ (c) $\overline{x} + \frac{n}{m}$ (d) $\overline{x} + \frac{m}{n}$

34. If $u_i = \frac{x_i - 25}{10}$, $\Sigma f_i u_i = 20$, $\Sigma f_i = 100$, then $\overline{x} =$
(a) 23 (b) 24 (c) 27 (d) 25

35. If 35 is removed from the data: 30, 34, 35, 36, 37, 38, 39, 40, then the median increases by
(a) 2 (b) 1.5 (c) 1 (d) 0.5

36. While computing mean of grouped data, we assume that the frequencies are
(a) evenly distributed over all the classes.
(b) centred at the class marks of the classes.
(c) centred at the upper limit of the classes.
(d) centred at the lower limit of the classes.

37. In the formula $\overline{X} = a + h\left(\frac{1}{N}\sum f_i\, u_i\right)$, for finding the mean of grouped frequency distribution $u_i =$
(a) $\frac{x_i + a}{h}$ (b) $h(x_i - a)$ (c) $\frac{x_i - a}{h}$ (d) $\frac{a - x_i}{h}$

38. For the following distribution:

Class:	0-5	5-10	10-15	15-20	20-25
Frequency:	10	15	12	20	9

the sum of the lower limits of the median and modal class is

(a) 15 (b) 25 (c) 30 (d) 35

39. For the following distribution:

Below:	10	20	30	40	50	60
Number of students:	3	12	27	57	75	80

the modal class is

(a) 10-20 (b) 20-30 (c) 30-40 (d) 50-60

40. Consider the following frequency distribution:

Class:	65-85	85-105	105-125	125-145	145-165	165-185	185-205
Frequency:	4	5	13	20	14	7	4

The difference of the upper limit of the median class and the lower limit of the modal class is

(a) 0 (b) 19 (c) 20 (d) 38

41. In the formula $\overline{X} = a + \frac{\Sigma f_i d_i}{\Sigma f_i}$, for finding the mean of grouped data d_i's are deviations from a of

(a) lower limits of classes (b) upper limits of classes

(c) mid-points of classes (d) frequency of the class marks

42. The abscissa of the point of intersection of less than type and of the more than type cumulative frequency curves of a grouped data gives its

(a) mean (b) median (c) mode (d) all the three above

43. Consider the following frequency distribution:

Class:	0-5	6-11	12-17	18-23	24-29
Frequency:	13	10	15	8	11

The upper limit of the median class is

(a) 17 (b) 17.5 (c) 18 (d) 18.5

ANSWERS

1. (d)	2. (c)	3. (a)	4. (c)	5. (a)	6. (d)
7. (a)	8. (c)	9. (c)	10. (c)	11. (b)	12. (d)
13. (a)	14. (d)	15. (b)	16. (c)	17. (b)	18. (c)
19. (a)	20. (d)	21. (d)	22. (b)	23. (b)	24. (c)
25. (c)	26. (c)	27. (b)	28. (a)	29. (c)	30. (b)
31. (d)	32. (c)	33. (a)	34. (c)	35. (d)	36. (b)
37. (c)	38. (b)	39. (c)	40. (c)	41. (c)	42. (b)
43. (b)					

SUMMARY

1. Three measures of central value are:

   (i) Mean (ii) Median and (iii) Mode

2. Mean is computed by following methods:

   (i) Direct method (ii) Short-cut Method (iii) Step-deviation method.

3. If a variate X takes values $x_1, x_2, \dots, x_n$ with corresponding frequencies $f_1, f_2, \dots, f_n$ respectively, then the arithmetic mean of these values is given by

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i, \text{ where } N = \sum_{i=1}^{n} f_i$$

Also, $\overline{X} = A + \frac{1}{N}\sum_{i=1}^{n} f_i d_i$, where $d_i = x_i - A$.

The number A is called the assumed mean.

If $u_i = \dfrac{x_i - A}{h}, i = 1, 2, \dots, n$. Then,

$$\overline{X} = A + h\left\{\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right\}$$

4. The median is the middle value of a distribution i.e. median of a distribution is the value of the variable which divides it into two equal parts.

   The median of a grouped or continuous frequency distribution may be computed by using the following formula:

$$\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h, \text{ where}$$

l = lower limit of the median class

f = frequency of the median class

h = width of the median class.

F = cumulative frequency of the class preceding the median class.

and, $N = \sum_{i=1}^{n} f_i$

5. Mode is the value of the variable which has the maximum frequency. The mode of a continuous or grouped frequency distribution may be computed by using the following formula:

$$\text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h, \text{ where}$$

l = lower limit of the modal class.

f = frequency of the modal class

h = width of the modal class

f_1 = frequency of the class preceding the modal class.

f_2 = frequency of the class following the modal class.

6. Three measures of central value are connected by the following relation:

   Mode = 3 Median – 2 Mean

7. Ogive(s) can be used to find the median of a frequency distribution.

CHAPTER 16

PROBABILITY

16.1 INTRODUCTION

The word 'probability' is commonly used in our day-to-day conversation and we generally use this word even without going into the details of its actual meaning. Generally, people have a rough idea about its meaning. In our day-to-day life we come across statements like:

(i) Probably it may rain today.

(ii) He may possibly join politics.

(iii) Indian Cricket team has good chances of winning World-Cup.

(iv) He is probably right.

In such statements, we generally use the terms: possible, probable, chance, likely etc. All these terms convey the same sense that the event is not certain to take place or, in other words, there is uncertainty about the occurrence (or happening) of the event in question. Thus, in layman's terminology the word 'probability', connotes that there is uncertainty about what has happened or what is going to happen? However, in the theory of probability we assign numerical value to the degree of uncertainty.

The concept of probability originated in the beginning of eighteenth century in problems pertaining to games of chance such as throwing a die, tossing a coin, drawing a card from a pack of cards etc. Starting with games of chance, 'probability' today has become one of the basic tools of Statistics and has vide range of applications in Science and Engineering.

16.2 THEORETICAL APPROACH TO PROBABILITY

What we have learnt in the chapter on probability in class IX was experimental or impirical approach to probability. In this approach, as we have seen, the probabilities were based on actual experiments and adequate recording of the happening of events. In this chapter and in higher classes, we will study about theoretical approach to probability. The basic difference between these two approaches to probability is that in the experimental approach to probability, the probability of an event is based on what has been actually happened while in theoretical approach to probability, we try to predict what will happen without actually performing the experiment.

It has been observed that the experimental probability of an event approaches to its theoretical probability if the number of trials of an experiment is very large.

In the theory of probability we deal with events which are outcomes of an experiment. The word 'experiment' means an operation which can produce some well defined outcome(s): There are two types of experiments: (i) Deterministic (ii) Random or Probabilistic.

Deterministic experiments are those experiments which when repeated under identical conditions produce the same result or outcome. When experiments in science and engineering are repeated under identical conditions, we obtain almost the same result every time.

If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes, then it is known as a random or probabilistic experiment. For example, in tossing of a coin one is not sure if a head or a tail will be obtained, so it is a random experiment. Similarly, rolling an unbiased die and drawing a card from a well shuffled pack of cards are examples of a random experiment.

Throughout this chapter we shall be discussing random experiments and the term experiment will stand for random experiment.

Let us now discuss various terms associated with a random experiment. These terms will help us in introducing the theoretical concept of probability.

ELEMENTARY EVENT *An outcome of a random experiment is called an elementary event.*

Consider the random experiment of tossing of a coin. The possible outcomes of this experiment are head (H) or tail (T).

Thus, if we define

E_1 = Getting head (H) on the upper face of the coin,

and,

E_2 = Getting (T) on the upper face of the coin.

Then, E_1 and E_2 are elementary events associated with the experiments of tossing of a coin.

Let us now consider the random experiment of tossing two coins simultaneously. The possible outcomes of this experiment are as under:

Head on first and Head on second,

Head on first and Tail on second,

Tail on first and Head on second,

Tail on first and Tail on second.

If we define

HH = Getting head on both the coins,

HT = Getting Head on first and tail on second,

TH = Getting tail on first and head on second,

and, TT = Getting tail on both coins.

Then, HH, HT, TH and TT are elementary events associated with the random experiment of tossing of two coins.

Similarly, if three coins are tossed simultaneously, then the elementary events associated with this experiment are $HHH, HHT, HTH, THH, HTT, THT, TTH, TTT$.

Let there be a cubical die marked with numbers 1, 2, 3, 4, 5 and 6 on its six faces. Consider now — the random experiment of throwing a cubical die. If the die is rolled, then any one of the six faces may come upward. So, there are six possible outcomes of this experiment, namely 1, 2, 3, 4, 5, 6. Thus, if we define

E_i = Getting a face marked with number i, where $i = 1, 2, \ldots 6$.

Then, $E_1, E_2 \ldots, E_6$ are six elementary events associated to this experiment.

In this experiment, elementary event E_i is generally denoted by i, where $i = 1, 2, 3, \ldots, 6$.

Now, consider the random experiment in which two six-faced dice are rolled together or a die is rolled twice. If (i, j) denotes the outcome of getting number i on first die and number j on second die, then possible outcomes of this experiment are:

$$\begin{array}{cccccc}
(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\
(3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\
(4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\
(5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\
(6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6).
\end{array}$$

Clearly, these outcomes are elementary events associated with the random experiment of throwing two six faced dice together. The total number of these elementary events is 36.

If a card is drawn from a well shuffled pack of 52 cards, then any one of 52 cards can be the outcome. So, there are 52 elementary events associated to the random experiment of drawing a card from a pack of 52 playing cards.

COMPOUND EVENT *An event associated to a random experiment is a compound event if it is obtained by combining two or more elementary events associated to the random experiment.*

In a single throw of a die, the event "Getting an even number" is a compound event as it is obtained by combining three elementary events, namely, 2, 4, 6.

Similarly, "Getting an odd number" is a compound event in a single throw of a die.

Consider the random experiment of tossing two coins simultaneously. If we define the event "Getting exactly one head", then HT and TH are two elementary events associated to it. So, it is a compound event.

OCCURRENCE OF AN EVENT *An event A associated to a random experiment is said to occur if any one of the elementary events associated to the event A is an outcome.*

Consider the random experiment of throwing an unbiased die. Let A denote the event "Getting an even number". Elementary events associated to this event are: 2, 4, 6. Now, suppose that in a trial the outcome is 4, then we say that the event A has occurred. In another trial, let the outcome be 3, then we say that the event A has not occurred.

Let a die be rolled and the outcome of the trial be 4. Then, we can say that each of the following events have occurred:

(i) Getting a number greater than or equal to 2,
(ii) Getting a number less than or equal to 5,
(iii) Getting an even number.

On the basis of the same outcome, we can also say that the following events have not occurred:

(i) Getting an odd number,
(ii) Getting a multiple of 3.

Let us now consider the random experiment of throwing a pair of dice. If (2, 6) is an outcome of a trial, then we can say that each of the following events have occurred.

(i) Getting an even number on first die.
(ii) Getting an even number on both dice
(iii) Getting 8 as the sum of the numbers on two dice.

However, on the basis of the same outcome, one can also say that the following events have not occurred:

(i) Getting a multiple of 3 on first die.

(ii) Getting an odd number on first die.

(iii) Getting a doublet.

FAVOURABLE ELEMENTARY EVENTS *An elementary event is said to be favourable to a compound event A, if it satisfies the definition of the compound event A.*

In other words, an elementary event E is favourable to a compound event A, if we say that the event A occurs when E is an outcome of a trial.

Consider the random experiment of throwing a pair of dice and the compound event A defined by "Getting 8 as the sum." We observe that the event A occurs if we get any one of the following elementary events as outcome:

$$(2, 6),\ (6, 2),\ (3, 5),\ (5, 3),\ (4,4)$$

So, there are 5 elementary events favourable to event A.

If two coins are tossed simultaneously and A is an event associated to it defined as "Getting exactly one head". We say that the event A occurs if we get either HT or TH as an outcome. So, there are two elementary events favourable to the event A.

NEGATION OF AN EVENT Corresponding to every event A associated with a random experiment we define an event "not A" which occurs when and only when A does not occur. The event "not A" is called the negation of event A and is denoted by $\overline{A}$.

Clearly, event A occurs if and only if $\overline{A}$ does not occur.

16.3 THEORETICAL PROBABILITY OF AN EVENT

DEFINITION *If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of event A is denoted by P (A) and is defined as the ratio* $\frac{m}{n}$.

Thus, $\quad P(A) = \frac{m}{n}$

Clearly, $\quad 0 \le m < n$.

$\therefore \quad 0 \le \frac{m}{n} \le 1$

$\Rightarrow \quad 0 \le P(A) \le 1$

If $P(A) = 1$, then A is called a certain event and A is called an impossible event, if $P(A) = 0$.

If m elementary events are favourable to an event A out of n elementary events, then the number of elementary events which ensure the non-occurrence of A *i.e.* the occurrence of $\overline{A}$ is $n - m$.

$$\therefore \quad P(\overline{A}) = \frac{n-m}{n}$$

$$\Rightarrow \quad P(\overline{A}) = 1 - \frac{m}{n}$$

$$\Rightarrow \quad P(\overline{A}) = 1 - P(A)$$

$$\Rightarrow \quad P(A) + P(\overline{A}) = 1$$

The odds in favour of the occurrence of the event A are defined by $m : n - m$ i.e., $P(A) : P(\overline{A})$ and the odds against the occurrence of A are defined by $(n - m) : m$ *i.e.*, $P(\overline{A}) : P(A)$.

Let us now discuss some problems to illustrate the above definition.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *An unbiased die is thrown. What is the probability of getting:*

(i) *an even number* (ii) *a multiple of 3* [CBSE 2008]

(iii) *an even number or a multiple of 3* [CBSE 2013]

(iv) *an even number and a multiple of 3* (v) *a number 3 or 4*

(vi) *an odd number* [NCERT] (vii) *a number less than 5*

(viii) *a number greater than 3* (ix) *a number between 3 and 6.*

SOLUTION In a single throw of a die we can get any one of the six numbers 1, 2, ..., 6 marked on its six faces. Therefore, the total number of elementary events associated with the random experiment of throwing a die is 6.

(i) Let A denote the event "Getting an even number"
Clearly, event A occurs if we obtain any one of 2, 4, 6 as an outcome.

$\therefore$ Favourable number of elementary events $= 3$

Hence, $P(A) = \frac{3}{6} = \frac{1}{2}$

(ii) Let A denote the event "Getting a multiple of 3"
We observe that the event A occurs if we obtain either 3 or 6 as an outcome.

$\therefore$ Favourable number of elementary events $= 2$

Hence, $P(A) = \frac{2}{6} = \frac{1}{3}$

(iii) An even number or a multiple of 3 is obtained if we obtain one of the numbers 2, 3, 4, 6 as an outcome.

$\therefore$ Favourable number of elementary events $= 4$

Hence, required probability $= \frac{4}{6} = \frac{2}{3}$

(iv) Let A denote the event "Getting an even number and a multiple of 3"
Clearly, Event A happens if we get 6 as an outcome

$\therefore$ Favourable number of elementary events $= 1$

Hence, required probability $= \frac{1}{6}$

(v) Let A denote the event "Getting 3 or 4"
Clearly, A occurs when we get either 3 or 4 as an outcome

$\therefore$ Favourable number of elementary events $= 2$

Hence, $P(A) = \frac{2}{6} = \frac{1}{3}$

(vi) Let A denote the event "Getting an odd number"
We observe that the event A occurs when we get 1 or 3 or 5 as an outcome.

$\therefore$ Favourable number of elementary events $= 3$

Hence, $P(A) = \frac{3}{6} = \frac{1}{2}$

(vii) The event "Getting a number less than 5" will occur if we get one of the numbers 1, 2, 3, 4 as an outcome.

$\therefore$ Favourable number of outcomes = 4

Hence, required probability $= \frac{4}{6} = \frac{2}{3}$

(viii) The event "Getting a number greater than 3" will occur if we obtain one of the numbers 4, 5, 6 as an outcome.

$\therefore$ Favourable number of outcomes = 3

Hence, required probability $= \frac{3}{6} = \frac{1}{2}$

(ix) The event "Getting a number between 3 and 6" occurs if we obtain either 4 or 5 as an outcome.

$\therefore$ Favourable number of outcomes = 2

Hence, required probability $= \frac{2}{6} = \frac{1}{3}$

EXAMPLE 2 *Two unbiased coins are tossed simultaneously. Find the probability of getting*

(i) *two heads* (ii) *one head*

(iii) *one tail* [CBSE 2014] (iv) *at least one head* [CBSE 2010, 2013]

(v) *at most one head.* (vi) *no head*

SOLUTION If two unbiased coins are tossed simultaneously we obtain any one of the following as an outcome.

HH, HT, TH, TT

$\therefore$ Total number of elementary events = 4

(i) Two heads are obtained if the elementary event HH occurs.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{4}$

(ii) One head is obtained if any one of the following elementary events occurs:

HT, TH

Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{4} = \frac{1}{2}$

(iii) One tail is obtained if any one of the following elementary events occurs: TH, HT

$\therefore$ Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{4} = \frac{1}{2}$

(iv) At least one head is obtained if any one of the following elementary events happens:

HH, HT, TH

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{4}$

(v) If one of the elementary events HT, TH, TT occurs, then we say that at most one head is obtained.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{4}$

(vi) No head is obtained if the elementary event TT occurs

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{4}$

EXAMPLE 3 *Three unbiased coins are tossed together. Find the probability of getting:*

(i) *all heads* (ii) *two heads*

(iii) *one head* (iv) *at least two heads* **[CBSE 2015]**

SOLUTION Elementary events associated to random experiment of tossing three coins are

$HHH, HHT, HTH, THH, HTT, THT, TTH, TTT$

$\therefore$ Total number of elementary events = 8.

(i) The event "Getting all heads" is said to occur, if the elementary event HHH occurs *i.e.* HHH is an outcome.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{8}$

(ii) The event "Getting two heads" will occur, if one of the elementary events HHT, THH, HTH occurs.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$

(iii) The events of getting one head, when three coins are tossed together, occurs if one of the elementary events HTT, THT, TTH happens.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$

(iv) If any of the elementary events HHH, HHT, HTH, and THH is an outcome, then we say that the event "Getting at least two heads" occurs.

$\therefore$ Favourable number of elementary events = 4

Hence, required probability $= \frac{4}{8} = \frac{1}{2}$

EXAMPLE 4 *Tickets numbered from 1 to 20 are mixed up together and then a ticket is drawn at random. What is the probability that the ticket has a number which is a multiple of 3 or 7?*

SOLUTION Out of 20 tickets numbered from 1 to 20, one can be chosen in 20 ways. So, total number of elementary events associated with the given random experiment is 20. Out of 20 tickets numbered 1 to 20, tickets bearing numbers which are multiple of 3 or 7 bear numbers 3, 6, 7, 9, 12, 14, 15 and 18.

$\therefore$ Favourable numbers of elementary events = 8

Hence, required probability $= \frac{8}{20} = \frac{2}{5}$

REMARK *A pack of playing cards consists of 52 cards which are divided into 4 suits of 13 cards each. Each suit consists of one ace, one king, one queen, one jack and 9 other cards numbered from 2 to 10. Four suits are named as spades (♠), hearts (♥), diamonds (♦), and clubs (♣).*

EXAMPLE 5 *One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is:*

(i) *an ace* (ii) *red* (iii) *either red or king*

(iv) *red and a king* (v) *a face card* (vi) *a red face card*

(vii) *'2' of spades* (viii) *'10' of a black suit*

SOLUTION Out of 52 cards, one card can be drawn in 52 ways.
So, total number of elementary events = 52.

(i) There are four ace cards in a pack of 52 cards. So, one ace can be chosen in 4 ways.

$\therefore$ Favourable number of elementary events = 4

Hence, required probability $= \frac{4}{52} = \frac{1}{13}$

(ii) There are 26 red cards in a pack of 52 cards. Out of 26 red cards one card can be chosen in 26 ways.

$\therefore$ Favourable number of elementary events = 26.

Hence, required probability $= \frac{26}{52} = \frac{1}{2}$

(iii) There are 26 red cards, including two red kings, in a pack of 52 playing cards. Also, there are 4 kings, two red and two black. Therefore, card drawn will be a red card or a king if it is any one of 28 cards (26 red cards and 2 black kings).

$\therefore$ Favourable number of elementary events = 28

Hence, required probability $= \frac{28}{52} = \frac{7}{13}$

(iv) A card drawn will be red as well as king, if it is a red king. There are 2 red kings in a pack of 52 playing cards.

$\therefore$ Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{52} = \frac{1}{26}$

(v) In a deck of 52 cards: aces, kings, queens, and jacks are called face cards. Thus, there are 12 face cards. So, one face card can be chosen in 12 ways.

$\therefore$ Favourable number of elementary events = 12

Hence, required probability $= \frac{12}{52} = \frac{3}{13}$

(vi) There are 6 red face cards 3 each from diamonds and hearts. Out of these 6 red face cards one card can be chosen in 6 ways.

$\therefore$ Favourable number of elementary events = 6

Hence, required probability $= \frac{6}{52} = \frac{3}{26}$

(vii) There is only one '2' of spades.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{52}$

(viii) There are two suits of black cards viz. spades and clubs. Each suit contains one card bearing number 10.

$\therefore$ Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{52} = \frac{1}{26}$

EXAMPLE 6 *A bag contains 3 red and 2 blue marbles. A marble is drawn at random. What is the probability of drawing a blue marble?*

SOLUTION There 5 marbles in the bag. Out of these 5 marbles one can be chosen in 5 ways.

$\therefore$ Total number of elementary events = 5

Since the bag contains 2 blue marbles. Therefore, one blue marble can be drawn in 2 ways.

$\therefore$ Favourable number of elementary events $= 2$

Hence, P (Getting a blue marble) $= \dfrac{2}{5}$

EXAMPLE 7 *It is know that a box of 600 electric bulbs contains 12 defective bulbs. One bulb is taken out at random from this box. What is the probability that it is a non-defective bulb?*

SOLUTION Out of 600 electric bulbs one bulb can be chosen in 600 ways.

$\therefore$ Total number of elementary events $= 600$

There are 588 ($= 600 - 12$) non-defective bulbs out of which one bulb can be chosen in 588 ways.

$\therefore$ Favourable number of elementary events $= 588$

Hence, P (Getting a non-defective bulb) $= \dfrac{588}{600} = \dfrac{49}{50} = 0.98$

EXAMPLE 8 *17 cards numbered 1, 2, 3, ..., 17 are put in a box and mixed thoroughly. One person draws a card from the box. Find the probability that the number on the card is:*

(i) *odd* (ii) *a prime*

(iii) *divisible by 3* (iv) *divisible by 3 and 2 both*

SOLUTION Out of 17 cards, in the box, one card can be drawn in 17 ways.

$\therefore$ Total number of elementary events $= 17$

(i) There 9 odd numbered cards, namely, 1, 3, 5, 7, 9, 11, 13, 15, 17. Out of these 9 cards one card can be drawn in 9 ways.

$\therefore$ Favourable number of elementary events $= 9$

Hence, required probability $= \dfrac{9}{17}$

(ii) There are 7 prime numbered cards, namely, 2, 3, 5, 7, 11, 13, 17. Out of these 7 cards one card can be chosen in 7 ways.

$\therefore$ Favourable number of elementary events $= 7$

Hence, P (Getting a prime number) $= \dfrac{7}{17}$

(iii) Let A denote the event of getting a card bearing a number divisible by 3.

Clearly, event A occurs if we get a card bearing one of the numbers 3, 6, 9, 12, 15.

$\therefore$ Favourable number of elementary events $= 5$

Hence, P (Getting a card bearing a number divisible by 3) $= \dfrac{5}{17}$

(iv) If a number is divisible by both 3 and 2, then it is a multiple of 6. In cards bearing number 1, 2, 3, ..., 17 there are only 2 cards which bear a number divisible by 3 and 2 both *i.e.* by 6. These cards bear numbers 6 and 12.

$\therefore$ Favourable number of elementary events $= 2$

Hence, P (Getting a card bearing a number divisible by 3 and 2) $= \dfrac{2}{17}$

EXAMPLE 9 *1000 tickets of a lottery were sold and there are 5 prizes on these tickets. If Saket has purchased one lottery ticket, what is the probability of winning a prize?*

SOLUTION Out of 1000 lottery tickets one ticket can be chosen in 1000 ways.

$\therefore$ Total number of elementary events $= 1000$

It is given that there are 5 prizes on these 1000 tickets. Therefore,

Number of ways of selecting a prize ticket = 5

$\therefore$ Favourable number of elementary events = 5

Hence, P (Winning a prize) $= \dfrac{5}{1000} = 0.005$

EXAMPLE 10 *A child has a block in the shape of a cube with one letter written on each face as shown below:*

A	B	C	D	E	A

The cube is thrown once. What is the probability of getting (i) A? (ii) D? [NCERT]

SOLUTION In throwing the cube any one of the six faces may come upward.

$\therefore$ Total number of elementary events = 6

(i) There are two faces bearing letter A.

$\therefore$ Favourable number of elementary events = 2

Hence, P (Getting A) $= \dfrac{2}{6} = \dfrac{1}{3}$

(ii) There is only one face bearing letter D.

$\therefore$ Favourable number of elementary events = 1

Hence, P (Getting D) $= \dfrac{1}{6}$

EXAMPLE 11 *A bag contains 5 red balls, 8 white balls, 4 green balls and 7 black balls. If one ball is drawn at random, find the probability that it is:*

(i) *black* (ii) *red* (iii) *not green.*

SOLUTION Total number of balls in the bag $= 5 + 8 + 4 + 7 = 24$

$\therefore$ Total number of elementary events = 24

(i) There are 7 black balls in the bag.

$\therefore$ Favourable number of elementary events = 7

Hence, P (Getting a black ball) $= \dfrac{7}{24}$

(ii) There are 5 red balls in the bag.

$\therefore$ Favourable number of elementary events = 5

Hence, P (Getting a red ball) $= \dfrac{5}{24}$

(iii) There are $5 + 8 + 7 = 20$ balls which are not green.

$\therefore$ Favourable number of elementary events = 20

Hence, P (Not getting a green ball) $= \dfrac{20}{24} = \dfrac{5}{6}$

EXAMPLE 12 *Savita and Hamida are friends. What is the probability that both will have (i) the same birthday? (ii) different birthdays? (ignoring a leap year).* [NCERT]

SOLUTION Savita may have any one of the 365 days of the year as her birthday. Similarly, Hamida may have any one of 365 days of the year as her birthday.

$\therefore$ Total number of ways in which Savita and Hamida may have their birthday $= 365 \times 365$

(i) Savita and Hamida may have same birthday on any one of 365 days of the year.

$\therefore$ Number of ways in which Savita and Hamida will have same birthday = 365

$\therefore$ Probability that Savita and Hamida will have the same birthday $= \dfrac{365}{365 \times 365} = \dfrac{1}{365}$

(ii) We have,

Probability that Savita and Hamida will have different birthdays

$= 1 -$ Probability that Savita and Hamida will have the same birthday

$$= 1 - \frac{1}{365} = \frac{364}{365}$$

EXAMPLE 13 *A letter is chosen at random from the letters of the word 'ASSASSINATION'. Find the probability that the letter chosen is a (i) vowel (ii) consonant.*

SOLUTION There are 13 letters in the word 'ASSASSINATION' out of which one letter can be chosen in 13 ways.

$\therefore$ Total number of elementary events = 13

(i) There are 6 vowels in the word 'ASSASSINATION'. So, there are 6 ways of selecting a vowel.

$\therefore$ Probability of selecting a vowel $= \frac{6}{13}$

(ii) We have,

Probability of selecting a consonant

$= 1 -$ Probability of selecting a vowel $= 1 - \frac{6}{13} = \frac{7}{13}$

EXAMPLE 14 *Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish. What is the probability that the fish taken out is a male fish?* [NCERT]

SOLUTION There are 13 (= 8 + 5) fish out of which one can be chosen in 13 ways.

$\therefore$ Total number of elementary events = 13

There are 5 male fish out of which one male fish can be chosen in 5 ways.

$\therefore$ Favourable number of elementary events = 5

Hence, required probability $= \frac{5}{13}$

EXAMPLE 15 *A piggy bank contains hundred 50 paisa coins, fifty ₹1 coins, twenty ₹2 coins and ten ₹5 coins. If it is equally likely that one of the coins will fallout when the bank is turned up side down, what is the probability that the coin (i) will be a 50 paisa coin? (ii) will not be a ₹ 5 coins?* [NCERT]

SOLUTION Total number of coins = 100 + 50 + 20 + 10 = 180

So, one coin can be chosen out of 180 coins in 180 ways.

$\therefore$ Total number of elementary events = 180

(i) There are 100 fifty paisa coins out of which one coin can be chosen in 100 ways.

$\therefore$ Probability that a 50 paisa coin will fall $= \frac{100}{180} = \frac{5}{9}$

(ii) Other than ₹ 5 coins there are 170 coins.

$\therefore$ Probability that coin fallen out is not a ₹ 5 coin $= \frac{170}{180} = \frac{17}{18}$

EXAMPLE 16 *A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (See Fig. 16.1), and these are equally likely outcomes. What is the probability that it will point at (i) 8? (ii) an odd number? (iii) a number greater than 2? (iv) a number less than 9?* [NCERT, CBSE 2016]

SOLUTION Since the arrow can come to rest at any one of the numbers 1, 2, 3, 4, 5, 6, 7, 8.

$\therefore$ Total number of elementary events = 8

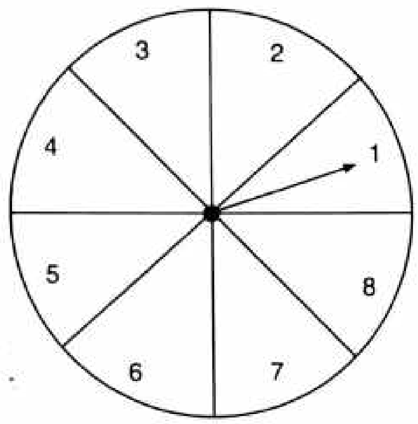

Fig. 16.1

(i) We have,

$$P\text{ (Arrow points at 8)} = \frac{1}{8}$$

(ii) There are 4 odd numbers, namely 1, 3, 5, 7.

$$\therefore \quad P\text{ (Arrow points at an odd number)} = \frac{4}{8} = \frac{1}{2}$$

(iii) There are 6 numbers greater than 2, namely, 3, 4, 5, 6, 7, 8.

$$\therefore \quad P\text{ (Arrow points at a number greater than 2)} = \frac{6}{8} = \frac{3}{4}$$

(iv) We have,

$$\therefore \quad P\text{ (Arrow points at a number less than 9)} = P\text{ (Arrow points at any number)} = \frac{8}{8} = 1.$$

EXAMPLE 17 *A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game.* [NCERT, CBSE 2016, 2017]

SOLUTION When a coin is tossed three times, possible outcomes are

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

$\therefore$ Total number of elementary events = 8

Hanif will lose the game if all the tosses do not give the same result i.e. all heads or all tails. So, favourable outcomes are: HHT, HTH, THH, TTH, HTT, THT,

$\therefore$ Favourable number of elementary events = 6

Hence, $P\text{ (Hanif will lose the game)} = \dfrac{6}{8} = \dfrac{3}{4}$

EXAMPLE 18 *A jar contains 24 marbles some are green are others are blue. If a marble is drawn at random from the jar, the probability that it is green is 2/3. Find the number of blue marbles in the jar.*

[NCERT]

SOLUTION Let there be b blue marbles and g green marbles in the jar. Then,

Total number of marbles in the jar $= b + g$

It is given that there are 24 marbles in the jar.

$$\therefore \quad b + g = 24 \qquad \ldots(i)$$

$$P\text{ (Getting a green marble from the jar)} = \frac{g}{24}$$

$$\Rightarrow \quad \frac{2}{3} = \frac{g}{24} \Rightarrow g = 24 \times \frac{2}{3} = 16$$

Putting $g = 16$ in (i), we get $b = 8$

Hence, there are 8 blue marbles in the jar.

EXAMPLE 19 *A jar contains 54 marbles each of which is blue, green or white. The probability of selecting a blue marble at random from the jar is $\frac{1}{3}$, and the probability of selecting a green marble at random is $\frac{4}{9}$. How many white marbles does the jar contain?*

SOLUTION Let there be b blue, g green and w white marbles in the jar. Then,

$$b + g + w = 54 \qquad \ldots(i)$$

$$\therefore \quad P\text{ (Selecting a blue marble)} = \frac{b}{54}$$

It is given that the probability of selecting a blue marble is $\frac{1}{3}$.

$$\therefore \quad \frac{1}{3} = \frac{b}{54} \Rightarrow b = 18$$

We have,

$$P\text{ (Selecting a green marble)} = \frac{4}{9}$$

$$\Rightarrow \quad \frac{g}{54} = \frac{4}{9} \qquad \left[\because P\text{ (Selecting a green marble)} = \frac{4}{9}\text{ (Given)}\right]$$

$$\Rightarrow \quad g = 24$$

Substituting the values of b and g in (i), we get

$$18 + 24 + w = 54 \Rightarrow w = 12$$

Hence, the jar contains 12 white marbles.

LEVEL-2

EXAMPLE 20 *Two dice are thrown simultaneously. Find the probability of getting:*

(i) *an even number as the sum*
(ii) *the sum as a prime number*
(iii) *a total of at least 10*
(iv) *a doublet of even number*
(v) *a multiple of 2 on one dice and a multiple of 3 on the other.*
(vi) *same number on both dice i.e. a doublet.* [CBSE 2008, 2013 2018]
(vii) *a multiple of 3 as the sum.*

SOLUTION Elementary events associated to the random experiment of throwing two dice are:

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6).

$\therefore$ Total number of elementary events $= 6 \times 6 = 36$

(i) Let A be the event of getting an even number as the sum *i.e.* 2, 4, 6, 8, 10, 12
Elementary events favourable to event A are:
(1, 1), (1, 3), (3, 1), (2, 2), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (2, 6), (6, 2), (4, 4), (5, 3), (3, 5), (5, 5), (6, 4), (4, 6) and (6, 6).
Clearly, favourable number of elementary events $= 18$

Hence, required probability $= \frac{18}{36} = \frac{1}{2}$

(ii) Let A be the event of getting the sum as a prime number *i.e.* 2, 3, 5, 7, 11
Elementary events favourable to event A are:
(1, 1), (1, 2), (2, 1), (1, 4), (4, 1), (2, 3), (3, 2), (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), (6, 5) and (5, 6).
$\therefore$ Favourable number of elementary events $= 15$

Hence, required probability $= \frac{15}{36} = \frac{5}{12}$

(iii) Let A be the event of getting a total of at least 10 *i.e.* 10, 11, 12. Then, the elementary events favourable to A are:
(6, 4), (4, 6), (5, 5), (6, 5), (5, 6) and (6, 6).
$\therefore$ Favourable number of elementary events $= 6$

Hence, required probability $= \frac{6}{36} = \frac{1}{6}$

(iv) Let A be the event of getting a doublet of even number. Then, the elementary events favourable to A are (2, 2), (4, 4) and (6, 6)
$\therefore$ Favourable number of elementary events $= 3$

Hence, required probability $= \frac{3}{36} = \frac{1}{12}$

(v) Let A be the event of getting a multiple of 2 on one die and a multiple of 3 on the other. Then, the elementary events favourable to A are:
(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2), (6, 4).
$\therefore$ Favourable number of elementary events $= 11$

Hence, required probability $= \frac{11}{36}$

(vi) Let A be the event of getting the same number on both dice. Then, elementary events favourable to A are:

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6)

$\therefore$ Favourable number of elementary events = 6

Hence, required probability $= \frac{6}{36} = \frac{1}{6}$

(vii) Let A be the event of getting a multiple of 3 as the sum *i.e.* 3, 6, 9, 12. Then, elementary events favourable to A are:

(1, 2), (2, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (3, 6), (6, 3), (5, 4), (4, 5), (6, 6).

$\therefore$ Favourable number of elementary events = 12.

Hence, required probability $= \frac{12}{36} = \frac{1}{3}$

EXAMPLE 21 *Peter throws two different dice together and finds the product of the two numbers obtained. Rina throws a die and squares the number obtained. Who has the better chance to get the number 25?* **[CBSE 2017]**

SOLUTION The person having higher probability of getting the number 25 has the better chance.

When a pair of dice is thrown, there are 36 elementary events as given below:

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

So, the product of numbers on two dice can take values 1, 2, 3, ..., 36.

The product of two numbers on two dice will be 25 if both the dice show number 5. So, there is only one elementary event, viz., (5, 5), favourable to getting 25.

$\therefore$ p_1 = Probability that Peter throws $25 = \frac{1}{36}$.

Rina throws a die on which she can get any one of the six numbers 1, 2, 3, 4, 5, 6 as an outcome. If she gets number 5 on the upper face of the die thrown, then the square of the number is 25.

$\therefore$ p_2 = Probability that the square of number obtained is $25 = \frac{1}{6}$.

Clearly, $p_2 > p_1$. So, Rina has better chance to get the number 25.

EXAMPLE 22 *Find the probability that a leap year selected at random will contain 53 Sundays.*

SOLUTION In a leap year there are 366 days and, 366 days = 52 weeks and 2 days.

Thus, a leap year has always 52 Sundays.

The remaining 2 days can be:

(i) Sunday and Monday
(ii) Monday and Tuesday
(iii) Tuesday and Wednesday
(iv) Wednesday and Thursday
(v) Thursday and Friday
(vi) Friday and Saturday
(vii) Saturday and Sunday.

Clearly, there are seven elementary events associated with this random experiment.

Let A be the event that a leap year has 53 Sundays.

Clearly, the event A will happen if the last two days of the leap year are either Sunday and Monday or Saturday and Sunday.

$\therefore$ Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{7}$

EXAMPLE 23 *What is the probability that a number selected from the numbers 1, 2, 3, ..., 25 is a prime number, when each of the given numbers is equally likely to be selected?*

SOLUTION Out of 25 numbers 1, 2, 3, ..., 25 one number can be chosen in 25 ways.

$\therefore$ Total number of elementary events = 25

The number selected will be a prime number if it is chosen from the numbers 2, 3, 5, 7, 11, 13, 17, 19, 23.

$\therefore$ Favourable number of elementary events = 9

Hence, required probability $= \frac{9}{25}$

EXAMPLE 24 *The king, queen and jack of clubs are removed from a deck of 52 playing cards and the well shuffled. One card is selected from the remaining cards. Find the probability of getting.*

(i) *a heart* (ii) *a king*

(iii) *a club* (iv) *the '10' of hearts.*

SOLUTION After removing king, queen and jack of clubs from a deck of 52 playing cards there are 49 cards left in the deck. Out of these 49 cards one card can be chosen in 49 ways.

$\therefore$ Total number of elementary events = 49

(i) There are 13 heart cards in the deck containing 49 cards out of which one heart card can be chose in 13 ways.

$\therefore$ Favourable number of elementary events = 13

Hence, P (Getting a heart) $= \frac{13}{49}$

(ii) There are 3 kings in the deck containing 49 cards. Out of these three kings one king can be chosen in 3 ways.

$\therefore$ Favourable number of elementary events = 3

Hence, P (Getting a king) $= \frac{3}{49}$

(iii) After removing king, queen and jack of clubs only 10 club cards are left in the deck. Out of these 10 club cards one club card is chosen in 10 ways.

$\therefore$ Favourable number of elementary events = 10

Hence, P (Getting a club) $= \frac{10}{49}$

(iv) There is only one '10' of hearts.

$\therefore$ Favourable number of elementary events = 1

Hence, P (Getting the '10' to hearts) $= \frac{1}{49}$

EXAMPLE 25 *A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that of a red ball, find the number of blue balls in the bag.* [NCERT]

SOLUTION Let there be x blue balls in the bag.

$\therefore$ Total number of balls in the bag $= (5 + x)$

Now,

$$p_1 = \text{Probability of drawing a blue ball} = \frac{x}{5+x}$$

$$p_2 = \text{Probability of drawing a red ball} = \frac{5}{5+x}$$

It is given that

$$p_1 = 2p_2$$

$$\Rightarrow \quad \frac{x}{5+x} = 2 \times \frac{5}{5+x} \Rightarrow \frac{x}{5+x} = \frac{10}{5+x} \Rightarrow x = 10$$

Hence, there are 10 blue balls in the bag.

EXAMPLE 26 *A bag contains 12 balls out of which x are white.*

(i) *If one ball is drawn at random, what is the probability that it will be a white ball?*

(ii) *If 6 more white balls are put in the bag, the probability of drawing a white ball will be double than that in (i). Find x.* NCERT]

SOLUTION (i) There are 12 balls in the bag. Out of these 12 balls one can be chosen in 12 ways.

$\therefore$ Total number of elementary events = 12

There are x white balls out of which one can be chosen in x ways.

$\therefore$ Favourable number of elementary events $= x$

Hence, $p_1 = P(\text{Getting a white ball}) = \dfrac{x}{12}$

(ii) If 6 more white balls are put in the bag, then

Total number of balls in the bag $= 12 + 6 = 18$

Number of white balls in the bags $= x + 6$

$$\therefore \quad p_2 = P\text{ (Getting a white ball)} = \frac{x+6}{18}$$

It is given that

$$p_2 = 2\,p_1$$

$$\Rightarrow \quad \frac{x+6}{18} = 2 \times \frac{x}{12}$$

$$\Rightarrow \quad \frac{x+6}{18} = \frac{x}{6} \Rightarrow 6(x+6) = 18x \Rightarrow 6x + 36 = 18x \Rightarrow 12x = 36 \Rightarrow x = \frac{36}{12} = 3$$

EXAMPLE 27 *Cards marked with the numbers 2 to 101 are placed in a box and mixed thoroughly. One card is drawn from this box. Find the probability that the number on the card is:*

(i) *an even number*

(ii) *a number less than 14*

(iii) *a number which is a perfect square*

(iv) *a prime number less than 20.*

SOLUTION There are 100 cards in the box out of which one card can be drawn in 100 ways.

$\therefore$ Total number of elementary events = 100

(i) From numbers 2 to 101, there are 50 even numbers, namely, 2, 4, 6, 8, ..., 100. Out of these 50 even numbered cards, one card can be chosen in 50 ways.

$\therefore$ Favourable number of elementary events = 50

Hence, P (Getting an even numbered card) $= \frac{50}{100} = \frac{1}{2}$

(ii) There are 12 cards bearing numbers less than 14 *i.e.* numbers 2, 3, 4, 5, ..., 13

$\therefore$ Favourable number of elementary events = 12

Hence, required probability $= \frac{12}{100} = \frac{3}{25}$

(iii) Those numbers from 2 to 101 which are perfect squares are 4, 9, 16, 25, 36, 49, 64, 81, 100 *i.e.* squares of 2, 3, 4, 5, ..., and 10 respectively. Therefore, there are 9 cards marked with the numbers which are perfect squares.

$\therefore$ Favourable number of elementary events = 9

Hence, P (Getting a card marked with a number which is a perfect square) $= \frac{9}{100}$

(iv) Prime numbers less than 20 in the numbers from 2 to 101 are 2, 3, 5, 7, 11, 13, 17 and 19. Thus, there are 8 cards marked with prime numbers less than 20. Out of these 8 cards one card can be chose in 8 ways.

$\therefore$ Favourable number of elementary events = 8

Hence, P (Getting a card marked with a prime number less than 20) $= \frac{8}{100} = \frac{2}{25}$

EXAMPLE 28 *There are 40 students in class X of a school of whom 25 are girls and 15 are boys. The class teacher has to select one student as a class representative. He writes the name of each student on a separate card, the cards being identical. Then she puts cards in a bag and stirs them thoroughly. She then draws one card from the bag. What is the probability that the name written on the card is the name of (i) a girl? (ii) a boy?* **[NCERT]**

SOLUTION Since there are 40 students and there is one card for each student. So, one card can be chosen out of 40 cards in 40 ways.

$\therefore$ Total number of elementary events = 40

(i) There are 25 girls and corresponding to each girl there is card of her name. Therefore, a card with the name of a girl can be chosen in 25 ways.

$\therefore$ Favourable number of elementary events = 25

Hence, P (Getting a card with the name of a girl) $= \frac{25}{40} = \frac{5}{8}$

(ii) We have,

P (Getting a card with name of a boy) = 1 − P (Getting a card with name of a girl)

$$= 1 - \frac{5}{8} = \frac{3}{8}$$

EXAMPLE 29 *A carton consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects. Jimmy, a trader, will only accept the shirts which are good, but Sujatha, another trader, will only reject the shirts which have major defects. One shirt is drawn at random from the carton. What is the probability that it is acceptable to (i) Jimmy? (ii) Sujatha?*

[NCERT]

SOLUTION One shirt is drawn at random from the carton of 100 shirts. This can be done in 100 ways

$\therefore$ Total number of elementary events = 100.

(i) Since Jimmy accepts only good shirts and the number of good shirts is 88

$\therefore$ Number of elementary events favourable to Jimmy = 88

So, probability that a shirt is acceptable to Jimmy $= \frac{88}{100} = 0.88$

(ii) Sujatha accepts good as well as shirts having minor defects. The number of such shirts is $88 + 8 = 96$

$\therefore$ Number of elementary events favourable to an event of selecting a good shirt or a shirt with minor defect is 96

Hence, probability that a shirt is acceptable to Sujatha $= \frac{96}{100} = 0.96$

EXAMPLE 30 *If a number x is chosen at random from the numbers* $-2, -1, 0, 1, 2$. *What is the probability that* $x^2 < 2$?

SOLUTION Clearly, number x can take any one of the five given values.

So, total number of elementary events = 5

We observe that $x^2 < 2$ when x takes any one of the following three values $-1, 0$ and 1.

So, favourable number of elementary events = 3

Hence, $P(x^2 < 2) = \frac{3}{5}$

EXAMPLE 31 *A number x is selected from the numbers 1, 2, 3 and then a second number y is randomly selected from the numbers 1, 4, 9. What is the probability that the product xy of the two numbers will be less than 9?*

SOLUTION Number x can be selected in three ways and corresponding to each such way there are three ways of selecting number y. Therefore, two numbers can be selected in 9 ways as listed below:

$$(1, 1), (1, 4), (1, 9), (2, 1), (2, 4), (2, 9), (3, 1), (3, 4), (3, 9)$$

So, total number of elementary events = 9

The product xy will be less than 9, if x, and y are chosen in one of the following ways:

$$(1, 1), (1, 4), (2, 1), (2, 4), (3, 1)$$

$\therefore$ Favourable number of elementary events = 5

Hence, required probability $= \frac{5}{9}$

LEVEL-3

EXAMPLE 32 *On the disc shown below, a player spins the arrow twice. The fraction* $\frac{a}{b}$ *is formed, where a is the number of the sector where the arrow stops after the first spin and b is the number of sector where the arrow stops after the second spin. On every spin each of the numbered sector has an equal probability of being the sector on which the arrow stops. What is the probability that the fraction* $\frac{a}{b}$ *is greater than 1?* **[CBSE 2016]**

SOLUTION Since the arrow can stop in any one of the six sectors. So, a and b both can assume values from 1 to 6. Thus, the ordered pair (a,b) can assume the following values:

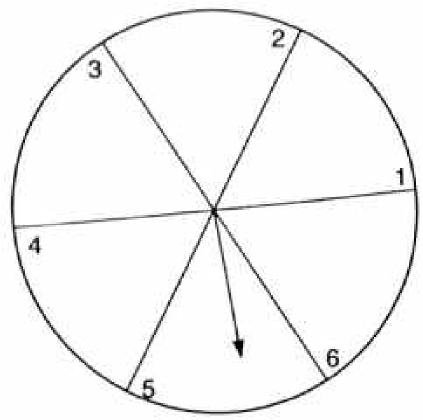

Fig. 16.2

15 ordere pairs (a, b) where $a > b$

$\therefore \quad \frac{a}{b} > 1$

$$\left\{\begin{array}{cccccc}(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6)\\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6)\\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6)\\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6)\\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6)\\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6)\end{array}\right\}$$

15 ordere pairs (a, b) where $a < b$

$\therefore \quad \frac{a}{b} < 1$

Clearly, there are 36 elementary events out of which we have 6 elementary events (a, b) in the diagonal for which $a = b$ and 15 elementary events (a, b) below the diagonal for which $a > b$.

$\therefore$ Favourable number of elementary events = 15

Hence, required probability $= \frac{15}{36} = \frac{5}{12}$

EXERCISE 16.1

LEVEL-1

1. The probability that it will rain tomorrow is 0.85. What is the probability that it will not rain tomorrow?
2. A die is thrown. Find the probability of getting:
   (i) a prime number [NCERT] (ii) 2 or 4
   (iii) a multiple of 2 or 3 (iv) an even prime number [CBSE 2008]
   (v) a number greater than 5 [CBSE 2008] (vi) a number lying between 2 and 6 [NCERT]
3. Three coins are tossed together. Find the probability of getting:
   (i) exactly two heads (ii) at most two heads
   (iii) at least one head and one tail. (iv) no tails
4. A and B throw a pair of dice. If A throws 9, find B's chance of throwing a higher number.
5. Two unbiased dice are thrown. Find the probability that the total of the numbers on the dice is greater than 10. [CBSE 2018]
6. A card is drawn at random from a pack of 52 cards. Find the probability that the card drawn is
   (i) a black king (ii) either a black card or a king

(iii) black and a king
(iv) a jack, queen or a king
(v) neither a heart nor a king
(vi) spade or an ace
(vii) neither an ace nor a king
(viii) neither a red card nor a queen. [CBSE 2005]
(ix) other than an ace
(x) a ten
(xi) a spade
(xii) a black card
(xiii) the seven of clubs
(xiv) jack
(xv) the ace of spades
(xvi) a queen
(xvii) a heart
(xviii) a red card
(xix)neither a king nor a queen [CBSE 20

7. In a lottery of 50 tickets numbered 1 to 50, one ticket is drawn. Find the probability that the drawn ticket bears a prime number.
8. An urn contains 10 red and 8 white balls. One ball is drawn at random. Find the probability that the ball drawn is white.
9. A bag contains 3 red balls, 5 black balls and 4 white balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is:
   (i) white? (ii) red? (iii) black? (iv) not red? [CBSE 2008]
10. What is the probability that a number selected from the numbers 1, 2, 3, ..., 15 is a multiple of 4?
11. A bag contains 6 red, 8 black and 4 white balls. A ball is drawn at random. What is the probability that ball drawn is not black?
12. A bag contains 5 white and 7 red balls. One ball is drawn at random. What is the probability that ball drawn is white?
13. Tickets numbered from 1 to 20 are mixed up and a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 7?
14. In a lottery there are 10 prizes and 25 blanks. What is the probability of getting a prize?
15. If the probability of winning a game is 0.3, what is the probability of loosing it?
16. A bag contains 5 black, 7 red and 3 white balls. A ball is drawn from the bag at random. Find the probability that the ball drawn is:
   (i) red (ii) black or white (iii) not black
17. A bag contains 4 red, 5 black and 6 white balls. A ball is drawn from the bag at random. Find the probability that the ball drawn is:
   (i) white (ii) red (iii) not black (iv) red or white [CBSE 2004]
18. One card is drawn from a well shuffled deck of 52 cards. Find the probability of getting:
   (i) a king of red suit (ii) a face card (iii) a red face card
   (iv) a queen of black suit (v) a jack of hearts (vi) a spade [NCERT]
19. Five cards—ten, jack, queen, king, and an ace of diamonds are shuffled face downwards. One card is picked at random.
   (i) What is the probability that the card is a queen?
   (ii) If a king is drawn first and put aside, what is the probability that the second card picked up is the (i) ace? (ii) king? [CBSE 2014, NCERT]
20. A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is:
   (i) red (ii) black [NCERT]
21. A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the number, 1, 2, 3, ..., 12 as shown in Fig. 16.3. What is the probability that it will point to:

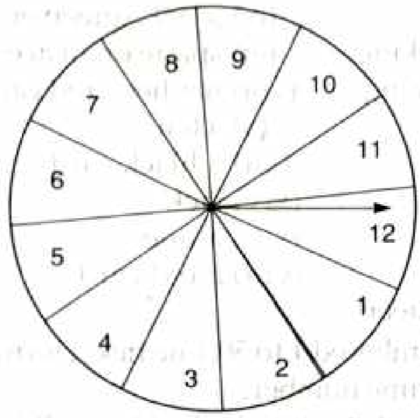

Fig. 16.3

(i) 10? (ii) an odd number?
(iii) a number which is multiple of 3? (iv) an even number?

22. In a class, there are 18 girls and 16 boys. The class teacher wants to choose one pupil for class monitor. What she does, she writes the name of each pupil on a card and puts them into a basket and mixes thoroughly. A child is asked to pick one card from the basket. What is the probability that the name written on the card is:
(i) the name of a girl (ii) the name of a boy?

23. Why is tossing a coin considered to be a fair way of deciding which team should choose ends in a game of cricket?

24. What is the probability that a number selected at random from the number 1, 2, 2, 3, 3, 3, 4, 4, 4, 4 will be their average?

25. There are 30 cards, of same size, in a bag on which numbers 1 to 30 are written. One card is taken out of the bag at random. Find the probability that the number on the selected card is not divisible by 3. [CBSE 2005]

26. A bag contains 5 red, 8 white and 7 black balls. A ball is drawn at random from the bag. Find the probability that the drawn ball is (i) red or white (ii) not black (iii) neither white nor black. [CBSE 2005]

27. Find the probability that a number selected from the number 1 to 25 is not a prime number when each of the given numbers is equally likely to be selected. [CBSE 2005]

28. A bag contains 8 red, 6 white and 4 black balls. A ball is drawn at random from the bag. Find the probability that the drawn ball is
(i) red or white (ii) not black (iii) neither white nor black.

29. Find the probability that a number selected at random from the numbers 1, 2, 3, ..., 35 is a
(i) prime number (ii) multiple of 7 (iii) a multiple of 3 or 5 [CBSE 2006C]

30. From a pack of 52 playing cards Jacks, queens, kings and aces of red colour are removed. From the remaining, a card is drawn at random. Find the probability that the card drawn is
(i) a black queen (ii) a red card (iii) a black jack
(iv) a picture card (Jacks, queens and kings are picture cards). [CBSE 2006C]

31. A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out
(i) an orange flavoured candy ?
(ii) a lemon flavoured candy? [NCERT]

32. It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday? [NCERT]

33. A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is (i) red? (ii) not red?

34. A box contains 5 red marbels, 8 white marbles and 4 green marbles. One marble is taken out of the box at random. What is the probability that the marble taken out will be (i) red? (ii) white? (iii) not green? [NCERT]

35. A lot consists of 144 ball pens of which 20 are defective and others good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

(i) She will buy it? (ii) She will not buy it? [NCERT]

36. 12 defective pens are accidently mixed with 132 good ones. It is not possible to just look at pen and tell whether or not it is defective. one pen is taken out at random from this lot. Determine the probability that the pen taken out is good one. [NCERT]

37. Five cards — the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.

(i) What is the probability that the card is the queen?

(ii) If the queen is drawn and put a side, what is the probability that the second card picked up is (a) an ace? (b) a queen?

38. Harpreet tosses two different coins simultaneously (say, one is of Re 1 and other of ₹ 2). What is the probability that he gets at least one head? [NCERT]

39. Cards marked with numbers 13, 14, 15, ..., 60 are placed in a box and mixed thoroughly. One card is drawn at random from the box. Find the probability that number on the card drawn is

(i) divisible by 5 (ii) a number is a perfect square [CBSE 2007]

40. A bag contains tickets numbered 11, 12, 13,..., 30. A ticket is taken out from the bag at random. Find the probability that the number on the drawn ticket (i) is a multiple of 7 (ii) is greater than 15 and a multiple of 5. [CBSE 2008]

41. Fill in the blanks:

(i) Probability of a sure event is ....

(ii) Probability of an impossible event is ....

(iii) The probability of an event (other than sure and impossible event) lies between ....

(iv) Every elementary event associated to a random experiment has .... probability.

(v) Probability of an event A + Probability of event 'not A'' = ...

(vi) Sum of the probabilities of each outcome in an experiment is .... [NCERT]

42. Examine each of the following statements and comment:

(i) If two coins are tossed at the same time, there are 3 possible outcomes—two heads, two tails, or one of each. Therefore, for each outcome, the probability of occurrence is $1/3$. [NCERT]

(ii) If a die is thrown once, there are two possible outcomes—an odd number or an even number. Therefore, the probability of obtaining an odd number is $1/2$ and the probability of obtaining an even number is $1/2$. [NCERT]

43. A box contains 100 red cards, 200 yellow cards and 50 blue cards. If a card is drawn at random from the box, then find the probability that it will be (i) a blue card (ii) not a yellow card (iii) neither yellow nor a blue card. [CBSE 2012]

44. A box contains cards numbered 3, 5, 7, 9, ..., 35, 37. A card is drawn at random from the box. Find the probability that the number on the drawn card is a prime number. [CBSE 2013]

45. A group consists of 12 persons, of which 3 are extremely patient, other 6 are extremely honest and rest are extremely kind. A person from the group is selected at random. Assuming that each person is equally likely to be selected, find the probability of selecting a person who is (i) extremely patient (ii) extremely kind or honest. Which of the above you prefer more. [CBSE 2013]

46. Cards numbered 1 to 30 are put in a bag. A card is drawn at random from this bag. Find the probability that the number on the drawn card is
   (i) not divisible by 3 (ii) a prime number greater than 7
   (iii) not a perfect square number. [CBSE 2014]

47. A piggy bank contains hundred 50 paise coins, fifty ₹ 1 coins, twenty ₹ 2 coins and ten ₹ 5 coins. If it is equally likely that one of the coins will fall out when the bank is turned upside down, find the probability that the coin which fell
   (i) will be a 50 paise win (ii) will be of value more than ₹ 1
   (iii) will be of value less than ₹ 5 (iv) will be a ₹ 1 or ₹ 2 coin [CBSE 2014]

48. A bag contains cards numbered from 1 to 49. A card is drawn from the bag at random, after mixing the card thoroughly. Find the probability that the number on the drawn card is
   (i) an odd number (ii) a multiple of 5
   (iii) a perfect square (iv) an even prime number. [CBSE 2014]

49. A box contains 20 cards numbered from 1 to 20. A card is drawn at random from the box. Find the probability that the number on the drawn card is
   (i) divisible by 2 or 3 (ii) a prime number [CBSE 2015]

LEVEL-2

50. In a simultaneous throw of a pair of dice, find the probability of getting:
   (i) 8 as the sum (ii) a doublet
   (iii) a doublet of prime numbers (iv) a doublet of odd numbers [CBSE 2017]
   (v) a sum greater than 9 (vi) an even number on first
   (vii) an even number on one and a multiple of 3 on the other
   (viii) neither 9 nor 11 as the sum of the numbers on the faces
   (ix) a sum less than 6 (x) a sum less than 7
   (xi) a sum more than 7 (xii) at least once
   (xiii) a number other than 5 on any dice. [CBSE 2014, 2015]
   (xiv) even number on each die [CBSE 2014, 2015]
   (xv) 5 as the sum [CBSE 2015]
   (xvi) 2 will come up at least once [CBSE 2015]
   (xvii) 2 will not come either time

51. What is the probability that an ordinary year has 53 Sundays?

52. What is the probability that a leap year has 53 Tuesdays and 53 Mondays? [CBSE 2015]

53. A black die and a white die are thrown at the same time. Write all the possible outcomes. What is the probability that:
   (i) the sum of the two numbers that turn up is 8? [CBSE 2014]
   (ii) of obtaining a total of 6?

(iii) of obtaining a total of 10? [CBSE 2014]
(iv) of obtaining the same number on both dice?
(v) of obtaining a total more than 9?
(vi) that the sum of the two numbers appearing on the top of the dice is 13?
(vii) that the sum of the numbers appearing on the top of the dice is less than or equal to 12?
(viii) that the product of numbers appearing on the top of the dice is less than 9. [CBSE 2014]
(ix) that the difference of the numbers appearing on the top of the two dice is 2. [CBSE 2014]
(x) that the numbers obtained have a product less than 16. [CBSE 2017]

54. A bag contains cards which are numberered from 2 to 90. A card is drawn at random from the bag. Find the probability that it bears
(i) a two digit number (ii) a number which is a perfect square [CBSE 2010]

55. The faces of a red cube and a yellow cube are numbered from 1 to 6. Both cubes are rolled. What is the probability that the top face of each cube will have the same number?

56. The probability of selecting a green marble at random from a jar that contains only green, white and yellow marbles is $1/4$. The probability of selecting a white marble at random from the same jar is $1/3$. If this jar contains 10 yellow marbles. What is the total number of marbles in the jar?

57. (i) A lot of 20 bulbs contain 4 defective ones. One bulb is drawn at random from the lot. What is the probability that this bulb is defective?
(ii) Suppose the bulb drawn in (i) is not defective and not replaced. Now bulb is drawn at random from the rest. What is the probability that this bulb is not defective? [NCERT]

58. A box contains 90 discs which are numbered from 1 to 90. If one discs is drawn at random from the box, find the probability that it bears (i) a two digit number (ii) a perfect square number (iii) a number divisible by 5. [CBSE 2017, NCERT]

59. Two dice, one blue and one grey, are thrown at the same time. Complete the following table:

Event: 'Sum on two dice'	2	3	4	5	6	7	8	9	10	11	12
Probability											

From the above table a student argues that there are 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Therefore, each of them has a probability $\frac{1}{11}$. Do you agree with this argument? [NCERT]

60. A bag contains 6 red balls and some blue balls. If the probability of drawing a blue ball from the bag is twice that of a red ball, find the number of blue balls in the bag. [CBSE 2007]

61. The king, queen and jack of clubs are removed from a deck of 52 playing cards and the remaining cards are shuffled. A card is drawn from the remaining cards. Find the probability of getting
(i) a card of heart (ii) a queen (iii) a card of clubs
(iv) a face card (v) a queen of diamond. [CBSE 2009, 2017]

62. Two dice are thrown simultaneously. What is the probability that:
(i) 5 will not come up on either of them? (ii) 5 will come up on at least one?
(iii) 5 will come up at both dice? [CBSE 2009]

63. A number is selected at random from first 50 natural numbers. Find the probability that it is a multiple of 3 and 4.

64. A dice is rolled twice. Find the probability that

(i) 5 will not come up either time. (ii) 5 will come up exactly one time.

[CBSE 2014]

65. All the black face cards are removed from a pack of 52 cards. The remaining cards are well shuffled and then a card is drawn at random. Find the probability of getting a

(i) face card (ii) red card (iii) black card (iv) king

66. Cards numbered from 11 to 60 are kept in a box. If a card is drawn at random from the box, find the probability that the number on the drawn cards is

(i) an odd number (ii) a perfect square number

(iii) divisible by 5 (iv) a prime number less than 20 [CBSE 2014]

67. All kings and queens are removed from a pack of 52 cards. The remaining cards are well-shuffled and then a card is randomly drawn from it. Find the probability that this card is (i) a red face card (ii) a black card. [CBSE 2014]

68. All jacks, queens and kings are removed from a pack of 52 cards. The remaining cards are well-shuffled and then a card is randomly drawn from it. Find the probability that this cards is (i) a black face card (ii) a red card [CBSE 2014]

69. Red queens and black jacks are removed from a pack of 52 playing cards. A cards is drawn at random from the remaining cards, after reshuffling them. Find the probability that the card drawn is

(i) a king (ii) of red colour (iii) a face card (iv) a queen [CBSE 2014]

70. In a bag there are 44 identical cards with figure of circle or square on them. There are 24 circles, of which 9 are blue and rest are green and 20 squares of which 11 are blue and rest are green. One card is drawn from the bag at random. Find the probability that it has the figure of (i) square (ii) green colour, (iii) blue circle and (iv) green square.

[CBSE 2015]

71. All red face cards are removed from a pack of playing cards. The remaining cards are well shuffled and then a card is drawn at random from them. Find the probability that the drawn card is (i) a red card (ii) a face card and (iii) a card of clubs.

[CBSE 2015]

LEVEL-3

72.. Two customers are visiting a particular shop in the same week (Monday to Saturday). Each is equally likely to visit the shop on any one day as on another. What is the probability that both will visit the shop on:

(i) the same day? (ii) different days? (iii) consecutive days? [NCERT]

ANSWERS

1. 0.15 2. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{2}{3}$ (iv) $\frac{1}{6}$ (v) $\frac{1}{6}$ (vi) $\frac{1}{2}$

3. (i) $\frac{3}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{3}{4}$ (iv) $\frac{1}{8}$ 4. $\frac{1}{6}$ 5. $\frac{1}{12}$

6. (i) $\frac{1}{26}$ (ii) $\frac{7}{13}$ (iii) $\frac{1}{26}$ (iv) $\frac{3}{13}$ (v) $\frac{9}{13}$ (vi) $\frac{9}{13}$
(vii) $\frac{11}{13}$ (viii) $\frac{7}{13}$ (ix) $\frac{12}{13}$ (x) $\frac{1}{13}$ (xi) $\frac{1}{4}$ (xii) $\frac{1}{2}$
(xiii) $\frac{1}{52}$ (xiv) $\frac{1}{13}$ (xv) $\frac{1}{52}$ (xvi) $\frac{1}{13}$ (xvii) $\frac{1}{4}$ (xviii) $\frac{1}{2}$
(xix) $\frac{11}{13}$ 7. $\frac{3}{10}$ 8. $\frac{4}{9}$ 9. (i) $\frac{1}{3}$ (ii) $\frac{1}{4}$ (iii) $\frac{5}{12}$
(iv) $\frac{3}{4}$ 10. $\frac{1}{5}$ 11. $\frac{5}{9}$ 12. $\frac{5}{12}$ 13. $\frac{2}{5}$ 14. $\frac{2}{5}$

15. 0.7 16. (i) $\frac{7}{15}$ (ii) $\frac{8}{15}$ (iii) $\frac{2}{3}$ 17. (i) $\frac{2}{5}$ (ii) $\frac{4}{15}$
(iii) $\frac{2}{3}$ (iv) $\frac{2}{3}$ 18. (i) $\frac{1}{26}$ (ii) $\frac{3}{13}$ (iii) $\frac{3}{26}$ (iv) $\frac{1}{26}$
(v) $\frac{1}{52}$ (vi) $\frac{1}{4}$ 19. (i) $\frac{1}{5}$ (ii) $\frac{1}{4}$, 0 20. (i) $\frac{3}{8}$ (ii) $\frac{5}{8}$

21. (i) $\frac{1}{12}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{3}$ (iv) $\frac{1}{3}$ 22. (i) $\frac{9}{17}$ (ii) $\frac{8}{17}$

23. When we toss a coin, the outcomes head and tail are equally likely. So, the result of an individual coin toss is completely unpredictable.

24. $\frac{3}{10}$ 25. $\frac{2}{3}$ 26. (i) $\frac{13}{20}$ (ii) $\frac{13}{20}$ (iii) $\frac{1}{4}$ 27. $\frac{16}{25}$

28. (i) $\frac{7}{9}$ (ii) $\frac{7}{9}$ (iii) $\frac{4}{9}$ 29. (i) $\frac{11}{35}$ (ii) $\frac{1}{7}$ (iii) $\frac{16}{35}$

30. (i) $\frac{1}{22}$ (ii) $\frac{9}{22}$ (iii) $\frac{1}{22}$ (iv) $\frac{3}{22}$ 31. (i) 0 (ii) 1

32. 0.008 33. (i) $\frac{3}{8}$ (ii) $\frac{5}{8}$ 34. (i) $\frac{5}{17}$ (ii) $\frac{8}{17}$ (iii) $\frac{13}{17}$

35. (i) $\frac{31}{36}$ (ii) $\frac{5}{36}$ 36. $\frac{11}{12}$ 37. (i) $\frac{1}{5}$ (ii) (a) $\frac{3}{4}$ (b) 0 38. $\frac{3}{4}$

39. (i) $\frac{5}{24}$ (ii) $\frac{1}{16}$ 40. (i) $\frac{3}{20}$ (ii) $\frac{3}{20}$

41. (i) 1 (ii) 0 (iii) 0 and 1 (iv) equal (v) 1 (vi) 1

42. (i) Incorrect:
We can classify the outcomes like this but in such a case they are not equally likely, because the event 'one of each' is twice as likely to occur as the remaining two.
(ii) Correct: The two outcomes considered in the question are equally likely.

43. (i) $\frac{1}{7}$ (ii) $\frac{3}{7}$ (iii) $\frac{2}{7}$ 44. $\frac{5}{9}$ 45. (i) $\frac{1}{4}$ (ii) $\frac{3}{4}$

46. (i) $\frac{2}{3}$ (ii) $\frac{1}{5}$ (iii) $\frac{5}{6}$ 47. (i) $\frac{5}{9}$ (ii) $\frac{1}{6}$ (iii) $\frac{17}{18}$
(iv) $\frac{7}{18}$ 48. (i) $\frac{25}{49}$ (ii) $\frac{9}{49}$ (iii) $\frac{1}{7}$ (iv) $\frac{1}{49}$

49. (i) $\frac{4}{5}$ (ii) $\frac{2}{5}$ 50. (i) $\frac{5}{36}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{12}$

(v) $\frac{1}{6}$ (vi) $\frac{1}{2}$ (vii) $\frac{11}{36}$ (viii) $\frac{5}{6}$ (ix) $\frac{5}{18}$ (x) $\frac{5}{12}$

(xi) $\frac{5}{12}$ (xii) $\frac{11}{36}$ (xiii) $\frac{25}{36}$ (xiv) $\frac{1}{4}$ (xv) $\frac{1}{9}$ (xvi) $\frac{11}{36}$

(xvii) $\frac{25}{36}$ 51. $\frac{1}{7}$ 52. $\frac{1}{7}$ 53. (i) $\frac{5}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{12}$

(iv) $\frac{1}{6}$ (v) $\frac{1}{6}$ (vi) 0 (vii) 1 (viii) $\frac{4}{9}$ (ix) $\frac{2}{9}$ (x) $\frac{25}{36}$

54. (i) $\frac{81}{89}$ (ii) $\frac{8}{89}$ (iii) $\frac{13}{17}$ 55. $\frac{1}{6}$ 56. 24

57. (i) $\frac{1}{5}$ (ii) $\frac{15}{19}$ 58. (i) $\frac{9}{10}$ (ii) $\frac{1}{10}$ (iii) $\frac{1}{5}$

59.

Event: 'Sum on two dice'	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

No, the outcomes are not equally likely.

60. 12 61. (i) $\frac{13}{49}$ (ii) $\frac{3}{49}$ (iii) $\frac{10}{49}$ (iv) $\frac{9}{49}$ (v) $\frac{1}{49}$

62. (i) $\frac{25}{36}$ (ii) $\frac{11}{36}$ (iii) $\frac{1}{36}$ 63. $\frac{2}{25}$ 64. (i) $\frac{11}{36}$ (ii) $\frac{5}{18}$

65. (i) $\frac{3}{23}$ (ii) $\frac{13}{23}$ (iii) $\frac{10}{23}$ (iv) $\frac{1}{23}$

66. (i) $\frac{1}{2}$ (ii) $\frac{2}{25}$ (iii) $\frac{1}{5}$ (iv) $\frac{1}{10}$ 67. (i) $\frac{1}{22}$ (ii) $\frac{1}{2}$

68. (i) 0 (ii) $\frac{1}{2}$ 69. (i) $\frac{1}{12}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{6}$ (iv) $\frac{1}{24}$

70. (i) $\frac{5}{11}$ (ii) $\frac{6}{11}$ (iii) $\frac{9}{44}$ (iv) $\frac{9}{44}$

71. (i) $\frac{10}{23}$ (ii) $\frac{3}{23}$ (iii) $\frac{13}{46}$ 72.. (i) $\frac{1}{6}$ (ii) $\frac{5}{6}$ (iii) $\frac{5}{36}$

HINTS TO SELECTED PROBLEMS

32. Required probability = 1 – Probability that they have the same birthday

$$= 1 - 0.992 = 0.008$$

33. (i) P (Getting red ball) $= \frac{3}{8}$ (ii) P (Not Getting red ball) $= 1 - \frac{3}{6} = \frac{5}{8}$

34. (i) P (Getting a red marble) $= \frac{5}{17}$ (ii) P (Getting a white marble) $= \frac{8}{17}$

(iii) P (Not getting a green marble $= \frac{13}{17}$

35. (i) Probability that Nuri buys the pen

$= $ Probability of selecting a good pen $= \frac{124}{144} = \frac{31}{36}$

(ii) P (Nuri will not buy pen) $= P$ (Selecting a defective pen) $= \frac{20}{144} = \frac{5}{36}$

36. Required probability $= \frac{132}{144}$

38. All possible outcomes are HH, HT, TH, TT

$\therefore$ Total number of elementary events = 4

At least one head is obtained if she gets any one of the following as an outcome: HH, HT, TH.

$\therefore$ Required probability $= \frac{3}{4}$

57. (i) Required probability $= \frac{4}{20}$, (ii) Required probability $= \frac{15}{19}$

58. (i) There are 81 two-digit numbers from 1 to 90.

$\therefore$ P(Getting a disc bearing a two digit number) $= \frac{81}{90} = \frac{9}{10}$

(ii) There are 9 perfect squares from 1 to 90, namely, 1, 4, 9, 16, 25, 36, ..., 81.

$\therefore$ P(Getting a disc bearing a perfect square) $= \frac{9}{90} = \frac{1}{10}$

(iii) P (Getting a disc bearing a number divisible by 5) $= \frac{18}{90} = \frac{1}{5}$

72. Two customers can visit the shop on two days in $6 \times 6 = 36$ ways.

$\therefore$ Total number of elementary events = 36.

(i) Two customers can visit the shop on the same day in one of the following ways.
Monday, Tuesday, Wednesday, Thursday, Friday, Saturday

$\therefore$ Favourable number of ways = 6

So, required probability $= \frac{6}{36} = \frac{1}{6}$

(ii) Required probability $= 1 - \frac{1}{6} = \frac{5}{6}$

(iii) Two customers can visit the shop on consecutive days in the following ways:
(Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday)

$\therefore$ Required probability $= \frac{5}{36}$

16.4 GEOMETRIC PROBABILITY

The definition of the probability of occurence of an event fails if the total number of outcomes (elementary events) of a trial in a random experiment is infinite. For example, if it is asked to find the probability that a point selected at random in a given region will lie in a specified part of that region, then the definition given in section 16.3 is unable to answer this. In such cases, the definition of probability is modified and extended to what is called geometric probability. In such cases, the formula for finding the probability p of occurrence of an event is given by

$$p = \frac{\text{Measure of the specified part of the region}}{\text{Measure of the whole region}}$$

Here 'Measure' means length, area or volume of the region or space.
Following examples will illustrate the use of the above definition.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A missing helicopter is reported to have crashed somewhere in the rectangular region in Fig. 16.4. What is the probability that it is crashed inside the lake shown in the figure?* [NCERT]

SOLUTION We have,

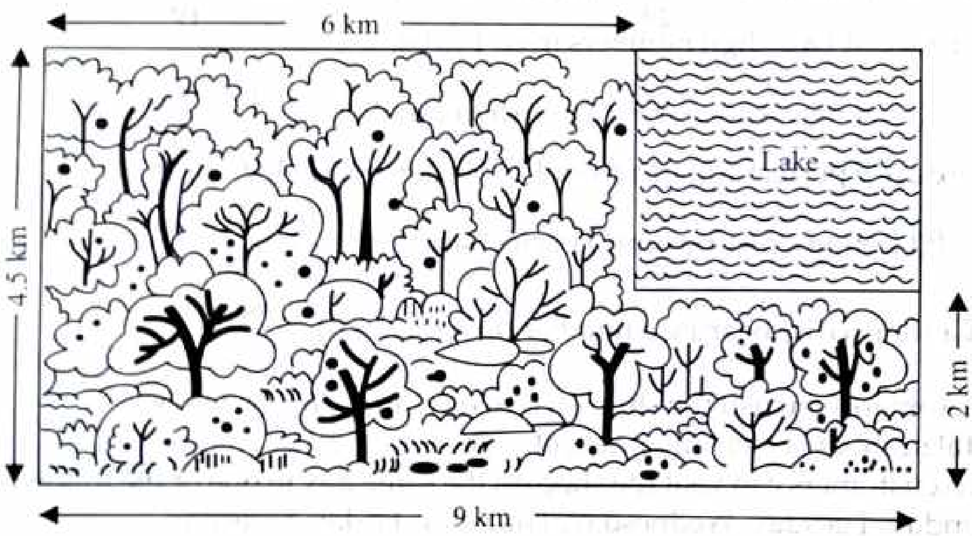

Fig. 16.4

Area of the entire region where the helicopter can crash $= (4.5 \times 9)\ \text{km}^2 = 40.5\ \text{km}^2$

Area of the lake $= (2.5 \times 3)\ \text{km}^2 = 7.5\ \text{km}^2$

$\therefore$ Probability that the helicopter crashed inside the lake $= \frac{7.5}{40.5} = \frac{75}{405} = \frac{5}{27}$

LEVEL-2

EXAMPLE 2 *In a musical chair game, the person playing the music has been advised to stop playing the music at any time within 2 minutes after she starts playing. What is the probability that the music will stop within the first half minute after starting?* [NCERT]

SOLUTION Here the possible outcomes are all the numbers between 0 and 2. This is the portion of the number line from 0 to 2 as shown in Fig. 16.5.

Let A be the event that 'the music is stopped within the first half minute'. Then, Outcomes favourable to the event A are all points on the number line from O to Q i.e. from 0 to 1/2

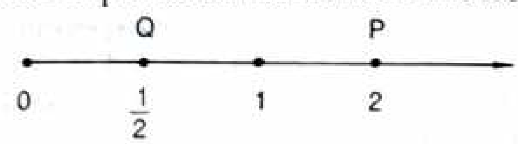

Fig. 16.5

The total number of outcomes are the points on the numberline from O to P i.e. from 0 to 2.

$$\therefore \qquad P(A) = \frac{\text{Length } OQ}{\text{Length } OP} = \frac{1/2}{2} = \frac{1}{4}$$

EXAMPLE 3 *In Fig. 16.6, a dart is thrown and lands in the interior of the circle. What is the probability that the dart will land in the shaded region?*

SOLUTION We have,

$$AB = CD = 8 \text{ and } AD = BC = 6$$

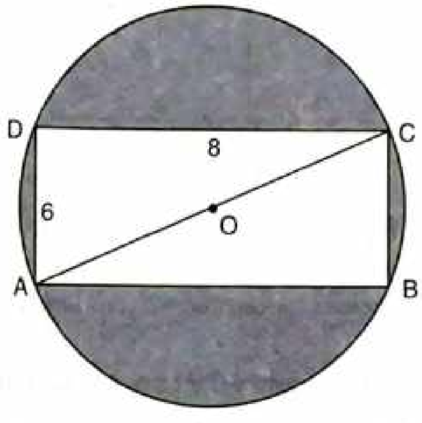

Fig. 16.6

Using Pythagoras theorem in ΔABC, we have

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \qquad AC^2 = 8^2 + 6^2 = 100$$

$$\Rightarrow \qquad AC = 10$$

$$\Rightarrow \qquad OA = OC = 5 \qquad [\because O \text{ is the mid-point of } AC]$$

$$\therefore \qquad \text{Area of the circle} = \pi (OA)^2 = 25\pi \text{ sq. units} \qquad [\because \text{Area} = \pi r^2]$$

$$\text{Area of rectangle } ABCD = AB \times BC = 8 \times 6 = 48 \text{ sq. units}$$

$$\therefore \qquad \text{Area of shaded region} = \text{Area of the circle} - \text{Area of rectangle } ABCD$$

$$\Rightarrow \qquad \text{Area of shaded region} = 25\pi - 48 \text{ sq. units}$$

Hence, $$P(\text{Dart lands in the shaded region}) = \frac{\text{Area of shaded region}}{\text{Area of circle}} = \frac{25\pi - 48}{25\pi}$$

LEVEL-3

EXAMPLE 4 *Figure 16.7, shows the top view of an open square box that is divided into 6 compartments with walls of equal height. Each of the rectangles D, E, F has twice the area of each of the squares A, B and C. When a marble is dropped into the box at random, it falls into one of the compartments. What is the probability that it will fall into compartment F?*

SOLUTION Let x square units be the area of the upper face of each of the compartments A, B and C. Then, area of the upper face of each compartment D, E and F is $2x$ sq. units.

$\therefore$ Area of the square box $= (x + x + x + 2x + 2x + 2x)$ sq. units $= 9x$ sq. units.

$$P\text{ (Marble falls in compartment } F) = \frac{\text{Area of compartment } F}{\text{Area of square box}} = \frac{2x}{9x} = \frac{2}{9}$$

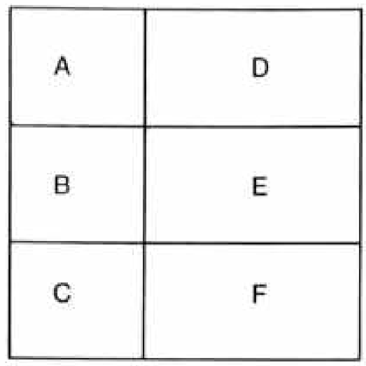

Fig. 16.7

EXAMPLE 5 *A square dart board is placed in the first quadrant from $x = 0$ to $x = 6$ and $y = 0$ to $y = 6$. A triangular region on the dart board is enclosed by the lines $y = 2$, $x = 6$ and $y = x$. Find the probability that a dart that randomly hits the dart board will land in the triangular region formed by the three lines.*

SOLUTION In Fig. 16.8, $OABC$ is the square dart board such that $OA = 6 = AB = BC$.

$\therefore$ Area of the square dart board $= (OA)^2 = 36$ square units.

ΔBED is a right triangle formed by the lines $y = 2$, $x = 6$ and $y = x$.

We have, $DE = 4$ and $BE = 4$

$\therefore$ Area of triangle $BED = \frac{1}{2}(DE \times BE) = \frac{1}{2}(4 \times 4) = 8$ sq. units.

Hence, $$P\text{ (Dart lands in the triangular region)} = \frac{\text{Area of } \Delta BED}{\text{Area of square } OABC} = \frac{8}{36} = \frac{2}{9}$$

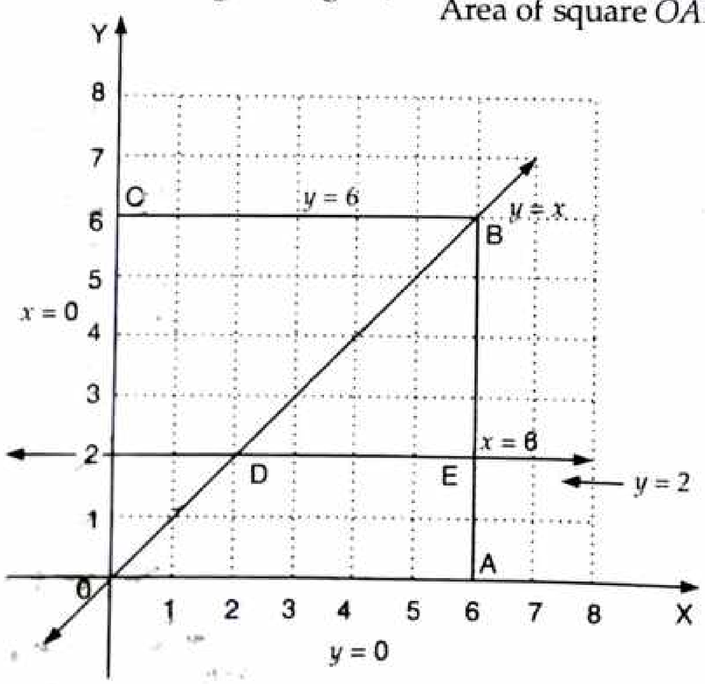

Fig. 16.8

EXERCISE 16.2

LEVEL-2

1. Suppose you drop a tie at random on the rectangular region shown in Fig. 16.9. What is the probability that it will land inside the circle with diameter 1 m? [NCERT]

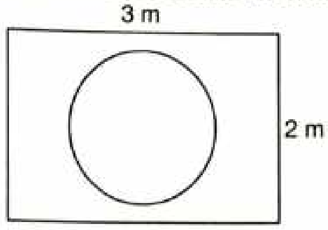

Fig. 16.9

LEVEL-3

2. In the accompanying diagram a fair spinner is placed at the centre O of the circle. Diameter AOB and radius OC divide the circle into three rigions labelled X, Y and Z. If $\angle BOC = 45°$. What is the probability that the spinner will land in the region X? (See Fig. 16.10).

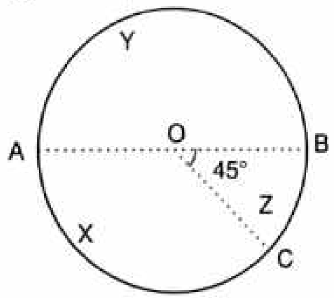

Fig. 16.10

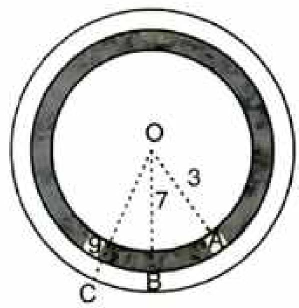

Fig. 16.11

3. A target shown in Fig. 16.11 consists of three concentric circles of radii 3, 7 and 9 cm respectively. A dart is thrown and lands on the target. What is the probability that the dart will land on the shaded region?

4. In Fig. 16.12, points A, B, C and D are the centres of four circles that each have a radius of length one unit. If a point is selected at random from the interior of square $ABCD$. What is the probability that the point will be chosen from the shaded region?

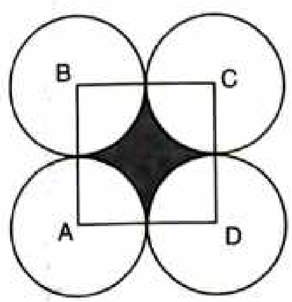

Fig. 16.12

5. In the Fig. 16.13, $JKLM$ is a square with sides of length 6 units. Points A and B are the mid-points of sides KL and LM respectively. If a point is selected at random from the interior of the square. What is the probability that the point will be chosen from the interior of ΔJAB?

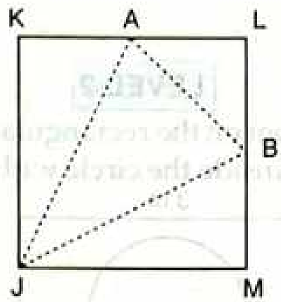

Fig. 16.13

6. In the Fig. 16.14, a square dart board is shown. The length of a side of the larger square is 1.5 times the length of a side of the smaller square. If a dart is thrown and lands on the larger square. What is the probability that it will land in the interior of the smaller square?

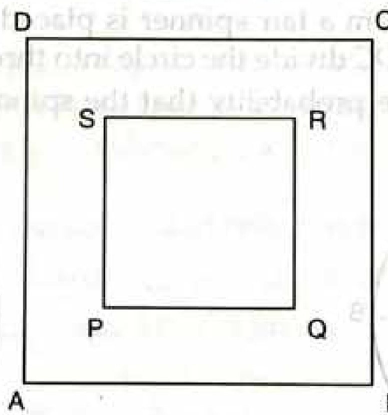

Fig. 16.14

ANSWERS

1. $\frac{\pi}{24}$ 2. $\frac{3}{8}$ 3. $\frac{40}{81}$

4. $\left(1-\frac{\pi}{4}\right)$ 5. $\frac{3}{8}$ 6. $\frac{4}{9}$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

1. Cards each marked with one of the numbers 4, 5, 6, ..., 20 are placed in a box and mixed thoroughly. One card is drawn at random from the box. What is the probability of getting an even number?
2. One card is drawn from a well shuffled deck of 52 playing cards. What is the probability of getting a non-face card?
3. A bag contains 5 red, 8 green and 7 white balls. One ball is drawn at random from the bag. What is the probability of getting a white ball or a green ball?
4. A die is thrown once. What is the probability of getting a prime number?
5. A die is thrown once. What is the probability of getting a number lying between 2 and 6?
6. A die is thrown once. What is the probability of getting an odd number?

7. If $\overline{E}$ denote the complement or negation of an even E, what is the value of $P(E) + P(\overline{E})$?
8. One card is drawn at random from a well shuffled deck of 52 cards. What is the probability of getting an ace?
9. Two coins are tossed simultaneously. What is the probability of getting at least one head?
10. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn bears a number which is a multiple of 3?
11. From a well shuffled pack of cards, a card is drawn at random. Find the probability of getting a black queen. [CBSE 2008]
12. A die is thrown once. Find the probability of getting a number less than 3. [CBSE 2008]
13. Two coins are tossed simultaneously. Find the probability of getting exactly one head. [CBSE 2009]
14. A die is thrown once. What is the probability of getting a number greater than 4? [CBSE 2010]
15. What is the probability that a number selected at random from the numbers 3, 4, 5, ..., 9 is a multiple of 4? [CBSE 2010]
16. A letter of English alphabet is chosen at random. Determine the probability that the chosen letter is a consonant. [CBSE 2015]
17. A bag contains 3 red and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is not red? [CBSE 2017]
18. A number is chosen at random from the numbers, $-3, -2, -1, 0, 1, 2, 3$. What will be the probability that the square of this number is less than or equal to 1? [CBSE 2017]

ANSWERS

1. $\frac{9}{17}$ 2. $\frac{10}{13}$ 3. $\frac{3}{4}$ 4. $\frac{1}{2}$ 5. $\frac{1}{2}$ 6. $\frac{1}{2}$ 7. 1

8. $\frac{1}{13}$ 9. $\frac{3}{4}$ 10. $\frac{3}{10}$ 11. $\frac{1}{26}$ 12. $\frac{1}{3}$ 13. $\frac{1}{2}$ 14. $\frac{1}{3}$

15. $\frac{2}{7}$ 16. $\frac{21}{26}$ 17. $\frac{5}{8}$ 18. $\frac{3}{7}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If a digit is chosen at random from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, then the probability that it is odd, is

   (a) $\frac{4}{9}$ (b) $\frac{5}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{3}$

2. In Q. No. 1, the probability that the digit is even, is

   (a) $\frac{4}{9}$ (b) $\frac{5}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{3}$

3. In Q. No. 1, the probability that the digit is a multiple of 3 is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$

4. If three coins are tossed simultaneously, then the probability of getting at least two heads, is

(a) $\frac{1}{4}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

5. In a single throw of a die, the probability of getting a multiple of 3 is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{2}{3}$

6. The probability of guessing the correct answer to a certain test questions is $\frac{x}{12}$. If the probability of not guessing the correct answer to this question is $\frac{2}{3}$, then $x =$

(a) 2 (b) 3 (c) 4 (d) 6

7. A bag contains three green marbles, four blue marbles, and two orange marbles. If a marble is picked at random, then the probability that it is not an orange marble is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{4}{9}$ (d) $\frac{7}{9}$

8. A number is selected at random from the numbers 3, 5, 5, 7, 7, 7, 9, 9, 9, 9
The probability that the selected number is their average is

(a) $\frac{1}{10}$ (b) $\frac{3}{10}$ (c) $\frac{7}{10}$ (d) $\frac{9}{10}$

9. The probability of throwing a number greater than 2 with a fair dice is

(a) $\frac{3}{5}$ (b) $\frac{2}{5}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

10. A card is accidently dropped from a pack of 52 playing cards. The probability that it is an ace is

(a) $\frac{1}{4}$ (b) $\frac{1}{13}$ (c) $\frac{1}{52}$ (d) $\frac{12}{13}$

11. A number is selected from numbers 1 to 25. The probability that it is prime is

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{5}{6}$

12. Which of the following cannot be the probability of an event?

(a) $\frac{2}{3}$ (b) -1.5 (c) 15% (d) 0.7

13. If $P(E) = 0.05$, then $P(\text{not } E) =$

(a) -0.05 (b) 0.5 (c) 0.9 (d) 0.95

14. Which of the following cannot be the probability of occurrence of an event?

(a) 0.2 (b) 0.4 (c) 0.8 (d) 1.6

15. The probability of a certain event is

(a) 0 (b) 1 (c) 1/2 (d) no existent

16. The probability of an impossible event is

(a) 0 (b) 1 (c) 1/2 (d) non-existent

17. Aarushi sold 100 lottery tickets in which 5 tickets carry prizes. If Priya purchased a ticket, what is the probability of Priya winning a prize?

(a) $\frac{19}{20}$ (b) $\frac{1}{25}$ (c) $\frac{1}{20}$ (d) $\frac{17}{20}$

18. A number is selected from first 50 natural numbers. What is the probability that it is a multiple of 3 or 5?

(a) $\frac{13}{25}$ (b) $\frac{21}{50}$ (c) $\frac{12}{25}$ (d) $\frac{23}{50}$

19. A month is selected at random in a year. The probability that it is March or October, is

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{3}{4}$ (d) None of these

20. From the letters of the word "MOBILE", a letter is selected. The probability that the letter is a vowel, is

(a) $\frac{1}{3}$ (b) $\frac{3}{7}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$

21. A die is thrown once. The probability of getting a prime number is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$ **[CBSE 2013]**

22. The probability of getting an even number, when a die is thrown once is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{5}{6}$ **[CBSE 2013]**

23. A box contains 90 discs, numbered from 1 to 90. If one disc is drawn at random from the box, the probability that it bears a prime number less than 23, is

(a) $\frac{7}{90}$ (b) $\frac{10}{90}$ (c) $\frac{4}{45}$ (d) $\frac{9}{89}$ **[CBSE 2013]**

24. The probability that a number selected at random from the numbers 1, 2, 3,...,15 is a multiple of 4, is

(a) $\frac{4}{15}$ (b) $\frac{2}{15}$ (c) $\frac{1}{5}$ (d) $\frac{1}{3}$ **[CBSE 2014]**

25. Two different coins are tossed simultaneously. The probability of getting at least one head is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{3}{4}$ (d) $\frac{7}{8}$ **[CBSE 2014]**

26. If two different dice are rolled together, the probability of getting an even number on both dice, is

(a) $\frac{1}{36}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$ **[CBSE 2014]**

27. A number is selected at random from the numbers 1 to 30. The probability that it is a prime number is

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{11}{30}$ **[CBSE 2014]**

28. A card is drawn at random from a pack of 52 cards. The probability that the drawn card is not an ace is

(a) $\frac{1}{13}$ (b) $\frac{9}{13}$ (c) $\frac{4}{13}$ (d) $\frac{12}{13}$ [CBSE 2014]

LEVEL-2

29. A number x is chosen at random from the numbers –3, –2, –1, 0, 1, 2, 3 the probability that $|x| < 2$ is

(a) $\frac{5}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{1}{7}$

30. If a number x is chosen from the numbers 1, 2, 3, and a number y is selected from the numbers 1, 4, 9. Then, $P(xy < 9)$

(a) $\frac{7}{9}$ (b) $\frac{5}{9}$ (c) $\frac{2}{3}$ (d) $\frac{1}{9}$

31. The probability that a non-leap year has 53 Sundays, is

(a) $\frac{2}{7}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) $\frac{1}{7}$

32. In a single throw of a pair of dice, the probability of getting the sum a perfect square is

(a) $\frac{1}{18}$ (b) $\frac{7}{36}$ (c) $\frac{1}{6}$ (d) $\frac{2}{9}$

33. What is the probability that a non-leap year has 53 Sundays?

(a) $\frac{6}{7}$ (b) $\frac{1}{7}$ (c) $\frac{5}{7}$ (d) None of these

34. Two numbers 'a' and 'b' are selected successively without replacement in that order from the integers 1 to 10. The probability that $\frac{a}{b}$ is an integer, is

(a) $\frac{17}{45}$ (b) $\frac{1}{5}$ (c) $\frac{17}{90}$ (d) $\frac{8}{45}$

35. Two dice are rolled simultaneously. The probability that they show different faces is

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{5}{6}$

36. What is the probability that a leap year has 52 Mondays?

(a) $\frac{2}{7}$ (b) $\frac{4}{7}$ (c) $\frac{5}{7}$ (d) $\frac{6}{7}$

37. If a two digit number is chosen at random, then the probability that the number chosen is a multiple of 3, is

(a) $\frac{3}{10}$ (b) $\frac{29}{100}$ (c) $\frac{1}{3}$ (d) $\frac{7}{25}$

38. Two dice are thrown together. The probability of getting the same number on both dice is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{12}$ [CBSE 2012]

39. In a family of 3 children, the probability of having at least one boy is

(a) $\frac{7}{8}$ (b) $\frac{1}{8}$ (c) $\frac{5}{8}$ (d) $\frac{3}{4}$ [CBSE 2014]

40. A bag contains cards numbered from 1 to 25. A card is drawn at random from the bag. The probability that the number on this card is divisible by both 2 and 3 is

(a) $\frac{1}{5}$ (b) $\frac{3}{25}$ (c) $\frac{4}{25}$ (d) $\frac{2}{25}$ [CBSE 2014]

ANSWERS

1. (b)	2. (a)	3. (a)	4. (c)	5. (b)	6. (c)
7. (d)	8. (b)	9. (c)	10. (b)	11. (c)	12. (b)
13. (d)	14. (d)	15. (b)	16. (a)	17. (c)	18. (d)
19. (b)	20. (d)	21. (c)	22. (a)	23. (c)	24. (c)
25. (c)	26. (d)	27. (c)	28. (d)	29. (c)	30. (b)
31. (d)	32. (b)	33. (b)	34. (c)	35. (d)	36. (c)
37. (c)	38. (c)	39. (a)	40. (c)		

SUMMARY

1. In the experimental approach to probability, we find the probability of the occurrence of an event by actually performing the experiment a number of times and adequate recording of the happening of event.
2. In the theoretical approach to probability, we try to predict what will happen without actually performing the experiment.
3. An outcome of a random experiment is called an elementary event.
4. An event associated to a random experiment is a compound event if it is obtained by combining two or more elementary events associated to the random experiment.
5. An event A associated to a random experiment is said to occur if any one of the elementary events associated to the event A is an outcome.
6. An elementary event is said to be favourable to a compound event A, if it satisfies the definition of the compound event.
7. If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening occurrence of event A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$

   i.e., $P(A) = \frac{\text{Favourable number of elementary events}}{\text{Total number of elementary events}}$
8. For any event A associated to a random experiment, we have

   (i) $0 \leq P(A) \leq 1$ (ii) $P(\overline{A}) = 1 - P(A)$
9. The probability of a sure event is 1.
10. The probability of an impossible event is 0.
11. The sum of the probabilities of all the outcomes (elementary events) of an experiment is 1.